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# ON NONAUTONOMOUS DIFFERENTIAL-DIFFERENCE AKP, BKP AND CKP EQUATIONS

WEI FU AND FRANK W. NIJHOFF

ABSTRACT. Based on the direct linearisation framework of the discrete Kadomtsev–Petviashvili-type equations presented in [Proc. R. Soc. A, **473** (2017) 20160915], six novel nonautonomous differential-difference equations are established, including three in the AKP class, two in the BKP class and one in the CKP class. In particular, one in the BKP class and the one in the CKP class are both in (2+2)-dimensional form. All the six models are integrable in the sense of having the same linear integral equation representations as those of their associated discrete Kadomtsev–Petviashvili-type equations, which guarantees the existence of soliton-type solutions and the multi-dimensional consistency of these new equations from the viewpoint of the direct linearisation.

## 1. INTRODUCTION

The study of discrete integrable systems has been becoming one of the most prominent branches in the theory of integrable systems in the past two decades, resulting in the establishment of many novel concepts and theories in modern mathematics, see e.g. [13]. There are several methods of constructing integrable discretisation of nonlinear differential equations. Among those, a very effective one is to construct the Bäcklund transform (BT) and the superposition formula of a partial differential equation (PDE), and consider them as the associated differential-difference equation (DΔE) and partial difference equation (PΔE), respectively (see [21] and [28]). Compared with integrable PDEs, the associated integrable PΔEs seem to possess a much richer structure, which reflects in the fact that the discrete equations encode the information of the whole hierarchy of the corresponding continuous equations in an implicit way.

Among the theory of discrete integrable systems, there are three important scalar PΔEs in three dimensions. The first one is the discrete Kadomtsev–Petviashvili (KP) equation (often known as the Hirota equation or the Hirota–Miwa (HM) equation)

$$(p_i - p_j)(T_{n_h} \tau)(T_{n_i} T_{n_j} \tau) + (p_j - p_h)(T_{n_i} \tau)(T_{n_j} T_{n_h} \tau) + (p_h - p_i)(T_{n_j} \tau)(T_{n_h} T_{n_i} \tau) = 0, \quad (1.1)$$

in which the dependent variable  $\tau$  is a function of the discrete arguments  $n_j$  and the lattice parameters  $p_j$  for  $j = 1, 2, \dots$ , the notation  $T_{n_j}$  stands for the forward shift operator with respect to the corresponding discrete argument  $n_j$ , and  $p_h, p_i$  and  $p_j$  are distinct. This equation was introduced by Hirota [15] as a discrete analogue of the generalised Toda equation, and here we adopt the form in (1.1) which possesses explicit discrete soliton solutions. Since equation (1.1) is associated with the infinite-dimensional algebra  $A_\infty$ , here we also refer to it as the discrete AKP equation, in order to distinguish it from the other two discrete KP-type equations below. The second one is the discrete BKP equation (also referred to as the Miwa equation)

$$(p_h - p_i)(p_i - p_j)(p_j - p_h)\tau(T_{n_h} T_{n_i} T_{n_j} \tau) + (p_h + p_i)(p_h + p_j)(p_i - p_j)(T_{n_h} \tau)(T_{n_i} T_{n_j} \tau) \\ + (p_i + p_j)(p_i + p_h)(p_j - p_h)(T_{n_i} \tau)(T_{n_h} T_{n_j} \tau) + (p_j + p_h)(p_j + p_i)(p_h - p_i)(T_{n_j} \tau)(T_{n_h} T_{n_i} \tau) = 0, \quad (1.2)$$

which was introduced by Miwa in [23]. Although (1.2) has an additional term in comparison with (1.1), it is actually a special reduction of the discrete AKP equation (algebraically its associated algebra  $B_\infty$  is a sub-algebra of  $A_\infty$ ). The third model is the discrete CKP equation (also called the hyperdeterminant equation)

$$(A_1 + A_2 - A_3 - A_4)^2 = 4B_1 B_2, \quad (1.3)$$

where the expressions  $A_i$  and  $B_i$  are given by

$$A_1 = (p_h - p_i)^2(p_i - p_j)^2(p_j - p_h)^2\tau(T_{n_h} T_{n_i} T_{n_j} \tau), \quad A_2 = (p_h + p_i)^2(p_h + p_j)^2(p_i - p_j)^2(T_{n_h} \tau)(T_{n_i} T_{n_j} \tau), \\ A_3 = (p_i + p_j)^2(p_i + p_h)^2(p_j - p_h)^2(T_{n_i} \tau)(T_{n_h} T_{n_j} \tau), \quad A_4 = (p_j + p_h)^2(p_j + p_i)^2(p_h - p_i)^2(T_{n_j} \tau)(T_{n_h} T_{n_i} \tau), \\ B_1 = (p_h^2 - p_i^2)(p_j^2 - p_h^2) [(p_i + p_j)^2(T_{n_i} \tau)(T_{n_j} \tau) - (p_i - p_j)^2\tau(T_{n_i} T_{n_j} \tau)], \quad \text{and} \quad B_2 = T_{n_h} B_1,$$

respectively, which is also a reduced equation from the discrete AKP equation. Equation (1.3) appeared in Kashaev's paper [17] for the first time from the star-triangle transform in the Ising model, and it was later identified by Schief [32] that such a model actually describes the superposition formula for the continuous KP equation of  $C_\infty$ -type. The form in (1.3) that contains lattice parameters in the coefficients was given by the authors in [11] in order to construct its soliton-type solution.

The discrete KP-type equations are not only remarkable in their own right in the integrable systems theory, since these equations, as higher-dimensional models, can reduce to many lower-dimensional integrable discrete systems, such as the famous discrete Korteweg–de Vries (KdV) and Boussinesq (BSQ) systems, see e.g. [4] and [27], but also play crucial roles in other subjects in modern mathematics, especially in geometry. It was shown by Konopelchenko and Schief that the discrete AKP, BKP and CKP equations are connected to fundamental theorems of plane geometry, i.e.

Menelaus' theorem [19], Reciprocal quadrangles [20] and Carnot's theorem [32], respectively. Besides, Doliwa pointed out that these models also arise in the multi-dimensional consistency (MDC) quadrilateral lattice theory in discrete geometry [5–7]. Very recently, it was also revealed that the discrete KP-type equations are closely related to discrete line complexes [2] and circle complexes [3].

In addition to PΔEs, there are also the so-called generating PDEs, where the terminology ‘generating’ follows from the fact that it generates a whole continuous integrable hierarchy. From the perspective of integrable systems, the key step of constructing generating PDEs is to take the lattice parameters in the associated PΔEs as independent variables. This technique is actually a reflection of introducing Miwa's coordinates, see [18]. To be more precise, the derivative with respect to a lattice parameter is equivalent to the higher-order derivatives in terms of the continuous flow variables, resulting in the higher-order symmetries in an integrable hierarchy. The first example was proposed in [26] for the KdV class in the form of a nonautonomous fourth-order nonlinear equation, and it is integrable in sense that it possesses soliton, Lax pair, Lagrangian structure, and Painlevé reduction, etc. More importantly, it was shown in [34] that a proper generalisation of this equation incorporates the hyperbolic Ernst equation for a Weyl neutrino field in general relativity. In spirit of this, a class of generating PDEs for the BSQ family, also as integrable nonlinear models, were constructed in [35], and they represent the hyperbolic Ernst equations for a source-free Maxwell field and a Weyl neutrino field, as a generalisation for the previous result of KdV.

It turns out that it is very difficult to derive closed-form equations for a scalar field that would be the analogues of the generating PDEs for the KP-type hierarchies. However, as we will show in this paper, it is possible to derive semi-discrete analogues (in form of DΔE) of these generating PDEs for the KP-type equations, which simultaneously contain lattice variables and lattice parameters as independent variables. Such DΔEs, as nonautonomous semi-discrete equations, are significant in integrable systems theory. In fact, on the two-dimensional (2D) level, these semi-discrete equations appear in similarity reductions to discrete Painlevé equations [30], and also play roles of master symmetries of 2D integrable difference equations, see e.g. [31, 37].

We aim to construct the associated nonautonomous DΔEs for the KP-type equations, based on the direct linearisation (DL) framework for the KP-type equations given in [11]. The DL method was proposed by Fokas and Ablowitz (see e.g. [8, 9]) to solve the initial value problems for the KdV and KP equations, as a generalisation of the well-known Riemann–Hilbert problem (RHP) [1]. Subsequently, it was developed into a powerful tool to systematically study integrable structures behind families of discrete and continuous nonlinear equations and their interrelations, see e.g. [24, 25, 27–30]. The key idea in the DL is to associate a nonlinear equation with a linear integral equation, and by introducing the infinite matrix structure, it allows us to observe the integrability and solution structures of a nonlinear system simultaneously. Recently, the link between the linear integral equation and several affine Lie algebras is further established. With the help of considering reductions on the measure of the linear integral equation, the DL scheme for the discrete AKP, BKP and CKP equations was proposed [11]. This makes it possible to further study the associated nonautonomous DΔEs for these KP-type equations from the general framework.

In this paper, we establish six nonautonomous DΔEs. They are

$$\frac{1}{2}(p_i - p_j)^2 D_{p_i} D_{p_j} \tau \cdot \tau = n_i n_j \left[ (T_{n_i} T_{n_j}^{-1} \tau)(T_{n_i}^{-1} T_{n_j} \tau) - \tau^2 \right], \quad (1.4a)$$

$$(p_i - p_j) D_{p_j} (T_{n_i} \tau) \cdot \tau = n_j \left[ \tau(T_{n_i} \tau) - (T_{n_j} \tau)(T_{n_i} T_{n_j}^{-1} \tau) \right], \quad (1.4b)$$

$$\begin{aligned} (p_i - p_j) D_{p_j} [\tau(T_{n_i} T_{n_j} \tau)] \cdot [(T_{n_i} \tau)(T_{n_j} \tau)] \\ = n_j (T_{n_j} \tau)^2 (T_{n_i} T_{n_j} \tau)(T_{n_i} T_{n_j}^{-1} \tau) - (n_j + 1) \tau (T_{n_i} \tau)^2 (T_{n_j}^2 \tau) + \tau (T_{n_i} \tau)(T_{n_j} \tau)(T_{n_i} T_{n_j} \tau) \end{aligned} \quad (1.4c)$$

in the AKP class<sup>1</sup>,

$$\begin{aligned} 2p_i p_j (p_i^2 - p_j^2) D_{p_i} D_{p_j} \tau \cdot \tau \\ = n_i n_j \left[ (p_i + q_j)^4 (T_{n_i} T_{n_j}^{-1} \tau)(T_{n_i}^{-1} T_{n_j} \tau) - (p_i - q_j)^4 (T_{n_i} T_{n_j} \tau)(T_{n_i}^{-1} T_{n_j}^{-1} \tau) - 8p_i p_j (p_i^2 + p_j^2) \tau^2 \right], \end{aligned} \quad (1.5a)$$

$$2p_j (p_i^2 - p_j^2) D_{p_j} (T_{n_i} \tau) \cdot \tau = n_j \left[ 4p_i p_j \tau (T_{n_i} \tau) + (p_i - p_j)^2 (T_{n_j}^{-1} \tau)(T_{n_i} T_{n_j} \tau) - (p_i + p_j)^2 (T_{n_j} \tau)(T_{n_i} T_{n_j}^{-1} \tau) \right] \quad (1.5b)$$

in the BKP class, and

$$p_i p_j (p_i^2 - p_j^2)^2 D_{p_i} D_{p_j} \tau \cdot \tau = n_i n_j \left[ (p_i - p_j)^2 E^{\frac{1}{2}} F^{\frac{1}{2}} + (p_i + p_j)^2 G^{\frac{1}{2}} H^{\frac{1}{2}} - 8p_i p_j (p_i^2 + p_j^2) \tau^2 \right] \quad (1.6)$$

in the CKP class, where  $E$ ,  $F$ ,  $G$  and  $H$  are given by

$$\begin{aligned} E &= (p_i + p_j)^2 (T_{n_i}^{-1} \tau)(T_{n_j}^{-1} \tau) - (p_i - p_j)^2 \tau (T_{n_i}^{-1} T_{n_j}^{-1} \tau), & F &= (p_i + p_j)^2 (T_{n_i} \tau)(T_{n_j} \tau) - (p_i - p_j)^2 \tau (T_{n_i} T_{n_j} \tau), \\ G &= (p_i - p_j)^2 (T_{n_i}^{-1} \tau)(T_{n_j} \tau) - (p_i + p_j)^2 \tau (T_{n_i}^{-1} T_{n_j} \tau) & \text{and} & & H &= (p_i - p_j)^2 (T_{n_i} \tau)(T_{n_j}^{-1} \tau) - (p_i + p_j)^2 \tau (T_{n_i} T_{n_j}^{-1} \tau), \end{aligned}$$

respectively, where the notation  $D$  stands for Hirota's bilinear derivative (see e.g. [16]) with respect to the corresponding arguments  $p_i$  and  $p_j$ , which is defined by

$$D_x f \cdot g = (\partial_x - \partial_{x'}) f(x) g(x')|_{x'=x},$$

for arbitrary differentiable functions  $f(x)$  and  $g(x)$ .

<sup>1</sup>Here by class we mean a family of equations possess the same solution from the perspective of the DL, see also section 6 for more details.

The tau functions for the above equations have their precise definitions in terms of infinite matrices and double integral with regard to the spectral variables, which will be given in the sections of the derivations of these equations. We note that an autonomous version of (1.5a) was given in [36] very recently, which plays the role of a higher-order semi-discrete BKP equation (i.e. a symmetry) and was referred to as the (2+2)-dimensional Toda lattice. But here equation (1.5a) is a nonautonomous equation having the lattice parameters as the continuous independent variables, potentially acting as the master symmetry of the discrete BKP equation.

The paper is organised as follows. Section 2 concerns the formal structure of the direct linearisation approach. In sections 3, 4 and 5, we provide derivations of (1.4a), (1.4b) and (1.4c) in the AKP class, (1.5a) and (1.5b) in the BKP class, as well as (1.6) in the CKP class, respectively. The brief discussion on the integrability of the six DΔEs is made in section 6.

## 2. FORMAL STRUCTURE OF THE DIRECT LINEARISATION

In [11], the discrete AKP, BKP and CKP equations were studied within a single framework, given by the direct linearising transform (DLT), which provides a dressing-type scheme for obtaining new solutions from given seed solutions for those integrable equations. In contrast, what we mean by DL is a special case where the seed corresponds to a ‘free’ solution (namely when the initial solution is trivial) in the DLT. The latter restriction is useful if we want to derive new equations from some basic assumptions about the initial solutions.

We start with introducing some fundamental infinite matrices and vectors and their properties which are needed in the DL framework. First we need a rank 1 projection matrix

$$\mathbf{O} = \begin{pmatrix} \ddots & & & & & \\ & 0 & & & & \\ & & \boxed{1} & & & \\ & & & 0 & & \\ & & & & \ddots & \end{pmatrix}, \quad (2.1)$$

where the ‘box’ denotes the location of the central element, namely the (0,0)-entry of the matrix. This infinite matrix has the property

$$(\mathbf{O}\mathbf{U})_{i,j} = \delta_{i,0}U_{0,j} \quad \text{and} \quad (\mathbf{U}\mathbf{O})_{i,j} = U_{i,0}\delta_{0,j}, \quad \text{with} \quad \delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \forall i, j \in \mathbb{Z},$$

for an arbitrary infinite matrix

$$\mathbf{U} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & U_{-1,-1} & U_{-1,0} & U_{-1,1} & \cdots \\ \cdots & U_{0,-1} & \boxed{U_{0,0}} & U_{0,1} & \cdots \\ \cdots & U_{1,-1} & U_{1,0} & U_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where the operation  $(\cdot)_{i,j}$  denotes taking the  $(i,j)$ -entry.

Next, we introduce two infinite matrices

$$\mathbf{\Lambda} = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & \boxed{0} & 1 & & \\ & & & 0 & \ddots & \\ & & & & & \ddots \end{pmatrix} \quad \text{and} \quad {}^t\mathbf{\Lambda} = \begin{pmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & \boxed{0} & & & \\ & & 1 & 0 & & \\ & & & & \ddots & \ddots \end{pmatrix}. \quad (2.2)$$

The matrices  $\mathbf{\Lambda}$  and  ${}^t\mathbf{\Lambda}$  are the transpose of each other, and have properties

$$(\mathbf{\Lambda}\mathbf{U})_{i,j} = U_{i+1,j} \quad \text{and} \quad (\mathbf{U}{}^t\mathbf{\Lambda})_{i,j} = U_{i,j+1},$$

namely the multiplications by  $\mathbf{\Lambda}$  and  ${}^t\mathbf{\Lambda}$  raise the row index and the column index of  $\mathbf{U}$ , respectively. For this reason, we refer to them as index-raising matrices. We also introduce the infinite-dimensional vectors

$$\mathbf{c}_k = (\cdots, k^{-1}, 1, k, \cdots)^T \quad \text{and} \quad {}^t\mathbf{c}_{k'} = (\cdots, k'^{-1}, 1, k', \cdots), \quad (2.3)$$

which obey the following identities:

$$\mathbf{\Lambda}\mathbf{c}_k = k\mathbf{c}_k, \quad {}^t\mathbf{c}_{k'}{}^t\mathbf{\Lambda} = k'{}^t\mathbf{c}_{k'}.$$

In the sections below, we also need the notions of trace and determinant for an infinite matrix. We give the formal definition of a trace of an arbitrary infinite matrix  $\mathbf{U}$  as follows:

$$\text{tr}\mathbf{U} = \sum_{i \in \mathbb{Z}} U_{i,i}. \quad (2.4)$$

Notice that this is a formal definition, since the infinite summation with respect to  $i$  over the integer ring may lead to divergence. In order to avoid this issue, in this paper we only deal with the trace of a matrix involving the projection matrix  $\mathbf{O}$ , in which case the trace is always convergent. For example, we have

$$\mathrm{tr}(\mathbf{O}\mathbf{U}) = \mathrm{tr}(\mathbf{U}\mathbf{O}) = U_{0,0} = (\mathbf{U})_{0,0}.$$

In the convergent case, the trace also satisfies  $\mathrm{tr}(\mathbf{U}\mathbf{V}) = \mathrm{tr}(\mathbf{V}\mathbf{U})$  for arbitrary infinite matrices  $\mathbf{U}$  and  $\mathbf{V}$ . The determinant of an infinite matrix is defined through

$$\ln(\det \mathbf{U}) = \mathrm{tr}(\ln \mathbf{U}). \quad (2.5)$$

Again, this is a formal definition which could result in divergence problem. But if we restrict ourselves to the infinite matrix  $1 + \mathbf{U}$ , where  $1$  is the identity infinite matrix and  $\mathbf{U}$  is an infinite matrix involving the projection matrix  $\mathbf{O}$ , the determinant is well-defined, since the right hand side of (2.5), namely

$$\mathrm{tr}[\ln(1 + \mathbf{U})] = \mathrm{tr} \left[ \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \mathbf{U}^i \right] = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \mathrm{tr}(\mathbf{U}^i),$$

has terms of convergent traces. We have the well-known Weinstein–Aronszajn formulas for the determinant. For instance, the following identities hold:

$$\det(1 + \mathbf{U}\mathbf{O}\mathbf{V}) = 1 + (\mathbf{V}\mathbf{U})_{0,0}, \quad \det(1 + \mathbf{U}(\mathbf{O}\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\mathbf{O})\mathbf{V}) = \det \begin{pmatrix} 1 + (\mathbf{\Lambda}\mathbf{V}\mathbf{U})_{0,0} & -(\mathbf{\Lambda}\mathbf{V}{}^t\mathbf{\Lambda})_{0,0} \\ (\mathbf{V}\mathbf{U})_{0,0} & 1 - (\mathbf{V}\mathbf{U}{}^t\mathbf{\Lambda})_{0,0} \end{pmatrix},$$

which are the cases of rank 1 and rank 2.

We provide the formal structure of the DL framework. The starting point is a linear integral equation

$$\mathbf{u}_k + \iint_D d\zeta(l, l') \rho_k \Omega_{k, l'} \sigma_{l'} \mathbf{u}_l = \rho_k \mathbf{c}_k, \quad (2.6)$$

in which the wave function  $\mathbf{u}_k$  is an infinite-dimensional column vector having its  $i$ -th component  $u_k^{(i)}$  (for  $i \in \mathbb{Z}$ ) being a function of the lattice variables  $n_j$  and the lattice parameters  $p_j$  for  $j = 1, 2, \dots$  as well as the spectral variable  $k$ ,  $\Omega_{k, l'}$  is the kernel of the linear integral equation, depending on the spectral variables  $k$  and  $l'$ ,  $d\zeta$  and  $D$  are the measure and the domain for integration, and  $\rho_k$  and  $\sigma_{l'}$  are the so-called plane wave factors depending on the discrete variables  $n_j$  and lattice parameters  $p_j$  as well as their respective spectral variables  $k$  and  $l'$ . Furthermore, to derive equations in given choice of independent variables (which could be either the discrete variables  $n_j$  or the continuous variables  $p_j$ ) we assume that the measure is independent of these chosen variables.

We present the infinite matrix representation of (2.6). We first introduce the infinite matrix  $\mathbf{\Omega}$  defined through

$$\Omega_{k, k'} = {}^t\mathbf{c}_{k'} \mathbf{\Omega} \mathbf{c}_k. \quad (2.7)$$

Following from the properties of the index-raising matrices and the projection matrix listed above, we can observe that  $\mathbf{\Omega}$  is actually an infinite matrix composed of  $\mathbf{\Lambda}$ ,  ${}^t\mathbf{\Lambda}$  and  $\mathbf{O}$ , which relies on the precise expression of  $\Omega_{k, k'}$ , namely it is an infinite matrix representation of the Cauchy kernel of the linear integral equation (2.6). If we replace  $\Omega_{k, l'}$  with the help of (2.7), the linear integral equation is reformulated as

$$\mathbf{u}_k = (1 - \mathbf{U}\mathbf{\Omega})\mathbf{c}_k \rho_k, \quad (2.8)$$

where the infinite matrix  $\mathbf{U}$  is given by

$$\mathbf{U} \doteq \iint_D d\zeta(k, k') \mathbf{u}_k {}^t\mathbf{c}_{k'} \sigma_{k'}. \quad (2.9)$$

Next, we consider the infinite matrix representation of the plane wave factors and introduce an infinite matrix

$$\mathbf{C} \doteq \iint_D d\zeta(k, k') \rho_k \mathbf{c}_k {}^t\mathbf{c}_{k'} \sigma_{k'}. \quad (2.10)$$

The key characteristic of  $\mathbf{C}$  is the product of the two plane wave factors, i.e.  $\rho_k \sigma_{k'}$ , which we normally refer to as the so-called effective plane wave factor. By acting the operation  $\iint_D d\zeta(k, k') \cdot {}^t\mathbf{c}_{k'} \sigma_{k'}$  on equation (2.10), we obtain

$$\mathbf{U} = (1 - \mathbf{U}\mathbf{\Omega})\mathbf{C}, \quad \text{or equivalently} \quad \mathbf{U} = \mathbf{C}(1 + \mathbf{\Omega}\mathbf{C})^{-1}. \quad (2.11)$$

The idea of the DL approach is to associate a nonlinear equation with a linear integral equation in the form of (2.6). Once the plane wave factors, the Cauchy kernel and the measure are given, the corresponding class of nonlinear integrable systems is fully determined. To be more precise, for a certain class of nonlinear integrable equations, the infinite matrix  $\mathbf{C}$  describes the linear dispersion, the infinite matrix  $\mathbf{\Omega}$  together with the measure governs the nonlinear structure of the corresponding integrable models, and  $\mathbf{u}_k$  and  $\mathbf{U}$  are corresponding to the wave function in the Lax pair and the nonlinear potential, respectively.

## 3. DERIVATION OF (1.4a), (1.4b) AND (1.4c)

In the AKP class, the discrete plane wave factors  $\rho_k$  and  $\sigma_{k'}$  are given by

$$\rho_k = \prod_{i=1}^{\infty} (p_i + k)^{n_i} \quad \text{and} \quad \sigma_{k'} = \prod_{i=1}^{\infty} (p_i - k')^{-n_i}, \quad (3.1)$$

respectively. Substituting these into the infinite matrix  $\mathbf{C}$  defined by (2.10) and considering the evolutions with regard to the discrete variables and lattice parameters, we obtain dynamical evolutions as follows:

$$(\mathbb{T}_{n_j} \mathbf{C})(p_j - {}^t\Lambda) = (p_j + \Lambda) \mathbf{C}, \quad (3.2a)$$

$$\partial_{p_j} \mathbf{C} = n_j \left( \frac{1}{p_j + \Lambda} \mathbf{C} - \mathbf{C} \frac{1}{p_j - {}^t\Lambda} \right), \quad (3.2b)$$

$$(\mathbb{T}_a^{-1} \mathbb{T}_b \mathbf{C}) \frac{b - {}^t\Lambda}{a - {}^t\Lambda} = \frac{b + \Lambda}{a + \Lambda} \mathbf{C}. \quad (3.2c)$$

Below we provide the derivation of (3.2b). By differentiating  $\rho_k \sigma_{k'}$  with respect to  $p_j$ , we have

$$\partial_{p_j} \rho_k \sigma_{k'} = \partial_{p_j} \prod_{i=1}^{\infty} \left( \frac{p_i + k}{p_i - k'} \right)^{n_i} = n_j \left( \frac{1}{p_j + k} - \frac{1}{p_j - k'} \right) \prod_{i=1}^{\infty} \left( \frac{p_i + k}{p_i - k'} \right)^{n_i} = n_j \left( \frac{1}{p_j + k} - \frac{1}{p_j - k'} \right) \rho_k \sigma_{k'},$$

and thus, the same operation on  $\mathbf{C}$  gives rise to

$$\begin{aligned} \partial_{p_j} \mathbf{C} &= \iint_D d\zeta(k, k') \mathbf{c}_k \partial_{p_j} (\rho_k \sigma_{k'}) {}^t \mathbf{c}_{k'} = \iint_D d\zeta(k, k') \mathbf{c}_k \partial_{p_j} (\rho_k \sigma_{k'}) {}^t \mathbf{c}_{k'} \\ &= n_j \left( \iint_D d\zeta(k, k') \frac{1}{p_j + k} \mathbf{c}_k \rho_k \sigma_{k'} {}^t \mathbf{c}_{k'} - \iint_D d\zeta(k, k') \mathbf{c}_k \rho_k \sigma_{k'} {}^t \mathbf{c}_{k'} \frac{1}{p_j - k'} \right) = n_j \left( \frac{1}{p_j + \Lambda} \mathbf{C} - \mathbf{C} \frac{1}{p_j - {}^t\Lambda} \right), \end{aligned}$$

where in the last step we have made use of the property of the operation of the index-raising matrices on  $\mathbf{c}_k$  and  ${}^t \mathbf{c}_{k'}$  given in section 2. The other two equations are derived similarly.

The Cauchy kernel in the linear integral equation for the discrete AKP equation takes the form of

$$\Omega_{k, k'} = \frac{1}{k + k'}, \quad (3.3)$$

and in this case we have no further requirement for the measure  $d\zeta(k, k')$  and the domain  $D$ , namely they are arbitrary. According to (2.7), we can observe that in this case  $\Omega = -\sum_{i=0}^{\infty} (-{}^t\Lambda)^{-i-1} \mathbf{O} \Lambda^i$ ; in other words, it satisfies

$$\Omega \Lambda + {}^t \Lambda \Omega = \mathbf{O}. \quad (3.4)$$

Equation (3.4) can also be written in other forms. Here we reformulate it in the following forms whose left hand sides are compatible with (3.2):

$$\Omega(p_j + \Lambda) + (p_j - {}^t\Lambda) \Omega = \mathbf{O}, \quad (3.5a)$$

$$\Omega \frac{1}{p_j + \Lambda} - \frac{1}{p_j - {}^t\Lambda} \Omega = \frac{1}{-p_j + {}^t\Lambda} \mathbf{O} \frac{1}{p_j + \Lambda}, \quad (3.5b)$$

$$\Omega \frac{b + \Lambda}{a + \Lambda} - \frac{b - {}^t\Lambda}{a - {}^t\Lambda} \Omega = -(a - b) \frac{1}{-a + {}^t\Lambda} \mathbf{O} \frac{1}{a + \Lambda}. \quad (3.5c)$$

Equations (3.5) together with (3.2) will provide the dynamical evolutions of the infinite matrix  $\mathbf{U}$  as follows:

$$(\mathbb{T}_{n_j} \mathbf{U})(p_j - {}^t\Lambda) = (p_j + \Lambda) \mathbf{U} - (\mathbb{T}_{n_j} \mathbf{U}) \mathbf{O} \mathbf{U}, \quad (3.6a)$$

$$\partial_{p_j} \mathbf{U} = n_j \left( \frac{1}{p_j + \Lambda} \mathbf{U} - \mathbf{U} \frac{1}{p_j - {}^t\Lambda} - \mathbf{U} \frac{1}{-p_j + {}^t\Lambda} \mathbf{O} \frac{1}{p_j + \Lambda} \mathbf{U} \right), \quad (3.6b)$$

$$(\mathbb{T}_a \mathbb{T}_b^{-1} \mathbf{U}) \frac{b - {}^t\Lambda}{a - {}^t\Lambda} = \frac{a + \Lambda}{b + \Lambda} \mathbf{U} + (a - b) (\mathbb{T}_a^{-1} \mathbb{T}_b \mathbf{U}) \frac{1}{-a + {}^t\Lambda} \mathbf{O} \frac{1}{a + \Lambda} \mathbf{U}. \quad (3.6c)$$

Again we only give the derivation of (3.6b) below and skip that of the other two, as the procedure is similar. Notice that the infinite matrix  $\mathbf{U}$  obeys (2.11) in the formal structure of the DL. Calculating the derivative of  $\mathbf{U}$  with respect to  $p_j$ , we obtain

$$\partial_{p_j} \mathbf{U} = -(\partial_{p_j} \mathbf{U}) \Omega \mathbf{C} + (1 - \mathbf{U} \Omega) (\partial_{p_j} \mathbf{C}),$$

which can equivalently be rewritten as

$$\begin{aligned} (\partial_{p_j} \mathbf{U})(1 + \Omega \mathbf{C}) &= n_j (1 - \mathbf{U} \Omega) \left( \frac{1}{p_j + \Lambda} \mathbf{C} - \mathbf{C} \frac{1}{p_j - {}^t\Lambda} \right) \\ &= n_j \left[ \frac{1}{p_j + \Lambda} \mathbf{C} - \mathbf{U} \frac{1}{p_j - {}^t\Lambda} - \mathbf{U} \Omega \frac{1}{p_j + \Lambda} \mathbf{C} \right] \\ &= n_j \left[ \frac{1}{p_j + \Lambda} \mathbf{C} - \mathbf{U} \frac{1}{p_j - {}^t\Lambda} (1 + \Omega \mathbf{C}) - \mathbf{U} \frac{1}{-p_j + {}^t\Lambda} \mathbf{O} \frac{1}{p_j + \Lambda} \mathbf{C} \right], \end{aligned}$$

where in the first and third equalities we have used (3.2b) and (3.5b), respectively. Multiplying the above equation by  $(1 + \mathbf{\Omega}\mathbf{C})^{-1}$  immediately gives rise to (3.6b).

Next, we introduce the tau function in this class, which is defined as

$$\tau \doteq \det(1 + \mathbf{\Omega}\mathbf{C}), \quad (3.7)$$

where 1 denotes the infinite unit matrix. As we have mentioned in section 2, the determinant of an infinite matrix should be understood as the formal expansion of  $\exp\{\text{tr}[\ln(1 + \mathbf{\Omega}\mathbf{C})]\}$ . Since  $\mathbf{\Omega} = -\sum_{i=0}^{\infty} (-{}^t\mathbf{\Lambda})^{-i-1} \mathbf{O}\mathbf{\Lambda}^i$  involves  $\mathbf{O}$  in every term, the trace action is always convergent. For convenience, we introduce quantities

$$u \doteq (U)_{0,0}, \quad V_a \doteq 1 - \left( U \frac{1}{a + {}^t\mathbf{\Lambda}} \right)_{0,0}, \quad W_a \doteq 1 - \left( \frac{1}{a + \mathbf{\Lambda}} U \right)_{0,0} \quad \text{and} \quad S_{a,b} \doteq \left( \frac{1}{a + \mathbf{\Lambda}} U \frac{1}{b + {}^t\mathbf{\Lambda}} \right)_{0,0}.$$

The tau function satisfies the dynamical evolutions with respect to the discrete variables and the lattices parameters as follows:

$$\frac{\mathbf{T}_{n_j} \tau}{\tau} = 1 - \left( U \frac{1}{-p_j + {}^t\mathbf{\Lambda}} \right)_{0,0} = V_{-p_j}, \quad \frac{\mathbf{T}_{n_j}^{-1} \tau}{\tau} = 1 - \left( \frac{1}{p_j + \mathbf{\Lambda}} U \right)_{0,0} = W_{p_j}, \quad (3.8a)$$

$$\partial_{p_j} \ln \tau = n_j S_{p_j, -p_j}, \quad (3.8b)$$

$$\frac{\mathbf{T}_a^{-1} \mathbf{T}_b \tau}{\tau} = 1 - (a - b) \left( \frac{1}{a + \mathbf{\Lambda}} U \frac{1}{-b + {}^t\mathbf{\Lambda}} \right)_{0,0} = 1 - (a - b) S_{a, -b}, \quad (3.8c)$$

$$(p_i - p_j) \frac{\tau(\mathbf{T}_{n_i} \mathbf{T}_{n_j} \tau)}{(\mathbf{T}_{n_i} \tau)(\mathbf{T}_{n_j} \tau)} = p_i - p_j + \mathbf{T}_{n_j} u - \mathbf{T}_{n_i} u. \quad (3.8d)$$

These equations are proven through direct computation in terms of infinite matrices. For instance, calculating the derivative of  $\ln \tau$  with respect to  $p_j$  yields

$$\begin{aligned} \partial_{p_j} \ln \tau &= \partial_{p_j} \ln[\det(1 + \mathbf{\Omega}\mathbf{C})] = \partial_{p_j} \text{tr}[\ln(1 + \mathbf{\Omega}\mathbf{C})] = \text{tr}[\partial_{p_j} \ln(1 + \mathbf{\Omega}\mathbf{C})] = \text{tr}[(1 + \mathbf{\Omega}\mathbf{C})^{-1} \mathbf{\Omega}(\partial_{p_j} \mathbf{C})] \\ &= n_j \text{tr} \left[ (1 + \mathbf{\Omega}\mathbf{C})^{-1} \mathbf{\Omega} \left( \frac{1}{p_j + \mathbf{\Lambda}} \mathbf{C} - \mathbf{C} \frac{1}{p_j - {}^t\mathbf{\Lambda}} \right) \right] = n_j \text{tr} \left[ \mathbf{C} (1 + \mathbf{\Omega}\mathbf{C})^{-1} \left( \mathbf{\Omega} \frac{1}{p_j + \mathbf{\Lambda}} - \frac{1}{p_j - {}^t\mathbf{\Lambda}} \mathbf{\Omega} \right) \right]. \end{aligned}$$

where in the last step the property of the cyclic permutation of the trace operation is used. With the help of (2.11) and (3.5b), this equation is reformulated as

$$\partial_{p_j} \ln \tau = n_j \text{tr} \left( U \frac{1}{-p_j + {}^t\mathbf{\Lambda}} \mathbf{O} \frac{1}{p_j + \mathbf{\Lambda}} \right) = n_j \text{tr} \left( \mathbf{O} \frac{1}{p_j + \mathbf{\Lambda}} U \frac{1}{-p_j + {}^t\mathbf{\Lambda}} \right) = n_j \left( \frac{1}{p_j + \mathbf{\Lambda}} U \frac{1}{-p_j + {}^t\mathbf{\Lambda}} \right)_{0,0},$$

namely  $\partial_{p_j} \ln \tau = n_j S_{p_j, -p_j}$ . To prove Equations (3.8a) and (3.8c), one needs to use (3.2a) and (3.5a) and (3.2c) and (3.5c), respectively. Since the idea of the proofs is similar to that of (3.8a), we skip them here. While equation (3.8d) is proven based on (3.8a) and (3.6), and it is a widely known relation which describes bilinear transformation for the discrete KP equation (see e.g. [11]).

Equations listed in (3.6) are the key formulas in the DL scheme to construct closed-form integrable equations, and they together with (3.8) can produce the bilinear equations in the AKP class. In [11], the well-known HM equation, i.e. equation (1.1), is derived from (3.6a) and (3.8a). Here we start from the dynamical relations in terms of  $p_j$  to construct the (2+2)-dimensional nonautonomous D $\Delta$ E in this class. Considering  $(a + \mathbf{\Lambda})^{-1}(3.6b)(b + {}^t\mathbf{\Lambda})^{-1}$  and taking the (0,0)-entry, we have the following equation for the quantity  $S_{a,b}$ :

$$\partial_{p_j} S_{a,b} = n_j \left[ \left( \frac{1}{p_j - a} - \frac{1}{p_j + b} \right) S_{a,b} - \frac{1}{p_j - a} S_{p_j, b} + \frac{1}{p_j + b} S_{a, -p_j} - S_{a, -p_j} S_{p_j, b} \right].$$

By setting  $a = p_i$  and  $b = -p_i$ , we reach to

$$\partial_{p_j} S_{p_i, -p_i} = \frac{n_j}{(p_i - p_j)^2} \left[ (1 - (p_j - p_i) S_{p_j, -p_i}) (1 - (p_i - p_j) S_{p_i, -p_j}) - 1 \right],$$

which only involves the  $S$ -variable. If we replace all the  $S$ -variables in this equation by the tau function via (3.8) and notice the identity

$$\partial_{p_i} \partial_{p_j} \ln \tau = \frac{\mathbf{D}_{p_i} \mathbf{D}_{p_j} \tau \cdot \tau}{2\tau^2}$$

for Hirota's bilinear operator  $\mathbf{D}$ ., the bilinear equation (1.4a) arises. This equation takes the form of a nonautonomous version of the 2D Toda equation, if we think of the discrete shift operations  $\mathbf{T}_{n_i} \mathbf{T}_{n_j}^{-1}$  and  $\mathbf{T}_{n_i}^{-1} \mathbf{T}_{n_j}$  as the forward and backward shifts along skew direction on the lattice. The difference is that here the lattice parameters  $p_i$  and  $p_j$  act the independent variables; while in the two-dimensional Toda lattice (2DTL), the bilinear derivatives are with respect to the continuous flow variables  $x_1$  and  $x_{-1}$  (cf. [10]), which leads to an autonomous equation.

From the scheme, we can also construct two more nonautonomous semi-discrete equations of  $\tau$ . Taking  $[(3.6b)(a + {}^t\mathbf{\Lambda})^{-1}]_{0,0}$ , one obtains

$$\partial_{p_j} V_a = -n_j \left[ V_{-p_j} S_{p_j, a} + \frac{1}{a + p_j} (V_a - V_{-p_j}) \right],$$

and its  $a = -p_i$  case gives rise to

$$\partial_{p_j} V_{-p_i} = \frac{n_j}{p_i - p_j} [V_{-p_i} - V_{-p_j} (1 - (p_j - p_i) S_{p_j, -p_i})].$$

Substituting the  $V$ - and  $S$ -variables with the tau function with the help of (3.8), we are able to construct the bilinear equation (1.4b).

Moreover, if we consider the equation for  $\partial_{p_j} u$  by evaluating the  $(0, 0)$ -entry of (3.6b), the following equation shows up:

$$\partial_{p_j} u = n_j (1 - V_{-p_j} W_{p_j}) = n_j \left[ 1 - \frac{(\mathbb{T}_{n_j} \tau)(\mathbb{T}_{n_j}^{-1} \tau)}{\tau^2} \right],$$

where formula (3.8) is used for the second equality, and therefore, we obtain from (3.8d) that

$$\partial_{p_j} \left[ (p_i - p_j) \frac{\tau(\mathbb{T}_{n_i} \mathbb{T}_{n_j} \tau)}{(\mathbb{T}_{n_i} \tau)(\mathbb{T}_{n_j} \tau)} \right] = \partial_{p_j} (p_i - p_j + \mathbb{T}_{n_j} u - \mathbb{T}_{n_i} u) = -1 + \mathbb{T}_{n_j} (\partial_{p_j} u) - \mathbb{T}_{n_i} (\partial_{p_j} u).$$

Finally, by substituting  $u$  with  $\tau$ , a closed-form quartic equation of  $\tau$  is constructed, which takes the form of equation (1.4c).

#### 4. DERIVATION OF (1.5a) AND (1.5b)

We select the following plane wave factors for the BKP class:

$$\rho_k = \prod_{i=1}^{\infty} \left( \frac{p_i + k}{p_i - k} \right)^{n_i}, \quad \sigma_{k'} = \rho_{k'} = \prod_{i=1}^{\infty} \left( \frac{p_i + k'}{p_i - k'} \right)^{n_i}, \quad (4.1)$$

which should be understood as discrete odd flows, compared with (3.1) in the AKP case. In this case, it is observed that the infinite matrix  $\mathbf{C}$  defined in (2.10) evolves with regard to the discrete variables and the lattice parameters in the following way:

$$(\mathbb{T}_{n_j} \mathbf{C}) \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} = \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{C}, \quad (4.2a)$$

$$\partial_{p_j} \mathbf{C} = n_j \left[ \left( \frac{1}{p_j + \Lambda} - \frac{1}{p_j - \Lambda} \right) \mathbf{C} + \mathbf{C} \left( \frac{1}{p_j + {}^t \Lambda} - \frac{1}{p_j - {}^t \Lambda} \right) \right]. \quad (4.2b)$$

The derivation of these formulas is straightforward, as they follow from the formal definition of  $\mathbf{C}$ , namely (2.10).

The Cauchy kernel in the BKP case takes the form of

$$\Omega_{k, k'} = \frac{1}{2} \frac{k - k'}{k + k'}, \quad (4.3)$$

which implies that the corresponding  $\Omega$  satisfies

$$\Omega \Lambda + {}^t \Lambda \Omega = \frac{1}{2} (\Omega \Lambda - {}^t \Lambda \Omega). \quad (4.4)$$

For the purpose of looking for compatible relations of  $\Omega$  with equations listed in (4.2), we reformulate (4.4) as

$$\Omega \frac{p_j + \Lambda}{p_j - \Lambda} - \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} \Omega = p_j \frac{1}{p_j + {}^t \Lambda} (\mathbf{O} \Lambda - {}^t \Lambda \mathbf{O}) \frac{1}{p_j - \Lambda} \quad (4.5a)$$

and

$$\begin{aligned} & \Omega \left( \frac{1}{p_j + \Lambda} - \frac{1}{p_j - \Lambda} \right) + \left( \frac{1}{p_j + {}^t \Lambda} - \frac{1}{p_j - {}^t \Lambda} \right) \Omega \\ &= -\frac{1}{2} \frac{1}{p_j - {}^t \Lambda} (\mathbf{O} \Lambda - {}^t \Lambda \mathbf{O}) \frac{1}{p_j + \Lambda} - \frac{1}{2} \frac{1}{p_j + {}^t \Lambda} (\mathbf{O} \Lambda - {}^t \Lambda \mathbf{O}) \frac{1}{p_j - \Lambda}, \end{aligned} \quad (4.5b)$$

respectively.

Equations (4.2) and (4.5) are the fundamental relations to construct the dynamical evolutions of the infinite matrix  $\mathbf{U}$ . By considering  $\mathbb{T}_{n_j} \mathbf{U}$  and  $\partial_{p_j} \mathbf{U}$  in (2.11), some straightforward calculation yields the following equations:

$$(\mathbb{T}_{n_j} \mathbf{U}) \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} = \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{U} - p_j (\mathbb{T}_{n_j} \mathbf{U}) \frac{1}{p_j + {}^t \Lambda} (\mathbf{O} \Lambda - {}^t \Lambda \mathbf{O}) \frac{1}{p_j - \Lambda} \mathbf{U}, \quad (4.6a)$$

$$\begin{aligned} \partial_{p_j} \mathbf{U} = n_j & \left[ \left( \frac{1}{p_j + \Lambda} - \frac{1}{p_j - \Lambda} \right) \mathbf{U} + \mathbf{U} \left( \frac{1}{p_j + {}^t \Lambda} - \frac{1}{p_j - {}^t \Lambda} \right) \right. \\ & \left. + \frac{1}{2} \mathbf{U} \frac{1}{p_j - {}^t \Lambda} (\mathbf{O} \Lambda - {}^t \Lambda \mathbf{O}) \frac{1}{p_j + \Lambda} \mathbf{U} + \frac{1}{2} \mathbf{U} \frac{1}{p_j + {}^t \Lambda} (\mathbf{O} \Lambda - {}^t \Lambda \mathbf{O}) \frac{1}{p_j - \Lambda} \mathbf{U} \right], \end{aligned} \quad (4.6b)$$

with the help of (4.2) and (4.5).



Besides the different kernel and plane wave factors, we also have to impose certain restrictions on the integration measure and the integration domain for the BKP equation. We require that the integration domain  $D$  is symmetric in terms of the spectral variables  $k$  and  $k'$ , and then the measure  $d\zeta(k, k')$  is antisymmetric, i.e.

$$d\zeta(k', k) = -d\zeta(k, k'). \quad (4.7)$$

These will together result in

$${}^t\mathbf{C} = \iint_D d\zeta(k, k') \rho_k \mathbf{c}_{k'} {}^t\mathbf{c}_k \rho_{k'} = - \iint_D d\zeta(k', k) \rho_{k'} \mathbf{c}_k {}^t\mathbf{c}_{k'} \rho_k = -\mathbf{C};$$

in other words, we have an antisymmetric infinite matrix  $\mathbf{C}$  in the discrete BKP. Notice that the kernel given in (4.3) is also antisymmetric, or equivalently  ${}^t\mathbf{\Omega} = -\mathbf{\Omega}$ . We can deduce from (2.11) that in this case the infinite matrix  $\mathbf{U}$  obeys the antisymmetry property

$${}^t\mathbf{U} = -\mathbf{U}. \quad (4.8)$$

Next, we present the dynamical evolutions of the tau function. For convenience we introduce quantities

$$V_a \doteq 1 - \left( \mathbf{U} \frac{a}{a - {}^t\mathbf{\Lambda}} \right)_{0,0}, \quad W_a \doteq 1 + \left( \frac{a}{a - \mathbf{\Lambda}} \mathbf{U} \right)_{0,0} \quad \text{and} \quad S_{a,b} \doteq \left( \frac{a}{a - \mathbf{\Lambda}} \mathbf{U} \frac{b}{b - {}^t\mathbf{\Lambda}} \right)_{0,0}.$$

Due to the antisymmetry property of the infinite matrix  $\mathbf{U}$ , it is obvious to see that

$$V_a = W_a \quad \text{and} \quad S_{a,b} = -S_{b,a}.$$

We define the tau function in this class by

$$\tau^2 = \det(1 + \mathbf{\Omega}\mathbf{C}), \quad (4.9)$$

since the antisymmetry of  $\mathbf{O}$  and  $\mathbf{C}$  will eventually make the above determinant a perfect square, namely the tau function itself is corresponding to a Pfaffian. We now consider the evolution of the tau function with respect to  $n_j$ , and this gives us

$$\begin{aligned} \mathbf{T}_{n_j} \tau^2 &= \det[1 + \mathbf{\Omega}(\mathbf{T}_{n_j} \mathbf{C})] = \det \left[ 1 + \mathbf{\Omega}\mathbf{C} + p_j \frac{1}{p_j - {}^t\mathbf{\Lambda}} (\mathbf{O}\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\mathbf{O}) \frac{1}{p_j - \mathbf{\Lambda}} \mathbf{C} \right] \\ &= \tau^2 \det \begin{bmatrix} 1 + p \left( \frac{\mathbf{\Lambda}}{p - \mathbf{\Lambda}} \mathbf{U} \frac{1}{p - {}^t\mathbf{\Lambda}} \right)_{0,0} & -p \left( \frac{\mathbf{\Lambda}}{p - \mathbf{\Lambda}} \mathbf{U} \frac{{}^t\mathbf{\Lambda}}{p - {}^t\mathbf{\Lambda}} \right)_{0,0} \\ p \left( \frac{1}{p - \mathbf{\Lambda}} \mathbf{U} \frac{1}{p - {}^t\mathbf{\Lambda}} \right)_{0,0} & 1 - p \left( \frac{1}{p - \mathbf{\Lambda}} \mathbf{U} \frac{{}^t\mathbf{\Lambda}}{p - {}^t\mathbf{\Lambda}} \right)_{0,0} \end{bmatrix} = \tau^2 V_{p_j}^2, \end{aligned}$$

where the rank 2 Weinstein–Aronszajn formula and equation (4.8) are used in the third and fourth equalities, respectively. Similarly, acting the backward shift on  $\tau$  gives us  $\mathbf{T}_{n_j}^{-1} \tau^2 / \tau = V_{-p_j}^2$ . In order to simplify this formula, we evaluate  $[(4.6a)]_{0,0}$ , which gives us  $V_{p_j}(\mathbf{T}_{n_j} V_{-p_j}) = 1$ . And thus, without loss of generality, we have

$$\frac{\mathbf{T}_{n_j} \tau}{\tau} = V_{p_j} \quad \text{and} \quad \frac{\mathbf{T}_{n_j}^{-1} \tau}{\tau} = V_{-p_j}. \quad (4.10a)$$

Next, we calculate  $[\frac{-p_i}{p_i - \mathbf{\Lambda}}(4.6a)]_{0,0}$ . The following equation is obtained:

$$1 + 2V_{p_j}(\mathbf{T}_{n_j} S_{-p_i, -p_j}) = \frac{p_i - p_j}{p_i + p_j} (V_{p_j} - V_{-p_i}) + V_{p_j}(\mathbf{T}_{n_j} V_{-p_i}).$$

This equation provides a way to express the  $S$ -variable by the  $V$ -variables, and consequently, the  $S$ -variable can be expressed by the tau function by making use of (4.10), and the formula is

$$S_{-p_i, -p_j} = \frac{1}{2} \left[ \frac{\mathbf{T}_{n_i}^{-1} \tau - \mathbf{T}_{n_j}^{-1} \tau}{\tau} + \frac{p_i - p_j}{p_i + p_j} \left( 1 - \frac{\mathbf{T}_{n_i}^{-1} \mathbf{T}_{n_j}^{-1} \tau}{\tau} \right) \right]. \quad (4.10b)$$

The expressions  $S_{p_i, -p_j}$ ,  $S_{-p_i, p_j}$  and  $S_{p_i, p_j}$  in terms of the tau function can also be derived from the above equation, with the help of  $(p_i, \mathbf{T}_{n_i}) \leftrightarrow (-p_i, \mathbf{T}_{n_i}^{-1})$ . Therefore, we have obtained the dynamical relations of  $\tau$  in terms of  $S_{a,b}$  for arbitrary  $a + b \neq 0$ . Furthermore, following the same idea of deriving (3.8b), we also have

$$2p_j \partial_{p_j} \ln \tau = n_j (V_{p_j} - V_{-p_j} - 2S_{p_j, -p_j}) \quad (4.10c)$$

for BKP. Equations listed in (4.10) establish the relations between the dynamics of the tau function and the  $V$ - and  $S$ -variables.

To construct equation (1.5a) in the DL framework, we compute  $[\frac{a}{a - \mathbf{\Lambda}}(4.6b)]_{0,0}$  and  $[\frac{a}{a - \mathbf{\Lambda}}(4.6b) \frac{b}{b - {}^t\mathbf{\Lambda}}]_{0,0}$ , which gives rise to

$$p_j \partial_{p_j} V_a = n_j \left[ V_a \left( \frac{2ap_j}{a^2 - p_j^2} - \frac{1}{2} V_{p_j} + \frac{1}{2} V_{-p_j} \right) + V_{p_j} \left( \frac{1}{2} \frac{p_j + a}{p_j - a} + S_{a, -p_j} \right) - V_{-p_j} \left( \frac{1}{2} \frac{p_j - a}{p_j + a} + S_{a, p_j} \right) \right], \quad (4.11a)$$

and

$$\begin{aligned}
p_j \partial_{p_j} S_{a,b} = n_j & \left[ \frac{1}{2} S_{p_j,b} \left( \frac{p_j+a}{p_j-a} - W_a \right) + \frac{1}{2} S_{-p_j,b} \left( \frac{a-p_j}{a+p_j} + W_a \right) \right. \\
& + \frac{1}{2} S_{a,p_j} \left( \frac{p_j+b}{p_j-b} - V_b \right) + \frac{1}{2} S_{a,-p_j} \left( \frac{b-p_j}{b+p_j} + V_b \right) \\
& \left. + S_{a,b} \left( \frac{2ap_j}{a^2-p_j^2} + \frac{2bp_j}{b^2-p_j^2} \right) - S_{a,p_j} S_{-p_j,b} + S_{a,-p_j} S_{p_j,b} \right], \quad (4.11b)
\end{aligned}$$

respectively. These two equations allow us to derive the closed-form equations in terms of the tau function. Notice that the logarithm derivative of  $\tau$  with respect to  $p_j$  satisfies (4.10c), and hence we have

$$p_i p_j \partial_{p_i} \partial_{p_i} \ln \tau = p_j \partial_{p_j} (p_i \partial_{p_i} \ln \tau) = p_j \partial_{p_j} [n_i (V_{p_i} - V_{-p_i} - 2S_{p_i,p_i})] = n_i [p_j (\partial_{p_j} V_{p_i}) - p_j (\partial_{p_j} V_{-p_i}) - 2p_j (\partial_{p_j} S_{p_i,p_i})],$$

in which the right hand side only involves the derivatives of the  $V$ - and  $S$ -variables with respect to  $p_j$ , and they can be further replaced by terms of  $V$ - and  $S$ -variables without derivatives with the help of (4.11a) and (4.11a) for special  $a$  and  $b$ . Finally, equations (4.10a) and (4.10b) help us to express every term on the right hand side of the above equation by the tau function. As a result, we derive (1.5a).

We can also derive an analogue of (1.4b) in the BKP class, from the DL scheme in this section. Setting  $a = p_i$  in (4.11a) yields

$$p_j \partial_{p_j} V_{p_i} = n_j \left[ V_{p_i} \left( \frac{2p_i p_j}{p_i^2 - p_j^2} - \frac{1}{2} V_{p_j} + \frac{1}{2} V_{-p_j} \right) + V_{p_j} \left( \frac{1}{2} \frac{p_j + p_i}{p_j - p_i} + S_{p_i, -p_j} \right) - V_{-p_j} \left( \frac{1}{2} \frac{p_j - p_i}{p_j + p_i} + S_{p_i, p_j} \right) \right].$$

Substituting  $V$  and  $S$  according to (4.10), we obtain (1.5b).

An interesting observation is that the right sides of (1.5a) and (1.5b) take the forms of Hirota's discrete-time Toda equation (a 5-point equation) [14] and the bilinear discrete KdV equation (a 6-point equation), respectively (see also chapter 8 of [13]), though the parametrisation here is entirely different.

## 5. DERIVATION OF (1.6)

We choose the same plane wave factors as those of BKP in the CKP class, namely

$$\rho_k = \prod_{i=1}^{\infty} \left( \frac{p_i + k}{p_i - k} \right)^{n_i} \quad \text{and} \quad \sigma_{k'} = \rho_{k'} = \prod_{i=1}^{\infty} \left( \frac{p_i + k'}{p_i - k'} \right)^{n_i}, \quad (5.1)$$

and these provide the same dynamical relations for the infinite matrix  $\mathbf{C}$  as follows:

$$(\mathbb{T}_{n_j} \mathbf{C}) \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} = \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{C}, \quad (5.2a)$$

$$\partial_{p_j} \mathbf{C} = n_j \left[ \left( \frac{1}{p_j + \Lambda} - \frac{1}{p_j - \Lambda} \right) \mathbf{C} + \mathbf{C} \left( \frac{1}{p_j + {}^t \Lambda} - \frac{1}{p_j - {}^t \Lambda} \right) \right], \quad (5.2b)$$

which describe the linear dispersion of the discrete CKP equation.

We select the Cauchy kernel

$$\Omega_{k,k'} = \frac{1}{k + k'} \quad (5.3)$$

for the CKP class, which is the same as that in the AKP case, and therefore we still have the fundamental relation for  $\Omega$  given by

$$\Omega \Lambda + {}^t \Lambda \Omega = \mathbf{O}. \quad (5.4)$$

Since the evolution of the infinite matrix  $\mathbf{C}$  is governed by (5.2), we reformulate (5.4) and present the following equations:

$$\Omega \frac{p_j + \Lambda}{p_j - \Lambda} - \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} \Omega = 2p_j \frac{1}{p_j + {}^t \Lambda} \mathbf{O} \frac{1}{p_j - \Lambda}, \quad (5.5a)$$

$$\Omega \left( \frac{1}{p_j + \Lambda} - \frac{1}{p_j - \Lambda} \right) + \left( \frac{1}{p_j + {}^t \Lambda} - \frac{1}{p_j - {}^t \Lambda} \right) \Omega = -\frac{1}{p_j - {}^t \Lambda} \mathbf{O} \frac{1}{p_j + \Lambda} - \frac{1}{p_j + {}^t \Lambda} \mathbf{O} \frac{1}{p_j - \Lambda}, \quad (5.5b)$$

in order to derive the dynamical relations of  $\mathbf{U}$  below.

Now we construct the dynamical evolutions of the infinite matrix  $\mathbf{U}$ . Acting the shift operator  $\mathbb{T}_{n_j}$  on equation (2.11) and taking (5.2a), we have

$$\mathbb{T}_{n_j} \mathbf{U} = [1 - (\mathbb{T}_{n_j} \mathbf{U}) \Omega] (\mathbb{T}_{n_j} \mathbf{C}) = [1 - (\mathbb{T}_{n_j} \mathbf{U}) \Omega] \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{C} \frac{p_j + {}^t \Lambda}{p_j - {}^t \Lambda}.$$

Notice that the infinite matrix  $\Omega$  obeys (5.5a). We can rewrite the above equation as

$$(\mathbb{T}_{n_j} \mathbf{U}) \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} = \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{C} - (\mathbb{T}_{n_j} \mathbf{U}) \Omega \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{C} = \frac{p_j + \Lambda}{p_j - \Lambda} \mathbf{C} - (\mathbb{T}_{n_j} \mathbf{U}) \left( 2p_j \frac{1}{p_j + {}^t \Lambda} \mathbf{O} \frac{1}{p_j - \Lambda} + \frac{p_j - {}^t \Lambda}{p_j + {}^t \Lambda} \Omega \right) \mathbf{C},$$

and this can be further simplified as

$$(\mathbb{T}_{n_j} \mathbf{U}) \frac{p_j - {}^t\mathbf{\Lambda}}{p_j + {}^t\mathbf{\Lambda}} = \frac{p_j + \mathbf{\Lambda}}{p_j - \mathbf{\Lambda}} \mathbf{U} - 2p_j (\mathbb{T}_{n_j} \mathbf{U}) \frac{1}{p_j + {}^t\mathbf{\Lambda}} \mathbf{O} \frac{1}{p_j - \mathbf{\Lambda}} \mathbf{U}. \quad (5.6a)$$

Equation (5.6a) is the first dynamical relation we need and it describes how  $\mathbf{U}$  evolves along the lattice directions  $n_j$ . The other dynamical relation we need is the one evolving with regard to  $p_j$ , and the derivation is very similar to those of (3.6b) and (4.6b), which reads

$$\partial_{p_j} \mathbf{U} = n_j \left[ \left( \frac{1}{p_j + \mathbf{\Lambda}} - \frac{1}{p_j - \mathbf{\Lambda}} \right) \mathbf{U} + \mathbf{U} \left( \frac{1}{p_j + {}^t\mathbf{\Lambda}} - \frac{1}{p_j - {}^t\mathbf{\Lambda}} \right) + \mathbf{U} \frac{1}{p_j - {}^t\mathbf{\Lambda}} \mathbf{O} \frac{1}{p_j + \mathbf{\Lambda}} \mathbf{U} + \mathbf{U} \frac{1}{p_j + {}^t\mathbf{\Lambda}} \mathbf{O} \frac{1}{p_j - \mathbf{\Lambda}} \mathbf{U} \right]. \quad (5.6b)$$

As is similar to the BKP case, we also need to impose a certain constraint on the spectral variables  $k$  and  $k'$ . This is realised by setting the integration domain  $D$  symmetric and simultaneously requiring integration measure satisfying

$$d\zeta(k', k) = d\zeta(k, k'). \quad (5.7)$$

Such a reduction results in a symmetric infinite matrix  $\mathbf{C}$  because

$${}^t\mathbf{C} = \iint_D d\zeta(k, k') \rho_k \mathbf{c}_{k'} {}^t\mathbf{c}_k \rho_{k'} = \iint_D d\zeta(k', k) \rho_{k'} \mathbf{c}_k {}^t\mathbf{c}_{k'} \rho_k = \mathbf{C}.$$

At the same time, we observe that the kernel (5.3) is symmetric in terms of  $k$  and  $k'$ , which implies that  ${}^t\mathbf{\Omega} = \mathbf{\Omega}$ . The symmetry properties of  $\mathbf{C}$  and  $\mathbf{\Omega}$  together guarantee that  $\mathbf{U}$  from (2.11) satisfies

$${}^t\mathbf{U} = \mathbf{U}, \quad (5.8)$$

i.e. it is a symmetric infinite matrix.

During the derivation of (1.6) below, for convenience we introduce variables

$$V_a \doteq 1 - \left( \mathbf{U} \frac{1}{a + {}^t\mathbf{\Lambda}} \right)_{0,0}, \quad W_a \doteq 1 - \left( \frac{1}{a + \mathbf{\Lambda}} \mathbf{U} \right)_{0,0} \quad \text{and} \quad S_{a,b} \doteq \left( \frac{1}{a + \mathbf{\Lambda}} \mathbf{U} \frac{1}{b + {}^t\mathbf{\Lambda}} \right)_{0,0}, \quad (5.9)$$

which satisfy

$$V_a = W_a \quad \text{and} \quad S_{a,b} = S_{b,a}, \quad (5.10)$$

respectively, due to the symmetry condition (5.8). We define the tau function in the CKP class as

$$\tau = \det(1 + \mathbf{\Omega}\mathbf{C}), \quad (5.11)$$

and after certain straightforward but relatively complex calculation based on (5.6a), we can derive

$$\frac{\mathbb{T}_{n_j} \tau}{\tau} = 1 + 2p_j S_{-p_j, -p_j} \quad \text{and} \quad \frac{\mathbb{T}_{n_j}^{-1} \tau}{\tau} = 1 - 2p_j S_{p_j, p_j}, \quad (5.12a)$$

as well as

$$[1 - (p_i + p_j) S_{p_i, p_j}]^2 = \frac{(p_i + p_j)^2 (\mathbb{T}_{n_i}^{-1} \tau) (\mathbb{T}_{n_j}^{-1} \tau) - (p_i - p_j)^2 \tau (\mathbb{T}_{n_i}^{-1} \mathbb{T}_{n_j}^{-1} \tau)}{4p_i p_j \tau^2}. \quad (5.12b)$$

Similar to the BKP case, the  $S_{-p_i, p_j}$ ,  $S_{p_i, -p_j}$  and  $S_{-p_i, -p_j}$  analogues of (5.12b) follow from the interchange relation  $(p_i, \mathbb{T}_{n_i}) \leftrightarrow (-p_i, \mathbb{T}_{n_i}^{-1})$ . Moreover, the dynamics of the tau function with respect to the lattice parameter  $p_j$  is derived by carrying the same calculation in the derivation of (3.8b), which establishes the link between  $\tau$  and  $S$  and takes the form of

$$\partial_{p_j} \ln \tau = 2n_j S_{p_j, -p_j}. \quad (5.12c)$$

Notice from (5.12) that the tau function is connected with the  $S$ -variable. We therefore multiply (5.6b) by  $(a + \mathbf{\Lambda})^{-1}$  from the left and  $(b + {}^t\mathbf{\Lambda})^{-1}$  from the right simultaneously and take the  $(0, 0)$ -entry, and as a result, an equation for  $S_{a,b}$  arises in the form of

$$\begin{aligned} \partial_{p_j} S_{a,b} = n_j & \left[ \frac{1}{p_j - a} (S_{a,b} - S_{p_j, b}) + \frac{1}{p_j + a} (S_{-p_j, b} - S_{a, b}) \right. \\ & \left. + \frac{1}{b - p_j} (S_{a, p_j} - S_{a, b}) + \frac{1}{p_j + b} (S_{a, -p_j} - S_{a, b}) - S_{a, -p_j} S_{p_j, b} - S_{a, p_j} S_{-p_j, b} \right]. \end{aligned}$$

By setting  $a = p_i$  and  $b = -p_i$ , it is rewritten as

$$\begin{aligned} \partial_{p_j} S_{p_i, -p_i} = \frac{n_j}{(p_i + p_j)^2} & [(1 - (p_i + p_j) S_{p_i, p_j}) (1 + (p_i + p_j) S_{-p_i, -p_j}) - 1] \\ & + \frac{n_j}{(p_i - p_j)^2} [(1 - (p_i - p_j) S_{p_i, -p_j}) (1 - (p_j - p_i) S_{-p_i, p_j}) - 1]. \end{aligned}$$

It is then possible to replace the  $S$ -variable by  $\tau$  through (5.12a) and (5.12b) in the above equation. As a result, it leads to the nonautonomous differential-difference CKP equation (1.6). We note that this equation is still in the form of the (2+2)-dimensional Toda-type, but here the bilinearity is broken (though the equation is still quadratic) compared with (1.4a) and (1.5a) since the square root operation is involved. This is not surprising as we have seen

the discrete CKP equation (1.3) is a quartic equation. Equation (1.6) can alternatively be written as an eighth-power equation

$$\left\{ \left[ \frac{p_i p_j (p_i^2 - p_j^2)^2}{n_i n_j} D_{p_i} D_{p_j} \tau \cdot \tau + 8 p_i p_j (p_i^2 + p_j^2) \tau^2 \right]^2 - (p_i - p_j)^4 EF - (p_i + p_j)^4 GH \right\}^2 = 4(p_i^2 - p_j^2)^4 EFGH$$

by taking square twice in order to eliminate the square root in the expression, where  $E$ ,  $F$ ,  $G$  and  $G$  are already given in section 1 below (1.3).

## 6. REMARKS ON SOLITON SOLUTIONS, MULTI-DIMENSIONAL CONSISTENCY AND HIGHER-ORDER SYMMETRIES

We have shown how the six nonautonomous DΔEs are derived from the DL scheme, by associating each equation with a certain linear integral equation in the form of (2.6). From the viewpoint of the DL approach, this guarantees the integrability of the resulting nonlinear DΔEs, as long as their respective linear integral equations are solvable. Here by integrability we mean the existence of an exact solution possessing an infinite number of free parameters (e.g. soliton-type solution), while the initial boundary value problem is not discussed. Below we present soliton solutions of these DΔEs, by performing a very special reduction on the measure in the respective linear integral equations.

We start with the formal tau functions (3.7), (4.9) and (5.11) in order to construct soliton-type solutions, instead of the linear integral equation (2.6), since the tau functions defined as such already contain the key ingredients of the linear integral equation, namely the integration domain and measure, the plane wave factors as well as the Cauchy kernel. The procedure is exactly the same as that given in [11]. Hence, we only provide a short guide and directly list the results below. For equations (1.4a), (1.4b) and (1.4c), we take the measure

$$d\zeta(k, k') = \sum_{i=1}^N \sum_{j=1}^{N'} \frac{A_{i,j}}{(2\pi i)^2} \frac{1}{k - k_i} \frac{1}{k' - k'_j} dk dk' \quad \text{with } i \text{ being the imaginary unit,} \quad (6.1)$$

which results in the determinant solution (i.e. soliton solution)

$$\tau = \det(1 + \mathbf{A}\mathbf{M}), \quad (6.2)$$

where  $1$  denotes the  $N \times N$  unit matrix,  $\mathbf{A} = (A_{i,j})_{N \times N'}$  is an arbitrary matrix and the entries of  $\mathbf{M} = (M_{j,i})_{N' \times N}$  are given by

$$M_{j,i} = \frac{\rho_{k_i} \sigma_{k'_j}}{k_i + k'_j}, \quad \text{with } \rho \text{ and } \sigma \text{ given by (3.1).}$$

Here  $N$  and  $N'$  are arbitrary positive integers. Similarly, for equations (1.5a) and (1.5b) we take a special measure

$$d\zeta(k, k') = \sum_{i,j=1}^{2N} \frac{A_{i,j}}{(2\pi i)^2} \frac{1}{k - k_i} \frac{1}{k' - k'_j} dk dk', \quad A_{i,j} = -A_{j,i}, \quad (6.3)$$

which obeys the antisymmetry condition in (4.7). In this case, the tau function is determined by

$$\tau^2 = \det(1 + \mathbf{A}\mathbf{M}), \quad (6.4)$$

where  $\mathbf{A} = (A_{i,j})_{2N \times 2N}$  is an antisymmetric matrix and the Cauchy matrix  $\mathbf{M} = (M_{j,i})_{2N \times 2N}$  has its  $(j, i)$ -entry

$$M_{j,i} = \rho_{k_i} \frac{1}{2} \frac{k_i - k'_j}{k_i + k'_j} \sigma_{k'_j}, \quad \text{in which } \rho \text{ and } \sigma \text{ are defined by (4.1).}$$

For the CKP class, we consider the following measure reduction:

$$d\zeta(k, k') = \sum_{i,j=1}^N \frac{A_{i,j}}{(2\pi i)^2} \frac{1}{k - k_i} \frac{1}{k' - k'_j} dk dk', \quad A_{i,j} = A_{j,i}, \quad (6.5)$$

where we have respected the condition in (5.7). Then the formal tau function (5.11) turns out to be

$$\tau = \det(1 + \mathbf{A}\mathbf{M}), \quad (6.6)$$

in which  $\mathbf{A} = (A_{i,j})_{N \times N}$  is a symmetric matrix and the  $(i, j)$ -entry of  $\mathbf{M} = (M_{j,i})_{N \times N}$  is defined as

$$M_{j,i} = \frac{\rho_{k_i} \sigma_{k'_j}}{k_i + k'_j}, \quad \text{where } \rho \text{ and } \sigma \text{ are defined as (5.1).}$$

The DL also provides a perspective to understand the MDC property of the six DΔEs, from their respective solution structures. We focus on the formal tau functions (3.7), (4.9) and (5.11), since the soliton-type expressions (6.2), (6.4) and (6.6) are the natural consequences arising from the measure reductions (6.1), (6.3) and (6.5), respectively. We take equation (1.5a) as an example. The formal solution (4.9) relies on its flow variables  $p_i$ ,  $p_j$ ,  $n_i$  and  $n_j$  through the infinite matrix  $\mathbf{C}$ , whose dynamics are completely determined by the plane wave factor  $\rho_k \sigma_{k'}$  given in (4.1). Observing that all independent variables are on the same footing<sup>2</sup> in (4.1), we conclude that DΔEs in the form of (1.5a) with regard to different lattice parameters and variables share the same nontrivial solution (4.9); in other words, these

<sup>2</sup>This is also the reason why we adopt the notations  $p_j$  and  $n_j$  for  $j = 1, 2, \dots$  rather than select fixed indices from the beginning.

equations are compatible with each other. This is the MDC property that we normally adopt in the theory of discrete integrable systems, and here it is applied to the differential-difference case.

Furthermore, the derivation of (1.4a), (1.4b) and (1.4c) shows that the three equations have the same tau function (3.7) as the common solution, and such a tau function simultaneously (cf. [11]) serves a solution of (1.1). This implies that these four equations are multi-dimensionally consistent, within the DL scheme; in other words, equations (1.4a), (1.4b), (1.4c) and (1.1) should be treated as compatible flows in the AKP class. Similarly, equations (1.5a), (1.5b) and (1.2) are also multi-dimensionally consistent with each other; while equation (1.6) is compatible with (1.3), from the same viewpoint. This is also the MDC property, namely equations in different forms can still be consistent with each other. As a remark, we would like to note that here the MDC is only verified on the solutions generated by the DL framework. The verification of the consistency on the level of equations still remains future work. We believe the assertion is correct, as the solutions following from the DL already possess an infinite number of degrees of freedom.

With the help of the solution structures of the AKP, BKP and CKP classes discussed above, we are now allowed to explain how the higher-order symmetries in the continuous KP-type hierarchies are generated from the lattice parameters. The idea to realise this is to compare the effective plane wave factors in each discrete KP class with its corresponding continuous analogues (which are given in [12]). In the AKP class, we require

$$\rho_k \sigma_{k'} = \prod_{i=1}^{\infty} \left( \frac{p_i + k}{p_i - k'} \right)^{n_i} = \exp \left\{ \sum_{j=1}^{\infty} (k^j - (-k')^j) \frac{(-1)^{j-1}}{j} \sum_{i=1}^{\infty} \frac{n_i}{p_i^j} \right\} = \exp \left\{ \sum_{j=1}^{\infty} (k^j - (-k')^j) x_j \right\}.$$

Therefore, the relationship between the continuous variables  $x_j$  and the lattice parameters  $p_i$  is given by

$$x_j = \frac{(-1)^{j-1}}{j} \sum_{i=1}^{\infty} \frac{n_i}{p_i^j} \quad \text{for } j = 1, 2, \dots, \quad \text{which results in } \partial_{p_i} = \sum_{j=1}^{\infty} (-1)^j \frac{n_i}{p_i^{j+1}} \partial_{x_j}. \quad (6.7)$$

Likewise, as the form of the plane wave factor is different, we require in the BKP and CKP classes

$$\rho_k \sigma_{k'} = \prod_{i=1}^{\infty} \left( \frac{p_i + k}{p_i - k} \frac{p_i + k'}{p_i - k'} \right)^{n_i} = \exp \left\{ \sum_{j=0}^{\infty} (k^{2j+1} + k'^{2j+1}) \frac{2}{2j+1} \sum_{i=1}^{\infty} \frac{n_i}{p_i^{2j+1}} \right\} = \exp \left\{ \sum_{j=0}^{\infty} (k^{2j+1} + k'^{2j+1}) x_{2j+1} \right\},$$

and subsequently we obtain

$$x_{2j+1} = \frac{2}{2j+1} \sum_{i=1}^{\infty} \frac{n_i}{p_i^{2j+1}} \quad \text{for } j = 0, 1, 2, \dots, \quad \text{which leads to } \partial_{p_i} = -2 \sum_{j=0}^{\infty} \frac{n_i}{p_i^{2j+2}} \partial_{x_{2j+1}}. \quad (6.8)$$

The transformations between the derivatives of lattice parameters  $p_i$  and continuous flow variables  $x_j$  listed in (6.7) and (6.8) provide a way of generating all the higher-order symmetries of the continuous KP-type from the lattice parameters. They are the counterparts of the result in the generating PDE of the discrete KdV equation given in [26] on the KP level, though the generating PDEs for the discrete KP-type equations are not yet clear so far.

## 7. CONCLUSIONS

Based on the DL framework for the discrete AKP, BKP and CKP equations, we introduce a new perspective to construct integrable nonlinear equations, by thinking of the lattice parameters as the independent variables and treating them together with the discrete variables equally. As a result, six novel nonautonomous differential-equations are proposed, including two (2+2)-dimensional nonautonomous DΔEs, namely (1.5a) and (1.6) from BKP and CKP classes, respectively, as well as four (2+1)-dimensional nonautonomous semi-discrete equations, i.e. (1.4a), (1.4b) and (1.4c) from the AKP class and (1.5b) from the BKP class.

It was already shown that nonautonomous DΔEs often play the role of master symmetries for lower-dimensional discrete integrable systems [33], while the master symmetry theory for higher-dimensional lattice equations is not yet clear. These new equations potentially provide us with an insight into understanding master symmetries of three-dimensional (3D) integrable discrete models. Furthermore, it seems that (2+2)-dimensional nonautonomous integrable models of Toda-type, such as equations (1.5a) and (1.6), have never appeared in the literature so far, to be best of the authors' knowledge. Their geometric interpretation still needs to be discovered.

In addition to symmetries and master symmetries, we also note that the integrability of discrete KP-type equations was also proven from the perspective of conservations laws. In [22], conservation laws for the discrete AKP and BKP equations were constructed, while it remains a problem for the discrete CKP equation.

The ultimate goal is to find closed-form generating PDEs for higher-dimensional integrable hierarchies and their integrability structures. This also remains an open problem for future study.

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## REFERENCES

- [1] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.
- [2] A.I. Bobenko and W.K. Schief, *Discrete line complexes and integrable evolution of minors*, Proc. R. Soc. A **471** (2015), 20140819.
- [3] ———, *Discrete line complexes and integrable evolution of minors*, Int. Math. Res. Notices **2017** (2017), 1504–1561.
- [4] E. Date, M. Jimbo, and T. Miwa, *Method for generating discrete soliton equations. III*, J. Phys. Soc. Jpn. **52** (1983), 388–393.
- [5] A. Doliwa, *The B-quadrilateral lattice, its transformations and the algebro-geometric construction*, J. Geom. Phys. **57** (2007), 1171–1192.
- [6] ———, *Desargues maps and the Hirota–Miwa equation*, Proc. R. Soc. A **466** (2010), 1177–1200.
- [7] ———, *The C-(symmetric) quadrilateral lattice, its transformations and the algebro-geometric construction*, J. Geom. Phys. **60** (2010), 690–707.
- [8] A.S. Fokas and M.J. Ablowitz, *Linearization of the Korteweg–de Vries and Painlevé II equations*, Phys. Rev. Lett. **47** (1981), 1096–1110.
- [9] ———, *On the inverse scattering and direct linearizing transforms for the Kadomtsev–Petviashvili equation*, Phys. Lett. A **94** (1983), 67–70.
- [10] W. Fu, *Direct linearisation of the discrete-time two-dimensional Toda lattices*, J. Phys. A: Math. Theor. **51** (2018), 334001.
- [11] W. Fu and F.W. Nijhoff, *Direct linearizing transform for three-dimensional discrete integrable systems: the lattice AKP, BKP and CKP equations*, Proc. R. Soc. A **473** (2017), 20160915.
- [12] ———, *Linear integral equations, infinite matrices, and soliton hierarchies*, J. Math. Phys. **59** (2018), 071101.
- [13] J. Hietarinta, N. Joshi, and F.W. Nijhoff, *Discrete Systems and Integrability*, Cambridge University Press, Cambridge, 2016.
- [14] R. Hirota, *Nonlinear partial difference equations II: Discrete-time Toda equation*, J. Phys. Soc. Jpn. **43** (1977), 2074–2078.
- [15] ———, *Discrete analogue of a generalized Toda equation*, J. Phys. Soc. Jpn. **50** (1981), 3785–3791.
- [16] ———, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, 2004.
- [17] R.M. Kashaev, *On discrete three-dimensional equations associated with the local Yang–Baxter relation*, Lett. Math. Phys. **38** (1996), 389–397.
- [18] B.G. Konopelchenko and L. Martínez Alonso, *The KP hierarchy in Miwa coordinates*, Phys. Lett. A **258** (1999), 272–278.
- [19] B.G. Konopelchenko and W.K. Schief, *Menelaus’ theorem, Clifford configurations and inversive geometry of the Schwarzian KP hierarchy*, J. Phys. A: Math. Gen. **35** (2002), 6125–6144.
- [20] ———, *Reciprocal figures, graphical statics and inversive geometry of the Schwarzian BKP hierarchy*, Stud. Appl. Math. **109** (2002), 89–124.
- [21] D. Levi and R. Benguria, *Bäcklund transformations and nonlinear differential difference equations*, Proc. Natl. Acad. Sci. **77** (1980), 5025–5027.
- [22] K. Maruno and G.R.W. Quispel, *Construction of integrals of higher-order mappings*, J. Phys. Soc. Jpn. **75** (2006), 123001.
- [23] T. Miwa, *On Hirota’s difference equations*, Proc. Jpn. Acad. **58A** (1982), 9–12.
- [24] F.W. Nijhoff and H.W. Capel, *The direct linearisation approach to hierarchies of integrable PDEs in  $2 + 1$  dimensions: I. Lattice equations and the differential-difference hierarchies*, Inverse Probl. **6** (1990), 567–590.
- [25] F.W. Nijhoff, H.W. Capel, G.L. Wiersma, and G.R.W. Quispel, *Bäcklund transformations and three-dimensional lattice equations*, Phys. Lett. A **105** (1984), 267–272.
- [26] F.W. Nijhoff, A. Hone, and N. Joshi, *On a Schwarzian PDE associated with the KdV hierarchy*, Phys. Lett. A **267** (2000), 147–156.
- [27] F.W. Nijhoff, V.G. Papageorgiou, H.W. Capel, and G.R.W. Quispel, *The lattice Gel’fand–Dikii hierarchy*, Inverse Probl. **8** (1992), 597–621.
- [28] F.W. Nijhoff, G.R.W. Quispel, and H.W. Capel, *Direct linearisation of difference-difference equations*, Phys. Lett. A **97** (1983), 125–128.
- [29] F.W. Nijhoff, G.R.W. Quispel, J. van der Linden, and H.W. Capel, *On some linear integral equations generating solutions of nonlinear partial differential equations*, Physica A **119** (1983), 101–142.
- [30] F.W. Nijhoff, A. Ramani, B. Grammaticos, and Y. Ohta, *On Discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation*, Stud. Appl. Math. **106** (2001), 261–314.
- [31] O.G. Rasin and P.E. Hydon, *Symmetries of integrable difference equations on the quad-graph*, Stud. Appl. Math. **119** (2007), 253–269.
- [32] W.K. Schief, *Lattice geometry of the discrete Darboux, KP, BKP and CKP equations Menelaus and Carnots theorem*, J. Nonlinear Math. Phys. **10** (suppl. 2) (2003), 194–208.
- [33] A. Tongas, D. Tsoubelis, and V. Papageorgiou, *Symmetries and group invariant reductions of integrable partial difference equations*, Proceedings of the 10th International Conference in Modern Group Analysis (N. H. Ibragimov, C. Sophocleous, and P. A. Damianou, eds.), 2005, pp. 222–230.
- [34] A. Tongas, D. Tsoubelis, and P. Xenitidis, *A family of integrable nonlinear equations of hyperbolic type*, J. Math. Phys. **42** (2001), 5762–5784.
- [35] A.S. Tongas and F.W. Nijhoff, *Generalized hyperbolic Ernst equations for an Einstein–Maxwell–Weyl field*, J. Phys. A: Math. Gen. **38** (2005), 895–906.
- [36] V.E. Vekslerchik, *Solitons of the  $(2+2)$ -dimensional Toda lattice*, J. Phys. A: Math. Theor. **52** (2019), 045202.
- [37] P. Xenitidis, F.W. Nijhoff, and S.B. Lobb, *On the Lagrangian formulation of multidimensionally consistent systems*, Proc. R. Soc. A **467** (2011), 20110124.

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