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Stochastic Camassa-Holm equation with convection type noise *

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Abstract

We consider a stochastic Camassa-Holm equation driven by a one-dimensional Wiener process with a first order differential operator as diffusion coefficient. We prove the existence and uniqueness of local strong solutions of this equation. In order to do so, we transform it into a random quasi-linear partial differential equation and apply Kato's operator theory methods. Some of the results have potential to find applications to other nonlinear stochastic partial differential equations.

1 Introduction

The (deterministic) Camassa-Holm (CH) equation is a non-local partial differential equation describing propagation of waves in shallow water. Although first introduced by Fuchssteiner and Fokas in [25] as part of a family of integrable Hamiltonian equations, it was rediscovered by Camassa and Holm [12], who gave its physical derivation and interpretation. In contrast to the Korteveg-de-Vries equation, the CH equation admits so-called peaked

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solutions describing wave breaking phenomena. Various aspects of the CH equation have been extensively studied, see literature reviews in, e.g., [17] and [3]. In particular, it is known that the CH equation is locally well-posed in Sobolev spaces $H^s := H^{s,2}(\mathbb{R})$, $s > 3/2$. Depending on the shape of the initial data, the solution can either exist globally or blow up in any Sobolev space, with its slope becoming vertical in finite time [17].

The CH equation has the form

$$\begin{aligned} &u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} \\ &\equiv (1-\partial_x^2)u_t + (1-\partial_x^2)\,[uu_x] + \partial_x \left[u^2 + \frac{1}{2}\,(u_x)^2\right] = 0,\ t > 0, x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $u(t,x)$ denotes the fluid velocity at time t. Here $\partial_x := \frac{\partial}{\partial x}$. Introducing a momentum density

$$y := u - u_{xx} \equiv (1-\partial_x^2)u =: Q^2 u,$$

one can rewrite equation (1.1) in a quasi-linear form

$$y_t(t) + A(y(t))y(t) = 0 \tag{1.2}$$

in any other suitable functional space. Here $A(v) := a(v)\partial_x + b(v)$ is the first-order differential operator with coefficients $a(v) = Q^{-2}v,\ b(v) = 2\,(\partial_x Q^{-2} v)$, that is,

$$[A(v)f]\,(x) = a(v)\partial_x f(x) + b(v)f(x),\ x \in \mathbb{R}. \tag{1.3}$$

Recently, Holm [27] proposed an approach for including stochastic perturbations in hydrodynamics equations. This approach is based on a stochastic extension of the variational principle in fluid dynamics. The corresponding stochastic version of the CH equation (1.1) was introduced by Crisan and Holm in [19]. It has the following form:

$$\begin{aligned} &dy(t) + A(y(t))y(t)dt + \sum_{k=1}^{n} \left(\partial_x y(t) + y(t)\partial_x\right)\xi_k \circ dw_k(t) \\ &\qquad \equiv dy(t) + A(y(t))y(t)dt + \sum_{k=1}^{n} D_k y(t) \circ dw_k(t) = 0,\ t > 0, y \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Here $D_k := \xi_k\partial_x + \partial_x\xi_k \equiv \xi_k\partial_x + 2\,(\partial_x\xi_k)\,,\ k = 1,..n$, are first-order differential operators associated with suitable functions (vector fields) $\xi_k : \mathbb{R} \to \mathbb{R}$, $w_k,\ k = 1,..n$, are independent Wiener processes and $\circ dw_k(t)$ stands for the Stratonovich stochastic differential (see Def. 2.1 below). For further developments from [27], [19] see, e.g., [28], [29]; these also relate to stochastic thermodynamics and turbulence, for which we refer to, e.g., [7], [13], [23], [24].

In this paper, we study the case of a single vector field ξ (with $n = 1$). In order to deal with the diffusion term of equation (1.4), we transform it into a partial differential equation with random coefficients. This approach goes back to the paper [36] by Sussman , see also Doss [22]. These works were concerned with stochastic ordinary differential equations and motivated by the control theory. In stochastic partial differential equations (SPDEs) theory, the Doss-Sussman method was first used in [1] and [6]. Both papers studied the Wong-Zakai approximations (or robustness) of linear SPDEs with drift being the generator of an analytic semigroup. The corresponding Banach space setting generalizations can be found in [10]. One should also mention the paper [21] by Da Prato and Tubaro, where fully nonlinear equations are considered. This paper only deals with the parabolic situation and therefore its results cannot be applied to our model.

Recently, the Doss-Sussman method was used in [26] to study the convergence of a finite element method for stochastic Landau-Lifshitz-Gilbert equations. The Wong-Zakai approximations to such equations were studied in [8]. Other related papers are [9] and [14], where it was noted that the Doss-Sussman method could lead to an alternative proof of the main result therein, and then applied to (nonlinear) stochastic compressible Euler equations. Another example of the use of the Doss-Sussman method is its application to the stochastic nonlinear Schrödinger equation, see [2].

After applying the Doss-Sussman method to equation (1.4) (with $n = 1$), we study the resulting partial differential equation (PDE) using a modified version of the approach of [17] based on Kato operator theory techniques. Our main result is the proof of the existence and uniqueness of local strong solutions of equation (1.4) in the Sobolev spaces $H^{2,p}$, $1 < p < \infty$. We hope that with an additional work it should also be possible to prove the robustness of this equation. Also, a modification of our method based on a paper [38] by Tubaro should allow for the study of the case of multiple (non-commuting) vector fields ξ_k on the right-hand side of (1.4).

Let us mention that a stochastic CH equation with additive noise was introduced and studied in [15]; the case of a multiplicative noise given by a one-dimensional Wiener process with H^s-continuous diffusion coefficient was considered in [16] and [37]. Those studies do not cover the case of the noise as in (1.4), where the diffusion coefficient is generated by an unbounded linear operator. The importance of studying equation (1.4) has been stressed by Crisan and Holm in [19] because of its geometric and physical motivations, and its relevance in geophysical applications.

The structure of the paper is as follows. In Section 2, we formulate the main result and derive the explicit form of the PDE obtained by the Doss-Sussman method. Section 3 is devoted to the general Kato method and its application to the latter PDE, which leads to the proof of our main result in Section 3.3. In Section 4.1 we provide the proofs of (auxiliary) technical results on the regularity of one-parameter groups generated by first order differential operators. Section 4.2 deals with the Doss-Sussman correspondence between SDEs and (random) ordinary differential equations in some Sobolev spaces, adapted to our setting. Finally, in Section 4.3 we give a brief account of the technical result from [35] on the boundedness of commutators, which is used in the proof of Theorem 3.4.

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2 Stochastic Camassa-Holm equation

2.1 Formulation of the main result

Throughout the paper, we will use the following standard notations: L^p – the space of (equivalence classes of) p-integrable functions on $\mathbb{R}$; $H^{n,p}$ – the Sobolev space on $\mathbb{R}$; C_b^n – the space of bounded n-times continuously differentiable functions on $\mathbb{R}$ with bounded derivatives; $\mathcal{L}(X,Y)$ – the space of bounded linear operators between generic Banach spaces X and Y; $\mathcal{L}(X) := \mathcal{L}(X,X)$.

We will consider a stochastic Camassa-Holm equation (SCH) of the form

$$dy(t)+F(y(t))dt+Dy(t)\circ dw(t) = 0,\ t \geq 0,\ F(y) := A(y)y,\ y \in H^{1,p}, \quad (2.1)$$

on a suitable filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, where $A(v)$ is given by formula (1.3), $D = \xi\partial_x + \eta$, $\xi \in C_b^4$, $\eta \in C_b^3$ and w is a one-dimensional Wiener process. We will be looking for a solution of this equation in $H^{2,p}$, $1 < p < \infty$.

Definition 2.1 *A strong solution of equation (2.1) is an $H^{2,p}$-valued continuous adapted process $y(t)$, $t \in [0,\theta]$, where θ is a finite stopping time, such that*

(i) the function $[0,\tau] \ni s \mapsto F(y(s)) \in L^p$ is integrable, $\mathbb{P}$-a.s.;

(ii) for every $t \geq 0$, $\mathbb{E}\int_0^{t\wedge\theta} |Dy(s)|^p_{L^p}\, ds < \infty$,

and the equality

$$y(t\wedge\theta) = y_0 + \int_0^{t\wedge\theta} F(y(s))ds + \int_0^{t\wedge\theta} Dy(s)\circ dw(s),\ t \geq 0,$$

is satisfied in L^p, $\mathbb{P}$*-a.s., for every* $t \geq 0$. *Here* $\circ dw(s)$ *stands for the Stratonovich stochastic differential, that is,*

$$\int_0^{t\wedge\theta} Dy(s)\circ dw(s) = \frac{1}{2}\int_0^{t\wedge\theta} D^2y(s)ds + \int_0^{t\wedge\theta} Dy(s)dw(s)$$

(the latter integral being the Itô stochastic integral).

We can now formulate the main result of this work.

Theorem 2.2 *For any* $y_0 \in H^{2,p}$, $1 < p < \infty$, *there exists a stopping time* $\theta > 0$ *and a strong solution* $y(t) \in H^{2,p}$, $t \in [0,\theta]$, *of equation (2.1) with initial condition* $y(0) = y_0$. *If* $y^1(t)$ *and* $y^2(t)$, $t \in [0,\theta]$, *are two such solutions then* $y^1 = y^2$.

The proof will go along the following lines: first, we reduce equation (2.1) to a PDE of a form similar to (1.2) but with time-dependent coefficients, and then apply the general Kato method, in a similar way as for the case of the deterministic Camassa-Holm equation, see [17].

2.2 Reduction to a random PDE

Let us fix $\xi \in C_b^4$, $\eta \in C_b^3$ and consider the one-parametric group $U = \left(U_t^{\xi,\eta}\right)_{t\in\mathbb{R}}$ of operators in the Lebesgue space L^p defined by the formula

$$\left[U_t^{\xi,\eta} f\right](x) = e^{c(t,x)} f(\varphi_{-t}(x)),\ f \in L^p,\ x \in \mathbb{R},\ t \geq 0, \tag{2.2}$$

where φ_t, $t \in \mathbb{R}$, is a diffeomorphism generated by the vector field $\xi\partial_x$ and $c(t,x) = \int_0^t \eta(\varphi_{s-t}(x))ds$, see Lemma 4.6 in Section 4.1 below.

According to the results of Section 4.1 (Lemma 4.5), U is a strongly continuous group not only in the Lebesgue space L^p but also in the Sobolev spaces $X := H^{1,p}$ and $Y := H^{2,p}$. Note that all these spaces are UMD Banach spaces. Thus there exist constants $C_1, C_2 < \infty$ such that

$$\left\|U_t^{\xi,\eta}\right\|_{\mathcal{L}(X)}, \left\|U_t^{\xi,\eta}\right\|_{\mathcal{L}(Y)} \leq C_1 e^{C_2|t|},\ t \in \mathbb{R}. \tag{2.3}$$

For the corresponding generators $(D^X, \mathrm{Dom}(D^X))$ and $(D^Y, \mathrm{Dom}(D^Y))$ we have

$$H^{2,p} \subset \mathrm{Dom}(D^X) \text{ and } H^{3,p} \subset \mathrm{Dom}(D^Y),$$

and the restrictions of D^X and D^Y on $H^{2,p}$ and $H^{3,p}$, respectively, coincide with the first order differential operator $D = \xi\partial_x + \eta$. It is clear that $D, D^2 \in \mathcal{L}(H^{2,p}, L^p)$. Note also that $\mathrm{Dom}(D^Y) \subset \mathrm{Dom}(D^X)$. In this section, we will write U_t in place of $U_t^{\xi,\eta}$, whenever possible.

According to the results of Section 4.2, with $Y = H^{2,p}$ and $\mathfrak{X} = L^p$, equation (2.1) is equivalent to the following random integral equation in L^p:

$$z(t) = z(0) - \int_0^t \widehat{F}(s, z(s))ds, \ t \geq 0, \tag{2.4}$$

where

$$\widehat{F}(t, z) := U_{w(t)} F\left(U_{w(t)}^{-1} z\right) \equiv \widehat{A}(w(t), z)z, \ t \geq 0, z \in H^{2,p},$$

and

$$\widehat{A}(t, v) := U_t A(U_t^{-1} v) U_t^{-1}, \ t \geq 0, v \in H^{2,p}. \tag{2.5}$$

Our next goal is to study the structure of operator $\widehat{A}(t, v)$.

Consider a generic first order differential operator $\mathcal{A} = a_0\partial_x + b_0$ with the coefficients $a_0 \in H^{3,p}$ and $b_0 \in H^{2,p}$ and define the operators

$$C(t) := U_t \mathcal{A} U_t^{-1}, \ t \geq 0, \tag{2.6}$$

on the domain $H^{2,p}$. Note that $C(t) \in \mathcal{L}(H^{2,p}, H^{1,p})$.

Lemma 2.3 *Assume that $a_0 \in H^{3,p}$, $b_0 \in H^{2,p}$. The operator $C(t)$ defined above by formula (2.6) has the form*

$$C(t)v = a(t, \cdot)\partial_x v + b(t, \cdot)v, \ v \in H^{2,p}, \ t \geq 0, \tag{2.7}$$

where $a(t, x)$ and $b(t, x)$ solve the system of first order partial differential equations

$$\begin{aligned} a_t(t, x) &= \xi(x)a_x(t, x) - \xi_x(x)a(t, x), \ a(0, x) = a_0(x), \\ b_t(t, x) &= \xi(x)b_x(t, x) - \eta_x(x)a(t, x), \ b(0, x) = b_0(x), \end{aligned}$$

(with subscript x denoting as usual the derivative ∂_x). Moreover, $a(t) := a(t, \cdot) \in H^{3,p}$ and $b(t) := b(t, \cdot) \in H^{2,p}$ and

$$\|a(t)\|_{H^{3,p}} \leq C_1 e^{tC_2} \|a_0\|_{H^{3,p}}, \ \|b(t)\|_{H^{2,p}} \leq C_1 e^{tC_2} \left(\|a_0\|_{H^{3,p}} + \|b_0\|_{H^{2,p}}\right) \tag{2.8}$$

for some constants $C_1, C_2 > 0$ (depending only on ξ and η).

Proof of Lemma 2.3. Let us fix $f \in H^{3,p}$ and consider the map $\mathbb{R} \ni t \mapsto C(t)f \in H^{1,p}$. Observe that $H^{3,p} \subset \mathrm{Dom}_Y(D)$ and $\mathcal{A}U_t^{-1} : H^{3,p} \to \mathrm{Dom}_X(D)$ for all t, which implies that the function $C(t)f$, $t \geq 0$, with $C(t)$ being given by (2.6), is differentiable and satisfies equation

$$\frac{d}{dt}C(t)f = [D, C(t)]\, f,$$

where $[\cdot,\cdot]$ stands for the commutator. The substitution of the explicit expression (2.2) in the formula $C(t) = U_t\mathcal{A}U_t^{-1}$ shows that $C(t)$ has the form (2.7). We can now compute the commutator:

$$[D, C(t)] = [\xi\partial_x + \eta, a(t)\partial_x + b(t)] = \alpha(t)\partial_x + \beta(t),$$

where

$$\alpha(t) = \xi a_x(t) - \xi_x a(t),\ \ \beta(t) = \xi b_x(t) - \eta_x a(t).$$

Observe that $f \in H^{3,p}$ belongs to the domain of the operators $DC(t)$ and $C(t)D$. Thus we have

$$\frac{d}{dt}C(t)f = (\xi a_x(t) - \xi_x a(t))\,\partial_x f + (\xi b_x(t) - \eta_x a(t))\, f,\ \ t \geq 0,$$

On the other hand, by (2.7),

$$\frac{d}{dt}C(t)f = a_t\partial_x f + b_t f,\ \ t \geq 0,$$

so that

$$\begin{aligned} a_t &= \xi a_x - \xi_x a,\ \ a(0) = a_0,\ \ t \geq 0, \\ b_t &= \xi b_x - \eta_x a,\ \ b(0) = b_0,\ \ t \geq 0. \end{aligned}$$

Thus for any $t \geq 0$ we have

$$C(t)f = a(t)\partial_x f + b(t)f,\ \ f \in H^{3,p}. \tag{2.9}$$

Observe, on the other hand, that the operators $C(t)$ and $a(t)\partial_x + b(t)$ belong to $\mathcal{L}(H^{2,p}, H^{1,p})$. Thus equality (2.9) can be extended to any $f \in H^{2,p}$.

Thus, recalling that $U_t^{\xi,-\xi'}$ is a one-parameter group generated by the operator $\xi\partial_x - \xi_x$, we have the representation

$$a(t) = U_t^{\xi,-\xi'}a_0,\ \ t \geq 0, \tag{2.10}$$

and

$$b(t) = U_t^{\xi,0}b_0 + \int_0^t U_{t-\tau}^{\xi,0}\left(\eta' a(\tau)\right) d\tau,\ \ t \geq 0. \tag{2.11}$$

Since by Lemma 4.5 below both $U_t^{\xi,-\xi'}$ and $U_t^{\xi,0}$ leave the spaces $H^{1,p}, H^{2,p}$ and $H^{3,p}$ invariant, we infer that $a(t) \in H^{3,p}$ and $b(t) \in H^{2,p}$. The bound (2.8) follows now easily from (2.3), (2.10) and (2.11). The proof is complete. ■

We can now return to the operator family $\widehat{A}(t,v)$ given by (2.5).

Proposition 2.4 *For any $t \geq 0$ and $v \in H^{1,p}$, the operator $\widehat{A}(t,v)$ has the form*

$$\widehat{A}(t,v) = a(t,v)\partial_x + b(t,v)$$

on the domain $H^{2,p}$, where $a(t,v) \in H^{3,p}$ and $b(t,v) \in H^{2,p}$ are given by formulae (2.10) and (2.11) with

$$a_0 = Q^{-2}U_t^{-1}v \text{ and } b_0 = 2\partial_x Q^{-2}U_t^{-1}v, \tag{2.12}$$

respectively, and satisfy the bound

$$\|a(t)\|_{H^{3,p}} \leq C_1 e^{tC_2} \|v\|_{H^{1,p}}, \ \|b(t)\|_{H^{2,p}} \leq C_1 e^{tC_2} \|v\|_{H^{1,p}}. \tag{2.13}$$

for some $C_1, C_2 < \infty$.

Proof. We can first fix any s and apply Lemma 2.3 to operator (2.6) with $\mathcal{A} : = A(U_s^{-1}v)$ and then set $s = t$. The bound (2.13) follows from (2.8) and estimate (4.14) of the norm of U_t. □

Corollary 2.5 *For any $v \in H^{1,p}$ we have $\widehat{A}(t,v) \in \mathcal{L}(H^{2,p}, H^{1,p})$ and the map $\mathbb{R} \ni t \mapsto \widehat{A}(t,v) \in \mathcal{L}(H^{2,p}, H^{1,p})$ is continuous.*

Proof. The result follows from formulae (2.10), (2.11), (2.12) and the strong continuity of the one-parameter groups $U_t^{\xi,-\xi'}$, $U_t^{\xi,0}$ and U_t. □

3 Quasi-linear equations via Kato's method

3.1 General Kato's method

Consider a pair of densely embedded Banach spaces $Y \subset X$ and a quasi-linear equation in X:

$$\frac{d}{dt}v + A(t,v)v = 0, \ v(0) = v_0 \in Y, \ t \in [0,T], \tag{3.1}$$

for some $T > 0$, where $A(t,v)$ is a linear (unbounded) operator in X with domain $D_{t,v} := \mathrm{Dom}(A(t,v)) \supset Y$.

We introduce the following condition, which is a version of the condition given in [31, page 34] adapted to our setting. Let $I \subset \mathbb{R}$ be an interval.

Condition 3.1 *There exists $R > 0$ such that the operator family $A(t,v)$, $v \in Y$, $t \in I$, satisfies the following:*

- *for any $v \in Y$ and $t \in I$ operator $-A(t,v)$ is quasi-m-accretive, that is, it generates a C_0-semigroup in X and there exists $\beta = \beta(R) \in \mathbb{R}$ such that*
$$\left\| e^{-sA(t,v)} \right\|_X \leq e^{\beta s}, \;\; s \geq 0, \;\; \|v\|_Y \leq R; \tag{3.2}$$

- *there exists an isomorphism $Q : Y \to X$ and $B(t,v) \in \mathcal{L}(X,X)$ such that, for all $v \in Y$ and $t \in I$, we have*
$$QA(t,v)Q^{-1} = A(t,v) + B(t,v); \tag{3.3}$$
the map $I \ni t \mapsto B(t,v) \in X$ is strongly measurable and
$$\lambda = \lambda(R) := \sup_{t \in I} \sup_{v : \|v\|_Y \leq R} \|B(t,v)\| < \infty; \tag{3.4}$$

- *for any $v \in Y$ and $t \in I$ we have $A(t,v) \in \mathcal{L}(Y,X)$ and the map*
$$I \ni t \mapsto A(t,v) \in \mathcal{L}(Y,X) \tag{3.5}$$
is continuous;

- *there exists $\mu_A = \mu_A(R)$ such that for all $u \in Y$ and $\|v_1\|_Y, \|v_2\|_Y \leq R$ we have*
$$\|(A(t,v_1) - A(t,v_2))\, u\|_X \leq \mu_A \|v_1 - v_2\|_X \|u\|_Y . \tag{3.6}$$

Theorem 3.2 *Let Condition 3.1 hold on the time interval $I = [0,T]$. Then for every $v_0 \in Y$ there exists $T' = T'(v_0) \leq T$ and a unique solution $v \in C([0,T'],Y) \cap C^1([0,T'],X)$ of equation (3.1).*

Proof. See [31, Theorem 6, page 36] □

Remark 3.3 *T' is an arbitrary number satisfying the following bounds:*
$$\begin{aligned} \exp\left((\beta + \lambda)\, T'\right) &< R\, \|v_0\|_Y^{-1}, \\ T' \exp\left(\beta T'\right) &< R^{-1} \mu_A^{-1}, \end{aligned}$$
Here the constants $\beta = \beta(R)$, $\lambda = \lambda(R)$ and $\mu_A = \mu_A(R)$ are defined in (3.2), (3.4) and (3.6), respectively, see [31, p. 45]. The corresponding solution of equation (3.1) will satisfy the bound $\|v(t)\|_Y \leq R$.

3.2 Kato's condition for first order differential operators

We set $X = H^{1,p}$, $Y = H^{2,p}$ and $Q = (1-\partial_x^2)^{1/2}$. It is known that $Q : H^{2,p} \to H^{1,p}$ is an isometric isomorphism. We first consider the family of first order differential operators

$$\mathcal{A}(y) = a(y)\partial_x + b(y), \; y \in H^{1,p}, \tag{3.7}$$

defined on $H^{2,p}$, with coefficients $a(y) \in H^{3,p}, b(y) \in H^{2,p}$, $y \in H^{1,p}$. We assume that the maps

$$a : H^{1,p} \to H^{3,p}, \; b : H^{1,p} \to H^{2,p} \text{ are Lipschitz continuous} \tag{3.8}$$

and bounded (uniformly in y), that is,

$$\sup_{y \in H^{1,p}} \|a(y)\|_{H^{3,p}} < \infty, \; \sup_{y \in H^{1,p}} \|b(y)\|_{H^{2,p}} < \infty. \tag{3.9}$$

It is clear that $\mathcal{A}(y) \in \mathcal{L}(Y, X)$ with the uniformly (in $y \in H^{1,p}$) bounded norm.

According to the results of Section 4.1 (Lemma 4.5 below), for any $y \in H^{1,p}$, there exists a one-parameter C_0-group in $H^{1,p}$ such that its generator contains $H^{2,p}$ in its domain and coincides with $\mathcal{A}(y)$ on $H^{2,p}$. We will preserve the notation $\mathcal{A}(y)$ for this operator. Observe that, again by Lemma 4.5, there exists an operator $\mathcal{A}^{(0)}(y)$ in L^p, which coincides with $\mathcal{A}(y)$ on $H^{2,p}$ and generates a one-parameter C_0-group in L^p.

Theorem 3.4 *The operator family (3.7) satisfies Condition 3.1 on the time interval $I = [-\tau, \tau]$, with arbitrary $R > 0$ and $\tau > 0$.*

Proof. (i) The first point in Condition 3.1 immediately follows from the results of Section 4.1 below. Indeed, the fact that $-\mathcal{A}(y)$ is the generator of the C_0-semigroup in X and estimate (3.2) follow from Lemma 4.5 below and the bound (3.9).

(ii) Condition (3.3) is proved in [17, Remark 2.6 b)] for $p = 2$, $a(y) = Q^{-2}v$ and $b(y) = 2\left(\partial_x Q^{-2}v\right), \; v = Qy \in L^p$, cf. (1.3). The proof does not use the explicit form of the coefficients and can be extended to the case of any $p \in [0, \infty)$. Here we give its main steps adapted to our setting.

We fix $y \in H^{1,p}$ and use the shorthand notation $\mathcal{A} := \mathcal{A}(y)$ and $\mathcal{A}^{(0)} := \mathcal{A}^{(0)}(y)$. The first step is to prove equality (3.3) for the operator $\mathcal{A}^{(0)}$ in the pair of spaces $H^{1,p} \subset L^p$. Denote by M_a and M_b the operators of multiplication by $a := a(y) \in H^{3,p}$ and $b := b(y) \in H^{2,p}$, respectively. Define an operator B by the equality $Bf := Q\mathcal{A}^{(0)}Q^{-1}f - \mathcal{A}^{(0)}f$ for $f \in \mathcal{S} := C^\infty \cap L^p$. Then on $\mathcal{S}$ we have the equality

$$B = [Q, M_a]\, \partial_x Q^{-1} + QM_bQ^{-1} - M_b,$$

because $\partial_x Q^{-1} f = Q^{-1}\partial_x f$ for $f \in \mathcal{S}$. The operators M_b, QM_bQ^{-1} and $\partial_x Q^{-1}$ are bounded in both spaces L^p and $H^{1,p}$, with $\|M_b\|_{\mathcal{L}(L^p)} \le \|b\|_{H^{1,p}}$ and

$$\left\|QM_bQ^{-1}\right\|_{\mathcal{L}(L^p)} = \|M_b\|_{\mathcal{L}(H^{1,p})} \le \|b\|_{H^{1,p}}\,, \tag{3.10}$$

$$\left\|QM_bQ^{-1}\right\|_{\mathcal{L}(H^{1,p})} = \|M_b\|_{\mathcal{L}(H^{2,p})} \le \|b\|_{H^{2,p}} \tag{3.11}$$

(because $H^{1,p}$ and $H^{2,p}$ are Banach algebras).

We denote by $k_1, k_2, .., k_5$ generic locally bounded functions $\mathbb{R}_+ \to \mathbb{R}_+$. It follows from the results of [35, Sections VII.3.5 and I.5.2] that the commutator $[Q, M_a]$ is bounded in L^p and the exists k_1 such that

$$\|[Q, M_a]\|_{\mathcal{L}(L^p)} \le k_1(\|\partial_x a\|_{H^{1,p}}), \tag{3.12}$$

see Section 4.3. This bound together with (3.10) implies that B is a bounded operator in L^p and

$$\|B\|_{\mathcal{L}(L^p)} \le k_2(\max(\|\partial_x a\|_{H^{1,p}}\,, \|b\|_{H^{1,p}})).$$

It is proved in [17, Proposition 2.3 a)] that $\mathcal{S}$ is a core for $\mathcal{A}^{(0)}$. Although the paper [17] deals with the case of $p = 2$ only, the proof can be directly extended to any $p > 1$.

This is sufficient for the equality

$$Q\mathcal{A}^{(0)}Q^{-1} = \mathcal{A}^{(0)} + B \tag{3.13}$$

to hold ([31, Remark 7.1.3.]).

We observe that the operator $\mathcal{A}$ coincides with the part of $\mathcal{A}^{(0)}$ in $H^{1,p}$ ([34, Theorem 4.5.5 and Lemma 5.4.4]). Thus, equality (3.3) for $\mathcal{A}$ will follow from (3.13) provided $B \in \mathcal{L}(H^{1,p})$. For this, it is sufficient to show that $[Q, M_a] \in \mathcal{L}(H^{1,p})$. Similar to [17, Remark 2.6 b)], we can write

$$\begin{aligned} \|[Q, M_a]\|^2_{\mathcal{L}(H^{1,p})} &= \left\|[Q, M_a]\, Q^{-1}\right\|^2_{\mathcal{L}(L^p, H^{1,p})} \\ &\le \left\|[Q, M_a]\, Q^{-1}\right\|^2_{\mathcal{L}(L^p)} + \left\|\partial_x\, [Q, M_a]\, Q^{-1}\right\|^2_{\mathcal{L}(L^p)}. \end{aligned}$$

The first term is bounded by $\|Q^{-1}\|^2_{\mathcal{L}(L^p)}\, k_1\left(\|\partial_x a\|^2_{H^{1,p}}\right) \le k_3\left(\|\partial_x a\|^2_{H^{2,p}}\right)$, cf. (3.12). For the second term we have

$$\partial_x\, [Q, M_a]\, Q^{-1} = QM_{\partial_x a}Q^{-1} - M_{\partial_x a} + [Q, M_a]\, \partial_x Q^{-1},$$

which, together with (3.11) applied to the operator $QM(\partial_x a)Q^{-1}$ and a new use of (3.12), leads to the bound

$$\left\|\partial_x \left[Q, M_a\right] Q^{-1}\right\|^2_{\mathcal{L}(L^p)} \leq k_4 \left(\|\partial_x a\|_{H^{2,p}}\right),$$

and so

$$\|B\|_{\mathcal{L}(H^{1,p})} \leq k_5 \left(\max\left(\|\partial_x a\|_{H^{2,p}}, \|b\|_{H^{2,p}}\right)\right). \tag{3.14}$$

Finally, estimate (3.4) follows now from assumption (3.8).

(iii) Condition (3.5) trivially holds because $A(v)$ is independent of t. Condition (3.6) can be checked directly using (3.8).

□

Remark 3.5 *We observe that (3.3) remains true if the coefficients a and b in (3.7) are t-dependent and such that $\|\partial_x a\|_{H^{2,p}}$ and $\|b\|_{H^{2,p}}$ are bounded uniformly in t (for every $y \in H^{1,p}$). For condition (3.5) to hold, it is sufficient that, for every $y \in H^{1,p}$, the maps $\mathbb{R} \ni t \mapsto a(t, y) \in H^{2,p}$ and $\mathbb{R} \ni t \mapsto b(t, y) \in H^{1,p}$ are continuous.*

Remark 3.6 *In [17, Remark 2.6 b)], the authors took a slightly different path (for $p = 2$). They proved Condition 3.1 for the pair $X = L^2$ and $Y = H^{1,2}$, which implies the existence of a solution of the Camassa-Holm equation (1.2) in $H^{1,2}$. Then they show that the solution actually belongs to $H^{2,2}$ provided the initial condition does so.*

Remark 3.7 *We do not know if Condition 3.1, which is required for an application of Kato's theory, holds for $X = H^{n,p}$ with $n \geq 2$. Indeed, we can prove the quasi-contractivity of the semigroup generated by $-\mathcal{A}(y)$ only in $H^{0,p} = L^p$ and $H^{1,p}$, see the proof of Lemma 4.3 below.*

3.3 Proof of the main result.

In this section we will show that Kato's theory can be applied to the integral equation (2.4). Recall that

$$\widehat{A}(t, v) = U_t A(U_t^{-1} v) U_t^{-1}, \; v \in H^{2,p}, \; t \in \mathbb{R}, \tag{3.15}$$

cf. (2.5). It has been proved in Proposition 2.4 that $\widehat{A}(t, v) = a(t, v)\partial_x + b(t, v)$ with $a(t, v) \in H^{3,p}$ and $b(t, v) \in H^{2,p}$. As before, we retain the same notation for the generator of the corresponding one-parameter C_0-group in L^p (see Lemma 4.5 below).

Theorem 3.8 *The operator family (3.15) satisfies Condition 3.1 with $X = H^{1,p}$ and $Y = H^{2,p}$ on the time interval $I = [-\tau, \tau]$, with arbitrary $R > 0$ and $\tau > 0$.*

Proof. It is clear that the coefficients $a(t,v) \in H^{3,p}$ and $b(t,v) \in H^{2,p}$ are bounded uniformly in t so that (3.14) is satisfied. Also, the Lipschitz condition (3.8) holds because of the explicit form (2.10), (2.11) of the coefficients and uniform in $t \in [-\tau, \tau]$ boundedness of the group U_t in both $\mathcal{L}(X,X)$ and $\mathcal{L}(Y,Y)$ (cf. (2.3)). Thus, according to Theorem 3.4 and Remark 3.5, the operator family $\widehat{A}(t,v)$, $t \in [-\tau, \tau]$, satisfies the first two parts of Condition 3.1 with arbitrary R.

The continuity condition (3.5) is proved in Corollary 2.5. Estimate (3.6) immediately follow from (3.15) and (2.3). □

Remark 3.9 (Change of time) *Let $f : [0,T] \to [-\tau, \tau]$ be a continuous function. It is clear that operator family $A_f(t,v) := \widehat{A}(f(t), v)$, $t \in [0,T]$, satisfies Condition 3.1. Moreover, since $\sup_{t\in[0,T]} \left\|U_{f(t)}\right\| \le \sup_{t\in[-\tau,\tau]} \|U_t\|$, the constants β, λ and μ_A remain unchanged.*

We return now to the stochastic Camassa-Holm equation (2.1), defined on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}\right)$.

Theorem 3.10 *For any $R > 0$ and $z_0 \in \mathbb{R}$ and each continuous Brownian path $w(t)$ there exists $\theta > 0$ and a unique solution $z \in C([0,\theta], H^{2,p})$, of the integral equation (2.4), such that $z(0) = z_0$ and $\|z(t)\|_{H^{2,p}} \le R$, $t \in [0,\theta]$.*

Proof. Fix $R > 0$ and a continuous Brownian path $w(t)$. Fix in addition $T > 0$ and define $\tau = \tau(w) := \inf\{t > 0 : |w(t)| \ge T\}$. According to Theorem 3.8 and Remark 3.9, the operator family $\widehat{A}(w(t), v)$, $t \in [0,\tau]$, satisfies Condition 3.1 with the constants β, λ and μ_A (depending on R and T).

Next, we choose any $T' > 0$ such that

$$\begin{aligned} \exp\left((\beta+\lambda)\, T'\right) &\le R\, \|v_0\|_{H^{2,p}}^{-1}, \\ T' \exp(\beta T') &< R^{-1}\mu_A^{-1}, \end{aligned}$$

and define $\theta := \min\{\tau, T'\}$. Then, by Theorem 3.2, there exists a solution $z \in C([0,\theta], H^{2,p})$ of the integral equation (2.4), such that $\|z(t)\|_{H^{2,p}} \le R$, $t \in [0,\theta]$. □

Remark 3.11 *It is clear that, for any $R > 0$, both τ and θ are stopping times.*

Proof of Theorem 2.2. The process $z(t)$ constructed in Theorem 3.10 satisfies the conditions of Theorem 4.8 with $Y = H^{2,p}$ and $\mathfrak{X} = L^p$, which implies that $y(t) := U_{w(t)}^{-1} z(t)$, $t \in [0,\theta]$, is the unique strong solution of equation (2.1). □

4 Auxiliary results

In this section we present some general results used in the main part of the paper.

4.1 One-parameter groups generated by first order differential operators

The aim of this section is to discuss properties of one-parameter groups in Sobolev spaces $H^{n,p}$, $n = 0, 1, 2, ...$, generated by first order differential operators. We will use the convention $H^{0,p} = L^p$.

We need some preparations. Recall that C_b^n, $n \geq 1$, denotes the Banach space of n-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$. We equip it with the norm

$$\|f\|^{(n)} := \max_{m=0,\dots,n} \sup_{x \in \mathbb{R}} \left| f^{(m)}(x) \right| < \infty,$$

where $f^{(m)}$ stands for the m-th derivative, $f^{(0)} \equiv f$. We set $C_b^0 := C_b$ endowed with the usual supremum norm.

Given a function $g(t, x)$, $t, x \in \mathbb{R}$, we will keep the notation $g^{(m)}(t, x) := \partial_x^m g(t, x)$ for the m-th derivative w.r.t. x. We will use, where possible, notations $g(t)$ and $g^{(m)}(t)$ for the mappings $x \mapsto g(t, x)$. and $x \mapsto g^{(m)}(t, x)$, respectively. Thus, we have, e.g., $g^{(1)}(t) : x \mapsto \partial_x g(t, x)$.

The following statement is essentially well-known and we give only a sketch of its proof.

Lemma 4.1 *Fix $n \in \mathbb{N}$ and assume that $\xi \in C_b^{n+1}$. Then there exist constants $c_1, c_2, c_3, c_4 > 0$ such that the following statements hold.*
(i) The equation

$$\frac{d}{dt}\psi(t, x) = -\xi(\psi(t, x)), \;\; \psi(0, x) = x, \;\; x \in \mathbb{R}, \tag{4.1}$$

has a unique solution $\psi(t)$, $t \in \mathbb{R}$. This solution satisfies the estimate

$$|\psi(t, x)| \leq c_1 e^{c_2|t|} \left(|x| + c_3\right). \tag{4.2}$$

(ii) The solution ψ is differentiable with respect to the x-variable; moreover, for any $t \in \mathbb{R}$, the derivative $\psi^{(1)}(t, \cdot) \in C_b^n$, and the map

$$\mathbb{R}_+ \ni t \mapsto \psi^{(1)}(t, \cdot) \in C_b^n$$

is continuously differentiable.
(iii) The following estimate holds:

$$\sup_{x \in \mathbb{R}} \left| \psi^{(1)}(t, \cdot) \right| \leq e^{c_4|t|}, \;\; t \in \mathbb{R}. \tag{4.3}$$

Proof of Lemma 4.1.

(i): For any fixed $x \in \mathbb{R}$, equation (4.1) has a unique solution because its right-hand side is globally Lipschitz. Estimate (4.2) follows in a standard way from the Gronwall Lemma.

(ii) and (iii): Consider the linear operator $\widehat{\xi}(t)$ acting on functions $u : \mathbb{R} \to \mathbb{R}$ by multiplication by $\xi^{(1)}(\psi(t, \cdot))$, that is,

$$(\widehat{\xi}(t)u)(x) := \xi^{(1)}(\psi(t,x))u(x).$$

It is immediate that $\psi^{(1)}(t) := \psi^{(1)}(t, \cdot)$ solves the equation

$$\frac{d}{dt}\psi^{(1)}(t) = -\widehat{\xi}(t)\psi^{(1)}(t),\ \ \psi^{(1)}(0) = 1. \tag{4.4}$$

A direct calculation shows that $\widehat{\xi}(t)$ is a bounded operator in C_b with norm

$$\left\|\widehat{\xi}(t)\right\|_{\mathcal{L}(C_b)} \le \|\xi\|^{(1)}.$$

Thus equation (4.4) has a unique solution in C_b, which satisfies the bound (4.3) and is continuously differentiable in t. In a similar way,

$$\left\|\widehat{\xi}(t)\right\|_{\mathcal{L}(C_b^1)} \le \left(\left\|\psi^{(1)}(t)\right\|^{(0)} + 1\right) \|\xi\|^{(2)} < \infty,$$

which shows that $\psi^{(1)}(t) \in C^1(\mathbb{R}, C_b^1)$. Statement (ii) for an arbitrary $n \in \mathbb{N}$ follows now by a repeated application of similar arguments. ■

Remark 4.2 *In particular, Lemma 4.1 implies in a standard way that* $(\psi(t))_{t\in\mathbb{R}}$ *is a one-parameter group of* C^{n+1}*-diffeomorphisms of* $\mathbb{R}^1$*, generated by the vector field* $-\xi\partial_x$*.*

Let us introduce an operator family U_t^ξ, $t \in \mathbb{R}$, by the formula $U_t^\xi f = f(\psi(t)),\ f : \mathbb{R} \to \mathbb{R}$.

Lemma 4.3 *Fix* $n \in \mathbb{N}$ *and assume that* $\xi \in C_b^{n+1}$*. Then* $U_t^\xi,\ t \in \mathbb{R}$*, is a strongly continuous one-parameter group of bounded operators in* $H^{n,p}$*. If* $n = 0$ *or* $n = 1$ *than* U_t^ξ *is quasi-contractive, that is, there exists a constant* $c < \infty$ *such that*

$$\left\|U_t^\xi\right\|_{\mathcal{L}(H^{n,p})} \le e^{c|t|},\ \ t \in \mathbb{R}. \tag{4.5}$$

The domain of the generator D_0 *of* U_t^ξ *contains* $H^{n+1,p}$ *and one has* $D_0 = \xi\partial_x$ *on* $H^{n+1,p}$*.*

Proof. In this proof, $c, c_1, c_2, ...$ will stand for universal positive constants (depending only on n and $\|\xi\|^{(n+1)}$).

1) Let us prove that the operators $U_t^\xi,\ t \in \mathbb{R}$, are bounded in $H^{n,p}$.

Consider first the case of $n = 0$. Then, for $f \in H^{0,p} \equiv L^p$, we have

$$\left\| U_t^\xi f \right\|_{L^p}^p = \int f(\psi(t,x))^p dx$$
$$\leq \sup_{x \in \mathbb{R}} \left| \psi^{(1)}(-t,x) \right|^p \|f\|_{L^p}^p \leq e^{2c_4|t|} \|f\|_{L^p}^p, \quad (4.6)$$

cf. (4.3), and estimate (4.5) holds.

Let now $n \geq 1$. By Faà di Bruno's theorem for any $k = 1, 2, ...n$ we have

$$\partial_x^k f(\psi(t,x)) = \sum_{m=1}^{k} f^{(m)}(\psi(t,x)) B_{k.m}(\psi^{(1)}(t,x), ..., \psi^{(k-m+1)}(t,x)), \quad (4.7)$$

where $B_{k.m}$ is the exponential Bell polynomial. Hölder's inequality implies that

$$\left| \partial_x^k f(\psi(t,x)) \right|^p$$
$$\leq k^{p-1} \max_{m=1,...,k} \sup_{x \in \mathbb{R}} \left| B_{k.m}(\psi^{(1)}(t,x), ..., \psi^{(k-m+1)}(t,x)) \right|^p \sum_{m=1}^{k} \left| f^{(m)}(\psi(t,x)) \right|^p. \quad (4.8)$$

It follows from Lemma 4.1 (ii) that $\sup_x \left| \psi^{(m)}(t,x) \right| < \infty$ for any $m = 1, ..., n$. Thus $\sup_{x \in \mathbb{R}} \left| B_{k.m}(\psi^{(1)}(t,x), ..., \psi^{(k-m+1)}(t,x)) \right| < \infty$ for any $k = 1, 2, ...n$, and we obtain the estimate

$$\|f(\psi(t))\|_{H^{n,p}} \leq c(t) \|f\|_{H^{n,p}}, \quad (4.9)$$

for some $c(t) < \infty$, which implies that $U_t^\xi \in \mathcal{L}(H^{n,p}),\ t \in \mathbb{R}$.

We observe that in the case of $n = 1$ formula (4.8) gets the form

$$|\partial_x f(\psi(t,x))|^p \leq \sup_{x \in \mathbb{R}} \left| \psi^{(1)}(t,x) \right|^p \left| f^{(1)}(\psi(t,x)) \right|^p.$$

It follows now from (4.3) and (4.6) that

$$\|f(\psi(t))\|_{H^{1,p}}^p \leq \int \left[|f(\psi(t,x))|^p + \sup_{x \in \mathbb{R}} \left| \psi^{(1)}(t,x) \right|^p \left| f^{(1)}(\psi(t,x)) \right|^p \right] dx$$
$$\leq \int \sup_{x \in \mathbb{R}} \left| \psi^{(1)}(-t,x) \right|^p \left[f(x)^p + \sup_{x \in \mathbb{R}} \left| \psi^{(1)}(t,x) \right|^p f^{(1)}(x)^p \right] dx$$
$$\leq e^{4|t|c_4} \|f\|_{H^{1,p}}^2,$$

and estimate (4.5) holds.

2) The fact that the operators $U_t^\xi \in \mathcal{L}(H^{n,p}),\ t \in \mathbb{R}$, form a one-parameter group, that is, $U_{t_1}^\xi U_{t_2}^\xi = U_{t_1+t_2}^\xi$, $t_1, t_2 \in \mathbb{R}$, and $U_0^\xi = I$, follows in a standard way from the group properties of the family of diffeomorphisms $\psi(t)$, $t \in \mathbb{R}$.

3) Now we shall prove that the map $\mathbb{R} \ni t \mapsto U_t^\xi$ is strongly continuous. Let $f \in C_0^\infty$. Observe that $\psi^{(m)}(t,x) \to x^{(m)} = \begin{cases} 1, & m=1 \\ 0, & m \geq 2 \end{cases}$, $t \to 0$, uniformly on compact sets. Thus for the r.h.s. of (4.7) we have

$$\sum_{m=1}^{k} f^{(m)}(\psi(t,x)) B_{k.m}(\psi^{(1)}(t,x), ..., \psi^{(k-m+1)}(t,x))$$
$$\rightrightarrows \sum_{m=1}^{k} f^{(m)}(x)) B_{k.m}(x^{(1)}, ..., x^{(k-m+1)}) = f^{(k)}(x),\ t \to 0,$$

where $\rightrightarrows$ stands for the uniform convergence in $x \in \mathbb{R}$. The last equality holds because $x^{(m)} = \begin{cases} 1, & m=1 \\ 0, & m \geq 2 \end{cases}$ and thus $B_{k.m}(x^{(1)}, ..., x^{(k-m+1)}) = \begin{cases} 1, & m=k \\ 0, & m \leq k-1 \end{cases}$. Therefore $\partial_x^k f(\psi(t)) \overset{L^p}{\to} f^{(k)},\ t \to 0$, for any $k \leq n$, which implies the $H^{n,p}$-convergence $f(\psi(t)) \to f$, as $t \to 0$.

Let now $u \in H^{n,p}$. We have the estimate

$$\left\| U_t^\xi u - u \right\|_{H^{k,p}} \leq \left\| U_t^\xi u - U_t^\xi f \right\|_{H^{k,p}} + \left\| U_t^\xi f - f \right\|_{H^{k,p}} + \|f-u\|_{H^{k,p}}$$
$$\leq \left\| U_t^\xi f - f \right\|_{H^{k,p}} + c\,\|f-u\|_{H^{k,p}},$$

and the required result follows from the fact that C_0^∞ is dense in $H^{n,p}$.

3) Let us prove that $t \mapsto U_t^\xi u \in H^{n,p}$ is differentiable for $u \in H^{n+1,p}$.

Let $v \in C_0^\infty$. Formula (4.7) implies that $t \mapsto \partial_x^k v(\psi(t,x))$ is continuously differentiable for any $x \in \mathbb{R}$. Denote

$$F(x) := \frac{d}{dt} \partial_x^k v(\psi(t,x))_{t=0}.$$

Then, because v has compact support,

$$\frac{\partial_x^k v(\psi(t,x)) - v^{(k)}(x)}{t} = \partial_x^k \frac{v(\psi(t,x)) - v(x)}{t} \rightrightarrows F(x),\ t \to 0,$$

and so we infer that

$$\frac{v(\psi(t)) - v}{t} \longrightarrow \frac{d}{dt} v(\psi(t))_{t=0},\ \text{in } H^{n,p} \text{ as } t \to 0. \tag{4.10}$$

We will prove now that (4.10) holds for any $u \in H^{n+1,p}$. Set $g(t,x) = \partial_x^k u(\psi(t,x))$ and $f(t,x) = \partial_x^k v(\psi(t,x))$. Then (denoting the derivative w.r.t. the first variable by "dot") we obtain

$$\begin{aligned}\dot{g}(s,x) &= \frac{d}{ds}\partial_x^k u(\psi(s,x)) = \partial_x^k \frac{d}{ds} u(\psi(s,x)) \\ &= \partial_x^k \left[\xi(x)\partial_x u(\psi(s,x))\right] = \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array}\right) \xi^{(m)}(x)\partial_x^{k-m}\partial_x u(\psi(s,x)). \end{aligned} \tag{4.11}$$

In particular,

$$\dot{g}(0,x) = \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array}\right) \xi^{(m)}(x)\partial_x^{k-m+1} u(x). \tag{4.12}$$

Of course, similar formulae hold for f.

Thus, applying Hölder's inequality, we obtain for any $t > 0$:

$$\begin{aligned}&\int_{\mathbb{R}} \left| \frac{1}{t}\int_0^t \left(\dot{g}(s,x) - \dot{f}(s,x)\right) ds \right|^p dx \\ &\qquad \leq \frac{1}{t}\int_{\mathbb{R}}\int_0^t \left|\dot{g}(s,x) - \dot{f}(s,x)\right|^p dsdx \\ &\leq \sum_m \left(\begin{array}{c} k \\ m \end{array}\right) \frac{1}{t}\int_0^t\int_{\mathbb{R}} \left|\xi^{(m)}(x)\right|^p \left|\partial_x^{k-m+1}\left(u(\psi(s,x)) - v(\psi(s,x))\right)\right|^p dxds \\ &\qquad \leq c\, \|\xi\|_{C_b^k}^p \\ &\qquad \sum_m \left(\begin{array}{c} k \\ m \end{array}\right) \frac{1}{t}\int_0^t\int_{\mathbb{R}} \left|u^{(k-m+1)}(\psi(s,x)) - v^{(k-m+1)}(\psi(s,x))\right|^p dxds.\end{aligned}$$

The last inequality is due to the formulae (4.7) and (4.9). Taking into account that $\int p(\psi(s,x))dx = \int |\partial_x \psi(s,x)^{-1}|\, p(x)dx$ for any integrable function p we obtain

$$\int_{\mathbb{R}} \left| \frac{1}{t}\int_0^t \left(\dot{g}(s,x) - \dot{f}(s,x)\right) ds \right|^p dx \leq c_1 \|u - v\|_{H^{k+1,p}}^2 .$$

Observe that (4.12) implies that

$$\int_{\mathbb{R}} \left|\dot{g}(0,x) - \dot{f}(0,x)\right|^p dx \leq c_2 \|u - v\|_{H^{k+1,p}}^p .$$

The following general relation holds for any $t > 0,\ x \in \mathbb{R}$:

$$\begin{aligned}
\frac{g(t,x)-g(0,x)}{t} - \dot{g}(0,x) &= \frac{1}{t}\int_0^t \dot{g}(s,x)ds - \dot{g}(0,x) \\
&= \frac{1}{t}\int_0^t \left(\dot{g}(s,x) - \dot{f}(s,x)\right) ds + \left[\frac{1}{t}\int_0^t \dot{f}(s,x)ds - \dot{f}(0,x)\right] \\
&\qquad + \left[\dot{f}(0,x) - \dot{g}(0,x)\right] \\
&= \frac{1}{t}\int_0^t \left(\dot{g}(s,x) - \dot{f}(s,x)\right) ds + \left[\frac{f(t,x)-f(0,x)}{t} - \dot{f}(0,x)\right] \\
&\qquad + \left[\dot{f}(0,x) - \dot{g}(0,x)\right].
\end{aligned}$$

Recalling that $g(t,x) = \partial_x^k u(\psi(t,x))$ and $f(t,x) = \partial_x^k v(\psi(t,x))$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\partial_x^k \left(\frac{u(\psi(t,x)) - u(x)}{t} - \frac{d}{dt}u(\psi(t,x))_{t=0}\right)\right|^p dx \\
&\qquad = \int_{\mathbb{R}} \left|\frac{g(t,x)-g(0,x)}{t} - \dot{g}(0,x)\right|^p dx \\
&\quad \leq 3^{p-1}c_1 \left\|u-v\right\|^p_{H^{k+1,p}} + 3^{p-1}c_3 \left\|\frac{v(\psi(t))-v}{t} - \frac{d}{dt}v(\psi(t))_{t=0}\right\|^p_{H^{k,p}} \\
&\qquad + 3^{p-1}c_2 \left\|u-v\right\|^p_{H^{k+1,p}}.
\end{aligned}$$

This estimate holds for all $k \leq n$, which implies that

$$\begin{aligned}
\left\|\frac{u(\psi(t))-u}{t} - \frac{d}{dt}u(\psi(t))_{t=0}\right\|^p_{H^{n,p}} &\leq c_4 \left\|u-v\right\|^p_{H^{n+1,p}} \\
&\quad + c_5 \left\|\frac{v(\psi(t))-v}{t} - \frac{d}{dt}v(\psi(t))_{t=0}\right\|^p_{H^{n,p}},
\end{aligned}$$

and the result follows from (4.10) and the fact that C_0^∞ is dense in $H^{n+1,p}$.

□

We will use the following well-known result.

Theorem 4.4 *([34, Theorem 3.1.1.]) Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X, satisfying $\|T(t)\|_X \leq Me^{\omega t}$ for some positive constants M and ω. If B is a bounded linear operator on X then $A+B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying*

$$\|S(t)\|_X \leq Me^{\left(\omega + M\|B\|_X\right)t}. \tag{4.13}$$

Let us now define for $\eta \in C_b^n$ the operator $D = D_0 + \eta$, $\mathrm{Dom}(D) = \mathrm{Dom}(D_0)$, so that $D = \xi\partial_x + \eta$ on $H^{n+1,p}$, $n \in \mathbb{N}$.

Lemma 4.5 *Fix $n \in \mathbb{N}$ and assume that $\xi \in C_b^{n+1}$ and $\eta \in C_b^n$. Then D generates a strongly continuous one-parameter group $\left(U_t^{\xi,\eta}\right)_{t\in\mathbb{R}}$ in $H^{n,p}$. If $n = 0$ or $n = 1$ then U_t^{ξ} is quasi-contractive,*

$$\left\| U_t^{\xi,\eta} \right\|_{\mathcal{L}(H^{n,p})} \leq e^{c|t|}, \; t \in \mathbb{R}, \tag{4.14}$$

for a constant $0 < c < \infty$.

Proof. We observe that the operator $D - D_0 = \eta$ is bounded in $H^{n,p}$. The statement follows now from Theorem 4.4 and Lemma 4.3. $\square$

The group $U_t^{\xi,\eta}$ has the following explicit form.

Lemma 4.6 *For any $f \in H^{n,p}$ we have*

$$U_t^{\xi,\eta} f(x) = e^{c(t,x)} f(\varphi_{-t}(x)), \; t, x \in \mathbb{R}, \tag{4.15}$$

where $(\varphi_t)_{t\in\mathbb{R}}$ is the diffeomorphism group generated by the vector field $\xi\partial_x$ and

$$c(t,x) = \int_0^t \eta(\varphi_{s-t}(x))ds, \; t, x \in \mathbb{R} \tag{4.16}$$

Proof. A direct calculation shows that the function $u(t,x) := e^{c(t,x)} f(\varphi_{-t}(x))$, $t, x \in \mathbb{R}$, is a solution of the initial value problem $u_t = Du, \; u(0,x) = f(x)$, if and only if $c(t,x)$ satisfies

$$c_t = \xi c_x + \eta, \; c(0,x) = 0.$$

Formula (4.16) can be obtained by the method of characteristics or checked directly (as in fact formula (4.15) itself). $\square$

4.2 From SDE to ODE

Let us consider a pair of densely embedded UMD Banach spaces $Y \subset \mathfrak{X}$, a continuous map $F : Y \to \mathfrak{X}$ and a bounded linear operator $D : Y \to \mathfrak{X}$ such that $D^2 \in \mathcal{L}(Y, \mathfrak{X})$. Our theory could be posed in a general setting of UMD Banach spaces but because our main application is for concrete Sobolev spaces, we assume that the spaces Y and $\mathfrak{X}$ are isomorphic to L^q spaces, see Remark A.7. In this way, stochastic integration is meaningful, see section

A.7, and an appropriate version of an Itô formula is available, see Theorem A.8.

Assume that $T > 0$ is fixed. Our aim is to study the stochastic differential equation

$$dy(t) + F(y(t))dt + Dy(t) \circ dw(t) = 0, \; y(0) = y_0 \in Y, \; t \in [0, T], \quad (4.17)$$

where w is an $\mathbb{R}$-valued Wiener process on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$, and $\circ$ means the Stratonovich stochastic differential. We suppose without loss of generality that all trajectories of w are continuous.

Definition 4.7 *A strong solution of equation (4.17) is a Y-valued continuous adapted process $y(t)$, $t \in [0, \theta]$, where θ is a stopping time, $0 < \theta \leq T$, such that*

$$\mathbb{E} \int_0^{t \wedge \theta} |Dy(s)|_{\mathfrak{X}}^2 \, ds < \infty$$

and the equality

$$y(t \wedge \theta) = y_0 + \int_0^{t \wedge \theta} F(y(s))ds + \frac{1}{2} \int_0^{t \wedge \theta} D^2 y(s)ds + \int_0^{t \wedge \theta} Dy(s)dw(s),$$

$t \geq 0$, *is satisfied in $\mathfrak{X}$, $\mathbb{P}$-a.s.*

Assume now that D is the generator of a one-parameter C_0 group $\{U(t)\}_{t \in \mathbb{R}}$ in $\mathfrak{X}$, which leaves Y invariant and satisfies the estimates

$$\|U(t)\|_{\mathcal{L}(\mathfrak{X})} \leq Me^{m|t|}, \; \|U(t)\|_{\mathcal{L}(Y)} \leq Me^{m|t|} \quad (4.18)$$

for some positive constants M and m. Let us define a (random) map $\widehat{F} : \mathbb{R}_+ \times Y \to \mathfrak{X}$

$$\widehat{F}(t, z) := U(w(t))F(U^{-1}(w(t))z), \; z \in Y, \; t \geq 0.$$

Obviously, for all $t \geq 0$, $\widehat{F}(t, \cdot)$ is a continuous map $Y \to \mathfrak{X}$. Observe also that the map $\mathbb{R} \ni t \mapsto \widehat{F}(t, z) \in \mathbb{R}$ is continuous for any trajectory $w(t)$ and $z \in Y$.

In what follows, we will impose the following integrability condition on a stopping time θ and Y-valued process $z(t)$, $t \in [0, T]$ such that for some $p > 1$

$$\mathbb{E} \int_0^{t \wedge \theta} \|z(s)\|_Y^{2p} \, ds < \infty, \; t \in [0, T]. \quad (4.19)$$

The next theorem is the main result of this section.

Theorem 4.8 *Assume that θ is a stopping time, $0 < \theta \leq T$. Let $z(t)$, $t \in [0, \theta]$, be a continuous adapted Y-valued process such that condition (4.19) holds. Then $z(t)$ satisfies the random integral equation*

$$z(t) = y_0 - \int_0^t \widehat{F}(s, z(s))ds, \ y_0 \in Y, \ t \in [0, \theta], \tag{4.20}$$

if and only if the process $y(t) := U(-w(t))z(t) \in Y$, $t \in [0, \theta]$, is a strong solution of (4.17).

Remark 4.9 *For technical reasons, we require strict inequality $p > 1$ in condition (4.19), cf. estimate (4.29) below.*

Remark 4.10 *Observe that condition (4.19) is obviously satisfied if there exists a constant $R > 0$ such that*

$$\|z(t)\|_Y \leq R \text{ for all } t \in [0, \theta], \ a.s. \tag{4.21}$$

To prove Theorem 4.8, we first need the following general result, which follows by an application of the Itô formula stated in Theorem A.8.

Lemma 4.11 *Assume that θ is a stopping time, $0 < \theta \leq T$.Let $\chi(t)$, $t \in [0, \theta]$, be a progressively measurable $\mathfrak{X}$-valued random process. Define a process $z(t)$, $t \in [0, \theta]$, by the formula*

$$z(t) := y_0 - \int_0^t \chi(s)ds, \ y_0 \in Y, \tag{4.22}$$

and assume that $z(t) \in Y$ for all $t \in [0, \theta]$ and condition (4.19) holds. Set

$$y(t) := U(-w(t))z(t) \in Y, \ t \in [0, \theta].$$

Then $y(t)$ satisfies the equation

$$\begin{aligned} y(t \wedge \theta) &= y_0 + \int_0^{t\wedge\theta} U(-w(s))\chi(s)ds + \int_0^{t\wedge\theta} DU(-w(s))z(s) \circ dw(s) \\ &= y_0 + \int_0^{t\wedge\theta} U(-w(s))\chi(s)ds + \int_0^{t\wedge\theta} Dy(s) \circ dw(s), \ t \geq 0, \end{aligned} \tag{4.23}$$

in $\mathfrak{X}$.

Proof of Lemma 4.11. Let us put $K := \mathbb{R}\times Y$ and observe that K satisfies the assumptions of Theorem A.8. Consider a time independent map

$$f : K \ni (\tau, y) \mapsto f(\tau, y) \in \mathfrak{X}.$$

and assume that it satisfies assumptions of Theorem A.8. Then $\frac{\partial f}{\partial y}(\tau, y) \in \mathcal{L}(Y, \mathfrak{X})$, $\frac{\partial f}{\partial \tau}(\tau, y) \in \mathfrak{X}$ and $\frac{\partial^2 f}{\partial \tau^2}(\tau, y) \in \mathfrak{X}$. Observe that $\frac{\partial f}{\partial \tau}(\tau, y)$ can be identified with a bounded linear operator $\mathbb{R} \to \mathfrak{X}$ acting on $h \in \mathbb{R}$ by

$$\frac{\partial f}{\partial \tau}(\tau, y)h := h\frac{\partial f}{\partial \tau}(\tau, y) \in \mathfrak{X},$$

with the norm equal to $\left\|\frac{\partial f}{\partial \tau}((\tau, y))\right\|_{\mathfrak{X}}$.

Define a stochastic process ξ by $\xi(t) = (w(t), z(t)) \in K$, $t \in [0, \theta]$. It is a K-valued Itô process such that

$$d\xi(t) = \alpha(t)dt + \beta dw(t),\ t \in [0, \theta]\,,$$

where $\alpha : [0, \theta] \ni t \mapsto (0, \chi(t)) \in K$ and $\beta := (1, 0_Y) \in K$. Here 0_Y stands for the zero element of Y.

Assume in addition that

$$\mathbb{E}\int_0^\theta \left\|\left(\frac{\partial}{\partial \tau}f\right)(\xi(s))\right\|_{\mathfrak{X}}^2 ds < \infty \text{ and } \mathbb{E}\int_0^\theta \left\|\left(\frac{\partial^2}{\partial \tau^2}f\right)(\xi(s))\right\|_{\mathfrak{X}}^2 ds < \infty. \tag{4.24}$$

Define now a K-valued process $\xi_\theta(t)$, $t \geq 0$, by setting

$$\xi_\theta(t) = \xi(0) + \int_0^t \alpha(s)\mathbf{1}_{[0,\theta]}(s)ds + \int_0^t \beta\mathbf{1}_{[0,\theta]}(s)dw(s),\ t \geq 0.$$

It is clear that $\xi_\theta(t) = \xi(t)$ for $t \in [0, \theta]$. Moreover, since θ is a stopping time, the process $\mathbf{1}_{[0,\theta]}(s)$, $s \geq 0$, is progressively measurable.

It follows then from the general Itô formula UMD Banach spaces, see [10] and Theorem A.8 for a version adapted to our purposes (and e.g. [20, Theorem VII.1.2] for a corresponding result in Hilbert spaces), that $f(\xi(t))$ is an $\mathfrak{X}$-valued Itô process such that

$$\begin{aligned} f(\xi_\theta(t)) = f(\xi(0)) &+ \int_0^t \frac{\partial f}{\partial y}(\xi_\theta(s))\chi(s)\mathbf{1}_{[0,\theta]}(s)ds \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial \tau^2}(\xi_\theta(s))ds + \int_0^t \frac{\partial f}{\partial \tau}(\xi_\theta(s))dw(s),\ t \geq 0. \end{aligned} \tag{4.25}$$

Here $\frac{\partial f}{\partial y}(\xi(s))$ is a bounded operator $Y \to \mathfrak{X}$ and $\frac{\partial f}{\partial \tau}(\xi(s))$ is a bounded linear operator $\mathbb{R} \to \mathfrak{X}$ acting on $h \in \mathbb{R}$ by

$$\frac{\partial f}{\partial \tau}(\xi(s))h := h\frac{\partial f}{\partial \tau}(\xi(s)) \in \mathfrak{X}.$$

Finally, $\frac{\partial^2 f}{\partial \tau^2}(\xi(s))$ can be identified with an element of $\mathfrak{X}$.

Set now

$$f(\tau, y) = U(-\tau)y \tag{4.26}$$

so that $y(t) = f(\xi(t))$. Taking into account that $Y \subset \mathrm{Dom}(D^2)$ we deduce that $f \in C^2(K, \mathfrak{X})$ and

$$\frac{\partial f}{\partial \tau}(\tau, y) = -DU(-\tau)y,\ \frac{\partial^2 f}{\partial \tau^2}(\tau, y) = D^2U(-\tau)y,\ \frac{\partial f}{\partial y}(\tau, y) = U(-\tau).$$

It follows now from (4.25) that

$$f(\xi_\theta(t)) = f(\xi(0)) + \int_0^t U(-w(s))\chi(s)\mathbf{1}_{[0,\theta]}ds$$
$$+\frac{1}{2}\int_0^t D^2U(-w(s))z(s)\mathbf{1}_{[0,\theta]}ds - \int_0^t DU(-w(s))z(s)\mathbf{1}_{[0,\theta]}dw(s),\ t \geq 0,$$

which implies (4.23).

Now it is only left to prove (4.24), which is equivalent to the pair of inequalities

$$\mathbb{E}\int_0^\theta \|DU(-w(s))z(s)\|_{\mathfrak{X}}^2\, ds < \infty,\ \mathbb{E}\int_0^\theta \left\|D^2U(-w(s))z(s)\right\|_{\mathfrak{X}}^2 ds < \infty. \tag{4.27}$$

Observe that both D and D^2 are bounded operators from Y to $\mathfrak{X}$, so that (4.27) becomes equivalent to the bound

$$\mathbb{E}\int_0^\theta \|U(-w(s))z(s)\|_Y^2\, ds < \infty. \tag{4.28}$$

The Hölder inequality implies that

$$\mathbb{E}\int_0^\theta \|U(-w(s))z(s)\|_Y^2\, ds$$
$$\leq \left(\mathbb{E}\int_0^\theta \|U(-w(s))\|_{\mathcal{L}(Y)}^{2q}\, ds\right)^{1/q}\left(\mathbb{E}\int_0^\theta \|z(s)\|_Y^{2p}\, ds\right)^{1/p} \tag{4.29}$$

for any $p, q > 1$ such that $p^{-1} + q^{-1} = 1$. Taking into account that, for any real x and λ, $e^{\lambda|x|} \leq e^{\lambda x} + e^{-\lambda x}$ and $\mathbb{E}e^{\lambda w(s)} = e^{\frac{1}{2}\lambda^2 s}$, we obtain the bound

$$\begin{aligned}\left(\mathbb{E}\int_0^\theta \|U(-w(s))\|^{2q}_{\mathcal{L}(Y)}\, ds\right)^{1/q} &\leq M^2\left(\int_0^T \mathbb{E}e^{2qm|w(s)|}ds\right)^{1/q} \\ &\leq 2M^2T^{1/q}e^{2qm^2T}\end{aligned}$$

with M and m from (4.18). Condition (4.19) shows now that (4.28) holds. The proof is complete. □

Remark 4.12 *It can be shown by similar arguments that, if a process $y(t) \in Y,\ t \in [0, \theta]$, is a solution of integral equation (4.23), then*

$$z(t) := U(w(t))y(t) \in Y$$

satisfies (4.22).

Now we can proceed with the proof of the main result of this section.

Proof of Theorem 4.8. Let θ be a finite stopping time and $z(t),\ t \in [0, \theta]$, a Y-valued process solving the integral equation (4.20). It is clear that $y(t) = U(-w(t))z(t) \in Y$ is a solution of the equation

$$y(t) = U(-w(t))\left(y_0 - \int_0^t U(w(s))F(y(s))ds\right),\ t \in [0, \theta]. \tag{4.30}$$

We can now apply Lemma 4.11 with $\chi(t) = U(w(t))F(y(t)),\ t \in [0, \theta]$, and obtain

$$\begin{aligned}y(t\wedge\theta) &= \int_0^{t\wedge\theta} U(-w(s))\chi(s)ds - \int_0^{t\wedge\theta} DU(-w(s))z(s)\circ dw(s) \\ &= \int_0^{t\wedge\theta} F(y(s))ds - \int_0^{t\wedge\theta} DU(-w(s))z(s)\circ dw(s) \\ &= \int_0^{t\wedge\theta} F(y(s))ds - \int_0^{t\wedge\theta} Dy(s)\circ dw(s).\end{aligned}$$

The converse implication can be shown by similar arguments, cf. Remark 4.12. ■

4.3 Boundedness of the commutator

In this section we explain that the general result on the boundedness of commutators stated in the book by Stein [35] in the L^2-setting, can be extended to L^p, $p > 1$, following the scheme outlined in that same book.

Let T be a pseudo-differential operator (PDO) of order $m = 1$, that is, with a smooth symbol h satisfying the bound

$$\left|\partial_x^\beta \partial_\xi^\alpha h(x,\xi)\right| \leq A_{\alpha\beta} \left(1 + |\xi|\right)^{m-\alpha}, \; m = 1,$$

for all $\alpha, \beta > 0$. Let $a : \mathbb{R} \to \mathbb{R}$ be such that $\partial_x a \in L^\infty$ and set $B := [T, M_a]$, defined on S. Observe that B is not a PDO of order $m = 0$ unless a is smooth with bounded derivatives of any order, which is insufficient for our purposes.

Remark 4.13 *In our case, $T = Q = \left(1 - \partial_x^2\right)^{1/2}$ so that $h(x,\xi) = \left(1+\xi^2\right)^{1/2}$, and $a \in H^{2,p}$.*

Theorem 4.14 *[35, VII.3.5, page 309] B extends to a bounded operator in L^2, and there exists a constant C such that*

$$\|B\|_{\mathcal{L}(L^2)} \leq A =: C \left\|\partial_x a\right\|_{L^\infty},$$

where C depends on T but is independent of a.

Idea towards the proof. B can be considered as the integral operator with a kernel $K(x,y)$ satisfying conditions of Theorem 3 of [35, VII.3.2, page 204], which states L^2-boundedness of such operators. □

Remark 4.15 *The proof of Theorem 3 [35, VII.3.2, page 294] is complicated, even for PDOs of order* 0*.*

Theorem 4.16 *B extends to a bounded operator in L^p for any $p > 1$, and there exists a locally bounded function $k_p : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|B\|_{\mathcal{L}(L^p)} \leq A_p =: k_p(A).$$

Proof. The proof follows the scheme outlined in [35, VI.5] for PDOs of order 0. The scheme is as follows: first prove L^2-boundedness and then extend this result to L^p using the theory of singular integrals discussed in [35, I.5].

Although, as stated above, B is not in general a PDO of order 0, the scheme still works. We can apply Theorem 3 and Corollary on page 22 of [35, I.5.2]. Indeed, it is straightforward that conditions (48), (49) of Theorem 3 [35, VII.3.2, page 294] imply that condition (10) of Theorem 3 [35, I.5.2, page 19] holds for both B and B^*. Thus both B and B^* are bounded in L^p, $1 < p < 2$, which also implies that they are bounded for $p > 2$. □

A Stochastic integration in L^q space

The first part of this appendix is based on Appendix from [11] which in turn is based on [33]. The second part contains a formulation of the Itô Lemma, which is based on [10], that is suitable for our purposes.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $w : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

Proposition A.1 *Let $(\mathcal{O}, \Sigma, \mu)$ be a σ-finite measure space. Let $p, q \in (1, \infty)$ and $T > 0$. For a progressively measurable process $\phi : [0, T] \times \Omega \to L^q(\mathcal{O})$ the following assertions are equivalent:*

(1) There exists a sequence $(\phi_n)_{n\geq 1}$ of adapted step processes such that

(i) $\lim_{n\to\infty} \|\phi - \phi_n\|_{L^p(\Omega;L^q(\mathcal{O};L^2(0,T)))} = 0$,

(ii) $(\int_0^T \phi_n(t)\, dw(t))_{n\geq 1}$ *is Cauchy sequence in* $L^p(\Omega; L^q(\mathcal{O}))$.

(2) There exists a random variable $\eta \in L^p(\Omega; L^q(\mathcal{O}))$ such that for all sets $A \in \Sigma$ with finite measure one has $(t, \omega) \mapsto \int_A \phi(t, \omega)\, d\mu \in L^p(\Omega; L^2(0, T))$, and

$$\int_A \eta\, d\mu = \int_0^T \int_A \phi(t)\, d\mu\, dw(t) \quad \text{in } L^p(\Omega).$$

(3) $\|\phi\|_{L^p(\Omega;L^q(\mathcal{O};L^2(0,T)))} < \infty$.

Moreover, in this situation one has $\lim_{n\to\infty} \int_0^T \phi_n(t)\, dw(t) = \eta$, *and*

$$c_{p,q}^{-1}\|\phi\|_{L^p(\Omega;L^q(\mathcal{O};L^2(0,T)))} \leq \|\eta\|_{L^p(\Omega;L^q(\mathcal{O}))} \leq C_{p,q}\|\phi\|_{L^p(\Omega;L^q(\mathcal{O};L^2(0,T)))}. \tag{A.1}$$

Definition A.2 *A progressively measurable process $\phi : [0, T] \times \Omega \to L^q(\mathcal{O})$ is called L^p-*stochastically integrable on $[0, T]$*, if and only if it satisfies any of the three equivalent conditions in Proposition A.1. In such a case we will write*

$$\int_0^T \phi(t)\, dw(t) = \eta.$$

Remark A.3 *If a progressively measurable process $\phi : [0, T] \times \Omega \to L^q(\mathcal{O})$ is L^p-stochastically integrable on $[0, T]$, then by (3) in Proposition A.1, ϕ is L^p-stochastically integrable on $[0, t]$ as well. Moreover, by the Doob maximal inequality, see [30, Proposition 7.16], one additionally gets*

$$c_{p,q}^{-1}\|\phi\|_{L^p(\Omega;L^q(\mathcal{O};L^2(0,T)))} \leq \left\| t \mapsto \int_0^t \phi(s)\, dw(s) \right\|_F \leq C_{p,q}\|\phi\|_{L^p(\Omega;L^q(\mathcal{O};L^2(0,T)))},$$

where $F = L^p(\Omega; C([0, T]; L^q(\mathcal{O})))$.

Corollary A.4 *Let $(\mathcal{O}, \Sigma, \mu)$ be a σ-finite measure space. Let $p \in (1, \infty)$ and $q \in [2, \infty)$. Let $T > 0$. Let $\phi : [0, T] \times \Omega \to L^q(\mathcal{O})$ be an adapted and strongly measurable process. If $\|\phi\|_{L^p(\Omega; L^2(0,T; L^q(\mathcal{O})))} < \infty$, then ϕ is L^p-stochastically integrable on $[0, T]$ and*

$$\Big\| \int_0^T \phi(t)\, dw(t) \Big\|_{L^p(\Omega; L^q(\mathcal{O}))} \leq C_{p,q} \|\phi\|_{L^p(\Omega; L^2(0,T; L^q(\mathcal{O})))}.$$

Lemma A.5 *Let $(\mathcal{O}, \Sigma, \mu)$ be a σ-finite measure space. Let $p \in (1, \infty)$ and $q \in (1, \infty)$. Let $T > 0$. Let $\phi : [0, T] \times \Omega \to L^q(\mathcal{O})$ be a progressively measurable process satisfying the following three conditions.*

(1) There exist a measurable function $\psi : [0, T] \times \Omega \times \mathcal{O} \to \mathbb{R}$ such that $\phi(t, \omega)(x) = \psi(t, \omega, x)$ for almost all $t \in [0, T]$, $\omega \in \Omega$ and $x \in \mathcal{O}$, and for all $x \in \mathcal{O}$, $\psi(\cdot, x)$ is adapted.

(2) For almost all $x \in \mathcal{O}$, $\psi(\cdot, x) \in L^p(\Omega; L^2(0, T))$.

(3) There is a $\eta \in L^p(\Omega; L^q(\mathcal{O}))$ such that

$$\eta(\omega)(x) = \Big(\int_0^T \psi(t, x)\, dw(t) \Big)(\omega) \quad \text{for almost all } \omega \in \Omega, \text{ and } x \in \mathcal{O}.$$

Then the process ϕ is L^p-stochastically integrable on $[0, T]$ and

$$\int_0^T \phi(t)\, dw(t) = \eta.$$

Definition A.6 *Assume that $\tau : \Omega \to [0, T]$ is a stopping time. If a progressively measurable process $\phi : [0, \tau] \times \Omega \to L^q(\mathcal{O})$ is such that the process $\mathbf{1}_{[0,\tau]}\phi : [0, T] \times \Omega \ni (s, \omega) \mapsto \mathbf{1}_{[0,\tau]}(s)\phi(s, \omega) \to L^q(\mathcal{O})$ is stochastically integrable, then we put*

$$\int_0^{t\wedge\tau} \phi(s)\, dw(s) := \int_0^t \mathbf{1}_{[0,\tau]}(s)\phi(s)\, dw(s), \quad t \in [0, T]. \tag{A.2}$$

If $\xi : [0, \tau] \times \Omega \to L^q(\mathcal{O})$ is a continuous and adapted process such that $\xi(t \wedge \tau) = \int_0^{t\wedge\tau} \phi(s)\, dw(s)$, for all $t \in [0, T]$, then we will write

$$d\xi(t) = \phi(t) dw(t), \quad t \in [0, \tau].$$

Remark A.7 *The above result can also be applied to the space $H^{s,q}(\mathbb{R})$ which is isomorphic to a $L^q(\mathbb{R})$. Let $J : H^{s,q}(\mathbb{R}) \to L^q(\mathbb{R})$ be an isomorphism. Then* *for a process $\phi : [0,T]\times\Omega \to H^{s,q}(\mathbb{R})$ let $\tilde{\phi} = J\phi$. The above results can be applied to $\tilde{\phi}$. Conversely, if $\tilde{\eta} = \int_0^T \tilde{\phi}(t)\,dw(t)$, then we define*

$$\eta = J^{-1}\tilde{\eta}.$$

Moreover, $\|\phi\|_{L^p(\Omega;H^{s,q}(\mathbb{R};L^2(0,T)))} < \infty$ is equivalent to stochastic integrability of ϕ. It is well-known (see [35, 8.24]) that J extends to a isomorphism from $H^{s,q}(\mathbb{R};L^2(0,T)))$ into $L^q(\mathbb{R};L^2(0,T))$.

In a similar way, the results extend to arbitrary X which are isomorphic to a closed subspace of any $L^q(\mathcal{O})$.

Theorem A.8 (Itô formula) *Let E and F be spaces which are isomorphic to appropriate $L^q(\mathcal{O})$ spaces. Assume that*

$$f : [0,T]\times E \to F$$

is of class $C^{1,2}$, i.e. f is differentiable in the first variable and twice Fréchet differentiable in the second variable and the functions f, $D_1 f$, $D_2 f$ and $D_2^2 f$ are continuous on $[0,T]\times E$. Here $D_1 f$ and $D_2 f$ are the derivatives with respect to the first and second variable, respectively. Let

$$\phi : [0,T]\times\Omega \to E$$

be a progressively measurable process which is stochastically integrable with respect to w and assume that the paths of ϕ belong to $L^2(0,T;E)$ almost surely. Let $\psi : [0,T]\times\Omega\to E$ be progressively measurable process with paths in $L^1(0,T;E)$ almost surely. Let $\xi : \Omega \to E$ be $\mathcal{F}_0$-measurable. Define a process $x : [0,T]\times\Omega\to E$ by

$$x(t) = \xi + \int_0^t \psi(s)\,ds + \int_0^t \phi(s)\,dw(s), \quad t\in[0,T].$$

Then the process $s\mapsto D_2 f(s,x(s))\phi(s)$ is stochastically integrable and almost surely we have, for all $t\in[0,T]$,

$$\begin{aligned} f(t,x(t)) - f(0,\xi) = &\int_0^t D_1 f(s,x(s))\,ds + \int_0^t D_2 f(s,x(s))\psi(s)\,ds \\ &+ \int_0^t D_2 f(s,x(s))\phi(s)\,dw(s) \\ &+ \tfrac{1}{2}\int_0^t \big(D_2^2 f(s,x(s))\big)(\phi(s),\phi(s))\,ds. \end{aligned} \tag{A.3}$$

Remark A.9 *For a related topic of a mild Itô formula in UMD Banach spaces one can consult a recent paper [18] by Cox et al.*

□

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