$\mathbf{Bad}\,(\mathbf{w})$ IS HYPERPLANE ABSOLUTE WINNING

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Abstract. In 1998 Kleinbock conjectured that any set of weighted badly approximable $d\times n$ real matrices is a winning subset in the sense of Schmidt's game. In this paper we prove this conjecture in full for vectors in $\mathbb{R}^d$ in arbitrary dimensions by showing that the corresponding set of weighted badly approximable vectors is hyperplane absolute winning. The proof uses the Cantor potential game played on the support of Ahlfors regular absolutely decaying measures and the quantitative nondivergence estimate for a class of fractal measures due to Kleinbock, Lindenstrauss and Weiss. To establish the existence of a relevant winning strategy in the Cantor potential game we introduce a new approach using two independent diagonal actions on the space of lattices.

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Dedicated to Anna Nesharim

1. Introduction

As is well known, the rational points are dense in the real space $\mathbb{R}^d$, meaning that $\mathbb{R}^d$ can be covered by cubes in $\mathbb{R}^d$ of an arbitrarily small fixed sidelength $\varepsilon > 0$ centred at rational points. Various quantitative aspects of this basic property are studied within the theory of Diophantine approximation. For instance, by Dirichlet's theorem, $\mathbb{R}^d$ can be covered by cubes in $\mathbb{R}^d$ of sidelength $2q^{-(d+1)/d}$ centred at rational points (not necessarily written in the lowest terms) with arbitrarily large denominators $q \in \mathbb{N}$. One of the fundamental concepts studied in Diophantine approximation is that of badly approximable points. These are precisely the points in $\mathbb{R}^d$ that cannot be covered by the cubes arising from Dirichlet's theorem when 2 is replaced by any positive constant. In the more general case one considers coverings by parallelepipeds with different sidelengths controlled by d real parameters referred to as weights. This more general setup gives rise to the notion of weighted badly approximable points that will be the main object of study in this paper.

In what follows $d \in \mathbb{N}$ and $\mathcal{W}_d$ denotes the collection of all *d-dimensional weights*:

$$\mathcal{W}_d = \Big\{ \mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d \,:\, w_1, \dots, w_d \geq 0,\ w_1 + \ldots + w_d = 1 \Big\}.$$

For $\mathbf{w} \in \mathcal{W}_d$, a vector $\mathbf{x}= (x_1, \dots, x_d) \in \mathbb{R}^d$ is called *badly approximable with respect to* $\mathbf{w}$ if there exists $c > 0$ such that for every $q \in \mathbb{N}$ and $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{Z}^d$ there exists $1 \leq i \leq d$ satisfying

$$\left| x_i - \frac{p_i}{q} \right| \geq \frac{c}{q^{1+w_i}}\,.$$

Let $\mathbf{Bad}\,(\mathbf{w})$ be the *set of badly approximable vectors in* $\mathbb{R}^d$ *with respect to* $\mathbf{w}$.

One of the motivations for studying the set of weighted badly approximable vectors comes from its connection to a conjecture of Littlewood – a famous open problem from the 1930s. Let us briefly recall this connection.

Conjecture 1 (Littlewood's conjecture, 1930s)**.** *Every* $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$ *satisfies*

$$\inf_{q\in\mathbb{N},\,\mathbf{p}\in\mathbb{Z}^2} q\,|qx_1-p_1|\,|qx_2-p_2|=0\,. \tag{1}$$

It was noted by Schmidt [Sch83] that if $\mathbf{x}\notin\mathbf{Bad}\,(\mathbf{w})$ for some $\mathbf{w}\in\mathcal{W}_2$ then $\mathbf{x}$ satisfies (1). In particular, if the intersection of the sets $\mathbf{Bad}\,(\mathbf{w})$ over all $\mathbf{w}\in\mathcal{W}_2$ was the empty set, then Littlewood's conjecture would follow. However, Schmidt doubted that using only two weights would be sufficient, if his observation can be used to verify (1) at all. Specifically, Schmidt formulated the following problem that has inspired many researchers in Diophantine approximation and homogeneous dynamics.

Conjecture 2 (Schmidt's conjecture, 1982)**.** *For every* $\mathbf{w}_1,\mathbf{w}_2\in\mathcal{W}_2$ *we have that*

$$\mathbf{Bad}(\mathbf{w}_1)\cap\mathbf{Bad}(\mathbf{w}_2)\neq\varnothing\,. \tag{2}$$

Almost three decades later Schmidt's conjecture was verified by Badziahin, Pollington and Velani in the tour de force [BPV11], which opened the way to many exciting new developments.

The more general version of Schmidt's conjecture deals with arbitrary finite and, furthermore, countable intersections of $\mathbf{Bad}\,(\mathbf{w})$. Already in [BPV11] arbitrary finite intersections were considered. In fact, the main result of [BPV11] implies that

$$\bigcap_{n=1}^{\infty}\mathbf{Bad}(\mathbf{w}_n)\neq\varnothing \tag{3}$$

if the countably many weights $\mathbf{w}_1,\mathbf{w}_2,\ldots\in\mathcal{W}_2$ satisfy the condition that

$$\liminf_{n\to\infty}\min\mathbf{w}_n>0\,. \tag{4}$$

Using different techniques condition (4) was independently removed by An [An13] and the second named author [Nes13], who both established (3) for arbitrary countable intersections. Indeed, An [An13] showed a stronger dimension statement.

Schmidt's conjecture can also be considered in higher dimensions. In this generality it was verified by the first named author [Ber15]. Similarly to the two dimensional result of [BPV11], (3) was established in [Ber15] for any sequence of weights $\mathbf{w}_1,\mathbf{w}_2,\ldots\in\mathcal{W}_d$ satisfying (4). Condition (4) was finally removed by the third named author in [Yan19]. In should be noted that these papers go the extra mile to give a full dimension statement for the intersection appearing in (3) and enable to restrict the left hand side of (3) to nondegenerate curves and manifolds.

Two natural frameworks for proving the countable intersection property of the sets $\mathbf{Bad}\,(\mathbf{w})$ are offered by topology and measure theory. Indeed, if X is a complete metric space or a measure space and $S_1,S_2,\ldots\subseteq X$ are G_δ dense, or, respectively, full measure sets, then $\bigcap_{n=1}^{\infty}S_n$ is G_δ dense, respectively, a set of full measure, and in particular, nonempty. However, the set $\mathbf{Bad}\,(\mathbf{w})$ is neither comeagre nor conull. In fact, $\mathbf{Bad}\,(\mathbf{w})$ is a countable union of closed sets whose Lebesgue measure is zero, hence it is both meagre and null.

An alternative framework to establish the countable intersection property is offered by game theory. This was first articulated by Schmidt [Sch66] who introduced a variant of the Banach-Mazur game, now called Schmidt's game, and its corresponding winning sets. Ever since other variants of Schmidt's game have been proposed by many authors for various purposes. We refer the reader to Section 2 for the definitions of *hyperplane absolute winning sets* (abbr. *HAW*) and *Cantor winning sets*, which will be used in this paper. We also refer the reader to [BHNS18, §2] or [BFK$^+$12, §2] for the definitions of *winning sets*, *α-winning sets* and *absolute winning sets* which will be mentioned in this paper.

The study of winning properties of $\mathbf{Bad}\,(\mathbf{w})$ has a long history. Schmidt proved in [Sch66] that $\mathbf{Bad}(1)$ is winning, where it was also mentioned that the analogous theorem holds for

$$\mathbf{w} = \mathbf{w}_d := \left(\tfrac{1}{d}, \ldots, \tfrac{1}{d}\right)$$

for every d. Indeed, the proof of this can be found in Schmidt's monograph [Sch80]. McMullen [McM10] proved that $\mathbf{Bad}(1)$ is absolute winning. Later Broderick, Fishman, Kleinbock, Reich and Weiss [BFK$^+$12] proved that $\mathbf{Bad}(\mathbf{w}_d)$ is HAW for any $d \geq 1$.

However, the study of weighted badly approximable points turned out to be much harder. Indeed, the following natural problem that was raised by Kleinbock [Kle98, Section 8] over two decades ago remains open with the exception of one special case that will shortly be mentioned.

Problem (Kleinbock, 1998)**.** Is it true that $\mathbf{Bad}\,(\mathbf{w})$ is winning for every weight $\mathbf{w}$?

The first breakthrough came about with the paper of An [An16] who settled it for $d = 2$. Based on [An16], Simmons and the second named author [NS14] proved that $\mathbf{Bad}\,(\mathbf{w})$ is HAW for any $\mathbf{w} \in \mathcal{W}_2$. In higher dimensions, the only known result towards Kleinbock's problem is due to Guan and Yu [GY19] who proved that for weights $\mathbf{w} \in \mathcal{W}_d$ satisfying the condition

$$w_1 = \cdots = w_{d-1} \geq w_d\,,$$

$\mathbf{Bad}\,(\mathbf{w})$ is HAW. The goal of this paper is to resolve Kleinbock's problem in full. Our main result reads as follows.

Theorem 3. *For any $\mathbf{w} \in \mathcal{W}_d$ the set $\mathbf{Bad}\,(\mathbf{w})$ is HAW. In particular, it is winning.*

The HAW property implies more than just the countable intersection property. For example, we have the following corollary, which follows from Theorem 3 on applying properties of HAW sets established in [BFK$^+$12] (see Section 2 for the definition of *Ahlfors regular* and *absolutely decaying* measures).

Corollary 4. *For any sequence of weights $\mathbf{w}_1, \mathbf{w}_2 \ldots \in \mathcal{W}_d$ and any sequence $f_1, f_2, \ldots$ of C^1 diffeomorphisms of $\mathbb{R}^d$, the set*

$$\bigcap_{n=1}^{\infty} f_n\left(\mathbf{Bad}\,(\mathbf{w}_n)\right)$$

is HAW. In particular, for every Ahlfors regular absolutely decaying measure μ on $\mathbb{R}^d$ we have that

$$\dim\left(\bigcap_{n=1}^{\infty} f_n\left(\mathbf{Bad}\,(\mathbf{w}_n)\right) \cap \operatorname{supp}\mu\right) = \dim\left(\operatorname{supp}\mu\right),$$

where dim *stands for Hausdorff dimension.*

Theorem 3 is proved by passing to the following equivalent formulation.

Theorem 5. *For any $\mathbf{w} \in \mathcal{W}_d$ and any compactly supported Ahlfors regular absolutely decaying measure μ on $\mathbb{R}^d$ we have that*

$$\mathbf{Bad}\,(\mathbf{w}) \cap \operatorname{supp}\mu \neq \varnothing\,. \tag{5}$$

Over the last two decades Schmidt's conjecture motivated significant amount of research concerning badly approximable points in fractals, starting with Pollington and Velani [PV02] and Kleinbock and Weiss [KW05]. Initial progress towards Theorem 5 was made in [KW05] for $\mathbf{w} = \mathbf{w}_d$ and in [KTV06], where (5) was proved for product measures $\mu = \mu_1 \times \cdots \times \mu_d$ with each μ_i being Ahlfors regular. Other notable developments include those by Fishman [Fis09] and Kleinbock and Weiss [KW10].

The tools used in the proof of Theorems 3 and 5 are the Cantor potential game which was introduced by Badziahin, Harrap, Simmons and the second named author [BHNS18], and the quantitative nondivergence estimate for "friendly" measures due to Kleinbock, Lindestrauss and Weiss [KLW04], albeit, within this paper, the latter is only applied in the context of Ahlfors regular absolutely decaying measures.

In order to shed some light on the new ideas involved in the proof of Theorem 3, it is useful to compare the results in this paper to those of [BNY20] and several preceding publications, which deal with badly approximable points on nondegenerate curves in $\mathbb{R}^d$. For simplicity we restrict our discussion to analytic nondegenerate curves. Let $\mathbf{f} : I_0 \to \mathbb{R}^d$ be an *analytic nondegenerate map* defined on an interval $I_0 \subseteq \mathbb{R}$. By definition, this means that the coordinate functions $f_1, \ldots, f_d$ are analytic and together with the constant function 1 are linearly independent over $\mathbb{R}$. The map $\mathbf{f}$ should be understood as the parameterisation of a curve $\mathcal{C}$ in $\mathbb{R}^d$, namely $\mathcal{C} = \mathbf{f}(I_0)$. In this case, the set $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{w}))$ precisely consists of the parameters $x \in I_0$ for which the corresponding point $\mathbf{f}(x)$ on the curve $\mathcal{C}$ is badly approximable with respect to the weight $\mathbf{w}$. For $d = 2$ Badziahin and Velani [BV14] proved that $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{w}))$ is Cantor winning for every $\mathbf{w} \in \mathcal{W}_2$. This property was then improved to 'winning' by An, Velani and the first named author [ABV18]. In fact, the 'winning' property can be strengthened to 'absolute winning' on applying [Nes13, Appendix B], see also [ABV18, Remark 7]. For higher dimensions, the first named author [Ber15] proved that for every $\mathbf{w} \in \mathcal{W}_d$ the set $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{w}))$ is Cantor winning (see also [BH17, Theorem B]). This result was then improved by the third named author [Yan19] in the following manner. By Definition 20, a Cantor winning set in $\mathbb{R}^d$ is α-Cantor winning for some $0 \leq \alpha < d$. In [Ber15] the parameter α depends on $\mathbf{w}$, while in [Yan19] it was shown that $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{w}))$ is α-Cantor winning for some $0 \leq \alpha < d$ that depends only on d. Eventually, the argument of [BNY20] strengthened the conclusions of [Yan19] to completely remove the dependence of α on d. While it does not do it explicitly, it does so essentially by allowing the Cantor potential game to be played on the support of any Ahlfors regular measure on I_0. By [BHNS18, Theorem 1.5] this implies that $\mathbf{f}^{-1}(\mathbf{Bad}(\mathbf{w}))$ is absolute winning.

Organisation of this paper. In Section 2 we recall the relevant variants of Schmidt's game, definitions in fractal measure theory and establish the equivalence between Theorem 3 and Theorem 5. In Section 3 we recall the Dani correspondence and the Kleinbock, Lindenstrauss, Weiss quantitative nondivergence. The proof of Theorem 5 is finally given in Section 4.

Notation and conventions. Throughout this paper, we will use the following notation. Given a metric space (X, d), any $S \subseteq X$ and $r > 0$, we denote the closed r neighborhood of S by

$$B(S, r) := \{x \in X \,:\, \mathrm{d}(x, S) \leq r\}.$$

A closed ball $B(x_0, r) := \{x \in X \,:\, \mathrm{d}(x, x_0) \leq r\}$ is defined by a fixed centre x_0 and radius $r > 0$, although these in general are not uniquely determined by the ball as a set. In view of the latter, when referring to a ball B we will mean the pair of its centre and radius and, with some abuse of notation, B will also mean the corresponding set of points when appearing in set theoretic expressions. The same will apply to the more general notion of r neighborhood of a set S. Finally, for any $\mathbf{x} \in \mathbb{R}^d$, $r > 0$ and $c > 0$, we let

$$cB(\mathbf{x}, r) := B(\mathbf{x}, cr).$$

2. Schmidt games and intersections with fractals

Schmidt's game is a quantitative version of the Banach-Mazur game played on a complete metric space. Its corresponding winning sets are dense and often have large Hausdorff dimension. Moreover, by definition, the collection of all α-winning sets is stable under taking countable intersections, where $\alpha \in (0,1)$ is a certain parameter of Schmidt's games. Schmidt's winning sets are also stable under affine transformations, although the parameter α may change. Schmidt's game was introduced in [Sch66] and used to strengthen and simplify earlier results in Diophantine approximation. There are several modifications of Schmidt's game resulting in alternative notions of winning sets. These include the notions of absolute winning sets [McM10], HAW sets [BFK$^+$12] and Cantor winning sets [BH17]. For a detailed survey of the various winning sets, their properties and the connections between them, see [BHNS18] and [BFK$^+$12]. In particular, in [BFK$^+$12] it is proved that HAW sets are α-winning for any $0 < \alpha < \frac{1}{2}$.

Definition 6 (Hyperplane absolute winning game and sets, [BFK$^+$12, §2])**.** The *hyperplane absolute game* on $\mathbb{R}^d$ is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a *parameter* $0 < \beta < 1/3$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$. Subsequently for $n = 0, 1, 2, \ldots$, first, Alice chooses a neighborhood A_{n+1} of any hyperplane in $\mathbb{R}^d$ of radius εr_n for some $0 < \varepsilon \leq \beta$; and second, Bob chooses a ball $B_{n+1} \subseteq B_n \setminus A_{n+1}$ of radius $r_{n+1} \geq \beta r_n$, where r_n is the radius of B_n.
A set $S \subseteq \mathbb{R}^d$ is called *hyperplane absolute winning* (abbr. *HAW*) if Alice has a strategy which ensures that $S \cap \bigcap_{n \geq 0} B_n \neq \varnothing$.

For the purposes of this paper it will be convenient to use the following modified version of the hyperplane absolute game.

Definition 7. The *restricted hyperplane absolute game* on $\mathbb{R}^d$ is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a *parameter* $0 < \beta < 1$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$. Subsequently for $n = 0, 1, 2, \ldots$, first, Alice chooses a neighborhood A_{n+1} of some hyperplane in $\mathbb{R}^d$ of radius $\beta r_n = \beta^{n+1} r_0$; and second, Bob chooses a ball $B_{n+1} \subseteq B_n \setminus A_{n+1}$ of radius $r_{n+1} = \beta r_n = \beta^{n+1} r_0$, where r_n is the radius of B_n. If there is no such ball the game stops and Alice wins by default. Otherwise, the *outcome* of the game is the unique point in $\bigcap_{n \geq 0} B_n$.
A set $S \subseteq \mathbb{R}^d$ will be called *restricted hyperplane absolute winning* if Alice has a strategy which ensures that she either wins by default or the outcome lies in S.

We note that there is no difference between HAW sets and restricted HAW sets and therefore throughout the rest of paper we will refer to the restricted hyperplane absolute game as the hyperplane absolute game, and to restricted hyperplane absolute winning sets as HAW sets. This fact was previously shown in [FSU18] in relation to the absolute game. Formally, we have the following statement.

Proposition 8. *Let $S \subseteq \mathbb{R}^d$. Then S is restricted hyperplane absolute winning if and only if S is hyperplane absolute winning.*

The proof of this proposition is essentially the same as that of Proposition 4.5 in [FSU18]. Indeed, the only change that is needed to obtain the proof of Proposition 8 from the proof of Proposition 4.5 in [FSU18] is to replace neighborhoods of balls, which are legal moves in the game played in [FSU18, Proposition 4.5] by neighborhoods of hyperplanes, which are Alice's legal moves in Definitions 6 and 7. However, as Proposition 8 forms a step in the argument towards our final goal, we give it a complete formal proof in the appendix at the end of this paper.

In order to reduce Theorem 3 to Theorem 5, let us recall the definitions of Ahlfors regular measures and absolute decaying measures, which can be found, for instance, in [BFK$^+$12].

Definition 9. Let X be a metric space. Given $\alpha > 0$, a Borel measure μ on X is *α-Ahlfors regular* if there exist $A, \rho_0 > 0$ such that for every $\mathbf{x} \in \operatorname{supp} \mu$

$$A^{-1} r^{\alpha} \leq \mu\left(B\left(\mathbf{x}, r\right)\right) \leq A r^{\alpha} \qquad \text{for all } 0 < r \leq \rho_0 \,. \tag{6}$$

We say that μ is *Ahlfors regular* if it is α-Ahlfors regular for some $\alpha > 0$.

Definition 10. A Borel measure μ on $\mathbb{R}^d$ is called *absolutely decaying* if there exist $D, \delta > 0$ and $r_0 > 0$ such that for every $\mathbf{x} \in \operatorname{supp} \mu$, $0 < r \leq r_0$, every hyperplane $H \subseteq \mathbb{R}^d$ and $r' > 0$ we have that

$$\mu\left(B\left(H, r'\right) \cap B\left(\mathbf{x}, r\right)\right) \leq D \left(\frac{r'}{r}\right)^{\delta} \mu\left(B\left(\mathbf{x}, r\right)\right). \tag{7}$$

The following proposition allows us to reduce Theorem 3 to Theorem 5. This proposition is already hinted in [BHNS18, Remark 4.5].

Proposition 11. *If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap \operatorname{supp} \mu \neq \varnothing$ for any Ahlfors regular absolutely decaying measure μ on $\mathbb{R}^d$. Conversely, if S is Borel and $S \cap \operatorname{supp} \mu \neq \varnothing$ for any compactly supported Ahlfors regular absolutely decaying measure μ on $\mathbb{R}^d$, then $S \subseteq \mathbb{R}^d$ is HAW.*

Proposition 11 has the following equivalent formulation, stated as Proposition 13, which does not use measures and is slightly easier to prove. First, recall the following definition appearing in [BFK$^+$12].

Definition 12. A nonempty closed subset $K \subseteq \mathbb{R}^d$ is called *hyperplane diffuse* if there exists $\beta > 0$ and $r_0 > 0$ such that for every $\mathbf{x} \in K$, $0 < r \leq r_0$ and every hyperplane $H \subseteq \mathbb{R}^d$ we have that

$$K \cap \left(B\left(\mathbf{x}, r\right) \setminus B\left(H, \beta r\right)\right) \neq \varnothing \,. \tag{8}$$

Proposition 13. *If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap K \neq \varnothing$ for any hyperplane diffuse set $K \subseteq \mathbb{R}^d$. Moreover, if S is Borel then the converse also holds.*

Only the second parts of Propositions 11 and 13 are new. Indeed, [BFK$^+$12, Proposition 5.5] further proves a lower bound on the dimension of the intersection of HAW sets with the support of decaying measures, and an analogous statement about the intersection of HAW sets with hyperplane diffuse sets follows from [BFK$^+$12, Proposition 5.5]. A direct proof of the first implication in Proposition 13 is provided below for the reader's convenience.

The equivalence between Propositions 11 and 13 follows from the fact that if μ is absolutely decaying then $\operatorname{supp}\mu$ is hyperplane diffuse [BFK+12, Proposition 5.1], and, on the other hand, if K is hyperplane diffuse then there exists an Ahlfors regular absolutely decaying measure μ for which $\operatorname{supp}\mu \subseteq K$. The latter can be shown on modifying the proof of [BFK+12, Proposition 5.5], where an absolutely decaying measure μ is constructed. Formally, we have the following statement.

Proposition 14. *Let $K \subseteq \mathbb{R}^d$ be hyperplane diffuse. Then there exists a compactly supported absolutely decaying Ahlfors regular measure μ on $\mathbb{R}^d$ such that* $\operatorname{supp}\mu \subseteq K$.

To summarise above discussion, in order to fully justify our claim that Theorem 3 follows from Theorem 5, it remains to give formal proofs to Propositions 13 and 14. To begin with, we deal with the former, and start by stating two auxiliary statements that will be used in the proof of Proposition 13.

Lemma 15. *For any $\beta > 0$ there exists $0 < \beta' < \beta$ and N such that, for every ball $B = B(\mathbf{x}, \rho) \subseteq \mathbb{R}^d$ there is a collection of at most N hyperplanes $\mathcal{H}_B$ such that for any hyperplane H' there exists $H \in \mathcal{H}_B$ for which*

$$B(\mathbf{x}, \rho) \cap B\left(H', \beta'\rho\right) \subseteq B\left(H, \beta\rho\right).$$

Proof. The statement of this lemma is a specific case of Assumption C.6 in [FSU18], where $\beta' = \frac{\beta}{3}$. In the case of hyperplanes (Lemma 15) it is verified as part (2) of Observation C.7. in [FSU18]. □

The following is a slightly simplified version of Lemma 4.3 in [BFK+12].

Lemma 16. *Let $K \subseteq \mathbb{R}^d$ be hyperplane diffuse. Then there exist $0 < \beta_0 < \frac{1}{3}$ and $r_0 > 0$ such that for any $0 < r \leq r_0$, any $\mathbf{x} \in K$ and any hyperplane H there exists $\mathbf{x}' \in K$ such that*

$$B\left(\mathbf{x}', \beta_0 r\right) \subseteq B(\mathbf{x}, r) \setminus B\left(H, \beta_0 r\right). \tag{9}$$

Proof of Proposition 13. Assume that S is hyperplane absolute winning and K is hyperplane diffuse. Let β_0 and r_0 be the same as in Lemma 16 and suppose that Alice and Bob play the restricted hyperplane absolute game according to Definition 7. Suppose that on the first move Bob chooses $\beta = \beta_0$ and a ball B_0 of radius r_0 centred in K. These are valid assumptions since $\beta_0 \in (0,1)$, K is non-empty and B_0 can be arbitrary. Let $n \geq 0$ and suppose that $B_0, \ldots, B_n$ are the balls arising from the restricted hyperplane absolute game that Alice and Bob play with Alice using her winning strategy. Suppose that all these balls are centred in K. Note that, by the choice of B_0, this is true for $n = 0$. Let A_{n+1} be the neighborhood of any hyperplane in $\mathbb{R}^d$ of radius $\beta^{n+1} r_0$ that Alice chooses according to Definition 7. Then, by Lemma 16, Bob can choose a ball B_{n+1} of radius $\beta^{n+1} r_0$ which is contained in $B_n \setminus A_{n+1}$ and centered in K. Indeed, B_{n+1} can be defined to be $B\left(\mathbf{x}', \beta_0 r\right)$ arising from Lemma 16 with $B(\mathbf{x}, r) = B_n$ and $B\left(H, \beta_0 r\right) = A_{n+1}$. Thus, for any n Bob has a legal move, which means that Alice cannot win by default, and the sequence $(B_n)_{n \geq 0}$ can be made infinite. Furthermore, as we have shown above Bob can play so that the centres of B_n are all in K and so the unique point in $\bigcap_{n \geq 0} B_n$ lies in K. At the same time Alice can play using her winning strategy so that the unique point in $\bigcap_{n \geq 0} B_n$ also lies in S. Therefore, $S \cap K \neq \varnothing$ and this completes the proof of the first part of Proposition 13.

For the converse, assume that S is a Borel set which is not HAW. Then, by Borel determinacy theorem for the absolute game appearing in [FLS14, Theorem 1.6], Bob has a winning strategy, which will be fixed for the rest of the proof. Let β and B_0 be chosen on the first move of Bob according to his winning strategy. Define $\mathcal{B}_0 = \{B_0\}$ and continue by induction to construct collections of closed balls $\mathcal{B}_n$ as follows. Given $\mathcal{B}_n$, for every $B \in \mathcal{B}_n$ let $\mathcal{H}_B$ be the collection of hyperplanes arising from Lemma 15. Define $\mathcal{B}_{n+1}(B)$ to be the collection of all of Bob's responses according to the winning strategy while considering the hyperplanes in $\mathcal{H}_B$ as possible moves of Alice. Note that $\mathcal{B}_{n+1}(B)$ is always nonempty. Define

$$K = \bigcap_{n\geq 0} \bigcup_{B\in\mathcal{B}_n} B\,. \tag{10}$$

By Lemma 15, every $\mathbf{x} \in K$ is an outcome of the hyperplane absolute game played according to Bob's winning strategy. Therefore, $\mathbf{x} \notin S$. Since $\mathbf{x}$ is an arbitrary point of K, we have that $K \cap S = \varnothing$.

It is left to verify that K is hyperplane diffuse. Indeed, we will show that it is $\frac{\beta'\beta}{2}$ hyperplane diffuse for β' as in Lemma 15. Assume $\mathbf{x} \in K$, $0 < r \leq r_0$ and $H' \subseteq \mathbb{R}^d$ is a hyperplane, where r_0 is the radius of B_0. Let n be the unique positive integer such that

$$2\beta^n r_0 \leq r < 2\beta^{n-1} r_0\,, \tag{11}$$

which clearly exists since $0 < \beta < 1$. Since $\mathbf{x} \in K$, by (10), there exists a ball $B = B(\mathbf{x}_0, \beta^n r_0) \in \mathcal{B}_n$ such that $\mathbf{x} \in B$. The left hand side of (11) implies that $B \subseteq B(\mathbf{x}, r)$. By Lemma 15 applied with $\rho = \beta^n r_0$, there exists $H \in \mathcal{H}_B$ such that

$$B \cap B\left(H', \beta'\beta^n r_0\right) \subseteq B\left(H, \beta^{n+1} r_0\right).$$

The right hand side of (11) implies that $\frac{\beta\beta'}{2} r < \beta'\beta^n r_0$ and hence

$$B \cap B\left(H', \frac{\beta\beta'}{2} r\right) \subseteq B\left(H, \beta^{n+1} r_0\right).$$

By the definition of $\mathcal{B}_{n+1}(B)$, there exists a ball $B' \in \mathcal{B}_{n+1}(B)$ such that $B' \cap B\left(H, \beta^{n+1} r_0\right) = \varnothing$. Since the collections $\mathcal{B}_{n+1}(B)$ are always nonempty, by (10), we have that $K \cap B' \neq \varnothing$. Since $\varnothing \neq K \cap B' \subseteq K \cap B \subseteq K \cap B(\mathbf{x}_0, r)$, we have $K \cap B(\mathbf{x}_0, r) \nsubseteq B\left(H', \frac{\beta\beta'}{2} r\right)$. Hence, $K \cap \left(B(\mathbf{x}_0, r) \setminus B\left(H', \frac{\beta\beta'}{2} r\right)\right) \neq \varnothing$. This verifies Definition 12 for the set K and thus completes the proof. □

2.1. **Proof of Proposition 14.** The proof of Proposition 14 relies on a standard construction of Ahlfors regular measures in $\mathbb{R}^d$ via decreasing collections of disjoint balls. For this construction we follow [KW05, Section 7.2]. Assume that $0 < \beta < 1$, $r_0 > 0$, and that $N > 1$ is some fixed integer. Assume that B_0 is a closed ball in $\mathbb{R}^d$ and that $(\mathcal{B}_n)_{n\geq 0}$ is a sequence of collections of closed balls such that $\mathcal{B}_0 = \{B_0\}$, any $B \in \mathcal{B}_n$ is a ball of radius $\beta^n r_0$, and

$$\mathcal{B}_{n+1} = \bigcup_{B\in\mathcal{B}_n} \mathcal{B}_{n+1}(B)\,, \tag{12}$$

where for every $B \in \mathcal{B}_n$ the collection

$$\mathcal{B}_{n+1}(B) = \left\{B' \in \mathcal{B}_{n+1} \,:\, B' \subseteq B\right\}$$

contains exactly N disjoint balls for any $n \geq 0$. Define

$$K = \bigcap_{n\geq 0} \bigcup_{B\in\mathcal{B}_n} B\,. \tag{13}$$

Also define the sequence of probability measures

$$\mu_n = \frac{1}{\#\mathcal{B}_n} \sum_{B \in \mathcal{B}_n} \lambda|_B \qquad (n \geq 0)$$

where λ is the Lebesgue measure on $\mathbb{R}^d$ and $\lambda|_B$ is the normalised restriction of λ to B which is defined by the formula $\lambda|_B(A) = \lambda(A \cap B)/\lambda(B)$ for any Lebesgue measurable set A. By (12), we have that $\operatorname{supp} \mu_n \supseteq \operatorname{supp} \mu_{n+1}$ for any $n \geq 0$. Let μ be the weak limit of μ_n and let

$$\alpha = -\frac{\log N}{\log \beta}\,. \tag{14}$$

Note that for every $n \geq 0$ and every $B \in \mathcal{B}_n$ we have that

$$\mu(B) = \mu_n(B) = \frac{1}{\#\mathcal{B}_n} = N^{-n}\,.$$

Proposition 17. *Let $K \subseteq \mathbb{R}^d$, μ and α be defined as above. Then* $\operatorname{supp} \mu = K$ *and μ is α-Ahlfors regular.*

Proposition 17 is proved in [KW05, Proposition 7.1] for $\mathbb{R}^d$ with the supremum norm. For completeness we repeat their proof with the Euclidean norm.

Proof. Assume $\mathbf{x} \in K$ and $0 < r \leq 2r_0$. Let n be the unique integer for which

$$2\beta^{n+1} r_0 < r \leq 2\beta^n r_0\,. \tag{15}$$

Since $\mathbf{x} \in K$, by (13) and the disjointness of the balls in $\mathcal{B}_{n+1}$, there exists a unique ball $B \in \mathcal{B}_{n+1}$ such that $\mathbf{x} \in B$. The left hand side of (15) implies that $B \subseteq B(\mathbf{x}, r)$. So, by (14) and the right hand side of (15) this implies that

$$\mu(B(\mathbf{x}, r)) \geq \mu(B) = \frac{1}{N^{n+1}} = \beta^{\alpha(n+1)} \geq \left(\frac{\beta}{2r_0}\right)^{\alpha} r^{\alpha}\,.$$

On the other hand, by the right hand side of (15), there exists a constant $M \geq 1$ depending only on β and d such that

$$\#\left\{B \in \mathcal{B}_n \,:\, B \cap B(\mathbf{x}, r) \neq \varnothing\right\} \leq M\,.$$

Therefore, by (14) and the left hand side of (15) this implies that

$$\mu(B(\mathbf{x}, r)) \leq \frac{M}{N^n} = M\beta^{\alpha n} < M\left(\frac{1}{2r_0\beta}\right)^{\alpha} r^{\alpha}\,.$$

So (6) is verified with $A = \max\left\{\left(\frac{2r_0}{\beta}\right)^{\alpha}, M\left(\frac{1}{2r_0\beta}\right)^{\alpha}\right\}$.

□

The proof of Proposition 14 is based on the construction described above, for a particular choice of balls which stay far from appropriate neighborhoods of hyperplanes in each level. The argument used for the proof of [BFK+12, Proposition 5.5] provides such a choice. It is based on the following lemma.

Definition 18. Say that d points in $\mathbb{R}^d$ are in *general position* if they lie on a unique hyperplane. If $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{R}^d$ are in general position denote this hyperplane by $H(\mathbf{x}_1, \ldots, \mathbf{x}_d)$.

Lemma 19 (See [BFK+12, Lemma 5.6]). *Given $\beta_0 > 0$, there exists a positive parameter $\beta' \leq \beta_0$ such that for every $\mathbf{x} \in \mathbb{R}^d$, $\rho > 0$, and $\mathbf{x}_1, \ldots, \mathbf{x}_d \in B(\mathbf{x}, \rho)$ in general position such that the balls $B(\mathbf{x}_i, \beta_0\rho)$ are contained in $B(\mathbf{x}, \rho)$ for every $1 \leq i \leq d$ and are pairwise disjoint, if a hyperplane H intersects $B(\mathbf{x}_i, \beta'\rho)$ for every $1 \leq i \leq d$ then*

$$B(\mathbf{x}, \rho) \cap B(H, \beta'\rho) \subseteq B(H(\mathbf{x}_1, \ldots, \mathbf{x}_d), \beta_0\rho).$$

Lemma 19 is stated in [BFK+12] with the general position assumption implicit. We repeat the proof that appears in [BFK+12] for completeness.

Proof. Without loss of generality assume that $\mathbf{x} = 0$ and $\rho = 1$. By contradiction, assume that for every integer $k \geq 1$ there are $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{d,k} \in B(0,1)$ in general position and a hyperplane H_k that intersects $B\left(\mathbf{x}_{i,k}, \frac{1}{k}\right)$ for each $1 \leq i \leq d$ but

$$B(0,1) \cap B\left(H_k, \frac{1}{k}\right) \not\subseteq B(H(\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{d,k}), \beta_0). \tag{16}$$

By the compactness of $B(0,1)$ there are subsequences $\left(\mathbf{x}_{1,k_j}, \ldots, \mathbf{x}_{d,k_j}\right)$ and H_{k_j} that converge, say to $(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ and H respectively. Then necessarily $\mathbf{x}_1, \ldots, \mathbf{x}_d \in H$ and, therefore, any j large enough satisfies

$$\begin{aligned} B(0,1) \cap B\left(H_{k_j}, \frac{\beta_0}{3}\right) &\subseteq B(0,1) \cap B\left(H, \frac{2\beta_0}{3}\right) \\ &\subseteq B(0,1) \cap B\left(H\left(\mathbf{x}_{1,k_j}, \ldots, \mathbf{x}_{d,k_j}\right), \beta_0\right). \end{aligned}$$

Choosing j large enough so that $\frac{1}{k_j} \leq \frac{\beta_0}{3}$ we obtain a contradiction to (16). $\square$

Proof of Proposition 14. We follow the proof of [BFK+12, Proposition 5.5]. Assume K is hyperplane diffuse. The goal is to construct an Ahlfors regular absolutely decaying measure supported on a subset of K. Let β_0 and r_0 be as in Lemma 16. Let β' be as in Lemma 19, and let

$$\beta = \frac{\beta'}{2}. \tag{17}$$

Let $\mathbf{x}_0 \in K$ be any point, and set $B_0 = B(\mathbf{x}_0, r_0)$ and $\mathcal{B}_0 = \{B_0\}$. Recursively construct the collections $\mathcal{B}_{n+1}(B)$ for every integer $n \geq 0$ and every $B \in \mathcal{B}_n$ as follows. Construct by recursion a collection of $d+1$ points in $K \cap B$. Assume $\mathbf{x}_1, \ldots, \mathbf{x}_i \in K \cap B$ are already defined for some $0 \leq i \leq d$, and let H be any hyperplane that passes through $\mathbf{x}_1, \ldots, \mathbf{x}_i$. By (9) there exists a point $\mathbf{x}_{i+1}$ such that

$$B(\mathbf{x}_{i+1}, \beta_0\beta^n r_0) \subseteq B \setminus B(H, \beta_0\beta^n r_0). \tag{18}$$

Define $\mathcal{B}_{n+1}(B) = \left\{B\left(\mathbf{x}_1, \beta^{n+1}r_0\right), \ldots, B\left(\mathbf{x}_{d+1}, \beta^{n+1}r_0\right)\right\}$. Since $\beta < \beta_0$ this is a collection of $d+1$ disjoint balls contained in B. Let μ be as defined in the beginning of this section. Then $\operatorname{supp}\mu \subseteq K$ since for every $n \geq 0$ every $B \in \mathcal{B}_n$ is a ball centered in K. Proposition 17 guarantees that μ is Ahlfors regular. It is left to verify that μ is absolutely decaying.

Assume $r \leq r_0$, $\mathbf{x} \in \operatorname{supp}\mu$ and $r' > 0$, and let H be any hyperplane. Let $n \geq 0$ be the unique integer satisfying

$$2\beta^{n+1}r_0 \leq r < 2\beta^n r_0. \tag{19}$$

Since $\mathbf{x} \in \operatorname{supp} \mu$ there are balls $B \subseteq B'$ with $B \in \mathcal{B}_{n+1}$ and $B' \in \mathcal{B}_n$ such that $\mathbf{x} \in B$. The left hand side of (19) implies $B \subseteq B(\mathbf{x}, r)$. On the other hand, equation (18) implies that

$$\mathrm{d}\left(B', B''\right) \geq 2\left(\beta_0 - \beta\right) \beta^{n-1} r_0$$

for any $B' \neq B'' \in \mathcal{B}_n$, therefore, since $\beta = \frac{\beta'}{2} \leq \frac{\beta_0}{2}$, the right hand side of (19) implies that $B(\mathbf{x}, r) \cap B'' = \varnothing$ for any $B' \neq B'' \in \mathcal{B}_n$. So, $B(\mathbf{x}, r) \cap \operatorname{supp} \mu \subseteq B'$.

It is enough to verify (7) for every r' small enough. Assume that $r' < \frac{1}{2}\beta r$ and let $m \geq 1$ be the unique integer satisfying

$$\frac{1}{2}\beta^{m+1} r \leq r' < \frac{1}{2}\beta^m r \,. \tag{20}$$

The right hand side of both (19) and (20) imply that $r' < \beta^{m+n} r_0$. Therefore, for every $1 \leq k \leq m$ and every $B'' \in \mathcal{B}_{n+k-1}$, the hyperplane neighborhood $B(H, r')$ intersects at most d balls from $\mathcal{B}_{n+k}(B'')$. Indeed, recall that

$$\mathcal{B}_{n+k}(B'') = \left\{ B\left(\mathbf{x}_1, \beta^{n+k} r_0\right), \ldots, B\left(\mathbf{x}_{d+1}, \beta^{n+k} r_0\right) \right\}.$$

If $B(H, r') \cap B\left(\mathbf{x}_i, \beta^{n+k} r_0\right) \neq \varnothing$ for every $1 \leq i \leq d+1$ then (17) implies that

$$H \cap B\left(\mathbf{x}_i, \beta' \beta^{n+k-1}\right) \neq \varnothing$$

for every $1 \leq i \leq d+1$. By construction, the points $\mathbf{x}_1, \ldots, \mathbf{x}_d$ are in general position, so Lemma 19 gives

$$B'' \cap B\left(H, \beta' \beta^{n+k-1} r_0\right) \subseteq B\left(H(\mathbf{x}_1, \ldots, \mathbf{x}_d), \beta_0 \beta^{n+k-1} r_0\right).$$

In particular, $\mathbf{x}_{d+1} \in B\left(H(\mathbf{x}_1, \ldots, \mathbf{x}_d), \beta_0 \beta^{n+k-1} r_0\right)$, which contradicts (18). The upshot is that $B(H, r')$ intersects at most d^m balls in $\mathcal{B}_{n+m}$, therefore,

$$\begin{aligned}\mu\left(B(\mathbf{x}, r) \cap B\left(H, r'\right)\right) &\leq \left(\frac{d}{d+1}\right)^m \mu\left(B'\right) \\ &= \left(\frac{d}{d+1}\right)^m (d+1)\mu\left(B\right) \leq \left(\frac{d}{d+1}\right)^m (d+1)\mu\left(B(\mathbf{x}, r)\right).\end{aligned}$$

By the left hand side of (20), this verifies (7) with $\delta = \frac{\log \frac{d}{d+1}}{\log \beta}$ and $D = \left(\frac{2}{\beta}\right)^{\delta} (d+1)$. □

2.2. **Cantor potential game.** In order to prove Theorem 5, we will use the Cantor potential game introduced in [BHNS18]. The game and its corresponding winning sets are defined as follows.

Definition 20. Let X be a complete metric space and $\alpha \geq 0$. The *α-Cantor potential game* is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a parameter $0 < \beta < 1$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$. Subsequently for $n = 0, 1, 2, \ldots$, first, Alice chooses collections $\mathcal{A}_{n+1,i}$ of at most $\beta^{-\alpha(i+1)}$ balls of radius $\beta^{n+1+i} r_0$ for every $i \geq 0$. Then, Bob chooses a ball B_{n+1} of radius $\beta^{n+1} r_0$ which is contained in B_n and disjoint from $\bigcup_{0 \leq \ell \leq n} \bigcup_{A \in \mathcal{A}_{n+1-\ell, \ell}} A$. If there is no such ball the game stops and Alice wins by default. Otherwise, the *outcome* of the game is the unique point in $\bigcap_{n \geq 0} B_n$.

A set $S \subseteq X$ is called *α-Cantor winning* if Alice has a strategy which ensures that she either wins by default or the outcome lies in S. If X is the support of an

α-Ahlfors regular measure then $S \subseteq X$ is called *Cantor winning* if it is α'-Cantor winning for some $0 \leq \alpha' < \alpha$.

It is proved in [BHNS18] that this definition of α-Cantor winning sets agrees with the original definition given in [BH17]. Here the convention regarding α is opposite to the one used in [BH17]. For example, in our convention 0-Cantor winning sets are absolute winning, see [BHNS18] or [BFK+12] for the definition of absolute winning. This convention allows the definition of the α-Cantor potential game to be independent of the space X. This comes at the price that some properties of Cantor winning subsets do depend on X. We will use the following fact about Cantor winning sets.

Theorem 21 (See [BHNS18, Theorems 3.4, 4.1])**.** *Let X be the support of an α-Ahlfors regular measure and let $S \subseteq X$ be Cantor winning. Then $S \neq \varnothing$.*

We finish this section by stating an auxiliary lemma about *efficient covers* for Ahlfors regular measures, which will be used in Section 4.

Lemma 22. *Let μ be an Ahlfors regular measure on a metric space X, let A, α, r_0 be as in Definition 9 and let $S \subseteq X$ be any subset. Suppose that $0 < r \leq r_0$ and $\mu(B(S, r)) < \infty$. Then there exists a cover of $S \cap \operatorname{supp} \mu$ with balls of radius $3r$ of cardinality at most*

$$\frac{A\mu\left(B\left(S,r\right)\right)}{r^{\alpha}}. \tag{21}$$

Proof. Assume $S \subseteq X$ and $r > 0$. Without loss of generality assume that $S \neq \varnothing$. Note that $B(S, r)$ is a Borel set. Choose any collection of points $\mathcal{U} \subseteq S \cap \operatorname{supp} \mu$ such that $\{B(x, r) : x \in \mathcal{U}\}$ is a collection of pairwise disjoint balls. By the pairwise disjointness and (6),

$$\#\mathcal{U} \times \frac{r^{\alpha}}{A} \leq \sum_{x \in \mathcal{U}} \mu\left(B\left(x, r\right)\right) = \mu\left(\bigcup_{x \in \mathcal{U}} B\left(x, r\right)\right) \leq \mu\left(B\left(S, r\right)\right). \tag{22}$$

Since $\mu(B(S, r)) < \infty$, any such collection $\mathcal{U}$ is finite. Therefore, of all the collections $\mathcal{U}$ as above there exists one, say $\mathcal{U}_{\max}$, with the maximal number of elements. Then, by its maximality, for any $x' \in S \cap \operatorname{supp} \mu$ there exists $x \in \mathcal{U}_{\max}$ such that $\mathrm{d}(x, x') \leq 2r$. Then, $x' \in B(x, 3r)$ and we conclude that $\{B(x, 3r) : x \in \mathcal{U}_{\max}\}$ is a cover of $S \cap \operatorname{supp} \mu$. Finally, (22) implies (21) and the proof is complete. □

3. Homogeneous dynamics and quantitative nondivergence

The connection between Diophantine approximation and homogeneous dynamics is well known as the Dani correspondence. In this context there is a beautiful relation between bounded orbits and badly approximable vectors. Throughout, $\operatorname{diag}(b_1, \ldots, b_d)$ denotes the $d \times d$ diagonal matrix with diagonal entries $b_1, \ldots, b_d$.

Let $G := \mathrm{SL}_{d+1}(\mathbb{R})$ and $\Gamma := \mathrm{SL}_{d+1}(\mathbb{Z})$. The homogeneous space $X_{d+1} := G/\Gamma$ can be identified with the moduli space of unimodular lattices in $\mathbb{R}^{d+1}$ via the following map:

$$g\Gamma \in X_{d+1} \mapsto g\mathbb{Z}^{d+1}.$$

Given $\mathbf{w} = (w_1, \ldots, w_d) \in \mathcal{W}_d$ and $b > 1$, for any $n \in \mathbb{Z}$ we let

$$a_n := \begin{pmatrix} b^n & & & \\ & b^{-w_1 n} & & \\ & & \ddots & \\ & & & b^{-w_d n} \end{pmatrix} \in G\,. \tag{23}$$

Further, for any $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$u_{\mathbf{x}} := \begin{pmatrix} 1 & x_1 & \cdots & x_d \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in G\,. \tag{24}$$

For $\varepsilon > 0$ define the set

$$K_\varepsilon := \{\Lambda \in X_{d+1} \,:\, \|\mathbf{v}\| \ge \varepsilon \text{ for any } \mathbf{v} \in \Lambda \setminus \{0\}\}\,,$$

where $\|\mathbf{v}\|$ is the Euclidean norm of $\mathbf{v}$. Then, as is well known for any $\mathbf{x} \in \mathbb{R}^d$ we have that

$$\mathbf{x} \in \mathbf{Bad}\,(\mathbf{w}) \iff \exists\, \varepsilon > 0 \text{ such that } a_n u_{\mathbf{x}} \mathbb{Z}^{d+1} \in K_\varepsilon \text{ for every } n \in \mathbb{N}\,. \tag{25}$$

See [BPV11, Appendix] and [Ber15, Appendix A] for detailed explanation of this equivalence.

Recall that by Mahler's criterion, the complements of the sets K_ε give a basis for the topology at ∞ in X_{d+1}, so (25) may be rephrased as $\mathbf{x} \in \mathbf{Bad}\,(\mathbf{w})$ if and only if $\left\{a_n u_{\mathbf{x}} \mathbb{Z}^{d+1} \,:\, n \in \mathbb{N}\right\}$ is bounded in X_{d+1}.

It is straightforward to verify that for every $\mathbf{x}' \in \mathbb{R}^d$ we have that

$$a_n u_{\mathbf{x}'} a_n^{-1} = u_{\mathrm{diag}\left(b^{(1+w_1)n}, \ldots, b^{(1+w_d)n}\right)\mathbf{x}'}\,. \tag{26}$$

Note that if $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}'$ then $u_{\mathbf{x}} = u_{\mathbf{x}'} u_{\mathbf{x}_0}$ and therefore

$$a_n u_{\mathbf{x}} \mathbb{Z}^{d+1} = a_n u_{\mathbf{x}'} a_n^{-1} a_n u_{\mathbf{x}_0} \mathbb{Z}^{d+1} \overset{(26)}{=} u_{\mathrm{diag}\left(b^{(1+w_1)n}, \ldots, b^{(1+w_d)n}\right)\mathbf{x}'} a_n u_{\mathbf{x}_0} \mathbb{Z}^{d+1}\,.$$

Thus, on letting $\Lambda = a_n u_{\mathbf{x}_0} \mathbb{Z}^{d+1}$ and $\mathbf{y} = \mathrm{diag}\left(b^{(1+w_1)n}, \ldots, b^{(1+w_d)n}\right)\mathbf{x}'$ we see that the set of parameters $\mathbf{y} \in \mathbb{R}^d$ for which $u_{\mathbf{y}}\Lambda \in K_\varepsilon$ plays a role in the study of bounded orbits of $u_{\mathbf{x}}\mathbb{Z}^{d+1}$ under the actions by a_n. The Dani-Kleinbock-Margulis quantitative nondivergence estimate (see [Dan86, KM98]) gives a sharp and uniform upper bound on the Lebesgue measure of the set of $\mathbf{y}$ for which $u_{\mathbf{y}}\Lambda \notin K_\varepsilon$ under some conditions on the lattice Λ. Later this was generalised to "friendly" measures by Kleinbock, Lindenstrauss and Weiss [KLW04]. Within this paper we will use the following direct consequence of Theorem 5.11 in [BNY20], which in turn is a consequence of the results of [KLW04].

Theorem 23. *Assume μ is an Ahlfors regular absolutely decaying measure on $\mathbb{R}^d$. Then for any $\mathbf{z} \in \operatorname{supp}\mu$ there exists an open ball $B(\mathbf{z})$ centred at $\mathbf{z}$ and constants $C, \gamma > 0$ such that for any ball $B \subseteq B(\mathbf{z})$ centred in $\operatorname{supp}\mu$, any diagonal matrix $g \in G$ and any $0 < \rho \le 1$ at least one of the following two conclusions holds:*

(i) *for all $\varepsilon > 0$*

$$\mu\left(\left\{\mathbf{x} \in B : g u_{\mathbf{x}} \mathbb{Z}^{d+1} \notin K_\varepsilon\right\}\right) \le C\left(\frac{\varepsilon}{\rho}\right)^{\gamma} \mu(B)\,; \tag{27}$$

(ii) *there exists* $\mathbf{0} \neq \mathbf{v} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \in \bigwedge^j \left(\mathbb{Z}^{d+1}\right)$ *with* $1 \leq j \leq d$ *such that*

$$\sup_{x \in B} \|g u_{\mathbf{x}} \mathbf{v}\| < \rho .$$

In order to use Theorem 23 some notation related to the action of G on the exterior algebra of $\mathbb{R}^{d+1}$ is set up in the rest of this section.

Let $\mathbf{e}_+ := (1, 0, \ldots, 0)$ and $\mathbf{e}_i := (0, \ldots, 1, \ldots, 0)$, where for every $1 \leq i \leq d$ the $i+1$st coordinate is one and the rest are zero, be the standard basis of $\mathbb{R}^{d+1}$. For any $I \subseteq \{+, 1 \ldots, d\}$ let $\mathbf{e}_I = \bigwedge_{i \in I} \mathbf{e}_i$ be the wedge product of basis elements with indices in I. For any $1 \leq j \leq d$ the collection $\{\mathbf{e}_I : \#I = j\}$ is a basis of $\bigwedge^j \left(\mathbb{R}^{d+1}\right)$. Define an inner product on $\bigwedge^j \left(\mathbb{R}^{d+1}\right)$ by setting $\langle \mathbf{e}_I, \mathbf{e}_J \rangle = \delta_{I,J}$ (where $\delta_{I,J} := 1$ if $I = J$ and $\delta_{I,J} := 0$ otherwise) and extending linearly. Let $\| \cdot \|$ be the Euclidean norm which is derived from this inner product. Note that this notation is consistent with that of Theorem 23.

For every $1 \leq j \leq d$ define the subspaces

$$V_+ := \operatorname{span}_{\mathbb{R}}\{\mathbf{e}_I : + \in I\} \qquad \text{and} \qquad V_- := \operatorname{span}_{\mathbb{R}}\{\mathbf{e}_I : I \subseteq \{1, \ldots, d\}\} .$$

Each vector $\mathbf{v} \in \bigwedge^j \left(\mathbb{R}^{d+1}\right)$ decomposes uniquely into $\mathbf{v} = \mathbf{v}_+ + \mathbf{v}_-$ with $\mathbf{v}_+ \in V_+$ and $\mathbf{v}_- \in V_-$.

Let G act on $\bigwedge^j \left(\mathbb{R}^{d+1}\right)$ by linear transformations defined on wedge products as follows: for any $g \in G$ and $\mathbf{v} = \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_j \in \bigwedge^j \left(\mathbb{R}^{d+1}\right)$ we define

$$g\mathbf{v} = g\mathbf{v}_1 \wedge \ldots \wedge g\mathbf{v}_j . \tag{28}$$

Proposition 24. *Assume* $\mathbf{x} \in \mathbb{R}^d$, $h \in \mathbb{Z}$, $h \geq 0$ *and* $\mathbf{v} = \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_j \in \bigwedge^j \left(\mathbb{R}^{d+1}\right)$, $1 \leq j \leq d$. *Then, assuming that* $\mathbf{v}_i = v_{i,+}\mathbf{e}_+ + v_{i,1}\mathbf{e}_1 + \cdots + v_{i,d}\mathbf{e}_d$, *we have that*

$$u_{\mathbf{x}}\mathbf{v} = \mathbf{v} + \mathbf{e}_+ \wedge \left(\sum_{i=1}^{j} (-1)^{i+1} \left(v_{i,1}x_1 + \ldots + v_{i,d}x_d\right) \bigwedge_{i' \neq i} \mathbf{v}_{i'} \right); \tag{29}$$

$$\|a_{-h}\mathbf{v}_+\| \leq b^{-w_d h}\|\mathbf{v}_+\| \quad \text{and} \quad \|a_{-h}\mathbf{v}_-\| \leq b^{h}\|\mathbf{v}_-\| , \tag{30}$$

where w_d *is assumed to be the smallest weight.*

Proof. Both (29) and (30) are elementary to prove. Indeed, (29) is an immediate consequence of definition (28) and the easily verified equation $u_{\mathbf{x}}\mathbf{v}_i = \mathbf{v}_i + \left(v_{i,1}x_1 + \ldots + v_{i,d}x_d\right)\mathbf{e}_+$ together with the alternating property of the wedge product and, in particular, the fact that $\mathbf{e}_+ \wedge \mathbf{e}_+ = \mathbf{0}$. In turn, since the standard basis $\mathbf{e}_I$ of $\bigwedge^j \left(\mathbb{R}^{d+1}\right)$, where $I \subseteq \{+, 1, \ldots, d\}$ and $\#I = j$, is orthonormal and each of $\mathbf{e}_I$ is an eigenvector of a_{-h}, it suffices to verify (30) for the basis vectors $\mathbf{e}_I$. The latter is a trivial job done by inspecting (30). We leave further computational details, which are straightforward, to the reader. □

When applying Theorem 23 in Section 4 we will use the following simple bound.

Lemma 25. *For every ball* $B \subseteq \mathbb{R}^d$, *every diagonal matrix* $g = \operatorname{diag}(b_+, b_1, \ldots, b_d)$ *such that* $b_+ \geq 1$ *and* $0 < b_1, \ldots b_d \leq 1$ *such that* $b_+ b_1 \cdots b_d = 1$, *and every* $\mathbf{v} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \in \bigwedge^j \left(\mathbb{Z}^{d+1}\right)$ *with* $1 \leq j \leq d+1$ *such that* $\mathbf{v} \neq \mathbf{0}$ *we have that*

$$\sup_{x \in B} \|g u_{\mathbf{x}} \mathbf{v}\| \geq \min\{1, r_B\} , \tag{31}$$

where r_B *is the Euclidean radius of* B.

Proof. The case of $j = d+1$ is trivial since in this case we have that $\|gu_{\mathbf{x}}\mathbf{v}\| = \|\mathbf{v}\| \geq 1$ for all $\mathbf{x}$. Let $1 \leq j \leq d$, $\mathbf{v} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \in \bigwedge^j \left(\mathbb{Z}^{d+1}\right)$ and $\mathbf{v} \neq \mathbf{0}$. Let $\tilde{\mathbf{x}} = (1, x_1, \ldots, x_d)$ and write each $\mathbf{v}_i = (v_{i,+}, v_{i,1}, \ldots, v_{i,d})$. Then

$$gu_{\mathbf{x}}\mathbf{v} = \bigwedge_{i=1}^{j} \begin{pmatrix} b_+\langle \tilde{\mathbf{x}}, \mathbf{v}_i \rangle \\ b_1 v_{i,1} \\ \vdots \\ b_d v_{i,d} \end{pmatrix}. \tag{32}$$

Since $\mathbf{v} \neq \mathbf{0}$, there exists a collection $\{\ell_2, \ldots, \ell_d\} \subseteq \{1, \ldots, d\}$ such that the rows $(b_{\ell_k} v_{1,\ell_k}, \ldots, b_{\ell_k} v_{j,\ell_k})$ $(2 \leq k \leq d)$ are linearly independent. It follows that the determinant

$$\det \begin{pmatrix} b_+\langle \tilde{\mathbf{x}}, \mathbf{v}_1 \rangle & \ldots & b_+\langle \tilde{\mathbf{x}}, \mathbf{v}_j \rangle \\ b_{\ell_2} v_{1,\ell_2} & \ldots & b_{\ell_2} v_{j,\ell_2} \\ \vdots & \ddots & \vdots \\ b_{\ell_j} v_{1,\ell_j} & \ldots & b_{\ell_j} v_{j,\ell_j} \end{pmatrix} = \prod_{k=1}^{j} b_{\ell_k} \times \det \begin{pmatrix} \langle \tilde{\mathbf{x}}, \mathbf{v}_1 \rangle & \ldots & \langle \tilde{\mathbf{x}}, \mathbf{v}_j \rangle \\ v_{1,\ell_2} & \ldots & v_{j,\ell_2} \\ \vdots & \ddots & \vdots \\ v_{1,\ell_j} & \ldots & v_{j,\ell_j} \end{pmatrix},$$

where $\ell_1 = +$, is not identically zero. Here we used the obvious fact that the functions $\langle \tilde{\mathbf{x}}, \mathbf{v}_1 \rangle, \ldots, \langle \tilde{\mathbf{x}}, \mathbf{v}_j \rangle$ are linearly independent over $\mathbb{R}$, which follows from the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_j$. Observe that the above determinant is one of the coordinates of $gu_{\mathbf{x}}\mathbf{v}$ (see [Sch80, Section IV.6 Lemma 6A]). Furthermore, since all the vectors $\mathbf{v}_i$ are integer, it is of the form $\prod_{k=1}^{j} b_{\ell_k} f(\mathbf{x})$, where $f(\mathbf{x}) = c_0 + c_1 x_1 + \cdots + c_d x_d$ for some integer coefficients $c_0, \ldots, c_d$, not all zeros. Since the norm of $gu_{\mathbf{x}}\mathbf{v}$ is at least the absolute value of any of its coordinates, using the assumptions that $b_+ \geq 1$, $0 < b_1, \ldots, b_d \leq 1$ and $b_+ b_1 \cdots b_d = 1$ gives

$$\|gu_{\mathbf{x}}\mathbf{v}\| \geq \left| \prod_{k=1}^{j} b_{\ell_k} f(\mathbf{x}) \right| \geq |f(\mathbf{x})|. \tag{33}$$

If $c_1 = \cdots = c_d = 0$, then the right hand side of (33) is a nonzero integer and therefore it is at least 1. Otherwise, $c_k \neq 0$ for some $1 \leq k \leq d$. Then, take the points $\mathbf{x}_{\pm 1} = \mathbf{x}_0 \pm r_B \mathbf{e}_k$, where $\mathbf{x}_0$ is the centre of B. Then, $|f(\mathbf{x}_{+1}) - f(\mathbf{x}_{-1})| = |2c_k r_B| \geq 2r_B$. Consequently, using the triangle inequality, we get that $\sup_{\mathbf{x} \in B} \|gu_{\mathbf{x}}\mathbf{v}\| \geq \max\{|f(\mathbf{x}_{+1})|, |f(\mathbf{x}_{-1})|\} \geq r_B$. The proof is complete. □

4. Proof of Theorem 5

Let $\mathbf{w}$ be any weight, μ be a compactly supported Ahlfors regular absolutely decaying measure on $\mathbb{R}^d$ and let A, α and ρ_0 be as in (6). For every $\mathbf{z} \in \operatorname{supp}\mu$ let $B(\mathbf{z})$ be the ball arising from Theorem 23. Clearly, $\left\{\frac{1}{2}B(\mathbf{z}) : \mathbf{z} \in \operatorname{supp}\mu\right\}$ is an open cover of $\operatorname{supp}\mu$. Since $\operatorname{supp}\mu$ is compact, there is a finite subcover $\left\{\frac{1}{2}B(\mathbf{z}_u) : 1 \leq u \leq U\right\}$ of $\operatorname{supp}\mu$. Thus,

$$\operatorname{supp}\mu \subseteq \bigcup_{u=1}^{U} \tfrac{1}{2}B(\mathbf{z}_u). \tag{34}$$

Proposition 26. *There exist positive constants C and γ with the following property. For every ball B centred in $\operatorname{supp}\mu$ that is contained in one of the balls $B(\mathbf{z}_u)$ with $1 \leq u \leq U$ and such that statement (ii) of Theorem 23 does not hold inequality* (27) *holds for all $\varepsilon > 0$.*

Proof. The existence of C and γ follows from Theorem 23 since we have a finite collection of balls $B(\mathbf{z}_u)$ and so C can be taken its maximal value over $B(\mathbf{z}_u)$ and γ can be taken its minimal value over $B(\mathbf{z}_u)$. $\square$

Recall that the ultimate goal is to show that $\mathbf{Bad}(\mathbf{w}) \cap \operatorname{supp}\mu \neq \varnothing$. Note that the support of a Borel measure is closed thus $X := \operatorname{supp}\mu \subseteq \mathbb{R}^d$ is complete. Then, by Theorem 21 with $S := \mathbf{Bad}(\mathbf{w}) \cap \operatorname{supp}\mu$, it suffices to show that S is α'-Cantor winning (in the sense of the Cantor potential game played on X) for some $0 \le \alpha' < \alpha$. The specific value of α' we use will be defined in (65) below.

We will describe a winning strategy for Alice for the α'-Cantor potential game. Assume Bob chooses B_0 and β on his first move. Recall that B_0 is a closed ball in $X = \operatorname{supp}\mu$ defined by its centre $\mathbf{x}_0 \in X$ and radius r_0, that is $B_0 = X \cap B(\mathbf{x}_0, r_0)$. Before describing Alice's strategy we start with several simplifying assumptions. Without loss of generality we can assume that r_0 is less than $\frac{1}{10}$ of the radius of every ball $\frac{1}{2}B(\mathbf{z}_u)$ $(1 \le u \le U)$ appearing in (34). This can be done as a result of Alice playing arbitrarily for several moves until the condition is met. Let u_0 be such that $\mathbf{x}_0 \in \frac{1}{2}B(\mathbf{z}_{u_0})$, which exists due to (34). Then using the triangle inequality and the above condition on r_0 we conclude that

$$B(\mathbf{x}_0, 5r_0) \subseteq B(\mathbf{z}), \qquad \text{where } \mathbf{z} = \mathbf{z}_{u_0}\,. \tag{35}$$

Here $B(\mathbf{x}_0, r_0)$ is the ball in $\mathbb{R}^d$ of radius r_0 centred at $\mathbf{x}_0$. Also without loss of generality we will assume that

$$r_0 \le \min\left\{\tfrac{1}{3}\rho_0, \frac{1}{\sqrt{d}},\ A^{-1/\alpha}\right\},$$

where ρ_0, A and α are as in (6). In particular, by (6), we have that

$$\mu(B_0) \le 1\,. \tag{36}$$

Without loss of generality we will assume that

$$w_1 \ge w_2 \ge \ldots \ge w_d > 0\,.$$

Define formally $w_{d+1} := 0$. Then there exists a unique integer t such that $1 \le t \le d$ and

$$w_1 = \ldots = w_t > w_{t+1}\,.$$

Without loss of generality we may assume that β is small (to be determined according to (66) and (67)). This can be done as a result of applying Alice's strategy described below with β^M for some integer $M \ge 1$ and letting Alice play arbitrarily on every step of the game which is not 1 modulo M. Let $b > 1$ be such that

$$\beta = b^{-(1+w_1)}\,.$$

Recall that b is a parameter appearing in the definition of a_n, see (23). For any $n \ge 0$ denote Bob's nth move by

$$B_n = X \cap B(\mathbf{x}_n, \beta^n r_0)\,.$$

Thus, B_n is a ball in X of radius $\beta^n r_0$ centred at $\mathbf{x}_n \in X$. For every integer $\ell \in \mathbb{Z}$ define the following diagonal matrix

$$d_\ell := \operatorname{diag}\Bigg(\beta^{\frac{t\ell}{d+1}},\ \underbrace{\beta^{-\frac{(d+1-t)\ell}{d+1}}, \ldots, \beta^{-\frac{(d+1-t)\ell}{d+1}}}_{t \text{ times}},\ \underbrace{\beta^{\frac{t\ell}{d+1}}, \ldots, \beta^{\frac{t\ell}{d+1}}}_{d-t \text{ times}}\Bigg) \in G\,. \tag{37}$$

Also, given any $\varepsilon > 0$, $B \subseteq \mathbb{R}^d$ and non-negative integers n, k and ℓ, let

$$A_\varepsilon^{k,\ell}(B) := \{\mathbf{x} \in B \,:\, d_\ell a_k u_{\mathbf{x}} \notin K_\varepsilon\}\,. \tag{38}$$

When $B = B_n$ we will write $A_\varepsilon^{k,\ell,n}$ for $A_\varepsilon^{k,\ell}(B_n)$. The sets $A_\varepsilon^{k,\ell,n}$ will be used to define Alice's winning strategy, see (68)–(71). It will be apparent from the definition of Alice's strategy and Definition 20 that our proof crucially depends on obtaining suitably precise upper bounds on the μ-measure of certain neighborhoods of the sets $A_\varepsilon^{k,\ell,n}$. The following lemma, which provides such bounds, is therefore the key step in defining Alice's winning strategy.

Lemma 27. *Let C and γ be the same as in Proposition 26 and let*

$$C' = CA^2 \max\left\{2^\alpha (2r_0)^{-\gamma},\, 3^\alpha\right\}.$$

Then for any quintuple of nonnegative integers (h, k, ℓ, m, n), if

$$d_{\ell+m} a_k u_{\tilde{\mathbf{x}}_n} \mathbb{Z}^{d+1} \in K_{\sqrt{d+1}\beta^{\frac{m}{d+1}} r_0^{-1}}\,, \tag{39}$$

and

$$d_\ell a_{k-h} u_{\tilde{\mathbf{x}}_n} \mathbb{Z}^{d+1} \in K_{\sqrt{2}d\beta^{\frac{\tau}{d}}}\,, \tag{40}$$

for some point $\tilde{\mathbf{x}}_n \in B_n$, where

$$\tau = \min\left\{k - \ell - m - n - \frac{h}{1+w_1}, \frac{hw_d}{1+w_1}\right\} \geq 0\,, \tag{41}$$

then for any $\varepsilon > 0$

$$\mu\left(A_\varepsilon^{k,\ell,n}\right) \leq C'\varepsilon^\gamma \mu\left(B_n\right). \tag{42}$$

If $n = 0$ and $k \geq \frac{1+w_1}{w_1}\ell$ then (42) *holds for any $\varepsilon > 0$ without assuming* (39) *and* (40)*. Moreover, if $0 < r \leq \beta^n r_0$ and*

$$\varepsilon' := \left(1 + \max\left\{\beta^{\ell-k}, b^{(1+w_{t+1})k}\right\} r\right)\varepsilon\,, \tag{43}$$

where $w_{t+1} = 0$ if $t = d$, then

$$\mu\left(B\left(A_\varepsilon^{k,\ell,n}, r\right)\right) \leq C'\varepsilon'^\gamma \mu\left(B_n\right), \tag{44}$$

which is valid for all n assuming (39) *and* (40) *and for $n = 0$ assuming $k \geq \frac{1+w_1}{w_1}\ell$.*

Proof. Let $n \geq 0$ and write $\mathbf{x} = \tilde{\mathbf{x}}_n + \mathbf{x}'$ with $\|\mathbf{x}'\| \leq \beta^n r_0$. Since $w_1 = \cdots = w_t$ and $\beta = b^{-(1+w_1)}$, conjugating $u_{\mathbf{x}'}$ by $d_\ell a_k$ in the equation $u_{\mathbf{x}} = u_{\mathbf{x}'} u_{\tilde{\mathbf{x}}_n}$ gives

$$d_\ell a_k u_{\mathbf{x}} = u_{\left(\beta^{\ell-k}x'_1,\ldots,\beta^{\ell-k}x'_t, b^{(1+w_{t+1})k}x'_{t+1},\ldots,b^{(1+w_d)k}x'_d\right)} d_\ell a_k u_{\tilde{\mathbf{x}}_n}\,. \tag{45}$$

Note that (35) and the fact $\tilde{\mathbf{x}}_n \in B_n = X \cap B\left(\mathbf{x}_n, \beta^n r_n\right)$ imply that

$$B\left(\mathbf{x}_n, 2\beta^n r_0\right) \subseteq B\left(\tilde{\mathbf{x}}_n, 3\beta^n r_0\right) \subseteq B\left(\mathbf{x}_n, 4\beta^n r_0\right) \subseteq B\left(\mathbf{x}_0, 5r_0\right) \subseteq B(\mathbf{z})\,, \tag{46}$$

where $B(\mathbf{z})$ is as in (35). Let $1 \leq j \leq d$ and $0 \neq \mathbf{v} = \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_j \in \bigwedge^j\left(\mathbb{Z}^{d+1}\right)$. Let $\mathbf{v}' = d_\ell a_k u_{\tilde{\mathbf{x}}_n}\mathbf{v}$ and $\mathbf{v}'_i = d_\ell a_k u_{\tilde{\mathbf{x}}_n}\mathbf{v}_i$ for every $1 \leq i \leq j$. Assume towards a contradiction that

$$\sup_{\|\mathbf{x}'\| \leq \beta^n r_0} \left\| u_{\left(\beta^{\ell-k}x'_1,\ldots,\beta^{\ell-k}x'_t, b^{(1+w_{t+1})k}x'_{t+1},\ldots,b^{(1+w_d)k}x'_d\right)} \mathbf{v}' \right\| < 1\,. \tag{47}$$

Inequality (47) applied at $\mathbf{x}' = 0$ implies that $\|\mathbf{v}'\| < 1$. Since $\| \ \|$ is the Euclidean norm, $\|\mathbf{v}'\| = \left\|\mathbf{v}'_1 \wedge \ldots \wedge \mathbf{v}'_j\right\|$ is the covolume of the lattice $\Lambda^* := \mathbb{Z}\mathbf{v}'_1 + \ldots + \mathbb{Z}\mathbf{v}'_j$ in the Euclidean subspace $W^* := \mathbb{R}\mathbf{v}'_1 + \ldots + \mathbb{R}\mathbf{v}'_j$ of $\mathbb{R}^{d+1}$. That is $\mathrm{covol}\,(\Lambda^*) < 1$. Next, as is well known the Euclidean ball $B^*_{\sqrt{j}}$ in W^* centred at 0 of radius $\sqrt{j}$ contains a cube $\mathcal{C}^*$ of sidelength 2, *i.e.* $\mathcal{C}^* := \left\{\sum_{i=1}^{j} \theta_i \mathbf{e}^*_i \ : \ |\theta_i| \le 1\right\}$, where $\mathbf{e}^*_1, \ldots, \mathbf{e}^*_j$ is any orthonormal basis in W^*. Therefore $B^*_{\sqrt{j}}$ has j-dimensional volume $> 2^j$. Then, by Minkowski's theorem for convex bodies, $B^*_{\sqrt{j}}$ contains a non-zero point of Λ^*. In other words, the shortest non-zero vector of Λ^*, say $\tilde{\mathbf{v}}'_1$, has Euclidean norm $\le \sqrt{j}$. Complete $\tilde{\mathbf{v}}'_1$ to a basis $\tilde{\mathbf{v}}'_1, \ldots, \tilde{\mathbf{v}}'_j$ of Λ^*, *e.g.* to a reduced Minkowski basis. Then $\mathbf{v}'_1 \wedge \ldots \wedge \mathbf{v}'_j = \pm \tilde{\mathbf{v}}'_1 \wedge \ldots \wedge \tilde{\mathbf{v}}'_j$, since the two bases span the same linear subspace of $\mathbb{R}^{d+1}$ and have the same Euclidean norm (equal to the covolume of Λ^*). To save on notation, without loss of generality we will assume that $\mathbf{v}'_i = \tilde{\mathbf{v}}'_i$ $(1 \le i \le j)$. Then, we have that

$$\left\|\mathbf{v}'_1\right\| < \sqrt{j} \le \sqrt{d} \le r_0^{-1} \,. \tag{48}$$

If $\left|v'_{1,i}\right| < \beta^m r_0^{-1}$ for all $1 \le i \le t$, where $\mathbf{v}'_1 = \left(v'_{1,+}, v'_{1,1}, \ldots, v'_{1,d}\right)$, then using (48) we get that

$$\|d_m \mathbf{v}'_1\| < \sqrt{d+1}\beta^{\frac{tm}{d+1}} r_0^{-1} \le \sqrt{d+1}\beta^{\frac{m}{d+1}} r_0^{-1} \,, \tag{49}$$

and so, since $\mathbf{v}'_1 = d_\ell a_k u_{\tilde{\mathbf{x}}_n} \mathbf{v}_1$ with $\mathbf{v}_1 \in \mathbb{Z}^{d+1} \setminus \{0\}$ and $d_m d_\ell = d_{\ell+m}$, we get that

$$d_{\ell+m} a_k u_{\tilde{\mathbf{x}}_n} \mathbb{Z}^{d+1} \notin K_{\sqrt{d+1}\beta^{\frac{m}{d+1}} r_0^{-1}} \,,$$

which contradicts assumption (39).

Otherwise, there exists $1 \le i_0 \le t$ for which

$$\left|v'_{1,i_0}\right| \ge \beta^m r_0^{-1} \,. \tag{50}$$

It is enough to use (47) for $\mathbf{x}'$ of the form $\mathbf{x}' = \left(0, \ldots, x'_{i_0}, \ldots, 0\right)$ where the only nonzero entry is in the i_0th coordinate. In this case, let

$$\tilde{\mathbf{v}} = \sum_{i=1}^{j} (-1)^{i+1} v'_{i,i_0} \bigwedge_{i' \ne i} \mathbf{v}'_{i'} \,. \tag{51}$$

Then, by (29), we have that

$$u_{\left(\beta^{\ell-k}x'_1, \ldots, \beta^{\ell-k}x'_t, b^{(1+w_{t+1})k}x'_{t+1}, \ldots, b^{(1+w_d)k}x'_d\right)} \mathbf{v}' = \mathbf{v}' + \beta^{\ell-k} x'_{i_0} \mathbf{e}_+ \wedge \tilde{\mathbf{v}} \tag{52}$$

for all $\mathbf{x}' = \left(0, \ldots, x'_{i_0}, \ldots, 0\right)$ with $|x'_{i_0}| \le \beta^n r_0$. By (47), $\left\|\mathbf{v}' \pm \beta^{\ell-k} x'_{i_0} \mathbf{e}_+ \wedge \tilde{\mathbf{v}}\right\| < 1$ when $x'_{i_0} = \beta^n r_0$. Then, by the triangle inequality,

$$\left\|2\beta^{\ell-k} x'_{i_0} \mathbf{e}_+ \wedge \tilde{\mathbf{v}}\right\| \le \left\|\mathbf{v}' + \beta^{\ell-k} x'_{i_0} \mathbf{e}_+ \wedge \tilde{\mathbf{v}}\right\| + \left\|\mathbf{v}' - \beta^{\ell-k} x'_{i_0} \mathbf{e}_+ \wedge \tilde{\mathbf{v}}\right\| < 2 \,.$$

Letting $x'_{i_0} = \beta^n r_0$ and dividing the above inequality through by $2\beta^{\ell-k} x'_{i_0}$ give

$$\|\mathbf{e}_+ \wedge \tilde{\mathbf{v}}\| < \beta^{k-\ell-n} r_0^{-1} \,. \tag{53}$$

Observe that $\mathbf{e}_+ \wedge \tilde{\mathbf{v}}_+ = 0$, and thus $\mathbf{e}_+ \wedge \tilde{\mathbf{v}} = \mathbf{e}_+ \wedge \tilde{\mathbf{v}}_-$. Furthermore, $\|\mathbf{e}_+ \wedge \tilde{\mathbf{v}}_-\| = \|\tilde{\mathbf{v}}_-\|$ and therefore (53) can be rewritten as

$$\|\tilde{\mathbf{v}}_-\| < \beta^{k-\ell-n} r_0^{-1} \,. \tag{54}$$

On the other hand, by (51), taking the wedge product of $\mathbf{v}'_1$ and $\tilde{\mathbf{v}}$ gives

$$\mathbf{v}'_1 \wedge \tilde{\mathbf{v}} = v'_{1,i_0}\mathbf{v}' , \tag{55}$$

so equations (48), (50), (54) and (55) yield

$$\left\|\mathbf{v}'_-\right\| \le \left|v'_{1,i_0}\right|^{-1} \left\|\mathbf{v}'_{1,-}\right\| \left\|\tilde{\mathbf{v}}_-\right\| < \sqrt{d}\beta^{k-\ell-m-n} .$$

Applying the right hand side of (30) gives

$$\left\|a_{-h}\mathbf{v}'_-\right\| < \sqrt{d}\beta^{k-\ell-m-n-\frac{h}{1+w_1}} . \tag{56}$$

Next, by (47), $\|\mathbf{v}'_+\| \le \|\mathbf{v}'\| < 1$, and so applying the left hand side of (30) gives

$$\left\|a_{-h}\mathbf{v}'_+\right\| < b^{-w_d h} . \tag{57}$$

Combining (56) and (57) together with the fact that $a_{-h}\mathbf{v}' = d_\ell a_{k-h} u_{\tilde{\mathbf{x}}_n}\mathbf{v}$ we obtain

$$\|d_\ell a_{k-h} u_{\tilde{\mathbf{x}}_n}\mathbf{v}\| = \left\|a_{-h}\mathbf{v}'\right\| < \sqrt{2d}\max\left\{\beta^{k-\ell-m-n-\frac{h}{1+w_1}}, b^{-hw_d}\right\} = \sqrt{2d}\beta^{\tau} , \tag{58}$$

where τ is given by (41). Using Minkowski's theorem for convex bodies in the same way as we did in the argument leading to (48), we deduce from (58) that the lattice $d_\ell a_{k-h} u\mathbb{Z}^{d+1}$ has a nonzero vector which Euclidean norm is smaller than

$$\sqrt{j}\left(\sqrt{2d}\beta^{\tau}\right)^{\frac{1}{j}} \le \sqrt{2}d\beta^{\frac{\tau}{j}} \le \sqrt{2}d\beta^{\frac{\tau}{d}}$$

since $\tau \ge 0$ and $0 < \beta < 1$, contrary to (40). Thus, (47) cannot hold and therefore, by (45), we have that Condition (ii) within Theorem 23 cannot hold with $\rho = 1$ and $B = B\left(\tilde{\mathbf{x}}_n, 3\beta^n r_0\right)$. Hence, by Theorem 23 with this choice of ρ and B, which is applicable in view of Proposition 26 and (46), we obtain that

$$\mu\left(\left\{\mathbf{x} \in B\left(\tilde{\mathbf{x}}_n, 3\beta^n r_0\right) \,:\, d_\ell a_k u_{\mathbf{x}} \notin K_\varepsilon\right\}\right) \le C\varepsilon^{\gamma}\mu\left(B(\tilde{\mathbf{x}}_n, 3\beta^n r_0)\right) . \tag{59}$$

Since $\mathbf{x}_n$ and $\tilde{\mathbf{x}}_n$ are both in the support of μ and $3r_0 \le \rho_0$, by (6), we have that

$$\begin{aligned} \mu\left(B(\tilde{\mathbf{x}}_n, 3\beta^n r_0)\right) &\le A\left(3\beta^n r_0\right)^{\alpha} = 3^{\alpha}A^2A^{-1}\left(\beta^n r_0\right)^{\alpha} \\ &\le 3^{\alpha}A^2\mu\left(B(\mathbf{x}_n, \beta^n r_0)\right) = 3^{\alpha}A^2\mu(B_n) . \end{aligned} \tag{60}$$

Further observe that, by (46), the left hand side of (59) is an upper bound for $\mu\left(A_\varepsilon^{k,\ell}(2B_n)\right)$. Hence, combining (59) and (60) gives

$$\mu\left(A_\varepsilon^{k,\ell}(2B_n)\right) \le C'\varepsilon^{\gamma}\mu\left(B_n\right) . \tag{61}$$

And since trivially we have that $A_\varepsilon^{k,\ell,n} \subseteq A_\varepsilon^{k,\ell}(2B_n)$, (61) implies (42), as required.

Regarding the case $n = 0$ first observe that, by (6), we have that

$$\mu(2B_n) \le A^2 2^{\alpha}\mu(B_n). \tag{62}$$

Now if $k \ge \frac{1+w_1}{w_1}\ell$ then $d_\ell a_k = \operatorname{diag}(b_+, b_1, \dots, b_d)$ with $b_+ \ge 1$ and $b_i \le 1$ for every $1 \le i \le d$, so Theorem 23 with $\rho = 2r_0$ together with Lemma 25 and (62) immediately imply (61) and consequently (42).

To see (44), assume that $0 < r \le \beta^n r_0$ and ε' is given by (43). If $\mathbf{x} \in A_\varepsilon^{k,\ell,n}$ then there exists $\mathbf{v} \in \mathbb{Z}^{d+1} \setminus \{0\}$ such that $\|d_\ell a_k u_{\mathbf{x}}\mathbf{v}\| < \varepsilon$. Suppose that $\|\mathbf{y} - \mathbf{x}\| < r$ and define $\mathbf{x}' = \mathbf{y} - \mathbf{x}$. Using the conjugation of $u_{\mathbf{x}'}$ by $d_\ell a_k$ as in (45) we get that

$$\begin{aligned} d_\ell a_k u_{\mathbf{y}}\mathbf{v} &= u_{\left(\beta^{\ell-k}x'_1, \dots, \beta^{\ell-k}x'_t, b^{(1+w_{t+1})k}x'_{t+1}, \dots, b^{(1+w_d)k}x'_d\right)} d_\ell a_k u_{\mathbf{x}}\mathbf{v} \\ &= d_\ell a_k u_{\mathbf{x}}\mathbf{v} + \langle \tilde{\mathbf{v}}, d_\ell a_k u_{\mathbf{x}}\mathbf{v}\rangle \mathbf{e}_+ , \end{aligned}$$

where $\tilde{\mathbf{v}} = \left(0, \beta^{\ell-k}x'_1, \dots, \beta^{\ell-k}x'_t, b^{(1+w_{t+1})k}x'_{t+1}, \dots, b^{(1+w_d)k}x'_d\right)$. Then on using the triangle and Cauchy-Schwarz inequalities we get that

$$\|d_\ell a_k u_{\mathbf{y}}\mathbf{v}\| \le \|d_\ell a_k u_{\mathbf{x}}\mathbf{v}\| + |\langle \tilde{\mathbf{v}}, d_\ell a_k u_{\mathbf{x}}\mathbf{v}\rangle| \le (1 + \|\tilde{\mathbf{v}}\|)\, \|d_\ell a_k u_{\mathbf{x}}\mathbf{v}\|\,.$$

Observe that

$$\|\tilde{\mathbf{v}}\| \le \max\left\{\beta^{\ell-k}, b^{(1+w_{t+1})k}\right\} \cdot \|\mathbf{x}'\| \le \max\left\{\beta^{\ell-k}, b^{(1+w_{t+1})k}\right\} r$$

and therefore

$$\|d_\ell a_k u_{\mathbf{y}}\mathbf{v}\| < \left(1 + \max\left\{\beta^{\ell-k}, b^{(1+w_{t+1})k}\right\} r\right)\varepsilon = \varepsilon'\,.$$

Further, since $\mathbf{x} \in A_\varepsilon^{k,\ell,n} \subseteq B_n$ and $\|\mathbf{y}-\mathbf{x}\| \le r \le \beta^n r_0$, we have that $\mathbf{y} \in 2B_n$. Then, $B\left(A_\varepsilon^{k,\ell,n}, r\right) \subseteq A_{\varepsilon'}^{k,\ell}(2B_n)$ and applying (61) with ε replaced by ε' gives (44). $\square$

To complete the proof of Theorem 5, let

$$s := \max\left\{5, \left\lceil \frac{1+w_1}{w_1 - w_{t+1}}\right\rceil, \left\lceil \frac{2(1+w_1)+1}{w_1}\right\rceil\right\}, \tag{63}$$

$$\eta := \min\left\{\frac{1}{4(d+1)}, \frac{w_d s}{2d(1+w_1)}, \frac{w_1 s - 2(1+w_1)}{2d(1+w_1)}, \frac{\alpha}{\gamma}\right\}, \tag{64}$$

$$\alpha' := \alpha - \frac{\gamma\eta}{4(s-1)}\,, \tag{65}$$

where it is agreed, if needed (i.e., in case $t = d$), that $w_{d+1} = 0$. Note that (63) and (64) imply that $\eta > 0$, $0 \le \alpha' < \alpha$ and $s \ge 5$. Assume that β is small enough so that it satisfies

$$\beta^{-\frac{w_1 s - 2(1+w_1)}{2d(1+w_1)}} \ge \sqrt{2}d, \quad \beta^{-\frac{w_d s}{2d(1+w_1)}} \ge \sqrt{2}d \quad \text{and} \quad \beta^{-\frac{1}{2(d+1)}} \ge \sqrt{d+1}\,r_0^{-1}\,, \tag{66}$$

$$\beta^{\frac{\gamma\eta}{s-1}} \le \min\left\{2^{-1}, \left(A^2 C' 2^{\gamma+1}(3r_0^{-1})^\alpha\right)^{-1}\right\}. \tag{67}$$

Recall that A and α are the parameters of μ appearing in (6), C' is given in Lemma 27, C and γ are as in Proposition 26 and r_0 is the radius of B_0 which choice is described after Proposition 26.

Now let us describe the winning strategy for Alice. We will keep notation used in Definition 20. Alice's strategy is understood as a sequence of maps $\mathcal{F}_n$ indexed by $n \in \mathbb{N}$ which assign a sequence $(\mathcal{A}_{n+1,i})_{i\ge 0}$ of collections of at most $\beta^{-\alpha(i+1)}$ balls of radius $\beta^{n+1+i}r_0$ to Bob's moves $\beta, B_0, \dots, B_n$, that is $(\mathcal{A}_{n+1,i})_{i\ge 0} = \mathcal{F}_n(\beta, B_0, \dots, B_n)$. The sets $\mathcal{A}_{n+1,i}$ will be defined in a 3-step process, which can be described as follows:

(i) Choose 'raw' subsets $A_{n+1,i} \subseteq B_n$ that Alice wishes to 'block out' as she plays the game; then

(ii) Refine $A_{n+1,i}$ to obtain subsets $\tilde{A}_{n+1,i} \subseteq A_{n+1,i}$ by removing overlaps between the sets $A_{n+1,i}$ that appear at different stages of the game; and

(iii) Finally 'convert' $\tilde{A}_{n+1,i}$ into the required collections of balls $\mathcal{A}_{n+1,i}$ by using the efficient covering of Lemma 22.

Naturally we start with Step (i) to define the raw sets $A_{n+1,i}$. To begin with, for every $i \ge 0$, let

$$A_{1,i} := \bigcup_{\substack{n'\ge 0,\ \ell \ge \frac{n'+1}{s} \\ n'+(s-1)\ell = i}} \left\{\mathbf{x} \in B_0 \,:\, d_\ell a_{n'+1+s\ell} u_{\mathbf{x}} \mathbb{Z}^{d+1} \notin K_{\beta^{\eta\ell}}\right\}, \tag{68}$$

where the union is taken over all possible values of integers ℓ and $n' \geq 0$. Note that if no such values of ℓ and n' exist, then $A_{1,i} = \varnothing$. Next, for $n \geq 1$ define

$$A_{n+1,i} := \left\{ \mathbf{x} \in B_n \,:\, d_\ell a_{n+1+s\ell} u_{\mathbf{x}} \mathbb{Z}^{d+1} \notin K_{\beta^{\eta\ell}} \right\} \tag{69}$$

if $i = (s-1)\ell$ for some integer $0 < \ell < \frac{n+1}{s}$, and we define $A_{n+1,i} := \varnothing$ otherwise, that is for $i \geq 0$ such that $i \notin \left\{ (s-1)\ell \,:\, \ell \in \mathbb{Z},\ 0 < \ell < \frac{n+1}{s} \right\}$.

Now moving on to Step (ii), define

$$\tilde{A}_{n+1,i} = A_{n+1,i} \setminus \bigcup_{0 \leq n' < n,\ i' \geq 0} A_{n'+1,i'}\,, \tag{70}$$

where in the case $n = 0$ the union (70) is empty and thus $\tilde{A}_{1,i} = A_{1,i}$.

Finally, moving on to Step (iii), with reference to Lemma 22,

(71) let $\mathcal{A}_{n+1,i}$ be an efficient cover of $\tilde{A}_{n+1,i}$ by balls of radius $\beta^{n+1+i} r_0$.

Since $\mathcal{A}_{n+1,i}$ is a cover of $\tilde{A}_{n+1,i}$, by Definition 20, when Bob makes his next move B_{n+1} it must be disjoint from $\tilde{A}_{n'+1,i'}$ for all $i', n' \geq 0$ such that $n'+i' = n$. Therefore, if $\mathbf{x}$ is an outcome of the game, that is $\mathbf{x} \in \cap_{n \geq 0} B_n$, then we necessarily have that

$$\mathbf{x} \notin \bigcup_{n,i \geq 0} \tilde{A}_{n+1,i}\,. \tag{72}$$

Using a standard inclusion-exclusion argument and (70), one readily verifies that

$$\bigcup_{n,i \geq 0} \tilde{A}_{n+1,i} = \bigcup_{n,i \geq 0} A_{n+1,i}$$

and therefore, by (72),

$$\mathbf{x} \notin \bigcup_{n,i \geq 0} A_{n+1,i}\,. \tag{73}$$

By (70), we have that the outcome $\mathbf{x}$ of the game satisfies

$$d_\ell a_{n+1+s\ell} u_{\mathbf{x}} \mathbb{Z}^{d+1} \in K_{\beta^{\eta\ell}}$$

for every $n \geq 0$ and $\ell \geq 1$. Using this for $\ell = 1$ we get that

$$a_n u_{\mathbf{x}} \mathbb{Z}^{d+1} \in d_{-1} a_{-s-1} K_{\beta^\eta} \subseteq K_{\beta^{\eta + \frac{s+1}{1+w_1} + \frac{d+1-t}{d+1}}}$$

for all $n \geq 0$. By Dani's correspondence (25), this means that $\mathbf{x} \in \mathbf{Bad}(\mathbf{w})$. In order to complete the proof that $\mathbf{Bad}(\mathbf{w})$ is Cantor winning in $\operatorname{supp}\mu$ it is left to show that Alice's strategy is legal, that is for all $n, i \geq 0$ we have that

$$\#\mathcal{A}_{n+1,i} \leq \beta^{-\alpha'(i+1)}\,. \tag{74}$$

The plan is to use Lemma 27 in order to get a measure estimate for small neighborhoods of the sets $\tilde{A}_{n+1,i}$ and then to apply Lemma 22 to derive (74). The definition of $\tilde{A}_{n+1,i}$ is designed to ensure that assumptions (39) and (40) hold when they are needed and so Lemma 27 is applicable. Note that in the case $n = 0$, which serves as the basis of the inductive argument, these assumptions are not needed. See Figure 4 for a more geometrical description of the definition of Alice's strategy.

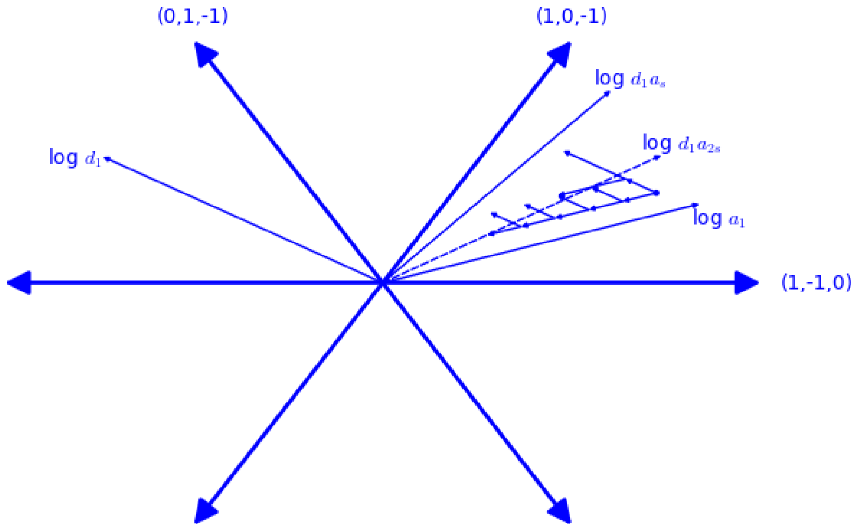

FIGURE 1. The plane represents all diagonal matrices in $\mathrm{SL}_3(\mathbb{R})$. Logarithm of a diagonal matrix is the vector whose coordinates are logarithms of the entries along the diagonal. Each blue point is the logarithm of a diagonal matrix $d_\ell a_{n+1+s\ell}$ for some parameters $n, \ell \geq 0$. An arrow from $d_\ell a_{n+1+s\ell}$ to $d_{\ell'} a_{n'+1+s\ell'}$ is drawn if estimating the measure of $A_\varepsilon^{n+1+s\ell,\ell,n}$ via applying Lemma 27 requires an assumption regarding $d_{\ell'} a_{n'+1+s\ell'} u_{\mathbf{x}_{n'}}$. Diagonal matrices in the region between the arrows which are labeled by $\log d_1 a_s$ and $\log d_1 a_{2s}$ are exactly those which satisfy $\ell \geq \frac{n+1}{s}$, and are dealt with at Alice's first turn. The parameters used to generate this figure are $\mathbf{w} = (2/3, 1/3)$, $s = 2$, $n = 5$, $\ell = 1$.

First we deal with $A_{1,i}$. Observe that the sets in the right hand side of (68) are precisely $A_\varepsilon^{k,\ell,0}$ given by (38) with $k = n' + 1 + s\ell$ and $\varepsilon = \beta^{\eta\ell}$. Therefore,

$$B\left(A_{1,i}, \tfrac{1}{3}\beta^{i+1} r_0\right) = \bigcup_{n' \geq 0,\ \ell \geq \frac{n'+1}{s},\ n'+(s-1)\ell = i} B\left(A_{\beta^{\eta\ell}}^{n'+1+s\ell,\ell,0}, \tfrac{1}{3}\beta^{i+1} r_0\right). \tag{75}$$

For any $i \geq 0$, for any $n' \geq 0$ and $\ell \geq \frac{n'+1}{s}$ such that $i = n' + (s-1)\ell$, apply Lemma 27 with the quintuple (h, k, ℓ, m, n) set to be $(0, n' + 1 + s\ell, \ell, 0, 0)$, $\varepsilon = \beta^{\eta\ell}$ and $r = \frac{1}{3}\beta^{i+1} r_0$. In this case using (43), (63), the equation $b = \beta^{-1/(1+w_1)}$ and the fact that $0 < \beta, r_0 \leq 1$ gives

$$\varepsilon' = \left(1 + \max\left\{\beta^{-(n'+1+(s-1)\ell)}, b^{(1+w_{t+1})(n'+1+s\ell)}\right\} \tfrac{1}{3}\beta^{i+1} r_0\right) \beta^{\eta\ell} \leq 2\beta^{\eta\ell}. \tag{76}$$

By (63) we have that $k \geq \frac{1+w_1}{w_1}\ell$ and thus Lemma 27 is applicable to each set on the right of (75). Therefore, using (44), (75) and (36) gives

$$\begin{aligned} \mu\left(B\left(A_{1,i}, \tfrac{1}{3}\beta^{i+1}r_0\right)\right) &\leq \sum_{n'\geq 0,\ \ell\geq\frac{n'+1}{s},\ n'+(s-1)\ell=i} C'2^\gamma\beta^{\gamma\eta\ell} \\ &\leq \sum_{\ell\geq\frac{i+1}{2s-1}} C'2^\gamma\beta^{\gamma\eta\ell} \leq \frac{C'2^\gamma}{1-\beta^{\gamma\eta}}\left(\beta^{\frac{\gamma\eta}{2s-1}}\right)^{i+1} \\ &\leq C'2^{\gamma+1}\left(\beta^{\frac{\gamma\eta}{2s-1}}\right)^{i+1} = C'2^{\gamma+1}\beta^{\frac{\gamma\eta}{2s-1}(i+1)}\beta^{\alpha n}, \end{aligned} \tag{77}$$

since $n=0$, where the last inequality holds due to (67).

Now let $n\geq 1$, $i\geq 0$ and let us assume without loss of generality that $\tilde{A}_{n+1,i}\neq\varnothing$ as otherwise $\mathcal{A}_{n+1,i}=\varnothing$ and there is nothing to prove. In particular, we have that $i=(s-1)\ell\geq 4$ for some $1\leq\ell<\frac{n+1}{s}$. Observe that $A_{n+1,i}=A^{n+1+s\ell,\ell,n}_{\beta^{\eta\ell}}$, where $A^{n+1+s\ell,\ell,n}_{\beta^{\eta\ell}}$ is given by (38). Therefore,

$$B\left(A_{n+1,i}, \tfrac{1}{3}\beta^{n+1+i}r_0\right) = B\left(A^{n+1+s\ell,\ell,n}_{\beta^{\eta\ell}}, \tfrac{1}{3}\beta^{n+1+i}r_0\right). \tag{78}$$

Note that $n+1+i=n+1+(s-1)\ell$. With the view to applying Lemma 27 let the quintuple (h,k,ℓ,m,n) be $(s\ell, n+1+s\ell, \ell, \ell, n)$, $\varepsilon=\beta^{\eta\ell}$ and $r=\frac{1}{3}\beta^{n+1+i}r_0=\frac{1}{3}\beta^{n+1+(s-1)\ell}r_0$. Using (43), (63), the equation $b=\beta^{-1/(1+w_1)}$ and the fact that $0<\beta, r_0\leq 1$ in the same way as in (76) we get that

$$\varepsilon' = \left(1+\max\left\{\beta^{-(n+1+(s-1)\ell)}, b^{(1+w_{t+1})(n+1+s\ell)}\right\}\tfrac{1}{3}\beta^{n+1+(s-1)\ell}r_0\right)\beta^{\eta\ell}\leq 2\beta^{\eta\ell}.$$

Further, since $\tilde{A}_{n+1,i}\neq\varnothing$, by (70), there exists a point $\tilde{\mathbf{x}}_n\in B_n$ such that

$$\tilde{\mathbf{x}}_n \notin \bigcup_{0\leq n'<n,\ i'\geq 0} A_{n'+1,i'}. \tag{79}$$

Recall that, by definition, B_n is a subset of $\operatorname{supp}\mu$ and therefore $\tilde{\mathbf{x}}_n\in\operatorname{supp}\mu$. By (79), we have that

$$\begin{aligned} &\tilde{\mathbf{x}}_n\notin A_{1,n+(s-2)\ell}\cup A_{1,n-\ell} && \text{if} \quad \frac{n+1}{2s}\leq\ell<\frac{n+1}{s}, \\ &\tilde{\mathbf{x}}_n\notin A_{1,n+(s-2)\ell}\cup A_{n+1-s\ell,(s-1)\ell} && \text{if} \quad \frac{n+1}{3s}\leq\ell<\frac{n+1}{2s}, \\ &\tilde{\mathbf{x}}_n\notin A_{n+1-s\ell,2(s-1)\ell}\cup A_{n+1-s\ell,(s-1)\ell} && \text{if} \quad \ell<\frac{n+1}{3s}. \end{aligned}$$

A routine inspection of each of the sets above gives that

$$d_{2\ell}a_{n+1+s\ell}u_{\tilde{\mathbf{x}}_n}\mathbb{Z}^{d+1}\in K_{\beta^{2\eta\ell}}\subseteq K_{\sqrt{d+1}\beta^{\frac{\ell}{d+1}}r_0^{-1}}, \tag{80}$$

and

$$d_\ell a_{n+1}u_{\tilde{\mathbf{x}}_n}\mathbb{Z}^{d+1}\in K_{\beta^{\eta\ell}}\subseteq K_{\sqrt{2}d\beta^{\frac{\tau}{d}}}, \tag{81}$$

where

$$\tau=\min\left\{k-\ell-m-n-\tfrac{h}{1+w_1}, \tfrac{hw_d}{1+w_1}\right\} = \min\left\{1+(s-2)\ell-\tfrac{s\ell}{1+w_1}, \tfrac{s\ell w_d}{1+w_1}\right\}>0.$$

and the containments on the right hand side follow from (66) together with (64). Thus, conditions (39) and (40) are satisfied. By (44) and (6) we have that

$$\mu\left(B\left(A^{n+1+s\ell,\ell,n}_{\beta^{\eta\ell}}, \tfrac{1}{3}\beta^{n+1+i}r_0\right)\right) \leq C'2^{\gamma}\beta^{\gamma\eta\ell}\mu(B_n) \leq C'2^{\gamma}\beta^{\gamma\eta\ell}A(\beta^n r_0)^{\alpha}.$$

Then, by (78) and the fact that $i=(s-1)\ell$, we get that

$$\begin{aligned}\mu\left(B\left(A_{n+1,i}, \tfrac{1}{3}\beta^{n+1+i}r_0\right)\right) &\leq C'2^{\gamma}r_0^{\alpha}A\beta^{\frac{\gamma\eta}{s-1}i}\beta^{\alpha n} \\ &= C'2^{\gamma}r_0^{\alpha}A\beta^{\frac{\gamma\eta}{2(s-1)}(i+1)}\beta^{\alpha n}\,.\end{aligned} \tag{82}$$

Combining (77) and (82) together with Lemma 22 applied with $r=\frac{1}{3}\beta^{n+i+1}r_0$ gives that for every $n\geq 0, i\geq 4$

$$\begin{aligned}\#\mathcal{A}_{n+1,i} &\leq \frac{A\mu\left(B\left(A_{n+1,i}, \frac{1}{3}\beta^{n+i+1}r_0\right)\right)}{\beta^{\alpha(n+i+1)}(\frac{1}{3}r_0)^{\alpha}} \\ &\leq \frac{A\max\left\{C'2^{\gamma+1}\beta^{\frac{\gamma\eta}{2s-1}(i+1)}\beta^{\alpha n},\ C'2^{\gamma}r_0^{\alpha}A\beta^{\frac{\gamma\eta}{2(s-1)}(i+1)}\beta^{\alpha n}\right\}}{\beta^{\alpha(n+i+1)}(\frac{1}{3}r_0)^{\alpha}} \\ &\leq A^2C'2^{\gamma+1}\left(3r_0^{-1}\right)^{\alpha}\beta^{-(i+1)\left(\alpha-\frac{\gamma\eta}{2(s-1)}\right)} \\ &= A^2C'2^{\gamma+1}\left(3r_0^{-1}\right)^{\alpha}\beta^{(i+1)\left(\frac{\gamma\eta}{4(s-1)}\right)}\beta^{-(i+1)\left(\alpha-\frac{\gamma\eta}{4(s-1)}\right)} \\ &\leq \beta^{-\alpha'(i+1)}\,,\end{aligned}$$

where the last inequality follows from (65) and (67) and on using the fact that $i\geq 4$. This shows that the collection $\{\mathcal{A}_{n+1,i} : n,i\in\mathbb{N}\}$ is a legal move for Alice. By the argument in the beginning of this section, this completes the proof.

Remark 28. The diagonal matrices d_ℓ arise naturally in the proof of Lemma 27, even while only considering the sets $A^{k,\ell,n}_{\varepsilon}$ with $\ell=0$. The general case is useful as the conditions (39) and (40) turn out to also be of the form $\mathbf{x}_n\notin A^{k',\ell',n'}_{\varepsilon'}$ for some parameters ε',k',ℓ',n'. Choosing d_ℓ as in (37) is natural, as it equally expands the 2nd to $(t+1)$st coordinates which are all contracted by a_k at the same rate. However, it is likely that d_1, say, can be replaced by any other unimodular diagonal matrix which expands the 2nd to $(t+1)$st coordinates and contracts the other directions. This property is necessary, in order to ensure that an inequality similar to (49) holds. Changing the definition of d_ℓ in this fashion will require a different choice of the parameter s, which is defined in (63). In this context, it should be noted that Lemma 27 is only applied with diagonal matrices of the form $d_\ell a_{n+1+s\ell} = (d_\ell a_{s\ell})\,a_{n+1}$, so it makes sense to also consider the one parameter group $d_\ell a_{s\ell}$. It is interesting to note that the above described strategy of Alice is in fact winning even if s is replaced by any integer larger than the one described in (63), and that $d_\ell a_{s\ell}$ becomes closer in direction to the direction of a_{n+1} as s becomes larger.

Appendix A. Proof of Proposition 8

As we mentioned earlier the proof essentially follows the argument of [FSU18, Proposition 4.5]. First of all, note that if $1/3\leq\beta<1$ then, with reference to Definition 7, Alice can win by default on her first move by taking A_1 to be the closed β-neighborhood of any hyperplane passing through the centre of B_0. Thus, without loss of generality we can assume throughout this proof that Bob always chooses $\beta < 1/3$ when he plays the restricted hyperplane absolute game. With this additional assumption the necessity part of Proposition 8 becomes obvious. Indeed, to win the restricted hyperplane absolute game Alice simply has to follow her strategy for the hyperplane absolute game and set ε to be exactly β on each of her moves. By increasing ε to its largest possible value Alice will only limit the possible choices for Bob's next moves. Additionally, the modified game has a greater restriction on Bob's moves, since the radii of his balls always satisfy $r_{n+1} = \beta r_n$. Note that since $\beta < 1/3$ the game does not stop at a finite step. Therefore, the outcome of the restricted hyperplane absolute game will lie in S and Alice will win.

To prove the sufficiency requires some work. Suppose that S is restricted HAW, which means that Alice has a strategy to win the restricted hyperplane absolute game. Following [FSU18] and indeed Schmidt [Sch66], by a strategy one can understand a sequence of maps $\mathcal{F}_{n+1}$, $n = 0, 1, 2, \dots$ which assign a legal move A_{n+1} for Alice depending on Bob's previous moves $\beta, B_0, \dots, B_n$. This strategy is winning if Alice can win when she uses it. As was demonstrated by Schmidt [Sch66, Theorem 7], Alice always has a positional winning strategy for every restricted hyperplane absolute winning set S. This means that for every $\beta \in \left(0, \frac{1}{3}\right)$ there exists a map $\mathcal{F}_\beta$ from the set of balls in $\mathbb{R}^d$ into the set of Alice's legal moves for the restricted hyperplane absolute game such that $\mathcal{F}_{n+1}(\beta, B_0, \dots, B_n) = \mathcal{F}_\beta(B_n)$ is Alice's winning strategy. That is Alice can make her move only using the knowledge of β and Bob's previous move B_n. From now on we fix a positional winning strategy for the restricted hyperplane absolute game, which we will use to define a winning strategy for the hyperplane absolute game. Also, note that since S is restricted HAW, it has to be dense in $\mathbb{R}^d$; otherwise Bob can win the restricted hyperplane absolute game by taking B_0 to contain no points of S.

Given any $0 < \beta < 1/3$, define the following map on balls $B(\mathbf{x}, r)$ in $\mathbb{R}^d$ into hyperplane neighborhoods as follows: first find the unique integer $m_r \in \mathbb{Z}$ satisfying

$$(\beta/2)^{2m_r+1} \le r < (\beta/2)^{2m_r-1}\,; \tag{83}$$

then write

$$\mathcal{F}_{(\beta/2)^2}\left(B\left(\mathbf{x}, (\beta/2)^{2m_r}\right)\right) = B\left(H, (\beta/2)^{2m_r+2}\right) \tag{84}$$

where H is a hyperplane in $\mathbb{R}^d$; and finally define

$$\mathcal{G}_\beta(B(\mathbf{x}, r)) := B\left(H, 2(\beta/2)^{2m_r+2}\right). \tag{85}$$

By the left hand side of (83), we have that

$$2(\beta/2)^{2m_r+2} \le \beta r$$

and therefore $\mathcal{G}_\beta(B(\mathbf{x}, r))$ represents a legal move of Alice for the hyperplane absolute game. We claim that the map $\mathcal{G}_\beta$ gives a positional winning strategy for Alice in the hyperplane absolute game.

Indeed, suppose that Bob chooses any $0 < \beta < 1/3$ on his first move and suppose that $B_n = B(x_n, r_n)$ $(n = 0, 1, 2, \dots)$ and $A_1, A_2, \dots$ are the moves taken by Bob and Alice respectively in the hyperplane absolute game such that

$$A_{n+1} = \mathcal{G}_\beta(B(\mathbf{x}_n, r_n)) = B\left(H_n, 2(\beta/2)^{2m_{r_n}+2}\right) \tag{86}$$

for all $n \ge 0$, where H_n is a hyperplane in $\mathbb{R}^d$ and m_{r_n} arises from (83) when $r = r_n$. Without loss of generality we can assume that $r_n \to 0$ as $n \to \infty$ as otherwise Alice wins because S is dense.

Now we extract a subsequence of B_n, say B_{n_k} such that the radii r_{n_k} are comparable to $(\beta/2)^{2k}$. To this end, first define $k_0 \in \mathbb{Z}$ such that $(\beta/2)^{2k_0} < r_1$. Next, for each $k \geq k_0$ define n_k such that

$$(\beta/2)^{2k+1} = \beta\,\tfrac{1}{2}(\beta/2)^{2k} \leq r_{n_k} < \tfrac{1}{2}(\beta/2)^{2k}\,. \tag{87}$$

The existence of n_k follows from the properties of the hyperplane absolute game, namely, the fact that $r_{n+1}/r_n \geq \beta$ for all $n \geq 0$. Furthermore, observe that n_k is increasing. Also, by (83),

$$m_{r_{n_k}} = k\,. \tag{88}$$

Now define

$$\tilde{B}_k = B\left(\mathbf{x}_{n_k}, (\beta/2)^{2k}\right) \qquad \text{and} \qquad \tilde{A}_{k+1} = \mathcal{F}_{(\beta/2)^2}\left(\tilde{B}_k\right) = B\left(\tilde{H}_k, (\beta/2)^{2k+2}\right)$$

for $k \geq k_0$, where $\tilde{H}_k$ are hyperplanes.
Claim: $\tilde{B}_{k_0}$, $\tilde{A}_{k_0+1}$, $\tilde{B}_{k_0+1}$, $\tilde{A}_{k_0+2}$, ... is a sequence of legal moves taken by Bob and Alice in the restricted hyperplane absolute game when Bob chooses $(\beta/2)^2$ as the parameter of the game.

The legality of Alice's moves $\tilde{A}_{k+1}$ is obvious from Definition 7. To determine the legality of Bob's moves we have to verify that

$$\tilde{B}_{k+1} \subseteq \tilde{B}_k \setminus \tilde{A}_{k+1} \tag{89}$$

for all $k \geq k_0$. First note that, since $B_n = B(x_n, r_n)$ for all $n = 0, 1, 2, \ldots$, we have that $\mathbf{x}_{n_{k+1}} \in B_{n_{k+1}}$. Further, since $n_{k+1} \geq n_k+1$ and B_n and A_{n+1} for $n = 0, 1, 2, \ldots$ are the legal moves of Bob and Alice for the hyperplane absolute game, we have that

$$B_{n_{k+1}} \subseteq B_{n_k+1} \subseteq B_{n_k} \setminus A_{n_k+1}\,.$$

Therefore,

$$\mathbf{x}_{n_{k+1}} \in B_{n_k} \setminus A_{n_k+1}\,.$$

Using the definitions of B_{n_k}, (86), (87) and (88), the above implies that

$$\mathrm{d}\left(\mathbf{x}_{n_{k+1}}, \mathbf{x}_{n_k}\right) \leq r_{n_k} < \tfrac{1}{2}(\beta/2)^{2k} \tag{90}$$

and

$$\mathrm{d}\left(\mathbf{x}_{n_{k+1}}, H_{n_k}\right) > 2(\beta/2)^{2m_{r_{n_k}}+2} = 2(\beta/2)^{2k+2}\,.$$

where $\mathrm{d}(\,\cdot\,)$ is the Euclidean distance on $\mathbb{R}^d$. Observe using (84), (85) and (86) that $H_{n_k} = \tilde{H}_k$. Then,

$$\mathrm{d}\left(\mathbf{x}_{n_{k+1}}, \tilde{H}_k\right) > 2(\beta/2)^{2k+2}\,. \tag{91}$$

Using the triangle inequality together with (90) and (91) gives

$$B\left(\mathbf{x}_{n_{k+1}}, (\beta/2)^{2k+2}\right) \subseteq B\left(\mathbf{x}_{n_k}, (\beta/2)^{2k}\right) \setminus B\left(\tilde{H}_k, (\beta/2)^{2k+2}\right),$$

which is precisely (89). This completes the proof of the above claim.

Finally, observe that $\cap_n B_n$ is the same as $\cap_k \tilde{B}_k$ which has to be in S by the above claim and the fact that Alice plays according to her winning strategy. Therefore, Alice wins the hyperplane absolute game, meaning that S is HAW. The proof of Proposition 8 is thus complete.

Remark 29. At no point in the above proof have we used the fact that H and H_n are hyperplanes. These could have been sets from any collection. Of course, on replacing hyperplanes with another collection of sets we would alter the hyperplane absolute game and the restricted hyperplane absolute game and the corresponding notions of winning. However, the fact that the nature of H and H_n is irrelevant to the above proof means that the altered notions of winning sets and restricted winning sets are the same. In particular, this comment applies to the k-dimensional absolute game, which is a more general version of the hyperplane absolute game, studied in [BFK$^+$12].

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