Article:

Guicherd, R. orcid.org/0000-0002-8755-1352, Mills, A., Trodden, P. orcid.org/0000-0002-8787-7432 et al. (1 more author) (2021) Simultaneous and sequential control design for discrete-time switched linear systems using semi-definite programming. IEEE Control Systems Letters, 5 (4). pp. 1393-1398. ISSN 2475-1456

https://doi.org/10.1109/LCSYS.2020.3037493

Simultaneous and Sequential Control Design for Discrete-Time Switched Linear Systems Using Semi-Definite Programming

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***Abstract*—The control of switched linear discrete-time systems occurs in multiple engineering fields, where it has been used to deal with complex and non-linear systems. This paper presents two strategies to design control laws for discrete-time switched linear systems, whilst guaranteeing asymptotic stability of the closed loop. Firstly, an arbitrary switching signal is considered. In this scenario a common quadratic Lyapunov function is used for stability, but subsystem Lyapunov functions are employed to improve local subsystem performance. Secondly, a constrained switching signal, associated with subsystem lower dwell time bounds is studied. In this case, a decrease in Lyapunov cost is achieved by design, based on dwell time constraints only, thus removing the need for both a common quadratic Lyapunov function or direct stable switches. It is shown in both cases that the control design problems can be formulated as one or a sequence of semi-definite programming problems, and therefore can be solved efficiently. Finally, two examples are provided in order to illustrate the different techniques presented.**

***Index Terms*—Switched systems, optimal control, optimization, semi-definite programming.**

I. Introduction

SWITCHED linear systems have great practical and theoretical interest. They are useful to model discrete parameter changes in real physical systems such as systems integrating logic, networked systems and power systems. They can model sudden failures of a system component, a sensor or an actuator [1]. In addition, it is known that some classes of non-linear systems can only be stabilized using a switching control strategy [2]. The control design for switched linear systems usually includes a collection of controllers as well as a switching rule, which orchestrates the switches between the different subsystems [3]. These switching rules can be time-dependent, state-dependent or controlled; for each of these cases the subsystem index values taken can be arbitrary or can belong to a known feasible set of switching sequences. All these different properties render switched systems non-trivial to analyse and design [4].

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A survey on stability and stabilizability of switched linear systems has detailed the multiple strategies to verify system stability under switching [2]. The stability of a set of control modes can be assessed using Lyapunov–Metzler inequalities resulting in bilinear matrix inequality (BMI) conditions. Mathematical tools such as the joint spectral radius have also been used to study the stability of autonomous systems under arbitrary switching [5]. Constrained switching has been studied based on joint spectral radius methods [6] as well as methods using Lie algebraic conditions [7]. Strategies relying on a common quadratic Lyapunov functions have been used for arbitrary switches [8]. Polytopic Lyapunov functions have been established as necessary and sufficient conditions for the stability of switched systems [9], [10]. It has been proved that the exponential stabilizability of switched linear discrete-time systems is equivalent to the existence of a piecewise quadratic Lyapunov function [11], [12]. Multiple Lyapunov functions have been used to study the stability of switched and hybrid systems [13]. Stability analysis of the closed-loop system can be defined by a decrease of the switched Lyapunov function, leading researchers to use approaches focusing on dwell time conditions. The main strategies have been minimum dwell time conditions [14], average dwell time [15] as well as mode-dependent minimum and average dwell times [16], [17]. The design of switched controlled systems can be divided into two main branches. The first branch considers the design of a family of control laws that will then be orchestrated by an independent switching signal. This applies to both the state- and time- dependent switching systems. The second branch considers the design of the control laws along with the switching signal, and in this case the switching signal is controlled.

This paper considers the problem of designing control modes in order to guarantee the asymptotic stability of a linear discrete-time switched system. Firstly, an arbitrary switching signal, unknown to the controller, is considered. For this scenario, we propose an approach where the modal control laws are synthesized such that they produce distinct Lyapunov functions, guaranteeing the individual stabilizability of modes, but at the same time guaranteeing the existence of a common Lyapunov function, ensuring stability despite mode-to-mode switching. The main advantage compared to [8] is to provide improved closed-loop performance, while maintaining stability of the switched system. Then, a constrained switching signal is considered, combined with given modal dwell-time lower bounds. It is shown in this case that control modes can be computed sequentially, ensuring asymptotic stability of the switched system. Consequently, subsystem dwell time information can be encoded in the design, removing the need for direct stable switches [12] between consecutive modes, or the necessity to add a large set of auxiliary design variables [18].

The paper is organised as follows. Section II defines the switched system dynamics and introduces the control problem tackled. Section III describes how to cast the control law design routines as convex optimization programming problems. Two cases are developed, respectively when the switching signal is arbitrary, and when the switching signal follows known feasible sequences with associated lower bounds on dwell time modes. Some numerical examples are included in Section IV, to show the efficacy of both switched control design approaches. Finally, Section V concludes the paper and proposes future research directions.

Notation: For $(a, b) \in \mathbb{N}^2$ such that $a < b$, the set $[\![a, b]\!]$ defines the set containing the integers from a to b included. For a set $\mathbb{N}$, the notation $\mathbb{N}^*$ defines $\mathbb{N} \setminus \{0\}$. The superscript $^\top$ represents the transpose of a vector or matrix. For all $n \in \mathbb{N}^*$ and $M \in \mathbb{R}^{n\times n}$, $\mathrm{trace}(M)$ denotes the sum of all the diagonal elements of M. The sets of symmetric positive semi-definite and symmetric positive definite matrices of size $n \in \mathbb{N}^*$ will be noted $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ respectively. The zero matrix of appropriate size will simply be noted 0, for conciseness. For $n \in \mathbb{N}^*$, the matrix I_n represents the identity matrix of dimension n. The symbol $\star$ denotes the symmetric element when used within a symmetric matrix. The generalized order on the semi-definite positive cone will be denoted with $\succ$ and $\succeq$ respectively for the strict and non-strict inequalities.

II. Problem statement

A. Discrete-time switched linear system dynamics

Consider the discrete-time switched linear system defined by the following dynamics

$$\forall k \in \mathbb{N},\, x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and input variables, respectively, at time step $k \in \mathbb{N}$. The function $\sigma : \mathbb{N} \to \mathcal{I}$ represents a switching signal, with $\mathcal{I} = [\![1, N]\!]$, where $N \in \mathbb{N}^*$ represents the number of possible distinct system dynamics. In other words, the discrete-time switched linear system dynamics can be selected from the following set of subsystem models $\{(A_1, B_1), \ldots, (A_N, B_N)\}$. For all $i \in \mathcal{I}, (A_i, B_i)$, the matrices representing the subsystem i are of appropriate dimensions. It is assumed that system (1) complies with the following two assumptions,

Assumption 1: For all $i \in \mathcal{I}$, the subsystem (A_i, B_i) is stabilizable.

Assumption 2: The switching signal σ is unknown a priori, but measurable in real time.

Assumption 1 implies the existence of a linear stabilizing state feedback controller $F_i \in \mathbb{R}^{m\times n}$, for all $i \in \mathcal{I}$. It is to be noted that Assumption 1 does not imply that the switched system (1) is stabilizable for an arbitrary switching signal. However, such a condition is necessary for the existence of a collection of stabilizable control modes.

B. Switched control problem

The aim of this paper is to design a collection of control laws F_i, $i \in \mathcal{I}$, for the subsystems of system (1), while guaranteeing asymptotic stability of the switched closed-loop system. The switched system dynamics under closed-loop therefore become,

$$\forall k \in \mathbb{N},\, x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k = A_{cl,\sigma_k} x_k, \tag{2a}$$

$$u_k = F_{\sigma_k} x_k,\, A_{cl,\sigma_k} = A_{\sigma_k} + B_{\sigma_k} F_{\sigma_k} \tag{2b}$$

where A_{cl,σ_k} and $F_{\sigma_k} \in \mathbb{R}^{m\times n}$ respectively represent the switched system closed-loop dynamics and the linear control mode, at time step $k \in \mathbb{N}$. Two different switching regimes are considered separately: (i) when the signal σ is arbitrary, and (ii) when σ is a constrained switching signal [17]. For all $i \in \mathcal{I}$, the control law F_i is synthesized optimally with regards to the infinite horizon linear quadratic regulator (LQR) cost, based on the state and input weighting matrices, $Q \in \mathbb{S}^n_+$ and $R \in \mathbb{S}^m_{++}$, respectively. This control cost is minimized offline based on the eigenvalues of the subsystem Lyapunov functions, correlated to the cost-to-go. The next section presents multiple switched control design methods, guaranteeing the asymptotic stability of the switched discrete-time linear system.

III. Optimal switched control methods

First, the arbitrary switching signal case is considered. The well known technique of constructing a common quadratic Lyapunov function (CQLF) is employed. However, rather than designing the control laws using a CQLF, multiple Lyapunov functions are used to allow for more flexibility in control mode performance tuning. Section III-A.1, the CQLF approach to stabilizing control law synthesis is briefly reviewed. Following this, in Section III-A.2, the approach based on multiple Lyapunov functions is developed. Finally, in Section III-B, the constrained switching scenario is addressed using given lower bounds on mode dwell time.

A. Stabilizing control design under arbitrary switching

1) Design based on a CQLF: A common quadratic Lyapunov function can be used in order to prove stability under arbitrary switching of a switched system [3], [19]. The common Lyapunov function is represented by a positive definite matrix $P \in \mathbb{S}^n_{++}$ which complies with the following matrix inequality, for all the subsystems of system (1),

$$\forall i \in \mathcal{I},\, A_{cl,i}^\top P A_{cl,i} - P \preceq -(Q + F_i^\top R F_i), \tag{3}$$

where $(Q, R) \in \mathbb{S}^n_+ \times \mathbb{S}^m_{++}$, such that the pair $(A_i, Q^{\frac{1}{2}})$ is observable for all $i \in \mathcal{I}$. It is well known that matrix inequalities (3) can be reformulated, losslessly, as a linear matrix inequality (LMI) problem via a change of variables

[20], [21], yielding the following semi-definite programming (SDP) problem,

$$\underset{X,K_i}{\text{minimize}} \quad -\operatorname{trace}(X) \tag{4a}$$

$$\text{subject to} \quad \forall i \in \mathcal{I},$$

$$\begin{bmatrix} X & \star & \star & \star \\ A_iX + B_iK_i & X & \star & \star \\ Q^{\frac{1}{2}}X & 0 & I_n & \star \\ R^{\frac{1}{2}}K & 0 & 0 & I_m \end{bmatrix} \succeq 0. \tag{4b}$$

Solving the SDP optimization presented in (4) provides the asymptotically stabilizing control modes F_i, for all $i \in \mathcal{I}$, under any arbitrary switching signal σ. However, such a control design technique can be conservative, as a single CQLF is used both to ensure the stability as well as to design the control modes. Theorem 3.1 summarizes the results linked to the SDP problem (4).

Theorem 3.1: If a solution to the LMI problem (4) exists then (i) for each $i \in \mathcal{I}$, $F_i = K_iX^{-1}$ stabilizes (A_i, B_i), (ii) the closed-loop system (1) with $u_k = F_{\sigma_k}x_k$ is asymptotically stable for all switching signals and (iii) there exists a CQLF shared by all the modes $A_{cl,i}$, $i \in \mathcal{I}$ of system (1).

Proof: The proof is not included for conciseness, but can be found in the prominent work [22]. ■

The next section includes the first contribution of this paper. It develops a strategy to simultaneously compute a collection of control laws stabilizing for the system (1), based on multiple subsystem Lyapunov functions as well as a CQLF. The mode dependent Lyapunov functions allow for more flexibility in the control design, whereas the existence of a CQLF ensures the stability for any switching signals.

2) Independent control law design guaranteeing the existence of a CQLF: The simultaneous design of a family of control modes can be achieved based on distinct Lyapunov functions allowing for the performance of any control mode to be not only guaranteed but also fine-tuned. However, in order to guarantee the stability of the switched system (1) under arbitrary switching, the existence of a CQLF is added as a constraint. The major difference with the strategy presented within the previous subsection is that the common quadratic Lyapunov function's purpose is only to guarantee the switched system stability. In a similar fashion, the control design problem can be formulated as an SDP problem as presented in (5).

$$\underset{X,X_i,K_i}{\text{minimize}} \quad -\sum_{i=1}^{N} \mu_i \operatorname{trace}(X_i) \tag{5a}$$

$$\text{subject to} \quad \forall i \in \mathcal{I},$$

$$\begin{bmatrix} X_i & \star & \star & \star \\ A_iX_i + B_iK_i & X_i & \star & \star \\ Q^{\frac{1}{2}}X_i & 0 & I_n & \star \\ R^{\frac{1}{2}}K_i & 0 & 0 & I_m \end{bmatrix} \succeq 0 \tag{5b}$$

$$\begin{bmatrix} X_i & \star & \star \\ A_iX_i + B_iK_i & X & \star \\ 0 & 0 & X_i - X \end{bmatrix} \succ 0. \tag{5c}$$

For all i in $\mathcal{I}$, μ_i is positively weighting the performance metric of the control mode associated to subsystem i.

Theorem 3.2: If a feasible solution to the LMI problem (5) exists, then (i) for each $i \in \mathcal{I}$, $F_i = K_iX_i^{-1}$ stabilizes (A_i, B_i), (ii) the closed-loop system (1) with $u_k = F_{\sigma_k}x_k$ is asymptotically stable for all switching signals and (iii) there exists a CQLF, shared by all the modes $A_{cl,i}$, $i \in \mathcal{I}$.

Proof: Taking the Schur complement of (5b) yields an equivalent relationship as the one provided in Theorem 3.1. This ensures the asymptotic stability of each control modes based on distinct Lyapunov functions $P_i = X_i^{-1}$, $i \in \mathcal{I}$. In addition, the LMI constraint provided in (5c) implies the following condition:

$$\forall i \in \mathcal{I}, \begin{bmatrix} X_i & \star \\ A_iX_i + B_iK_i & X \end{bmatrix} \succ 0. \tag{6}$$

Consequently, taking the Schur complement [20] yields the following equivalent matrix inequality:

$$\forall i \in \mathcal{I}, (A_iX_i + B_iK_i)^\top X^{-1}(A_iX_i + B_iK_i) \prec X_i. \tag{7}$$

The pre- and post-multiplication by X_i^{-1}, along with the change of variable $F_i = K_iX_i^{-1}$ leads to,

$$\forall i \in \mathcal{I}, (A_i + B_iF_i)^\top X^{-1}(A_i + B_iF_i) \prec X_i^{-1}. \tag{8}$$

The LMI condition (5c) also ensures that the following linear matrix inequality is respected,

$$\forall i \in \mathcal{I}, X_i \succ X \succ 0 \Leftrightarrow 0 \prec X_i^{-1} \prec X^{-1}. \tag{9}$$

Combining the last two conditions guarantees the existence of a common quadratic Lyapunov function $P = X^{-1}$ and subsequently concludes the proof. ■

A necessary and sufficient condition for feasibility of problem (5) is the existence of a CQLF; expressing this in terms of properties of the system and cost is a topic of on-going research. The use of multiple Lyapunov functions allows for the optimization cost function to be a weighted combination of the performance metric of each control mode, and ultimately permits more flexible tuning. For instance, as discussed in [21], the objective function weights can be set according to the probabilities of occurrence for the different subsystems, or ranked according to the relative importance of different control modes in the overall system.

The next section proposes a strategy to design control laws when the switching signal is unknown a priori, but constrained to a subset of sequences, and subsystem dwell time lower bound information is provided.

B. Stabilizing control design under constrained switching

Certain switched systems have a constrained switching signal. In this case, the switching signal only allows switches within a given set of subsystem pairs. An admissible constrained switching signal can be represented equivalently by a directed graph or by a set of ordered mode index pairs. Within this subsection, the switching signal is assumed to be constrained and the set of admissible switching sequences will be denoted $\mathcal{I}_s$, such that,

$$\mathcal{I}_s = \left\{(i,j) \in \mathcal{I}^2 | \exists (k,\sigma) \in \mathbb{N} \times \Omega,\, \sigma_k = i,\, \sigma_{k+1} = j\right\}, \tag{10}$$

where Ω represents the set of admissible switching signals for the switched system (1). Even if the switching signal is not known a priori, the set $\mathcal{I}_s$ is provided to the controller. Consequently, not all control mode switches have to ensure system closed-loop stability. Enforcing system stability for the set of constrained switching sequences $\mathcal{I}_s$ has previously been studied in the context of state feedback synthesis for piecewise-affine systems [12]. Such a constraint can be implemented during the design of a collection of control laws, based on the following LMI

$$\forall (i,j) \in \mathcal{I}_s,\; \begin{bmatrix} X_i & \star \\ A_i X_i + B_i K_i & X_j \end{bmatrix} \succ 0. \tag{11}$$

Replacing the LMI constraint (5c) within the optimization problem (5) by the LMI (11) with the appropriate set of decision variables, guarantees direct stable switches between the control mode pairs in $\mathcal{I}_s$. The inequality (11) guarantees stability with the minimum dwell time of 1 for the admissible switches [14]. It is well known that dwell time constraints can be used in order to design control modes able to switch in a stable fashion [3]. In particular, ensuring stability between two control modes with the minimum possible dwell time of 1, ensures a stable switch for any greater dwell time, for the same two control modes [23]. Enforcing direct stable switches between any two admissible control modes can be conservative, and even lead to infeasible problems. In some applications, information on subsystem dwell time lower bounds is given and can be exploited to guarantee a strict Lyapunov cost decrease as per equation (12).

$$\forall (i,j) \in \mathcal{I}_s,\; P_i \succ (A_i + B_i F_i)^{\Delta_i \top} P_j \,(A_i + B_i F_i)^{\Delta_i} \tag{12}$$

The parameters $\Delta_i \in \mathbb{N}^*$, for all $i \in \mathcal{I}$, represent lower bounds on mode dwell times. In other words, it is assumed that the switched system remains in mode i, for at least Δ_i time steps.

The main contribution developed in this section is to extend the work of [12], [18], by adding dwell time lower bounds information in the control design. Providing a change of variables is done, the constraints (11) and (12) are equivalent when all Δ_i are set to 1. However, using mode dwell time bounds strictly greater than 1 triggers a loss of convexity [18]. This leads to the LMI problem formulation given in (13). This SDP computes the control law F_j and Lyapunov function P_j, for $j \in \mathcal{I}$, based on the already existing control laws F_i, Lyapunov matrices P_i and dwell time bounds Δ_i, for all i in the set $\mathcal{I}_s^j = \{i | (i,j) \in \mathcal{I}_s\}$. It can be highlighted that the feasibility of problem (13) depends on the control modes indexed by the set $\mathcal{I}_s^j$, and thus it is not trivial to evaluate it a priori; this is a topic of future research. Problem (13) is inspired by the LMI problem used for fixed dwell time and system dynamics, when the control modes are design sequentially online [23], [24].

Theorem 3.3: If a feasible solution to the LMI problem (13) exists, then (i) $F_j = K_j X_j^{-1}$ stabilizes the subsystem (A_j, B_j) and (ii) asymptotic stability is guaranteed for a switch between mode i and mode j after Δ_i time steps have elapsed in mode i, for all $i \in \mathcal{I}_s^j$.

$$\underset{X_j, K_j}{\text{minimize}} \quad -\operatorname{trace}(X_j) \tag{13a}$$

$$\text{subject to} \quad \forall i \in \mathcal{I}_s^j,$$

$$\begin{bmatrix} X_j & \star & \star & \star \\ A_j X_j + B_j K_j & X_j & \star & \star \\ Q^{\frac{1}{2}} X_j & 0 & I_n & \star \\ R^{\frac{1}{2}} K_j & 0 & 0 & I_m \end{bmatrix} \succeq 0 \tag{13b}$$

$$\begin{bmatrix} P_i & \star \\ (A_i + B_i F_i)^{\Delta_i} & X_j \end{bmatrix} \succ 0 \tag{13c}$$

Proof: The constraints provided in equation (13b) ensures that the control mode $j \in \mathcal{I}$ is asymptotically stabilizing for the associated subsystem mode. Taking the Schur complement of the equation (13c) yields the following,

$$\forall i \in \mathcal{I}_s^j,\; P_i \succ \left(A_{cl,i}^{\Delta_i}\right)^\top X_j^{-1} \left(A_{cl,i}^{\Delta_i}\right). \tag{14}$$

Pre- and post-multiplying this inequality by the system state at time k, $x_k^\top$ and x_k respectively, different from the zero vector, and applying the change of variable $P_j = X_j^{-1}$, yields the following inequality,

$$\begin{aligned} &\forall i \in \mathcal{I}_s^j,\; x_k^\top P_i x_k > x_k^\top \left(A_{cl,i}^{\Delta_i}\right)^\top P_j \left(A_{cl,i}^{\Delta_i}\right) x_k, \\ \Leftrightarrow\; &\forall i \in \mathcal{I}_s^j,\; x_k^\top P_i x_k > x_{k+\Delta_i}^\top P_j x_{k+\Delta_i}. \end{aligned} \tag{15}$$

This relationship ensures a strict cost decrease when switching from mode i to mode j, after Δ_i time steps have elapsed, for any i in $\mathcal{I}_s^j$, which concludes the proof. ■

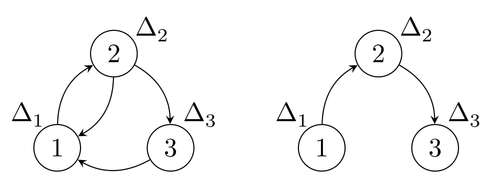

Fig. 1. Feasible switching sequences between three different system modes, representation of a cyclic switching cycle and an acyclic switching cycle, respectively on the left and on the right.

Figure 1 represents different possible constrained switching sets $\mathcal{I}_s$, along with the corresponding mode dwell time lower bounds, for a simple three-mode system. The control design routine for any acyclic graph will be initialized by designing the control laws and Lyapunov functions of the starting modes. These modes correspond to the vertices without any parent vertices, for example, mode 1 on the right graph in Figure 1. Following this initialization step, the SDP problem (13) can be solved sequentially in order to design the control laws of all the remaining modes of the graph. The control modes are propagated throughout the graph from parents to children nodes, until all the vertices have been explored. In case the switching graph contains cycles, a minimum feedback arc set problem is solved first, allowing to generate an acyclic subgraph, by removing the smallest amount of edges. For instance, removing the edge from mode 1 to mode 2 on the left graph in Figure 1, yields the maximum acyclic subgraph. Afterwards, the control design procedure linked to acyclic digraphs can be performed on the subgraph obtained. Finally, the stability of all the edges that have been removed is checked, based on the computation of the minimum dwell time Δ_i, such that equation (12) holds from parent i to child j [14].

The next section presents examples to show the efficacy of the two design methods developed in this paper.

IV. Numerical examples

The examples are solved in Python using CVXPY [25] along with the CVXOPT and SCS solvers [26], [27]. These solvers respectively rely on interior-point algorithms [28] and operator splitting methods [29]. The two examples in this section respectively illustrates the two methods presented in the SDP problem (5) and (13).

A. Example 1

This first example is composed of a switched system with three subsystems, which are as follows,

$$A_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{16a}$$

$$A_2 = \begin{bmatrix} -1 & -\frac{3}{4} \\ \frac{1}{2} & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & -\frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}, \tag{16b}$$

$$A_3 = \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}, B_3 = \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{4} & -1 \end{bmatrix}. \tag{16c}$$

The switching signal is defined to be arbitrary, hence it can select any indices in the set $\mathcal{I} = [\![1, 3]\!]$. The weights associated to the modes is respectively encoded by the entries of the following vector $\mu = [0.1, 0.1, 0.8]$. In this example, three controllers are designed and their performance compared, a robust controller, a collection of control laws based on a common Lyapunov function and a collection of control laws enforcing the existence of a common Lyapunov function. The robust control law design is performed using an offline version of the work presented in [22]. The two other design strategies are the ones included within this paper. The control gains obtained in each of these three scenarios are given in equations (17), (18) and (19) respectively.

$$F_{\text{robust}} = \begin{bmatrix} -0.0233 & 0.0924 \\ -0.3612 & -1.02091 \end{bmatrix} \tag{17}$$

$$\begin{gathered} F_{c,1} = \begin{bmatrix} -0.6527 & -0.3711 \\ 0.2824 & -0.7356 \end{bmatrix} F_{c,2} = \begin{bmatrix} 0.5602 & 0.2963 \\ -0.4612 & -0.7424 \end{bmatrix} \\ F_{c,3} = \begin{bmatrix} -0.6105 & -0.2296 \\ -0.1345 & -0.3151 \end{bmatrix} \end{gathered} \tag{18}$$

$$\begin{gathered} F_{s,1} = \begin{bmatrix} -0.6434 & -0.3216 \\ 0.3217 & -0.6433 \end{bmatrix} F_{s,2} = \begin{bmatrix} 0.4837 & 0.2178 \\ -0.3524 & -0.6263 \end{bmatrix} \\ F_{s,3} = \begin{bmatrix} -0.5509 & -0.2245 \\ -0.1567 & -0.2668 \end{bmatrix} \end{gathered} \tag{19}$$

The design of the collection of control laws $\{F_{s,1}, F_{s,2}, F_{s,3}\}$ is performed using the entries of the vector μ in order to weight the mode performance objectives within the SDP problem (5).

Table I compares the eigenvalues of the Lyapunov functions for each of the switched system modes as well as the performance metric considered within the optimization formulation.

TABLE I

Comparison of the Lyapunov function eigenvalues and control costs for the different switched control laws.

	Robust	Common	Single
Mode 1	$\{6.710, 2.129\}$	$\{1.830, 2.657\}$	$\{1.804, 1.804\}$
Mode 2	$\{6.710, 2.129\}$	$\{1.830, 2.657\}$	$\{1.098, 2.351\}$
Mode 3	$\{6.710, 2.129\}$	$\{1.830, 2.657\}$	$\{1.656, 1.073\}$
Control Costs	8.839	4.487	2.889

The performance metric is taken as the weighted sum of the trace of the mode Lyapunov functions, where the mode associated weights are the entries of the vector μ. This example demonstrates that the robust strategy is the most conservative in term of control cost, followed by the control design using a common Lyapunov function. Finally, the last design method simply relying on the existence of a common Lyapunov function offers the best control cost. Interestingly, the eigenvalues of the common Lyapunov function added for stability are higher than the eigenvalues of the common Lyapunov function used for design purposes $\{2.354, 3.42\}$.

B. Example 2

This second example establishes the feasibility of designing a sequence of control laws with known switching sequences as well as dwell time lower bounds. The switched linear system is composed of three subsystems inspired by a triple integrator system, defined such that,

$$A_1 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{20a}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{20b}$$

Figure 2 represents the switching graph of the set $\mathcal{I}_s$ for the system (20). It displays the possible switches along with the associated subsystem lower bound dwell times.

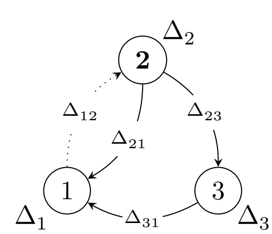

Fig. 2. Feasible switching sequences between three different system modes and subsystem dwell time lower bounds for the switched system (20).

The dwell time parameters with a single subscript represent subsystem dwell time lower bounds, such that $\Delta_1 = 1$, $\Delta_2 = 12$ and $\Delta_3 = 5$. The dwell time parameters with double subscripts represent the minimum dwell times required for a strict cost decrease between two control modes [14]. First, the graph presented in 2 is changed into an acyclic graph by removing the directional link (i.e. dotted edge) from mode 1 to mode 2. Following this, the design procedure is initiated with mode 2 using an LQR technique. Then, mode 3 and 1 are designed using the formulation provided in (13), based on Δ_2 and $\{\Delta_2, \Delta_3\}$ respectively. Since this system has a cyclic switching graph, a check is performed for the previously removed edge using condition (12). The minimum dwell time results between the admissible modes are summarized in Table II. It can be seen that this switched control design complies with all the dwell time bounds. The control design strategy in [12], not accounting for dwell times, yields an infeasible optimization problem.

TABLE II

COMPARISON OF THE MODE DWELL TIME LOWER BOUNDS WITH THE MINIMUM DWELL TIMES OBTAINED FOR THE ADMISSIBLE SWITCHES.

	Mode dwell time bound	Maximum of minimum dwell times
Δ_1	1	$\Delta_{12} = 1$
Δ_2	12	$\max\{\Delta_{21}, \Delta_{23}\} = 7$
Δ_3	5	$\Delta_{31} = 1$

The switching control laws and associated Lyapunov function matrices obtained are as follows,

$$F_1^\top = \begin{bmatrix} -0.42 & 0.17 \\ -0.80 & -0.03 \\ 0.39 & -0.96 \end{bmatrix}, P_1 = \begin{bmatrix} 1.70 & 0.42 & -0.88 \\ 0.42 & 1.80 & -0.39 \\ -0.88 & -0.39 & 2.85 \end{bmatrix}, \quad \text{(21a)}$$

$$F_2^\top = \begin{bmatrix} -0.43 & -0.04 \\ -1.23 & 0.02 \\ 0.87 & -0.76 \end{bmatrix}, P_2 = \begin{bmatrix} 2.92 & 2.35 & -0.38 \\ 2.35 & 4.59 & -1.26 \\ -0.38 & -1.26 & 2.64 \end{bmatrix}, \quad \text{(21b)}$$

$$F_3^\top = \begin{bmatrix} 0.03 & -0.39 \\ 1.48 & 1.73 \\ 0.03 & -0.33 \end{bmatrix}, P_3 = \begin{bmatrix} 38 & -1.0e3 & -37 \\ -1.0e3 & 4.0e4 & 1.0e3 \\ -37 & 1.0e3 & 38 \end{bmatrix}. \quad \text{(21c)}$$

V. CONCLUSION

Two different strategies proposing a solution to the control design problem linked with discrete-time switched linear systems have been developed. Two switching scenarios were investigated: in the first case an arbitrary signal was considered, whereas in the second case a constrained switching signal was studied. In each case, it is shown how a stabilizing optimal control synthesis may be formulated as an LMI problem. These problems are solved either once or in a sequence, depending on the switching signal scenario, and ensure by design that the control modes render the closed-loop switched system asymptotically stable. Each scenario has been illustrated by a numerical example, highlighting the difference in performance or feasibility with current approaches. Future research directions could explore the influence of the control mode initialization in the sequential design case.

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