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Porous media equations with multiplicative space–time white noise

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Abstract. The existence of martingale solutions for stochastic porous media equations driven by nonlinear multiplicative space–time white noise is established in spatial dimension one. The Stroock–Varopoulos inequality is identified as a key tool in the derivation of the corresponding estimates.

Résumé. L'existence d'une solution martingale pour l'équation stochastique des milieux poreux, dirigée par un bruit non-linéaire multiplicatif en espace et en temps, est établie dans le cas spatial unidimensionnel. L'inégalité de Stroock–Varopoulos est identifiée comme un outil clé dans l'obtention des estimées correspondantes.

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1. Introduction

Let T be a positive real number and let $I=(0,1)$. We consider the equation

$$\begin{aligned} &\partial_t u=\partial_x^2\big(u^{[m]}\big)+\sigma(x,u)\xi, \quad \text{in } Q:=(0,T)\times I,\\ &u=u^{(0)} \quad \text{on } \{0\}\times I, \end{aligned} \tag{1.1}$$

with homogeneous Dirichlet boundary conditions, where ξ is noise which is white in both space and time, and $u^{[m]}:=|u|^{m-1}u$, $m\in(1,\infty)$.

While stochastic porous media equations attracted significant attention, all available results concerning multiplicative noise pose strong spatial colouring conditions on the noise. Indeed, the monotone operator approach, for which we refer the reader to the monographs [2,27], requires noise with Cameron–Martin space $L^2([0,T],\mathcal{H})$, where $\mathcal{H}=H^{3/2+\varepsilon}$. Here and throughout the introduction, one may choose arbitrarily small but positive $\varepsilon>0$. Another notable restriction of the monotone operator approach is that when the mapping $u\mapsto\sigma(x,u)$ is given by a pointwise composition, only affine linear diffusion coefficients are covered. In the case of nonlinear diffusion coefficients, the existence of martingale solutions was shown in [20], again restricting to spatially colored noise with $\mathcal{H}=H^{1/2+\varepsilon}$. Recent development of an L^1-based theory, see for example [4,9,12,13,18,19] and references therein, has lead to the (probabilistically) strong existence and uniqueness for a large class of nonlinear diffusions σ with spatially colored noise. The most lenient conditions are from [9], which corresponds to $\sigma\in C^{1/2+\varepsilon}$ and $\mathcal{H}=H^{1/2+1/(m\vee 2)+\varepsilon}$. Needless to say, all of these results are quite far from the space–time white noise case $\mathcal{H}=L^2$.

Equation (1.1) can also be seen as an example of a singular SPDE. The theory of these equations have seen major advances recently thanks to the theories of [21,22]. Quasilinear singular SPDEs, first studied by [30], have recently been also solved with space–time white noise via these theories [1,15,16]. However, the degeneracy of the leading order operator prevents any of these works to apply for the study of (1.1). The additional (Itô-) structure of the equation, however, allows for stochastic analytic tools.

In the main result of this work, Theorem 2.5 below, we establish the probabilistically weak existence of solutions for a class of diffusion nonlinearities. The scope of Theorem 2.5 is quite large: any continuous σ satisfying a mild growth condition fits in the framework. This in particular covers the case $\sigma(r) = \sqrt{r}$ which is known to be relevant in scaling limits of interacting branching particle systems (see Section 1.1 below).

In order to prove the existence of solutions, we obtain estimates for solutions of viscous approximations of (1.1) with finitely many modes of noise, the well-posedness of which is guaranteed by [10, Theorem 3.1]. These estimates should be in spaces of positive regularity in order to guarantee compactness in some L^p space (in space time). At the same time the regularity exponent should be relatively small in order to avoid blow ups appearing due to the irregularity of the noise. We identify the Stroock–Varopoulos inequality as a key ingredient in obtaining such estimates that are compatible with the non-linear nature of (1.1). It is remarkable that this inequality, which originates in the analysis of non-local porous media equations, proves to be vital to the local but irregular setting of (1.1).

In such generality, no strong uniqueness is expected to hold for (1.1), since it is not even true in the semilinear case $m = 1$, see [29]. It is, however, reasonable to expect strong uniqueness when σ is, say, Lipschitz continuous, which remains an open question. In this article we show that strong existence and uniqueness hold when, roughly speaking, $\sigma(x, u)$ behaves like $u^{[(m+1)/2]}$ around the origin (see Proposition 2.8 below).

The rest of the article is structured as follows. In Section 1.1 we give an example on how a heuristic scaling of a simple system of interacting particles gives rise to an SPDE of the form (1.1). In Section 2 we state the main results. The proof of Theorem 2.5 is divided into a priori estimates for some approximating equations in Section 3 and the passage to the limit in Section 4. The proof of Proposition 2.8 is given in Section 5.

1.1. *A heuristic derivation*

In $\mathbb{R}$, let us consider particles $(X_t^i)_{t\geq 0}$ (of negligible mass), for $i \in I_t$, interacting through a potential V. Here, I_t is an index set depending on time with $|I_0| \sim N$, The system undergoes critical branching: each particle, with rate one, dies and leaves behind offspring with the expected number of offspring being one. During their lifetime the particles X_t^i evolve under the dynamics

$$dX_t^i = \sum_{j\in I_t} \partial_x V\big(X_t^i - X_t^j\big)\,dt,$$

where $V : \mathbb{R} \to \mathbb{R}$ is a compactly supported, smooth, non-negative function integrating to one. After introducing the rescaling $Y_t^{i,N} = N^{-2/3} X_{Nt}^i$, one has that

$$dY_t^{i,N} = \frac{1}{N} \sum_{j\in \tilde{I}_t} \partial_x V_{N^{-2/3}}\big(Y_t^{i,N} - Y_t^{j,N}\big),$$

where $\tilde{I}_t = I_{Nt}$, $V_\alpha(\cdot) := \alpha^{-1} V(\alpha^{-1}\cdot)$ for $\alpha > 0$, and the particles Y_t^i branch with rate N. The ultimate goal is to let $N \to \infty$. Since $N^{2/3} \ll N$ we make the following simplification: we consider the system with the same branching mechanism but dynamics given by

$$dY_t^{i,N,\varepsilon} = \frac{1}{N} \sum_{j\in \tilde{I}_t} \partial_x V_\varepsilon\big(Y_t^{i,N,\varepsilon} - Y_t^{j,N,\varepsilon}\big).$$

Let us denote by $\mu_t^{N,\varepsilon}$ the empirical measure of the above system at time t, that is,

$$\mu_t^{N,\varepsilon} = \frac{1}{N} \sum_{i\in \tilde{I}_t} \delta_{Y_t^{i,N,\varepsilon}}.$$

It follows from [28, Theorem 1] that for $N \uparrow \infty$, $(\mu_t^{N,\varepsilon})_{t\in[0,T]}$ converges – in an appropriate sense – to a non-negative measure valued stochastic process $(\mu_t^\varepsilon)_{t\in[0,T]}$ which satisfies

$$\partial_t \mu^\varepsilon = \partial_x\big(\mu^\varepsilon\big(\partial_x V_\varepsilon * \mu^\varepsilon\big)\big) + c\sqrt{\mu^\varepsilon}\,\xi,$$

where c is a constant depending on the variance of the branching mechanism and ξ is space time white noise. The interpretation of the term $c\sqrt{\mu^\varepsilon}\xi$ is in a weak sense, namely, for every test function ϕ, the process

$$\eta_t(\phi) := \mu_t^\varepsilon(\phi) - \mu_0^\varepsilon(\phi) + \int_0^t \mu_s^\varepsilon\big(\big(\partial_x V_\varepsilon * \mu_s^\varepsilon\big)\partial_x\phi\big)\,ds,$$

is a martingale with quadratic variation given by

$$\langle \eta(\phi)\rangle_t = c^2 \int_0^t \mu_s^\varepsilon(\phi^2)\, ds.$$

Informally, since V_ε tends to δ_0, passing to the limit $\varepsilon \to 0$ in the above equation leads to

$$\partial_t \mu = \frac{1}{2}\partial_x^2(\mu^2) + c\sqrt{\mu}\xi.$$

In the deterministic case, that is if $c = 0$, the limit $\varepsilon \to 0$ has been rigorously justified in [26].

1.2. *Notation*

Due to the low regularity we always work with weak (in the terminology of e.g. [6], 'weak dual') solutions in the PDE sense, and consider both strong and weak solutions in the probabilistic sense. For the former, fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the space–time white noise ξ is given. Recall that this means a collection of jointly Gaussian centred random variables $\xi(\varphi)$, $\varphi \in L^2(Q)$, with covariance $\mathbb{E}(\xi(\varphi)\xi(\bar\varphi)) = (\varphi, \bar\varphi)_{L^2(Q)}$. For the remainder of the article we set $e^k(x) = \sqrt{2}\sin(\pi k x)$ for $k \in \mathbb{N} = \{1, 2, \dots\}$. We have that $(e^k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(I)$. For each $k \in \mathbb{N}$, set $(w_t^k)_{t\in[0,T]}$ to be a continuous modification of the collection of random variables $(\xi(\mathbf{1}_{[0,t]}e^k))_{t\in[0,T]}$. It is well-known that such modifications exist, as is the fact that $w^1, w^2, \dots$ is a sequence of independent Wiener processes. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the right continuous completion of the filtration generated by them.

Function spaces in the spatial variable $x \in I$ are denoted by the lower index x. For notational simplicity we do not make the x-dependency of $\sigma(x, u)$ explicit when convenient. The set $(e^k)_{k\in\mathbb{N}}$ consists of the eigenvectors of the $(-\Delta)$ with Dirichlet boundary condition on ∂I, with corresponding eigenvalues $\lambda_k = (\pi k)^2$. For $\gamma \geq 0$ we introduce the space

$$H_x^\gamma = \left\{ v \in L_x^2 : \sum_{k\in\mathbb{N}} \lambda_k^\gamma \left|(v, e^k)_{L_x^2}\right|^2 < \infty \right\},$$

endowed with the norm

$$\|v\|_{H_x^\gamma}^2 := \sum_{k\in\mathbb{N}} \lambda_k^\gamma \left|(v, e^k)_{L_x^2}\right|^2.$$

Here and in the sequel if H is a Hilbert space, $(\cdot,\cdot)_H$ stands for the inner product in H. We define $H_x^{-\gamma}$ to be the dual of H_x^γ in the Gelfand triple $H_x^\gamma \subset L_x^2 \equiv (L_x^2)^* \subset (H_x^\gamma)^*$. Extending $(\cdot,\cdot)_{L_x^2}$ to the $H_x^\gamma - H_x^{-\gamma}$ duality denoted by $\langle\cdot,\cdot\rangle$, the norm of an element $v^* \in H_x^{-\gamma}$ is given by

$$\|v^*\|_{H_x^{-\gamma}}^2 = \sum_{k\in\mathbb{N}} \lambda_k^{-\gamma} \left|\langle v^*, e^k\rangle\right|^2. \tag{1.2}$$

It is easy to see that L_x^2 is dense in $H_x^{-\gamma}$, and therefore so is $C_c^\infty(I)$. It is also easy to verify that for $\gamma > 0$ the embedding $L_x^2 \subset H_x^{-\gamma}$ is compact. For $\beta \in \mathbb{R}$ we define

$$(-\Delta)^{\beta/2}\phi := \sum_{k\in\mathbb{N}} \lambda_k^{\beta/2}(\phi, e^k)_{L_x^2} e^k, \quad \text{for } \phi \in C_c^\infty(I).$$

In the sequel we drop the lower index from $(\cdot,\cdot)$ and we always mean

$$(f, g) = \int_I f(x)g(x)\, dx. \tag{1.3}$$

Note that this expression is meaningful whenever $f \in L_x^p$, $g \in L_x^{p/(p-1)}$, $p \in [1, \infty]$. Whenever the scalar product is taken in another Hilbert space H, this is reflected in the notation $(\cdot,\cdot)_H$.

For any $\gamma \in \mathbb{R}$ the operator $(-\Delta)^{\beta/2}$ extends to an isometry

$$(-\Delta)^{\beta/2} : H_x^{\gamma} \to H_x^{\gamma-\beta}.$$

It follows that for $\gamma_1 > \gamma_2$ the embedding $H_x^{\gamma_1} \subset H_x^{\gamma_2}$ is compact. For all $\gamma \in \mathbb{R}$, H_x^{γ} is a Hilbert space. Using the inner product of H_x^{-1} to identify it with its own dual, one can consider the Gelfand triple $L_x^{m+1} \subset H_x^{-1} \equiv (H_x^{-1})^* \subset (L_x^{m+1})^*$. The operator $u \mapsto \Delta u^{[m]}$ maps L_x^{m+1} to $(L_x^{m+1})^*$ and the action of $\Delta u^{[m]}$ on an element $\phi \in L_x^{m+1}$ is given by

$${}_{(L_x^{m+1})^*}\langle \Delta u^{[m]}, \phi \rangle_{L_x^{m+1}} = -\big(u^{[m]}, \phi\big).$$

For more details we refer to [32, Ex. 4.1.11].

Function spaces in the temporal variable t, whenever given on the whole time horizon $[0, T]$, are denoted by the lower index t. For instance, $L_t^2 H_x^{\gamma}$ stands for $L^2([0, T], H_x^{\gamma})$. Occasionally the time horizon will be different, in these cases we specify the domains. In the temporal and spatial variable we will also consider the Sobolev–Slobodeckij spaces $W^{\gamma,p}$ (see e.g. [34, Section 4.2]) and the usual Hölder spaces C^{γ} (see e.g. [34, Section 4.5]). Their relevant properties are stated in Proposition 1.1 below. By $\dot{W}_x^{\gamma,p}$ we denote the closure of $C_c^{\infty}(I)$ in $W_x^{\gamma,p}$. Finally, spaces of functions on Ω (which will always be L^p spaces) are denoted by the lower index ω. When L^p spaces are considered on Q or $\Omega \times Q$, we write $L_{t,x}^p$ or $L_{\omega,t,x}^p$.

Proposition 1.1.

(i) [34, *Rmk* 4.4.2/2]. *Let* $p \in (1, \infty)$, $\gamma \in (0, 1)$. *Then an equivalent norm in* $W_x^{\gamma,p}$ *is given by*

$$\left(\|v\|_{L_x^p}^p + \int_{I \times I} \frac{|v(x) - v(y)|^p}{|x - y|^{1+\gamma p}} \, dx \, dy \right)^{1/p}; \tag{1.4}$$

(ii) [34, *Thm* 4.3.2/1]. *Let* $p \in (1, \infty)$, $\gamma \in (-\infty, 1/p]$. *Then* $\dot{W}_x^{\gamma,p} = W_x^{\gamma,p}$;

(iii) [6, *Eq* 2.11]. *Let* $\gamma \in (0, 1) \setminus \{1/2\}$. *Then* $\dot{W}_x^{\gamma,2} = H_x^{\gamma}$;

(iv) [34, *Thm* 4.8.2]. *Let* $p \in (1, \infty)$, $\gamma \in [0, \infty)$ *such that* $\gamma - 1/p \notin \mathbb{Z}$. *Then the dual of* $\dot{W}_x^{\gamma,p}$, *viewed as a subset of distributions, is* $W_x^{-\gamma,p'}$, *where* $1/p + 1/p' = 1$;

(v) [34, *Thm* 4.3.1/1], [34, *Eq* 2.4.1/(8)]. *Let* $-\infty < \gamma_0 < \gamma_1 < \infty$, $1 < p_0, p_1 < \infty$, $\theta \in (0, 1)$, *and define*

$$\gamma = (1 - \theta)\gamma_0 + \theta\gamma_1, \qquad p = \big((1 - \theta)p_0^{-1} + \theta p_1^{-1}\big)^{-1}.$$

Suppose $\gamma_0, \gamma_1, \gamma \notin \mathbb{N}$. *Then one has*

$$\|v\|_{W_x^{\gamma,p}} \le \|v\|_{W_x^{\gamma_0,p_0}}^{1-\theta} \|v\|_{W_x^{\gamma_1,p_1}}^{\theta}. \tag{1.5}$$

Remark 1.2. Interpolation between H_x^{γ} spaces is straightforward from the definition.

2. Formulation and main results

Definition 2.1. Suppose that $u^{(0)}$ is an $\mathcal{F}_0$-measurable H_x^{-1}-valued random variable. A *strong solution* of (1.1) is an H_x^{-1}-valued continuous $\mathbb{F}$-adapted process u, that furthermore belongs to $L_{t,x}^{m+1}$ almost surely and such that for all $\phi \in L_x^{m+1}$ almost surely the equality

$$\big(u(t), \phi\big)_{H_x^{-1}} = \big(u^{(0)}, \phi\big)_{H_x^{-1}} - \int_0^t \big(u^{[m]}(s), \phi\big) \, ds + \sum_{k \in \mathbb{N}} \int_0^t \big(\sigma\big(u(s)\big)e^k, \phi\big)_{H_x^{-1}} \, dw_s^k$$

holds for all $t \in [0, T]$.

Remark 2.2. Notice that in the second term at the right-hand side of the above equality, it is not the H_x^{-1}-inner product but rather the usual duality between $L_x^{(m+1)/m}$ and L_x^m given by (1.3).

Definition 2.3. Suppose that $u^{(0)}$ is an $\mathcal{F}_0$-measurable H_x^{-1}-valued random variable. A *weak solution* of (1.1) is a collection $\{(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}), \bar{\mathbb{F}}, (\bar{w}^k)_{k \in \mathbb{N}}, \bar{u}\}$, such that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is a probability space, $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \in [0,T]}$ is a complete filtration of $\bar{\mathcal{F}}$,

$(\bar{w}^k)_{k\in\mathbb{N}}$ is a sequence of independent $\bar{\mathbb{F}}$-Wiener processes, and $\bar{u}$ is an H_x^{-1}-valued continuous $\bar{\mathbb{F}}$-adapted process, that furthermore belongs to $L_{t,x}^{m+1}$ almost surely and such that for all $\phi \in L_x^{m+1}$ almost surely the equality

$$\big(\bar{u}(t), \phi\big)_{H_x^{-1}} = \big(\bar{u}(0), \phi\big)_{H_x^{-1}} - \int_0^t \big(\bar{u}^{[m]}(s), \phi\big)\, ds + \sum_{k\in\mathbb{N}} \int_0^t \big(\sigma\big(\bar{u}(s)\big)e^k, \phi\big)_{H_x^{-1}}\, dw_s^k$$

holds for all $t \in [0, T]$, and $\bar{u}(0) \stackrel{d}{=} u^{(0)}$.

For the definition to be meaningful, some assumption of σ has to be imposed. It is a consequence of Lemma 3.4 below that if σ is of polynomial growth with order at most $(m+1)/2$, then for all $t \in [0, T]$ the series of stochastic integrals converge in probability.

Remark 2.4 (On the notion of solution). Let u be a solution in the sense of Definition 2.1. On the subset of $\Omega \times [0, T]$ where $u(t)$ belongs to L_x^{m+1}, we see by virtue of the definition of the inner product in H_x^{-1} (induced by (1.2))

$$\big(u(t), \phi\big)_{H_x^{-1}} = \sum_{k\in\mathbb{N}} \lambda_k^{-1}\big(u(t), e^k\big)\big(\phi, e^k\big) = \big(u(t), (-\Delta)^{-1}\phi\big),$$

and similarly

$$\big(\sigma\big(u(s)\big)e^k, \phi\big)_{H_x^{-1}} = \big(\sigma\big(u(s)\big)e^k, (-\Delta)^{-1}\phi\big),$$

provided, say, that σ has polynomial growth with order at most $(m+1)/2$. This, combined with the fact that $(-\Delta)^{-1}(-\Delta)\psi = \psi$ whenever $\psi \in H_x^\gamma$, for any $\gamma \in \mathbb{R}$, implies that if we choose $\phi = -\Delta\psi$ with $\psi \in C_c^\infty(I)$ in Definition 2.1, we get

$$\big(u(t), \psi\big) = \big(u(s), \psi\big) + \int_s^t \big(u^{[m]}(s), \Delta\psi\big)\, ds + \sum_{k\in\mathbb{N}} \int_s^t \big(\sigma\big(u(s)\big)e^k, \psi\big)\, dw_s^k$$

for almost all $\omega \in \Omega$ and almost all $(s, t) \in [0, T]^2$ with $s < t$. Therefore, u is a distributional solution of (1.1). Moreover, the Dirichlet boundary condition is encoded in the formulation in the following weak sense: for all $s < t$, almost surely

$$(-\Delta)^{-1}\left(u(t) - u(s) - \sum_{k\in\mathbb{N}} \int_s^t \sigma\big(u(r)\big)e^k\, dw_r^k\right) = \int_s^t u^{[m]}(r)\, dr.$$

Notice that the left hand side of the above equality is an element of H_x^1 (in particular it vanishes on ∂I) and therefore so is the right hand side.

The assumption on the diffusion coefficient is formulated via the following notation. For $\delta \ge 0$, let $S(\delta, m)$ be the set of continuous functions $\sigma : I \times \mathbb{R} \to \mathbb{R}$ such that there exists a $K \ge 0$ such that for all $x \in I$, $r \in \mathbb{R}$,

$$\big|\sigma(x, r)\big| \le K + \delta|r|^{(m+1)/2}. \tag{2.1}$$

Our first main result reads as follows.

Theorem 2.5. *For any $\gamma \in (-1, -1/2)$ there exists a $\delta_0 = \delta_0(\gamma, m) > 0$ such that the following holds. Let $\sigma \in S(\delta, m)$ with $\delta \le \delta_0$ and let $u^{(0)} \in L_\omega^{m+1}H_x^\gamma$ be $\mathcal{F}_0$-measurable. Then, there exists a weak solution $\{(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}), \bar{\mathbb{F}}, (\bar{w}^k)_{k\in\mathbb{N}}, \bar{u}\}$ of equation (1.1). Moreover, $\bar{u}$ satisfies the following bounds*:

(i) (*Energy estimates.*) *For all $p \in [0, m+1]$ there exists a constant $N = N(\gamma, p, m, K, T)$ such that*

$$\bar{\mathbb{E}}\|\bar{u}\|_{L_t^\infty H_x^\gamma}^p + \bar{\mathbb{E}}\big\|\bar{u}^{[\frac{m+1}{2}]}\big\|_{L_t^2 H_x^{1+\gamma}}^p + \bar{\mathbb{E}}\|\bar{u}\|_{L_t^{m+1} W_x^{\gamma',m+1}}^{p(m+1)/2} \le N\big(\mathbb{E}\big\|u^{(0)}\big\|_{H^\gamma}^p + 1\big), \tag{2.2}$$

where $\gamma' = \frac{2(1+\gamma)}{m+1}$.

(ii) (*Coming down from infinity.*) *There exists a constant* $N = N(m, T)$ (*in particular, independent of the initial condition*) *such that for all* $t \in [0, T]$

$$\bar{\mathbb{E}}\big\|\bar{u}(t)\big\|^2_{H^\gamma_x} \le N t^{-2/(m-1)}. \tag{2.3}$$

(iii) (*Temporal regularity.*) *For all* $\varepsilon > 0$ *there exists an* $\varepsilon' > 0$ *and a constant* $N = N(\gamma, m, K, T)$ *such that*

$$\bar{\mathbb{E}}\|\bar{u}\|^{\frac{m+1}{m}}_{C^{\varepsilon'}_t H^{\gamma-\varepsilon}_x} \le N\big(\mathbb{E}\big\|u^{(0)}\big\|^{m+1}_{H^\gamma} + 1\big).$$

Remark 2.6. In light of the smallness assumptions on δ above, it is worth noting that if σ is continuous and has polynomial growth with exponent $m' < (m+1)/2$, then $\sigma \in S(\delta, m)$ for all $\delta > 0$.

Remark 2.7. The reader may notice that the estimates (2.2) are stronger than what the standard theory [25] yields for *nondegenerate* quasilinear space–time white noise driven equations

$$\partial_t u = \partial^2_x\big(A(u)\big) + \sigma(x, u)\xi. \tag{2.4}$$

Indeed, if A' takes values in $[\lambda, \lambda^{-1}]$ for some $\lambda > 0$ and σ is sufficiently smooth and small, then [25] provides well-posedness in the Gelfand triple $L^2_x \subset H^{-1}_x \equiv (H^{-1}_x)^* \subset (L^2_x)^*$, and therefore gives bounds in $C_t H^{-1}_x \cap L^2_{t,x}$. We leave it as an exercise to the reader to check that our argument carries through for (2.4) and one can obtain the estimates (2.2) with $m = 1$, thus gaining (almost) $1/2$ regularity compared to [25].

In Proposition 2.8 below we show that the monotone operator approach of [25,31] can be applied to a small but nontrivial class of nonlinear diffusion coefficients to obtain (probabilistically) strong well-posedness. Let us point out a key difference to [2]: Therein, assumptions on the drift operator to be coercive/monotone and the diffusion operator to be bounded/Lipschitz are considered separately. In contrast, by virtue of the elementary estimate from Lemma 3.4 below, we use joint coercivity/monotonicity conditions for the drift and diffusion, in the spirit of the so-called stochastic parabolicity (see, e.g. [25,31]). Apart from the additive noise case, which directly follows from the monotone operator theory, the class of diffusion coefficients considered here contains some interesting cases – for example, $\sigma(r) = \lambda r^{[\frac{m+1}{2}]}$ with sufficiently small λ – but the conditions on σ are certainly restrictive. The exponent $(m+1)/2$ guarantees that the noise shuts down sufficiently fast when the solution approaches zero, the region where the regularizing effect of the second order operator fades. For $\bar{\delta} \ge 0$, we set $\bar{S}(\bar{\delta}, m)$ to be set of continuous functions $\sigma : I \times \mathbb{R} \to \mathbb{R}$ such that for all $x \in I$, $r, \bar{r} \in \mathbb{R}$,

$$\big|\sigma(x, r) - \sigma(x, \bar{r})\big| \le \bar{\delta}\big|r^{[\frac{m+1}{2}]} - \bar{r}^{[\frac{m+1}{2}]}\big|. \tag{2.5}$$

Under the additional assumption that σ also belongs to $\bar{S}(\bar{\delta}, m)$, we have the next theorem.

Proposition 2.8. *Let* $\sigma \in S(\delta, m) \cap \bar{S}(\bar{\delta}, m)$ *with* $\delta < 6$ *and* $\bar{\delta} \le 24\frac{m}{(m+1)^2}$, *and let* $u^{(0)} \in L^2_\omega H^{-1}_x$ *be* $\mathcal{F}_0$*-measurable. Then there exists a unique strong solution to* (1.1).

Remark 2.9. In continuation of Remark 2.6, $\sigma \in \bar{S}(\bar{\delta}, m)$ requires higher exponents: if $\sigma(r) = r^{[m']}$ around $r = 0$, then one needs $m' \ge (m+1)/2$. For example, this excludes the linear multiplicative case $\sigma(u) = u$.

3. A priori estimates

In this section we derive a priori estimates for approximations of (1.1). Take some $\nu \in (0, 1]$, $n \in \mathbb{N}$, and consider the equation

$$dv = \partial^2_x\big(\nu v + v^{[m]}\big)\,dt + \sum_{k=1}^{n} \sigma(x, v)e^k\,dw^k_t \tag{3.1}$$

on Q with homogeneous Dirichlet boundary conditions and with initial condition $v_0 = v^{(0)}$.

Assumption 3.1. The initial condition $v^{(0)}$ belongs to the space $L^{m+1}_{\omega,x}$, is $\mathcal{F}_0$-measurable, and σ is Lipschitz continuous.

Definition 3.2. An L^2-solution of equation (3.1) is an $\mathbb{F}$-adapted, continuous L_x^2-valued process v, such that almost surely $v, v^{[m]} \in L_t^2 H_x^1$ and such that for all $\phi \in H_x^1$ we have with probability one

$$\big(v(t),\phi\big) = \big(v^{(0)},\phi\big) - \int_0^t \big(\partial_x\big(\nu v(s) + v^{[m]}(s)\big), \partial_x\phi\big)\,ds + \sum_{k=1}^n \int_0^t \big(\sigma\big(v(s)\big)e^k,\phi\big)\,dw_s^k.$$

for all $t \in [0,T]$.

By [10, Theorem 3.1, Remark 5.6], under Assumption 3.1, equation (3.1) admits an L^2-solution v. Moreover, on the basis of Itô's formula for the function $u \mapsto \|u\|_{L_x^2}^2$, one can easily derive (see in the Appendix) that for $p \in [0, m+1]$

$$\mathbb{E}\|v\|_{C_t L_x^2}^p + \mathbb{E}\|v\|_{L_t^2 H_x^1}^p + \mathbb{E}\big\|v^{[\frac{m+1}{2}]}\big\|_{L_t^2 H_x^1}^p < \infty. \tag{3.2}$$

The main result of this section is the following.

Lemma 3.3. *For any $\gamma \in [-1,-1/2)$ there exists a $\delta_0 = \delta_0(\gamma,m)$ such that the following holds. Let σ and $v^{(0)}$ satisfy Assumption 3.1 and $\sigma \in S(\delta,m)$ with $\delta \le \delta_0$. Then, for all $p \in [0,m+1]$ the solution v of (3.1) satisfies the bound*

$$\mathbb{E}\|v\|_{L_t^\infty H_x^\gamma}^p + \mathbb{E}\big\|v^{[\frac{m+1}{2}]}\big\|_{L_t^2 H_x^{1+\gamma}}^p \le N\big(\mathbb{E}\big\|v^{(0)}\big\|_{H_x^\gamma}^p + 1\big), \tag{3.3}$$

with some $N = N(\gamma,p,m,K,T)$. Moreover, with the notation $\gamma' = \frac{2(1+\gamma)}{m+1}$, one also has the bound

$$\mathbb{E}\|v\|_{L_t^{m+1} W_x^{\gamma',m+1}}^{p(m+1)/2} \le N\big(\mathbb{E}\big\|v^{(0)}\big\|_{H_x^\gamma}^p + 1\big). \tag{3.4}$$

First we collect some auxiliary statements. The following lemma is part of [24, Lem 8.4] with $\mathbb{R}$ in place of I and $(1-\Delta)$ in place of $(-\Delta)$. In our setting the proof is particularly short, so we include it for the sake of completeness.

Lemma 3.4. *For all $\tilde\gamma < -1/2$ there exists a constant $N = N(\tilde\gamma)$ such that for all $u \in L_x^2$ one has*

$$\sum_{k\in\mathbb{N}} \big\|ue^k\big\|_{H_x^{\tilde\gamma}}^2 \le N\|u\|_{L_x^2}^2. \tag{3.5}$$

Moreover, one has $N(-1) = 1/3$.

Proof. By the definition of the norm in $H_x^{\tilde\gamma}$ and Parseval's identity we get

$$\sum_{k\in\mathbb{N}} \big\|ue^k\big\|_{H_x^{\tilde\gamma}}^2 = \sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}} \lambda_l^{\tilde\gamma}\big|\big(ue^k,e^l\big)\big|^2 = \sum_{l\in\mathbb{N}} \lambda_l^{\tilde\gamma}\big\|ue^l\big\|_{L_x^2}^2 \le N\|u\|_{L_x^2}^2,$$

where we have used that for all $l \in \mathbb{N}$, $\|e^l\|_{L_x^\infty} = \sqrt{2}$ and $\lambda_l = (\pi l)^2$. □

Next is a bound to deduce the regularity of a function from the regularity of its monotone power.

Lemma 3.5. *Let u be such that $u^{[\tilde m]} \in H_x^{\tilde\gamma}$ with $\tilde\gamma \in [0,1/2)$ and $\tilde m \in (1,\infty)$. Then $u \in W_x^{\tilde\gamma/\tilde m, 2\tilde m}$ and there exists a constant $N = N(\tilde m)$ such that the following bound holds*

$$\|u\|_{W_x^{\tilde\gamma/\tilde m,2\tilde m}}^{2\tilde m} \le N\big\|u^{[\tilde m]}\big\|_{H_x^{\tilde\gamma}}^2. \tag{3.6}$$

Proof. The $\tilde\gamma = 0$ case is trivial. Recall the elementary inequality $|a-b|^{2\tilde m} \le 2^{2\tilde m}|a^{[\tilde m]} - b^{[\tilde m]}|^2$. Indeed, this can be seen by noticing that the function $\phi(r) = r^{[\tilde m]}$ is odd, increasing, and has increasing derivative on $[0,\infty)$. Therefore for $0 \le a \le b$, one has $\phi(b) - \phi(a) = \int_a^b \phi'(r)\,dr \ge \int_0^{b-a}\phi'(r)\,dr = \phi(b-a)$, while for $a \le 0 \le b$ one has $\phi(b-a) \le 2^{\tilde m}\max(\phi(b),\phi(-a)) \le 2^{\tilde m}(\phi(b)-\phi(a))$. Therefore, in the case $\tilde\gamma \in (0,1/2)$, using the equivalent norm (1.4), to write

$$\|u\|_{W_x^{\tilde\gamma/\tilde m,2\tilde m}}^{2\tilde m} \le N\|u\|_{L_x^{2\tilde m}}^{2\tilde m} + N\int_{I\times I} \frac{|u(x)-u(y)|^{2\tilde m}}{|x-y|^{1+2\tilde m(\tilde\gamma/\tilde m)}}\,dy\,dx$$

$$\leq N\|u^{[\tilde{m}]}\|^2_{L^2_x} + N\int_{I\times I}\frac{|u(x)^{[\tilde{m}]}-u(y)^{[\tilde{m}]}|^2}{|x-y|^{1+2\tilde{\gamma}}}\,dy\,dx$$

$$\leq N\|u^{[\tilde{m}]}\|^2_{W^{\tilde{\gamma},2}_x} \leq N\|u^{[\tilde{m}]}\|^2_{H^{\tilde{\gamma}}_x}. \qquad \square$$

The next tool is the Stroock–Varopoulos lemma. It appears and is proved in various forms in the literature, so for the convenience of the reader we give the proof in the appendix, following the standard proof strategy (see e.g. [11, Section 5]) based on the Cafarelli–Silvestre extension [8]. An alternative, more elementary strategy can be found in e.g. [5, App B2].

Lemma 3.6 (Stroock–Varopoulos). *Let $\beta\in(0,1/2)$ and let $f,g\in C^1(\mathbb{R})$ satisfy $f'=(g')^2$. Then, for any $u\in H^1_x$ such that $f(u),g(u)\in H^1_x$, we have*

$$\int_I f(u)(-\Delta)^\beta u\,dx \geq \int_I\left|(-\Delta)^{\beta/2}g(u)\right|^2(x)\,dx. \tag{3.7}$$

The particular form that we use below is

$$\int_I u^{[m]}(-\Delta)^\beta u\,dx \geq \frac{4m}{(m+1)^2}\int_I\left|(-\Delta)^{\beta/2}u^{[\frac{m+1}{2}]}\right|^2(x)\,dx, \tag{3.8}$$

the more general form (3.7) can be useful for general nonlinear leading operators, for example in the context of Remark 2.7. We can now prove the a priori bounds (3.3)–(3.4).

Proof of Lemma 3.3. We test the equation with e^l and apply Itô's formula for the square to obtain

$$\begin{aligned}\left|\left(v(s),e^l\right)\right|^2 &= \left|\left(v^{(0)},e^l\right)\right|^2 - \int_0^s 2\lambda_l\left(\nu v(r)+v^{[m]}(r),e^l\right)\left(v(r),e^l\right)dr\\ &\quad+\sum_{k=1}^n\int_0^s\left|\left(\sigma\left(v(r)\right)e^k,e^l\right)\right|^2dr\\ &\quad+\sum_{k=1}^n 2\int_0^s\left(\sigma\left(v(r)\right)e^k,e^l\right)\left(v(r),e^l\right)dw^k_r.\end{aligned}$$

We multiply the above equality with λ_l^γ and we sum over l to obtain

$$\begin{aligned}\|v(s)\|^2_{H^\gamma_x} &= \|v^{(0)}\|^2_{H^\gamma_x} - 2\nu\int_0^s\left(v(r),(-\Delta)^{1+\gamma}v(r)\right)dr\\ &\quad-2\int_0^s\left(v^{[m]}(s),(-\Delta)^{1+\gamma}v(r)\right)dr+\int_0^s\sum_{k=1}^n\left\|\sigma\left(v(r)\right)e^k\right\|^2_{H^\gamma_x}dr\\ &\quad+\sum_{k=1}^n 2\int_0^s\left((-\Delta)^{\gamma/2}v(r),(-\Delta)^{\gamma/2}\left(\sigma\left(v(r)\right)e^k\right)\right)dw^k_r.\end{aligned} \tag{3.9}$$

Denote the last term by M_s. The first integral on the right-hand side is nonnegative so we simply bound it by 0. For the second one we apply (3.8) (with $\beta=\gamma$) and for the third we use Lemma 3.4 (with $\tilde{\gamma}=\gamma$). We therefore get

$$\begin{aligned}\|v(s)\|^2_{H^\gamma_x} &\leq \|v^{(0)}\|^2_{H^\gamma_x} - \frac{8m}{(m+1)^2}\left\|v^{[\frac{m+1}{2}]}\right\|^2_{L^2([0,s];H^{1+\gamma}_x)} + \bar{N}(\gamma)\|\sigma(v)\|^2_{L^2([0,s];L^2_x)} + M_s\\ &\leq \|v^{(0)}\|^2_{H^\gamma_x} - \frac{8m}{(m+1)^2}\left\|v^{[\frac{m+1}{2}]}\right\|^2_{L^2([0,s];H^{1+\gamma}_x)}\\ &\quad+\bar{N}(\gamma)\delta\|v\|^{m+1}_{L^{m+1}([0,s];L^{m+1}_x)}+N+M_s.\end{aligned}$$

Notice that $\frac{8m}{(m+1)^2} > \frac{2}{m}$. Assuming δ is small enough so that $\bar{N}(\gamma)\delta \le \frac{1}{m}$, we obtain

$$\|v(s)\|^2_{H^\gamma_x} + \frac{1}{m}\|v^{[\frac{m+1}{2}]}\|^2_{L^2([0,s];H^{1+\gamma}_x)} \le \|v^{(0)}\|^2_{H^\gamma_x} + N + M_s. \tag{3.10}$$

The quadratic variation process $\langle M\rangle_s$ of the local martingale M_s is given by

$$\begin{aligned}\langle M\rangle_s &= 4\int_0^s \sum_{k=1}^n \big((-\Delta)^{\gamma/2}v(r), (-\Delta)^{\gamma/2}\big(\sigma\big(v(r)\big)e^k\big)\big)^2\,dr \\ &\le 4\int_0^s \|v(r)\|^2_{H^\gamma_x}\sum_{k\in\mathbb{N}}\|\sigma\big(v(r)\big)e^k\|^2_{H^\gamma_x}\,dr \\ &\le 4\bar{N}(\gamma)\int_0^s \|v(r)\|^2_{H^\gamma_x}\|\sigma\big(v(r)\big)\|^2_{L^2_x}\,dr,\end{aligned}$$

where we used Lemma 3.4 as before. Using the bound (2.1) to bound $\|\sigma(v(r))\|^2_{L^2_x}$, we obtain the bound

$$\langle M\rangle_s \le 4\bar{N}(\gamma)\delta\|v\|^2_{L^\infty([0,s];H^\gamma_x)}\|v\|^{m+1}_{L^{m+1}([0,s];L^{m+1}_x)} + N\|v\|^2_{L^2([0,s];H^\gamma_x)} \tag{3.11}$$

In particular, by (3.2), M_s is a martingale. Denote $X_s = \|v\|_{L^\infty([0,s],H^\gamma_x)}$ and $Y_s = \|v^{[\frac{m+1}{2}]}\|_{L^2([0,s];H^{1+\gamma}_x)}$. By (3.10), (3.11) and the Burkholder–Gundy–Davis and Jensen inequalities, we have for any stopping time $\tau \le T$,

$$\begin{aligned}\mathbb{E}X^{m+1}_{s\wedge\tau} + \frac{1}{m^{(m+1)/2}}\mathbb{E}Y^{m+1}_{s\wedge\tau} &\le N\mathbb{E}\|v^{(0)}\|^{m+1}_{H^\gamma_x} + N + N\int_0^s \mathbb{E}X^{m+1}_{r\wedge\tau}\,dr \\ &\quad + \tilde{N}(\gamma,m)\delta^{(m+1)/4}\mathbb{E}\bigg(X^{(m+1)/2}_{s\wedge\tau}\frac{1}{m^{(m+1)/4}}Y^{(m+1)/2}_{s\wedge\tau}\bigg).\end{aligned}$$

By (3.2), we have that $\mathbb{E}X^{m+1}_{s\wedge\tau} + \mathbb{E}Y^{m+1}_{s\wedge\tau} < \infty$. Therefore, assuming δ is small enough so that $\tilde{N}(p,m)\delta^{(m+1)/4} \le 1$, we can apply Young's inequality for the last term and absorb it in the left-hand side. We get

$$\mathbb{E}X^{m+1}_{s\wedge\tau} + \mathbb{E}Y^{m+1}_{s\wedge\tau} \le N\mathbb{E}\|v^{(0)}\|^{m+1}_{H^\gamma_x} + N + N\int_0^s \mathbb{E}X^{m+1}_{r\wedge\tau}\,dr.$$

Applying Gronwall's inequality for the function $s \mapsto \mathbb{E}X^{m+1}_{s\wedge\tau} + \mathbb{E}Y^{m+1}_{s\wedge\tau}$, yields

$$\mathbb{E}X^{m+1}_{\tau} + \mathbb{E}Y^{m+1}_{\tau} \le N\mathbb{E}\|v^{(0)}\|^{m+1}_{H^\gamma_x} + N. \tag{3.12}$$

By choosing $\tau = T$ we have (3.3) for $p = m+1$.

The case $p < m+1$ then follows by Lenglart's inequality (see, e.g., [33, Prop IV.4.7 and Ex IV.4.31/1]). To avoid clash of notation, we use boldface for the objects in [33], and then chose $\mathbf{X}_t = X^{m+1}_{t\wedge T} + Y^{m+1}_{t\wedge T}$, $\mathbf{A}_t = N(\|v^{(0)}\|^{m+1}_{H^\gamma_x} + 1)$, and $\mathbf{k} = p/(m+1)$. The bound (3.12) then yields $\mathbb{E}\mathbf{X}_\tau \le \mathbb{E}\mathbf{A}_\tau$, and in fact one can easily obtain $\mathbb{E}\mathbf{1}_B\mathbf{X}_\tau \le \mathbb{E}\mathbf{1}_B\mathbf{A}_\tau$ for any $\mathcal{F}_0$-measurable event B by following the exact same argument. Therefore, $\mathbb{E}(\mathbf{X}_\tau|\mathcal{F}_0) \le \mathbb{E}(\mathbf{A}_\tau|\mathcal{F}_0)$, which is precisely the condition of Lenglart's inequality, and the conclusion is $\mathbb{E}\mathbf{X}^{\mathbf{k}}_T \le N\mathbb{E}\mathbf{A}^{\mathbf{k}}_T$, which is (3.3) in disguise.

Finally, the bound (3.4) follows by applying Lemma 3.5 with $\tilde{\gamma} = 1+\gamma$, $\tilde{m} = (m+1)/2$. □

Corollary 3.7. *Take $\gamma \in (-1,-1/2)$. Let σ and $v^{(0)}$ satisfy Assumption 3.1 and $\sigma \in S(\delta,m)$ with $\delta \le \delta_0$. Then, there exists a $c = c(\gamma,m) > 1$ such that*

$$\|v\|_{L^{c(m+1)}_{\omega,t,x}} \le N\big(\|v^{(0)}\|_{L^{m+1}_\omega H^\gamma_x} + 1\big) \tag{3.13}$$

with some $N = N(\gamma,p,m,K,T)$.

Proof. Recall the notation $\gamma' = \frac{2(1+\gamma)}{m+1}$ from Lemma 3.3. By the standard interpolation properties of L^p spaces and (1.5)

$$\|v\|_{L_\omega^{\frac{(m+1)^2}{2}-\varepsilon_1(\theta)}L_t^{m+1+\varepsilon_2(\theta)}W_x^{\gamma'-\varepsilon_3(\theta),m+1-\varepsilon_4(\theta)}} \le N\|v\|^{1-\theta}_{L^{m+1}_\omega L^\infty_t W^{\gamma,2}_x}\|v\|^\theta_{L_\omega^{\frac{(m+1)^2}{2}}L^{m+1}_t W^{\gamma',m+1}_x}$$

$$= N\|v\|^{1-\theta}_{L^{m+1}_\omega L^\infty_t H^\gamma_x}\|v\|^{\theta}_{L_\omega^{\frac{(m+1)^2}{2}} L_t^{m+1} W_x^{\gamma',m+1}},$$

where in the last step we used Proposition 1.1(iii) and (iv). Here $\theta \in (0,1)$ and $\varepsilon_i(\theta) > 0$ such that $\varepsilon_i(\theta) \to 0$ as $\theta \to 1$. Since $\gamma > -1$ implies $\gamma' > 0$, we can choose θ sufficiently close to 1 such that

$$\frac{(m+1)^2}{2} - \varepsilon_1(\theta) > m+1, \qquad \gamma' - \varepsilon_3(\theta) - \frac{1}{m+1-\varepsilon_4(\theta)} > -\frac{1}{m+1}.$$

By Sobolev's embedding we then see that for some $c > 1$,

$$\|v\|_{L^{c(m+1)}_{\omega,t,x}} \le N\|v\|_{L_\omega^{\frac{(m+1)2}{2}-\varepsilon_1(\theta)} L_t^{m+1+\varepsilon_2(\theta)} W_x^{\gamma'-\varepsilon_3(\theta),m+1-\varepsilon_4(\theta)}}.$$

On the other hand, (3.3)–(3.4) yields

$$\|v\|^{1-\theta}_{L^{m+1}_\omega L^\infty_t H^\gamma_x}\|v\|^{\theta}_{L_\omega^{\frac{(m+1)2}{2}} L_t^{m+1} W_x^{\gamma',m+1}} \le N\big(\big\|v^{(0)}\big\|_{L^{m+1}_\omega H^\gamma_x} + 1\big)^{(1-\theta)+\theta\frac{2}{m+1}},$$

and putting the above bounds together we readily get (3.13). □

3.1. *Time regularity*

The following is a simple variant of [14, Lem 2.1]. While in [14] only the $q' = q$ case is stated, the form below is easily obtained via Lenglart's inequality similarly to the end of the proof of Lemma 3.3, with the choice $\mathbf{X}_r = \|t \mapsto \sum_{k\in\mathbb{N}} \int_0^t f^k(s)\,dw^k_s\|^q_{W^{\alpha,q}([0,r\wedge T],H)}$, $\mathbf{A}_r = \|f\|^q_{L^q([0,r\wedge T],\ell^2(H))}$.

Lemma 3.8. *Let H be a separable Hilbert space, $q \ge 2$, and $f = (f^k)_{k=1}^\infty$ be a progressively measurable $\ell^2(H)$-valued process such that $f \in L^q(\Omega\times[0,T],\ell^2(H))$. Then, for all $\alpha < 1/2$ and $q' \in [0,q]$ there exists $N = N(\alpha,q,q')$ such that*

$$\mathbb{E}\|t \mapsto \sum_{k\in\mathbb{N}} \int_0^t f^k(s)\,dw^k_s\|^{q'}_{W^{\alpha,q}([0,T],H)} \le N\mathbb{E}\|f\|^{q'}_{L^q([0,T],\ell^2(H))}.$$

Corollary 3.9. *Take $\gamma \in (-1,1/2)$. Let σ and $v^{(0)}$ satisfy Assumption 3.1 and $\sigma \in S(\delta,m)$ with $\delta \le \delta_0$. Let, furthermore, $\alpha < 1/2$, $\beta > 5/2$, and define the space*

$$\mathcal{X} = W^{1,\frac{m+1}{m}}\big([0,T];H_x^{-\beta}\big) + W^{\alpha,2c}\big([0,T];H_x^{-1}\big), \tag{3.14}$$

where c is as in Corollary 3.7. Then, the solution v of (3.1) satisfies the bound

$$\mathbb{E}\|v\|_{\mathcal{X}}^{\frac{m+1}{m}} \le N\big(\mathbb{E}\big\|v^{(0)}\big\|^{m+1}_{H^\gamma_x} + 1\big) \tag{3.15}$$

with some $N = N(\alpha,\beta,m,K,T)$.

Proof. We apply Lemma 3.8 with $f^k(s) = \sigma(v(s))e^k\mathbf{1}_{k\le n}$, $H = H_x^{-1}$, $q = 2c$, and $q' = 2$, to get that

$$\begin{aligned}
&\mathbb{E}\|s \mapsto \sum_{k=1}^n \int_0^s \sigma\big(v(r)\big)e^k\,dw^k_r\|^2_{W^{\alpha,2c}_t H^{-1}_x} \\
&\quad\le N\mathbb{E}\left(\int_0^T \left(\sum_{k=1}^n \big\|\sigma\big(v(s)\big)e^k\big\|^2_{H^{-1}_x}\right)^c ds\right)^{1/c} \\
&\quad\le N\big(\mathbb{E}\big\|v^{[\frac{m+1}{2}]}\big\|^2_{L^{2c}_t L^2_x} + 1\big) \\
&\quad= N\big(\mathbb{E}\|v\|^{m+1}_{L^{c(m+1)}_t L^{m+1}_x} + 1\big),
\end{aligned} \tag{3.16}$$

where we have used Lemma 3.4 and the growth of σ in the last inequality. On the other hand, one easily sees that

$$\begin{aligned}
&\mathbb{E}\|s\mapsto\int_0^s \Delta\big(\nu v(r)+v^{[m]}(r)\big)\,dr\|^{\frac{m+1}{m}}_{W_t^{1,\frac{m+1}{m}}H_x^{-\beta}} \\
&\quad\le N\mathbb{E}\int_0^T \big\|\Delta\big(\nu v(r)+v^{[m]}(r)\big)\big\|^{\frac{m+1}{m}}_{H_x^{-\beta}}\,dt \\
&\quad\le N\mathbb{E}\int_0^T\Big(\sum_{k\in\mathbb{N}}\lambda_k^{2-\beta}\big(\nu v(r)+v^{[m]}(r),e^k\big)^2\Big)^{\frac{m+1}{2m}}\,dt \\
&\quad\le N\big(\mathbb{E}\|v^{[\frac{m+1}{2}]}\|^2_{L^2_{t,x}}+1\big)\Big(\sum_{k\in\mathbb{N}}k^{4-2\beta}\Big)^{\frac{m+1}{2m}}.
\end{aligned}\tag{3.17}$$

Since $\beta>5/2$, the last sum is finite. By (3.13), the right-hand-side of both (3.16) and (3.17) are bounded as in (3.15), hence the proof is finished. □

4. Limiting procedure

Proof of Theorem 2.5. Let $\sigma_n : I\times\mathbb{R}\to\mathbb{R}$ be bounded smooth functions with bounded derivatives such that $\sigma_n\to\sigma$ uniformly on compacts as $n\to\infty$ and for all $x\in I$, $r\in\mathbb{R}$

$$\sup_n\big|\sigma_n(x,r)\big|\le K+\delta|r|^{(m+1)/2},\tag{4.1}$$

and let $u_n^{(0)}$ be $\mathcal{F}_0$-measurable random variables with $\mathbb{E}\|u_n^{(0)}\|^{m+1}_{L_x^{m+1}}<\infty$ and

$$\lim_{n\to\infty}\mathbb{E}\big\|u_n^{(0)}-u^{(0)}\big\|^{m+1}_{H_x^\gamma}=0.\tag{4.2}$$

Let u_n be an L^2-solution of

$$\begin{aligned}
du_n&=\partial_x^2\big(n^{-1}u_n+u_n^{[m]}\big)\,dt+\sum_{k=1}^n\sigma_n(x,u_n)e^k\,dw^k,\\
u_n(0)&=u_n^{(0)}.
\end{aligned}\tag{4.3}$$

Take $\alpha\in(1/(2c),1/2)$, where c is as in Corollary 3.7, $\beta\in(5/2,3)$, and set $\mathcal{X}$ as in (3.14). Let us set

$$\mathcal{Y}=L_t^{m+1}W_x^{\gamma',m+1}\cap W_t^{\alpha,\frac{m+1}{m}}H_x^{-\beta},$$

where $\gamma'=2(1+\gamma)/(m+1)$, as in Lemma 3.3. By [14, Thms 2.1–2.2] we have the compact embeddings

$$\mathcal{Y}\Subset L^{m+1}_{t,x},\qquad \mathcal{X}\Subset C_tH_x^{-3}.\tag{4.4}$$

Therefore,

$$\mathcal{Y}\cap\mathcal{X}\Subset L^{m+1}_{t,x}\cap C_tH_x^{-3}=:\mathcal{Z}.$$

Notice that $\mathcal{X}\subset W_t^{\alpha,\frac{m+1}{m}}H_x^{-\beta}$. Therefore, by (3.4) and (3.15) we have the estimate

$$\mathbb{E}\|u_n\|^{\frac{m+1}{m}}_{\mathcal{X}\cap\mathcal{Y}}\le N\big(\mathbb{E}\|u_n^{(0)}\|^{m+1}_{H_x^\gamma}+1\big)\le N\big(\mathbb{E}\|u^{(0)}\|^{m+1}_{H_x^\gamma}+1\big),\tag{4.5}$$

which in turn implies that the laws of $(u_n)_{n\in\mathbb{N}}$ on $\mathcal{Z}$ are tight. Let us set

$$w(t)=\sum_{k\in\mathbb{N}}\frac{1}{\sqrt{2^k}}w^k(t)\mathfrak{e}_k,$$

where $(\mathfrak{e}_k)_{k\in\mathbb{N}}$ is the standard orthonormal basis of ℓ^2. By Prokhorov's theorem, there exists a (non-relabelled) subsequence $(u_n)_n$ such that the laws of (u_n, w) on $\mathcal{Z} \times C([0,T];\ell^2)$ are weakly convergent. By Skorohod's representation theorem, there exist $\mathcal{Z} \times C([0,T];\ell^2)$-valued random variables $(\bar{u}, \bar{w})$, $(\bar{u}_n, \bar{w}_n)$, for $n \in \mathbb{N}$, on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, such that in $\mathcal{Z} \times C([0,T];\ell^2)$, $\bar{\mathbb{P}}$-almost surely

$$(\bar{u}_n, \bar{w}_n) \to (\bar{u}, \bar{w}), \tag{4.6}$$

as $n \to \infty$, and for each $n \in \mathbb{N}$, as random variables in $\mathcal{Z} \times C([0,T];\ell^2)$

$$(\bar{u}_n, \bar{w}_n) \stackrel{d}{=} (u_n, w). \tag{4.7}$$

Moreover, upon passing to a subsequence, we may assume that

$$\bar{u}_n \to \bar{u} \quad \text{for almost all } (\bar{\omega}, t, x). \tag{4.8}$$

Let $(\bar{\mathcal{F}}_t)_{t\in[0,T]}$ be the augmented filtration of $\mathcal{G}_t := \sigma(\bar{u}(s), \bar{w}(s); s \le t)$, and let $\bar{w}^k(t) := \sqrt{2^k}(\bar{w}(t), \mathfrak{e}_k)_{\ell^2}$. It is easy to see that $\bar{w}^k$, $k \in \mathbb{N}$, are independent, standard, real-valued $\bar{\mathcal{F}}_t$-Wiener processes (see for example the argument in [10, Proof of Prop. 5.5]).

We now show that $\bar{u}$ is a weak solution. Notice that by virtue of the a priori estimates (3.3)–(3.4) and (4.7) we have

$$\bar{\mathbb{E}}\|\bar{u}_n\|^2_{L^\infty_t H^\gamma_x} + \bar{\mathbb{E}}\big\|\bar{u}_n^{[\frac{m+1}{2}]}\big\|^2_{L^2_t H^{1+\gamma}_x} + \bar{\mathbb{E}}\|\bar{u}_n\|^{m+1}_{L^{m+1}_t W^{\gamma',m+1}_x} \le N\big(\mathbb{E}\big\|u^{(0)}\big\|^2_{H^\gamma_x} + 1\big), \tag{4.9}$$

which, by the lower semicontinuity of the norms, gives

$$\bar{\mathbb{E}}\|\bar{u}\|^2_{L^\infty_t H^\gamma_x} + \bar{\mathbb{E}}\big\|\bar{u}^{[\frac{m+1}{2}]}\big\|^2_{L^2_t H^{1+\gamma}_x} + \bar{\mathbb{E}}\|\bar{u}\|^{m+1}_{L^{m+1}_t W^{\gamma',m+1}_x} \le N\big(\mathbb{E}\big\|u^{(0)}\big\|^2_{H^\gamma_x} + 1\big). \tag{4.10}$$

Let us set

$$M(\bar{u}, t) := \bar{u}(t) - \bar{u}(0) - \int_0^t \Delta\big(\bar{u}^{[m]}(s)\big)\,ds,$$

and for $v \in \{u_n, \bar{u}_n\}$,

$$M_n(v, t) := v(t) - v(0) - \int_0^t \Delta\big(n^{-1}\bar{v}(s) + \bar{v}^{[m]}(s)\big)\,ds.$$

Fix an arbitrary $l \in \mathbb{N}$. We will show that for any $\phi \in H^{-3}_x$, the processes

$$\begin{aligned}
M^1(\bar{u}, t) &:= \big(M(\bar{u}, t), \phi\big)_{H^{-3}_x},\\
M^2(\bar{u}, t) &:= \big(M(\bar{u}, t), \phi\big)^2_{H^{-3}_x} - \int_0^t \sum_{k\in\mathbb{N}} \big|\big(\sigma\big(\bar{u}(s)\big)e^k, \phi\big)_{H^{-3}_x}\big|^2\,ds,\\
\bar{M}^3(\bar{u}, t) &:= \bar{w}^l(t)\big(M(\bar{u}, t), \phi\big)_{H^{-3}_x} - \int_0^t \big(\sigma\big(\bar{u}(s)\big)e^l, \phi\big)_{H^{-3}_x}\,ds
\end{aligned} \tag{4.11}$$

are continuous $\bar{\mathcal{F}}_t$-martingales. We first show that they are continuous $\mathcal{G}_t$-martingales. Assume for now that $\phi \in C^\infty_c(I)$. For, $i = 1, 2, 3$ and $v \in \{u_n, \bar{u}_n\}$, let us also define the processes $M^i_n(v, t)$ similarly to $M^i(\bar{u}, t)$, but with $\bar{u}$, $M(\bar{u}, t)$, $\sigma(\bar{u})$ replaced by v, $M_n(v, t)$, $\sigma_n(v)$, and the corresponding summation in (4.11) going only up to n. Let us fix $s < t$ and let V be a bounded, continuous function on $C([0,s]; H^{-3}_x) \times C([0,s];\ell^2)$. We have that

$$\big(M_n(u_n, t), \phi\big)_{H^{-3}_x} = \sum_{k=1}^n \int_0^t \big(\sigma_n\big(u_n(s)\big)e^k, \phi\big)_{H^{-3}_x}\,dw^k_s.$$

It follows that $M^i_n(u_n, t)$ are continuous $\mathcal{F}_t$-martingales. Hence, for $i = 1, 2, 3$,

$$\mathbb{E}V(u_n|_{[0,s]}, w|_{[0,s]})\big(M^i_n(u_n, t) - M^i_n(u_n, s)\big) = 0,$$

which combined with (4.7) gives

$$\bar{\mathbb{E}}V(\bar{u}_n|_{[0,s]}, \bar{w}_n|_{[0,s]})\big(M_n^i(\bar{u}_n, t) - M_n^i(\bar{u}_n, s)\big) = 0. \tag{4.12}$$

Next, notice that $\bar{\mathbb{P}}$-almost surely

$$\begin{aligned}
&\int_0^T \big|\big(\Delta\big(n^{-1}\bar{u}_n(t) + \bar{u}_n^{[m]}(t) - \bar{u}^{[m]}(t)\big), \phi\big)_{H_x^{-3}}\big|\, dt \\
&\quad = \int_0^T \Big|\sum_{k\in\mathbb{N}} \lambda_k^{1-3}\big(n^{-1}\bar{u}_n(t) + \bar{u}_n^{[m]}(t) - \bar{u}^{[m]}(t), e^k\big)\big(\phi, e^k\big)\Big|\, dt \\
&\quad \le N\|\phi\|_{L_x^\infty}\big(n^{-1}\|\bar{u}_n\|_{L^1_{t,x}} + \big\|\bar{u}_n^{[m]} - \bar{u}^{[m]}\big\|_{L^1_{t,x}}\big) \to 0,
\end{aligned} \tag{4.13}$$

where the convergence follows from (4.8) and the bounds (4.9). Hence, by (4.13) and (4.6) we see that for each $t \in [0, T]$, $\bar{\mathbb{P}}$-almost surely

$$\big(M_n(\bar{u}_n, t), \phi\big)_{H_x^{-3}} \to \big(M(\bar{u}, t), \phi\big)_{H_x^{-3}}. \tag{4.14}$$

In addition, it is easy to see that

$$\begin{aligned}
&\bar{\mathbb{E}}\int_0^T \Big|\sum_{k=1}^n \big(\sigma_n\big(\bar{u}_n(t)\big)e^k, \phi\big)^2_{H_x^{-3}} - \sum_{k\in\mathbb{N}} \big(\sigma\big(\bar{u}(t)\big)e^k, \phi\big)^2_{H_x^{-3}}\Big|\, dt \\
&\quad \le N\|\phi\|^2_{H_x^{-3}}\bar{\mathbb{E}}\int_0^T \sum_{k=n+1}^\infty \big\|\sigma\big(\bar{u}(t)\big)e^k\big\|^2_{H_x^{-3}}\, dt \\
&\qquad + N\|\phi\|^2_{H_x^{-3}}\bar{\mathbb{E}}\int_0^T \big\|\sigma\big(\bar{u}(t)\big) - \sigma_n\big(\bar{u}_n(t)\big)\big\|_{L_x^2}\big\|\sigma\big(\bar{u}(t)\big) + \sigma_n\big(\bar{u}_n(t)\big)\big\|_{L_x^2}\, dt,
\end{aligned} \tag{4.15}$$

where we have used also Lemma 3.4. The first term of the right hand side converges to zero as $n \to \infty$ by virtue of Lemma 3.4, (2.1) and (4.10). For the second term we have

$$\begin{aligned}
&\Big|\bar{\mathbb{E}}\int_0^T \big\|\sigma\big(\bar{u}(t)\big) - \sigma_n\big(\bar{u}_n(t)\big)\big\|_{L_x^2}\big\|\sigma\big(\bar{u}(t)\big) + \sigma_n\big(\bar{u}_n(t)\big)\big\|_{L_x^2}\, dt\Big|^2 \\
&\quad \le N\bar{\mathbb{E}}\big\|\sigma(\bar{u}) - \sigma_n(\bar{u}_n)\big\|^2_{L^2_{t,x}}\big(\bar{\mathbb{E}}\|\bar{u}\|^{m+1}_{L^{m+1}_{t,x}} + \bar{\mathbb{E}}\|\bar{u}_n\|^{m+1}_{L^{m+1}_{t,x}} + 1\big) \\
&\quad \le N\bar{\mathbb{E}}\big\|\sigma(\bar{u}) - \sigma_n(\bar{u}_n)\big\|^2_{L^2_{t,x}}\big(\mathbb{E}\big\|u^{(0)}\big\|^2_{H_x^\gamma} + 1\big),
\end{aligned}$$

where we have used (2.1), (4.1) and the bounds (4.9)–(4.10). By (4.8), the uniform convergence on compacts of σ_n to σ and the continuity of σ we have that $|\sigma_n(\bar{u}_n) - \sigma(\bar{u})|^2 \to 0$ for almost every $(\bar{\omega}, t, x)$. Moreover, by (2.1) and (4.1), we have

$$\big|\sigma_n(\bar{u}_n) - \sigma(\bar{u})\big|^2 \le N\big(1 + |\bar{u}_n|^{m+1} + |\bar{u}|^{m+1}\big).$$

Hence, to conclude that the right hand side of (4.15) converges to zero, it suffices to check that $|\bar{u}_n|^{m+1}$ are uniformly integrable in $(\bar{\omega}, t, x)$. This follows immediately from (3.13) and (4.7). Using (4.14) we can conclude that $M_n^2(\bar{u}_n, t) \to M^2(\bar{u}, t)$ in probability. Similarly one shows that $M_n^3(\bar{u}_n, t) \to M^3(\bar{u}, t)$. Therefore, for each $t \in [0, T]$ we have that $M_n^i(\bar{u}_n, t) \to M^i(\bar{u}, t)$ in probability. Moreover, with $c > 1$ from Corollary 3.7, we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \bar{\mathbb{E}}\big|\big(M(\bar{u}_n, t), \phi\big)_{H_x^{-3}}\big|^{2c} &= \sup_{n\in\mathbb{N}} \mathbb{E}\Big|\sum_{k=1}^n \int_0^t \big(\sigma\big(u_n(s)\big)e^k, \phi\big)_{H_x^{-3}}\, dw_s^k\Big|^{2c} \\
&\le N\|\phi\|^{2c}_{H_x^{-3}} \sup_{n\in\mathbb{N}}\big(1 + \|u_n\|^{c(m+1)}_{L^{c(m+1)}_{\omega,t,x}}\big) < \infty
\end{aligned}$$

and

$$\sup_{n\in\mathbb{N}}\bar{\mathbb{E}}\left|\int_0^t\sum_{k=1}^n\big(\sigma\big(\bar{u}_n(s)\big)e^k,\phi\big)^2_{H_x^{-3}}\,ds\right|^c\le N\|\phi\|^{2c}_{H_x^{-3}}\sup_{n\in\mathbb{N}}\big(1+\|\bar{u}_n\|^{c(m+1)}_{L^{c(m+1)}_{\bar{\omega},t,x}}\big)<\infty,$$

from which one deduces that for each $i=1,2,3$ and $t\in[0,T]$, $M_n^i(\bar{u}_n,t)$ are uniformly integrable in $\bar{\omega}$. Hence, we can pass to the limit in (4.12) to obtain, for $i=1,2,3$,

$$\bar{\mathbb{E}}V(\bar{u}|_{[0,s]},\bar{w}|_{[0,s]})\big(M^i(\bar{u},t)-M^i(\bar{u},s)\big)=0. \tag{4.16}$$

In addition, using the continuity of $M^i(\bar{u},t)$ in ϕ, uniform integrability, and the fact that $C_c^\infty(I)$ is dense in H_x^{-3}, it follows that (4.16) holds also for all $\phi\in H_x^{-3}$. Hence, for all $\phi\in H_x^{-3}(I)$, $i=1,2,3$, one can see that $\bar{M}^i(\bar{u},t)$ are continuous $\mathcal{G}_t$-martingales having finite c-moments. In particular, by Doob's maximal inequality, they are uniformly integrable (in t), which combined with continuity (in t) implies that they are also $\bar{\mathcal{F}}_t$-martingales. By [23, Prop. A.1] we obtain that almost surely, for all $\phi\in H_x^{-3}$, $t\in[0,T]$

$$\begin{aligned}\big(\bar{u}(t),\phi\big)_{H_x^{-3}}&=\big(\bar{u}(0),\phi\big)_{H_x^{-3}}+\int_0^t\big(\Delta\big(\bar{u}^{[m]}(s)\big),\phi\big)_{H_x^{-3}}\,ds\\&\quad+\sum_{k\in\mathbb{N}}\int_0^t\big(\sigma\big(\bar{u}(s)\big)e^k,\phi\big)_{H_x^{-3}}\,d\bar{w}_s^k.\end{aligned}\tag{4.17}$$

Notice that by (4.2), (4.7) and (4.6), it follows that $\bar{u}(0)\overset{d}{=}u^{(0)}$ and consequently $\bar{u}(0)\in L^{m+1}(\bar{\Omega};H_x^\gamma)$. Also, from (4.10) it follows that $\bar{u}\in L^{m+1}(\bar{\Omega}_T;L_x^{m+1})$. Choosing $\phi=(-\Delta)^2\psi$ in (4.17) for $\psi\in C_c^\infty(I)$, we obtain that for almost all $(\bar{\omega},t)$

$$\begin{aligned}\big(\bar{u}(t),\psi\big)_{H_x^{-1}}&=\big(\bar{u}(0),\psi\big)_{H_x^{-1}}-\int_0^t\big(\bar{u}^{[m]}(s),\psi\big)_{L_x^2}\,ds\\&\quad+\sum_{k\in\mathbb{N}}\int_0^t\big(\sigma\big(\bar{u}(s)\big)e^k,\psi\big)_{H_x^{-1}}\,d\bar{w}_s^k.\end{aligned}$$

By [25, Thm. 3.2] we have that $\bar{u}$ is an $\bar{\mathbb{F}}$-adapted, continuous H_x^{-1}-valued process. This shows that $\{(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}}),\bar{\mathbb{F}},(\bar{w}^k)_{k\in\mathbb{N}},\bar{u}\}$ is a weak solution.

Concerning the claimed bounds:

(i) Estimate (2.2) is obtained in (4.10).

(ii) For (2.3) we have the following. Notice that due to (3.9), the quantity $\mathbb{E}\|\bar{u}_n(t)\|^2_{H_x^\gamma}$ is differentiable in t, and similarly to the argumentation for (3.10), one sees that it satisfies

$$\partial_t\mathbb{E}\big\|u_n(t)\big\|^2_{H_x^\gamma}\le-\frac{1}{m}\mathbb{E}\big\|u_n^{[\frac{m+1}{2}]}(t)\big\|^2_{H_x^{1+\gamma}}+N,$$

where N depends on γ,m,K and T. By the inequalities

$$\mathbb{E}\big\|u_n^{[\frac{m+1}{2}]}(t)\big\|^2_{H_x^{1+\gamma}}\ge\mathbb{E}\big\|u_n^{[\frac{m+1}{2}]}(t)\big\|^2_{L_x^2}\ge\big(\mathbb{E}\big\|u_n(t)\big\|^2_{L_x^2}\big)^{(m+1)/2}\ge\big(\mathbb{E}\big\|u_n(t)\big\|^2_{H_x^\gamma}\big)^{(m+1)/2},$$

it follows that $g(t):=\mathbb{E}\|u_n(t)\|^2_{H_x^\gamma}$ satisfies for almost all t

$$\partial_t g(t)+\frac{1}{m}\big|g(t)\big|^{(m+1)/2}\le N.$$

This implies that (see, e.g., [17, Lemma 5.1]) with a constant N, depending only on γ,m,K and T, we have for all $t\in[0,T]$

$$\mathbb{E}\big\|u_n(t)\big\|^2_{H_x^\gamma}\le Nt^{-2/(m-1)}$$

Inequality (2.3) follows from the above, again by (4.7) and lower semicontinuity of the norms.

(iii) As before, it suffices to check the bound for u_n. Since $\mathcal{X}$ embeds continuously into $C_t^{\varepsilon_0}H_x^{-3}$ for some $\varepsilon_0 > 0$, for $\varepsilon \geq \gamma + 3$ the statement follows from (4.5), with $\varepsilon' = \varepsilon_0$. Otherwise let $\theta \in (0,1)$ be the number defined by $\gamma - \varepsilon = (1-\theta)\gamma - 3\theta$. Then by interpolation (see Remark 1.2)

$$\begin{aligned}\|u_n(t) - u_n(s)\|_{H_x^{\gamma-\varepsilon}} &\leq \|u_n(t) - u_n(s)\|_{H_x^{\gamma}}^{1-\theta} \|u_n(t) - u_n(s)\|_{H_x^{-3}}^{\theta} \\ &\leq \|u_n\|_{L_t^\infty H_x^\gamma}^{1-\theta} |t-s|^{\theta\varepsilon_0} \|u_n\|_{C_t^{\varepsilon_0} H_x^{-3}}^{\theta}.\end{aligned}$$

The statement therefore follows once again from (4.5) with $\varepsilon' = \theta\varepsilon_0$, and (4.9), (4.7). □

5. Strong well-posedness in H^{-1}

Proof of Proposition 2.8. In the following we denote $c_0 = 1/3$, which is $N(-1)$ from Lemma 3.4, so we have $\delta < \frac{2}{c_0}$ and $\bar{\delta} \leq \frac{8m}{c_0(m+1)^2}$. We verify the assumptions of [25]. Consider the Gelfand triple $L_x^{m+1} \subset H_x^{-1} \equiv (H_x^{-1})^* \subset (L_x^{m+1})^*$. The inner product in H_x^{-1} as well as the duality between L_x^{m+1} and $(L_x^{m+1})^*$ is denoted by $\langle\cdot,\cdot\rangle$, so that the two possible interpretations of $\langle f, g\rangle$ with $f \in L_x^{m+1}$ and $g \in H_x^{-1}$ agree. The operator $A : u \mapsto \Delta u^{[m]}$ maps L_x^{m+1} to $(L_x^{m+1})^*$ and $B = (B^k)_{k\in\mathbb{N}} : u \mapsto (\sigma(\cdot,u)e^k)_{k\in\mathbb{N}}$ maps L_x^{m+1} to $\ell^2(H_x^{-1})$. We now recall and verify the assumptions from [25] in a somewhat more restrictive form than therein, which will suffice for our purposes. It is assumed that there exist $\mu > 0$, $M \in \mathbb{R}$, such that for all $v, v_1, v_2 \in L_x^{m+1}$ the properties (A_1)–(A_5) below hold:

(A_1) *Semicontinuity of A: the function $\langle v, A(v_1 + \lambda v_2)\rangle$ is continuous in $\lambda \in \mathbb{R}$.*
This is a standard fact for the porous medium operator, see [32, Ex. 4.1.11].

(A_2) *Monotonicity of* (A, B):

$$2\langle v_1 - v_2, Av_1 - Av_2\rangle + \sum_{k\in\mathbb{N}} \|B^k v_1 - B^k v_2\|_{H_x^{-1}}^2 \leq 0.$$

First we use Lemma 3.4 to write

$$\sum_{k\in\mathbb{N}} \|B^k v_1 - B^k v_2\|_{H_x^{-1}}^2 = \sum_{k\in\mathbb{N}} \|(\sigma(v_1) - \sigma(v_2))e^k\|_{H_x^{-1}}^2 \leq c_0\|\sigma(v_1) - \sigma(v_2)\|_{L_x^2}^2. \tag{5.1}$$

Next, observe the elementary inequality: for $f, g : \mathbb{R} \to \mathbb{R}$ satisfying $f' = (g')^2$

$$\begin{aligned}(a-b)\big(f(a) - f(b)\big) &= (a-b)^2 \frac{1}{a-b}\int_b^a f'(c)\,dc = (a-b)^2\frac{1}{a-b}\int_b^a \big(g'(c)\big)^2\,dc \\ &\geq (a-b)^2\left(\frac{1}{(a-b)}\int_b^a g'(c)\,dc\right)^2 = |g(a) - g(b)|^2\end{aligned}$$

for $a > b$, and by symmetry, for all $a, b \in \mathbb{R}$. This in particular implies

$$(a-b)\big(a^{[m]} - b^{[m]}\big) \geq \frac{4m}{(m+1)^2}\big|a^{[\frac{m+1}{2}]} - b^{[\frac{m+1}{2}]}\big|^2, \quad a, b \in \mathbb{R}. \tag{5.2}$$

By (5.1), (2.5), and (5.2) we therefore have

$$\begin{aligned}&2\langle v_1 - v_2, Av_1 - Av_2\rangle + \sum_{k\in\mathbb{N}} \|B^k v_1 - B^k v_2\|_{H_x^{-1}}^2 \\ &\quad\leq \int_I -2(v_1 - v_2)\big(v_1^{[m]} - v_2^{[m]}\big) + c_0|\sigma(v_1) - \sigma(v_2)|^2\,dx \\ &\quad\leq \int_I -2(v_1 - v_2)\big(v_1^{[m]} - v_2^{[m]}\big) + c_0\bar{\delta}\big|v_1^{[\frac{m+1}{2}]} - v_2^{[\frac{m+1}{2}]}\big|^2\,dx \\ &\quad\leq 0,\end{aligned}$$

where in the last step we used $\bar{\delta} \leq \frac{8m}{c_0(m+1)^2}$.

(A_3) *Coercivity of* (A, B):

$$2\langle v, Av\rangle + \sum_{k\in\mathbb{N}} \|B^k v\|_{H_x^{-1}}^2 \le -\mu \|v\|_{L_x^{m+1}}^{m+1} + M.$$

Using Lemma 3.4 similarly as above, we can write

$$2\langle v, Av\rangle + \sum_{k\in\mathbb{N}} \|B^k v\|_{H_x^{-1}}^2 \le \int_I \left(-2|v|^{m+1} + c_0\left|\sigma(v)\right|^2\right) dx$$

$$\le (-2 + c_0\delta)\|v\|_{L_x^{m+1}}^{m+1} + 2c_0 K.$$

We can therefore set $\mu = 2 - c_0\delta$, which is positive by assumption.

(A_4) *Boundedness of the growth of* A:

$$\|Av\|_{(L_x^{m+1})^*} \le M\|v\|_{L_x^{m+1}}^m.$$

This is also standard, see [32, Ex. 4.1.11].

(A_5) $\mathbb{E}\|u^{(0)}\|_{H_x^{-1}}^2 < \infty$. This holds by assumption.

Invoking [25, Thms 2.1–2.2], the proof is complete. □

Appendix

Proof of (3.2). Let $R > 0$ and set

$$\tau := \left\{t > 0 : \left\|v(t)\right\|_{L_x^2} > R\right\} \wedge T.$$

By Itô's formula for the function $v \mapsto \|v\|_{L_x^2}^2$, we have

$$\left\|v(t\wedge\tau)\right\|_{L_x^2}^2 = \left\|v^{(0)}\right\|_{L_x^2}^2 + \sum_{k=1}^n \int_0^{t\wedge\tau} 2\left(v(s), \sigma\left(v(s)\right)e^k\right) dw_s^k$$

$$- \int_0^{t\wedge\tau} 2\nu\left\|\partial_x v(s)\right\|_{L_x^2}^2 + 2\left\|\partial_x v^{[\frac{m+1}{2}]}(s)\right\|_{L_x^2}^2 - \sum_{k=1}^n \left\|\sigma\left(v(s)\right)e^k\right\|_{L_x^2}^2 ds.$$

Upon rearranging, raising to the power $(m+1)/2$, and using the linear growth of σ, we get

$$\left\|v(t\wedge\tau)\right\|_{L_x^2}^{m+1} + \|\mathbf{1}_{[0,t\wedge\tau]}\partial_x v\|_{L_{t,x}^2}^{m+1} + \left\|\mathbf{1}_{[0,t\wedge\tau]}\partial_x v^{[\frac{m+1}{2}]}\right\|_{L_x^2}^{m+1} \le N\left\|v^{(0)}\right\|_{L_x^2}^{m+1}$$

$$+ N\int_0^{t\wedge\tau}\left(1 + \left\|v(s)\right\|_{L_x^2}^{m+1}\right) ds + \sum_{k=1}^n \left|\int_0^{t\wedge\tau} 2\left(v(s), \sigma\left(v(s)\right)e^k\right) dw_s^k\right|^{\frac{m+1}{2}},$$

where N depends only on ν, m, n, T and the Lipschitz constant of σ. Taking suprema in time up to t', expectations, and using the Burkholder–Gundy–Davis inequality and the linear growth of σ, we get

$$\mathbb{E}\sup_{t\le t'}\left\|v(t\wedge\tau)\right\|_{L_x^2}^{m+1} + \mathbb{E}\|\mathbf{1}_{[0,t'\wedge\tau]}\partial_x v\|_{L_{t,x}^2}^{m+1} + \left\|\mathbf{1}_{[0,t'\wedge\tau]}\partial_x v^{[\frac{m+1}{2}]}\right\|_{L_x^2}^{m+1}$$

$$\le N\mathbb{E}\left\|v^{(0)}\right\|_{L_x^2}^{m+1} + N\mathbb{E}\int_0^{t'\wedge\tau}\left(1 + \left\|v(s)\right\|_{L_x^2}^{m+1}\right) ds < \infty.$$

Gronwall's lemma gives

$$\mathbb{E}\sup_{t\le T}\left\|v(t\wedge\tau)\right\|_{L_x^2}^{m+1} + \mathbb{E}\|\mathbf{1}_{[0,\tau]}\partial_x v\|_{L_{t,x}^2}^{m+1} + \left\|\mathbf{1}_{[0,\tau]}\partial_x v^{[\frac{m+1}{2}]}\right\|_{L_x^2}^{m+1}$$

$$\le N\mathbb{E}\left\|v^{(0)}\right\|_{L_x^2}^{m+1} \le N\mathbb{E}\left\|v^{(0)}\right\|_{L_x^{m+1}}^{m+1}.$$

Since N does not depend on R, by virtue of Fatou's lemma we can replace τ by T, which shows (3.2) for $p = m + 1$. Consequently, (3.2) also holds with any $p < m + 1$. □

Proof of Lemma 3.6. We assume that $f' = (g')^2$ is bounded, the assumptions of the lemma guarantee that the general case follows from a standard approximation argument. Denote by $X_0^{2\beta}$ the completion of $C_c^\infty(I \times [0, \infty))$ under the norm

$$\|\phi\|^2_{X_0^{2\beta}} := C_\beta \int_0^\infty \int_I y^{1-2\beta} \big|\nabla\phi(x, y)\big|^2 \, dx \, dy,$$

where $C_\beta > 0$ is a normalizing constant such that (ii) below holds. For $\psi \in H_x^\beta$ we denote by $E(\psi) \in X_0^{2\beta}$ the unique solution of

$$\begin{cases} -\nabla \cdot (y^{1-2\beta} \nabla w) = 0 & \text{in } I \times (0, \infty), \\ w = 0 & \text{on } \partial I \times (0, \infty), \\ w = \psi & \text{on } I \times \{0\}. \end{cases}$$

The following facts are well known (see, [3,7]):

(i) The map $E : H_x^\beta \to X_0^{2\beta}$ is an isometry and for all $\phi \in X_0^{2\beta}$ we have

$$\| \operatorname{Tr} \phi \|_{H_x^\beta} \le \|\phi\|_{X_0^{2\beta}},$$

where Tr is the closure of the operator Tr_0 defined on $C_c^\infty(I \times [0, \infty))$ by $(\operatorname{Tr}_0 \phi)(x) = \phi(x, 0)$.

(ii) For $u \in H_x^\beta$ we have

$$C_\beta \int_0^\infty \int_I y^{1-2\beta} \nabla E(u) \nabla \phi \, dx \, dy = \int_I (-\Delta)^{\beta/2} u (-\Delta)^{\beta/2} \operatorname{Tr} \phi \, dx,$$

for all $\phi \in X_0^{2\beta}$.

Let u be as in the statement of Lemma 3.6. Since $f(u) = \operatorname{Tr} f(E(u))$, we get by applying (ii) with $\phi = f(E(u))$

$$\begin{aligned} \int_I f(u)(-\Delta)^\beta u \, dx &= \int_I \big((-\Delta)^{\beta/2} \operatorname{Tr} f\big(E(u)\big)\big)\big((-\Delta)^{\beta/2} u\big) \, dx \\ &= C_\beta \int_0^\infty \int_I y^{1-2\beta} \nabla f\big(E(u)\big) \nabla E(u) \, dx \, dy \\ &= C_\beta \int_0^\infty \int_I y^{1-2\beta} f'\big(E(u)\big) \big|\nabla E(u)\big|^2 \, dx \, dy. \end{aligned} \tag{A.1}$$

Using $f' = (g')^2$ and (i) we get

$$\begin{aligned} C_\beta \int_0^\infty \int_I y^{1-2\beta} f'\big(E(u)\big)\big|\nabla E(u)\big|^2 \, dx \, dy &= C_\beta \int_0^\infty \int_I y^{1-2\beta} \big|\nabla g\big(E(u)\big)\big|^2 \, dx \, dy \\ &\ge \big\| \operatorname{Tr} g\big(E(u)\big)\big\|^2_{H_x^\beta}. \end{aligned} \tag{A.2}$$

By (A.1), (A.2), and the equality $\operatorname{Tr} g(E(u)) = g(u)$, we get the claim. □

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