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Simpson, D. and Ruderman, M.S. (2005) Absolute and convective instabilities of parallel propagating circularly polarized Alfven waves: Beat instability. Physics of Plasmas, 12 (6). Art. No. 062103. ISSN: 1089-7674

https://doi.org/10.1063/1.1919407

Absolute and convective instabilities of parallel propagating circularly polarized Alfvén waves: Beat instability

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(Received 25 January 2005; accepted 30 March 2005; published online 26 May 2005)

Ruderman and Simpson [Phys. Plasmas **11**, 4178 (2004)] studied the absolute and convective decay instabilities of parallel propagating circularly polarized Alfvén waves in plasmas where the sound speed c_S is smaller than the Alfvén speed v_A. We extend their analysis for the beat instability which occurs in plasmas with $c_S > v_A$. We assume that the dimensionless amplitude of the circularly polarized Alfvén wave (pump wave), a, is small. Applying Briggs' method we study the problem analytically using expansions in power series with respect to a. It is shown that the pump wave is absolutely unstable in a reference frame moving with the velocity U with respect to the rest plasma if $U_l < U < U_r$, where $U_l = -v_A + \mathcal{O}(a)$ and $U_r = v_A + \mathcal{O}(a)$. When $U < U_l$ or $U > U_r$, the instability is convective. The signaling problem is studied in a reference frame where the pump wave is convectively unstable. It is shown that the spatially amplifying waves exist only when the signaling frequency is in two narrow symmetric frequency bands with the widths of the order of a^3. These results enable us to extend for the case when $c_S > v_A$ the conclusions, previously made for the case when $c_S < v_A$, that circularly polarized Alfvén waves propagating in the solar wind are convectively unstable in a reference frame of any spacecraft moving with the velocity not exceeding a few tens of km/s in the solar reference frame. The characteristic scale of spatial amplification for these waves exceeds 1 a.u. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1919407]

I. INTRODUCTION

A finite amplitude circularly polarized, parallel propagating Alfvén wave is an exact solution of the nonlinear magnetohydrodynamic (MHD) equations. Since the 1960s this solution has been known to be unstable with respect to harmonic perturbations in the density and magnetic field.[1,2] Circularly polarized Alfvén waves are commonly observed in the solar wind and are thought to exist in other astrophysical plasmas. Their stability has attracted ample attention of plasma physicists in an attempt to explain observed phenomena. Galeev and Oraevskii[1] were the first to study this problem. Their analysis was based on the ideal MHD equations and they assumed that the pump Alfvén wave amplitude and the plasma β were small parameters. They obtained the result that the pump wave can decay into a forward propagating sound wave and a backward propagating Alfvén wave. Derby[3] and Goldstein[4] extended the work of Galeev and Oraevskii for arbitrary pump-wave amplitude and plasma β. They discovered that the decay products were no longer normal modes of the plasma and that a forward propagating transverse wave is also involved in the process. In the following studies the dispersive and kinetic effects as well as the effects related to the oblique propagation of perturbations were investigated (for references see, e.g., Ruderman and Simpson[5]).

Although the stability of circularly polarized Alfvén waves has been studied for more than four decades, it still remains among the hot topics in plasma physics, which is confirmed by recent publications. The nonlinear evolution of linearly unstable circularly polarized Alfvén waves is intensively studied numerically. Turkman and Torkelsson studied the nonlinear evolution of circularly polarized Alfvén waves both in homogeneous[6] and stratified[7] plasmas using a one-dimensional numerical code, and applied their results to the acceleration of the solar wind. Del Zanna *et al.*[8,9] and Del Zanna and Velli[10] developed a three-dimensional MHD code to study the stability and nonlinear evolution of Alfvén waves. They applied their numerical results to the evolution of Alfvén wave spectra in the solar wind, and to plasma heating in coronal holes. Shevchenko *et al.*[11] used the derivative nonlinear Schrödinger (DNLS) equation to study the parametric decay instability of Alfvén packets propagating in the opposide directions. Hertzberg *et al.*[12–14] and Cramer *et al.*[15] extended the linear theory of parametric instabilities of Alfvén waves to multicomponent and dusty plasmas. Matsukiyo and Hada[16] studied the parametric instabilities of circularly polarized Alfvén waves in a relativistic electron-positron plasma.

Ruderman and Simpson[17] have recently addressed the problem of whether an unstable Alfvén wave appears to give rise to growing modes in a fixed reference frame. This is an important problem from the point of view of observations, as a system with unstable modes will only appear unstable to an observer if the instability grows in time in the observer's reference frame. This occurs only when the instability is absolute. Normal-mode analysis is not enough to study this problem. Ruderman and Simpson[17] used the method formulated by Briggs[18] and Bers[19] to study the absolute and convective natures of the instability. They restricted their analysis to the decay instability which occurs when the sound speed c_S in the unperturbed plasma is smaller than the Alfvén speed v_A. When $c_S > v_A$, the instability becomes a beat insta-

1070-664X/2005/12(6)/062103/9/$22.50

bility and the analysis needs to be modified. The aim of the present paper is to study the absolute and convective beat instabilities of circularly polarized Alfvén waves. Our analysis closely follows the analysis of Ruderman and Simpson,[17] so that we refer the reader to this paper (hereafter referred to as "Paper 1") for a detailed method description.

In Sec. II we formulate the problem and briefly describe the application of Briggs' method. In Sec. III we study the absolute and convective beat instabilities of small-amplitude circularly polarized Alfvén waves. In Sec. IV we consider the signaling problem for convectively unstable Alfvén waves and find the criterion for the existence of spatially amplifying waves. Section V contains the summary of our result and the discussion of their possible implication for interpretation of observation obtained in space missions.

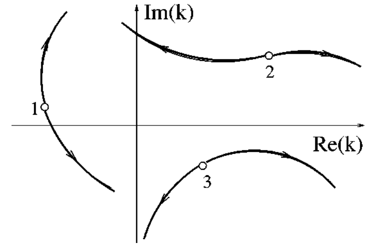

FIG. 1. The trajectories of the k roots of the dispersion equation that start at a double k root. The double k roots are shown by the circles. The arrows show the direction of motion along a trajectory when $\Re(\omega)$ is fixed while $\Im(\omega)$ increases from ω_i to $\gamma_M+\epsilon$. The double root 1 is pinching, while the double roots 2 and 3 are nonpinching.

II. FORMULATION AND METHOD DESCRIPTION

We are studying the linear stability of a circularly polarized Alfvén wave propagating along the mean magnetic field in the approximation of ideal MHD. The finite-amplitude Alfvén wave (pump wave) is an exact solution of the nonlinear ideal MHD equations. This solution is unstable with respect to small perturbations. Using the linearized MHD equations the following dispersion equation describing the stability of the pump wave can be derived:[3–5]

$$D(\omega,k) \equiv (\omega^2 - b^2k^2)(\omega - k)[(\omega + k)^2 - 4] - a^2k^2(\omega^3 + \omega^2 k - 3\omega + k) = 0. \quad (1)$$

Here $\omega=\Omega/\omega_0$ and $k=K/k_0$, where Ω and K are the frequency and wave number of the density perturbation, and ω_0 and k_0 are the frequency and wave number of the pump wave; $b=c_s/v_A$, where c_s is the sound speed and v_A is the Alfvén speed calculated using the mean magnetic field; a is the dimensionless amplitude of the pump wave given by $a=B_\perp/B_0$, where $B_\perp$ is the amplitude of the magnetic field in the pump wave and B_0 is the magnitude of the ambient magnetic field.

The aim of our work is to study the absolute and convective instabilities of the pump wave when $c_s > v_A$. The detailed description of the method for studying absolute and convective instabilities is given by Briggs[18] (see also Paper 1). For a particular problem studied in this paper the analysis is reduced to the investigation of the asymptotic behavior of the integral[17]

$$\delta\rho(x,t) = \int_{i\tau-\infty}^{i\tau+\infty} e^{-i\omega t} d\omega \int_{-\infty}^{\infty} \frac{T(\omega,k)}{\tilde{D}(\omega,k)} e^{ikx} dk, \quad (2)$$

as $t\to\infty$, where $\delta\rho$ is the density perturbation. The function $T(\omega,k)$ is determined by initial conditions so that it is not important for studying the asymptotic response; $\tilde{D}(\omega,k)=D(\tilde{\omega},k)$, where $\tilde{\omega}=\omega+kU$ is the Doppler-shifted frequency, $U=\bar{U}/v_A$ and $\bar{U}$ is the velocity of the observer's reference frame in the direction of the ambient magnetic field. The Bromwich integration contour $\Im(\omega)=\tau$ (where $\Im$ denotes the imaginary part of a quantity) is taken to be above all zeros of $\tilde{D}(\omega,k)$ considered as a function of ω.

Similar to Paper 1 we carry out the analysis of the asymptotic behavior of $\delta\rho$ in five steps:

(i) First, we calculate the maximum growth rate of the instability γ_M.

(ii) Then we calculate all double k roots of Eq. (1) by solving the system of equations

$$\tilde{D}(\omega,k) = 0, \quad \frac{\partial\tilde{D}}{\partial k} = 0. \quad (3)$$

(iii) Now we consider all pairs of solutions to Eq. (3), (ω,k), and choose only those with ω satisfying the inequality

$$0 < \omega_i \leqslant \gamma_M, \quad (4)$$

where $\omega=\omega_r+i\omega_i$. Pinching roots causing the absolute instability can only arise for pairs (ω,k) satisfying Eq. (4).

(iv) From all pairs (ω,k) satisfying Eq. (4) we choose only those with k being a pinching root. To do this, we fix $\Re(\omega)$ (where $\Re$ indicates the real part of a quantity) and increase $\Im(\omega)$ from ω_i to $\gamma_M+\epsilon$, where ϵ is an arbitrary positive quantity. As a result we map the trajectories of the two roots in the complex k plane which merge to form the double root k. If these trajectories end on different sides of the real axis in the complex k plane, then the double root k is pinching. Otherwise it is nonpinching (see Fig. 1).

(v) Finally, among all the solutions (ω,k) to Eq. (3) such that ω satisfies Eq. (4) and k is pinching we choose one with the largest ω_i. Using the notation ω_m for ω in this solution, we find that the asymptotic behavior of the density perturbation is given by

$$\delta\rho \propto t^{-1/2} \exp[t(\omega_{\rm im} - i\omega_{\rm rm})], \quad (5)$$

which implies that the instability is absolute. If there are no solutions of Eq. (3) with k pinching and ω satisfying Eq. (4), then the instability is convective.

III. ABSOLUTE AND CONVECTIVE BEAT INSTABILITIES OF SMALL-AMPLITUDE ALFVÉN WAVES

For the beat instability, Jayanti and Hollweg[20] have calculated the maximum growth rate of the instability for small pump-wave amplitude:

$$\gamma_M = \frac{a^3}{4\sqrt{2}(b^2-1)^{3/2}}. \quad (6)$$

This expression is valid for b not close to unity and we assume this in what follows. It is worth noting that $\gamma_M \propto a^3$ while $\gamma_m \propto a$ in the case of the decay instability ($b<1$).

To find the double k roots of the dispersion equation we solve the system of equations (3). The explicit form of these equations is given in Paper 1 [see Eqs. (8) and (9) in that paper]. Our analysis here is identical to that of Paper 1 so we may omit some details. It is shown in Paper 1 that the system of equations (3) can be rewritten in terms of $c=\tilde{\omega}/k$ and k as

$$\begin{aligned} &4(1+U)(c+1)(c-1)^2(c^2-b^2)^2 \\ &\quad - a^2\{[c^6+4c^5-3c^4-2(1+3b^2)c^3+3b^2c^2+4b^2c-b^2] \\ &\quad + U[6c^5-2c^4-(5+7b^2)c^3+4b^2c^2+(1+5b^2)c-2b^2]\} \\ &\quad + a^4[2c^3+U(3c^3-c)]=0, \end{aligned} \quad (7)$$

$$k^2 = \frac{4(c-1)(c^2-b^2)-a^2(3c-1)}{(c+1)[c^4-(1+a^2+b^2)c^2+b^2]}. \quad (8)$$

When $a=0$, Eq. (7) has one simple root, $c_1=-1$ and three double roots: $c_{2,3}=1$, $c_{4,5}=b$, and $c_{6,7}=-b$. The approximate solution of Eq. (7) close to -1 is given by

$$c_1 = -1 - \frac{a^2}{4(b^2-1)} + \mathcal{O}(a^3), \quad (9)$$

and it is straightforward to see that c_1 remains real in any order approximation with respect to a. The approximate solutions of Eq. (7) close to 1 are given by

$$c_{2,3} = 1 - \frac{a^2}{2(b^2-1)} \pm \frac{a^3}{8}\left[\frac{2(U-1)}{(1+U)(b^2-1)^3}\right]^{1/2} + \mathcal{O}(a^4), \quad (10)$$

where the "+" and "−" signs correspond to c_2 and c_3, respectively. Let us now find the solutions of Eq. (7) close to b. We are looking for these solutions in the form of an expansion in power series $c=b+\sum_{n=1}^{\infty}u_n a^n$. Substituting this expression in Eq. (7) we obtain

$$(c-b)^2 = a^2\left[\frac{(b-1)(b-U)}{16b(1+U)} + \sum_{n=1}^{\infty} v_n a^n\right]. \quad (11)$$

The coefficients v_n are expressed in terms of u_n. We do not give these expressions because they are not used in what follows. It follows from Eq. (11) that the solutions close to b are given by

$$c_{4,5} = b \pm \frac{a}{4}\left[\frac{(b-1)(b-U)}{b(1+U)}\right]^{1/2} + \mathcal{O}(a^2), \quad (12)$$

where the + and − signs correspond to c_4 and c_5, respectively. In addition, Eq. (11) implies that c_4 and c_5 are real in any order approximation with respect to a if the first term in square brackets in Eq. (11) is positive and much larger than a. In the same way we obtain that the solutions of Eq. (7) close to $-b$ are given by

$$c_{6,7} = -b \pm \frac{a}{4}\left[\frac{-(1+b)(U+b)}{b(1+U)}\right]^{1/2} + \mathcal{O}(a^2). \quad (13)$$

The quantities c_6 and c_7 are real in any order approximation with respect to a if the expression in square brackets is positive and much larger than a. Note also that the expansions (10), (12), and (13) are only valid when $|1+U| \gg a$. We assume that this inequality is satisfied in what follows.

Now we use Eq. (8) to calculate the double roots of the dispersion equation considered as an equation for k:

$$k_{1\pm} = \pm \frac{8i(b^2-1)}{a^2} + \mathcal{O}(1), \quad (14)$$

$$k_{2,3\pm} = \pm\left\{1 + \frac{a^2}{4(b^2-1)} \mp \frac{a^3 U}{4(b^2-1)^{3/2}[2(U^2-1)]^{1/2}}\right\} + \mathcal{O}(a^4), \quad (15)$$

$$k_{4,5\pm} = \pm\left\{\frac{2}{1+b} \mp \frac{a(b-1)^{1/2}(1-b+2U)}{2(1+b)^2[b(U+1)(b-U)]^{1/2}}\right\} + \mathcal{O}(a^2), \quad (16)$$

$$k_{6,7\pm} = \pm\left\{\frac{2}{b-1} \pm \frac{a(1+b)^{1/2}(1+b+2U)}{2(b-1)^2[-b(U+1)(U+b)]^{1/2}}\right\} + \mathcal{O}(a^2). \quad (17)$$

The corresponding values of ω are given by

$$\omega_{1\pm} = \pm \frac{8i(b^2-1)(1+U)}{a^2} + \mathcal{O}(1), \quad (18)$$

$$\omega_{2,3\pm} = \pm\left\{1 - U - \frac{a^2(1+U)}{4(b^2-1)} \pm \frac{a^3}{8}\left[\frac{2(U^2-1)}{(b^2-1)^3}\right]^{1/2}\right\} + \mathcal{O}(a^4), \quad (19)$$

$$\omega_{4,5\pm} = \pm\left\{\frac{2(b-U)}{1+b} \pm \frac{a[(b-1)(U+1)(b-U)]^{1/2}}{b^{1/2}(1+b)^2}\right\} + \mathcal{O}(a^2), \quad (20)$$

$$\omega_{6,7\pm} = \pm\left\{-\frac{2(b+U)}{b-1} \pm \frac{a[-(1+b)(U+1)(U+b)]^{1/2}}{b^{1/2}(b-1)^2}\right\} + \mathcal{O}(a^2). \quad (21)$$

In these expressions the number subscripts correspond to the upper and lower signs inside the curly brackets. For example, we choose the upper sign in the curly brackets to calculate $k_{2\pm}$ and the lower sign to calculate $k_{3\pm}$. The $\pm$ subscripts correspond to the $\pm$ signs outside of the curly brackets. The results obtained for c_4 and c_5 imply that $k_{4,5\pm}$ and $\omega_{4,5\pm}$ are real in any order approximation with respect to a if the second terms in the curly brackets in Eqs. (16) and (20) are real and $|b-U| \gg a$. Similarly, $k_{6,7\pm}$ and $\omega_{6,7\pm}$ are real in any order approximation with respect to a if the second terms in the curly brackets in Eqs. (17) and (21) are real and $|U+b| \gg a$.

We must now select values of omega which satisfy the inequality $0 < \omega_i \leq \gamma_M$. It is here that our analysis starts to differ from the decay instability case due to the much smaller maximum growth rate of the beat instability. In what follows we assume that U is not very close either to ± 1 or to $\pm b$, and take $|U^2-1| \gg a$, $|U^2-b^2| \gg a$. We consider the case where $|U^2-b^2| \lesssim a$ separately. Now we can immediately reject $\omega_{1\pm}$ as it is obvious that $\Im(\omega_{1\pm})$ is either greater than γ_M or negative.

$\Im(\omega_{2,3\pm}) \neq 0$ when $-1 < U < 1$ but in this case $\Im(\omega_{2-}) = \Im(\omega_{3+}) < 0$ so we can reject k_{2-} and k_{3+}. For k_{2+} and k_{3-} the inequality $0 < \omega_i < \gamma_M$ reduces to $U^2 \geq 0$ which is always true so we retain these roots.

When $-1 < U < b$, the second term in the curly brackets in Eq. (20) is real. Since we assume that $|1+U| \gg a$ and $|U-b| \gg a$, it follows that in this case $\omega_{4,5\pm}$ are real in any order approximation with respect to a. When either $U < -1$ or $U > b$, it follows from the assumptions $|1+U| \gg a$ and $|U-b| \gg a$ that $|\Im(\omega_{4,5\pm})| \gg \gamma_M \sim a^3$. Hence, we reject $k_{4,5\pm}$.

Similarly, either $\omega_{6,7\pm}$ is real in any order approximation with respect to a, or $|\Im(\omega_{6,7\pm})| \gg \gamma_M$, so we reject $k_{6,7\pm}$.

Now we must determine whether the roots which we have retained are pinching. We can simplify our analysis by noting that $k_{3-} = -k^*_{2+}$ and $\omega_{3-} = -\omega^*_{2+}$. This implies that the trajectories of the roots that collide to form the double root k_{2+} and the trajectories of the roots that collide to form the double root k_{3-} are symmetric to each other with respect to the imaginary axis in the complex k plane. Hence, the roots k_{2+} and k_{3-} are either both pinching or both nonpinching. This observation enables us to restrict the analysis to the root k_{2+}. We take $\omega = \omega_{2+} + ia^3\sigma$, where σ varies from 0 to $\sigma_2 + \epsilon$ with $\sigma_2 = [\gamma_M - \Im(\omega_{2+})]/a^3$ and $\epsilon > 0$. We now let $k = 1 + a\bar{k}$ and substitute this expression into the equation $\widetilde{D}(k,\omega) = 0$. Collecting terms of the lowest order with respect to a we obtain

$$\bar{k}^2(U^2-1)(b^2-1) = \mathcal{O}(a^2), \qquad (22)$$

so we let $\bar{k} = a\hat{k}$. Once again collecting terms of the lowest order with respect to a yields

$$4\hat{k}^2(b^2-1) - 2\hat{k} + \frac{1}{4(b^2-1)} = \mathcal{O}(a). \qquad (23)$$

This equation has a double root $\hat{k} = [4(b^2-1)]^{-1}$ so we let $k = 1 + a^2[4(b^2-1)]^{-1} + a^3\tilde{k}$ and substitute this into the equation $\widetilde{D}(k,\omega) = 0$ again. This gives us the equation

$$4(1-U^2)(b^2-1)\tilde{k}^2 - (b^2-1)U \times \left\{ \left[\frac{2(1-U^2)}{(b^2-1)^3} \right]^{1/2} i + 8i\sigma \right\} \tilde{k} - \frac{U^2}{8(b^2-1)^2} + \sigma(b^2-1) \left\{ \left[\frac{2(1-U^2)}{(b^2-1)^3} \right]^{1/2} + 4\sigma \right\} = \mathcal{O}(a). \qquad (24)$$

Solving this quadratic equation we obtain the expressions determining the trajectories of the two roots k^+ and k^-, that collide as $\omega = \omega_{2+}$,

$$k^{\pm} = k_{2+} + \frac{ia^3}{1-U^2}\left(-\sigma U \pm \left\{ \sigma^2 + \frac{\sigma}{4}\left[\frac{2(1-U^2)}{(b^2-1)^3} \right]^{1/2} \right\}^{1/2} \right). \qquad (25)$$

When $\sigma = 0$ we obtain $k^+ = k^- = k_{2+}$ as expected. We find that $\Im(k^-)$ is a monotonically decreasing function of σ for all U and $\Im(k^+)$ is a monotonically increasing function of σ for all U (recall that $|U| < 1$). When $U < 0$, $\Im(k_{2+}) > 0$ so the whole trajectory of k^+ is above the real axis. If we let $\sigma = \sigma_2$ then we see that $\Im(k^-) = \mathcal{O}(a^4)$. This means that if ϵ is taken to be large enough, then the trajectory of k^- will cross the real axis and we have a pinching root. If $U > 0$ then $\Im(k_{2+}) < 0$, so the whole trajectory of k^- is below the real axis. When $\sigma = \sigma_2$ we obtain $\Im(k^+) = \mathcal{O}(a^4)$, so that if ϵ is taken to be large enough, the trajectory of k^+ will cross the real axis. Hence, once again k_{2+} is a pinching root.

This analysis shows that we have pinching roots corresponding to ω satisfying Eq. (4) when $-1 < U < 1$. This means that the instability is absolute in a reference frame moving with the dimensionless velocity U with respect to the rest plasma if $|U| < 1$. It is worth recalling, however, that this result is only valid up to a certain accuracy. We have performed our analysis assuming that $|U^2-1| \gg a$ and $|U^2-b^2| \gg a$ so, to be precise, we only can claim that the instability is absolute when $U_l < U < U_r$, where $U_l = -1 + \mathcal{O}(a)$ and $U_r = 1 + \mathcal{O}(a)$.

Our analysis has to be modified when U is close to ± 1 or $\pm b$. However, we will ignore the case when U is close to ± 1 since this analysis would provide only small corrections to the boundaries of the absolute instability. We must consider the case when U is close to $\pm b$, however, as these values are outside of the found boundaries of the absolute instability. Hence, if pinching roots would exist for U close to $\pm b$, it would have a serious physical implication.

When $U = b + \mathcal{O}(a)$ the expressions for $c_{4,5}$, $k_{4,5\pm}$, and $\omega_{4,5\pm}$ are invalid and we need to modify the analysis for these roots. This modified analysis is presented in Appendix. It shows that the instability is convective when $U = b + \mathcal{O}(a)$.

When $U = -b + \mathcal{O}(a)$, we have to modify the analysis for $c_{6,7}$, $k_{6,7\pm}$, and $\omega_{6,7\pm}$. The modified analysis in this case is almost identical to that in the case when $U = b + \mathcal{O}(a)$, so that we do not present it in this paper. The result of this analysis is the same: the instability is convective when $U = -b + \mathcal{O}(a)$.

Summarizing the results obtained in this section and in the Appendix we conclude that the instability is absolute when $U_l < U < U_r$ and convective otherwise. The instability increment is given by

$$\gamma = \Im(\omega_{2+}) = \Im(\omega_{3-}) = \frac{a^3}{8}\left[\frac{2(1-U^2)}{(b^2-1)^3}\right]^{1/2} + \mathcal{O}(a^4). \quad (26)$$

It takes its maximum value, $\gamma=\gamma_M$, when $U=0$, which is equal to the group velocity of the unstable wave mode.[20]

IV. SPATIALLY AMPLIFYING WAVES

A. Theory

When we have a convective instability, i.e., when $U < U_l$ or $U > U_r$, we can obtain spatially amplifying solutions which can be excited by imposing perturbations periodic in time.[18] We now determine the frequencies for which spatially amplifying waves can exist. We again refer readers to Paper 1 and Briggs[18] for a detailed description of the method. Here we briefly describe how we will apply this theory to our particular problem.

The problem reduces to evaluating the asymptotic response of the density perturbation given by

$$\delta\rho(x,t) = \int_{i\tau-\infty}^{i\tau+\infty} \frac{e^{-i\omega_d t}}{\omega-\omega_d} d\omega \int_{-\infty}^{\infty} \frac{S(\omega,k)}{\tilde{D}(\omega,k)} e^{ikx} dk, \quad (27)$$

as $t\to\infty$ and $x\to\infty$ or $x\to-\infty$. $S(\omega,k)$ is an analytic function of k and ω which depends on the initial conditions and the amplitude of the external perturbation, and it is not important for our analysis; ω_d is real and $\tilde{D}(\omega,k)=D(\tilde{\omega},k)$ where $\tilde{\omega}=\omega+kU$ is the Doppler-shifted frequency. Now we outline the step-by-step method by which we look for spatially amplifying waves.

(i) Spatially amplifying waves can only occur when there is a root of $\tilde{D}(\omega,k)=0$ considered as an equation for k with $\Im(k)\neq 0$ for real ω. Hence, the first step is to find all values of ω, which we denote as ω_d, for which this condition is satisfied.

(ii) When we have found a pair (ω_d,k) which satisfies the previous condition, we substitute $\omega=\omega_d+ia^3\sigma$ in the equation $\tilde{D}(\omega,k)=0$ and solve for k. Then we decrease σ from $\tau>\gamma_M/a^3$ to zero. If the trajectory of k crosses the real k axis then that solution can give rise to spatially amplifying waves.

(iii) As $x\to-\infty$ the asymptotic response is determined by the root starting in the lower complex k plane with the largest imaginary part at the end of its trajectory. We denote this root as k_l. As $x\to\infty$ the asymptotic response is determined by the root starting in the upper complex k plane with the smallest imaginary part at the end of its trajectory. We denote this root as k_r. Then the asymptotic response is given by $F(x)\sim e^{ik_l x}$ as $x\to-\infty$ and $F(x)\sim e^{ik_r x}$ as $x\to\infty$. The corresponding spatial amplification rates are $\gamma_l^s=\Im(k_l)$ and $\gamma_r^s=-\Im(k_r)$, respectively.

B. Calculations

To begin, we assume that we have a convective instability, so that $U>U_r=1+\mathcal{O}(a)$ or $U<U_l=-1+\mathcal{O}(a)$. In what follows we impose a slightly stronger restriction that $|U^2-1|\gg a$. We first need to find all real values of ω such that, when we solve the equation $\tilde{D}(\omega,k)=0$, we obtain solutions for k with $\Im(k)\neq 0$. Similarly to Paper 1, it is easy to see that roots of this equation will remain real in any order approximation with respect to a if the differences between them in the zero-order approximation are of the order of unity. Hence we can only obtain complex roots for k if the difference between at least two roots is of the order of a or smaller. This can happen only when $\omega_d=\bar{\omega}_{dj}+a\lambda$, where λ is real and $\bar{\omega}_{dj}$ $(j=1,\ldots,6)$ are given by

$$\bar{\omega}_{d1,2} = \pm(1-U), \quad \bar{\omega}_{d3,4} = \pm\frac{2(b-U)}{b+1},$$
$$\bar{\omega}_{d5,6} = \pm\frac{2(U+b)}{b-1}. \quad (28)$$

The upper sign on the right-hand side of the first equation refers to $\bar{\omega}_{d1}$ and the lower sign refers to $\bar{\omega}_{d2}$. A similar rule applies to the other two equations. When $\omega_d=\bar{\omega}_{dj}$ the equation $\tilde{D}(\omega,k)=0$ has a double root $\bar{k}_j$ in the zero-order approximation with respect to a, where

$$\bar{k}_{d1,2} = \pm 1, \quad \bar{k}_{d3,4} = \pm\frac{2}{b+1}, \quad \bar{k}_{d5,6} = \mp\frac{2}{b-1}. \quad (29)$$

When $|\bar{\omega}_{dj}-\bar{\omega}_{dl}|\sim 1$ for any $j\neq l$, the is exactly one double root and three simple roots in the zero-order approximation with respect to a. Then it is straightforward to show that the simple roots remain real in any order approximation with respect to a when $\sigma=0$. Hence, only the two roots close to the double roots of the zero-order approximation can have nonzero imaginary parts. There are particular values of U when two out of six quantities $\bar{\omega}_{dj}$ coincide. For these values of U there are two double roots in the zero-order approximation with respect to a. However, there is no need to modify the analysis in this case because we always look for roots close to the double roots of the zero-order approximation, no matter if there is one such a root or there are two such roots.

Now we substitute $\omega=\bar{\omega}_{dj}+a\lambda+ia^3\sigma$ $(j=1,\ldots,6)$ in the equation $\tilde{D}(\omega,k)=0$ considered as an equation for k, and look for the solutions in the form $k=\bar{k}_j+a\bar{k}$. We start with $j=1$ and substitute $\omega=1-U+a\bar{\omega}$ and $k=1+a\bar{k}$ in the equation $\tilde{D}(\omega,k)=0$, where $\bar{\omega}=\lambda+ia^2\sigma$. Terms of the order of a cancel each other. Collecting terms of the order of a^2 we obtain a quadratic equation with respect to $\bar{k}$ with the solutions $\bar{k}_1^+=-\bar{\omega}/(U-1)$ and $\bar{k}_1^-=-\bar{\omega}/(U+1)$, which are real when $\sigma=0$. It is straightforward to show that $\bar{k}_1^+$ and $\bar{k}_1^-$ remain real in any order approximation with respect to a when $\sigma=0$ if $\lambda\sim 1$. Hence, we take $\lambda=a\lambda_1$, $\bar{k}=a\hat{k}$, and repeat the procedure to obtain a quadratic equation for $\hat{k}$. This equation has two roots, $\hat{k}_1^+$ and $\hat{k}_1^-$, which are real when $\sigma=0$. In addition, $|\hat{k}_1^+-\hat{k}_1^-|\sim 1$ unless λ_1 is close to $-\frac{1}{4}(U+1)/(b^2-1)$. Then once again it can be shown that $\hat{k}_1^+$ and $\hat{k}_1^-$ and, consequently, k_1^+ and k_1^-, remain real in any order approximation with respect to a when $\sigma=0$. So, to obtain k roots with nonzero imaginary parts when $\sigma=0$ we have to take $\lambda_1=-\frac{1}{4}(U+1)/(b^2-1)+a\xi$, giving the following expression for ω_d:

$$\omega_d = 1 - U - \frac{a^2(U+1)}{4(b^2-1)} + a^3\xi. \tag{30}$$

In this case $\hat{k}_1^+=\hat{k}_1^-=[4(b^2-1)]^{-1}$ in the lowest-order approximation with respect to a. Substituting $\omega=\omega_d+ia^3\sigma$ and $k=1+a^2[4(b^2-1)]^{-1}+a^3\tilde{k}$ in the equation $\tilde{D}(\omega,k)=0$ we obtain a quadratic equation for $\tilde{k}$. Solving this equation we obtain the expressions for the two roots close to unity, k_1^+ and k_1^-, with the accuracy up to terms of the order of a^3,

$$k_1^{\pm} = 1 + \frac{a^2}{4(b^2-1)} - \frac{a^3U(\xi+i\sigma)}{U^2-1} \pm \frac{a^3}{U^2-1}\left[(\xi+i\sigma)^2 - \frac{U^2-1}{32(b^2-1)^3}\right]^{1/2}. \tag{31}$$

We see that $\Im(k_1^{\pm})\neq 0$ when $\sigma=0$ if ξ satisfies the inequality

$$\xi^2 < \frac{U^2-1}{32(b^2-1)^3}. \tag{32}$$

The imaginary parts of $k_1^{\pm}$ are given by

$$\frac{\Im(k_1^{\pm})}{a^3} = \frac{-U\sigma}{U^2-1} \pm \frac{1}{U^2-1}\left(\frac{1}{2}\left[\xi^2-\sigma^2-\frac{(U^2-1)}{32(b^2-1)^3}\right] + \frac{1}{2}\left\{\left[\xi^2-\sigma^2-\frac{U^2-1}{32(b^2-1)^3}\right]^2 + 4\xi^2\sigma^2\right\}^{1/2}\right)^{1/2}. \tag{33}$$

Let us consider $U>1$. In this case $\Im(k_1^+)>0$ and $\Im(k_1^-)<0$ when $\sigma=0$. It is straightforward to show that $\Im(k_1^-)<0$ for all $\sigma>0$, so that k_1^- does not give rise to spatially amplifying waves. By calculating the product $\Im(k_1^-)\Im(k_1^+)$ we can show that $\Im(k_1^+)>0$ when

$$\sigma^2 < \max\left\{\frac{(\gamma_M/a^3)^2-\xi^2}{U^2}, \quad \frac{(U^2-1)(\gamma_M/a^3)^2-\xi^2}{2U^2-1}\right\}, \tag{34}$$

and $\Im(k_1^+)<0$ otherwise. This result implies that $\Im(k_1^+)<0$ when $\sigma=\gamma_M/a^3$. Since $\Im(k_1^+)>0$ when $\sigma=0$, the trajectory of k_1^+ starts in the lower half of the complex k plane and ends in the upper half of the complex k plane, so that it gives rise to spatially amplifying waves as $x\to-\infty$.

When $U<-1$ it immediately follows from Eq. (33) that $\Im(k_1^+)>0$ for all $\sigma>0$. Now Eq. (34) gives the condition that $\Im(k_1^-)<0$. It is easy to see that the trajectory of k_1^- starts in the upper half of the complex k plane and ends in the lower half of the complex k plane. Hence it gives rise to spatially amplifying waves as $x\to\infty$.

Now that we have studied the case where $j=1$, we do not need to do the analysis for $j=2$. Instead we notice that if (ω,k) is a solution of the equations $\tilde{D}(\omega,k)=0$, then so is $(-\omega^*,-k^*)$. This implies that the trajectories of $k_2^{\pm}$ are symmetric to those of $k_1^{\pm}$ with respect to the imaginary axis. Hence k_2^+ gives rise to spatially amplifying waves as $x\to-\infty$ when $U>1$, and k_2^- gives rise to spatially amplifying waves as $x\to\infty$ when $U<-1$.

Let us proceed to $j=3$. Now we substitute $\omega=2(b-U)/(b+1)+a\bar{\omega}$ and $k=2/(b+1)+a\bar{k}$ in the equation $\tilde{D}(\omega,k)=0$. Terms of the order of a cancel each other. Collecting terms of the order of a^2 we obtain the quadratic equation for $\bar{k}$

$$\bar{k}^2(U-b)(U+1) + \bar{k}\lambda(2U+1-b) + \lambda^2 - \frac{b-1}{4b(b+1)^2} = 0. \tag{35}$$

We obtain from this equation that $\Im(k)\neq 0$ when $\sigma=0$ if the following inequality is satisfied:

$$\lambda^2 < \frac{(b-1)(b-U)(U+1)}{b(1+b)^4} \equiv \lambda_0^2, \tag{36}$$

where λ_0 can be either positive or negative. The inequality (36) can be satisfied only for $-1<U<b$ as λ_0^2 is negative otherwise. If $\lambda-\lambda_0=\mathcal{O}(1)$ then the imaginary part of k is of order a and its sign is determined by λ only. The variation of σ affects only terms of higher-order approximation in the expansion of k in power series with respect to a. This implies that the whole trajectory of the k root that we obtain by varying σ from zero to γ_M/a^3 is either below or above the real k axis. Hence such a root cannot give rise to spatially amplifying waves.

It follows from this analysis that we can obtain a k root with the trajectory crossing the real k axis only if we take $\lambda=\lambda_0+a^{1/2}\tilde{\lambda}_1$. Then, in the lowest-order approximation with respect to a, Eq. (35) has the repeated root

$$\bar{k}_0 = \frac{\lambda_0(2U+1-b)}{2(b-U)(U+1)}. \tag{37}$$

Now we look for the solution in the form $\bar{k}=\bar{k}_0+a^{1/2}\tilde{k}_1$. Substituting this expression in the equation $\tilde{D}(\omega,k)=0$ we obtain in the lowest-order approximation with respect to a the equation for $\bar{k}_1$,

$$\tilde{k}_1^2 = \frac{\lambda_0(\tilde{\lambda}_1-\lambda_1)(1+b)^2}{2(b-U)^2(U+1)^2}, \tag{38}$$

where

$$\lambda_1 = \frac{(20b^3-11b^2+6b+1)U - b^2(3b^2-26b+7)}{16(1+b)^3(b-1)b^2}. \tag{39}$$

In order to have $\Im(k)\neq 0$, the following inequality $\tilde{\lambda}_1<\lambda_1$ must be satisfied. If $\tilde{\lambda}_1-\lambda_1=\mathcal{O}(1)$ then the imaginary part of k is of order $a^{3/2}$ and its sign is determined by $\tilde{\lambda}_1$ only. Once again the variation of σ affects only terms of higher-order approximation in the expansion of k in power series with respect to a. And once again this implies that the whole trajectory of the k root that we obtain by varying σ from zero to γ_M/a^3 is either below or above the real k axis, so that such a root cannot give rise to spatially amplifying waves.

On the basis of this analysis we conclude that, in order to have a k root with the trajectory crossing the real k axis, we have to take $\tilde{\lambda}_1 = \lambda_1 + a\tilde{\lambda}_2$. Then it follows from Eq. (38) that $\tilde{k}_1 = a^{1/2}\tilde{k}_2$, so that $k = 2/(b+1) + a\bar{k}_0 + a^2\tilde{k}_2$. Now we substitute this expression for k and $\omega = 2(b-U)/(b+1) + a\lambda_0 + a^2\lambda_1 + a^3(\tilde{\lambda}_2 + i\sigma)$ in the equation $\tilde{D}(\omega, k) = 0$ to obtain, in the lowest-order approximation with respect to a, a quadratic equation for $\tilde{k}_2$. The roots of this equation are given by

$$\tilde{k}_2^{\pm} = \frac{1+6b-11b^2-20b^3}{16b^2(1+b)^3(b-1)} \pm \frac{[H + 128ib^4(1+b)^6\lambda_0\sigma]^{1/2}}{16b^2(1+b)^4(b-U)(U+1)}, \tag{40}$$

where H is a real quantity expressed in terms of b, U, λ_0, and $\tilde{\lambda}_2$. We do not give this expression because it is not used in what follows. $\Im(\tilde{k}_2^{\pm}) \neq 0$ when $\sigma = 0$ if $H < 0$. Then it is straightforward to see that

$$\mathrm{sgn}[\Im(\tilde{k}_2^{+})] = \mathrm{sgn}[\lambda_0(b-U)(U+1)],$$

$$\mathrm{sgn}[\Im(\tilde{k}_2^{-})] = -\,\mathrm{sgn}[\lambda_0(b-U)(U+1)],$$

for any $\sigma \geqslant 0$. This means that the two trajectories of the two k roots close to $2/(b+1)$ do not cross the real k axis when σ varies from zero to γ_M/a^3. Hence, these roots cannot give rise to spatially amplifying waves.

Once again there is no need to do the analysis for $j=4$ because the trajectories of the roots $k_4^{\pm}$ close to $\bar{k}_4$ are symmetric to the trajectories of the roots $k_3^{\pm}$ close to $\bar{k}_3$ with respect to the imaginary k axis. Hence that the roots $k_4^{\pm}$ also do not give rise to spatially amplifying waves.

Finally, we consider $j=5,6$. The analysis for these cases is very similar to that for $j=3,4$, so we omit it and only present the final result: the roots $k_5^{\pm}$ close to $\bar{k}_5$ and $k_6^{\pm}$ close to $\bar{k}_6$ do not give rise to spatially amplifying waves.

When $j=3$, the expressions for $\bar{k}_0$, $\tilde{k}_1$, and $\tilde{k}_2$ contain $b-U$ in the denominator. It can be shown that the same is true when $j=4$, and similar expressions contain $b+U$ in the denominator when $j=5,6$. This means that the analysis for $j=3,\ldots,6$ is only valid when $|U^2-b^2| \gg \mathcal{O}(a)$. When $|U^2-b^2| = \mathcal{O}(a)$ we need to modify it. We repeated the analysis taking $U = \pm b + \mathcal{O}(a)$ and arrived at the same results: none of the roots $k_3^{\pm}$, $k_4^{\pm}$, $k_5^{\pm}$, and $k_6^{\pm}$ gives rise to spatially amplifying waves.

To summarize, we have shown that spatially amplifying waves only exist if $\omega = \pm\omega_d$ where ω_d is given by Eq. (30) with ξ satisfying the inequality (32). The corresponding wave numbers are given by

$$k = \pm\left\{1 + \frac{a^2}{4(b^2-1)}\right\} + \mathcal{O}(a^3). \tag{41}$$

When $U < -1 + \mathcal{O}(a)$ there is a spatially amplifying wave traveling in the positive x direction, and when $U > 1 + \mathcal{O}(a)$ there is a spatially amplifying wave traveling in the negative x direction. The spatial amplification rate is given by

$$|\Im[k_1^{\pm}(\sigma=0)]| = \frac{a^3}{U^2-1}\left\{\frac{U^2-1}{32(b^2-1)^3} - \xi^2\right\}^{1/2}. \tag{42}$$

V. SUMMARY AND CONCLUSIONS

In this paper we have considered the beat instability of a circularly polarized Alfvén wave (pump wave) which occurs when the sound speed is bigger than the Alfvén speed ($b = c_S/v_A > 1$). We have studied the absolute and convective nature of this instability. The nature of the instability is determined by the dimensionless parameter U, which is the ratio of the speed of the reference frame with respect to the rest plasma to the Alfvén speed. We restricted our analysis to pump waves with the small amplitude a. Our main result is that the instability if absolute when $U_l < U < U_r$ and convective otherwise, where $U_l = -1 + \mathcal{O}(a)$ and $U_r = 1 + \mathcal{O}(a)$. Hence, the instability is absolute in a reference frame moving with a velocity bigger than $-v_A + \mathcal{O}(a)$ and smaller than $v_A + \mathcal{O}(a)$ with respect to the rest plasma. We can give a simple physical interpretation of this result. Jayanti and Hollweg[20] have shown that the beat instability primarily involves forward and backward propagating Alfvén waves with the dispersion equations $\omega = \omega_{fA}(k)$ and $\omega = \omega_{bA}(k)$, respectively. For small a we have $\Re(\omega_{fA}) \approx v_A k$ and $\Re(\omega_{bA}) \approx v_A(2-k)$, so that $d\Re(\omega_{fA})/dk \approx v_A$ and $d\Re(\omega_{bA})/dk \approx -v_A$. This implies that the wave energy is transported with velocity $-v_A$ by the backwards propagating Alfvén wave and with velocity v_A by the forwards propagating Alfvén wave. Using the method outlined in Paper 1 we obtain that, if x' is the spatial coordinate in the reference frame moving with velocity $\tilde{U} = v_A U$ parallel to the direction of Alfvén wave propagation, then the perturbed portion of the spatial domain after time t is given by the inequality $-t(v_A + \tilde{U}) < x' < (v_A - \tilde{U})t$. This shows that if $\tilde{U} < -v_A$ then the left boundary is moving forward, and if $\tilde{U} > v_A$ then the right boundary is moving backwards. In these two situations the perturbations are swept away and we have a convective instability. This leaves us with the result that we have absolute instability if $-v_A < \tilde{U} < v_A$.

We have also studied the signaling problem when either $U < U_l$ or $U > U_r$, so that the instability is convective. We have found that signaling drives spatially amplifying waves only if the signaling frequency is equal to $\pm\omega_d$, where ω_d is given by Eq. (30) with ξ satisfying the inequality (32). The spatial amplification rate is given by Eq. (42)

Similar to Paper 1 we apply our results to circularly polarized Alfvén waves propagating in the solar wind. Both the Alfvén and sound speed are of order 50 km/s at the Earth orbit. The solar wind speed, v_{sol}, is of order 500 km/s. In the solar reference frame the speed of any realistic space station is much smaller than v_{sol}. This imply that the space station reference frame moves relative to the rest plasma with the speed approximately equal to v_{sol}, i.e., $|\tilde{U}| \approx v_{\mathrm{sol}}$. Hence, we obtain $|U| \sim 10$, $|U| > U_l$, U_r, and the instability of any pump wave is convective in the space station reference frame.

Let us assume that a wave packet is created in the solar wind at the initial moment of time. We estimate the distance that this convectively unstable wave packet will travel in the space station reference frame before its amplitude increases by e-times. Since, in accordance with the results obtained in Sec. III, the wave packet has the maximum increment when

$U=0$, which corresponds to the rest plasma reference frame, we conclude that this reference frame travels with the unstable wave packet. Taking $a=0.1$ and assuming that $b-1\sim 1$ we obtain from Eq. (26) $\gamma\approx 2\times 10^{-4}$. Let us consider a pump wave with the period T_0 in the solar reference frame. The period of this wave in the solar wind reference frame is approximately $T_0 v_{sol}/v_A$ and its frequency is $\omega_0=2\pi v_A(T_0 v_{sol})^{-1}\approx 0.6/T_0$. Then the dimensional increment is $\tilde{\gamma}=0.6\gamma/T_0\approx 1.2\times 10^{-4}/T_0$, and the e-folding time is $\tilde{\gamma}^{-1}\approx 10^4 T_0$ s. This implies that the unstable wave packet will travel the distance $v_{sol}\tilde{\gamma}^{-1}\approx 5\times 10^6 T_0$ km before its amplitude increases by e-times. Taking T_0 equal to 1 h, which is the typical period of Alfvén waves observed in the solar wind, we obtain that this distance is approximately equal to 2×10^{10} km ≈ 130 a.u. Hence, if a wave packet is excited by a small perturbation near the Sun, it is unlikely that this packet will have large enough amplitude at the Earth orbit to be observable.

Now we consider the signaling problem. Once again we take $v_A\approx 50$ km/s and $v_{sol}\approx 500$ km/s, so that $U\approx 10$. Then it follows from Eq. (42) that, for $b-1\sim 1$, the maximum amplification rate is approximately equal to $0.02a^3$, so that the spatial amplification scale is $L\gtrsim 50 k_0^{-1}a^{-3}$. Let us once again take a pump wave with the period T_0 in the solar reference frame. Then once again the period of this wave is approximately equal to $T_0 v_{sol}/v_A$ in the solar wind reference frame, and its frequency is $\omega_0=2\pi v_A(T_0 v_{sol})^{-1}\approx 0.6/T_0$. Using the relation $\omega_0=v_A k_0$, we obtain $L\gtrsim 80 v_A T_0 a^{-3}$. Once again taking $a=0.1$ and $T_0=1$ h, we eventually arrive at $L\gtrsim 1.5\times 10^{10}$ km $=100$ a.u. This result implies that it is highly improbable to observe spatially amplifying waves at the Earth orbit in the solar wind with the sound speed bigger than the Alfvén speed ($b>1$).

ACKNOWLEDGMENTS

One of the authors (M.S.R.) acknowledges the support of the U.K.'s PPARC (Particle Physics and Astronomy Research Council). Another author (D.S.) acknowledges the support by the University of Sheffield Endowment Fellowship.

APPENDIX: STUDY OF ABSOLUTE AND CONVECTIVE INSTABILITIES WHEN *U* IS CLOSE TO *b*

In this appendix we study the absolute and convective instabilities of the pump wave in the case when U is close to b. Let us take $U=b+aU_1$. In this case the expressions for $c_{4,5}$, $k_{4,5\pm}$, and $\omega_{4,5\pm}$ are invalid and we need to modify our analysis for these roots. We carry out the same procedure as before to calculate c, k, and ω with the only difference that $U=b+aU_1$ in Eq. (7). It is easy to show that when U_1 is of the order of unity then either $\Im(\omega)<0$ or $\Im(\omega)>\gamma_M$. Therefore we let $U_1=aU_2$ and repeat the procedure. In this case we obtain

$$c_{4,5}=b+a^2\left\{\frac{(5b-1)(3b+1)}{32b(b^2-1)}\pm x\right\}+\mathcal{O}(a^4), \tag{A1}$$

where

$$x=\frac{[33b^2-2b+1-64b(b^2-1)U_2]^{1/2}}{32b(b+1)}. \tag{A2}$$

The corresponding values of k and ω are given by

$$k_{4,5\pm}=\frac{\pm 2}{b+1}\left\{\frac{\pm 32b(b+1)x+3(b-1)}{\pm 32b(b+1)x-(b-1)}\right\}^{1/2}+\mathcal{O}(a^2), \tag{A3}$$

$$\omega_{4,5\pm}=\frac{a^2k_{4,5\pm}[\pm 32b(b+1)x+3(b-1)]}{64b(b^2-1)}\times[\pm 32b(b+1)x-(b-1)]+\mathcal{O}(a^4), \tag{A4}$$

where the $\pm$ signs inside the curly brackets correspond to the subscripts 4 and 5, respectively, and the $\pm$ signs outside the brackets correspond to the "+" and "−" subscripts. Equations (A3) and (A4) are only valid if the denominator in the curly brackets in Eq. (A3) is not close to zero. It is straightforward to show that either $\Im(\omega_{4,5\pm})<0$ or $\Im(\omega_{4,5\pm})>\gamma_M$ if x is purely imaginary. Hence, in what follows, we assume that U_2 satisfies

$$U_2<\frac{33b^2-2b+1}{64b(b^2-1)}. \tag{A5}$$

Then it follows from Eq. (A4) that we can only obtain ω with $0<\Im(\omega)<\gamma_M\sim a^3$ if we let

$$x=\frac{3(b-1)}{32b(b+1)}+\mathcal{O}(a^{2/3})$$

or

$$x=\frac{b-1}{32b(b+1)}+\mathcal{O}(a^2). \tag{A6}$$

In the first case we take the − sign in the curly brackets in Eqs. (A3) and (A4), and in the second case we take the + sign. Note that in the second case the condition that the denominator in Eq. (A3) is not close to zero is not satisfied. This implies that in this case we cannot use Eqs. (A3) and (A4) and, as we will see, in this case we also need to calculate c with better accuracy. Hence, in the case when x is given by the second formula in Eq. (A6) we will calculate c, k, and ω directly from Eqs. (7) and (8) and the relation $\tilde{\omega}=ck$.

Using Eq. (A2) we find the corresponding values of U_2,

$$U_2=U_{21}\equiv\frac{3b-1}{8b(b-1)}+a^{2/3}\chi, \tag{A7}$$

$$U_2=U_{22}\equiv\frac{b}{2(b^2-1)}+a^2\chi, \tag{A8}$$

where χ is a free parameter that has to be positive in order to have $\Im(\omega)\neq 0$. It is easy to see that these values of U_2 satisfy Eq. (A5). When $U_2=U_{21}$,

$$c_5=b+\frac{a^2(3b-1)}{8b(b-1)}+\frac{a^{8/3}\chi}{3}, \tag{A9}$$

$$k_{5\pm} = \pm 4ia^{1/3}\left[\frac{2\chi b}{3(b^2-1)}\right]^{1/2}, \tag{A10}$$

$$\omega_{5\pm} = \mp \frac{8ia^3}{3}\left[\frac{2b\chi^3}{3(b^2-1)}\right]^{1/2}. \tag{A11}$$

We do not give the expressions for c_4, $k_{4\pm}$, and $\omega_{4\pm}$ because either $\Im(\omega_{4\pm})<0$ or $\Im(\omega_{4\pm})>\gamma_M$. We obtain $0<\Im(\omega_{5-})<\gamma_M$ if χ satisfies

$$0<\chi<\frac{3}{16b^{1/3}(b^2-1)^{2/3}}. \tag{A12}$$

Now we substitute $\omega=\omega_{5-}+ia^3\sigma$ and $k=a^{1/3}\bar{k}$ in the equation $\tilde{D}(\omega,k)=0$ and solve it for $\bar{k}$ in the lowest-order approximation with respect to a, so that we can map the trajectories of the k roots as we increase σ from 0 to $[\gamma_M-\Im(\omega_{5-})]/a^3$. We obtain the cubic equation

$$\left\{y-8\left[\frac{2\chi b}{3(b^2-1)}\right]^{1/2}\right\}\left\{y+4\left[\frac{2\chi b}{3(b^2-1)}\right]^{1/2}\right\}^2=\frac{32b\sigma}{b^2-1}, \tag{A13}$$

where $y=-i\bar{k}$. When $\sigma=0$ we obtain that $y=-ik_{5-}$ is a double root as expected. A graphical investigation of Eq. (A13) shows that for any $\sigma>0$ there is only one real root, y_1, which is always greater than 0. The other two roots that collide to form the double root $y=-ik_{5-}$ when $\sigma=0$ are complex conjugate and given by $y_{2,3}=y_r\pm iy_i$. Consequently, the two k roots that collide when $\sigma=0$ are given by $\bar{k}^{\pm}=iy_r\pm y_i$, so that $\Im(\bar{k}^+)=\Im(\bar{k}^-)$. We see that $\bar{k}^+$ and $\bar{k}^-$ are always at the same side of the real axis, so that k_{5-} is not a pinching root. Hence, the instability is convective when $U=b+a^2U_{21}$.

Now we take $U_2=U_{22}$. As we have already pointed out, in this case the expression for c given by Eq. (A1) has an insufficient accuracy, so that we have to calculate c using Eq. (7). Also Eqs. (A3) and (A4) are not valid and we have to use Eq. (8) and the relation $\tilde{\omega}=ck$ to calculate k and ω. Omiting all calculations we write down the final result:

$$c_4=b+\frac{a^2b}{2(b^2-1)}-a^4\left\{\chi+\frac{b(b^2+3)}{4(b^2-1)^3}\right\}, \tag{A14}$$

$$k_{4\pm}=\pm\frac{2i(b-1)^2}{a\mu b^{1/2}}, \tag{A15}$$

$$\omega_{4\pm}=\mp\frac{ia^3\mu}{2b^{1/2}(b-1)(b+1)^3}, \tag{A16}$$

where $\mu=[8\chi(b^2-1)^3+b(b^2+3)]^{1/2}$. We do not give the expressions for c_5, $k_{5\pm}$, and $\omega_{5\pm}$ because either $\Im(\omega_{5\pm})<0$ or $\Im(\omega_{5\pm})>\gamma_M$. To have $\Im(\omega)>0$ we choose the pair (ω_{4-},k_{4-}). The condition $\omega_{4-}<\gamma_M$ reduces to

$$\chi<\frac{25-21b+11b^2-b^3}{64(b-1)(b^2-1)^3}. \tag{A17}$$

Since $\chi>0$, this inequality can be satisfied only if its right-hand side is positive. The right-hand side of Eq. (A17) is positive when $b<b_c\approx 1.33$, and it is negative otherwise. In what follows we assume that $b<b_c$ and Eq. (A17) is satisfied.

Now we take $\omega=\omega_{4-}+ia^3\sigma$ and $k=\bar{k}/a$ and substitute these expressions into the equation $\tilde{D}(\omega,k)=0$ to verify whether k_{4-} is a pinching root. This gives us a quadratic equation for $\bar{k}$,

$$b\mu^2\bar{k}^2+4i(b-1)^2[\mu b^{1/2}+2b(b-1)(b+1)^3\sigma]\bar{k}-4(b-1)^4=0. \tag{A18}$$

When $\sigma>0$ this equation has two purely imaginary roots, $\bar{k}^+$ and $\bar{k}^-$. If we multiply the imaginary parts of the roots, we obtain

$$\Im(\bar{k}^+)\Im(\bar{k}^-)=\frac{4(b-1)^4}{\mu^2 b}. \tag{A19}$$

We see that the imaginary parts of $\bar{k}^+$ and $\bar{k}^-$ always have the same signs, so that $\bar{k}^+$ and $\bar{k}^-$ are on the same side of the real axis. This implies that k_{4-} is not pinching. Thus the instability is always convective when $U=b+U_{21}$.

Summarizing the results obtained in this appendix we conclude that the instability is convective when $U=b+\mathcal{O}(a)$.

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