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NEARLY INVARIANT SUBSPACES FOR OPERATORS IN HILBERT SPACES

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ABSTRACT. For a shift operator T with finite multiplicity acting on a separable infinite dimensional Hilbert space we represent its nearly T^{-1} invariant subspaces in Hilbert space in terms of invariant subspaces under the backward shift. Going further, given any finite Blaschke product B, we give a description of the nearly T_B^{-1} invariant subspaces for the operator T_B of multiplication by B in a scale of Dirichlet-type spaces.

1. INTRODUCTION

Given α a real number, the Dirichlet-type space $\mathcal{D}_\alpha(\mathbb{D})$ consists of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathbb{D}$ such that its norm

$$\|f\|_\alpha := \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^\alpha \right)^{1/2} < +\infty.$$

If $\alpha = -1$, $\mathcal{D}_{-1} = A^2(\mathbb{D})$ the classical Bergman space, for $\alpha = 0$, $\mathcal{D}_0 = H^2(\mathbb{D})$ the Hardy space, and for $\alpha = 1$, $\mathcal{D}_1 = \mathcal{D}(\mathbb{D})$ the classical Dirichlet space, systematically investigated in the book [6]. These are particular instances of separable infinite dimensional Hilbert spaces, to be denoted by $\mathcal{H}$ in this paper. We let $\mathcal{B}(\mathcal{H})$ denote the collection of all bounded linear operators acting on $\mathcal{H}$.

The notations $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of all nonnegative integers and positive integers, respectively. Here we recall the $\mathbb{C}^l$-vector-valued Hardy space $H^2(\mathbb{D}, \mathbb{C}^l)$ consists of all analytic $F : \mathbb{D} \to \mathbb{C}^l$ such that the norm

$$\|F\| = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|F(re^{iw})\|^2 dw \right)^{\frac{1}{2}} < \infty.$$

Writing $F = [f_1, f_2, \cdots, f_l]$ with $f_i : \mathbb{D} \to \mathbb{C}$, it is clear that $F \in H^2(\mathbb{D}, \mathbb{C}^l)$ if and only if $f_i \in H^2(\mathbb{D})$ for $i = 1, 2, \cdots, l$ with $l \in \mathbb{N}$.

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Inner functions play important role in describing the invariant subspaces of the unilateral shift $Sf(z) = zf(z)$ (multiplication by the independent variable) on $H^2(\mathbb{D})$. Beurling's Theorem states that a nontrivial closed subspace $\mathcal{M} \subset H^2(\mathbb{D})$ satisfies $S\mathcal{M} \subset \mathcal{M}$ if and only if $\mathcal{M} = \theta H^2(\mathbb{D})$ with θ is inner. The simplest nontrivial inner function is an automorphism of $\mathbb{D}$ mapping $\mathbb{T}$ onto $\mathbb{T}$. More generally, if $\{a_n\}_{n\geq 1}$ is a sequence of points in $\mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$, then we can construct the corresponding Blaschke product

$$B(z) := z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}, \ m \in \mathbb{N}_0.$$

The Toeplitz operator T_g on $H^2(\mathbb{D})$ is defined by

$$T_g f = P(gf) \text{ with } g \in L^\infty(\mathbb{T})$$

and P is the orthogonal projection from $L^2(\mathbb{T})$ on $H^2(\mathbb{D})$. It is well-known that the kernel of $T_{\overline{\theta}}$ on $H^2(\mathbb{D})$ is a model space $K_\theta := H^2 \ominus \theta H^2$ with θ an inner function (cf. [8, 9, 15]).

Given a Blaschke product B, the Wold Decomposition Theorem implies that every $f \in H^2(\mathbb{D})$ has an expression

$$f(z) = \sum_{k=0}^{\infty} B^k(z) h_k(z)$$

with h_k are functions in $K_B = H^2 \ominus BH^2$. An analogous theorem for Dirichlet-type space $\mathcal{D}_\alpha(\mathbb{D})$ has been proved as follows.

Theorem 1.1. [5, Theorem 3.1], [4, Theorem 2.1] *Let $\alpha \in [-1, 1]$ and B a finite Blaschke product. Then $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f = \sum_{k=0}^{\infty} B^k h_k$ (convergence in $\mathcal{D}_\alpha$ norm) with $h_k \in K_B$ and*

$$\sum_{k=0}^{\infty} (k+1)^\alpha \|h_k\|_{H^2}^2 < \infty. \tag{1.1}$$

Since K_B is finite-dimensional, we may take other (equivalent) norms here, such as $\|h_k\|_{\mathcal{D}_\alpha}$.

A concept commonly appearing in operator theory and complex function theory is that of near invariance, which arises in the investigations of (almost) invariant subspaces. At the beginning, nearly S^* invariant subspaces of $H^2(\mathbb{D})$ were introduced by Hayashi [9], Hitt [10], and then Sarason [14] in the context of kernels of Toeplitz operators. There are also many other contributions related with this topic; for example the case of backwards shifts on vector-valued Hardy spaces was analysed in [3]. The interested reader can also refer to [1, 2, 12] and the references therein. Roughly speaking, a subspace $\mathcal{M} \subset \mathcal{H}$ is said to be nearly S^* invariant if the zeros of functions in $\mathcal{M}$ can be divided out without leaving the space. The following definition presents a nearly T^{-1} invariant subspace for any left invertible $T \in \mathcal{B}(\mathcal{H})$.

Definition 1.2. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be left invertible. Then a subspace $\mathcal{M} \subset \mathcal{H}$ is said to be nearly T^{-1} invariant if for every $g \in \mathcal{H}$ such that $Tg \in \mathcal{M}$, it holds that $g \in \mathcal{M}$.

A shift operator acting on a separable infinite dimensional Hilbert space is the direct generalization of the unilateral shift S and multiplication operator T_B on $H^2(\mathbb{D}, \mathbb{C}^l)$. It is isometric and left invertible, and was abstractly defined in [13, Chapter 1] as below.

Definition 1.3. An operator $T \in \mathcal{B}(\mathcal{H})$ is a shift operator if T is an isometry and $\|T^{*n} f\| \to 0$ for all $f \in \mathcal{H}$ as $n \to \infty$.

There are also other equivalent descriptions for a shift operator. For example, an isometry $T \in \mathcal{B}(\mathcal{H})$ is a *shift operator* if and only if T is *pure*. Here an isometry T on $\mathcal{H}$ is *pure* whenever $\bigcap_{n\geq 0} T^n\mathcal{H} = \{0\}$. Furthermore, a pure isometry $T \in \mathcal{B}(\mathcal{H})$ has *multiplicity* m if the dimension of the subspace $\mathcal{K} := \mathcal{H} \ominus T\mathcal{H} = \operatorname{Ker} T^*$ is m.

Besides, an operator $A \in \mathcal{B}(\mathcal{H})$ is T-inner if A is analytic (that is, $AT = TA$) and partially isometric. Based on the concept of T-inner operator, the Beurling-Lax Theorem for invariant subspaces under a shift operator $T \in \mathcal{B}(\mathcal{H})$ was given in [13, Section 1.12], as follows.

Theorem 1.4. *A subspace $\mathcal{F}$ of $\mathcal{H}$ is invariant under the shift operator $T \in \mathcal{B}(\mathcal{H})$ if and only if $\mathcal{F} = A\mathcal{H}$ for some T-inner operator A on $\mathcal{H}$.*

It is natural to ask how to give an expression for nearly T^{-1} invariant subspaces in $\mathcal{H}$. Motivated by this question, we organize the rest of the paper as follows. In Section 2, using the formulae of S^* invariant subspaces in vector-valued Hardy space, we present a characterization for nearly T^{-1} invariant subspaces when $T \in \mathcal{B}(\mathcal{H})$ is a shift operator with *multiplicity* m. Going beyond this, noting that a finite Blaschke product B is a multiplier of Dirichlet-type spaces, we give some descriptions for nearly T_B^{-1} invariant subspaces in Section 3, extending considerably some work of Erard [7].

2. NEARLY T^{-1} INVARIANT SUBSPACES

In this section, we always suppose $T \in \mathcal{B}(\mathcal{H})$ is a shift operator with multiplicity m. This gives

$$1 \leq l := \dim[\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})] \leq m \tag{2.1}$$

for every nonzero nearly T^{-1} invariant subspace $\mathcal{M} \subset \mathcal{H}$. Denote an orthonormal basis of $\mathcal{K} := \mathcal{H} \ominus T\mathcal{H}$ by $e_1, \cdots, e_m$. And let $\delta_j^m = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the j-th place be an orthonormal basis of $K := H^2(\mathbb{D}, \mathbb{C}^m) \ominus zH^2(\mathbb{D}, \mathbb{C}^m)$ for $j = 1, 2, \cdots, m$. Based on the following two orthogonal decompositions

$$\mathcal{H} = \bigoplus_{i=0}^{\infty} T^i \mathcal{K} \text{ and } H^2(\mathbb{D}, \mathbb{C}^m) = \bigoplus_{i=0}^{\infty} z^i K,$$

there exists a unitary mapping $U: \mathcal{H} \to H^2(\mathbb{D}, \mathbb{C}^m)$ defined by

$$U(T^i e_j) = z^i \delta_j^m. \tag{2.2}$$

So the following commutative diagram (2.3) holds for the unilateral shift $S: H^2(\mathbb{D}, \mathbb{C}^m) \to H^2(\mathbb{D}, \mathbb{C}^m)$ and the shift operator $T: \mathcal{H} \to \mathcal{H}$ with multiplicity m.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{T} & \mathcal{H} \\ \Big\downarrow U & & \Big\downarrow U \\ H^2(\mathbb{D}, \mathbb{C}^m) & \xrightarrow{S} & H^2(\mathbb{D}, \mathbb{C}^m), \end{array} \tag{2.3}$$

which implies the following equations

$$S^n U = U T^n \text{ for } n \in \mathbb{N}_0. \tag{2.4}$$

Using the unitary mapping $U : \mathcal{H} \to H^2(\mathbb{D}, \mathbb{C}^m)$, the following lemma holds, where the superscript 't' means the transpose of a matrix.

Lemma 2.1. *Let $\mathcal{M} \subset \mathcal{H}$ be a nonzero nearly T^{-1} invariant subspace and $G_0 := [g_1, g_2, \cdots, g_l]^t$ be a matrix containing an orthonormal basis $(g_i)_{i \in \{1, \cdots, l\}}$ of $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$. Then*

$$F_0 := [Ug_1, Ug_2, \cdots, Ug_l]^t \tag{2.5}$$

is a matrix containing an orthonormal basis $(Ug_i)_{i \in \{1, \cdots, l\}}$ of $U\mathcal{M} \ominus (U\mathcal{M} \cap zH^2(\mathbb{D}, \mathbb{C}^m))$.

The lemma below shows the link of nearly invariant subspaces between similar operators.

Lemma 2.2. *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ are two left invertible operators. Assume there exists an invertible operator $V : \mathcal{H}_1 \to \mathcal{H}_2$ so that $T_2 = VT_1V^{-1}$. Let $\mathcal{M}$ be a nearly T_1^{-1} invariant subspace in $\mathcal{H}_1$, then $V(\mathcal{M})$ is a nearly T_2^{-1} invariant subspace in $\mathcal{H}_2$.*

Proof. For any $h \in \mathcal{H}_2$, if $T_2h \in V(\mathcal{M})$, we need to show $h \in V(\mathcal{M})$. Since $T_2h = VT_1V^{-1}h \in V(\mathcal{M})$, it follows that $T_1V^{-1}h \in \mathcal{M}$, so that $V^{-1}h \in \mathcal{M}$. This means $h \in V(\mathcal{M})$, ending the proof. □

Lemma 2.3. *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a shift operator, and let U be as* (2.2)*. Then*

$$U^*[(Ug)h] = h(T)g, \tag{2.6}$$

for any $g \in \mathcal{H},\ h \in H^2(\mathbb{D})$ such that $(Ug)h \in H^2(\mathbb{D}, \mathbb{C}^m)$.

Proof. From the commutative diagram (2.3), we have $Ug \in H^2(\mathbb{D}, \mathbb{C}^m)$ and $(Ug)z^n = S^n(Ug)$, and then (2.4) implies that

$$U^*[(Ug)z^n] = U^*S^n(Ug) = T^nU^*Ug = T^ng, \text{ for } n \in \mathbb{N}_0.$$

For any polynomial $p(z) = \sum_{k=0}^n a_kz^k \in H^2(\mathbb{D})$, since U^* is a linear operator, we have

$$U^*[(Ug)p(z)] = \sum_{k=0}^n a_kT^kg = p(T)g,$$

where the operator $p(T) = \sum_{k=0}^n a_kT^k$. So for any $h(z) = \sum_{k=0}^\infty h_kz^k \in H^2(\mathbb{D})$ such that $(Ug)h \in H^2(\mathbb{D}, \mathbb{C}^m)$, there exists a sequence of polynomials $q_n(z) = \sum_{k=0}^n h_kz^k \in H^2(\mathbb{D})$ such that $q_n \to h$ in $H^2(\mathbb{D})$, as $n \to \infty$. Then we deduce that

$$U^*[(Ug)h(z)] = U^*[(Ug)(\lim_{n\to\infty} q_n(z))] = \lim_{n\to\infty} q_n(T)g = h(T)g,$$

with the operator $h(T) = \sum_{k=0}^\infty h_kT^k$. □

We are now in a position to state a theorem for nearly T^{-1} invariant subspaces in $\mathcal{H}$. Recall that $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{l'}, \mathbb{C}^l))$ is an operator-valued inner function if $\Phi(e^{iw})$ is an isometry a.e. on $\mathbb{T}$.

Theorem 2.4. *Suppose $T : \mathcal{H} \to \mathcal{H}$ is a shift operator with multiplicity m and $\mathcal{M} \subset \mathcal{H}$ is a nonzero nearly T^{-1} invariant subspace. Let $G_0 := \left[g_1, g_2, \cdots, g_l\right]^t$ be a matrix containing an orthonormal basis $(g_i)_{i\in\{1,\cdots,l\}}$ of $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$. Then there exist a nonnegative integer $l' \leq l$ and an operator-valued inner function Φ belonging to $H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{l'}, \mathbb{C}^l))$, unique up to unitary equivalence, such that*

$$\mathcal{M} = \{f \in \mathcal{H} :\ \exists h \in H^2(\mathbb{D}, \mathbb{C}^l) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{l'}),\ f = h(T)G_0\}.$$

Besides, there is an isometric mapping

$$Q:\ \mathcal{M} \to H^2(\mathbb{D},\mathbb{C}^l) \ominus \Phi H^2(\mathbb{D},\mathbb{C}^{l'}) \text{ given by } Q(f) = h. \tag{2.7}$$

Proof. From Lemma 2.2 and the commutative diagram (2.3), it follows $U\mathcal{M}$ is a nearly S^* invariant subspace in $H^2(\mathbb{D},\mathbb{C}^m)$. From [3, Theorem 4.4], there exists an isometric mapping

$$J:\ U\mathcal{M} \to \mathcal{F}' \text{ given by } J(hF_0) = h, \tag{2.8}$$

with a subspace

$$\mathcal{F}' := \{h \in H^2(\mathbb{D},\mathbb{C}^l):\ \exists\, Uf \in U\mathcal{M},\ \ Uf = hF_0\}$$

and F_0 given in (2.5). Moreover, $\mathcal{F}'$ is an S^* invariant subspace in $H^2(\mathbb{D},\mathbb{C}^l)$. The Beurling-Lax Theorem on $H^2(\mathbb{D},\mathbb{C}^l)$ implies there exist a nonnegative integer $l' \le l$ and an inner function $\Phi \in H^\infty(\mathbb{D},\mathcal{L}(\mathbb{C}^{l'},\mathbb{C}^l))$, unique to unitary equivalence, such that

$$\mathcal{F}' = H^2(\mathbb{D},\mathbb{C}^l) \ominus \Phi H^2(\mathbb{D},\mathbb{C}^{l'}).$$

Given any $f \in \mathcal{M}$, the isometric mapping (2.8) implies there exists $h = [h_1,\cdots,h_l] \in \mathcal{F}'$ such that

$$Uf = hF_0 = [h_1,\cdots,h_l][Ug_1,\cdots,Ug_l]^t = \sum_{i=1}^{l}(Ug_i)h_i,$$

with $\|f\| = \|Uf\| = \|h\|$. Then the formula (2.6) gives

$$f = \sum_{i=1}^{l} U^*[(Ug_i)h_i] = \sum_{i=1}^{l} h_i(T)g_i = h(T)G_0,$$

with $h(T) = [h_1(T),\cdots,h_l(T)]$. We therefore have

$$\mathcal{M} = \{f \in \mathcal{H}:\ \exists h \in H^2(\mathbb{D},\mathbb{C}^l) \ominus \Phi H^2(\mathbb{D},\mathbb{C}^{l'}),\ f = h(T)G_0\},$$

and there is an isometric mapping $Q : \mathcal{M} \to H^2(\mathbb{D},\mathbb{C}^l) \ominus \Phi H^2(\mathbb{D},\mathbb{C}^{l'})$ given by $Q = JU$ satisfying (2.7). □

Remark 2.5. Let

$$\mathcal{M}' = U^*\mathcal{F}' = U^*[H^2(\mathbb{D},\mathbb{C}^l) \ominus \Phi H^2(\mathbb{D},\mathbb{C}^{l'})] \subset \mathcal{H}.$$

Then $\mathcal{M}'$ is a T^* invariant subspace and there also exists an isometric mapping $\widetilde{Q} = U^*JU : \mathcal{M} \to \mathcal{M}'$ defined by $f \mapsto U^*h$.

Given a degree-m Blaschke product B, $T_B : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is a shift operator with multiplicity m, so we can deduce a corollary.

Corollary 2.6. *Let $\mathcal{M} \subset H^2(\mathbb{D})$ is a nonzero nearly T_B^{-1} invariant subspace with a degree-m Blaschke product B. Let the matrix $G_0(z) := [g_1(z), g_2(z), \cdots, g_l(z)]^t$ contain an orthonormal basis $(g_i(z))_{i\in\{1,\cdots,l\}}$ of $\mathcal{M} \ominus (\mathcal{M} \cap BH^2(\mathbb{D}))$. Then there exist a nonnegative integer $l' \leq l$ and an operator-valued inner function $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{l'}, \mathbb{C}^l))$, unique up to unitary equivalence, such that*

$$\mathcal{M} = \{f \in H^2(\mathbb{D}) : \exists h \in H^2(\mathbb{D}, \mathbb{C}^l) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{l'}), f(z) = h(T_B)G_0(z)\}.$$

Here we show an example to illustrate Corollary 2.6.

Example 2.7. Given $a \in \mathbb{D} \setminus \{0\}$ and some $m \in \mathbb{N}_0$, denote a subspace

$$\mathcal{M} = \varphi_a(z) \cdot \left(\bigvee \{z^{2k} : \ k \in \mathbb{N}\} \oplus \bigvee \{z, z^3, \cdots, z^{2m+1}\} \right),$$

which is nearly $T_{z^2}^{-1}$ invariant in $H^2(\mathbb{D})$. It holds that

$$\mathcal{M} \ominus (\mathcal{M} \cap z^2\mathcal{M}) = \langle g_1(z), g_2(z) \rangle$$

with the function matrix $G_0(z) := [g_1(z), g_2(z)]^t = \varphi_a(z) \cdot \left[1, z\right]^t$.

For any $f \in \mathcal{M}$, it turns out

$$f(z) = \left[\sum_{i=0}^{\infty} a_{i1} z^{2i}, \sum_{i=0}^{m} a_{i,2} z^{2i} \right] G_0(z),$$

where $(a_{i,j})_{i,j}$ satisfies

$$[a_{i,1}, a_{i,2}] \in \begin{cases} \mathbb{C} \times \mathbb{C}, & i = 0, 1, \cdots, m, \\ \mathbb{C} \times \{0\}, & i \geq m+1. \end{cases}$$

The formula in Corollary 2.6 together the above facts imply

$$\mathcal{M} = \left\{ f \in H^2(\mathbb{D}) : \exists h \in H^2(\mathbb{D}, \mathbb{C}^2) \ominus \Phi(z)H^2(\mathbb{D}), f(z) = h(T_{z^2})G_0(z) \right\},$$

with an operator-valued inner function $\Phi(z) = z^{m+1}(0, 1) \in \mathbb{C}^2$.

3. Nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$

In this section we address the more difficult question of near invariance for the operator $T_B : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ with a degree-m Blaschke product B, which is not isometric but simply bounded below, extending the methods of Erard [7].

It is known that the special multiplication operator T_B is always bounded on Dirichlet-type space $\mathcal{D}_\alpha := \mathcal{D}_\alpha(\mathbb{D})$ for any finite Blaschke product B. The study of multiplication invariant subspaces of Hardy spaces can be traced back to Lance and Stessin's work [11]; they described closed subspaces of the Hardy spaces $H^p(\mathbb{D})$ which are inner-invariant. In 2004, Erard investigated the nearly invariant subspaces related to lower bounded multiplication operator M_u on a Hilbert space $\mathcal{H}$ in [7]. There are four conditions on the pairs $(\mathcal{H}, u)$ as below:

(i) $\mathcal{H}$ is a Hilbert space and a linear submanifold of

$$\mathcal{O}(\mathcal{W}) := \{f : \mathcal{W} \to \mathbb{C} \mid f \text{ is analytic}\},$$

where $\mathcal{W}$ is an open subset of $\mathbb{C}^d$ $(d \in \mathbb{N})$,

(ii) $u \in \mathcal{O}(\mathcal{W})$ satisfies $uh \in \mathcal{H}$ for all $h \in \mathcal{H}$,

(iii) for all $w \in \mathcal{W}$, the evaluation $\mathcal{H} \to \mathbb{C}$, $h \to h(w)$ is continuous,

(iv) there exists $c > 0$ such that for all $h \in \mathcal{H}$, $c\|h\|_{\mathcal{H}} \leq \|uh\|_{\mathcal{H}}$.

In the sequel, a "subspace" means a closed linear subspace, and a "linear manifold" is an algebraic subspace that is not necessarily closed.

For the above $(\mathcal{H}, u)$, the *lower bound* of M_u relative to the norm $\|\cdot\|_{\mathcal{H}}$ is defined by

$$\gamma_{\mathcal{H},M_u} = \sup\{c > 0 : \ \forall\, h \in \mathcal{H},\ c\|h\|_{\mathcal{H}} \leq \|uh\|_{\mathcal{H}}\} \in]0, \infty[. \tag{3.1}$$

Erard gave the definition of "nearly invariant under division by u", which is same as "nearly M_u^{-1} invariant", a special case of Definition 1.2. Considering the importance of the backward shift M_z, Erard proved the following theorem on nearly S^* invariant subspaces in $\mathcal{H}$, under the assumption that $M_z : \mathcal{H} \to \mathcal{H}$ is bounded below.

Theorem 3.1. [7, Theorem 5.1] *Assume that $\mathcal{H}$ satisfies (i)-(iv) with $u(z) = z$, and*

$$dim(\mathcal{H} \ominus M_z\mathcal{H}) = 1 \text{ and } \|h\|_{\mathcal{H}} \leq \|M_z h\|_{\mathcal{H}} \text{ for all } h \in \mathcal{H}.$$

Assume also that there exists $f \in \mathcal{H}$ with $f(0) \neq 0$. Let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$ which is nearly invariant under the backward shift M_z. Let g be any unit vector of $\mathcal{M} \ominus (\mathcal{M} \cap M_z\mathcal{H})$. Then there exists a linear submanifold $\mathcal{N}$ of $H^2(\mathbb{D})$ such that $\mathcal{M} = g\mathcal{N}$ and for all $h \in \mathcal{M}$, we have

$$\|h\|_{\mathcal{H}} \geq \left\|\frac{h}{g}\right\|_{H^2(\mathbb{D})}.$$

Besides, $\mathcal{N}$ is invariant under the backward shift and $g(0) \neq 0$.

We note the operator $T_B : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is more general than $M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, so we seek characterizations for nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [-1, 1]$ and a degree-m Blaschke product B. The following lemma is the factorization of elements in a nearly T^{-1} invariant subspace with a bounded below $T \in \mathcal{B}(\mathcal{H})$. (We use the notation $P^{\mathcal{N}}$ for the orthogonal projection onto a subspace $\mathcal{N}$.)

Lemma 3.2. [7, Lemma 2.1] *Let $\mathcal{H}$ be a Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator such that for all $h \in \mathcal{H}$, $\|h\|_{\mathcal{H}} \leq \|Th\|_{\mathcal{H}}$ and $\mathcal{M}$ be* *a nearly T^{-1} invariant subspace of $\mathcal{H}$. Set*

$$R = (T^*T)^{-1}T^*P^{\mathcal{M}\cap T\mathcal{H}} \text{ and } Q = P^{\mathcal{M}\ominus(\mathcal{M}\cap T\mathcal{H})}. \tag{3.2}$$

Then for all $h \in \mathcal{M}$ and $p \in \mathbb{N}_0$, we have

$$h = \sum_{k=0}^{p} T^k Q R^k h + T^{p+1}R^{p+1}h. \tag{3.3}$$

In this section, the range of the symbol l in (2.1) is also true for $T = T_B$, a nonzero nearly T_B^{-1} invariant subspace $\mathcal{M}$ of $\mathcal{H} = \mathcal{D}_\alpha$ with $\alpha \in [-1, 1]$ and a degree-m Blaschke product B. In the sequel, we will endow the space $\mathcal{D}_\alpha$ with two different equivalent norms according to the cases $\alpha \in [0, 1]$ and $\alpha \in [-1, 0)$, so we divide the discussion into two subsections.

3.1. **Nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [0, 1]$.** In this subsection, $\mathcal{D}_\alpha$ is endowed with an equivalent norm as in (1.1) denoted by $\|\cdot\|_1$, that is,

$$\|f\|_1^2 := \sum_{k=0}^{\infty}(k+1)^\alpha \|h_k\|_{H^2}^2 \tag{3.4}$$

for any $f = \sum_{k=0}^{\infty} B^k h_k$ with $h_k \in K_B$. Then it holds that

$$\|T_B f\|_1^2 = \|Bf\|_1^2 = \sum_{k=0}^{\infty}(k+2)^\alpha \|h_k\|_{H^2}^2 \geq \|f\|_1^2,$$

which implies the operator $T_B : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is lower bounded. This confirms that $(\mathcal{D}_\alpha, T_B)$ satisfies *Conditions* (i)-(iv) and the lower bound of T_B relative the norm $\|\cdot\|_1$ is $\gamma_1 := 1$. And it is also true that $B^{-1}(D(0,1)) = B^{-1}(\mathbb{D}) = \mathbb{D}$. The above facts together with

$$\bigcap_{n\in\mathbb{N}} B^n \mathcal{D}_\alpha = \{0\} \text{ on } \mathbb{D}$$

imply the following lemma, which can be deduced from [7, Theorem 3.2] with $\mathcal{H} = \mathcal{D}_\alpha$, $u = B$, $\gamma = \gamma_1 := 1$ and the index set $I = \{1, 2, \cdots, l\}$.

Lemma 3.3. *For $\alpha \in [0, 1]$, let $\mathcal{M}$ be a nonzero nearly T_B^{-1} invariant subspace of $\mathcal{D}_\alpha$ endowed with the norm $\|\cdot\|_1$ in (3.4) and $(g_i)_{i\in\{1,\cdots,l\}}$ be a hilbertian basis of $\mathcal{M} \ominus (\mathcal{M} \cap T_B\mathcal{D}_\alpha)$. Then for all $h \in \mathcal{M}$, there exists $(q_i)_{i\in\{1,\cdots,l\}}$ in $\mathcal{O}(\mathbb{D})$ such that*

$$h = \sum_{i=1}^{l} g_i q_i \text{ on } \mathbb{D}, \tag{3.5}$$

and for all $i \in \{1, \cdots, l\}$, there exists a sequence $\{c_{ki}\}_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with

$$q_i = \sum_{k=0}^{\infty} c_{ki} B^k, \tag{3.6}$$

$$\sum_{i=1}^{l} \sum_{k=0}^{\infty} |c_{ki}|^2 \leq \|h\|_1^2. \tag{3.7}$$

Now we are ready to state a theorem on nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [0, 1]$.

Theorem 3.4. *For $\alpha \in [0, 1]$, let $\mathcal{M}$ be a nonzero nearly T_B^{-1} invariant subspace of $\mathcal{D}_\alpha$ endowed with the norm $\|\cdot\|_1$ in (3.4) and $G_0 := [g_1, g_2, \cdots, g_l]^t$ be a matrix containing an orthonormal basis $(g_i)_{i\in\{1,\cdots,l\}}$ of $\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)$. Then there exists a linear submanifold $\mathcal{N} \subset H^2(\mathbb{D}, \mathbb{C}^l)$ such that $\mathcal{M} = \mathcal{N} G_0$ and for all $h \in \mathcal{M}$, there exists $q \in \mathcal{N}$ such that $h = qG_0$ and*

$$\|h\|_1 \geq \|q\|_{H^2(\mathbb{D},\mathbb{C}^l)}.$$

Moreover, $\mathcal{N}$ is invariant under $T_{\overline{B}}$.

Proof. For $\alpha \in [0, 1]$, the equation (3.5) implies that every $h \in \mathcal{M}$ has the form

$$h = \sum_{i=1}^{l} g_i q_i = qG_0 \text{ on } \mathbb{D}, \tag{3.8}$$

with $q = [q_1, q_2, \cdots, q_l]$. Combining the series of q_i in (3.6), we deduce

$$\|q_i\|_{H^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} |c_{ki}|^2 \text{ for all } i \in \{1, \cdots, l\}.$$

And then the norm estimation in (3.7) implies

$$\|q\|_{H^2(\mathbb{D},\mathbb{C}^l)}^2 = \sum_{i=1}^{l} \|q_i\|_{H^2(\mathbb{D})}^2 = \sum_{i=1}^{l} \sum_{k=0}^{\infty} |c_{ki}|^2 \leq \|h\|_1^2, \tag{3.9}$$

with

$$q = [q_1, q_2, \cdots, q_l] = \sum_{k=0}^{\infty} B^k C_k \in H^2(\mathbb{D}, \mathbb{C}^l)$$

where $C_k = (c_{k1}, \cdots, c_{kl}) \in \mathbb{C}^l$. Then there exists a linear submanifold

$$\mathcal{N} := \{q \in H^2(\mathbb{D}, \mathbb{C}^l) : \exists h \in \mathcal{M},\ h = qG_0\},$$

satisfying $\mathcal{M} = \mathcal{N} G_0$. For all $h \in \mathcal{M}$, (3.9) implies

$$\|h\|_1 \geq \|q\|_{H^2(\mathbb{D},\mathbb{C}^l)}.$$

Next we show $\mathcal{N}$ is invariant under $T_{\overline{B}}$. Let $T = T_B$ and $\mathcal{H} = \mathcal{D}_\alpha$ with $\alpha \in [0, 1]$ in Lemma 3.2, and then the operators in (3.2) become

$$R = (T_B^* T_B)^{-1} T_B^* P^{\mathcal{M} \cap T_B \mathcal{D}_\alpha} \text{ and } Q = P^{\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)}.$$

Hence the equation (3.3) with $p = 0$ gives $h = Qh + T_B R h$, which together with (3.8) entail

$$qG_0 = Q(qG_0) + T_B R(qG_0) = C_0 G_0 + BR(qG_0).$$

And then we obtain

$$BR(qG_0) = (q - C_0)G_0 = \left(\sum_{k=1}^{\infty} B^k C_k \right) G_0,$$

which implies

$$\begin{aligned} R(qG_0) &= \left(\sum_{k=1}^{\infty} B^{k-1} C_k \right) G_0 \\ &= \left(T_{\overline{B}} \left(\sum_{k=0}^{\infty} B^k C_k \right) \right) G_0 \\ &= (T_{\overline{B}}(q)) G_0. \end{aligned}$$

The formula $T_B R h = P^{\mathcal{M} \cap T_B \mathcal{D}_\alpha} h \in \mathcal{M}$ together with the fact that $\mathcal{M}$ is a nearly T_B^{-1} invariant subspace imply $R(\mathcal{M}) \subset \mathcal{M}$. In particular, $R(qG_0) \in \mathcal{M}$ and then $T_{\overline{B}}(q) \in \mathcal{N}$ from the definition of $\mathcal{N}$. This means $\mathcal{N}$ is a $T_{\overline{B}}$ invariant submanifold of $H^2(\mathbb{D}, \mathbb{C}^l)$. □

Now consider the following special case of (2.3):

$$\begin{array}{ccc} H^2(\mathbb{D}, \mathbb{C}^l) & \xrightarrow{T_B} & H^2(\mathbb{D}, \mathbb{C}^l) \\ \big\downarrow U & & \big\downarrow U \\ H^2(\mathbb{D}, \mathbb{C}^{ml}) & \xrightarrow{S} & H^2(\mathbb{D}, \mathbb{C}^{ml}). \end{array} \tag{3.10}$$

Here $SU = UT_B$ holds for the unilateral shift $S : H^2(\mathbb{D}, \mathbb{C}^{ml}) \to H^2(\mathbb{D}, \mathbb{C}^{ml})$ and $T_B : H^2(\mathbb{D}, \mathbb{C}^l) \to H^2(\mathbb{D}, \mathbb{C}^l)$ with multiplicity ml. This leads to the following remark for finite-dimensional nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [0, 1]$.

Remark 3.5. In Theorem 3.4, if $\mathcal{M}$ is finite-dimensional, then $\mathcal{N} \subset H^2(\mathbb{D}, \mathbb{C}^l)$ is also finite-dimensional and hence closed. From the Beurling-Lax Theorem and the commutative diagram (3.10), we deduce that

$$\mathcal{N} = U^*(H^2(\mathbb{D}, \mathbb{C}^{ml}) \ominus \Psi H^2(\mathbb{D}, \mathbb{C}^r))$$

with $0 \le r \le ml$ and an inner function $\Psi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^r, \mathbb{C}^{ml}))$. Then $\mathcal{M} = [U^*(H^2(\mathbb{D}, \mathbb{C}^{ml}) \ominus \Psi H^2(\mathbb{D}, \mathbb{C}^r))]G_0$.

3.2. **Nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [-1, 0)$.** In this subsection, we cannot make T_B into an expansive operator, but it is possible to achieve a good enough lower bound by taking an equivalent norm. So we endow $\mathcal{D}_\alpha$ with the modified equivalent norm denoted by $\|\cdot\|_2$ as follows: for any $f = \sum_{k=0}^\infty B^k h_k$ with $h_k \in K_B$,

$$\|f\|_2^2 := \sum_{k=0}^{N-1} N^\alpha \|h_k\|_{H^2}^2 + \sum_{k=N}^{\infty} (k+1)^\alpha \|h_k\|_{H^2}^2, \tag{3.11}$$

where N is a fixed and sufficiently large positive integer, to be specified below. With respect to the norm $\|\cdot\|_2$, the lower bound of T_B defined in (3.1) is

$$\gamma_2 := \left(1 - \frac{1}{N+1}\right)^{-\alpha/2}. \tag{3.12}$$

Then it holds that $\|T_B f\|_2^2 = \|Bf\|_2^2 \ge \gamma_2^2 \|f\|_2^2$ for any $f \in \mathcal{D}_\alpha$, implying that the operator $T := \gamma_2^{-1} T_B : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ satisfies

$$\|Tf\|_2 = \|\gamma_2^{-1} T_B f\|_2 \ge \|f\|_2 \text{ for any } f \in \mathcal{D}_\alpha.$$

So $(\mathcal{D}_\alpha, T_B)$ also satisfies *Conditions* (i)-(iv) with the lower bound γ_2 given in (3.12) for $\alpha \in [-1, 0)$. Here we choose N large enough such that γ_2 satisfies $B^{-1}(D(0, \gamma_2)) \supset s\mathbb{D}$ with $s\mathbb{D}$ a disc containing all the zeros of B. This ensures that

$$\|\gamma_2^{-1} B\|_{H^\infty(s\mathbb{D})} < 1. \tag{3.13}$$

Furthermore, $T := \gamma_2^{-1} T_B$ satisfies the assumptions in Lemma 3.2 and

$$\bigcap_{n\in\mathbb{N}} (B^n \mathcal{D}_\alpha)\,|_{s\mathbb{D}} = \bigcap_{n\in\mathbb{N}} (T^n \mathcal{D}_\alpha)\,|_{s\mathbb{D}} = \{0\}.$$

Based on the above facts and [7, Theorem 3.2], a lemma similar to Lemma 3.3 holds for the case $\alpha \in [-1, 0)$ with γ_2 in (3.12).

Lemma 3.6. *For $\alpha \in [-1, 0)$ and $\gamma_2 := \left(1 - \frac{1}{N+1}\right)^{-\alpha/2}$ with large enough N, let $\mathcal{M}$ be a nonzero nearly T_B^{-1} invariant subspace of $\mathcal{D}_\alpha$ endowed with the norm $\|\cdot\|_2$ in* (3.11) *and $(g_i)_{i\in\{1,\cdots,l\}}$ be a hilbertian basis of $\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)$. Then for all $h \in \mathcal{M}$, there exists $(q_i)_{i\in\{1,\cdots,l\}}$ in $\mathcal{O}(s\mathbb{D})$ such that*

$$h = \sum_{i=1}^{l} g_i q_i \text{ on } s\mathbb{D}, \tag{3.14}$$

and for all $i \in \{1, \cdots, l\}$, there exists a sequence $\{d_{ki}\}_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with

$$q_i = \sum_{k=0}^{\infty} d_{ki} \left(\gamma_2^{-1} B\right)^k \quad \text{on } s\mathbb{D}, \tag{3.15}$$

$$\sum_{i=1}^{l} \sum_{k=0}^{\infty} |d_{ki}|^2 \leq \|h\|_2^2. \tag{3.16}$$

In order to use the submanifold in $H^2(\mathbb{D}, \mathbb{C}^l)$ to describe nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [-1, 0)$, here we introduce an unitary mapping $U_s : H^2(s\mathbb{D}, \mathbb{C}^l) \to H^2(\mathbb{D}, \mathbb{C}^l)$ by

$$(U_s f)(z) = f(sz).$$

Then the diagram (3.17) commutes, with $T_s^* := U_s T_{B^{-1}} U_s^*$,

$$\begin{array}{ccc} H^2(s\mathbb{D}, \mathbb{C}^l) & \xrightarrow{T_{B^{-1}}} & H^2(s\mathbb{D}, \mathbb{C}^l) \\ \Big\downarrow U_s & & \Big\downarrow U_s \\ H^2(\mathbb{D}, \mathbb{C}^l) & \xrightarrow{T_s^*} & H^2(\mathbb{D}, \mathbb{C}^l). \end{array} \tag{3.17}$$

Since the disc $s\mathbb{D}$ contains all zeros of B, the symbol B^{-1} lies in $L^\infty(s\mathbb{T})$, and thus

$$\begin{aligned} (T_s^* f)(z) &= (U_s T_{B^{-1}} U_s^* f)(z) \\ &= (U_s T_{B^{-1}}) f(s^{-1} z) \\ &= U_s \left[P_{H^2(s\mathbb{D}, \mathbb{C}^l)} \left(\frac{1}{B(z)} f(s^{-1} z) \right) \right] \\ &= P_{H^2(\mathbb{D}, \mathbb{C}^l)} \left(\frac{1}{B(sz)} f(z) \right) \\ &= T_{\frac{1}{B(sz)}} f(z) = T_{\overline{B(s^{-1} z)}} f(z), \end{aligned} \tag{3.18}$$

due to the fact $B^{-1}(sz) = \overline{B(s^{-1} z)}$ for $z \in \mathbb{T}$.

Based on the above notations, we present a theorem for nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ with $\alpha \in [-1, 0)$ and γ_2 in (3.12).

Theorem 3.7. *For $\alpha \in [-1, 0)$ and $\gamma_2 := \left(1 - \frac{1}{N+1}\right)^{-\alpha/2}$ with large enough N, let $\mathcal{M}$ be a nonzero nearly T_B^{-1} invariant subspace of $\mathcal{D}_\alpha$ endowed with the norm $\|\cdot\|_2$ in (3.11) and $G_0 := [g_1, g_2, \cdots, g_l]^t$ be a matrix containing an orthonormal basis $(g_i)_{i\in\{1,\cdots,l\}}$ of $\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)$. Then there exists a linear submanifold $\mathcal{N} \subset H^2(s\mathbb{D}, \mathbb{C}^l)$ such that $\mathcal{M} = \mathcal{N} G_0$ on $s\mathbb{D}$ and for all $h \in \mathcal{M}$ there exists $q \in \mathcal{N}$ such that*

$h = qG_0$ on $s\mathbb{D}$ with

$$\|h\|_2 \geq \left(1 - \left\|\gamma_2^{-1}B\right\|^2_{H^\infty(s\mathbb{D})}\right)^{1/2} \|q\|_{H^2(s\mathbb{D},\mathbb{C}^l)}.$$

Moreover, $\mathcal{N}$ is invariant under $T_{B^{-1}}$ and then $U_s(\mathcal{N})$ is invariant under $T_s^* := U_s T_{B^{-1}} U_s^*$ in $H^2(\mathbb{D}, \mathbb{C}^l)$.

Proof. For $\alpha \in [-1, 0)$, the equation (3.14) implies

$$h = \sum_{i=1}^{l} g_i q_i = qG_0 \text{ on } s\mathbb{D} \tag{3.19}$$

with $q = [q_1, q_2, \cdots, q_l]$. The display (3.13) and q_i in (3.15) entail

$$\begin{aligned}
\|q_i\|_{H^2(s\mathbb{D})} &= \left\|\sum_{k=0}^{\infty} d_{ki} \left(\gamma_2^{-1}B\right)^k\right\|_{H^2(s\mathbb{D})} \\
&\leq \sum_{k=0}^{\infty} |d_{ki}| \left\|\gamma_2^{-1}B\right\|^k_{H^\infty(s\mathbb{D})} \\
&\leq \left(\sum_{k=0}^{\infty} \left\|\gamma_2^{-1}B\right\|^{2k}_{H^\infty(s\mathbb{D})}\right)^{1/2} \left(\sum_{k=0}^{\infty} |d_{ki}|^2\right)^{1/2} \\
&\leq \left(1 - \left\|\gamma_2^{-1}B\right\|^2_{H^\infty(s\mathbb{D})}\right)^{-1/2} \left(\sum_{k=0}^{\infty} |d_{ki}|^2\right)^{1/2},
\end{aligned}$$

for all $i = 1, \cdots, l$. Thus the above estimations and (3.16) imply

$$\begin{aligned}
\|q\|^2_{H^2(s\mathbb{D},\mathbb{C}^l)} &= \sum_{i=1}^{l} \|q_i\|^2_{H^2(s\mathbb{D})} \\
&\leq \left(1 - \left\|\gamma_2^{-1}B\right\|^2_{H^\infty(s\mathbb{D})}\right)^{-1} \sum_{i=1}^{l} \sum_{k=0}^{\infty} |d_{ki}|^2 \\
&\leq \left(1 - \left\|\gamma_2^{-1}B\right\|^2_{H^\infty(s\mathbb{D})}\right)^{-1} \|h\|_2^2 < +\infty.
\end{aligned} \tag{3.20}$$

This implies

$$q = \sum_{k=0}^{\infty} \left(\gamma_2^{-1}B\right)^k D_k \in H^2(s\mathbb{D}, \mathbb{C}^l),$$

where $D_k = (d_{k1}, d_{k2}, \cdots, d_{kl}) \in \mathbb{C}^l$. Hence define a linear submanifold

$$\mathcal{N} := \{q \in H^2(s\mathbb{D}, \mathbb{C}^l) : \exists h \in \mathcal{M}, \ \ h = qG_0 \text{ on } s\mathbb{D}\}, \tag{3.21}$$

satisfying $\mathcal{M} = \mathcal{N}G_0$ on $s\mathbb{D}$. For all $h \in \mathcal{M}$, (3.20) gives

$$\|h\|_2 \geq \left(1 - \left\|\gamma_2^{-1}B\right\|^2_{H^\infty(s\mathbb{D})}\right)^{1/2} \|q\|_{H^2(s\mathbb{D},\mathbb{C}^l)}.$$

Next we show $\mathcal{N}$ is invariant under $T_{B^{-1}}$. Let $T = \gamma_2^{-1}T_B$ in the display (3.2), and then the equation (3.3) with $p = 0$ gives

$$h = Qh + TRh = Qh + \gamma_2^{-1}T_B Rh.$$

On $s\mathbb{D}$, the above equation together with (3.19) entail

$$qG_0 = Q(qG_0) + \gamma_2^{-1}T_B R(qG_0) = D_0G_0 + \gamma_2^{-1}BR(qG_0),$$

which further verifies

$$\begin{aligned}\gamma_2^{-1}BR(qG_0) &= (q - D_0)G_0 \\ &= \left(\sum_{k=1}^{\infty} \left(\gamma_2^{-1}B\right)^k D_k\right) G_0 \text{ on } s\mathbb{D}.\end{aligned}$$

Letting B^{-1} act on both sides, it yields that

$$\begin{aligned}\gamma_2^{-1}R(qG_0) &= \gamma_2^{-1}\left(\sum_{k=1}^{\infty} \left(\gamma_2^{-1}B\right)^{k-1} D_k\right) G_0 \\ &= \gamma_2^{-1}\left(\gamma_2 T_{B^{-1}}(q)\right) G_0 \\ &= \left(T_{B^{-1}}(q)\right) G_0,\end{aligned}$$

which together with $\gamma_2^{-1}R(qG_0) \in \mathcal{M}$ entail $T_{B^{-1}}(q) \in \mathcal{N}$ from the definition of $\mathcal{N}$ in (3.21). That means $\mathcal{N}$ is $T_{B^{-1}}$ invariant in $H^2(s\mathbb{D}, \mathbb{C}^l)$. Finally, the commutative diagram (3.17) ensures that $T_s^*(U_s(\mathcal{N})) \subset U_s(\mathcal{N})$, ending the proof. □

In order to illustrate the operator T_s^* given in (3.18), we firstly take a degree-1 Blaschke product $B(z) = (a - z)(1 - \overline{a}z)^{-1}$ with $a \in \mathbb{D}$, it is easy to obtain

$$B(s^{-1}z) = \frac{a - s^{-1}z}{1 - \overline{a}s^{-1}z} = \frac{1}{s}\frac{as - z}{1 - \overline{a}sz}\frac{1 - \overline{a}sz}{1 - \overline{a}s^{-1}z}. \tag{3.22}$$

Since the disc $s\mathbb{D}$ contains the zero of B, that means $|\overline{a}s^{-1}| < 1$, and then the last term in (3.22) is an invertible analytic function on $\mathbb{D}$. This is to say that $B(s^{-1}z)$ can be written as a degree-1 Blaschke product times an invertible analytic function.

Generally, if B is a degree-m Blaschke product, it can be similarly calculated that

$$B(s^{-1}z) = b(z)F_s(z)$$

with a degree-m Blaschke product $b(z)$ and an invertible analytic function $F_s(z)$ on $\mathbb{D}$. So $T_s^* = T_{\overline{bF_s}}$ and then $T_s = T_{bF_s}$ on $H^2(\mathbb{D}, \mathbb{C}^l)$.

Suppose $\mathcal{F} \subset H^2(\mathbb{D}, \mathbb{C}^l)$ is a T_b invariant subspace, Theorem 1.4 implies $\mathcal{F} = AH^2(\mathbb{D}, \mathbb{C}^l)$ with some T_b-inner operator A. For the special case $l = 1$, it follows that $\mathcal{F} = \theta H^2(\mathbb{D})$ with an inner function θ. The fact $F_s H^2(\mathbb{D}) = H^2(\mathbb{D})$ entails $\theta H^2(\mathbb{D})$ is also a T_{bF_s}-invariant subspace in $H^2(\mathbb{D})$. So the model space K_θ is T_s^* invariant.

In general, there is no simple description for T_s^*-invariant subspaces of $H^2(\mathbb{D})$, although in the finite-dimensional case, elementary linear algebra tells us that they are spanned by generalized eigenvectors of T_s^*, that is, elements of Toeplitz kernels. The paper [2] is relevant here.

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References

[1] Câmara, M.C., Partington, J.R.: Near invariance and kernels of Toeplitz operators. Journal d'Analyse Math. 124, 235–260 (2014)

[2] Câmara, M.C., Partington, J.R.: Finite-dimensional Toeplitz kernels and nearly-invariant subspaces. J. Operator Theory. 75(1), 75–90 (2016)

[3] Chalendar, I., Chevrot, N., Partington, J.R.: Nearly invariant subspaces for backwards shifts on vector-valued Hardy spaces. J. Operator Theory. 63(2), 403–415 (2010)

[4] Chalendar, I., Gallardo-Gutiérrez, E.A., Partington, J.R.: Weighted composition operators on the Dirichlet space: boundedness and spectral properties. Math. Ann. 363, 1265–1279 (2015)

[5] Gallardo-Gutiérrez, E.A., Partington J.R., Seco, D.: On the wandering property in Dirichlet spaces. Integral Equations and Operator Theory. 92, no. 2, paper no. 16 (2020)

[6] El-Fallah, O., Kellay, K., Mashreghi J., Ransford, T.: A Primer on the Dirichlet Space. Cambridge Tracts in Mathematics, 203, Cambridge University Press, Cambridge (2014)

[7] Erard, C.: Nearly invariant subspaces related to multiplication operators in Hilbert spaces of analytic functions. Integral Equations and Operator Theory. 50, 197–210 (2004)

[8] Garcia, S., Mashreghi, J., Ross, W.T.: Introduction to Model Spaces and Their Operators. Cambridge: Cambridge University Press (2016)

[9] Hayashi, E.: The kernel of a Toeplitz operator. Integral Equations and Operator Theory. 9(4), 588–591 (1986)

[10] Hitt, D.: Invariant subspaces of $\mathcal{H}^2$ of an annulus. Pacific J. Math. 134(1), 101–120 (1988)

[11] Lance, T.L., Stessin, M.I.: Multiplication invariant subspaces of Hardy spaces. Can. J. Math. 49(1), 100–118 (1997)

[12] Liang, Y., Partington, J.R.: Representing kernels of perturbations of Toeplitz operators by backward shift-invariant subspaces. Integral Equations and Operator Theory. 92, no. 4, paper no. 35 (2020)

[13] Rosenblum, M., Rovnyak, J.: Hardy classes and operator theory. Oxford University Press, New York (1985)

[14] Sarason, D.: Nearly invariant subspaces of the backward shift, Contributions to operator theory and its applications (Mesa, AZ, 1987), 481-493, Oper. Theory Adv. Appl. 35, Birkhäuser, Basel (1988)

[15] Sarason, D.: Kernels of Toeplitz operators. Oper. Theory: Adv. Appl. 71, 153–164 (1994)

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