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Propagation of solitons of the Derivative Nonlinear Schrödinger equation in a plasma with fluctuating density

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The propagation of quasi-parallel nonlinear small-amplitude magnetohydrodynamic waves in a cold Hall plasma with fluctuating density is studied. The density is assumed to be a homogeneous random function of one spatial variable. The modified Derivative Nonlinear Schrödinger equation (DNLS) is derived with the use of the mean waveform method developed by Gurevich, Jeffrey, and Pelinovsky [Wave Motion **17**, 287 (1993)], which is the generalization of the reductive perturbation method for nonlinear waves propagating in random media. This equation differs from the standard DNLS equation by one additional term describing the interaction of nonlinear waves with random density fluctuations. As an example of the use of the modified DNLS equation, the quasi-adiabatic evolution of a one-parametric DNLS soliton propagating through a plasma with fluctuating density is studied. [DOI: 10.1063/1.1482764]

I. INTRODUCTION

The Derivative Nonlinear Schrödinger equation (DNLS) describes the propagation of small-amplitude nonlinear magnetohydrodynamic (MHD) waves at small angles with respect to the equilibrium magnetic field. It was first derived by Rogister[1] starting with the Vlasov kinetic description for the particle species. Later it was derived by Mjϕlhus[2] and Mio *et al.*[3] on the basis of Hall magnetohydrodynamics for cold plasmas, and by Spangler and Sheerin[4] and by Sakai and Sonnerup[5] from fluid models for warm plasmas. A comprehensive review of the theory of quasi-parallel small-amplitude MHD waves based on the use of the DNLS equation and its extensions has been given by Mjϕlhus and Hada.[6] The comparison of this theory with the observation of nonlinear MHD waves at the Earth's bow shock has been considered by Spangler.[7]

In the majority of papers on the DNLS equation the equilibrium plasma was assumed to be homogeneous. However, Buti[8] derived the modified DNLS equation for waves in a Hall plasma with a homogeneous equilibrium magnetic field and plasma density varying along the magnetic field. Buti *et al.*[9] extended this derivation for the radial equilibrium magnetic field.

The aim of this paper is also to derive the modified DNLS equation for a Hall plasma with inhomogeneous density. However, in contrast to Buti[8] and Buti *et al.*,[9] we consider random fluctuation of the equilibrium density. The paper is organized as follows. In the next section we describe the equilibrium state and present the governing equations and boundary conditions. In Sec. III we derive the modified DNLS equation describing the quasi-parallel propagation of small-amplitude nonlinear MHD waves in a cold plasma with fluctuating density. In Sec. IV we study the evolution of one-parameter solitons due to density fluctuations. Section V contains our conclusions.

II. BASIC EQUATIONS

We consider wave propagation in a cold Hall plasma. The plasma motions are described by the system of Hall MHD equations,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + l\left(\frac{\rho_{00}}{\mu}\right)^{1/2}\nabla\times\frac{1}{\rho}(\mathbf{B}\times\nabla\times\mathbf{B}). \tag{3}$$

Here ρ is the density, $\mathbf{v}$ the velocity, $\mathbf{B}$ the magnetic field, μ the magnetic permeability of empty space, and $\rho_{00}=\text{const}$ is the averaged equilibrium density. The ion inertia length l is given by

$$l = \left(\frac{m_i^2}{\mu e^2 \rho_{00}}\right)^{1/2}, \tag{4}$$

where m_i is the ion mass and e the elemental electric charge.

In what follows we consider perturbations that depend only on x in Cartesian coordinates x, y, z. We assume that the characteristic scale of perturbations is $\epsilon^2 l$, where $\epsilon \ll 1$. In accordance with this we introduce the scaled ion inertia length $\overline{l} = \epsilon^{-2} l$. We consider the fluctuating equilibrium density ρ_0 with the fluctuation amplitude of order ϵ, so that

$$\rho_0 = \rho_{00}[1 + \epsilon\theta(x)], \tag{5}$$

where $\theta(x)$ is a homogeneous random function with the zero average ($\langle\theta(x)\rangle = 0$, the angular brackets indicating the stochastic averaging). The autocorrelation function is given by

$$R(x_1 - x_2) = \langle\theta(x_1)\theta(x_2)\rangle. \tag{6}$$

Since all variables depend only on one spatial variable x, the system of Eqs. (1)–(3) can be rewritten as

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \tag{7}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{1}{2\mu}\frac{\partial|\mathbf{B}_\perp|^2}{\partial x}, \tag{8}$$

$$\rho\left(\frac{\partial \mathbf{v}_\perp}{\partial t}+u\frac{\partial \mathbf{v}_\perp}{\partial x}\right)=\frac{B_x}{\mu}\frac{\partial\mathbf{B}_\perp}{\partial x}, \tag{9}$$

$$\frac{\partial\mathbf{B}_\perp}{\partial\tau}=B_x\frac{\partial\mathbf{v}_\perp}{\partial x}-\frac{\partial(u\mathbf{B}_\perp)}{\partial x}+\epsilon^2\overline{l}B_x\left(\frac{\rho_{00}}{\mu}\right)^{1/2}\hat{\mathbf{x}}\times\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial\mathbf{B}_\perp}{\partial x}\right). \tag{10}$$

Here u and $B_x=\text{const}$ are the x-components of the velocity and the magnetic field, $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$ are the components of the velocity and the magnetic field perpendicular to the x-direction, and $\hat{\mathbf{x}}$ is the unit vector in the x-direction. In what follows we look for solutions to Eqs. (7)–(10) in the region $x>0$ for all moments of time (positive and negative), and assume that $u,\mathbf{v}_\perp\to 0$, $\rho\to\rho_0$, and $\mathbf{B}_\perp\to\mathbf{B}_0$ as $|t|\to\infty$.

The system of Eqs. (7)–(10) will be used in the next section to derive the governing equation for $\mathbf{B}_\perp$.

III. DERIVATION OF THE GOVERNING EQUATION

To derive the governing equation for quasi-parallel small-amplitude waves we use the mean waveform method developed by Gurevich *et al.*[10] (see also Refs. 11 and 12). This method is a generalization of the reductive perturbation method[13–16] for nonlinear waves propagating in random media. Following Gurevich *et al.*,[10] we introduce the running variable,

$$\tau=t-\int_0^x\lambda(\xi)d\xi, \tag{11}$$

and the sequence of stretched variables,

$$X_1=\epsilon x,\quad X_2=\epsilon^2 x,\ \ldots\ . \tag{12}$$

In the new variables the system of Eqs. (7)–(10) is rewritten as

$$\frac{\partial\rho}{\partial\tau}-\lambda\frac{\partial(\rho u)}{\partial\tau}+\mathcal{L}[\rho u]=0, \tag{13}$$

$$\rho\left(\frac{\partial u}{\partial\tau}-\lambda u\frac{\partial u}{\partial\tau}+u\mathcal{L}[u]\right)=\frac{\lambda}{2\mu}\frac{\partial|\mathbf{B}_\perp|^2}{\partial\tau}-\frac{1}{2\mu}\mathcal{L}[|\mathbf{B}_\perp|^2], \tag{14}$$

$$\rho\left(\frac{\partial\mathbf{v}_\perp}{\partial\tau}-\lambda u\frac{\partial\mathbf{v}_\perp}{\partial\tau}+u\mathcal{L}[\mathbf{v}_\perp]\right)=-\frac{\lambda B_x}{\mu}\frac{\partial\mathbf{B}_\perp}{\partial\tau}+\frac{B_x}{\mu}\mathcal{L}[\mathbf{B}_\perp], \tag{15}$$

$$\begin{aligned}\frac{\partial\mathbf{B}_\perp}{\partial\tau}&=\lambda\frac{\partial}{\partial\xi}(u\mathbf{B}_\perp-B_x\mathbf{v}_\perp)-\mathcal{L}[u\mathbf{B}_\perp-B_x\mathbf{v}_\perp]\\&\quad+\epsilon^2\overline{l}B_x\left(\frac{\rho_{00}}{\mu}\right)^{1/2}\hat{\mathbf{x}}\times\left(\lambda\frac{\partial}{\partial\tau}-\mathcal{L}\right)\\&\quad\times\left\{\frac{1}{\rho}\left(\lambda\frac{\partial\mathbf{B}_\perp}{\partial\tau}-\mathcal{L}[\mathbf{B}_\perp]\right)\right\},\end{aligned} \tag{16}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L}=\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial X_1}+\epsilon^2\frac{\partial}{\partial X_2}+\cdots. \tag{17}$$

We look for the solution to the system of Eqs. (13)–(16) in the form of expansions,

$$\begin{aligned}&\rho=\rho_{00}(1+\epsilon\theta)+\epsilon^2\rho_2+\cdots,\\&u=\epsilon^2u_2+\cdots,\\&\mathbf{v}_\perp=\epsilon\mathbf{v}_{\perp 1}+\epsilon^2\mathbf{v}_{\perp 2}+\epsilon^3\mathbf{v}_{\perp 3}+\cdots,\\&\mathbf{B}_\perp=\epsilon\mathbf{B}_{\perp 1}+\epsilon^2\mathbf{B}_{\perp 2}+\epsilon^3\mathbf{B}_{\perp 3}+\cdots\ .\end{aligned} \tag{18}$$

The inverse spatial-dependent velocity $\lambda(x)$ is also expanded as

$$\lambda=\lambda_0+\epsilon\lambda_1+\epsilon^2\lambda_2+\cdots. \tag{19}$$

In what follows we assume that all quantities except $\mathbf{v}_{\perp 1}$ and $\mathbf{B}_{\perp 1}$ are zero at $x=0$.

A. First-order approximation

In the first-order approximation we collect terms of order ϵ in Eqs. (13)–(16). Since such terms are present only in Eqs. (15) and (16), we obtain

$$\begin{aligned}&\frac{\partial\mathbf{v}_{\perp 1}}{\partial\tau}=\frac{B_x}{\mu\rho_{00}}\left(\frac{\partial\mathbf{B}_{\perp 1}}{\partial x}-\lambda_0\frac{\partial\mathbf{B}_{\perp 1}}{\partial\tau}\right),\\&\frac{\partial\mathbf{B}_{\perp 1}}{\partial\tau}=B_x\left(\frac{\partial\mathbf{v}_{\perp 1}}{\partial x}-\lambda_0\frac{\partial\mathbf{v}_{\perp 1}}{\partial\tau}\right).\end{aligned} \tag{20}$$

In what follows we consider a particular solution to this system of equations corresponding to the wave propagating in the positive x-direction. This solution is given by

$$\lambda_0=\frac{(\mu\rho_{00})^{1/2}}{|B_x|}\equiv\frac{1}{V},\quad \mathbf{v}_{\perp 1}=-\frac{V}{B_x}(\mathbf{B}_{\perp 1}-\mathbf{B}_0), \tag{21}$$

where V is the Alfvén speed and $\mathbf{B}_{\perp 1}$ is independent of x.

B. Second-order approximation

In the second-order approximation we collect terms of order ϵ^2 in Eqs. (13)–(16). This yields

$$\frac{\partial\rho_2}{\partial\tau}-\frac{\rho_{00}}{V}\frac{\partial u_2}{\partial\tau}+\rho_{00}\frac{\partial u_2}{\partial x}=0, \tag{22}$$

$$\frac{\partial u_2}{\partial \tau} = \frac{V}{2B_x^2}\frac{\partial |\mathbf{B}_{\perp 1}|^2}{\partial \tau}, \tag{23}$$

$$\frac{\partial \mathbf{v}_{\perp 2}}{\partial \tau} + \frac{V}{B_x}\frac{\partial \mathbf{B}_{\perp 2}}{\partial \tau} - \frac{V^2}{B_x}\frac{\partial \mathbf{B}_{\perp 2}}{\partial x} = \frac{V^2}{B_x}\left(\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_1} - \lambda_1 \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}\right) - \theta \frac{\partial \mathbf{v}_{\perp 1}}{\partial x}, \tag{24}$$

$$\frac{\partial \mathbf{B}_{\perp 2}}{\partial \tau} + \frac{B_x}{V}\frac{\partial \mathbf{v}_{\perp 2}}{\partial \tau} - B_x \frac{\partial \mathbf{v}_{\perp 2}}{\partial x} = B_x \frac{\partial \mathbf{v}_{\perp 1}}{\partial X_1} - B_x \lambda_1 \frac{\partial \mathbf{v}_{\perp 1}}{\partial \tau}. \tag{25}$$

It follows from Eqs. (22) and (23) and the conditions at $|t| \to \infty$ that

$$\frac{u_2}{V} = \frac{\rho_2}{\rho_{00}} = \frac{|\mathbf{B}_{\perp 1}|^2 - |\mathbf{B}_0|^2}{2B_x^2}. \tag{26}$$

When obtaining this result we have taken into account that u_2 is independent of x and, as a result, the last term in Eq. (22) is zero. With the account of Eq. (21) we obtain from Eqs. (24) and (25),

$$\frac{\partial}{\partial x}\left(\mathbf{B}_{\perp 2} - \frac{B_x}{V}\mathbf{v}_{\perp 2}\right) = \left(2\lambda_1 - \frac{\theta}{V}\right)\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} - 2\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_1}. \tag{27}$$

It follows from this equation that

$$\frac{\partial}{\partial x}\left\langle \mathbf{B}_{\perp 2} - \frac{B_x}{V}\mathbf{v}_{\perp 2}\right\rangle = 2\langle \lambda_1 \rangle \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} - 2\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_1}. \tag{28}$$

Since the right-hand side of Eq. (28) is independent of x, this equation implies that the quantity $\langle \mathbf{B}_{\perp 2} - B_x \mathbf{v}_{\perp 2}/V\rangle$ is a linear function of x. However, this quantity must be bounded for $x \to \infty$, which is only possible if the right-hand side of Eq. (28) is zero, i.e.,

$$\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_1} = \langle \lambda_1 \rangle \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}, \tag{29}$$

which yields

$$\mathbf{B}_{\perp 1} = \mathbf{B}_{\perp 1}(\tau + \langle \lambda_1 \rangle X_1, X_2, \dots). \tag{30}$$

Using Eq. (29) we rewrite Eq. (27) as

$$\frac{\partial}{\partial x}\left(\mathbf{B}_{\perp 2} - \frac{B_x}{V}\mathbf{v}_{\perp 2}\right) = g(x)\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}, \tag{31}$$

where

$$g(x) = 2\lambda_1(x) - 2\langle \lambda_1 \rangle - \frac{\theta(x)}{V}. \tag{32}$$

The solution to Eq. (31) is

$$\mathbf{B}_{\perp 2} - \frac{B_x}{V}\mathbf{v}_{\perp 2} = \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}\int_0^x g(\xi)\,d\xi. \tag{33}$$

It is easy to obtain

$$\left\langle \left(\int_0^x g(\xi)\,d\xi\right)^2 \right\rangle = 2\int_0^x (x-\xi) R_g(\xi)\,d\xi, \tag{34}$$

where $R_g(x) = \langle g(\xi) g(x+\xi)\rangle$ is the autocorrelation function of the homogeneous random function $g(x)$. Usually $R_g(x)$ tends to zero fast enough as $x \to \infty$, so that

$$\left|\int_0^\infty \xi R_g(\xi)\,d\xi\right| < \infty. \tag{35}$$

Then, assuming that $\int_0^\infty R_g(\xi)\,d\xi \neq 0$, we obtain that the mean value of the square of the left-hand side of Eq. (33) grows as x when $x \to \infty$, i.e., it behaves similar to the Brownian particle. However, it must be bounded when $|x| \to \infty$, which is only possible if $g(x) = 0$. Hence, we obtain

$$\lambda_1(x) = \langle \lambda_1 \rangle + \frac{\theta(x)}{2V}, \quad \mathbf{v}_{\perp 2} = \frac{V}{B_x}\mathbf{B}_{\perp 2}. \tag{36}$$

Using Eqs. (21), (30), and (36), we rewrite Eq. (24) as

$$\frac{\partial \mathbf{B}_{\perp 2}}{\partial x} - \frac{2}{V}\frac{\partial \mathbf{B}_{\perp 2}}{\partial \tau} = -\frac{\theta(x)}{2V}\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}. \tag{37}$$

The solution to this equation is

$$\mathbf{B}_{\perp 2} = -\int_0^x \frac{\theta(\xi)}{2V}\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}(\tau + 2(x-\xi)/V + \langle \lambda_1 \rangle X_1, X_2, \dots)\,d\xi. \tag{38}$$

In order to eliminate transitional effects which arise when the random process is switched on at a given spatial position, it is convenient to shift the x value at which the boundary conditions are given from 0 to $-\infty$. Then we rewrite Eq. (38) as

$$\mathbf{B}_{\perp 2} = -\int_0^\infty \frac{\theta(x-\xi)}{2V}\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}(\tau + 2\xi/V + \langle \lambda_1 \rangle X_1, X_2, \dots)\,d\xi. \tag{39}$$

C. Third-order approximation

In the third-order approximation we only use Eqs. (15) and (16). Collect terms of order ϵ^3 in these equations and using Eqs. (21), (26), (30), and (36), we obtain

$$\frac{\partial \mathbf{v}_{\perp 3}}{\partial \tau} + \frac{V}{B_x}\frac{\partial \mathbf{B}_{\perp 3}}{\partial \tau} - \frac{V^2}{B_x}\frac{\partial \mathbf{B}_{\perp 3}}{\partial x} = \frac{V^2}{B_x}\left\{\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_2} - \lambda_2 \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} + \frac{\partial \mathbf{B}_{\perp 2}}{\partial X_1} - \left(\langle \lambda_1 \rangle - \frac{\theta}{2V}\right)\frac{\partial \mathbf{B}_{\perp 2}}{\partial \tau}\right\}, \tag{40}$$

$$\frac{\partial \mathbf{B}_{\perp 3}}{\partial \tau} + \frac{B_x}{V}\frac{\partial \mathbf{v}_{\perp 3}}{\partial \tau} - B_x \frac{\partial \mathbf{v}_{\perp 3}}{\partial x} = V\left\{\lambda_2 \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} - \frac{\partial \mathbf{B}_{\perp 1}}{\partial X_2} + \frac{\partial \mathbf{B}_{\perp 2}}{\partial X_1} - \left(\langle \lambda_1 \rangle + \frac{\theta}{2V}\right)\frac{\partial \mathbf{B}_{\perp 2}}{\partial \tau}\right\} + \frac{1}{V}\frac{\partial}{\partial \tau}(u_2 \mathbf{B}_{\perp 1}) + 2\chi\kappa V \hat{\mathbf{x}} \times \frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \tau^2}, \tag{41}$$

where $\kappa=\overline{l}/2V^2$ and $\chi=\mathrm{sign}(B_x)$. It follows from these equations that

$$\frac{\partial}{\partial x}\left(\mathbf{B}_{\perp 3}-\frac{B_x}{V}\mathbf{v}_{\perp 3}\right)=2\lambda_2\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}-2\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_2}-\frac{\theta}{V}\frac{\partial \mathbf{B}_{\perp 2}}{\partial \tau}+\frac{1}{V}\frac{\partial}{\partial \tau}(u_2\mathbf{B}_{\perp 1})+2\chi\kappa\hat{\mathbf{x}}\times\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \tau^2}. \quad (42)$$

Averaging this equation and taking into account that the quantity $\langle \mathbf{B}_{\perp 3}-B_x\mathbf{v}_{\perp 3}/V\rangle$ is bounded as $x\to\infty$, we arrive at

$$\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_2}=\langle\lambda_2\rangle\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}-\frac{1}{V}\frac{\partial}{\partial \tau}(\langle \theta \mathbf{B}_{\perp 2}\rangle-u_2\mathbf{B}_{\perp 1})+\chi\kappa\hat{\mathbf{x}}\times\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \tau^2}. \quad (43)$$

With the aid of Eqs. (6) and (39) we obtain

$$\langle \theta \mathbf{B}_{\perp 2}\rangle=-\int_0^\infty \frac{R(\xi)}{2V}\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}(\tau+2\xi/V+\langle\lambda_1\rangle X_1,X_2,\ldots)d\xi. \quad (44)$$

The substitution of Eqs. (26) and (44) into Eq. (43) yields

$$\frac{\partial \mathbf{B}_{\perp 1}}{\partial X_2}=\langle\lambda_2\rangle\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}+\frac{1}{4VB_x^2}\frac{\partial}{\partial \tau}\{\mathbf{B}_{\perp 1}(|\mathbf{B}_{\perp 1}|^2-|\mathbf{B}_0|^2)\}+\chi\kappa\hat{\mathbf{x}}\times\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \tau^2}+\int_0^\infty \frac{R(\xi)}{4V^2}\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \tau^2}(\tau+2\xi/V+\langle\lambda_1\rangle X_1,X_2,\ldots)d\xi. \quad (45)$$

Let us introduce the new variables,

$$\tilde{\mathbf{B}}_{\perp 1}(\tau+\langle\lambda_2\rangle X_2,X_2,\ldots)=\mathbf{B}_{\perp 1}(\tau,X_2,\ldots),$$
$$b=(\tilde{B}_{y1}+i\tilde{B}_{z1})/|B_x|,\quad b_0=(B_{y0}+iB_{z0})/|B_x|, \quad (46)$$
$$\tau'=\tau+\langle\lambda_1\rangle X_1+\langle\lambda_2\rangle X_2,$$

and the notation $\sigma=X_2$. Then, dropping the prime at τ', we rewrite Eq. (45) as

$$\frac{\partial b}{\partial \sigma}-\frac{1}{4V}\frac{\partial}{\partial \tau}\{b(|b|^2-|b_0|^2)\}-i\chi\kappa\frac{\partial^2 b}{\partial \tau^2}=\frac{1}{4V^2}\frac{\partial^2}{\partial \tau^2}\int_0^\infty R(\xi)b(\tau+2\xi/V,\sigma)d\xi. \quad (47)$$

When there are no density fluctuations $[R(\xi)=0]$ this equation coincides (with the accuracy up to the notation) with the DNLS equation for a cold plasma.[2,3,6]

IV. SOLITON EVOLUTION DUE TO DENSITY FLUCTUATIONS

The DNLS equation describes a few different types of solitons. Here we consider only one-parameter solitons. The analytic expression for these solitons can be given either in a modulational form,[17–19] or as a rational expression of exponential functions.[6] There are two types of one-parametric solitons, bright and dark.[20,6] In bright solitons $|b|>|b_0|$, while in dark solitons $|b|<|b_0|$. Since dark solitons are unstable to transverse perturbations,[19] it does not make very much sense to consider their evolution due to density fluctuations. Therefore, we concentrate on bright solitons in what follows. We write the rational expression of exponential functions for bright one-parameter solitons in a form slightly different from that in Miϕlhus and Hada:[6]

$$b=b_0\left\{1+\frac{4\sin^2\alpha\exp(-\gamma\Theta-i\alpha)}{[1-\exp(-\gamma\Theta-i\alpha)]^2}\right\}, \quad (48)$$

where

$$\Theta=\chi\tau+\Lambda\sigma,\quad \gamma=\frac{|b_0|^2\sin 2\alpha}{4\kappa V},\quad \Lambda=\frac{|b_0|^2\cos^2\alpha}{2V}. \quad (49)$$

The angle α varies from 0 to $\pi/2$. The soliton amplitude is given by

$$A=\max(|b|-|b_0|)=2|b_0|\cos\,\alpha. \quad (50)$$

We see that the amplitude decreases when α increases.

Multiplying Eq. (47) by b^*, where the asterisk indicates a complex conjugate quantity, taking the real part of the result, and integrating with respect to τ, we obtain

$$\frac{\partial}{\partial \sigma}\int_{-\infty}^{\infty}(|b|^2-|b_0|^2)d\tau=-\frac{1}{2V^2}\int_0^\infty R(\xi)F(\xi)d\xi, \quad (51)$$

where

$$F(\xi)=\Re\left\{\int_{-\infty}^{\infty}\frac{\partial b^*}{\partial \tau}(\tau,\sigma)\frac{\partial b}{\partial \tau}(\tau+2\xi/V,\sigma)d\tau\right\}, \quad (52)$$

with $\Re$ indicating the real part of a quantity. Now we assume that the right-hand side of Eq. (47) is small in comparison with the left-hand side and can be considered as a perturbation. Then we can look for a solution for Eq. (47) in the form of the one-parametric soliton given by Eqs. (48) and (49) with the parameter α slowly varying with σ. If b is given by Eq. (48) then, after a long but straightforward calculation, we obtain

$$\int_{-\infty}^{\infty}(|b|^2-|b_0|^2)d\tau=16\kappa V(\pi-\alpha), \quad (53)$$

$$F(\xi)=\frac{4|b_0|^4}{\kappa V}\sin^4\alpha\sin 2\alpha\,\Re\left\{\psi\left[4\frac{1+4\psi+\psi^2}{(\psi-1)^4}-\phi\frac{(1+\psi)(1+10\psi+\psi^2)}{(\psi-1)^5}\right]\right\}, \quad (54)$$

where

$$\phi=2i(\alpha-\pi)+\frac{\chi\xi|b_0|^2\sin 2\alpha}{2\kappa V^2},\quad \psi=e^{\phi}. \quad (55)$$

Let us introduce the characteristic length of the function's $F(\xi)$ variation,

$$L=\frac{2\kappa V^2}{|b_0|^2}=(\epsilon|b_0|)^{-2}l. \tag{56}$$

For α not very close to either 0 or $\pi/2$, this quantity is of order of the soliton width γ^{-1}. When $|\xi|\sin 2\alpha \gg L$, the function $F(\xi)$ can be approximated by

$$F(\xi)\approx-\frac{2|\xi||b_0|^4}{\kappa VL}\sin^4\alpha\sin 2\alpha\sin 4\alpha\exp(-|\xi|\sin 2\alpha/L), \tag{57}$$

i.e., $F(\xi)$ exponentially decays on the spatial scale L. We also introduce the characteristic correlation scale l_{cor} determined by the condition that $R(\xi)$ is of order of $R(0)$ for $|\xi|\lesssim l_{cor}$, while $R(\xi)\approx 0$ for $|\xi|\gg l_{cor}$. Now we assume that $l_{cor}\ll L$. In this case,

$$\int_0^\infty R(\xi)F(\xi)d\xi\approx F(0)\int_0^\infty R(\xi)d\xi$$

$$=\frac{2VK}{L}\sin\alpha\sin 2\alpha\{2\sin\alpha(2+\cos 2\alpha)+(\pi-\alpha)\cos\alpha(5+\cos 2\alpha)\}, \tag{58}$$

where

$$K=\int_0^\infty R(\xi)d\xi. \tag{59}$$

We now can make qualitative conclusions about the soliton behavior even without solving the equation for α that is obtained by the substitution of Eqs. (53) and (58) into Eq. (51). When $K>0$, it follows that $d\alpha/d\sigma>0$, which means that the soliton amplitude decreases. If $K<0$, then $d\alpha/d\sigma<0$, which means that the soliton amplitude increases. These results are not surprising at all because, for $l_{cor}\ll L$, the right-hand side of Eq. (47) reduces to the second derivative of b with respect to τ, the proportionality coefficient having the same sign as K. Hence, for $K>0$, it describes conventional diffusion, while for $K<0$ it describes negative diffusion.

In Fig. 1 the dependence of α on the dimensionless distance $\bar{\sigma}=\sigma|K|(\kappa LV^2)^{-1}$ is shown for different values of α at $\sigma=0$. The solid lines correspond to $K>0$, and the dashed lines to $K<0$. We see that $\alpha\to\pi/2$ as $\bar{\sigma}\to\infty$ for $K>0$, while $\alpha\to 0$ as $\bar{\sigma}\to\infty$ for $K<0$. In accordance with Eqs. (49) and (50) this implies that the soliton amplitude A tends to zero as $\bar{\sigma}\to\infty$ for $K>0$, and to $2|b_0|$ for $K<0$. The soliton width γ^{-1} tends to infinity in both cases.

V. DISCUSSION AND CONCLUSIONS

In this paper we have studied the quasi-longitudinal with respect to the magnetic field propagation of nonlinear MHD waves in a cold plasma with fluctuating background density. The main result of the paper is the modified DNLS equation (47). In comparison to the standard DNLS equation it contains one additional term that describes the effect of density fluctuation. Under the assumption that the density is a homogeneous random process, the effect of density fluctuation is completely determined by the autocorrelation function of the density.

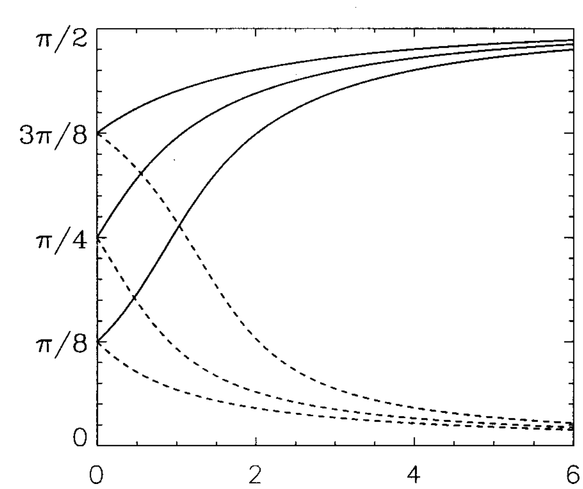

FIG. 1. The dependence of the soliton parameter α on the dimensionless distance $\bar{\sigma}=\sigma|K|(\kappa LV^2)^{-1}$. The solid lines correspond to $K>0$, and the dashed lines to $K<0$.

Equation (47) has been derived under a few simplifying assumptions. The most important one is that the density is a homogeneous random process. We considered a boundary problem in a region $x>0$ with all quantities given at $x=0$. At first sight this contradicts the assumption that the density is a homogeneous random function of x. However, this contradiction can be easily circumvented if we assume that, although we consider the wave only in the region $x>0$, the density is given for all x, both positive and negative.

As an example of the use of the modified DNLS equation we considered propagation of one-parametric solitons. We assumed that the term describing the effect of the density fluctuation is small in comparison with other term in the modified DNLS equation. This assumption enabled us to consider quasi-adiabatic evolution of the soliton and derive the equation describing this evolution. When the correlation length of the density fluctuations is much smaller than the soliton width, the term describing the effect of density fluctuation corresponds either to conventional or to negative diffusion. In the first case the soliton decays, and in the second its amplitude tends to its maximum value, while the soliton energy tends to infinity.

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