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Slow solitary waves in multi-layered magnetic structures

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The propagation of slow sausage surface waves in a multi-layered magnetic configuration is considered. The magnetic configuration consists of a central magnetic slab sandwiched between two identical magnetic slabs (with equilibrium quantities different from those in the central slab) which in turn are embedded between two identical semi-infinite regions. The dispersion equation is obtained in the linear approximation. The nonlinear governing equation describing waves with a characteristic wavelength along the central slab much larger than the slab thickness is derived. Solitary wave solutions to this equation are obtained in the case where these solutions deviate only slightly from the algebraic soliton of the Benjamin–Ono equation. [DOI: 10.1063/1.1371520]

I. INTRODUCTION

Magnetized plasmas in geophysical and astrophysical conditions are highly inhomogeneous. Very often they consist of filament structures stretched along the magnetic field lines, and strongly inhomogeneous in a direction perpendicular to the magnetic field. Such structures are usually called magnetic flux tubes (see, e.g., Ref. 1). Flux tubes can support new types of magnetohydrodynamic (MHD) waves with the wave energy confined to the tube or its vicinity. The waves are called surface and body modes.[2–6]

The simplest model of a magnetic tube is a magnetic cylinder with homogeneous plasmas inside and outside the cylinder, and with the surface of the cylinder being a MHD tangential discontinuity. The analysis is even simpler when a magnetic slab, which consists of two parallel identical MHD tangential discontinuities with a homogeneous plasma between them, and identical homogeneous plasmas in the two outer regions, is used as a planar model of a magnetic tube. The properties of sausage (axisymmetric) wave modes in a magnetic cylinder and those of sausage modes in a magnetic slab are quite similar.[3,6] It is this similarity together with the relative simplicity of the analysis that has made the magnetic slab a popular model for studying wave propagation in a magnetically structured plasma. In sausage modes, oscillations are symmetric about the central axis of the magnetic configuration; the central axis remains undisturbed in the motions.

The linear theory of waves in magnetic slabs has been developed in Refs. 3, 6–9. Different types of nonlinear waves in magnetic slabs have been studied in Refs. 10–19. In particular, it has been shown[10,11,13] that, in the long-wavelength approximation, the propagation of nonlinear slow sausage waves in a magnetic slab is described by the Benjamin–Ono (BO) equation, previously derived for waves in fluids with the infinite depth.[20,21]

Although the representation of magnetic filaments by magnetic slabs enabled us to understand many important properties of magnetic flux tube oscillations, this approximation is not particularly realistic. Real magnetic flux tubes are inhomogeneous and, in particular, characterized by a continuous dependence of equilibrium quantities on the radial distance. Hence, more sophisticated models are needed for a comprehensive study of magnetic flux tube oscillations.

In this paper we aim to narrow the gap between the model of magnetic tube or slab and the reality. It studies the propagation of nonlinear slow sausage waves in a magnetic configuration more complicated than a simple magnetic slab. The configuration consists of a central magnetic slab sandwiched between two identical magnetic slabs with equilibrium quantities different from those in the central slab. This three-slab configuration is, in turn, embedded between two semi-infinite regions with identical equilibrium quantities. Such a magnetic configuration can be considered as a model of a number magnetic structures in astrophysical plasmas, e.g., as a model of a magnetic loop with a thin core in the solar corona.

The paper is organized as follows. In the next section we describe the equilibrium state and present the governing equations and boundary conditions. In Sec. III we study wave propagation in the linear approximation. In Sec. IV we derive the governing equation for nonlinear slow sausage surface waves; this is a new result. We go on in Sec. V to obtain an approximate solution to the nonlinear governing equation in the form of a solitary wave in the case where the wave differs only slightly from the algebraic soliton of the

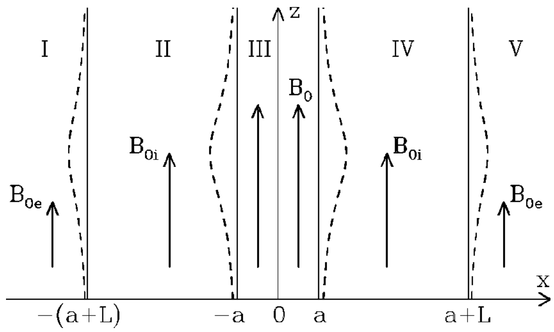

FIG. 1. A sketch of the equilibrium configuration. The equilibrium quantities are the same in regions I and V, and also in regions II and IV. The dashed lines show the perturbed boundaries separating the regions with different equilibrium quantities.

BO equation. A summary of our results is presented in Sec. VI.

II. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

To describe the plasma motion we use the ideal MHD equations,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\mu\rho}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma}\right) = 0. \tag{4}$$

Here ρ is the plasma density, p the pressure, $\mathbf{v}$ the velocity, $\mathbf{B}$ the magnetic induction, μ the magnetic permeability of a vacuum, and γ the ratio of specific heats.

In what follows we study wave motion in the equilibrium configuration shown in Fig. 1. This configuration consists of five regions, separated by tangential MHD discontinuities parallel to the yz-plane in the Cartesian coordinates x, y, z. The equilibrium magnetic field is in the z-direction. There is no equilibrium flow. The equilibrium quantities are the same in regions I and V; in regions II and IV they are also the same (and in general distinct from regions I and V). Quantities in regions I and V are labeled with the subscript "$0e$," and quantities in regions II and IV with the subscript "$0i$." Equilibrium quantities in the inner region (region III) are labeled with the subscript "0."

A number of special cases of particular interest are included in our treatment. For example, the case of an isolated magnetic slab corresponds to choosing $B_{0i} = B_{0e} = 0$, and is of particular interest in solar photospheric studies (e.g., Refs. 3–9). The case of B_{0i}, B_{0e}, and B_0 all broadly comparable corresponds to solar coronal circumstances.[4–6]

We consider only planar motions where perturbations of all quantities are independent of y, and the y-component of the velocity and the magnetic induction is zero, so that the velocity and the perturbation of the magnetic induction are $(u,0,w)$ and $(b_x,0,b_z)$, respectively. This eliminates Alfvén waves. In addition, we restrict our analysis to motions symmetric with respect to the z-axis. This restriction enables us to consider Eqs. (1)–(4) in the region $x>0$ only, and to impose symmetry conditions at $x=0$:

$$u = b_x = 0, \quad \frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial b_z}{\partial x} = 0. \tag{5}$$

Let the equations of the perturbed boundaries between regions III and IV, and between regions IV and V, be $x = a + \eta(t,z)$ and $x = a + L + \zeta(t,z)$ respectively. At these boundaries, the kinematic boundary conditions and the conditions of total pressure balance have to be satisfied:

$$u = \frac{\partial \eta}{\partial t} + w\frac{\partial \eta}{\partial z}, \quad u_i = \frac{\partial \eta}{\partial t} + w_i\frac{\partial \eta}{\partial z}, \quad P = P_i, \quad \text{at } x = a + \eta, \tag{6}$$

$$u_i = \frac{\partial \zeta}{\partial t} + w_i\frac{\partial \zeta}{\partial z}, \quad u_e = \frac{\partial \zeta}{\partial t} + w_e\frac{\partial \zeta}{\partial z}, \quad P_i = P_e, \quad \text{at } x = a + L + \zeta, \tag{7}$$

where $P = p + B^2/2\mu$ is the total pressure in a plasma with magnetic field strength $B = |\mathbf{B}|$. In addition, perturbations of all variables are assumed to vanish as $x \to \infty$.

Equations (1)–(4) and the boundary conditions (5)–(7) will be used in what follows to study the propagation of a sausage wave.

III. LINEAR THEORY

In this section we study the propagation of sausage waves in the linear approximation. We linearize Eqs. (1)–(4) and take perturbations of all quantities to be proportional to $\exp[i(kz-\omega t)]$ with real wavenumber k and frequency ω. Eliminating perturbations of all quantities in favor of u and $P' = p' + B_0 b_z/\mu$ (the prime indicates the perturbation of a quantity), we obtain the system of equations (see Refs. 2, 3) describing the wave motion in the central region (region III),

$$\frac{du}{dx} = \frac{i\omega(\omega^2 - c_S^2 k^2)}{\rho_0(c_S^2 + v_A^2)(\omega^2 - c_T^2 k^2)} P', \qquad \frac{dP'}{dx} = \frac{i\rho_0(\omega^2 - v_A^2 k^2)}{\omega} u, \tag{8}$$

where the squares of the Alfvén, sound, and cusp speeds are given by

$$v_A^2 = \frac{B_0^2}{\mu\rho_0}, \quad c_S^2 = \frac{\gamma p_0}{\rho_0}, \quad c_T^2 = \frac{c_S^2 v_A^2}{c_S^2 + v_A^2}. \tag{9}$$

The system of equations describing the motions in regions IV and V are obtained from Eq. (8) by substituting the unlabeled quantities by the corresponding quantities with the labels "i" and "e," respectively.

The linearized boundary conditions are

$$u = -i\omega\eta, \quad u_i = -i\omega\eta, \quad P' = P_i', \quad \text{at } x = a, \tag{10}$$

$$u_i = -i\omega\zeta, \quad u_e = -i\omega\zeta, \quad P_i' = P_e', \quad \text{at} \quad x = a + L. \tag{11}$$

The solution to Eq. (8) in region III, satisfying the boundary conditions (5), is

$$u = \omega m_0 A_0 \sinh(m_0 x),$$
$$P' = i\rho_0 A_0(\omega^2 - v_A^2 k^2)\cosh(m_0 x), \quad 0 < x < a, \tag{12}$$

where

$$m_0^2 = \frac{(c_S^2 k^2 - \omega^2)(v_\mathrm{A}^2 k^2 - \omega^2)}{(c_S^2 + v_\mathrm{A}^2)(c_T^2 k^2 - \omega^2)}, \tag{13}$$

and A_0 is an arbitrary constant. The quantity m_0^2 can be either positive or negative. In the latter case we take $m_0 = i|m_0|$.

The solution to Eq. (8) in region IV is

$$u_i = \omega m_i (A_+ e^{m_i x} + A_- e^{-m_i x}),$$
$$P_i' = i\rho_{0i}(\omega^2 - v_{Ai}^2 k^2)(A_+ e^{m_i x} - A_- e^{-m_i x}),$$
$$a < x < a + L, \tag{14}$$

where A_+ and A_- are arbitrary constants, and m_i^2 is given by Eq. (13) with the equilibrium quantities labeled with "i." Once again m_i^2 can be either positive or negative, and in the latter case we take $m_i = i|m_i|$.

The solution to Eq. (8) in region V, vanishing as $x \to \infty$, is

$$u_e = \omega m_e A_e e^{-m_e x},$$
$$P_e' = -i\rho_{0e}(\omega^2 - v_{Ae}^2 k^2) A_e e^{-m_e x}, \quad x > a + L, \tag{15}$$

where m_e^2 is given by Eq. (13) with the equilibrium quantities labeled with "e." In order that this solution vanishes at infinity, the quantity m_e^2 must be positive.

Substituting Eqs. (12), (14), and (15) into Eqs. (10) and (11), we obtain a homogeneous system of six linear algebraic equations for η, ζ, A_0, A_+, A_-, and A_e. The condition for the existence of a nontrivial solution to this system gives the dispersion relation. Writing ρ_e in place of ρ_{0e} and ρ_i in place of ρ_{0i}, we obtain

$$\frac{\rho_i m_0 (v_\mathrm{Ai}^2 k^2 - \omega^2)}{\rho_0 m_i (v_\mathrm{A}^2 k^2 - \omega^2)} \tanh(m_0 a)$$
$$= -\frac{\rho_i m_e (v_\mathrm{Ai}^2 k^2 - \omega^2) + \rho_e m_i (v_\mathrm{Ae}^2 k^2 - \omega^2)\tanh(m_i L)}{\rho_e m_i (v_\mathrm{Ae}^2 k^2 - \omega^2) + \rho_i m_e (v_\mathrm{Ai}^2 k^2 - \omega^2)\tanh(m_i L)}. \tag{16}$$

In what follows we concentrate on the long wavelength approximation where $|ak| \ll 1$. Note that $|kL|$ can be arbitrary. In the case of an isolated slab there is a surface sausage wave propagating with the velocity c_T along the slab.[3] For this wave the condition $|am_0| \ll 1$ is satisfied. We look for a solution of the same type to Eq. (16). A straightforward calculation yields

$$\omega = c_T k + k|k|\varphi(k), \quad \varphi(k) = \beta_i \frac{1 - \lambda \exp(-2\kappa_i |k| L)}{1 + \lambda \exp(-2\kappa_i |k| L)}, \tag{17}$$

where

$$\beta_\alpha = \frac{a\rho_\alpha c_T^3 (v_{\mathrm{A}\alpha}^2 - c_T^2)}{2\rho_0 \kappa_\alpha v_\mathrm{A}^4},$$
$$\kappa_\alpha^2 = \frac{(c_{S\alpha}^2 - c_T^2)(v_{\mathrm{A}\alpha}^2 - c_T^2)}{(c_{S\alpha}^2 + v_{\mathrm{A}\alpha}^2)(c_{T\alpha}^2 - c_T^2)}, \tag{18}$$
$$\lambda = \frac{\rho_i \kappa_e (v_\mathrm{Ai}^2 - c_T^2) - \rho_e \kappa_i (v_\mathrm{Ae}^2 - c_T^2)}{\rho_i \kappa_e (v_\mathrm{Ai}^2 - c_T^2) + \rho_e \kappa_i (v_\mathrm{Ae}^2 - c_T^2)},$$

and the suffix $\alpha = i, e$. The quantity κ_i^2 can be either positive or negative. In the latter case we take $\kappa_i = i|\kappa_i|$. It is straightforward to check that $\varphi(k)$ is real when $\kappa_i = i|\kappa_i|$. In order that perturbations vanish at infinity, the quantity κ_e^2 must be positive. This implies that one of the following two inequalities has to be satisfied:

$$c_T < c_{Te} \quad \text{or} \quad \min(c_{Se}, v_\mathrm{Ae}) < c_T < \max(c_{Se}, v_\mathrm{Ae}). \tag{19}$$

The second inequality is readily satisfied in applications to the solar corona, where the Alfvén speed is much larger than the sound or cusp speed in all three regions. In the solar photosphere with $v_\mathrm{Ae} = 0$ the condition (19) is reduced to $c_T < c_{Se}$.

In what follows we assume that the denominator of the expression for λ is nonzero, and so is the denominator of the expression for $\varphi(k)$. The first condition is readily satisfied in applications to the solar corona; however, the second condition can be broken if κ_i is purely imaginary.

When $L = 0$, there is no region IV, and the considered configuration is a magnetic slab. In this case the dispersion relation (17) coincides with that given in Ref. 6. When $L \to \infty$, region V disappears. In this case we once again have the slab configuration, and Eq. (17) coincides with that given Ref. 6 (with the external equilibrium quantities labeled with "i" instead of "e").

It is convenient to introduce the dimensionless amplitude of perturbations, writing $\epsilon = \max|\eta|/a$. Then, using the linearized MHD equations, the solution to the linear problem and the dispersion relation (17), it is straightforward to show that the dimensionless amplitudes of ρ', p', w, b_z, and ζ are all of order ϵ, while the dimensionless amplitudes of u and b_x are of order $\epsilon a|k|$ ($\ll \epsilon$). The dimensionless amplitudes of perturbations of all quantities in regions IV and V are of order $\epsilon a|k|$. One additional and very important observation is that the characteristic scale of variation of ρ', p', w, and b_z in the x-direction is k^{-1}. Hence, they are almost independent of x in the long wavelength approximation, where $|ak| \ll 1$.

IV. DERIVATION OF THE NONLINEAR GOVERNING EQUATION

To derive the nonlinear equation governing the propagation of slow surface sausage waves in the central magnetic slab (region III) we use the reductive perturbation method (e.g., Refs. 22, 23). The procedure is similar to that used in Ref. 13 to derive the BO equation for slow waves in a magnetic slab. We consider nonlinear waves with amplitudes of the order ϵ, propagating in the positive z-direction with phase velocity close to c_T. Our aim is to obtain the equation describing the competition between nonlinearity and dispersion. The effect of dispersion is comparable with the effect of nonlinearity when the ratio of the dispersion correction, which is the second term on the right-hand side of the dispersion relation (17), to the main term, which is the first term, is of the order ϵ. This implies that $a|k|\sim\epsilon$, and the characteristic scale in the z-direction is $\epsilon^{-1}a$. To take into account this estimate, and the fact that perturbations propagate in the z-direction with a phase speed that is close to c_T, we introduce the running variable $\theta=\epsilon(z-c_T t)$.

Since the nonlinear and dispersive effects are both of order ϵ, they cause the evolution of the wave shape on a time-scale of order ϵ^{-1} multiplied by the characteristic wave period. To take this slow evolution into account, we introduce the ''slow'' time $\tau=\epsilon^2 t$.

The thickness L of region IV can be much larger than the thickness $2a$ of region III, so that a cannot be considered as a characteristic scale in the x-direction in region IV. Since region V is semi-infinite, the characteristic wavelength is the only spatial scale present in this region. This observation inspires us to introduce the stretched variable $X=\epsilon x$ in regions IV and V.

In accordance with the reductive perturbation method, we look for the solution in the form of asymptotic expansions with respect to ϵ. Taking into account our estimates for the order of magnitude of perturbations of different quantities, obtained at the end of the previous section, and the estimate $a|k|\sim\epsilon$, we write the expansions in the form

$$f=\epsilon f^{(1)}+\epsilon^2 f^{(2)}\dots, \tag{20}$$

for ρ', p', w, b_z, η, and ζ, and in the form

$$g=\epsilon^2 g^{(1)}+\epsilon^3 g^{(2)}\dots, \tag{21}$$

for u, b_x, and for the perturbations of all quantities in regions IV and V. In addition, in accordance with our note at the end of the previous section, we assume that $f^{(1)}$ is independent of x. We solve the MHD equations in regions III, IV, and V separately, and then match the solutions at the boundaries. The solutions in region III and regions IV and V are given in Appendices A and B, respectively. To obtain the governing equation we substitute Eq. (B9) into Eq. (A16), return to the original variables t and z, and take $\eta\approx\epsilon\eta^{(1)}$. As a result, we finally obtain

$$\frac{\partial\eta}{\partial t}+c_T\frac{\partial\eta}{\partial z}+q\,\eta\frac{\partial\eta}{\partial z}-\beta_i\mathcal{H}\left(\frac{\partial^2\eta}{\partial z^2}\right)-\beta_i\mathcal{L}\left(\frac{\partial^2\eta}{\partial z^2}\right)=0, \tag{22}$$

where the Hilbert transform $\mathcal{H}$ is given by (B10), and the operator $\mathcal{L}$ is determined by

$$\mathcal{L}(f)=\int_{-\infty}^{\infty}G(z-s)f(s)ds,$$
$$G(z)=\frac{\lambda}{\pi L}\int_0^{\infty}\frac{\sin(rz/2L)}{\lambda+e^{r\kappa_i}}\,dr. \tag{23}$$

Equation (22) is our main goal in this paper. It describes the weakly nonlinear, weakly dispersive behavior of slow magnetosonic sausage waves in a slab configuration.

When $L\to\infty$, $\mathcal{L}(f)\to 0$ and Eq. (22) reduces to the BO equation previously derived in Refs. 10, 11, 13. When $L\approx a$, it is straightforward to show that $G(z)\to 2\lambda(1+\lambda)^{-1}\mathcal{P}(1/z)$, and then $\mathcal{L}(f)\to -2\lambda(1+\lambda)^{-1}\mathcal{H}(f)$ and Eq. (22) once again reduces to the BO equation, however, with β_e substituted for β_i. All these results are in agreement with those obtained in the previous section in the linear approximation.

V. SOLITARY WAVE

In this section we obtain a solution to Eq. (22) in the form of a solitary wave. We look for a solution where η vanishes at infinity, and η depends on $\vartheta=z-(c_T+V)t$ rather than on z and t separately. In this case Eq. (22) reduces to

$$V\eta-\frac{q}{2}\eta^2+\beta_i\mathcal{H}\left(\frac{d\eta}{d\vartheta}\right)+\beta_i\mathcal{L}\left(\frac{d\eta}{d\vartheta}\right)=0. \tag{24}$$

Equation (24) is a complicated integro-differential equation, and at present we are unable to find its solution. To make analytical progress we assume that $|\lambda|\ll 1$. This condition is, in particular, satisfied if the equilibrium quantities in region IV differ only slightly from those in region V, which is relevant for the solar corona. We now use the regular perturbation method to obtain the solution. We expand the denominator in the expression (23) for $G(z)$ in the power series with respect to λ. The corresponding integrals are easily calculated, and we arrive at the following expression for the operator $\mathcal{L}$:

$$\mathcal{L}=\sum_{n=1}^{\infty}\lambda^n\mathcal{L}_n,$$
$$\mathcal{L}_n(f(z))=\frac{2(-1)^{n+1}}{\pi}\int_{-\infty}^{\infty}\frac{(z-s)f(s)\,ds}{(z-s)^2+n^2\sigma^2}, \tag{25}$$

where $\sigma=2\kappa_i L$. Now we look for a solution in the form of a power series $\eta=\eta_0+\lambda\eta_1+\lambda^2\eta_2+\cdots$. In the zeroth order approximation we obtain the BO equation for η_0. It possesses a solution in the form of an algebraic soliton (e.g., Refs. 20, 21, 24),

$$\eta_0=\frac{-4\beta_i l}{q(l^2+\vartheta^2)},\qquad Vl=-\beta_i,\qquad l>0. \tag{26}$$

The quantity l is the half-width of the soliton, and $-4\beta_i/lq$ is its amplitude.

In the first order approximation we obtain

$$-V\eta_1+q\,\eta_0\eta_1-\beta_i\mathcal{H}\left(\frac{d\eta_1}{d\vartheta}\right)=\beta_i F_1(\vartheta), \tag{27}$$

$$F_1(\vartheta)=\mathcal{L}_1\left(\frac{d\eta_0}{d\vartheta}\right)=\frac{8Vl}{q}\frac{d}{d\vartheta}\frac{\vartheta}{\vartheta^2+(l+\sigma)^2}. \tag{28}$$

Using the identity

$$\int_{-\infty}^{\infty}f\mathcal{H}(g)\,d\vartheta=-\int_{-\infty}^{\infty}g\mathcal{H}(f)\,d\vartheta,$$

it is straightforward to show that the operator on the left-hand side of Eq. (27) is self-adjoint in L_2. Differentiating the equation for η_0 we show that $d\eta_0/d\vartheta$ is an eigenfunction of this operator corresponding to the zero eigenvalue. It can be also shown that the eigenspace corresponding to the zero eigenvalue is one-dimensional, so that $d\eta_0/d\vartheta$ is the only eigenfunction (apart from a multiplicative constant). This implies that Eq. (27) has a solution from L_2 if and only if $F_1(\vartheta)$ is orthogonal to $d\eta_0/d\vartheta$, i.e., if $\int_{-\infty}^{\infty}F_1(\vartheta)\times(d\eta_0/d\vartheta)\,d\vartheta=0$. Since $F_1(\vartheta)$ is even and $d\eta_0/d\vartheta$ is odd, this condition is obviously satisfied.

Proceeding to the second order approximation, we obtain an equation similar to Eq. (27), but with η_2 and $F_2=(d/d\vartheta)[\mathcal{L}_1(\eta_1)+\mathcal{L}_2(\eta_0)]$ substituted for η_1 and F_1, respectively. Once again, the solvability condition for this equation is the orthogonality of $F_2(\vartheta)$ and $d\eta_0/d\vartheta$. The expression for $\mathcal{L}_2(d\eta_0/d\vartheta)$ is obtained by the substitution of 2σ for σ in Eq. (28), so the solvability condition of the second order approximation reduces to the orthogonality condition of $\mathcal{L}_1(d\eta_1/d\vartheta)$ and $d\eta_0/d\vartheta$. Let $\eta_1(\vartheta)$ be a solution to Eq. (27), satisfying the solvability condition of the second order approximation. Since Eq. (27) is invariant with respect to the substitution $-\vartheta\to\vartheta$, $\overline{\eta_1}(\vartheta)=\eta_1(-\vartheta)$ is also a solution to Eq. (27), and, obviously, $\mathcal{L}_1(d\overline{\eta_1}/d\vartheta)$ and $d\eta_0/d\vartheta$ are also orthogonal. Then $\tilde{\eta}_1(\vartheta)=\eta_1(\vartheta)-\overline{\eta_1}(\vartheta)$ is a solution to Eq. (27) with $F_1\equiv 0$ and, consequently, $\tilde{\eta}_1(\vartheta)=C\,d\eta_0/d\vartheta$, where C is constant. And, of course, $\mathcal{L}_1(d\tilde{\eta}_1/d\vartheta)$ and $d\eta_0/d\vartheta$ are orthogonal. This orthogonality condition reduces to

$$C\int_{-\infty}^{\infty}\frac{d\eta_0}{d\vartheta}\,\frac{d^2\mathcal{L}_1(\eta_0)}{d\vartheta^2}\,d\vartheta=0. \tag{29}$$

A direct calculation with using Eq. (28) shows that the integral in this expression is nonzero, which implies that $C=0$. Hence $\eta_1(-\vartheta)=\eta_1(\vartheta)$, i.e., $\eta_1(\vartheta)$ is an even function.

In what follows we consider a more general problem. Let $F_1(\vartheta)$ be an arbitrary continuously differentiable even function, $F_1(\vartheta)\in L_2$. Our task is to find an even function $\eta_1(\vartheta)$ satisfying Eq. (27). The perturbation theory for the BO equation has been studied by many authors (see, e.g., Ref. 25 and references therein). In principle, the solution to the considered problem can be obtained on the basis of the general theory developed in Ref. 25. However, from our point of view, it is convenient to give a direct solution to the problem based on a very simple analysis. In addition to its simplicity, our approach has another advantage in comparison with that used in Ref. 25. The theory developed in Ref. 25 is based on the complete integrability of the BO equation. Our approach does not use the integrability and can also be applied to nonintegrable BO-type equations.

We start our analysis by introducing the Cauchy integral,

$$\tilde{\eta}(\vartheta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\eta_1(s)\,ds}{s-\vartheta}, \tag{30}$$

where ϑ is now considered a complex variable, and $\mathfrak{I}(\vartheta)\neq 0$, where $\mathfrak{I}$ indicates the imaginary part of a quantity. Equation (30) determines two analytical functions of the complex variable ϑ. The function $\eta_+=\tilde{\eta}$ is analytical in the upper half of the complex plane $[\mathfrak{I}(\vartheta)>0]$, and the function $\eta_-=\tilde{\eta}$ is analytical in the lower half of the complex plane $[\mathfrak{I}(\vartheta)<0]$. The boundary values of these two functions at the real axis $[\mathfrak{I}(\vartheta)\to\pm 0]$ are given by the Plemelj formulas,

$$\eta_\pm(\vartheta)=\pm\tfrac{1}{2}\eta_1(\vartheta)-\tfrac{1}{2}i\mathcal{H}(\eta_1). \tag{31}$$

The relation $\eta_-(\vartheta)=-[\eta_+(\vartheta^*)]^*$ (the asterisk indicates a complex conjugate value) enables us to consider only η_+ in what follows.

Let us now apply the Hilbert transform to Eq. (27). Using the formulas

$$\mathcal{H}^2=-1,\qquad \mathcal{H}\left(\frac{f(\vartheta)}{\vartheta-\nu}\right)=\frac{\mathcal{H}(f)}{\vartheta-\nu}-\frac{1}{\pi(\vartheta-\nu)}\int_{-\infty}^{\infty}\frac{f(s)\,ds}{s-\nu},$$

where ϑ is real and ν is an arbitrary complex quantity with the nonzero imaginary part $[\mathfrak{I}(\nu)\neq 0]$, we obtain

$$-V\mathcal{H}(\eta_1)+q\eta_0\mathcal{H}(\eta_1)+\beta_i\frac{d\eta_1}{d\vartheta}=\beta_i\mathcal{H}(F_1)-8\beta_i\frac{\vartheta\eta_+(il)}{\vartheta^2+l^2}. \tag{32}$$

Note that $\eta_+(il)$ is real. Now we multiply Eq. (27) by i, and add the result to Eq. (32). As a result we obtain an equation for $\eta_+(\vartheta)$,

$$\beta_i\frac{d\eta_+}{d\vartheta}-i\eta_+(V-q\eta_0)=i\beta_iF_+-4\beta_i\frac{\vartheta\eta_+(il)}{\vartheta^2+l^2}\equiv\beta_ih, \tag{33}$$

where the function $F_+(\vartheta)$ is determined in terms of $F_1(\vartheta)$ in the same manner as $\eta_+(\vartheta)$ is determined in terms of $\eta_1(\vartheta)$. Since $\eta_0(\vartheta)$ can be analytically continued on the whole complex plane, we can consider Eq. (33) in the upper half of the complex plane. The solution to Eq. (33) is straightforward:

$$\eta_+(\vartheta)=e^{-i\vartheta/l}\left(\frac{\vartheta-il}{\vartheta+il}\right)^2\left\{C_1+\int_0^{\vartheta}h(s)e^{is/l}\left(\frac{s+il}{s-il}\right)^2ds\right\}, \tag{34}$$

where C_1 is constant. It is easy to check that $\eta_+(\vartheta)$ determined by Eq. (34) tends to $\eta_+(il)$ as $\vartheta\to il$, so that the solution (34) is self-consistent. In general, the point il is a branch point for $\eta_+(\vartheta)$, which contradicts the property of $\eta_+(\vartheta)$ to be analytic in the upper half of the complex plane. This point is regular only if the expansion of the integrand in the vicinity of il does not contain the term $(\vartheta-il)^{-1}$. This condition results in the relation

$$\eta_+(il)=il^2F_+'(il), \tag{35}$$

where the prime denotes the derivative with respect to the argument. The fact that $\eta_1(\vartheta)$ is even implies that $\eta_+(-\vartheta)=\eta_+^*(\vartheta)$ for real ϑ. This relation is satisfied only when C_1 is real. The condition that $\eta_+(\vartheta)\to 0$ as $\vartheta\to-\infty$ leads to

$$C_1=\int_{-\infty}^{0}h(\vartheta)e^{i\vartheta/l}\left(\frac{\vartheta+il}{\vartheta-il}\right)^2d\vartheta. \tag{36}$$

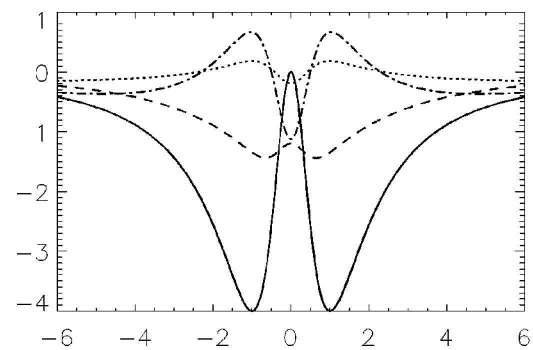

FIG. 2. The dependence of $q\eta_1/V$ on ϑ/l for different values of χ. The solid, dashed, dotted, and dashed–dotted curves correspond to $\chi=1$, $\chi=1.5$, $\chi=5$, and $\chi=40$, respectively.

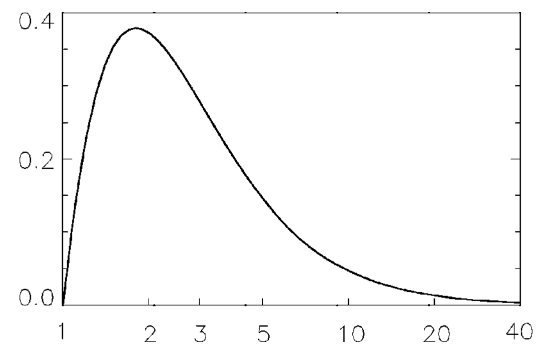

FIG. 3. The dependence of $-\eta_1(0)/\eta_0(0)$ on χ. Note the logarithmic scale used in the horizontal axis.

The condition that C_1 is real requires that

$$\int_{-\infty}^{\infty} h(\vartheta) e^{i\vartheta/l}\left(\frac{\vartheta+il}{\vartheta-il}\right)^2 d\vartheta = 0. \tag{37}$$

In accordance with the Jordan Lemma and the Residue Theorem, the integral on the left-hand side of Eq. (37) is equal to $2\pi i$ times the residue of the integrand with respect to the pole at $\vartheta=il$. Equation (35) guarantees that this residue is zero, so that Eq. (36) is satisfied. Hence, eventually, we arrive at

$$\eta_+(\vartheta) = e^{-i\vartheta/l}\left(\frac{\vartheta-il}{\vartheta+il}\right)^2 \int_{-\infty}^{\vartheta} h(s) e^{is/l}\left(\frac{s+il}{s-il}\right)^2 ds, \tag{38}$$

where $h(\vartheta)$ is given by Eqs. (33) and (35). Equation (37) guarantees that $\eta_+(\vartheta)\to 0$ as $\vartheta\to\infty$. The function $\eta_1(\vartheta)$ is given by $\eta_1 = 2\Re(\eta_+)$, where $\Re$ indicates the real part.

When the function $F_1(\vartheta)$ is given by Eq. (28), direct calculation yields

$$\eta_+ = \frac{-4Vl(\vartheta-il)^2}{q(\sigma+2l)^3(\vartheta+il)^2}\left\{\frac{4il^2[(2l-\sigma)\vartheta+il\sigma]}{(\vartheta-il)^2} - \frac{i\sigma^2(\sigma+2l)}{\vartheta+i(l+\sigma)} - \frac{\sigma}{l}[(\sigma+l)^2+3l^2]e^{-i\vartheta/l} \times \int_{-\infty}^{\vartheta}\frac{e^{is/l}ds}{s+i(l+\sigma)}\right\}. \tag{39}$$

It is easy to check that $\eta_1 = \mathcal{O}(\vartheta^{-2})$ as $|\vartheta|\to\infty$, so that η_1/η_0 is uniformly bounded on the real axis.

Let us study how this expression agrees with the two limiting cases considered at the end of the previous section. The limiting case $L\to\infty$ corresponds to $\sigma\to\infty$. In this case $\eta_+\to 0$ as it should, because η_0 is an exact solution to Eq. (22). In the second limiting case $L\to 0$, which corresponds to $\sigma\to 0$, the exact solution to Eq. (22) is given by Eq. (26), with β_e and $l_e = -\beta_e/V$ substituted for β_i and l, respectively (recall that we fix V). Using the relation $\beta_e = \beta_i(1-2\lambda)+\mathcal{O}(\lambda^2)$, we immediately obtain that the solution to Eq. (23) can be written as

$$\eta = \eta_0 + \lambda\frac{16\beta_i l\vartheta^2}{q(\vartheta^2+l^2)^2} + \mathcal{O}(\lambda^2). \tag{40}$$

The use of Eq. (39) gives the same result.

It is straightforward to see that the quantity $q\eta_1/V$, considered as a function of ϑ/l, depends only on one parameter, $\chi = 1+\sigma/l$. This is shown in Fig. 2 for different values of χ. The first order correction to the solitary wave amplitude is $\lambda\eta_1(0)$, where $\eta_1(0)$ is given by

$$\eta_1(0) = \frac{8V(\chi-1)[\chi(\chi^2+3)e^{\chi}\,\mathrm{Ei}(-\chi)+\chi^2-4\chi-1]}{q\chi(\chi+1)^3}, \tag{41}$$

with $\mathrm{Ei}(x) = \int_{-\infty}^{x} s^{-1}e^{s}\,ds$ $(x<0)$ the integral exponent. The dependence of $-\eta_1(0)/\eta_0(0)$ on χ is shown in Fig. 3. For small and large values of $(\chi-1)$ the quantity $\eta_1(0)/\eta_0(0)$ is given by the asymptotic expressions $\eta_1(0)/\eta_0(0) \sim -4.2(\chi-1)$ as $\chi\to 1$, and $\eta_1(0)/\eta_0(0)\sim -6\chi^{-2}$ as $\chi\to\infty$.

Finally, we calculate the first order correction to the solitary wave width, H. We determine H by the equation $\eta(H) = \frac{1}{2}\eta(0)$. In the zeroth order approximation we have $H = H_0 = l$. Then it is straightforward to obtain that in the first order approximation,

$$H = H_0 + \lambda H_1, \quad H_1 = \frac{\eta_1(0)-2\eta_1(l)}{2\eta_0'(l)}, \tag{42}$$

where the prime indicates the derivative. The explicit expression for H_1 is very complicated, and we do not write it down. In Fig. 4 the quantity $-H_1/l$ is shown as a function of χ.

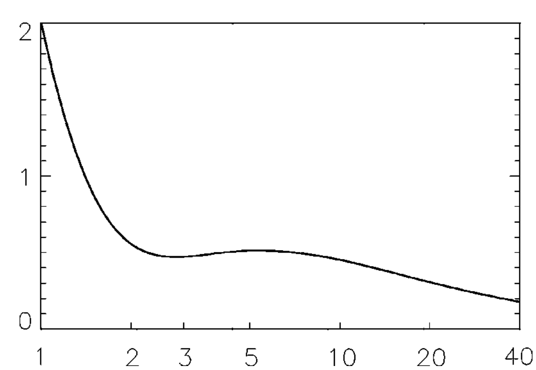

FIG. 4. The dependence of $-H_1/l$ on χ. Note the logarithmic scale is used in the horizontal axis.

VI. SUMMARY

In this paper we have studied wave propagation in a planar symmetric magnetic configuration consisting of a central magnetic slab and two side magnetic slabs embedded in two semi-infinite regions. Equilibrium quantities are constant in all five regions, and they have jumps at the boundaries of the regions. We have derived a dispersion relation for linear waves propagating in this configuration, and shown that in the long wavelength approximation it possesses a solution describing slow sausage waves in the central slab. When the thickness of the side slabs tends either to zero or to infinity, this solution reduces to the dispersion relation for slow sausage waves in a single magnetic slab.

Using the method of multiple scales we have derived an equation governing the nonlinear evolution of slow sausage waves in the central slab. This equation reduces to the Benjamin–Ono (BO) equation when the thickness of the side slab tends either to zero or to infinity. Using the regular perturbation method we have studied the solution to this equation in the form of a solitary wave in the case where it differs only slightly from the BO equation. This condition is, in particular, satisfied if the equilibrium quantities in region IV differ only slightly from those in region V, which is relevent in, for example, solar coronal loops with a thin core. We have used the BO soliton as a zeroth order approximation, and then calculated the small correction to it. In particular, we have calculated the corrections to the soliton amplitude and width as functions of the thickness of the side slabs. The study of such magnetic configurations is of particular importance in solar and magnetospheric physics.

In keeping with the regular perturbation method, the solution describing a solitary wave in the multi-layered configuration differs only slightly from the BO algebraic soliton. Nevertheless, on the basis of this solution, we can make an interesting conclusion about solitary waves in coronal loops with a thin core. Assuming that the equilibrium quantities in regions IV and V are only slightly different from one another, and that the density and temperature in region IV are larger than those in region V, and the variations of these quantities are of the same order, we immediately conclude that, for typical coronal conditions, $\lambda<0$. It then follows from Figs. 3 and 4 that the presence of the side slabs (regions II and IV) increases the amplitude and width of the solitary wave, so that its amplitude and width are larger than those of the BO soliton propagating with the same velocity.

ACKNOWLEDGMENTS

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APPENDIX A: SOLUTION IN REGION III

In this appendix we obtain the solution in region III. Using the new variables θ and τ, we rewrite the system of MHD equations (1)–(4) as

$$\epsilon\frac{\partial\rho}{\partial\tau}-c_T\frac{\partial\rho}{\partial\theta}+\epsilon^{-1}\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho w)}{\partial\theta}=0, \tag{A1}$$

$$\epsilon^2\frac{\partial u}{\partial\tau}-\epsilon c_T\frac{\partial u}{\partial\theta}+u\frac{\partial u}{\partial x}+\epsilon w\frac{\partial u}{\partial\theta} =-\frac{1}{\rho}\frac{\partial}{\partial x}\left(p+\frac{B_z^2}{2\mu}\right)+\epsilon\frac{B_z}{\mu\rho}\frac{\partial b_x}{\partial\theta}, \tag{A2}$$

$$\epsilon\frac{\partial w}{\partial\tau}-c_T\frac{\partial w}{\partial\theta}+\epsilon^{-1}u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial\theta} =-\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(p+\frac{b_x^2}{2\mu}\right)+\epsilon^{-1}\frac{b_x}{\mu\rho}\frac{\partial b_z}{\partial x}, \tag{A3}$$

$$\frac{\partial b_x}{\partial\tau}-\epsilon^{-1}c_T\frac{\partial b_x}{\partial\theta}=\epsilon^{-1}\frac{\partial}{\partial\theta}(uB_z-wb_x), \tag{A4}$$

$$\epsilon\frac{\partial b_z}{\partial\tau}-c_T\frac{\partial b_z}{\partial\theta}=\epsilon^{-1}\frac{\partial}{\partial x}(wb_x-uB_z), \tag{A5}$$

$$\epsilon\frac{\partial}{\partial\tau}\left(\frac{p}{\rho^\gamma}\right)-c_T\frac{\partial}{\partial\theta}\left(\frac{p}{\rho^\gamma}\right)+\epsilon^{-1}u\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right)+w\frac{\partial}{\partial\theta}\left(\frac{p}{\rho^\gamma}\right)=0. \tag{A6}$$

The boundary conditions (6) give

$$\epsilon^{-1}u=\epsilon\frac{\partial\eta}{\partial\tau}-c_T\frac{\partial\eta}{\partial\theta}+w\frac{\partial\eta}{\partial\theta},\quad p+\frac{B^2}{2\mu}=p_i+\frac{B_i^2}{2\mu}, \quad \text{at } x=a+\eta. \tag{A7}$$

We substitute the expansions (20) and (21) into Eqs. (A1)–(A7), and collect terms of order ϵ. As a result we obtain a system of equations and boundary conditions for the variables of the first order approximation, which are the variables with the superscript ''1.'' The solution to this system is straightforward. It gives the expressions of all variables in terms of $u^{(1)}$ and $w^{(1)}$,

$$\rho^{(1)}=\frac{\rho_0 c_T}{c_S^2}w^{(1)},\quad p^{(1)}=\rho_0 c_T w^{(1)}, \quad b_z^{(1)}=-\frac{B_0 c_T}{v_A^2}w^{(1)},\quad b_x^{(1)}=-\frac{B_0}{c_T}u^{(1)}, \tag{A8}$$

and the relations for $u^{(1)}$, $w^{(1)}$, and $\eta^{(1)}$,

$$u^{(1)}=-\frac{c_T^2 x}{v_A^2}\frac{\partial w^{(1)}}{\partial\theta},\quad w^{(1)}=\frac{v_A^2\eta^{(1)}}{c_T a}. \tag{A9}$$

When deriving Eq. (A9) we have used the symmetry conditions (5), and the fact that $w^{(1)}$ is independent of x.

In the second order approximation we do not use Eq. (A4). Collecting terms of the order of ϵ^2 in Eqs. (A1)–(A3), (A5), and (A6), we obtain the system of equations of the second order approximation. Using Eqs. (A8) and (A9), we write this system in the form

$$c_T\frac{\partial\rho^{(2)}}{\partial\theta}-\rho_0\frac{\partial u^{(2)}}{\partial x}-\rho_0\frac{\partial w^{(2)}}{\partial\theta} =\frac{\rho_0 c_T^2}{c_S^2}\left(\frac{\partial w^{(1)}}{\partial\tau}+\frac{\partial(u^{(1)}w^{(1)})}{\partial x}+2w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}\right), \tag{A10}$$

$$\frac{\partial}{\partial x}\left(p^{(2)}+\frac{B_0}{\mu}b_z^{(2)}\right)=0, \tag{A11}$$

$$c_T\frac{\partial w^{(2)}}{\partial\theta}-\frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial\theta}=\frac{\partial w^{(1)}}{\partial\tau}+\frac{c_T^2}{v_A^2}w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}, \tag{A12}$$

$$c_T\frac{\partial b_z^{(2)}}{\partial\theta}-B_0\frac{\partial u^{(2)}}{\partial x}=-\frac{B_0c_T}{v_A^2}\frac{\partial w^{(1)}}{\partial\tau}+\frac{B_0c_T}{c_S^2}\frac{\partial(u^{(1)}w^{(1)})}{\partial x}, \tag{A13}$$

$$\frac{\partial}{\partial\theta}(p^{(2)}-c_S^2\rho^{(2)})=\frac{(\gamma-1)\rho_0c_T^2}{c_S^2}w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}. \tag{A14}$$

The second boundary condition (A7) gives at $x=a$,

$$p^{(2)}+\frac{B_0}{\mu}b_z^{(2)}=p_i^{(1)}+\frac{B_{0i}}{\mu}b_{zi}^{(1)}-\frac{1}{2\mu}(b_z^{(1)})^2. \tag{A15}$$

Since $b_z^{(1)}$ is independent of x, it follows from Eq. (A11) that Eq. (A15) is valid for $x\leqslant a$ (however $p_i^{(1)}$ and $b_{zi}^{(1)}$ are calculated at $x=a$).

The homogeneous system of equations of the second order approximation, obtained by taking the right-hand parts of Eqs. (A10)–(A15) equal to zero, coincides with the system of the first order approximation. The system of the first order approximation admits a nontrivial solution and, consequently, so does the homogeneous system of the second order approximation. This implies that the inhomogeneous system of the second order approximation has a solution only when the right-hand sides of Eqs. (A10)–(A15) satisfy a compatibility condition. To derive this condition we eliminate all variables of the second order approximation from Eqs. (A10)–(A15). As a result, using Eq. (A9), we arrive at

$$\frac{\partial\eta^{(1)}}{\partial\tau}+q\,\eta^{(1)}\frac{\partial\eta^{(1)}}{\partial\theta}=-\frac{ac_T^3}{2\rho_0v_A^4}\frac{\partial}{\partial\theta}\left(p_i^{(1)}+\frac{B_{0i}}{\mu}b_{zi}^{(1)}\right), \tag{A16}$$

where the right-hand side is calculated at $x=a$, and coefficient q is given by

$$q=\frac{v_A^4[3c_S^2+(\gamma+1)v_A^2]}{2ac_T(c_S^2+v_A^2)^2}. \tag{A17}$$

APPENDIX B: SOLUTION IN REGIONS IV AND V

In this appendix we obtain the solution in regions IV and V. In these regions we use the system of MHD equations written in the variables θ, τ, and X. This system is obtained from Eqs. (A1)–(A6) by substituting the operator $\epsilon\partial/\partial X$ for the operator $\partial/\partial x$. Collecting terms of the order ϵ^2 in this system, we obtain the linear system of equations for the variables with the superscript ''1.'' Eliminating all variables from this system except $u_\alpha^{(1)}$ and $P_\alpha^{(1)}=p_\alpha^{(1)}+B_{0\alpha}b_{z\alpha}^{(1)}/\mu$, we arrive at the system of two equations:

$$\begin{aligned}&c_T\frac{\partial P_\alpha^{(1)}}{\partial X}=\rho_\alpha(c_T^2-v_{A\alpha}^2)\frac{\partial u_\alpha^{(1)}}{\partial\theta},\\ &\frac{\partial u_\alpha^{(1)}}{\partial X}=\frac{c_T(c_{S\alpha}^2-c_T^2)}{\rho_\alpha(c_{S\alpha}^2+v_{A\alpha}^2)(c_{T\alpha}^2-c_T^2)}\frac{\partial P_\alpha^{(1)}}{\partial\theta},\end{aligned} \tag{B1}$$

where $\alpha=i,e$. The boundary conditions (6) and (7) give in the first order approximation,

$$u_i^{(1)}=-c_T\frac{\partial\eta^{(1)}}{\partial\theta},\quad\text{at}\quad X=0, \tag{B2}$$

$$u_i^{(1)}=u_e^{(1)}=-c_T\frac{\partial\zeta^{(1)}}{\partial\theta},\quad P_i^{(1)}=P_e^{(1)},\ \text{at}\ X=\epsilon L. \tag{B3}$$

In addition, $u_e^{(1)}$ and $P_e^{(1)}$ has to vanish as $X\to\infty$. Note that the boundary condition (B2) should be written at $X=\epsilon a$; however, with the standard approach of linear theory, we have moved it to $X=0$. The quantity ϵL is arbitrary, so that we cannot do the same with the boundary condition (B3).

To solve the system of equations (B1) with the boundary conditions (B2) and (B3), we introduce the Fourier transform with respect to θ,

$$\hat f(k)=\int_{-\infty}^{\infty}f(\theta)e^{-ik\theta}d\theta,\quad f(\theta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(k)e^{ik\theta}dk. \tag{B4}$$

The Fourier-transformed equations (B1)–(B3) take the form

$$\begin{aligned}&c_T\frac{\partial\hat P_\alpha^{(1)}}{\partial X}=i\rho_\alpha(c_T^2-v_{A\alpha}^2)k\hat u_\alpha^{(1)},\\ &\frac{\partial\hat u_\alpha^{(1)}}{\partial X}=\frac{ic_T(c_{S\alpha}^2-c_T^2)k\hat P_\alpha^{(1)}}{\rho_\alpha(c_{S\alpha}^2+v_{A\alpha}^2)(c_{T\alpha}^2-c_T^2)},\end{aligned} \tag{B5}$$

$$\hat u_i^{(1)}=-ic_Tk\hat\eta^{(1)},\ \text{at}\ X=0, \tag{B6}$$

$$\hat u_i^{(1)}=\hat u_e^{(1)}=-ic_Tk\hat\zeta^{(1)},\quad\hat P_i^{(1)}=\hat P_e^{(1)},\ \text{at}\ X=\epsilon L. \tag{B7}$$

The solution to this system of equations and boundary conditions is straightforward, so that we omit details and give only the expression for $\hat P_i^{(1)}$ at $X=0$, which is the only quantity that we need in what follows:

$$\hat P_i^{(1)}|_{X=0}=\frac{2\rho_0v_A^4|k|\varphi(\epsilon k)\,\hat\eta^{(1)}}{ac_T^3}, \tag{B8}$$

where λ and $\varphi(k)$ are given by Eqs. (17) and (19). The inverse Fourier transform gives

$$P_i^{(1)}|_{X=0}=-\frac{2\rho_0v_A^4\beta_i}{ac_T^3}\left\{\mathcal{H}\left(\frac{\partial\eta^{(1)}}{\partial\theta}\right)-\frac{\lambda}{\pi\epsilon L}\int_{-\infty}^{\infty}\frac{\partial\eta^{(1)}}{\partial s}\times ds\int_0^\infty\frac{\sin[r(2\epsilon L)^{-1}(\theta-s)]}{\lambda+e^{r\kappa_i}}dr\right\}, \tag{B9}$$

where the Hilbert transform is determined by (see, e.g., Sneddon[26])

$$\mathcal{H}(f)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(s)\,ds}{s-\theta}, \tag{B10}$$

with $\mathcal{P}$ indicating the principal Cauchy part of an integral. When deriving Eq. (B9) we have used the formula for the Fourier transform of the Hilbert transform[26] $\widehat{\mathcal{H}(f)}=i\hat f\,\mathrm{sign}(k)$.

Note that $|\lambda|=1$ when $\kappa_i=i|\kappa_i|$. This implies that the denominator in the second integrand in Eq. (B9) may be zero for some values of k. These values of k correspond to resonances between slow surface waves in region III and body waves in region IV. To avoid this difficulty we assumed that $\kappa_i^2>0$.

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