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TWISTED ADJOINT L-VALUES, DIHEDRAL CONGRUENCE PRIMES AND THE BLOCH-KATO CONJECTURE

NEIL DUMMIGAN

Abstract. We show that a dihedral congruence prime for a normalised Hecke eigenform f in $S_k(\Gamma_0(D), \chi_D)$, where χ_D is a real quadratic character, appears in the denominator of the Bloch-Kato conjectural formula for the value at 1 of the twisted adjoint L-function of f. We then use a formula of Zagier to prove that it appears in the denominator of a suitably normalised $L(1, \mathrm{ad}^0(g) \otimes \chi_D)$ for *some* $g \in S_k(\Gamma_0(D), \chi_D)$.

1. Introduction

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$. Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, where $k \geq 2$ and χ_D is the Legendre symbol attached to $F/\mathbb{Q}$. Say $f = \sum_{m=1}^{\infty} a_m(f) q^m$. Let K_f be the CM subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f, with real subfield K_f^+, ring of integers $\mathcal{O}_f$, and let $\mathcal{O}(f)$ be the order in $\mathcal{O}_f$ generated by the $a_m(f)$. Let $f_c = \sum_{m=1}^{\infty} \overline{a_m(f)} q^m$ be the complex conjugate eigenform, and note that f_c is the newform associated to the twisted form f_{χ_D}. (This is because for each prime $q \nmid D$, T_q and $\langle q \rangle^{-1} T_q$ are adjoints for the Petersson inner product, so $a_q(f)$ is real or purely imaginary according as $\chi_D(q) = 1$ or -1, respectively.) Note that because the conductor of χ_D is D, $S_k(\Gamma_0(D), \chi_D)$ contains no old forms. The following is very easy to prove. For reference, it is a trivial modification of a special case of [BG, Lemma 3.1].

Lemma 1.1. *Let $\mathfrak{P} \mid (p)$ be a prime divisor in K_f. Suppose that $p \nmid [\mathcal{O}_f : \mathcal{O}(f)]$. We have $f \equiv f_c \pmod{\mathfrak{P}}$, i.e. $a_m(f) \equiv \overline{a_m(f)} \pmod{\mathfrak{P}}$ $\forall m$, if and only if $\mathfrak{P}$ is ramified in K_f/K_f^+.*

Congruences between the Hecke eigenvalues of automorphic forms often produce non-zero elements in groups whose orders appear in the Bloch-Kato conjecture on special values of L-functions. When the L-values in question are amenable to analysis or to computation, this can provide an opportunity to test the conjecture, by proving a consequence or computing data that support it. In order to examine the consequences of the congruences, one has to interpret them in terms of Galois representations.

Given $f, \mathfrak{P}$ as above, let

$$\rho_{f,\mathfrak{P}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\mathfrak{P}})$$

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be the continuous linear representation attached to f by Deligne [De2]. For every prime $q \nmid Dp$, $\rho_{f,\mathfrak{P}}$ is unramified at q, and if $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\overline{\mathbb{F}}_q$ as $x \mapsto x^q$ then

$$\det(I - \rho_{f,\mathfrak{P}}(\mathrm{Frob}_q^{-1})X) = 1 - a_q(f)X + \chi_D(q)q^{k-1}X^2.$$

Choosing a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant $\mathcal{O}_{f,\mathfrak{P}}$-lattice in the 2-dimensional $K_{f,\mathfrak{P}}$-vector space on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via $\rho_{f,\mathfrak{P}}$, then reducing modulo $\mathfrak{P}$, one obtains a residual representation

$$\overline{\rho}_{f,\mathfrak{P}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_{\mathfrak{P}}),$$

where $\mathbb{F}_{\mathfrak{P}} := \mathcal{O}_{f,\mathfrak{P}}/\mathfrak{P}$.

Proposition 1.2. *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$. Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, $\mathfrak{P} \mid (p)$ a prime divisor in K_f such that $f \equiv f_c \pmod{\mathfrak{P}}$. Suppose that $p \nmid 2D$, that $\mathfrak{P} \nmid a_f(p)$ (i.e. f is ordinary at $\mathfrak{P}$) and that $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible. Then*

1. *$\overline{\rho}_{f,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D$, where χ_D is viewed as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*
2. *The restriction of $\overline{\rho}_{f,\mathfrak{P}}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ is reducible.*
3. *The prime p splits in F, say as $\mathfrak{p}\mathfrak{p}^{\sigma}$. The representation $\overline{\rho}_{f,\mathfrak{P}}$ is induced from a character $\phi_{\mathfrak{p}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, coming via class field theory from an idele class character whose finite part is the $(1-k)$-power of the identity character $(\mathcal{O}_F/\mathfrak{p})^{\times} \to \overline{\mathbb{F}}_p^{\times}$. Equally it is induced by $\phi_{\mathfrak{p}^c}$, similarly defined but of conductor $\mathfrak{p}^c$ in place of $\mathfrak{p}$.*
4. *$p \mid \mathrm{Norm}_{F/\mathbb{Q}}((\epsilon_+)^{k-1} - 1)$, where ϵ_+ is a generator for the group of totally positive units of $\mathcal{O}_F$.*

Conversely, if $p \nmid 6D$ is a prime that splits, and if $p \mid \mathrm{Norm}_{F/\mathbb{Q}}((\epsilon_+)^{k-1} - 1)$ then there exists a normalised Hecke eigenform $f \in S_k(\Gamma_0(D), \chi_D)$, ordinary at some $\mathfrak{P} \mid (p)$, such that $f \equiv f_c \pmod{\mathfrak{P}}$ and $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible.

A convenient reference for the proof is [BG], where (1)-(4) are covered by Theorem 2.1 and the converse part is covered by Theorem 2.11, both of which are more general statements. I have largely adopted their notation, though their $\rho_{f,\mathfrak{P}}$ is the dual of ours. (4) is a consequence of class field theory, that the character $\phi_{\mathfrak{p}}$ in (3) must kill the totally positive unit ϵ_+. It was proved in the case $k = 2$ by Ohta [O], confirming an experimental observation of Shimura [Sh, before Proposition 7.34]. For general k it is part of Theorem 1 in a paper of Hida [H1]. The converse part was proved by Koike in the case $k = 2$, which was all he needed [K, Proposition 4.1], and in general it is again part of Hida's Theorem 1. Though $\mathfrak{P}$ is said to be a dihedral congruence prime for f, it is not the image of $\overline{\rho}_{f,\mathfrak{P}}$ in $\mathrm{GL}_2(\mathbb{F}_{\mathfrak{P}})$, rather its projection to $\mathrm{PGL}_2(\mathbb{F}_{\mathfrak{P}})$, that is isomorphic to a dihedral group.

A consequence of $\overline{\rho}_{f,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D$ is the existence of a non-zero element of $H^0(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$ (Lemma 3.1). As we shall see in §3, this contributes to the denominator of a conjectural formula (1) for the value at $s = 1$ of a "twisted adjoint" L-function $L(s, \mathrm{ad}^0(f) \otimes \chi_D)$, whose Euler factor at any prime $q \nmid D$ is

$$L_q(s, \mathrm{ad}^0(f) \otimes \chi_D) = [(1-(\alpha_q/\beta_q)\chi_D(q)q^{-s})(1-\chi_D(q)q^{-s})(1-(\beta_q/\alpha_q)\chi_D(q)q^{-s})]^{-1},$$

where the Euler factor at q of the Hecke L-function $L(s, f)$ is

$$L_q(s, f) = [(1 - \alpha_q q^{-s})(1 - \beta_q q^{-s})]^{-1}.$$

Since $\alpha_q\beta_q = \chi_D(q)q^{k-1}$, we also have $L(s, \mathrm{ad}^0(f)\otimes\chi_D) = L(s+k-1, \mathrm{Sym}^2(f))$, so we are equally looking at the value $L(k, \mathrm{Sym}^2(f))$. The conjectural formula is an instance of the Bloch-Kato conjecture. It is in fact a formula for the factorisation of the algebraic number

$$\frac{L(1, \mathrm{ad}^0(f)\otimes\chi_D)}{\Omega},$$

where Ω is a suitably normalised Deligne period [De1], and we are looking at the $\mathfrak{P}$-part. There is also a term in the numerator of the conjectural formula (1), the order of a certain Selmer group. In Lemma 3.2 we are able to show, under certain hypotheses, that this Selmer group contributes nothing at $\mathfrak{P}$, so we expect that $\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\Omega}\right) < 0$.

Zagier [Z] proved a formula for the critical values of $L(s, \mathrm{ad}^0(f)\otimes\chi_D)$, in particular for the algebraic number $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}$, where (f, f) is the Petersson norm. (It shows that this algebraic number lies, as expected, in K_f. In fact it lies in K_f^+, since it is easy to check that for any Hecke eigenform $f \in S_k(\Gamma_0(D), \chi_D)$ the coefficients of the Dirichlet series $L(s, \mathrm{ad}^0(f)\otimes\chi_D)$ are real.) So we need to make use of the relation between Ω and the Petersson norm, between which intervenes a certain cohomological congruence ideal η_f, which is the subject of §2. For the very special type of simple congruence we are looking at, we are able, with the help of a "multiplicity one" theorem of Faltings and Jordan [FJ], to say (under mild hypotheses) exactly what the $\mathfrak{P}$-part of η_f is; see Proposition 2.2. We use this in §3, both in proving triviality of the Selmer group (Lemma 3.2) and in producing a definite prediction that

$$\mathrm{ord}_{\mathfrak{P}^+}\left(\frac{L(1, \mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}\right) < 0,$$

where $\mathfrak{P}^+$ is the divisor of K_f^+ below K_f.

In §4 we seek to use Zagier's formula to confirm this, but have to settle for showing that it is true for *some* normalised Hecke eigenform $f \in S_k(\Gamma_0(D), \chi_D)$, not necessarily one satisfying $f \equiv f_c \pmod{\mathfrak{P}}$; see Theorem 4.1. (We also need conditions $D \equiv 1 \pmod 4$ and $k > 2$.) Remarkably, the required contribution of $\mathfrak{P}^+$ to the denominator comes from $((\epsilon_+)^{k-1}-1)$, after summing a geometric series. The occurrence of divisors of $((\epsilon_+)^{k-1}-1)$ in the denominator of $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}$ was observed in [DHI, 2.2] in numerical examples, which Doi and Ishii computed using Zagier's formula, so presumably they likewise summed this series.

I am grateful to an anonymous referee for raising the question of whether in certain cases we can see that the f produced by Theorem 4.1 *does* necessarily satisfy $f \equiv f_c \pmod{\mathfrak{P}}$. This is certainly true in the examples $D = 5$ and $(k, p) = (8, 29)$ or $(6, 11)$, where $S_k(\Gamma_0(D), \chi_D)$ is 2-dimensional.

Ghate [G, §10, Remark 4] has an alternative explanation for the appearance of dihedral congruence primes in the denominator of $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\Omega}$. This is based on the fact that, as a congruence prime for f, $\mathfrak{P}$ appears in the numerator of a suitably normalised $L(1, \mathrm{ad}^0(f))$, by a theorem of Hida [H3, Theorem 5.16], but because f and f_c have the same Doi-Naganuma lift $\hat{f} = \hat{f}_c$ (base change to F), $f \equiv f_c \pmod{\mathfrak{P}}$ does not make $\mathfrak{P}$ a congruence prime for $\hat{f}$, so it is not expected to appear in the numerator of a suitably normalised $L(1, \mathrm{ad}^0(\hat{f})) = L(1, \mathrm{ad}^0(f))L(1, \mathrm{ad}^0(f)\otimes$

χ_D). Thus it is required in the denominator of the second factor, subject to a conjectured period relation making the normalisations compatible. Note that the recent proof by Tilouine and Urban [TU] of such a period relation is currently only for trivial nebentypus, so does not apply here.

The viewpoint taken here is that of [DH], where however the situation was a little different. We had a Hecke eigenform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with $\overline{\rho}_{f,\mathfrak{P}}$ dihedral, whose existence depended on non-triviality of the class group of $\mathbb{Q}(\sqrt{-p})$, with $p \equiv 3 \pmod 4)$), and $k = (p+1)/2$. Again, we confirmed a prediction of the Bloch-Kato conjecture, this time proving that $\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(\mathrm{Sym}^2(g),2k-2)}{\Omega}\right) < 0$ (a rightmost, rather than near-central, critical value) for *some* Hecke eigenform $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. (Strictly speaking, since we did not prove triviality of the Selmer group, we had to reverse this logic, predicting and then proving the existence of f after showing that of g, an approach we could have taken here too.)

The main new proved results of the paper are Propositions 2.2 and 3.3, and Theorem 4.1.

2. The congruence ideal

In this section we prove a technical result ready for use in the following section. Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$, $f \in S_k(\Gamma_0(D), \chi_D)$ a normalised Hecke eigenform. Let K_f be the CM subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f, with ring of integers $\mathcal{O}_f$, maximal real subfield K_f^+, with its ring of integers $\mathcal{O}_f^+$. Let $\mathbb{T}$ be the ring generated over $\mathcal{O}_f^+$ by the endomorphisms of $S_k(\Gamma_0(D), \chi_D)$ given by all the Hecke operators T_q for all primes q. Let $\theta_f : \mathbb{T} \to \mathcal{O}_f$ be the homomorphism such that $T(f) = \theta_f(T)f\ \forall T \in \mathbb{T}$. Let S be the set of primes dividing $D(k!)$.

We consider the premotivic structure (with coefficients in $\mathbb{Q}$) $M(D, \chi_D)_!$ constructed in [DFG1, §1.4.2]. (See [DFG1, §§1.1.1,1.1.2] for generalities on premotivic and S-integral premotivic structures.) It has realisations $M(D, \chi_D)_{!,B}$, $M(D, \chi_D)_{!,\mathrm{dR}}$, $M(D, \chi_D)_{!,\ell}$ and $M(D, \chi_D)_{!,\ell\text{-crys}}$ (for each prime $\ell \notin S$). (Actually, even for $\ell \in S$ we have $M(D, \chi_D)_{!,\ell}$, but strictly speaking it is not part of the structure.) The first two are $\mathbb{Q}$-vector spaces, subspaces of the first singular and algebraic de Rham cohomologies of the modular curve $X_1(D)$ with coefficients in a local system depending on k. The second two are $\mathbb{Q}_\ell$-vector spaces, coming from ℓ-adic (étale) and crystalline cohomology. This gives various extra structures and comparison isomorphisms. For instance, $M(D, \chi_D)_{!,\ell}$ has a continuous linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $M(D, \chi_D)_{!,\mathrm{dR}}$ has a filtration, with $\mathrm{Fil}^{k-1}M(D, \chi_D)_{!,\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq S_k(\Gamma_0(D), \chi_D)$. In this sense, $M(D, \chi_D)_!$ is the premotivic structure associated to $S_k(\Gamma_0(D), \chi_D)$. By [DFG1, §1.5.3, Proposition 1.3], there is a Poincaré duality isomorphism

$$\hat{\delta}_! : M(D, \chi_D)_! \to \mathrm{Hom}_{\mathbb{Q}}(M(D, \chi_D)_!, M_{\chi_D}(1-k)),$$

where $M_{\chi_D}(1-k)$ is a Tate twist of a rank-1 premotivic structure M_{χ_D} attached to the Dirichlet character χ_D. This duality isomorphism is compatible with natural actions of $\mathbb{T}$, and the associated perfect pairing is alternating. Thus

$$[,] : \wedge^2 M(D, \chi_D)_! \simeq M_{\chi_D}(1-k).$$

We have also an S-integral premotivic structure $\mathcal{M}(D, \chi_D)_!$. Among its realisations, $\mathcal{M}(D, \chi_D)_{!,B}$ is a $\mathbb{Z}$-lattice in $M(D, \chi_D)_{!,B}$, $\mathcal{M}(D, \chi_D)_{!,\mathrm{dR}}$ is a $\mathbb{Z}_S$-lattice in

$M(D,\chi_D)_{!,\mathrm{dR}}$, $\mathcal{M}(D,\chi_D)_{!,\ell}$ is a $\mathbb{Z}_\ell$-lattice in $M(D,\chi_D)_{!,\ell}$, preserved by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $\mathcal{M}(D,\chi_D)_{!,\ell\text{-crys}}$ is a $\mathbb{Z}_\ell$-lattice in $M(D,\chi_D)_{!,\ell\text{-crys}}$.

We also have a premotivic structure M_f with coefficients in K_f [DFG1, §1.6.2]. It is a substructure of $M(D,\chi_D)_! \otimes_{\mathbb{Q}} K_f$, with $\mathrm{Fil}^{k-1}M_f = K_f f$. For any prime divisor λ of K_f, the λ-adic realisation $M_{f,\lambda}$ is a 2-dimensional $K_{f,\lambda}$-vector space with continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. This is the Galois representation attached to f. The S-integral premotivic structure $\mathcal{M}_f$ has $\mathrm{Fil}^{k-1}\mathcal{M}_{f,\mathrm{dR}} = \mathcal{O}_{f,S}f$. The $\mathcal{O}_{f,\lambda}$-lattice $\mathcal{M}_{f,\lambda}$ in $M_{f,\lambda}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, and $\overline{\mathcal{M}}_{f,\lambda} := \mathcal{M}_{f,\lambda}/\lambda\mathcal{M}_{f,\lambda}$ is the residual representation.

The isomorphism $\hat{\delta}_! : M(D,\chi_D)_! \to \mathrm{Hom}_{\mathbb{Q}}(M(D,\chi_D)_!, M_{\chi_D}(1-k))$ restricts (after extension of scalars) to an isomorphism

$$\hat{\delta}_f : M_f \to \mathrm{Hom}_{K_f}(M_f, M_{\chi_D}(1-k)\otimes K_f),$$

i.e. $[,] : \wedge^2_{K_f} M_f \simeq M_{\chi_D}(1-k)\otimes K_f$. However, although the duality pairing gives $\hat{\delta}_! : \mathcal{M}(D,\chi_D)_! \to \mathrm{Hom}_{\mathbb{Z}_S}(\mathcal{M}(D,\chi_D)_!, \mathcal{M}_{\chi_D}(1-k)\otimes\mathcal{O}_{f,S})$, it does not restrict to $[,] : \wedge^2_{\mathcal{O}_{f,S}}\mathcal{M}_f \simeq \mathcal{M}_{\chi_D}(1-k)\otimes\mathcal{O}_{f,S}$, rather

$$[,] : \wedge^2_{\mathcal{O}_{f,S}}\mathcal{M}_f \simeq \eta_f\mathcal{M}_{\chi_D}(1-k)\otimes\mathcal{O}_{f,S},$$

for some integral ideal η_f, as noted in [DFG2, §2]; see also [DFG1, §1.7.3].

Definition 2.1. *This η_f is the* congruence ideal *for f.*

Proposition 2.2. *Fix $\mathfrak{P}$ a prime divisor in K_f, with $\mathfrak{P} \nmid D(k!)[\mathcal{O}_f : \theta_f(\mathbb{T})]$. Suppose that given $g \in S_k(\Gamma_0(D),\chi_D)$ a normalised Hecke eigenform, we have a congruence $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}$ $\forall T \in \mathbb{T}$, if and only if $g = f$ or $g = f_c$, the complex conjugate eigenform. Then*

$$\mathrm{ord}_{\mathfrak{P}}(\eta_f) = 1.$$

Note that $\theta_f(\mathbb{T})$ is the same thing as $\mathcal{O}(f)$. To prove the proposition, we need two lemmas. Let $\overline{\theta_f} : \mathbb{T} \to \mathbb{F}_{\mathfrak{P}}$ be the composition of θ_f with the reduction map $\mathcal{O}_f \to \mathcal{O}_f/\mathfrak{P} =: \mathbb{F}_{\mathfrak{P}}$, and let $\mathfrak{m} := \ker\overline{\theta_f}$, $\mathbb{T}_{\mathfrak{m}}$ the local completion at $\mathfrak{m}$. We may define a premotivic structure $M_{[f]}$, with coefficients in K_f^+, associated with the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit $[f] := \{f, f_c\}$, as the kernel of the appropriate ideal of $\mathbb{T}$ acting on $M(D,\chi_D)_! \otimes_{\mathbb{Q}} K_f^+$, so that $\mathrm{Fil}^{k-1}M_{[f]} \otimes_{K_f^+} \mathbb{C} \simeq \mathbb{C}f \oplus \mathbb{C}f_c$, and similarly an S-integral premotivic structure $\mathcal{M}_{[f]}$ (with coefficients in $O^+_{f,S}$). Let $\mathfrak{P}$ be as in the proposition, with $\mathfrak{P}^+$ the divisor below it in K_f^+.

Lemma 2.3. *$\mathcal{M}_{[f],\mathfrak{P}^+} \simeq \mathbb{T}_{\mathfrak{m}}^2$ as a $\mathbb{T}_{\mathfrak{m}}$-module.*

Proof. First note that, because of the congruence $\theta_{f_c}(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}$ $\forall T \in \mathbb{T}$, $\mathcal{M}_{[f],\mathfrak{P}^+}$ is a $\mathbb{T}_{\mathfrak{m}}$-module. One may prove, just as in the proof of [FJ, Theorem 2.1], the "multiplicity one" formula

$$\overline{\mathcal{M}}_{[f],\mathfrak{P}^+}[\mathfrak{m}] \simeq (\mathbb{T}/\mathfrak{m})^2.$$

The lemma then follows by a standard application of Nakayama's Lemma. $\square$

Lemma 2.4. *Suppose that $\mathfrak{P}$ is ramified in K_f/K_f^+. Consider the map ψ, K_f^+-linear in the first factor, K_f-linear in the second factor, given by*

$$\psi : K_f \otimes_{K_f^+} K_f \simeq K_f^2 : \qquad \alpha\otimes\beta \mapsto (\alpha\beta, \overline{\alpha}\beta).$$

Then

$$\psi(\mathcal{O}_{f,\mathfrak{P}} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}}) = \{(z,w) \in \mathcal{O}^2_{f,\mathfrak{P}} :\ z \equiv w \pmod{\mathfrak{P}}\}.$$

Proof. Choosing an $\mathcal{O}^+_{f,\mathfrak{P}^+}$-basis $\{1,\pi\}$ for $\mathcal{O}_{f,\mathfrak{P}}$, where π is a uniformiser for $\mathfrak{P}$ and π^2 a uniformiser for $\mathfrak{P}^+$, every element of $\mathcal{O}_{f,\mathfrak{P}} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}}$ is of the form $1 \otimes (a+b\pi) + \pi \otimes (c+d\pi)$, with $a,b,c,d \in \mathcal{O}^+_{f,\mathfrak{P}^+}$. Now

$$\psi(1\otimes(a+b\pi)+\pi\otimes(c+d\pi)) = (a+d\pi^2+(b+c)\pi, a-d\pi^2+(b-c)\pi) = (x+y\pi, u+v\pi),$$

where

$$a = \frac{x+u}{2},\ b = \frac{y+v}{2},\ c = \frac{y-v}{2},\ d = \frac{x-u}{2\pi^2}.$$

The condition $x \equiv u \pmod{\pi^2}$ is equivalent to $z \equiv w \pmod{\mathfrak{P}}$. □

Proof of Proposition 2.2. By Lemma 1.1, $\mathfrak{P}$ is ramified in K_f/K_f^+. By Lemma 2.3, $\mathcal{M}_{[f],\mathfrak{P}^+} \simeq \mathbb{T}^2_{\mathfrak{m}}$ as a $\mathbb{T}_{\mathfrak{m}}$-module. Since $p \nmid [\mathcal{O}_f : \theta_f(\mathbb{T})]$, θ_f induces an isomorphism between $\mathbb{T}_{\mathfrak{m}}$ and $\mathcal{O}_{f,\mathfrak{P}}$, hence $\mathcal{M}_{[f],\mathfrak{P}^+} \simeq (\mathcal{O}_{f,\mathfrak{P}})^2$. Inside $\mathcal{M}_{[f]} \otimes_{\mathcal{O}_f^+} \mathcal{O}_f$, the substructure $\mathcal{M}_f$ is defined by the condition that the action of any $T \in \mathbb{T}$ via the first factor matches the action of $\theta_f(T)$ via the second factor. For $\mathcal{M}_{f_c}$ we just replace θ_f by θ_{f_c}. Identifying $\mathcal{M}_{[f],\mathfrak{P}^+} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}}$ with $\mathcal{O}^2_{f,\mathfrak{P}} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}}$, and applying Lemma 2.4, we find that

$$\mathcal{M}_{f,\mathfrak{P}} \simeq \{(z_1,0,z_2,0) \in \mathcal{O}^4_{f,\mathfrak{P}} :\ \ z_1,z_2 \in \mathfrak{P}\}$$

and

$$\mathcal{M}_{f_c,\mathfrak{P}} \simeq \{(0,w_1,0,w_2) \in \mathcal{O}^4_{f,\mathfrak{P}} :\ \ w_1,w_2 \in \mathfrak{P}\}.$$

Hence

$$(\mathcal{M}_{[f],\mathfrak{P}^+} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}})/(\mathcal{M}_{f,\mathfrak{P}} \oplus \mathcal{M}_{f_c,\mathfrak{P}}) \simeq \mathcal{O}_{f,\mathfrak{P}}/\mathfrak{P}^2.$$

This $\mathcal{M}_{f,\mathfrak{P}} \oplus \mathcal{M}_{f_c,\mathfrak{P}}$ is an orthogonal direct sum for the pairing $[,]$. Recall that

$$[,] : \wedge^2_{\mathcal{O}_{f,\mathfrak{P}}} \mathcal{M}_{f,\mathfrak{P}} \simeq \eta_f \mathcal{M}_{\chi_D}(1-k) \otimes \mathcal{O}_{f,\mathfrak{P}},$$

and similarly $[,] : \wedge^2_{\mathcal{O}_{f,\mathfrak{P}}} \mathcal{M}_{f_c,\mathfrak{P}} \simeq \eta_{f_c} \mathcal{M}_{\chi_D}(1-k) \otimes \mathcal{O}_{f,\mathfrak{P}}$. But the condition that $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}\ \forall T \in \mathbb{T}$, only for $g = f$ or $g = f_c$, implies that

$$[,] : \wedge^2_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{M}_{[f],\mathfrak{P}^+} \simeq \mathcal{M}_{\chi_D}(1-k) \otimes \mathcal{O}^+_{f,\mathfrak{P}^+}.$$

It follows (using also symmetry between f and f_c) that $\mathrm{ord}_{\mathfrak{P}}(\eta_f) = \mathrm{ord}_{\mathfrak{P}}(\eta_{f_c}) = 1$. □

3. The Bloch-Kato conjecture

As before, let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, K_f the CM subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f. We saw the premotivic structure M_f, with coefficients in K_f, and the S-integral premotivic structure $\mathcal{M}_f$, where S is the set of primes dividing $D(k!)$. Following [DFG1, §1.7.1], we consider the adjoint premotivic structure $A_f = \mathrm{ad}^0(M_f)$, the kernel of the trace morphism $\mathrm{Hom}_{K_f}(M_f, M_f) \to K_f$, and the associated S-integral premotivic structure $\mathcal{A}_f$. We will need also $A_{f,\chi_D} := A_f \otimes M_{\chi_D}$ and $\mathcal{A}_{f,\chi_D} := \mathcal{A}_f \otimes \mathcal{M}_{\chi_D}$. We can recover the Hecke L-function $L(s,f) = \sum_{m=1}^{\infty} a_f(m) m^{-s}$ in the following way. For each finite prime q, choose any $\ell \neq q$, and $\lambda \mid \ell$ in K_f. Let $F_p(X) = \det(I - \rho|_{V^{I_q}}(\mathrm{Frob}_q^{-1})X)$, where $V = M_{f,\lambda}$. Then $L(s,f) = \prod_q L_q(s,f)$, where $L_q(s,f)^{-1} = F_q(q^{-s})$. We may also define an adjoint L-function $L(s,\mathrm{ad}^0(f))$, and a twisted adjoint L-function $L(s,\mathrm{ad}^0(f)\otimes\chi_D)$, by using $V=A_{f,\lambda}$ and $V=A_{f,\chi_D,\lambda}$, respectively. The Euler factors at "bad" primes $q\mid D$ are as follows:

$$\begin{aligned} L_q(s,f) &= (1-a_f(q)q^{-s})^{-1}, \text{ with } a_f(q)\overline{a_f(q)}=q^{k-1};\\ L_q(s,\mathrm{ad}^0(f)) &= (1-q^{-s})^{-1};\\ L_q(s,\mathrm{ad}^0(f)\otimes\chi_D) &= ((1-a_f(q)^2q^{1-k-s})(1-\overline{a_f(q)}^2q^{1-k-s}))^{-1}. \end{aligned}$$

Our $L(s,\mathrm{ad}^0(f)\otimes\chi_D)$ is the same as Zagier's $D_f(s+k-1)$ in [Z, §6], but note that in Ghate's $L(s,\mathrm{Ad}(f))$ and $L(s,\mathrm{Ad}(f)\otimes\chi_D)$ [G, §5], Euler factors at primes $q\mid D$ are omitted. Since the dual of M_f is $M_f\otimes M_{\chi_D}(1-k)$, $L(s,\mathrm{ad}^0(f)\otimes\chi_D)$ can also be described as $L(s+k-1,\mathrm{Sym}^2(f))$, i.e. $D_f(s)=L(s,\mathrm{Sym}^2(f))$.

Lemma 3.1. *Let $f\in S_k(\Gamma_0(D),\chi_D)$ be a normalised Hecke eigenform, $\mathfrak{P}$ a prime divisor in K_f such that $f\equiv f_c \pmod{\mathfrak{P}}$ and $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible. Then $H^0(\mathbb{Q},\mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ is non-trivial.*

Proof. By (1) of Proposition 1.2, $\overline{\rho}_{f,\mathfrak{P}}\simeq\overline{\rho}_{f,\mathfrak{P}}\otimes\chi_D$. (For just this part, we do not need the additional conditions of the proposition.) At any $q\mid D$, the space of $\overline{\rho}_{f,\mathfrak{P}}$ has an unramified line and a line on which I_q acts via χ_D, by a Theorem of Langlands and Carayol [H2, Theorem 4.2.7(3)(a)]. Tensoring with χ_D swaps those lines, so an isomorphism from $\overline{\rho}_{f,\mathfrak{P}}$ to $\overline{\rho}_{f,\mathfrak{P}}\otimes\chi_D$ must have trace zero, and gives us a non-zero element of $\mathfrak{P}$-torsion in $H^0(\mathbb{Q},\mathcal{A}_{f,\chi_D}/A_{f,\chi_D})$. □

This is the key Galois-theoretical consequence of the congruence $f\equiv f_c \pmod{\mathfrak{P}}$, since the order of $H^0(\mathbb{Q},\mathcal{A}_{f,\chi_D}/A_{f,\chi_D})$ appears in the denominator of the conjectural formula (1) for $L(1,\mathrm{ad}^0(f)\otimes\chi_D)$ given by the Bloch-Kato conjecture. We must prepare ourselves to look at other terms in the formula.

Given a field F and a continuous $\mathrm{Gal}(\overline{F}/F)$-module M, $H^1(F,M)$ will mean for us $H^1_{\mathrm{cont.}}(F,M)$ (the quotient of continuous cocycles by continuous coboundaries). Given a finite-dimensional continuous representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Q}_p$, unramified outside a finite set of primes, following Bloch and Kato [BK] we define

$$H^1_f(\mathbb{Q}_q,V):=\begin{cases} H^1_{\mathrm{ur}}(\mathbb{Q}_q,V) & q\neq p\\ \ker(H^1(\mathbb{Q}_q,V)\to H^1(\mathbb{Q}_q,V\otimes B_{\mathrm{crys}})) & q=p\end{cases},$$

where I_q is the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, B_{crys} is Fontaine's ring, as defined in [BK, §1], and

$$H^1_{\mathrm{ur}}(\mathbb{Q}_q,M):=\ker(H^1(\mathbb{Q}_q,M)\to H^1(I_q,M)).$$

Now let $T\subset V$ be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbb{Z}_p$-lattice, and $W:=V/T$. Further define

$$H^1_f(\mathbb{Q}_q,W):=\mathrm{im}(H^1_f(\mathbb{Q}_q,V)\to H^1(\mathbb{Q}_q,W)),$$

and for any finite set of primes Σ not containing p let $H^1_\Sigma(\mathbb{Q},W)$ be the subgroup of elements of $H^1(\mathbb{Q},W)$ whose images in $H^1(\mathbb{Q}_q,W)$ lie in $H^1_f(\mathbb{Q}_q,W)$, for all (finite) primes $q\notin\Sigma$. As noted in [DFG1, §2.1] if V is unramified at q (with $q\neq p$) then $H^1_f(\mathbb{Q}_q,W)=H^1_{\mathrm{ur}}(\mathbb{Q}_q,W)$.

Lemma 3.2. *Let $f\in S_k(\Gamma_0(D),\chi_D)$ be a normalised Hecke eigenform, $\mathfrak{P}\mid p$ a prime divisor in K_f with $p\nmid D(2k-1)(2k-3)(k!)[\mathcal{O}_f:\theta_f(\mathbb{T})]$. Suppose that* *given $g \in S_k(\Gamma_0(D), \chi_D)$ a normalised Hecke eigenform, we have a congruence $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}$ $\forall T \in \mathbb{T}$, if and only if $g = f$ or $g = f_c$, the complex conjugate eigenform, and that $\rho_{f,\mathfrak{P}} \pmod{\mathfrak{P}^2} \not\simeq \rho_{f_c,\mathfrak{P}} \pmod{\mathfrak{P}^2}$. Suppose that $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible and that $\mathfrak{P} \nmid a_f(p)$. Let Σ be the set of primes dividing D. Suppose that for all primes $q \mid D$, $q \not\equiv 1 \pmod{p}$. Then $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ is trivial.*

Proof. We consider the long exact sequence in Galois cohomology arising from the short exact sequence

$$0 \longrightarrow \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D \longrightarrow \frac{\mathcal{A}_{f,\chi_D,\mathfrak{P}}}{A_{f,\chi_D,\mathfrak{P}}} \xrightarrow{\pi} \frac{\mathcal{A}_{f,\chi_D,\mathfrak{P}}}{A_{f,\chi_D,\mathfrak{P}}} \longrightarrow 0,$$

where the third map from the left is multiplication by some uniformising element π for $\mathfrak{P}$. If $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ were non-trivial, there would be a non-zero element killed by $\mathfrak{P}$, which is necessarily in the image of $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$, say coming from an element α. By Lemma 3.1 we have a non-zero element killed by $\mathfrak{P}$ in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$. By the condition $\rho_{f,\mathfrak{P}} \pmod{\mathfrak{P}^2} \not\simeq \rho_{f_c,\mathfrak{P}} \pmod{\mathfrak{P}^2}$, there is no element of exact annihilator $\mathfrak{P}^2$ in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$. Hence our element of annihilator $\mathfrak{P}$ in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ maps to a non-zero element β of $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$. Since β maps to 0 in $H^1(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ (by exactness), while α maps to a non-zero element, α and β must be linearly independent elements of $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$. Since $H^0(\mathbb{Q}, \mathbb{F}_{\mathfrak{P}} \otimes \chi_D)$ is trivial, $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$ injects into $H^1(\mathbb{Q}, \mathrm{ad}(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$. Composing with the isomorphism $\overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D \simeq \overline{\rho}_{f,\mathfrak{P}}$, we obtain independent non-zero elements α', β' of $H^1(\mathbb{Q}, \mathrm{ad}(\overline{\rho}_{f,\mathfrak{P}}))$.

Actually, viewing $\rho_{f_c,\mathfrak{P}}$ as representing a deformation of $\overline{\rho}_{f,\mathfrak{P}}$, we have obtained β' by the standard construction in disguise: if (using bases compatible with $\overline{\rho}_{f_c,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}}$), $\rho_{f_c,\mathfrak{P}}(g) \equiv \rho_{f,\mathfrak{P}}(g)(I + \pi c(g)) \pmod{\mathfrak{P}^2}$, where π is a uniformiser at $\mathfrak{P}$, then the cocycle $g \mapsto c(g)$ represents β'. Since $\rho_{f,\mathfrak{P}}$ and $\rho_{f,\mathfrak{P}} \otimes \chi_D$ have the same determinant, β' actually lives in (the image of) $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}))$.

Since α' comes from $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$, its image in $H^1(\mathbb{Q}, \mathbb{F}_{\mathfrak{P}})$ by the trace map, composed with any linear map $\mathbb{F}_{\mathfrak{P}} \to \mathbb{F}_p$, produces either 0 or an element of $\mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{F}_p)$ whose kernel has fixed field a degree p extension of $\mathbb{Q}$, unramified for any $q \nmid D$. (That it is unramified at p is addressed by [BK, Example 3.9].) Such an extension does not exist, given our assumptions that $p \nmid D$ and $q \not\equiv 1 \pmod{p}$ for all $q \mid D$. Hence the image of α' in $H^1(\mathbb{Q}, \mathbb{F}_{\mathfrak{P}})$ is 0, so α' also lives in $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}))$.

By Proposition 1.2, $\rho_{f,\mathfrak{P}}$ is dihedral, from which it easily follows that $H^0(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})$ is trivial. Hence α', β' map to independent non-zero elements α'', β'' of $H^1(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})$. Using [DFG1, Proposition 2.2] we see that α'' (having come from $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$) satisfies the local conditions to lie in $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})$. So does β'', since $\rho_{f_c,\mathfrak{P}}$ is unramified at $q \nmid pD$ and crystalline at p.

We have now that $\mathfrak{P}^2$ divides the Fitting ideal $\mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}}))$. Since $p \nmid D(2k-1)(2k-3)(k!)$, the restriction of $\overline{\rho}_{f,\mathfrak{P}}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}))$ is absolutely irreducible, by [DFG1, Lemma 2.5]. Then by [DFG1, Theorem 3.7, Proposition 1.4(c)],

$$\mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})) = \eta_f \prod_{q \mid D} L_q(1, \mathrm{ad}^0(f))^{-1} = \eta_f \prod_{q \mid D}(1 - q^{-1}).$$

By our assumption that $q \not\equiv 1 \pmod p$ for all $q \mid D$, we have $\mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_\Sigma(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})) = \eta_f$, but by Proposition 2.2, $\mathrm{ord}_{\mathfrak{P}}(\eta_f) = 1$, contradicting $\mathfrak{P}^2 \mid \mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_\Sigma(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}}))$. □

Since $\Sigma \neq \emptyset$, the $\mathfrak{P}$-part of the Bloch-Kato conjecture, applied to the critical value $L(1, \mathrm{ad}^0(f) \otimes \chi_D)$, may be formulated as follows, following [DFG1, (59)], and using the exact sequence in their Lemma 2.1.

$$(1)\qquad \mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\Omega}\right) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{\mathrm{Fitt}_{\mathcal{O}_{f,\mathcal{P}}} H^1_\Sigma(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})}{\mathrm{Fitt}_{\mathcal{O}_{f,\mathcal{P}}} H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})}\right),$$

where Ω is a certain Deligne period normalised by the integral structure $\mathcal{A}_f$. (We are retaining the condition $p \nmid D(2k-1)(2k-3)(k!)$, hence as in [DFG1, Proposition 2.16] the Tamagawa factor is trivial, so does not appear.) Note that Deligne's conjecture [De1, Conjecture 2.8] already says that $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\Omega}$ should be an element of the coefficient field K_f. A corollary of Lemmas 3.1 and 3.2 is the following.

Proposition 3.3. *Subject to the conditions of Lemma 3.2, the right hand side of (1) is negative.*

We predict then that (subject to the conditions of Lemma 3.2)

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\Omega}\right) < 0.$$

As in [Du, §5], up to $\mathfrak{P}$-units (where our Ω is the $(2\pi i)^{2k}\Omega$ there),

$$\Omega = \pi^{k+1}(f, f)\eta_f^{-1}.$$

(For the type of argument leading to the relation between the Petersson norm (f, f), periods $\Omega^\pm$ of M_f, and η_f, as in [Du, (4)], a good additional reference is [H3, (5.18)]. The $\langle \zeta_+, \zeta_- \rangle$ in [H3, Theorem 5.16] is our η_f.) So the Bloch-Kato conjecture leads to the prediction that (subject to the conditions of Lemma 3.2)

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\pi^{k+1}(f, f)}\right) < -\mathrm{ord}_{\mathfrak{P}}(\eta_f).$$

Using Proposition 2.2 we may reformulate this again as

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\pi^{k+1}(f, f)}\right) \leq -2.$$

As already noted in the introduction, in fact $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)} \in K_f^+$, and since $\mathfrak{P}^+ = \mathfrak{P}^2$, it becomes

$$\mathrm{ord}_{\mathfrak{P}^+}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\pi^{k+1}(f, f)}\right) < 0.$$

In the following section we shall prove something slightly weaker, that if $p \mid \mathrm{Norm}_{F/\mathbb{Q}}((\epsilon_+)^{k-1} - 1)$ then $\mathrm{ord}_{\mathfrak{P}^+}\left(\frac{L(1,\mathrm{ad}^0(g)\otimes\chi_D)}{\pi^{k+1}(g,g)}\right) < 0$ for *some* normalised Hecke eigenform $g \in S_k(\Gamma_0(D), \chi_D)$ (and $\mathfrak{P}$ now a divisor of p in K_g). Of course we expect it to be f satisfying $f \equiv f_c \pmod{\mathfrak{P}}$, with $\mathfrak{P}$ ramified in K_f/K_f^+, but we cannot eliminate the possibility that it is only some other g. Note that if $\deg(\mathfrak{P}^+) > 1$ then applying a non-trivial element of its decomposition group to the pair f, f_c will produce another pair g, g_c congruent to each other mod $\mathfrak{P}$, for whom we should also see $\mathfrak{P}^+$ in the denominator.

One might question the condition that $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}\ \forall T \in \mathbb{T}$, if and only if $g = f$ or $g = f_c$. How strong is this? In twelve out of the thirteen numerical examples in [DHI, Table 1], the normalised Hecke eigenforms in $S_k(\Gamma_0(D), \chi_D)$ form a single Galois orbit. Assuming also that $p \nmid [\mathcal{O}_f : \theta_f(\mathbb{T})]$, if the condition failed then an automorphism taking f to g (in addition to one taking f to f_c) would be in the inertia group for $\mathfrak{P}$, so p would be ramified in $K_f^+/\mathbb{Q}$, which seems unlikely. Such a p would be listed in both the second and third columns of the table, for a given row, but in none of those twelve examples does this happen.

4. The denominator of the twisted adjoint L-value

Theorem 4.1. *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$, $D \equiv 1 \pmod 4$. Fixing an even $k > 2$, let ϵ_+ be a generator for the group of totally positive units of $\mathcal{O}_F$, and let $\mathfrak{p}$ be any prime divisor of $(\epsilon_+)^{k-1} - 1$ in $\mathcal{O}_F$, with $p \nmid D(k!)$, where $\mathfrak{p}$ divides a rational prime p. Let v be any extension to $\overline{\mathbb{Q}}$ of the valuation associated to $\mathfrak{p}$. There exists a normalised Hecke eigenform $f \in S_k(\Gamma_0(D), \chi_D)$, such that if K_f is the subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f (maximal real subfield K_f^+) and $\mathfrak{P}^+$ is the divisor in K_f^+ associated with the restriction of v, then*

$$\mathrm{ord}_{\mathfrak{P}^+}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\pi^{k+1}(f,f)}\right) < 0.$$

Proof. By work of Zagier [Z, (91),(92)], $L(1, \mathrm{ad}^0(f) \otimes \chi_D) = -\frac{\pi}{4}\frac{(4\pi)^k}{\Gamma(k)}(C_{k,1,D}, f)$, where $C_{k,1,D}(z) :=$

$$\sum_{m=0}^{\infty}\left(\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4m \\ t^2 \equiv 4m \pmod D}} p_{k,1}(t,m) H\left(\frac{4m - t^2}{D}\right) + \frac{1}{\sqrt{D}} \sum_{\substack{\lambda \in \mathcal{O}_F \\ \lambda > 0 \\ \lambda\lambda' = m}} \min(\lambda, \lambda')^{k-1}\right) e^{2\pi i m z}.$$

Here $p_{k,1}(t,m)$, the coefficient of x^{k-2} in $(1 - tx + mx^2)^{-1}$, is an integer, and $H(n)$, the Hurwitz class number, is integral away from 2 and 3. Also, we are thinking of F as embedded in $\mathbb{R}$ in a fixed way, but λ' means the Galois conjugate of λ, i.e. the result of applying the other embedding. Now if $\epsilon \in F$ is a totally positive unit then $\epsilon' = 1/\epsilon$, so given a factorisation $m = \lambda\lambda'$ appearing in the sum, $m = (\epsilon\lambda)(\epsilon'\lambda')$ is another one. Let ϵ_+ be a generator for the group of totally positive units, chosen with $\epsilon_+ < 1$ and $(\epsilon_+)' > 1$. Choosing $m = q^2$, with q a prime number inert in F,

$$\sum_{\substack{\lambda \in \mathcal{O}_F \\ \lambda > 0 \\ \lambda\lambda' = m}} \min(\lambda, \lambda')^{k-1} = q^{k-1}(1 + 2\epsilon_+^{k-1} + 2\epsilon_+^{2(k-1)} + \ldots) = q^{k-1}\left(\frac{2}{1 - \epsilon_+^{k-1}} - 1\right).$$

Let $\mathfrak{p}$ be as in the theorem, in particular a prime divisor of $(\epsilon_+)^{k-1} - 1$ in $\mathcal{O}_F$. Then for $m = q^2$ with q inert (in particular $q \neq p$),

$$\mathrm{ord}_{\mathfrak{p}}\left(\sum_{\substack{\lambda \in \mathcal{O}_F \\ \lambda > 0 \\ \lambda\lambda' = m}} \min(\lambda, \lambda')^{k-1}\right) = -\mathrm{ord}_{\mathfrak{p}}((\epsilon_+)^{k-1} - 1).$$

Letting c_m denote the coefficient of $q^m = e^{2\pi imz}$ in $C_{k,1,D}$, we see that $\mathrm{ord}_{\mathfrak{p}} c_m = -\mathrm{ord}_{\mathfrak{p}}((\epsilon_+)^{k-1} - 1)$, for any $m = q^2$ with q inert.

Since $k > 2$, $C_{k,1,D}$ is a cusp form, and may be expressed as a linear combination of normalised Hecke eigenforms in $S_k(\Gamma_0(D), \chi_D)$. These eigenforms may be divided into $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits, and the contributions to the linear combination coming from any particular orbit (conjugate pair) may be combined. Let $B_{[g]}$ be the contribution from the orbit of g, so that $C_{k,1,D} = \sum_{[g]} B_{[g]}$. The coefficients of the Dirichlet series $L(s, \mathrm{ad}^0(g) \otimes \chi_D)$ are real, and the same as those of $L(s, \mathrm{ad}^0(g_c) \otimes \chi_D)$, while it is easy to show that $(g, g) = (g_c, g_c)$. Zagier's formula then implies that $(C_{k,1,D}, g)/(g, g)$ is real, and the same as $(C_{k,1,D}, g_c)/(g_c, g_c)$, so that $B_{[g]} = \alpha_g(g + g_c)$ for some real α_g.

In fact, since the Fourier coefficients of $C_{k,1,D}$ are rational (as noted near [Z, (98)], and cf. remark below) and the coefficients α_g are unique, any element of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ fixing the Fourier coefficients of $g + g_c$ must fix α_g, so $\alpha_g \in K_g^+$. Since $\mathrm{ord}_{\mathfrak{p}} c_m = -\mathrm{ord}_{\mathfrak{p}}((\epsilon_+)^{k-1} - 1) < 0$, for infinitely many m, there must exist a normalised eigenform f such that if $B_{[f]} = \sum_{m=1}^{\infty} b_m q^m$ then $\mathrm{ord}_{\mathfrak{P}^+} b_m < 0$, for infinitely many m. It follows that $\mathrm{ord}_{\mathfrak{P}^+}(\alpha_f) < 0$, and since

$$\alpha_f = \frac{(C_{k,1,D}, f)}{(f, f)} = \frac{4\Gamma(k) L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{(4\pi)^{k+1}(f, f)},$$

we obtain the proposition. □

Remark 4.2. In case it does not look like c_{q^2} is rational, note that $\frac{2}{1-\epsilon_+^{k-1}} - 1 = \frac{1+\epsilon_+^{k-1}}{1-\epsilon_+^{k-1}}$. Recalling that $(\epsilon_+)' = 1/\epsilon_+$, one sees that this expression is mapped to minus itself by Galois conjugation, so is necessarily a rational multiple of $\sqrt{D}$.

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