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A Review of Closed-Form Cramér-Rao Bounds for DOA Estimation in the Presence of Gaussian Noise Under a Unified Framework

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ABSTRACT The Cramér-Rao Bound (CRB) for direction of arrival (DOA) estimation has been extensively studied over the past four decades, with a plethora of CRB expressions reported for various parametric models. In the literature, there are different methods to derive a closed-form CRB expression, but many derivations tend to involve intricate matrix manipulations which appear difficult to understand. Starting from the Slepian-Bangs formula and following the simplest derivation approach, this paper reviews a number of closed-form Gaussian CRB expressions for the DOA parameter under a unified framework, based on which all the specific CRB presentations can be derived concisely. The results cover three scenarios: narrowband complex circular signals, narrowband complex noncircular signals, and wideband signals. Three signal models are considered: the deterministic model, the stochastic Gaussian model, and the stochastic Gaussian model with the *a priori* knowledge that the sources are spatially uncorrelated. Moreover, three Gaussian noise models distinguished by the structure of the noise covariance matrix are concerned: spatially uncorrelated noise with unknown either identical or distinct variances at different sensors, and arbitrary unknown noise. In each scenario, a unified framework for the DOA-related block of the deterministic/stochastic CRB is developed, which encompasses one class of closed-form deterministic CRB expressions and two classes of stochastic ones under the three noise models. Comparisons among different CRBs across classes and scenarios are presented, yielding a series of equalities and inequalities which reflect the benchmark for the estimation efficiency under various situations. Furthermore, validity of all CRB expressions are examined, with some specific results for linear arrays provided, leading to several upper bounds on the number of resolvable Gaussian sources in the underdetermined case.

INDEX TERMS Circular and noncircular, Cramér-Rao bound, direction of arrival estimation, narrowband and wideband, underdetermined and overdetermined.

I. INTRODUCTION

The Cramér-Rao Bound (CRB), which provides a lower bound on the variance of any unbiased estimator, has been extensively studied in the context of direction of arrival (DOA) estimation using sensor arrays during the past four decades, and it still attracts substantial research interest with the development of novel DOA estimation methods and array design techniques. This topic covers a broad range of results which have been published separately in the open literature, including many celebrated papers.

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The CRB depends implicitly on the data properties via the probability density function (p.d.f.). Since the Gaussian distribution, whose p.d.f. is mathematically tractable, is frequently encountered in practice, the Gaussian CRB is by far the most popular one. Another reason for the popularity is that the Gaussian CRB would be the largest of all CRBs corresponding to different congruous distributions [1, p. 363], [2]. Moreover, the CRB depends on the parametric model instead of a specific algorithm or estimator. Traditionally, two kinds of signal models are widely adopted, i.e., the deterministic (conditional) model and the stochastic (unconditional) model [3]. The former assumes the signals to be deterministic but unknown, whereas the latter assumes them to be stochastic,

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usually Gaussian distributed. Compared to the deterministic model, the detection and estimation schemes derived from the stochastic Gaussian model is usually found to yield superior performance, regardless of the actual distribution of emitter signals [4].

The signals received by each sensor are often corrupted by an additive noise. For sparsely placed sensors, the noise is spatially uncorrelated. In the ideal case, the noise covariance matrix is assumed to be diagonal with identical variances across sensors, which is known as the uniform noise (UN). Due to variation of the manufacturing process or the imperfection of array calibration, the noise variances may be different [5], which is called the nonuniform noise (NUN). More practically, the noise can be correlated from sensor to sensor. Various modeling schemes are developed to characterize the spatially colored noise, e.g., [6]–[9], and they can be summarized by an arbitrary unknown noise (AUN) model, where the noise is parameterized by a set of arbitrary unknowns depending on a specific modeling scheme.

As from its definition, the CRB can be calculated from the inverse of the Fisher information matrix (FIM) [10], but the computation is rather complicated due to the derivatives of the log-likelihood function of the data samples with respect to (w.r.t.) all unknown parameters involved. In many applications, only the DOA-related block of the CRB matrix is of interest. A closed-form CRB expression not only offers a clear interpretation of the CRB, but also allows the comparison with the asymptotic covariance matrix of estimation errors. It also supports the understanding of the source/array configuration and provides physical insights into the underlying problem.

A. CRBs FOR NARROWBAND COMPLEX CIRCULAR SIGNALS

Most CRB expressions are derived based on the p.d.f. of a complex circular Gaussian distribution, under which the additive noise is also Gaussian distributed. For the deterministic model, the closed-form CRB expression for DOA estimation in the presence of UN was derived in [11], [12], along with its worst/best version under different criteria presented in [13]. To simplify the intricate derivations, a linearization and decoupling technique was proposed in [14]. Inspired by this idea, the most compact derivation was presented in [15] by means of transforming the FIM into a block diagonal form. In the presence of NUN, two closed-form CRB expressions were derived in [16] and [17], respectively, in the single-source case and the multi-source case. In the presence of AUN, the closed-form CRB expression was derived in [18], with specific results concerning an autoregressive noise provided. Furthermore, a unified closed-form CRB expression based on known signal structures was reported in [19], which also accounts for AUN.

For the stochastic model, the closed-form CRB in the presence of UN was indirectly derived in [2], [20], [21] through asymptotic covariance matrices of estimation errors of different DOA estimators. The first direct derivation was presented in [22] by writing all submatrices of the FIM explicitly and then applying the partitioned matrix inversion lemma. Almost a decade later, a much more simplified direct derivation was given in [23], which avoids the complicated calculation of all submatrices of the FIM. Under various noise fields, the DOA estimation problems were investigated in [7]–[9], [16], [17], [24], with the corresponding CRB expressions provided. A common conclusion is that if the noise covariance matrix is parameterized by more than one unknowns, then the corresponding stochastic CRB for DOAs will be no less than that in the presence of UN, whereas the deterministic CRBs for DOAs will be identical in both cases [17], [24].

The deterministic and stochastic CRBs play an important role in asymptotic performance studies. The term ‘‘asymptotic’’ can refer to different cases where one or a combination of the following factors tend to infinity, including the number of snapshots, the number of sensors, and the signal-to-noise ratio (SNR). When the number of snapshots alone tends to infinity, the stochastic CRB can be asymptotically achieved by the stochastic maximum likelihood (ML) estimator [2], [20], the method of direction estimation (MODE) estimator [20], and the weight subspace fitting (WSF) estimator [2], [21], whereas the deterministic CRB cannot be asymptotically achieved by the deterministic ML estimator unless the number of sensors also tends to infinity [11], [20]. When the SNR alone tends to infinity, the deterministic ML estimator attains the CRB [25], but the stochastic one does not [26]. If both the number of sensors and the SNR are sufficiently large, both the deterministic and stochastic ML estimators will attain the respective CRBs [27]. These asymptotic properties are mainly studied in the presence of UN. For NUN and AUN, a number of extended estimators (mostly the ML ones) are shown to asymptotically (w.r.t. snapshots) achieve the corresponding CRBs [7], [9], [16], [17].

B. CRBs FOR NARROWBAND COMPLEX CIRCULAR SIGNALS IN THE UNDERDETERMINED CASE

The results outlined above are only applicable to the overdetermined case, where the number of physical sensors is larger than that of the sources. In the past decade, a family of sparse linear arrays (SLAs) with closed-form sensor positions have attracted renewed research interest [28]–[36]. Assume that the sources are known *a priori* to be spatially uncorrelated, many effective techniques, such as the spatial smoothing based method [28], the compressive sensing based method [37], and the ML method [38], can be applied to resolve more sources than sensors (the underdetermined case) with the assistance of SLAs.

A decade earlier than the flourish of underdetermined DOA estimation methods, the CRB employing the *a priori* knowledge of uncorrelated sources was derived in [39], but only limited insights were gained in the underdetermined case. After one and a half decades, this CRB was studied again in [40]–[44], with the role of the virtual difference co-array highlighted. The condition under which this CRB exists was examined in [40], [41], which leads to an upper bound on the number of resolvable Gaussian sources by a specific SLA. Note that these CRB expressions are derived for the stochastic model only, because the deterministic CRB does not exist in the underdetermined case [40].

The achievability of the underdetermined stochastic CRB has not been fully investigated at present, but there exist some insightful results. In [42], a closed-form asymptotic mean square error (MSE) for the co-array based Mutiple Signal Classification (MUSIC) algorithm was derived. The asymptotic (w.r.t. SNR) analysis therein showed that neither the direct augmentation [45] based MUSIC nor the spatial smoothing [28] based MUSIC are efficient when the number of sources is larger than one and less than the number of sensors.

C. CRBs FOR NARROWBAND COMPLEX NONCIRCULAR SIGNALS

In applications such as digital communications, signals generated by modulation schemes, such as binary phase shift keying (BPSK) and quaternary phase-shift keying (QPSK), are no longer circularly symmetric. The DOA estimation techniques for noncircular signals have been extensively studied [46]–[51], with the CRBs derived. The closed-form stochastic CRB expression for complex noncircular Gaussian signals in the presence of UN was derived in [52] by two approaches. The direct one starts from the noncircular Slepian-Bangs formula, whereas the indirect one is based on the asymptotic covariance matrix of the ML estimation errors. The authors further extended there results to the case of NUN and AUN [53]. It was demonstrated that the noncircular Gaussian CRB is upper bounded by the circular Gaussian one. Specifically, for discrete distributed BPSK and QPSK modulated signals, the corresponding stochastic CRBs were derived in [54], which indicates that the stochastic CRBs under the noncircular and circular complex Gaussian distributions are tight upper bounds on those under the discrete BPSK and QPSK distributions, respectively, at very low and very high SNRs only.

On the other hand, the deterministic noncircular Gaussian CRB is simply shown to be identical with the circular Gaussian one [53]. If the signals are known to have a strictly noncircular structure, the rotation phase angles will be considered as unknown parameters instead of imaginary parts of the signal waveforms. This feature leads to some specialized closed-form deterministic CRB expressions, see, e.g., [55] for a mixture of circular and strictly noncircular signals, and [55], [56] for strictly noncircular signals only. It was proved that the strictly noncircular deterministic CRB degenerates to the circular one in some special cases [56].

D. CRBs FOR WIDEBAND SIGNALS BASED ON FREQUENCY DECOMPOSITION

Different from the narrowband scenario, for wideband signals, the phase difference between sensor pairs depends on not only the DOAs but also the signal frequencies. Mathematically, the array sampling process for wideband signals involves matrix convolution instead of direct multiplication [57]. In an effort to deal with this problem, the observation interval can be divided into nonoverlapping subintervals and then transformed into the frequency domain via the discrete Fourier transform (DFT) or a filter bank [58]. The processing bandwidth is therefore decomposed into a set of frequency bins that resemble narrowband settings, based on which signal subspace methods [59]–[65], ML methods [66]–[68], and compressive sensing (CS) based methods [69], [70] can be implemented to produce high-resolution DOA estimates.

If the wideband signals are Gaussian random processes or the observation duration is sufficiently long, the Fourier coefficients will be (asymptotically in the latter case) Gaussian distributed [71, p. 94]. Accordingly, both the deterministic and stochastic models apply to the Fourier coefficients of the source signals, and the concept of UN, NUN, and AUN applies to the noise Fourier coefficients. Thus, the narrowband Gaussian CRB can be extended to the wideband scenario. If the duration of each subinterval is much longer than the correlation time, the Fourier coefficients will be asymptotically uncorrelated across frequency. Consequently, the wideband FIM is a superposition of those at all frequency bins, and the wideband CRB can be evaluated numerically from the inverse of the wideband FIM [60], [72], [73]. Note that other wideband models which are not established via frequency decomposition lead to different wideband CRBs, such as [74], [75].

For the wideband deterministic model, the closed-form CRB expressions in the presence of UN and NUN were derived in [68], [76] and [5], respectively, together with the corresponding ML estimators that asymptotically (w.r.t. SNR) approach these CRBs proposed. For the wideband stochastic model, a direct examination of the multi-source CRB is more challenging, and early analytical expressions were either obtained approximately [66], [77] or written in an intermediate form [78]. In particular, the stochastic ML estimator employing the spectra smoothness condition is asymptotically (w.r.t. snapshots) efficient [66]. Afterwards, the first closed-form expression for the wideband stochastic CRB was presented in [79], but detailed proof was unavailable in the published paper. Note that these CRB results are only valid in the overdetermined case, which implies that when no *a priori* knowledge on the source spectra is available, the wideband model based on frequency decomposition shares the same resolution capacity with the narrowband one [78], [79].

In the past few years, narrowband underdetermined DOA estimation techniques have been extended to the wideband scenario, see, e.g., [65], [69], [70], [80]. Similar to their narrowband counterparts, most wideband underdetermined methods also employ the *a priori* knowledge of uncorrelated sources. A few years later, the closed-form expression for the wideband stochastic CRB accounting for this *a priori* knowledge was derived in [81], which shows that the narrowband limitation on the number of resolvable Gaussian sources can be exceeded. Consequently, the assistance of special array structures is no longer necessary for wideband underdetermined DOA estimation, where as a nonuniform linear array (NULA) is indispensable to the narrowband scenario [40], [41].

E. CRBs FOR OTHER EXTENDED PARAMETRIC MODELS

Based on the aforementioned results, a myriad of closed-form CRB expressions have been proposed under extended parametric models that involve, e.g., modeling errors [82]–[86], a time-varying source [87], multiple noncoherent subarrays [88]–[90], and coherent signals with mutual coupling [91]. Taking into account the *a priori* knowledge of spatially uncorrelated sources, the underdetermined CRB has been generalized to many cases, e.g., two sources with coprime frequencies [92], the compressed sparse array scheme [32], sensor location errors [93], noncircular signals [94], and modeling errors [95].

In applications such as 3-D source localization problems, the location of a single source is parameterized by more than one unknowns, whereas the other ones associated with the source signals, the source covariance matrix, and the noise covariance matrix, may be known. The corresponding CRBs provide insightful guidance on array design in a particular scenario, see, e.g., [68], [96]–[100] and the references therein.

In applications such as sonar, radar, and communication systems, the noise distribution may be far from Gaussian. If the p.d.f. of the nonGaussian noise can be specified, the CRB will also be tractable. A number of related DOA estimation techniques and closed-form CRB expressions can be found in [101]–[105].

F. MOTIVATION AND CONTRIBUTION

There exist some relevant works that offer a comprehensive overview of typical performance bounds (including the CRB) and the asymptotic distributions of DOA estimates produced by many celebrated algorithms, see, e.g., [106], [107], but they did not elaborate in detail how a valuable analytical CRB expression can be reached. As emphasized in [23], a detailed and direct derivation of a closed-form CRB expression is important and requires painstaking efforts. Different approaches have been used by the literature to derive a closed-form CRB, which may start from the FIM, the Slepian-Bangs formula, the asymptotic covariance matrix of estimation errors of a specific estimator, and so forth. It would be time-consuming for a novice researcher seeking for and trying to understand those intricate derivations that appear in scattered publications, some of which could even be oversimplified. Moreover, sometimes those well-known CRB expressions turn out to be inapplicable to a particular problem for which a novel algorithm or array structure is designed. This difficulty was encountered when dealing with underdetermined problems for narrowband circular/noncircular and wideband signals. In such cases, comprehending the existing derivations will help derive the correct CRB in need and also benefit future studies.

This paper is devoted to illustrating a direct and concise way to derive the rich closed-form CRB expressions in the literature by reviewing a number of typical results in different scenarios under a unified framework. This will build the bridge between the general CRB formula and its many specific presentations. Furthermore, original supplementary materials, especially in the noncircular scenario with uncorrelated sources and the wideband scenario, are provided to shed light on some important points that have not been investigated in the past. In addition, the recent research developments in underdetermined DOA estimation, which were not covered by earlier reviewing works, are also visited in this paper.

First, we illustrate the probability distribution model of the narrowband data samples under the deterministic/stochastic model. Based on the p.d.f. of the complex circular Gaussian distribution, the Slepian-Bangs formula is presented. Following the shortest derivation in the literature, we then show how this general formula evolves into a unified framework for the DOA-related block of the deterministic/stochastic CRB. This framework indicates that the explicit deterministic CRB expression is distinguished by the noise covariance matrix, whereas the stochastic one depends on the derivatives of the source and noise covariance matrices w.r.t. all nuisance parameters.

By specifying the noise covariance matrix under three different models (UN, NUN, and AUN), one class of closed-form deterministic CRB expressions and two classes of stochastic ones (one without *a priori* knowledge, and the other employs the *a priori* knowledge of uncorrelated sources) are derived based on the developed framework. Then, comparisons are conducted among these CRBs and the asymptotic covariance matrix of estimation errors of the deterministic/stochastic ML estimator, leading to a series of equalities and order relationships.

The results for narrowband circular Gaussian signals are further extended to two scenarios, i.e., complex noncircular Gaussian signals and wideband signals. In each scenario, we elaborate the signal model and the extended Slepian-Bangs formula, based on which the corresponding closed-form deterministic/stochastic CRB framework is developed. Then, a class of extended closed-form deterministic CRB expressions and two stochastic ones are presented. The noncircular stochastic CRB with uncorrelated sources are derived in this paper, and its difference with the recently reported result in [94] is explained. From a general perspective, we demonstrate how the noncircular deterministic/stochastic CRB degenerates to the circular ones in a special case. Furthermore, we show that the wideband deterministic/stochastic CRB for DOAs can be interpreted as a combination of the CRBs for DOAs at all frequencies.

The deterministic CRBs and the stochastic ones without *a priori* knowledge exist only in the overdetermined case, regardless of the array geometry. However, those stochastic CRBs employing the *a priori* knowledge of uncorrelated sources can exist in the underdetermined case. Since the validity of the Gaussian CRB is connected with identifiability of the unknown parameters, some further results based on linear arrays are presented. The co-array concept is first reviewed, and its connection with the circular/noncircular/wideband stochastic CRB employing uncorrelated sources is discussed. We extend the rank condition, a condition under which the circular CRB exists, to the noncircular/wideband scenario, and examine the number of resolvable Gaussian noncircular/wideband sources by a given linear array. We show that the information contained in either the conjugate part of noncircular signals or the multiple frequency components of wideband signals can significantly increase the number of resolvable Gaussian sources, compared to the circular scenario.

G. ORGANIZATION

The rest of this paper is organized as follows. In Section II, the narrowband deterministic/stochastic model and the general CRB formula are first introduced, and then the unified framework for the DOA-related block of the deterministic/stochastic CRB is developed. Based on this framework, Section III presents a class of closed-form deterministic CRB expressions and two classes of stochastic ones in the presence of UN, NUN, and AUN, and then conduct comparisons among these results. Similarly, the extensions to noncircular complex Gaussian signals and wideband signals based on frequency decomposition are provided in Section IV and Section V, respectively. In Section VI, the existence of all the CRBs is examined, with the corresponding upper bounds on the number of resolvable Gaussian sources discussed based on linear arrays.

II. PRELIMINARIES ON THE CRAMÉR-RAO BOUND

A. NARROWBAND SIGNAL MODEL

Consider an array consisting of M omnidirectional sensors with identical responses receiving narrowband signals from K far-field sources. Assume that all the sources are located at distinct directions, and there is only one angular parameter to be estimated for each source. Then, the unknown DOAs are denoted by $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_K]^T$, where $(\cdot)^T$ is the transpose operation. After sampling, the array output signals can be modeled as

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t), \tag{1}$$

where $t = 1, 2, \ldots, N$ is the snapshot index. $\boldsymbol{x}(t)$, $\boldsymbol{s}(t)$, and $\boldsymbol{n}(t)$ collect the samples of the sensor output signals, the source signals, and additive noise, respectively:

$$\begin{aligned}\boldsymbol{x}(t) &= [x_1(t), x_2(t), \ldots, x_M(t)]^T \in \mathbb{C}^{M\times 1},\\ \boldsymbol{s}(t) &= [s_1(t), s_2(t), \ldots, s_K(t)]^T \in \mathbb{C}^{K\times 1},\\ \boldsymbol{n}(t) &= [n_1(t), n_2(t), \ldots, n_M(t)]^T \in \mathbb{C}^{M\times 1},\end{aligned}$$

where $\mathbb{C}^{M\times 1}$ denotes the space of M-by-1 complex-valued vectors. $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \ldots, \boldsymbol{a}(\theta_K)] \in \mathbb{C}^{M\times K}$ is the array manifold matrix with $\boldsymbol{a}(\theta_k)$ denoting the steering vector associated with the k-th source. The explicit form of $\boldsymbol{a}(\theta_k)$ will not be specified here, so that the following discussions are applicable to different array geometries.

The noise is assumed to be a zero-mean circular Gaussian process, both temporally and spatially uncorrelated with the source signals. Under the deterministic model, the source signals are assumed to be deterministic but unknown, leading to [20]

$$\boldsymbol{x}(t) \sim \mathcal{CN}[\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t), \boldsymbol{Q}], \tag{2}$$

where $\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Gamma})$ stands for the multidimensional complex Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Gamma}$. The noise covariance matrix is defined as

$$\boldsymbol{Q} \triangleq E[\boldsymbol{n}(t)\boldsymbol{n}^H(t)],$$

where $E[\cdot]$ is the expectation operator, and $(\cdot)^H$ represents the conjugate transpose operation.

On the other hand, under the stochastic model with zero-mean and wide-sense stationary sources, the output signal at each snapshot is an observation of a zero-mean complex Gaussian process [20], and thus

$$\boldsymbol{x}(t) \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{R}), \tag{3}$$

where

$$\begin{aligned}\boldsymbol{R} &\triangleq E[\boldsymbol{x}(t)\boldsymbol{x}^H(t)] = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^H(\boldsymbol{\theta}) + \boldsymbol{Q},\\ \boldsymbol{P} &\triangleq E[\boldsymbol{s}(t)\boldsymbol{s}^H(t)].\end{aligned} \tag{4}$$

Note that $\boldsymbol{R}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$ are all assumed to be Hermitian positive definite.

B. CRB FORMULA

Denote $\bar{\boldsymbol{x}} = [\boldsymbol{x}^T(1), \boldsymbol{x}^T(2), \ldots, \boldsymbol{x}^T(N)]^T$ as the overall data vector containing N independent and identically distributed (i.i.d.) snapshots, and thus $\bar{\boldsymbol{x}}$ follows an MN-variate complex Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Gamma}$, both of which are determined by a real-valued vector $\boldsymbol{\alpha}$ containing all unknown parameters. Let $f(\bar{\boldsymbol{x}}; \boldsymbol{\alpha})$ represent the p.d.f. of $\bar{\boldsymbol{x}}$ which depends on $\boldsymbol{\alpha}$. Under certain regularity conditions (a precise summary of all required regularity conditions can be found in [108]), the FIM is defined as [109]

$$\boldsymbol{\mathcal{F}} \triangleq -E\left[\frac{\partial^2 \ln f(\bar{\boldsymbol{x}}; \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\, \partial \boldsymbol{\alpha}^T}\right], \tag{5}$$

where $\partial f(\boldsymbol{\alpha})/\partial \boldsymbol{\alpha}$ denotes the partial derivative of $f(\boldsymbol{\alpha})$ w.r.t. the variable vector $\boldsymbol{\alpha}$. If $\boldsymbol{\mathcal{F}}$ is positive definite, then the CRB for $\boldsymbol{\alpha}$, denoted by $\boldsymbol{B}(\boldsymbol{\alpha})$, is given by

$$\boldsymbol{B}(\boldsymbol{\alpha}) = \boldsymbol{\mathcal{F}}^{-1}. \tag{6}$$

Since $\boldsymbol{\mathcal{F}}$ is nonnegative definite by definition, (6) is valid if and only if $\boldsymbol{\mathcal{F}}$ is nonsingular.

In most cases, $\bar{\boldsymbol{x}}$ is modeled to be circularly symmetric Gaussian distributed with a p.d.f. given by [110]

$$f(\bar{\boldsymbol{x}}; \boldsymbol{\alpha}) = \frac{1}{\pi^{MN}\det(\boldsymbol{\Gamma})} \mathrm{e}^{-[\bar{\boldsymbol{x}}-\boldsymbol{\mu}]^H \boldsymbol{\Gamma}^{-1}[\bar{\boldsymbol{x}}-\boldsymbol{\mu}]}, \tag{7}$$

where $\det(\cdot)$ denotes the determinant of a matrix. It follows from (5), (6), and (7) that the (i,j)-th element of the CRB matrix is given by [1, p. 363], [111, p. 927]

$$\langle \boldsymbol{B}^{-1}(\boldsymbol{\alpha})\rangle_{i,j} = \operatorname{tr}\left(\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \langle\boldsymbol{\alpha}\rangle_i}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \langle\boldsymbol{\alpha}\rangle_j}\right) + 2\operatorname{Re}\left(\frac{\partial \boldsymbol{\mu}^H}{\partial \langle\boldsymbol{\alpha}\rangle_i}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\mu}}{\partial \langle\boldsymbol{\alpha}\rangle_j}\right), \tag{8}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a square matrix, and $\operatorname{Re}(\cdot)$ is the real part of the input argument.

The general CRB formula in (6) is applicable to different probability distributions, whereas that in (8), which is also known as the Slepian-Bangs formula [112], [113], is a specialized version for the Gaussian distribution. The Gaussian CRB can be numerically evaluated from both formulas, but a closed-form CRB expression for DOA parameters alone, denoted by $\boldsymbol{B}(\boldsymbol{\theta})$, is more desirable, as stated in Section I. In what follows, we first examine the geometrical interpretation of the DOA-related block of the deterministic/stochastic CRB, and then develop a unified framework encompassing most closed-form deterministic/stochastic CRB expressions. Throughout the remainder of this paper, $\boldsymbol{B}(\boldsymbol{\theta})$, $\boldsymbol{A}(\boldsymbol{\theta})$ and $\boldsymbol{a}(\theta_k)$ will be written briefly as $\boldsymbol{B}$, $\boldsymbol{A}$, and $\boldsymbol{a}_k$, respectively.

C. UNIFIED DETERMINISTIC CRB FRAMEWORK

For the deterministic model in (2), we have

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\mathrm{det}} = \boldsymbol{\Delta}_{\mathrm{det}}\bar{\boldsymbol{s}}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\mathrm{det}} = \boldsymbol{I}_N \otimes \boldsymbol{Q}, \tag{9}$$

where

$$\boldsymbol{\Delta}_{\mathrm{det}} = \boldsymbol{I}_N \otimes \boldsymbol{A}, \quad \bar{\boldsymbol{s}} = [\boldsymbol{s}^T(1), \boldsymbol{s}^T(2), \ldots, \boldsymbol{s}^T(N)]^T,$$

the symbol $\otimes$ stands for the Kronecker product, and $\boldsymbol{I}_N$ is an N-by-N identity matrix. The unknown parameter vector is expressed as

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathrm{det}} = \left[\boldsymbol{\theta}^T, \operatorname{Re}(\bar{\boldsymbol{s}}^T), \operatorname{Im}(\bar{\boldsymbol{s}}^T), \boldsymbol{\sigma}^T\right]^T, \tag{10}$$

where $\operatorname{Im}(\cdot)$ is the imaginary part of the input argument, and $\boldsymbol{\sigma}$ consists of all real-valued unknown parameters that determine $\boldsymbol{Q}$.

Substituting (9) and (10) into (8), we can write the deterministic CRB in a partitioned form:

$$\boldsymbol{B}(\boldsymbol{\alpha}_{\mathrm{det}}) = \left(\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\mathcal{F}}_{\sigma\sigma}\end{bmatrix} + \begin{bmatrix} \bar{\boldsymbol{\mathcal{F}}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)^{-1}, \tag{11}$$

where $(\cdot)^{-1}$ is the inverse operation. $\bar{\boldsymbol{\mathcal{F}}}$ is the submatrix associated with the DOAs and the covariance matrix of the source signals. It can be partitioned as

$$\bar{\boldsymbol{\mathcal{F}}} = \begin{bmatrix} \boldsymbol{\mathcal{F}}_{\theta\theta} & \boldsymbol{\mathcal{F}}_{\theta\operatorname{Re}(\bar{s})} & \boldsymbol{\mathcal{F}}_{\theta\operatorname{Im}(\bar{s})} \\ \boldsymbol{\mathcal{F}}_{\operatorname{Re}(\bar{s})\theta} & \boldsymbol{\mathcal{F}}_{\operatorname{Re}(\bar{s})\operatorname{Re}(\bar{s})} & \boldsymbol{\mathcal{F}}_{\operatorname{Re}(\bar{s})\operatorname{Im}(\bar{s})} \\ \boldsymbol{\mathcal{F}}_{\operatorname{Im}(\bar{s})\theta} & \boldsymbol{\mathcal{F}}_{\operatorname{Im}(\bar{s})\operatorname{Re}(\bar{s})} & \boldsymbol{\mathcal{F}}_{\operatorname{Im}(\bar{s})\operatorname{Im}(\bar{s})}\end{bmatrix},$$

where $\boldsymbol{\mathcal{F}}_{\theta\theta}$ refers to the DOA-related block, while the other ones are associated with the unknown parameters specified by their subscripts. Note that $\boldsymbol{B}(\boldsymbol{\theta})$ can be extracted from $\bar{\boldsymbol{\mathcal{F}}}^{-1}$, which is the inverse of the second term on the right hand side of (8).

Computing the derivatives of $\boldsymbol{\mu}_{\mathrm{det}}$ w.r.t. $\boldsymbol{\alpha}_{\mathrm{det}}^T$ yields

$$\begin{aligned} \frac{\partial \boldsymbol{\mu}_{\mathrm{det}}}{\partial \boldsymbol{\alpha}_{\mathrm{det}}^T} &= [\boldsymbol{G}_{\mathrm{det}}, \boldsymbol{\Delta}_{\mathrm{det}}, \mathrm{j}\boldsymbol{\Delta}_{\mathrm{det}}, \boldsymbol{0}], \\ \boldsymbol{G}_{\mathrm{det}} &= [\boldsymbol{s}(1), \boldsymbol{s}(2), \ldots, \boldsymbol{s}(N)]^T \odot \boldsymbol{A}', \\ \boldsymbol{A}' &= \left[\boldsymbol{a}_1', \boldsymbol{a}_2', \ldots, \boldsymbol{a}_K'\right], \quad \boldsymbol{a}_k' = \frac{\partial \boldsymbol{a}_k}{\partial \theta_k}, \end{aligned} \tag{12}$$

where $\odot$ represents the Khatri-Rao product. Thus, we can rewrite $\bar{\boldsymbol{\mathcal{F}}}$ as

$$\bar{\boldsymbol{\mathcal{F}}} = 2\operatorname{Re}\left\{\begin{bmatrix} \bar{\boldsymbol{G}}_{\mathrm{det}}^H \\ \bar{\boldsymbol{\Delta}}_{\mathrm{det}}^H \\ -\mathrm{j}\bar{\boldsymbol{\Delta}}_{\mathrm{det}}^H \end{bmatrix}[\bar{\boldsymbol{G}}_{\mathrm{det}}, \bar{\boldsymbol{\Delta}}_{\mathrm{det}}, \mathrm{j}\bar{\boldsymbol{\Delta}}_{\mathrm{det}}]\right\},$$

with

$$\begin{aligned} \bar{\boldsymbol{G}}_{\mathrm{det}} &= [\boldsymbol{s}(1), \boldsymbol{s}(2), \ldots, \boldsymbol{s}(N)]^T \odot \bar{\boldsymbol{A}}', \\ \bar{\boldsymbol{\Delta}}_{\mathrm{det}} &= \boldsymbol{I}_N \otimes \bar{\boldsymbol{A}}, \quad \bar{\boldsymbol{A}} = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}, \quad \bar{\boldsymbol{A}}' = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}'. \end{aligned} \tag{13}$$

Furthermore, $\bar{\boldsymbol{\mathcal{F}}}$ can be rewritten in a block-diagonal form using the matrix manipulation introduced in [1, p. 370], [15]. As a result, the DOAs are decoupled with nuisance parameters, and $\bar{\boldsymbol{\mathcal{F}}}^{-1}$ can be calculated neatly. Following this approach, we can obtain the *geometrical interpretation* of the DOA-related block of the deterministic CRB:

$$\boldsymbol{B}_{\mathrm{det}} = \frac{1}{2}\operatorname{Re}\left(\bar{\boldsymbol{G}}_{\mathrm{det}}^H \boldsymbol{\Pi}_{\bar{\boldsymbol{\Delta}}_{\mathrm{det}}}^{\perp}\bar{\boldsymbol{G}}_{\mathrm{det}}\right)^{-1}, \tag{14}$$

where $\boldsymbol{\Pi}_{\bar{\boldsymbol{\Delta}}_{\mathrm{det}}}^{\perp} = \boldsymbol{I} - \bar{\boldsymbol{\Delta}}_{\mathrm{det}}(\bar{\boldsymbol{\Delta}}_{\mathrm{det}}^H\bar{\boldsymbol{\Delta}}_{\mathrm{det}})^{-1}\bar{\boldsymbol{\Delta}}_{\mathrm{det}}^H$ stands for the orthogonal projector onto the null space of $\bar{\boldsymbol{\Delta}}_{\mathrm{det}}^H$. Throughout the rest of this paper, other orthogonal projectors will be defined similarly, distinguished by their subscripts.

On one hand, $\bar{\boldsymbol{\mathcal{F}}}$ is nonsingular only if $\bar{\boldsymbol{\Delta}}_{\mathrm{det}}$ has full column rank. On the other hand, the validity of (14) requires $\bar{\boldsymbol{\Delta}}_{\mathrm{det}}^H\bar{\boldsymbol{\Delta}}_{\mathrm{det}}$ and $\bar{\boldsymbol{G}}_{\mathrm{det}}^H\boldsymbol{\Pi}_{\bar{\boldsymbol{\Delta}}_{\mathrm{det}}}^{\perp}\bar{\boldsymbol{G}}_{\mathrm{det}}$ to be positive definite, which needs $\boldsymbol{\Pi}_{\bar{\boldsymbol{\Delta}}_{\mathrm{det}}}^{\perp} \neq \boldsymbol{0}$. From the definition of $\bar{\boldsymbol{G}}_{\mathrm{det}}$ and $\bar{\boldsymbol{\Delta}}_{\mathrm{det}}$ in (13), these conditions lead to $K<M$. Suppose that $K<M$, and then the following result can be obtained by block-wise matrix computation.

$$\bar{\boldsymbol{G}}_{\mathrm{det}}^H\boldsymbol{\Pi}_{\bar{\boldsymbol{\Delta}}_{\mathrm{det}}}^{\perp}\bar{\boldsymbol{G}}_{\mathrm{det}} = N(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\bar{\boldsymbol{A}}') \circ \hat{\boldsymbol{P}}^T, \tag{15}$$

where

$$\hat{\boldsymbol{P}} = \frac{1}{N}\sum_{t=1}^{N}\boldsymbol{s}(t)\boldsymbol{s}^H(t),$$

and $\circ$ stands for the Hadamard product. With (15), (14) can be transformed into a unified framework for the DOA-related block of the deterministic CRB:

$$\boldsymbol{B}_{\mathrm{det}} = \frac{1}{2N}\left\{\operatorname{Re}\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\bar{\boldsymbol{A}}') \circ \hat{\boldsymbol{P}}^T\right]\right\}^{-1}. \tag{16}$$

D. UNIFIED STOCHASTIC CRB FRAMEWORK

For the stochastic model in (3), we have

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\rm sto} = \mathbf{0}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\rm sto} = \boldsymbol{I}_N \otimes \boldsymbol{R}. \tag{17}$$

The unknown parameter vector is expressed as

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\rm sto} = \left[\boldsymbol{\theta}^T, \boldsymbol{p}^T, \boldsymbol{\sigma}^T\right]^T, \tag{18}$$

with $\boldsymbol{p}$ holding the real-valued unknown parameters related to the real and imaginary parts of all unknown entries in $\boldsymbol{P}$.

Substituting (17), (18) into (8) and utilizing the following identities [111]:

$$\begin{aligned} {\rm tr}(\mathcal{ABCD}) &= {\rm vec}(\mathcal{B}^H)^H(\mathcal{A}^T \otimes \mathcal{C}){\rm vec}(\mathcal{D}),\\ {\rm vec}(\mathcal{ABC}) &= (\mathcal{C}^T \otimes \mathcal{A}){\rm vec}(\mathcal{B}),\\ (\mathcal{A}\otimes\mathcal{B})^{-1} &= \mathcal{A}^{-1}\otimes\mathcal{B}^{-1}, \end{aligned} \tag{19}$$

we can obtain

$$\boldsymbol{B}^{-1}(\boldsymbol{\alpha}_{\rm sto}) = N\left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\alpha}_{\rm sto}^T}\right)^H (\boldsymbol{R}^{-T}\otimes\boldsymbol{R}^{-1})\left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\alpha}_{\rm sto}^T}\right), \tag{20}$$

where $\boldsymbol{r} = {\rm vec}(\boldsymbol{R})$ represents the vectorization of $(\boldsymbol{R})$.

Introducing the following notations:

$$\begin{aligned} \boldsymbol{G}_{\rm sto} &= \boldsymbol{W}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\theta}^T}, \quad \boldsymbol{\Delta}_{\rm sto} = [\boldsymbol{V}, \boldsymbol{U}], \quad \boldsymbol{V} = \boldsymbol{W}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{p}^T},\\ \boldsymbol{U} &= \boldsymbol{W}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\sigma}^T}, \quad \boldsymbol{W} = \boldsymbol{R}^{-\frac{T}{2}}\otimes\boldsymbol{R}^{-\frac{1}{2}}, \end{aligned} \tag{21}$$

we can rewrite (20) in a partitioned form:

$$\boldsymbol{B}^{-1}(\boldsymbol{\alpha}_{\rm sto}) = N\begin{bmatrix}\boldsymbol{G}_{\rm sto}^H\\ \boldsymbol{\Delta}_{\rm sto}^H\end{bmatrix}[\boldsymbol{G}_{\rm sto}, \boldsymbol{\Delta}_{\rm sto}]. \tag{22}$$

Using the standard result on the inverse of a partitioned matrix yields the *geometrical interpretation* of the DOA-related block of the stochastic CRB [23]:

$$\boldsymbol{B}_{\rm sto} = \frac{1}{N}\left(\boldsymbol{G}_{\rm sto}^H\boldsymbol{\Pi}_{\boldsymbol{\Delta}_{\rm sto}}^\perp\boldsymbol{G}_{\rm sto}\right)^{-1}. \tag{23}$$

According to (22), a necessary condition for the FIM to be nonsingular is that $[\boldsymbol{G}_{\rm sto}, \boldsymbol{\Delta}_{\rm sto}]$ should have full column rank, which is true only if $\boldsymbol{V}$ has full column rank. Meanwhile, (23) is valid if and only if $\boldsymbol{\Delta}_{\rm sto}^H\boldsymbol{\Delta}_{\rm sto}$ and $\boldsymbol{G}_{\rm sto}^H\boldsymbol{\Pi}_{\boldsymbol{\Delta}_{\rm sto}}^\perp\boldsymbol{G}_{\rm sto}$ are positive definite.

To derive a more explicit CRB framework, we assume that $\boldsymbol{V}$ have fewer columns than rows, so that $\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp \neq \mathbf{0}$. Since $\boldsymbol{\Delta}_{\rm sto}$ shares the same range space with $[\boldsymbol{V}, \ \boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{U}]$, it follows from the projection decomposition theorem that [23], [39]

$$\begin{aligned} \boldsymbol{\Pi}_{\boldsymbol{\Delta}_{\rm sto}}^\perp &= \boldsymbol{\Pi}_{\boldsymbol{V}}^\perp - \boldsymbol{\Pi}_{\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{U}}\\ &= \boldsymbol{\Pi}_{\boldsymbol{V}}^\perp - \boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{U}\left[\boldsymbol{U}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{U}\right]^{-1}\boldsymbol{U}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp. \end{aligned} \tag{24}$$

Substituting (24) into (23) gives a unified framework for the DOA-related block of the stochastic CRB:

$$\boldsymbol{B}_{\rm sto} = \frac{1}{N}\left(\boldsymbol{C} - \boldsymbol{D}\boldsymbol{F}^{-1}\boldsymbol{D}^H\right)^{-1}, \tag{25}$$

where

$$\boldsymbol{C} = \boldsymbol{G}_{\rm sto}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{G}_{\rm sto}, \quad \boldsymbol{D} = \boldsymbol{G}_{\rm sto}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{U}, \ \boldsymbol{F} = \boldsymbol{U}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^\perp\boldsymbol{U}. \tag{26}$$

Remark 1: Comparing (14) with (23), we find that the *geometrical interpretations* of the deterministic and stochastic CRBs share a similar form. $\bar{\boldsymbol{G}}_{\rm det}$ and $\boldsymbol{G}_{\rm sto}$ are associated with the DOAs, whereas $\bar{\boldsymbol{\Delta}}_{\rm det}$ and $\boldsymbol{\Delta}_{\rm sto}$ correspond to nuisance parameters. The difference is that the deterministic CRB depends on $\boldsymbol{Q}^{-1/2}$ instead of $\partial\boldsymbol{Q}/\partial\boldsymbol{\sigma}^T$, whereas the stochastic one is not only determined by $\partial\boldsymbol{r}/\partial\boldsymbol{\sigma}^T$ but also $\partial\boldsymbol{r}/\partial\boldsymbol{p}^T$. Consequently, more variants of the stochastic CRB will be produced with different choices of $\boldsymbol{p}$ and $\boldsymbol{\sigma}$, as will be illustrated in Section III.

All the expressions derived in this section are based on the mean and covariance of the overall data vector $\bar{\boldsymbol{x}}$ containing N i.i.d. snapshots. The log-likelihood function of $\bar{\boldsymbol{x}}$ can be written as a multiplication of N log-likelihood functions for each snapshot: $\ln f(\bar{\boldsymbol{x}}; \boldsymbol{\alpha}) = \prod_{t=1}^N \ln f[\boldsymbol{x}(t); \boldsymbol{\alpha}]$. It follows from (5) that $\boldsymbol{\mathcal{F}} = \sum_{t=1}^N \boldsymbol{\mathcal{F}}_t$ with $\boldsymbol{\mathcal{F}}_t$ denoting the FIM for the t-th snapshot. Since the N snapshots are i.i.d., $\{\boldsymbol{\mathcal{F}}(t)\}_{t=1}^N$ are identical. As a result, the multi-snapshot CRB can be alternatively obtained by multiplying $1/N$ to the single-snapshot CRB, which can be derived in the same way based on the mean and covariance matrix given in either (2) or (3). In other words, multiplying N to (16) and (23), respectively, yields the deterministic and stochastic CRB geometrical interpretations for the single-snapshot case [111, p. 932].

III. CLOSED-FORM NARROWBAND CRB EXPRESSIONS IN DIFFERENT CASES

In this section, we review a number of typical closed-form CRB expressions for DOAs by specifying the nuisance parameters under three noise models, namely, UN, NUN, and AUN. We will show how the results in (16) and (25) evolve into explicit closed-form expressions in different cases.

We start from the deterministic CRB, whose explicit expression is distinguished by $\boldsymbol{Q}^{-1/2}$. Since the stochastic CRB depends on $\boldsymbol{p}$ and $\boldsymbol{\sigma}$, in addition to the three noise models, two classes of closed-form stochastic CRB expressions with different $\boldsymbol{p}$ are also provided.

A. DETERMINISTIC CRB WITH UNIFORM NOISE

In the presence of UN, we have

$$\boldsymbol{Q} = \sigma\boldsymbol{I}_M, \quad \boldsymbol{\sigma} = \sigma, \tag{27}$$

where $\sigma \in \mathbb{R}^+$ represents the power of the noise, and $\mathbb{R}^+$ denotes the set of positive real-valued numbers. Substituting (27) into (16), we immediately obtain the following *Result* 1.

Result 1: Denote the DOA-related block of the deterministic CRB in the presence of UN as $\boldsymbol{B}_{\rm det}^{\rm un}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}_{\rm det}^{\rm un}$ is given by

$$\boldsymbol{B}_{\rm det}^{\rm un} = \frac{\sigma}{2N}\left\{{\rm Re}\left[(\boldsymbol{A}'^H\boldsymbol{\Pi}_{\boldsymbol{A}}^\perp\boldsymbol{A}')\circ\hat{\boldsymbol{P}}^T\right]\right\}^{-1}. \tag{28}$$

This result was independently derived in [11], [12]. Although both of them started from the general CRB formula in (5), the subsequent derivation was carried out in a different manner. In [11], all blocks of the FIM were explicitly derived by differentiating the log-likelihood function and taking the expectation, and then (28) was obtained by an intricate calculation of the inverse of the partitioned FIM. In [12], the log-likelihood function was written in a reduced form depending on the estimated DOAs $\hat{\boldsymbol{\theta}}$ and the noise power $\hat{\sigma}$:

$$\ln f(\bar{\boldsymbol{x}}; \hat{\boldsymbol{\theta}}, \hat{\sigma}) = -MN\ln(\pi\hat{\sigma}) - \frac{\sigma N}{\hat{\sigma}}(M-N) - \frac{N}{\hat{\sigma}}\mathrm{tr}(\boldsymbol{A}^H \boldsymbol{\Pi}_{\hat{\boldsymbol{A}}}^{\perp} \boldsymbol{A}\boldsymbol{P}), \quad (29)$$

where $\hat{\boldsymbol{A}}$ stands for the array manifold matrix associated with $\hat{\boldsymbol{\theta}}$. It was then demonstrated that

$$\boldsymbol{A}^H \hat{\boldsymbol{\Pi}}_{\boldsymbol{A}}^{\perp} \boldsymbol{A} = \hat{\boldsymbol{A}}^H \boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp} \hat{\boldsymbol{A}} + O(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^3),$$

which implies that the second-order derivatives of $\boldsymbol{A}^H \hat{\boldsymbol{\Pi}}_{\boldsymbol{A}}^{\perp} \boldsymbol{A}$ and $\hat{\boldsymbol{A}}^H \boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp} \hat{\boldsymbol{A}}$ w.r.t. $\boldsymbol{\theta}^T$ are identical for $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$. Finally, (28) was obtained with reduced matrix computation by substituting (29) into (6).

B. DETERMINISTIC CRB WITH NONUNIFORM NOISE OR ARBITRARY UNKNOWN NOISE

The covariance matrix of NUN takes the following form

$$\boldsymbol{Q} = \mathrm{diag}(\boldsymbol{\sigma}), \quad \boldsymbol{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_M]^T, \quad (30)$$

where $\mathrm{diag}(\cdot)$ refers to a diagonal matrix whose diagonal entries are listed inside the brackets, and $\{\sigma_m\}_{m=1}^M \in \mathbb{R}^+$ are the noise variances at different sensors. Since the diagonal elements of $\boldsymbol{Q}$ are different, it seems infeasible to rewrite (16) more explicitly as in (28).

Moreover, the covariance matrix of AUN is modeled to be determined by L unknown real-valued parameters [24]:

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{\sigma}), \quad \boldsymbol{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_L]^T, \quad (31)$$

where $\{\sigma_l\}_{l=1}^L \in \mathbb{R}$ with $\mathbb{R}$ denoting the set of real-valued numbers. For the same reason as in the NUN case, the closed-form deterministic CRB expression in the presence of AUN cannot be written more explicitly than that in (16).

Result 2: Denote the DOA-related block of the deterministic CRB in the presence of NUN and AUN as $\boldsymbol{B}_{\mathrm{det}}^{\mathrm{nun}}$ and $\boldsymbol{B}_{\mathrm{det}}^{\mathrm{aun}}$, respectively. If $K<M$, then the closed-form expression for $\boldsymbol{B}_{\mathrm{det}}^{\mathrm{nun}}$ or $\boldsymbol{B}_{\mathrm{det}}^{\mathrm{aun}}$ is given by (16).

Similar to the UN case, this result was originally obtained by evaluating all submatrices of the partitioned FIM in (11) and then applying the partitioned matrix formula [17]. The difference is that the derivation in [17] is based on (8) instead of (6), so that fewer matrix manipulations are involved compared to [11].

C. STOCHASTIC CRB WITH UNIFORM NOISE

For the stochastic model without *a priori* knowledge, the source covariance matrix $\boldsymbol{P}$ is determined by its K^2 upper triangular elements collected by

$$\boldsymbol{p} = [p_{11}, \mathrm{Re}(p_{12}), \mathrm{Im}(p_{12}), \ldots, \mathrm{Re}(p_{1K}), \mathrm{Im}(p_{1K}), p_{22}, \mathrm{Re}(p_{23}), \mathrm{Im}(p_{23}), \ldots, p_{KK}]^T, \quad (32)$$

Applying the vectorization operator to (4) leads to

$$\boldsymbol{r} = (\boldsymbol{A}^* \otimes \boldsymbol{A})\boldsymbol{J}_1\boldsymbol{p} + \mathrm{vec}(\boldsymbol{Q}), \quad (33)$$

where $\boldsymbol{J}_1 \in \mathbb{C}^{K^2\times K^2}$ is a nonsingular matrix satisfying $\mathrm{vec}(\boldsymbol{P}) = \boldsymbol{J}_1\boldsymbol{p}$, and $(\cdot)^*$ denotes the conjugate operation.

To derive a closed-form expression for the stochastic CRB, we need to specify $\boldsymbol{G}_{\mathrm{sto}}$, $\boldsymbol{V}$, and $\boldsymbol{U}$ in (21). From (19) and (33), the k-th ($k = 1, 2, \cdots, K$) column of $\boldsymbol{G}_{\mathrm{sto}}$ is given by

$$\boldsymbol{g}_k = \mathrm{vec}(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}'\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{P}\boldsymbol{A}^H\boldsymbol{R}^{-\frac{1}{2}} + \boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{A}'^H\boldsymbol{R}^{-\frac{1}{2}}), \quad (34)$$

where $\boldsymbol{e}_k \in \mathbb{R}^{K\times 1}$ contains one at the k-th position and zeros elsewhere. By (33), $\boldsymbol{V}$ is expressed as

$$\boldsymbol{V} = \boldsymbol{W}(\boldsymbol{A}^* \otimes \boldsymbol{A})\boldsymbol{J}_1 = \left[(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})^* \otimes (\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})\right]\boldsymbol{J}_1. \quad (35)$$

As assumed previously when deriving (25), $\boldsymbol{V}$ has fewer rows than columns, leading to $K<M$. In the presence of UN, it follows from (27) that

$$\boldsymbol{U} = \boldsymbol{W}\mathrm{vec}(\boldsymbol{I}_M) = \mathrm{vec}(\boldsymbol{R}^{-1}). \quad (36)$$

Next, we show the key derivations of $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{F}$ in (25). Since $\boldsymbol{J}_1$ is nonsingular, we first obtain the following important result.

$$\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp} = \boldsymbol{\Pi}_{(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})^*\otimes(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})}^{\perp} = \boldsymbol{I}_M \otimes \boldsymbol{\Pi}_{(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})^*}^{\perp} + \boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp} \otimes \boldsymbol{I}_M - \boldsymbol{\Pi}_{(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})^*}^{\perp} \otimes \boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp}, \quad (37)$$

where the identity below is used

$$\boldsymbol{\Pi}_{\mathcal{A}\otimes\mathcal{B}}^{\perp} = \boldsymbol{I} \otimes \boldsymbol{\Pi}_{\mathcal{A}}^{\perp} + \boldsymbol{\Pi}_{\mathcal{B}}^{\perp} \otimes \boldsymbol{I} - \boldsymbol{\Pi}_{\mathcal{A}}^{\perp} \otimes \boldsymbol{\Pi}_{\mathcal{B}}^{\perp}.$$

From (19), (34) and (37), we have

$$\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{g}_k = \mathrm{vec}(\boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp}\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}'\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{P}\boldsymbol{A}^H\boldsymbol{R}^{-\frac{1}{2}}) + \mathrm{vec}(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{A}'^H\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp}). \quad (38)$$

With (19), (34), and (38), the (k_1, k_2)-th ($k_1, k_2 = 1, 2, \cdots, K$) element of $\boldsymbol{C}$ is expressed as

$$\langle\boldsymbol{C}\rangle_{k_1,k_2} = \boldsymbol{g}_{k_1}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{g}_{k_2} = \boldsymbol{g}_{k_1}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{g}_{k_2} = 2\mathrm{Re}\Big[(\boldsymbol{e}_{k_1}^T\boldsymbol{A}'^H\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp}\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}'\boldsymbol{e}_{k_2}) \cdot(\boldsymbol{e}_{k_2}^T\boldsymbol{P}\boldsymbol{A}^H\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_{k_1})\Big]. \quad (39)$$

In this case, $\boldsymbol{D}$ becomes a vector whose k-th element is

$$\langle\boldsymbol{D}\rangle_k = \boldsymbol{g}_k^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{U} = 2\mathrm{Re}\left[\mathrm{tr}(\boldsymbol{R}^{-1}\boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp}\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{A}'^H\boldsymbol{R}^{-\frac{1}{2}})\right] = 0. \quad (40)$$

Then, introduce the following identity [23]:

$$\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}^{\perp}\boldsymbol{R}^{-\frac{1}{2}} = \frac{1}{\sigma}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}. \quad (41)$$

Combining (39), (40), (41), and (25) gives *Result* 3 as shown below.

Result 3: Denote the DOA-related block of the stochastic CRB in the presence of UN as $\boldsymbol{B}_{\text{sto}}^{\text{un}}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}_{\text{sto}}^{\text{un}}$ is given by

$$\boldsymbol{B}_{\text{sto}}^{\text{un}} = \frac{\sigma}{2N}\left\{\text{Re}\left[(\boldsymbol{A}'^H\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{A}')\circ(\boldsymbol{P}\boldsymbol{A}^H\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{P})^T\right]\right\}^{-1}. \quad (42)$$

This well-known result was initially derived indirectly through the asymptotic covariance matrix of estimation errors of the stochastic ML estimator [2], [20] or the optimal subspace fitting estimator [21]. Then, this result was derived directly in [22]. Explicit expressions of all submatrices of the FIM were derived therein, and then the partitioned matrix inversion formula was used to obtain (42). A few years later, a compact derivation of (42) was presented in [23], which incorporates the key steps to reach (23) and also the subsequent derivation in this subsection.

Remark 2: From (40), it is clear that $\boldsymbol{D} = \boldsymbol{0}$. Thus, (25) is simplified to $\boldsymbol{B}_{\text{sto}} = (N\boldsymbol{C})^{-1}$. It is easy to verify that this simplified stochastic CRB framework and (42) are applicable to the case where the noise covariance matrix is completely known, indicating that the DOA estimation accuracy will not be affected by whether the power of UN is known or not.

D. STOCHASTIC CRB WITH NONUNIFORM NOISE

In the presence of NUN, it follows from (30) that the m-th ($m = 1, 2, \ldots, M$) column of $\boldsymbol{U}$ is expressed as

$$\boldsymbol{u}_m = (\boldsymbol{R}^{-\frac{T}{2}} \otimes \boldsymbol{R}^{-\frac{1}{2}})\text{vec}(\boldsymbol{e}_m\boldsymbol{e}_m^T). \quad (43)$$

Let us consider a full column-rank matrix $\boldsymbol{\Theta} \in \mathbb{C}^{M\times(M-K)}$ whose columns span the null space of $\boldsymbol{A}^H$. This leads to $\boldsymbol{\Theta}^H\boldsymbol{A} = \boldsymbol{0}$, and hence

$$\boldsymbol{\Pi}^{\perp}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}} = \boldsymbol{\Pi}_{\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{\Theta}}, \quad (44)$$

where $\boldsymbol{\Pi}_{\boldsymbol{R}^{1/2}\boldsymbol{\Theta}}$ is the pseudo inverse of $\boldsymbol{R}^{1/2}\boldsymbol{\Theta}$. The identity below can be deduced from (44), and it plays an important role in the subsequent derivation [24]

$$\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{\Pi}^{\perp}_{\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A}}\boldsymbol{R}^{-\frac{1}{2}} = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}\boldsymbol{Q}^{-\frac{1}{2}}. \quad (45)$$

Note that (45) is a generalized version of (41).

In this case, $\boldsymbol{V}$ is the same as that with UN, so that (34), (38), and (39) are preserved. Meanwhile, with (19), (38), and (45), the (k, m)-th element of $\boldsymbol{D}$ is expressed as

$$\begin{aligned}\langle\boldsymbol{D}\rangle_{k,m} &= \boldsymbol{g}_k^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{u}_m \\ &= 2\text{Re}\left[(\boldsymbol{e}_k^T\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{e}_m)(\boldsymbol{e}_m^T\boldsymbol{Q}^{-\frac{1}{2}}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{A}}\boldsymbol{P}\boldsymbol{e}_k)\right], \quad (46)\end{aligned}$$

where

$$\bar{\boldsymbol{R}} = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{R}\boldsymbol{Q}^{-\frac{1}{2}}. \quad (47)$$

From (19), (37), (43), and (45), the (m_1, m_2)-th ($m_1, m_2 = 1, 2, \ldots, M$) element of $\boldsymbol{F}$ is given by

$$\begin{aligned}&\langle\boldsymbol{F}\rangle_{m_1,m_2} \\ &= \boldsymbol{u}_{m_1}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{u}_{m_2} \\ &= 2\text{Re}\left[(\boldsymbol{e}_{m_1}^T\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{e}_{m_2})(\boldsymbol{e}_{m_2}^T\boldsymbol{Q}^{-\frac{1}{2}}\bar{\boldsymbol{R}}^{-1}\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{e}_{m_1})\right] \\ &\quad -(\boldsymbol{e}_{m_1}^T\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{e}_{m_2})(\boldsymbol{e}_{m_2}^T\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{e}_{m_1}), \quad (48)\end{aligned}$$

Substituting (39), (46), (48) into (25) yields the following *Result* 4.

Result 4: Denote the DOA-related block of the stochastic CRB in the presence of NUN as $\boldsymbol{B}_{\text{sto}}^{\text{nun}}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}_{\text{sto}}^{\text{nun}}$ is given by (25), with

$$\boldsymbol{C} = 2\text{Re}\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\bar{\boldsymbol{A}}')\circ(\boldsymbol{P}\bar{\boldsymbol{A}}^H\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{A}}\boldsymbol{P})^T\right], \quad (49)$$

$$\boldsymbol{D} = 2\text{Re}\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})\circ(\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{A}}\boldsymbol{P})^T\right], \quad (50)$$

$$\boldsymbol{F} = 2\text{Re}(\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\circ\bar{\boldsymbol{R}}^{-T}) - \boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\circ(\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})^T. \quad (51)$$

This result was reported in [17] with a detailed but complicated derivation. The authors started from (8) and provided closed-form expressions for each submatrix in the FIM. Then, the partitioned matrix inversion formula was applied to reach the final result. Note that in [17], the matrix $\boldsymbol{F}$ is written differently as

$$\boldsymbol{F} = \bar{\boldsymbol{R}}^{-H}\circ\bar{\boldsymbol{R}}^{-1} - (\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}\bar{\boldsymbol{R}}^{-1})^H\circ(\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}\bar{\boldsymbol{R}}^{-1}). \quad (52)$$

Since $\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\bar{\boldsymbol{R}}^{-1} = \boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}$, it can be verified that the two forms of $\boldsymbol{F}$ in (51) and (52) are equivalent [24].

E. STOCHASTIC CRB WITH ARBITRARY UNKNOWN NOISE

Under the AUN model in (31), the l-th ($l = 1, 2, \ldots, L$) column of $\boldsymbol{U}$ is given by

$$\boldsymbol{u}_l = (\boldsymbol{R}^{-\frac{T}{2}} \otimes \boldsymbol{R}^{-\frac{1}{2}})\text{vec}(\boldsymbol{Q}_l'), \quad (53)$$

where

$$\boldsymbol{Q}_l' = \frac{\partial\boldsymbol{Q}}{\partial\sigma_l}. \quad (54)$$

The matrix $\boldsymbol{C}$ remains (49). Meanwhile, with (19), (38), (45), and (53), the (k, l)-th element of $\boldsymbol{D}$ is expressed as

$$\begin{aligned}&\langle\boldsymbol{D}\rangle_{k,l} \\ &= \boldsymbol{g}_k^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{u}_l \\ &= 2\text{Re}\left[\text{tr}(\boldsymbol{Q}_l'\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{A}'^H\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}})\right] \\ &= 2\text{Re}\left\{\text{vec}(\boldsymbol{e}_k\boldsymbol{e}_k^T)^T\left[(\bar{\boldsymbol{A}}^H\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})\otimes(\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{A}}\boldsymbol{P})^T\right]\text{vec}(\bar{\boldsymbol{Q}}_l')^*\right\}, \quad (55)\end{aligned}$$

where

$$\bar{\boldsymbol{Q}}_l' = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{Q}_l'\boldsymbol{Q}^{-\frac{1}{2}}. \quad (56)$$

It follows from (19), (37), (45), and (53) that the (l_1, l_2)-th ($l_1, l_2 = 1, 2, \ldots, L$) element of $\boldsymbol{F}$ is given by

$$\begin{aligned}\langle\boldsymbol{F}\rangle_{l_1,l_2} &= \boldsymbol{u}_{l_1}^H\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}\boldsymbol{u}_{l_2} \\ &= 2\text{Re}\left[\text{tr}(\bar{\boldsymbol{Q}}_{l_1}'\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}\bar{\boldsymbol{Q}}_{l_2}'\boldsymbol{R}^{-1})\right] \\ &\quad -\text{tr}(\bar{\boldsymbol{Q}}_{l_1}'\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}\bar{\boldsymbol{Q}}_{l_2}'\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\boldsymbol{Q}^{-\frac{1}{2}}) \\ &= 2\text{Re}\left[\text{vec}(\bar{\boldsymbol{Q}}_{l_1}')^H(\bar{\boldsymbol{R}}^{-T}\otimes\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})\text{vec}(\bar{\boldsymbol{Q}}_{l_2}')\right] \\ &\quad -\text{vec}(\bar{\boldsymbol{Q}}_{l_1}')^H\left[(\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})^T\otimes\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\right]\text{vec}(\bar{\boldsymbol{Q}}_{l_2}'). \quad (57)\end{aligned}$$

Combining (39), (55), (57), and (25) gives the following *Result* 5.

Result 5: Denote the DOA-related block of the stochastic CRB in the presence of AUN as $\boldsymbol{B}_{\text{sto}}^{\text{aun}}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}_{\text{sto}}^{\text{aun}}$ is given by (25), with

$$\begin{aligned}
\boldsymbol{C} &= 2\text{Re}\left[(\bar{\boldsymbol{A}}'^{H}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\bar{\boldsymbol{A}}') \circ (\boldsymbol{P}\bar{\boldsymbol{A}}^{H}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{A}}\boldsymbol{P})^{T}\right],\\
\boldsymbol{D} &= 2\text{Re}\left\{\boldsymbol{\mathcal{E}}_K^T\left[(\bar{\boldsymbol{A}}'^{H}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}) \otimes (\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{A}}\boldsymbol{P})^{T}\right]\bar{\boldsymbol{\mathcal{Q}}}^{*}\right\},\\
\boldsymbol{F} &= 2\text{Re}\left[\bar{\boldsymbol{\mathcal{Q}}}^{H}(\bar{\boldsymbol{R}}^{-T} \otimes \boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})\bar{\boldsymbol{\mathcal{Q}}}\right]\\
&\quad -\bar{\boldsymbol{\mathcal{Q}}}^{H}\left[(\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp})^{T} \otimes \boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\right]\bar{\boldsymbol{\mathcal{Q}}},\\
\boldsymbol{\mathcal{E}}_K &= \left[\text{vec}(\boldsymbol{e}_1\boldsymbol{e}_1^T), \text{vec}(\boldsymbol{e}_2\boldsymbol{e}_2^T), \ldots, \text{vec}(\boldsymbol{e}_K\boldsymbol{e}_K^T)\right],\\
\bar{\boldsymbol{\mathcal{Q}}} &= \left[\text{vec}(\bar{\boldsymbol{Q}}_1'), \text{vec}(\bar{\boldsymbol{Q}}_2'), \ldots, \text{vec}(\bar{\boldsymbol{Q}}_L')\right].
\end{aligned}$$

This result was obtained in [24] through the derivation above, and the closed-form expressions for $\boldsymbol{B}_{\text{sto}}^{\text{un}}$ and $\boldsymbol{B}_{\text{sto}}^{\text{nun}}$ were also derived therein as special cases of AUN.

As a special case of AUN, the partially unknown noise field was studied in [7]–[9] with the noise model given by

$$\boldsymbol{Q}(\boldsymbol{\sigma}) = \sum_{l=1}^{L}\sigma_l\boldsymbol{\Psi}_l, \tag{58}$$

where $\boldsymbol{\Psi}_l \in \mathbb{C}^{M\times M}$ represents a known matrix. As pointed out in [7], $\{\sigma_l\}_{l=1}^{L}$ are determined by the intensity and the spatial distribution of the noise, whereas $\{\boldsymbol{\Psi}_l\}_{l=1}^{L}$ depend on the array configuration. The corresponding CRB was derived by the inverse of the partitioned FIM in [7], but a closed-form expression was not available. Then, a CRB expression in the presence of partially unknown noise, which is similar to that in (23), was derived from the limiting Hessian matrix [9], but it can be written more explicitly by substituting $\boldsymbol{Q}_l' = \boldsymbol{\Psi}_l$ into *Result* 5.

F. STOCHASTIC CRB WITH SPATIALLY UNCORRELATED SOURCES

When $K \geq M$, the CRB expressions in *Results* 3-5 are no longer applicable. This difficulty can be overcome by employing the *a priori* knowledge of uncorrelated sources, which reduces the unknown parameters in $\boldsymbol{P}$. In this case, we have

$$\boldsymbol{P} = \text{diag}(\boldsymbol{p}), \quad \boldsymbol{p} = [p_1, p_2, \ldots, p_K]^T, \tag{59}$$

where $p_k \in \mathbb{R}^{+}$. Therefore, $\boldsymbol{R}$ can be written as

$$\boldsymbol{R} = \sum_{k=1}^{K}p_k\boldsymbol{a}_k\boldsymbol{a}_k^H + \boldsymbol{Q}. \tag{60}$$

In contrast to (33), the vectorization of $\boldsymbol{R}$ is expressed as

$$\boldsymbol{r} = \boldsymbol{T}\boldsymbol{p} + \text{vec}(\boldsymbol{Q}), \tag{61}$$

where

$$\boldsymbol{T} = \boldsymbol{A}^{*} \odot \boldsymbol{A}. \tag{62}$$

Notice that $\boldsymbol{P}\boldsymbol{e}_k = p_k\boldsymbol{e}_k$, and thus

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \theta_k} &= \text{vec}(\boldsymbol{A}'\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{P}\boldsymbol{A}^H + \boldsymbol{A}\boldsymbol{P}\boldsymbol{e}_k\boldsymbol{e}_k^T\boldsymbol{A}'^H)\\
&= \left[(\boldsymbol{A}^{*}\boldsymbol{e}_k) \otimes (\boldsymbol{A}'\boldsymbol{e}_k) + (\boldsymbol{A}'^{*}\boldsymbol{e}_k) \otimes (\boldsymbol{A}\boldsymbol{e}_k)\right]p_k.
\end{aligned} \tag{63}$$

By (63), $\boldsymbol{G}_{\text{sto}}$ becomes

$$\boldsymbol{G}_{\text{sto}} = \boldsymbol{W}\boldsymbol{T}'\boldsymbol{P}, \tag{64}$$

where

$$\boldsymbol{T}' = \boldsymbol{A}^{*} \odot \boldsymbol{A}' + \boldsymbol{A}'^{*} \odot \boldsymbol{A}. \tag{65}$$

According to the relationship between the Khatri-Rao product and the Kronecker product [114], we can rewrite $\boldsymbol{V}$ as

$$\boldsymbol{V} = \boldsymbol{W}\boldsymbol{T} = [(\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})^{*} \otimes (\boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{A})]\boldsymbol{J}_2, \tag{66}$$

where $\boldsymbol{J}_2 \in \mathbb{R}^{K^2\times K}$ is a singular selection matrix.

Remark 3: Compared with (35), the number of columns of $\boldsymbol{V}$ in (66) is reduced from K^2 to K, whereas the number of rows remains M^2. Consequently, the necessary condition for the FIM to be nonsingular is relaxed from $K \leq M$ to $K \leq M^2$. This explains why the *a priori* knowledge of uncorrelated sources allows (23) to be valid in the range $K \geq M$. However, in this case, $\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp}$ cannot be written in a form similar to (37), so that explicit expressions for $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{F}$ are unavailable. The following *Result* 6 gives a class of closed-form expressions for the DOA-related blocks of the stochastic CRBs with uncorrelated sources under different noise models.

Result 6: Consider the *a priori* knowledge that the sources are spatially uncorrelated. Denote the DOA-related block of the stochastic CRB in the presence of UN, NUN, and AUN as $\boldsymbol{B}_{\text{unc}}^{\text{un}}$, $\boldsymbol{B}_{\text{unc}}^{\text{nun}}$, and $\boldsymbol{B}_{\text{unc}}^{\text{aun}}$, respectively. The closed-form expressions for $\boldsymbol{B}_{\text{unc}}^{\text{un}}$, $\boldsymbol{B}_{\text{unc}}^{\text{nun}}$, and $\boldsymbol{B}_{\text{unc}}^{\text{aun}}$ are given by (23), with the same $\boldsymbol{G}_{\text{sto}}$ and $\boldsymbol{V}$ shown in (64) and (66), but with different $\boldsymbol{U}$ given by

$$\begin{aligned}
\text{UN}: \ \boldsymbol{U} &= \text{vec}(\boldsymbol{R}^{-1}),\\
\text{NUN}: \ \boldsymbol{U} &= \boldsymbol{W}\boldsymbol{\mathcal{E}}_M,\\
\text{AUN}: \ \boldsymbol{U} &= \boldsymbol{W}\boldsymbol{\mathcal{Q}},
\end{aligned} \tag{67}$$

where

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_M &= \left[\text{vec}(\boldsymbol{e}_1\boldsymbol{e}_1^T), \text{vec}(\boldsymbol{e}_2\boldsymbol{e}_2^T), \ldots, \text{vec}(\boldsymbol{e}_M\boldsymbol{e}_M^T)\right],\\
\boldsymbol{\mathcal{Q}} &= \left[\text{vec}(\boldsymbol{Q}_1'), \text{vec}(\boldsymbol{Q}_2'), \ldots, \text{vec}(\boldsymbol{Q}_L')\right].
\end{aligned} \tag{68}$$

The closed-form expression for $\boldsymbol{B}_{\text{unc}}^{\text{un}}$ was first presented in [39], and it was further rewritten in a more explicit form to analyze the performance of the proposed DOA estimator. The key steps are carried out as follows.

Assume that $K<M$, and then

$$\text{vec}(\boldsymbol{I}_M) = \text{vec}(\boldsymbol{\Pi}_{\boldsymbol{A}}) + \text{vec}(\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}). \tag{69}$$

Introduce a full column-rank matrix $\boldsymbol{\Lambda} \in \mathbb{C}^{M^2\times(M^2-K)}$ whose columns span the null space of $\boldsymbol{T}^H$. Similar to (44), we have

$$\boldsymbol{\Pi}_{\boldsymbol{V}}^{\perp} = \boldsymbol{\Pi}_{\boldsymbol{W}^{-1}\boldsymbol{\Lambda}}. \tag{70}$$

Substituting (64), (66), (67), (69) and (70) into (25) yields

$$\boldsymbol{B}_{\text{unc}}^{\text{un}} = \frac{1}{N}\left[\boldsymbol{P}\boldsymbol{T}'^{H}\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{Y}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\boldsymbol{T}'\boldsymbol{P}\right]^{-1}, \tag{71}$$

where

$$\boldsymbol{Y} = (\boldsymbol{R}^{T}\otimes\boldsymbol{R}) + \frac{\sigma^{2}}{M-K}\text{vec}(\boldsymbol{\Pi}_{\boldsymbol{A}})\text{vec}(\boldsymbol{\Pi}_{\boldsymbol{A}})^{H}. \tag{72}$$

G. COMPARISON AND DISCUSSION

When a new CRB expression was derived, it was often compared with the existing ones, see, e.g., [2], [17], [20], [24]. Now we have obtained three closed-form deterministic CRB expressions, i.e., $\boldsymbol{B}_{\text{det}}^{\text{un}}$, $\boldsymbol{B}_{\text{det}}^{\text{nun}}$, and $\boldsymbol{B}_{\text{det}}^{\text{aun}}$, as well as six more stochastic ones, namely, $\boldsymbol{B}_{\text{sto}}^{\text{un}}$, $\boldsymbol{B}_{\text{sto}}^{\text{nun}}$, $\boldsymbol{B}_{\text{sto}}^{\text{aun}}$, $\boldsymbol{B}_{\text{unc}}^{\text{un}}$, $\boldsymbol{B}_{\text{unc}}^{\text{nun}}$, and $\boldsymbol{B}_{\text{unc}}^{\text{aun}}$. Comparisons among these CRBs lead to many interesting results which offer valuable insights into the estimation efficiency under different model assumptions. In what follows, the symbols $\succeq$, $\succ$, $\preceq$, $\prec$ are used to describe the partial order between two matrices, which means subtracting the matrix on the right from that on the left produces a matrix that is nonnegative definite, positive definite, nonpositive definite, and negative definite, respectively.

1) DETERMINISTIC CRB VERSUS DETERMINISTIC CRB

In the deterministic case, the signal parameters and noise are decoupled in the FIM. From *Results* 1 and 2, the following equality holds [17], [24].

Property 1: If the true noise covariance matrix is described by the UN model, then the deterministic CRBs derived under the assumptions of UN, NUN, and AUN are identical, yielding

$$\boldsymbol{B}_{\text{det}}^{\text{aun}}|_{\boldsymbol{Q}=\sigma\boldsymbol{I}_M} = \boldsymbol{B}_{\text{det}}^{\text{nun}}|_{\boldsymbol{Q}=\sigma\boldsymbol{I}_M} = \boldsymbol{B}_{\text{det}}^{\text{un}}. \tag{73}$$

Moreover, if the true noise covariance matrix is described by the NUN model, then the deterministic CRBs derived under the assumptions of NUN and AUN are identical, leading to

$$\boldsymbol{B}_{\text{det}}^{\text{aun}}|_{\boldsymbol{Q}=\text{diag}(\boldsymbol{\sigma})} = \boldsymbol{B}_{\text{det}}^{\text{nun}}. \tag{74}$$

2) STOCHASTIC CRB VERSUS STOCHASTIC CRB

Intuitively, adding extra nuisance parameters will expand the dimension of the subspace of $\boldsymbol{\Delta}_{\text{sto}}$, and hence increase the stochastic CRB for DOAs [24]. We restate this conclusion in the following theorem and give a rigorous proof.

Theorem 1: Let the vector $\boldsymbol{\omega}$ ($\boldsymbol{\omega} \neq \boldsymbol{0}$) collect a group of extra unknown parameters.

a) If $\boldsymbol{\omega}$, $\boldsymbol{p}$, and $\boldsymbol{\sigma}$ are linearly independent, adding $\boldsymbol{\omega}$ to $\boldsymbol{p}$ or $\boldsymbol{\sigma}$ will increase the stochastic CRB for DOAs.
b) If $\boldsymbol{\omega}, \boldsymbol{p}$, and $\boldsymbol{\sigma}$ are linearly dependent, adding $\boldsymbol{\omega}$ to $\boldsymbol{p}$ or $\boldsymbol{\sigma}$ will not change the the stochastic CRB for DOAs.
c) An arbitrary permutation of nuisance parameters will not change the stochastic CRB for DOAs.

Proof: See Appendix A. ■

Based on *Theorem* 1, a number of order relationships among the stochastic CRBs under different model assumptions can be immediately obtained by comparing the specific presentations of $\boldsymbol{\alpha}_{\text{sto}}$.

Property 2: Under the AUN model, assume that $\boldsymbol{\sigma}$ contains not only the M unknown parameters on the diagonal of $\boldsymbol{Q}$ but also some other ones. Then, we have

$$\boldsymbol{B}_{\text{sto}}^{\text{aun}} \succ \boldsymbol{B}_{\text{sto}}^{\text{nun}} \succ \boldsymbol{B}_{\text{sto}}^{\text{un}}, \quad \boldsymbol{B}_{\text{unc}}^{\text{aun}} \succ \boldsymbol{B}_{\text{unc}}^{\text{nun}} \succ \boldsymbol{B}_{\text{unc}}^{\text{un}}.$$

Property 3: The stochastic CRB for DOAs with the *a priori* knowledge of uncorrelated sources is always smaller than that without *a priori* knowledge, i.e.,

$$\boldsymbol{B}_{\text{sto}}^{\text{un}} \succ \boldsymbol{B}_{\text{unc}}^{\text{un}}, \quad \boldsymbol{B}_{\text{sto}}^{\text{nun}} \succ \boldsymbol{B}_{\text{unc}}^{\text{nun}}, \; \boldsymbol{B}_{\text{sto}}^{\text{aun}} \succ \boldsymbol{B}_{\text{unc}}^{\text{aun}}.$$

In [17], [24], the first inequality in *Property* 2 was written as $\boldsymbol{B}_{\text{sto}}^{\text{nun}}|_{\boldsymbol{Q}=\boldsymbol{I}_M} \succeq \boldsymbol{B}_{\text{sto}}^{\text{un}}$ and $\boldsymbol{B}_{\text{sto}}^{\text{aun}}|_{\boldsymbol{Q}=\boldsymbol{I}_M} \succeq \boldsymbol{B}_{\text{sto}}^{\text{un}}$, respectively. These results were derived based on the fact that $(N\boldsymbol{C})^{-1}|_{\boldsymbol{Q}=\boldsymbol{I}_M} = \boldsymbol{B}_{\text{sto}}^{\text{un}}$ and also $\boldsymbol{D}\boldsymbol{F}^{-1}\boldsymbol{D}^{H}$ is nonnegative definite. However, the condition under which the equality holds was not given, and *Theorem* 1 indicates that these inequalities are strict.

3) DETERMINISTIC CRB VERSUS STOCHASTIC CRB

To compare the deterministic and stochastic CRBs, the following asymptotic (w.r.t. snapshots) deterministic CRB expressions for a sufficiently large N will be useful [20].

$$\boldsymbol{B}_{\text{det,asy}}^{\text{un}} = \frac{\sigma}{2N}\left\{\text{Re}\left[(\boldsymbol{A}'^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{A}')\circ\hat{\boldsymbol{P}}_{\text{asy}}^{T}\right]\right\}^{-1}, \tag{75}$$

$$\boldsymbol{B}_{\text{det,asy}}^{\text{nun}} = \boldsymbol{B}_{\text{det,asy}}^{\text{aun}} = \frac{1}{2N}\left\{\text{Re}\left[(\bar{\boldsymbol{A}}'^{H}\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}^{\perp}\bar{\boldsymbol{A}}')\circ\hat{\boldsymbol{P}}_{\text{asy}}^{T}\right]\right\}^{-1}, \tag{76}$$

where

$$\hat{\boldsymbol{P}}_{\text{asy}} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\boldsymbol{s}(t)\boldsymbol{s}^{H}(t).$$

Then, another property is given as follows.

Property 4: For a finite number of sensors, the stochastic CRB is always larger than the deterministic one.

$$\boldsymbol{B}_{\text{sto}}^{\text{un}} \succ \boldsymbol{B}_{\text{det,asy}}^{\text{un}}, \quad \boldsymbol{B}_{\text{sto}}^{\text{nun}} \succ \boldsymbol{B}_{\text{det,asy}}^{\text{nun}}, \; \boldsymbol{B}_{\text{sto}}^{\text{aun}} \succ \boldsymbol{B}_{\text{det,asy}}^{\text{aun}}.$$

The first inequality can be demonstrated by subtracting (42) from (75), which results in a positive definite matrix for any finite M [20]. Besides, this inequality was shown to be strict when $\boldsymbol{A}'^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\boldsymbol{A}'$ and $\boldsymbol{P}$ are both positive definite [2]. Similarly, the other two inequalities can be obtained via comparing (49) and (76), which leads to $\boldsymbol{B}_{\text{sto}}^{\text{nun}} \succ (N\boldsymbol{C})^{-1} \succ \boldsymbol{B}_{\text{det,asy}}^{\text{nun}}$ and $\boldsymbol{B}_{\text{sto}}^{\text{aun}} \succ (N\boldsymbol{C})^{-1} \succ \boldsymbol{B}_{\text{det,asy}}^{\text{aun}}$. In addition, The third inequality was originally written as $\boldsymbol{B}_{\text{sto}}^{\text{aun}} \succeq (N\boldsymbol{C})^{-1} \succeq \boldsymbol{B}_{\text{det,asy}}^{\text{aun}}$ [24], but the condition under which the equality holds was not given.

4) CRB VERSUS ASYMPTOTIC COVARIANCE MATRIX

In the literature, some of the aforementioned inequalities are accompanied by the asymptotic covariance matrices of estimation errors of specific estimators [2], [4], [20], [21], [107]. For instance, an interesting result is given below [4].

$$\frac{1}{N}\boldsymbol{C}^{\text{un}}_{\text{DML,asy}} = \boldsymbol{B}^{\text{un}}_{\text{det,asy}} + 2N\boldsymbol{B}^{\text{un}}_{\text{det,asy}}\text{Re}\left[(\boldsymbol{A}'^H\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}}\boldsymbol{A}') \circ(\boldsymbol{A}^H\boldsymbol{A})^{-T}\right]\boldsymbol{B}^{\text{un}}_{\text{det,asy}},$$
$$\frac{1}{N}\boldsymbol{C}^{\text{un}}_{\text{DML,asy}} \succeq \frac{1}{N}\boldsymbol{C}^{\text{un}}_{\text{SML,asy}} = \boldsymbol{B}^{\text{un}}_{\text{sto}} \succeq \boldsymbol{B}^{\text{un}}_{\text{det,asy}}, \quad (77)$$

where $\boldsymbol{C}^{\text{un}}_{\text{DML,asy}}$ and $\boldsymbol{C}^{\text{un}}_{\text{SML,asy}}$ denote the asymptotic covariance matrices of estimation errors of the deterministic and stochastic ML estimators, respectively, with UN.

Other similar results in the literature offer valuable insights into the asymptotic performance of various estimators, but they involve a broad range of discussions which cannot be fully covered in this paper. For detail, the reader can refer to the original work where a specific inequality is presented.

IV. EXTENSION TO COMPLEX NONCIRCULAR GAUSSIAN SIGNALS

For complex noncircular Gaussian signals, in addition to $\boldsymbol{\Gamma}$, the covariance matrix of the conjugate part of the signals is also required to describe the second-order statistical property of $\bar{\boldsymbol{x}}$. Taking into account the conjugate part of the samples, we denote the extended signal vector and its mean as

$$\dot{\boldsymbol{x}} = \begin{bmatrix}\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{x}}^*\end{bmatrix}, \quad \dot{\boldsymbol{\mu}} = \begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\mu}^*\end{bmatrix}.$$

Then, the covariance matrix of $\dot{\boldsymbol{x}}$ is defined as

$$\dot{\boldsymbol{\Gamma}} \triangleq E[(\dot{\boldsymbol{x}}-\dot{\boldsymbol{\mu}})(\dot{\boldsymbol{x}}-\dot{\boldsymbol{\mu}})^H] = \begin{bmatrix}\boldsymbol{\Gamma} & \boldsymbol{\Gamma}_{\text{c}}\\ \boldsymbol{\Gamma}^*_{\text{c}} & \boldsymbol{\Gamma}^*\end{bmatrix}, \quad (78)$$

where

$$\boldsymbol{\Gamma}_{\text{c}} \triangleq E[(\bar{\boldsymbol{x}}-\boldsymbol{\mu})(\bar{\boldsymbol{x}}^*-\boldsymbol{\mu}^*)^H].$$

Here, $\dot{\boldsymbol{\mu}}$ and $\dot{\boldsymbol{\Gamma}}$ are determined by the extended unknown parameter vector $\dot{\boldsymbol{\alpha}}$. Thus, $f(\bar{\boldsymbol{x}};\dot{\boldsymbol{\alpha}})$ can be rewritten as a function of $\dot{\boldsymbol{x}}$, $\dot{\boldsymbol{\mu}}$ and $\dot{\boldsymbol{\Gamma}}$ [110], [115], which can be used to derive the Slepian-Bangs formula for complex noncircular Gaussian signals [52]:

$$\langle \boldsymbol{B}^{-1}(\dot{\boldsymbol{\alpha}})\rangle_{i,j} = \frac{1}{2}\text{tr}\left(\dot{\boldsymbol{\Gamma}}^{-1}\frac{\partial\dot{\boldsymbol{\Gamma}}}{\partial\langle\dot{\boldsymbol{\alpha}}\rangle_i}\dot{\boldsymbol{\Gamma}}^{-1}\frac{\partial\dot{\boldsymbol{\Gamma}}}{\partial\langle\dot{\boldsymbol{\alpha}}\rangle_j}\right) + \text{Re}\left(\frac{\partial\dot{\boldsymbol{\mu}}^H}{\partial\langle\dot{\boldsymbol{\alpha}}\rangle_i}\dot{\boldsymbol{\Gamma}}^{-1}\frac{\partial\dot{\boldsymbol{\mu}}}{\partial\langle\dot{\boldsymbol{\alpha}}\rangle_j}\right). \quad (79)$$

For circular signals, $\boldsymbol{\Gamma}_{\text{c}} = \boldsymbol{0}$, so that (79) reduces to (8).

A. CLOSED-FORM NONCIRCULAR DETERMINISTIC CRB EXPRESSIONS WITH DIFFERENT NOISE MODELS

Under the deterministic model, the signals are assumed to be deterministic, and the noise is assumed to be circularly symmetric Gaussian distributed. Therefore, we have

$$\dot{\boldsymbol{\mu}} = \dot{\boldsymbol{\mu}}_{\text{det}} = \begin{bmatrix}\boldsymbol{\Delta}_{\text{det}}\bar{\boldsymbol{s}}\\ \boldsymbol{\Delta}^*_{\text{det}}\bar{\boldsymbol{s}}^*\end{bmatrix},$$
$$\dot{\boldsymbol{\Gamma}} = \dot{\boldsymbol{\Gamma}}_{\text{det}} = \begin{bmatrix}\boldsymbol{I}_N\otimes\boldsymbol{Q} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_N\otimes\boldsymbol{Q}^*\end{bmatrix},$$
$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}_{\text{det}} = \boldsymbol{\alpha}_{\text{det}}. \quad (80)$$

Substituting (80) into (79) and following the derivation in Section II-C, we finally reach the unified framework for the circular deterministic CRB in (16). This indicates that the DOA-related block of the deterministic CRB for complex circular and noncircular Gaussian signals are identical. This conclusion also follows intuitively from the fact that the deterministic model does not account for the second-order statistics of the source signals, so that noncircularity will not affect the deterministic CRB. Consequently, the circular deterministic CRB expressions in *Results* 1 and 2 are all applicable to the noncircular scenario.

Result 7: Denote the DOA-related block of the noncircular deterministic CRB in the presence of UN, NUN, and AUN as $\boldsymbol{B}^{\text{un}}_{\text{det,nc}}$, $\boldsymbol{B}^{\text{nun}}_{\text{det,nc}}$, and $\boldsymbol{B}^{\text{aun}}_{\text{det,nc}}$, respectively. If $K<M$, then the closed-form expression for $\boldsymbol{B}^{\text{un}}_{\text{det,nc}}$ is given by (28), and those for $\boldsymbol{B}^{\text{nun}}_{\text{det,nc}}$ or $\boldsymbol{B}^{\text{aun}}_{\text{det,nc}}$ are given by (16).

B. UNIFIED NONCIRCULAR STOCHASTIC CRB FRAMEWORK

For the stochastic model, we define the following covariance matrices

$$\boldsymbol{P}_{\text{c}} \triangleq E[\boldsymbol{s}(t)\boldsymbol{s}^T(t)],$$
$$\boldsymbol{R}_{\text{c}} \triangleq E[\boldsymbol{x}(t)\boldsymbol{x}^T(t)] = \boldsymbol{A}\boldsymbol{P}_{\text{c}}\boldsymbol{A}^T,$$

and thereby

$$\dot{\boldsymbol{\mu}} = \dot{\boldsymbol{\mu}}_{\text{sto}} = \boldsymbol{0},\ \dot{\boldsymbol{\Gamma}} = \dot{\boldsymbol{\Gamma}}_{\text{sto}} = \begin{bmatrix}\boldsymbol{I}_N\otimes\boldsymbol{R} & \boldsymbol{I}_N\otimes\boldsymbol{R}_{\text{c}}\\ \boldsymbol{I}_N\otimes\boldsymbol{R}^*_{\text{c}} & \boldsymbol{I}_N\otimes\boldsymbol{R}^*\end{bmatrix}. \quad (81)$$

Introduce the following notations:

$$\dot{\boldsymbol{A}} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}^*\end{bmatrix},\quad \dot{\boldsymbol{P}} = \begin{bmatrix}\boldsymbol{P} & \boldsymbol{P}_{\text{c}}\\ \boldsymbol{P}^*_{\text{c}} & \boldsymbol{P}^*\end{bmatrix},\quad \dot{\boldsymbol{Q}} = \begin{bmatrix}\boldsymbol{Q} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q}^*\end{bmatrix},$$
$$\dot{\boldsymbol{R}} = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{R}_{\text{c}}\\ \boldsymbol{R}^*_{\text{c}} & \boldsymbol{R}^*\end{bmatrix} = \dot{\boldsymbol{A}}\dot{\boldsymbol{P}}\dot{\boldsymbol{A}}^H + \dot{\boldsymbol{Q}}, \quad (82)$$

and notice that

$$\dot{\boldsymbol{\Gamma}}_{\text{sto}} = \boldsymbol{O}_2(\boldsymbol{I}_N\otimes\dot{\boldsymbol{R}})\boldsymbol{O}_2^T, \quad (83)$$

where $\boldsymbol{O}_2 \in \mathbb{R}^{2MN\times 2MN}$ is a permutation matrix.

The extended unknown parameter vector is expressed as

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}_{\text{sto}} = \left[\boldsymbol{\theta}^T, \dot{\boldsymbol{p}}^T, \boldsymbol{\sigma}^T\right]^T, \quad (84)$$

where $\dot{\boldsymbol{p}}$ holds the real and imaginary parts of the unknown entries in $\dot{\boldsymbol{P}}$.

Substituting (81), (83), and (84) into (79) and using (19), we obtain

$$\boldsymbol{B}^{-1}(\dot{\boldsymbol{\alpha}}_{\text{sto}}) = \frac{N}{2}\left(\frac{\partial\dot{\boldsymbol{r}}}{\partial\dot{\boldsymbol{\alpha}}^T_{\text{sto}}}\right)^H(\dot{\boldsymbol{R}}^{-T}\otimes\dot{\boldsymbol{R}}^{-1})\left(\frac{\partial\dot{\boldsymbol{r}}}{\partial\dot{\boldsymbol{\alpha}}^T_{\text{sto}}}\right), \quad (85)$$

where $\dot{\boldsymbol{r}} = \text{vec}(\dot{\boldsymbol{R}})$. Introduce the following notations:

$$\dot{\boldsymbol{G}}_{\text{sto}} = \dot{\boldsymbol{W}}\frac{\partial\dot{\boldsymbol{r}}}{\partial\boldsymbol{\theta}^T},\quad \dot{\boldsymbol{\Delta}}_{\text{sto}} = \left[\dot{\boldsymbol{V}}, \dot{\boldsymbol{U}}\right],\ \dot{\boldsymbol{V}} = \dot{\boldsymbol{W}}\frac{\partial\dot{\boldsymbol{r}}}{\partial\dot{\boldsymbol{p}}^T},$$

$$\dot{\boldsymbol{U}} = \dot{\boldsymbol{W}}\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{\sigma}^T}, \quad \dot{\boldsymbol{W}} = \dot{\boldsymbol{R}}^{-\frac{T}{2}} \otimes \dot{\boldsymbol{R}}^{-\frac{1}{2}}, \tag{86}$$

and then (23) can be straightforwardly extended to the complex noncircular Gaussian scenario, leading to the *geometrical interpretation* of the noncircular stochastic CRB:

$$\boldsymbol{B}_{\text{sto,nc}} = \frac{2}{N}\left(\dot{\boldsymbol{G}}_{\text{sto}}^H \boldsymbol{\Pi}_{\dot{\boldsymbol{\Delta}}_{\text{sto}}}^{\perp} \dot{\boldsymbol{G}}_{\text{sto}}\right)^{-1}. \tag{87}$$

Furthermore, assume that $\dot{\boldsymbol{V}}$ has more rows than columns, then the unified framework for the DOA-related block of the noncircular stochastic CRB is given by

$$\boldsymbol{B}_{\text{sto,nc}} = \frac{2}{N}\left(\dot{\boldsymbol{C}} - \dot{\boldsymbol{D}}\dot{\boldsymbol{F}}^{-1}\dot{\boldsymbol{D}}^H\right)^{-1}. \tag{88}$$

C. CLOSED-FORM NONCIRCULAR STOCHASTIC CRB EXPRESSIONS WITH DIFFERENT NOISE MODELS

First, consider the general case without *a priori* knowledge about the sources. According to (82), $\dot{\boldsymbol{P}}$ is parameterized by the following $2K^2 + K$ real-valued unknowns:

$$\Big[\text{Re}(\langle \boldsymbol{P}\rangle_{i,j}), \text{Im}(\langle \boldsymbol{P}\rangle_{i,j}), \text{Re}(\langle \boldsymbol{P}_{\text{c}}\rangle_{i,j}), \text{Im}(\langle \boldsymbol{P}_{\text{c}}\rangle_{i,j})\Big]_{1\leq i<j\leq K}, \quad \Big[\langle \boldsymbol{P}\rangle_{i,i}, \text{Re}(\langle \boldsymbol{P}_{\text{c}}\rangle_{i,i}), \text{Im}(\langle \boldsymbol{P}_{\text{c}}\rangle_{i,i})\Big]_{i=1,2,\ldots,K}. \tag{89}$$

It has been proved that the ML estimate of $\boldsymbol{\theta}$ is invariant to the constrained structure of $\dot{\boldsymbol{P}}$ [52], [53]. Therefore, we can treat $\dot{\boldsymbol{P}}$ as an arbitrary Hermitian matrix which is parameterized by its $4K^2$ upper triangular elements collected by

$$\dot{\boldsymbol{p}} = \left[\langle \dot{\boldsymbol{P}}\rangle_{1,1}, \text{Re}(\langle \dot{\boldsymbol{P}}\rangle_{1,2}), \text{Im}(\langle \dot{\boldsymbol{P}}\rangle_{1,2}), \ldots, \langle \dot{\boldsymbol{P}}\rangle_{2K,2K}\right]^T,$$

Remark 4: The underlying equivalence can also be explained by *Theorem* 1. Notice that $\dot{\boldsymbol{p}}$ contains all parameters in (89), whereas the rest are their duplicates. Thus, replacing the original unknown parameters in (89) with $\dot{\boldsymbol{p}}$ will not change the DOA-related block of the stochastic CRB. This shows that in some cases, adding redundant or, more generally, linearly dependent nuisance parameters will benefit the derivation of the CRB for DOAs, without impairing the correctness.

Consequently, the key relationship (33) is preserved for complex noncircular Gaussian signals

$$\dot{\boldsymbol{r}} = (\dot{\boldsymbol{A}}^* \otimes \dot{\boldsymbol{A}})\dot{\boldsymbol{J}}_1\dot{\boldsymbol{p}} + \text{vec}(\dot{\boldsymbol{Q}}), \tag{90}$$

where $\dot{\boldsymbol{J}}_1 \in \mathbb{C}^{4K^2\times 4K^2}$ is a nonsingular matrix. Since $\dot{\boldsymbol{V}}$ is assumed to have more rows than columns, i.e., $K<M$, it can be verified that (45) becomes

$$\dot{\boldsymbol{R}}^{-\frac{1}{2}}\boldsymbol{\Pi}^{\perp}_{\dot{\boldsymbol{R}}^{-\frac{1}{2}}\dot{\boldsymbol{A}}}\dot{\boldsymbol{R}}^{-\frac{1}{2}} = \dot{\boldsymbol{Q}}^{-\frac{1}{2}}\boldsymbol{\Pi}^{\perp}_{\bar{\dot{\boldsymbol{A}}}}\dot{\boldsymbol{Q}}^{-\frac{1}{2}}. \tag{91}$$

Introduce the following notations:

$$\bar{\dot{\boldsymbol{R}}} = \dot{\boldsymbol{Q}}^{-\frac{1}{2}}\dot{\boldsymbol{R}}\dot{\boldsymbol{Q}}^{-\frac{1}{2}}, \quad \bar{\boldsymbol{R}}_{\text{c}} = \boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{R}_{\text{c}}\boldsymbol{Q}^{*-\frac{1}{2}}. \tag{92}$$

From (47) and (92), we have $\bar{\dot{\boldsymbol{R}}} = \begin{bmatrix}\bar{\boldsymbol{R}} & \bar{\boldsymbol{R}}_{\text{c}}\\ \bar{\boldsymbol{R}}_{\text{c}}^* & \bar{\boldsymbol{R}}^*\end{bmatrix}$, and thereby $\bar{\dot{\boldsymbol{R}}}^{-1}$ takes the form

$$\bar{\dot{\boldsymbol{R}}}^{-1} = \begin{bmatrix}\boldsymbol{Z} & \boldsymbol{Z}_{\text{c}}\\ \boldsymbol{Z}_{\text{c}}^* & \boldsymbol{Z}^*\end{bmatrix}, \tag{93}$$

where

$$\boldsymbol{Z} = (\bar{\boldsymbol{R}} - \bar{\boldsymbol{R}}_{\text{c}}\bar{\boldsymbol{R}}^{*-1}\bar{\boldsymbol{R}}_{\text{c}}^*)^{-1}, \quad \boldsymbol{Z}_{\text{c}} = -\boldsymbol{Z}\bar{\boldsymbol{R}}_{\text{c}}\bar{\boldsymbol{R}}^{*-1}.$$

Following the derivations from Section III-C to Section III-E and using (90), (91), (92) and (93), we can obtain the following results:

Result 8: Denote the DOA-related block of the noncircular stochastic CRB in the presence of UN as $\boldsymbol{B}^{\text{un}}_{\text{sto,nc}}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}^{\text{un}}_{\text{sto,nc}}$ is given by

$$\boldsymbol{B}^{\text{un}}_{\text{sto,nc}} = \frac{\sigma}{2N}\left\{\text{Re}\left[(\boldsymbol{A}'^H\boldsymbol{\Pi}^{\perp}_{\boldsymbol{A}}\boldsymbol{A}') \circ \left([\boldsymbol{P}\boldsymbol{A}^H, \boldsymbol{P}_{\text{c}}\boldsymbol{A}^T]\dot{\boldsymbol{R}}^{-1}\begin{bmatrix}\boldsymbol{A}\boldsymbol{P}\\ \boldsymbol{A}^*\boldsymbol{P}_{\text{c}}^*\end{bmatrix}\right)^T\right]\right\}^{-1}. \tag{94}$$

Result 9: Denote the DOA-related block of the noncircular stochastic CRB in the presence of NUN as $\boldsymbol{B}^{\text{nun}}_{\text{sto,nc}}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}^{\text{nun}}_{\text{sto,nc}}$ is given by (88), with

$$\begin{aligned}
\dot{\boldsymbol{C}} &= 4\text{Re}\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}\bar{\boldsymbol{A}}') \circ \left([\boldsymbol{P}\bar{\boldsymbol{A}}^H, \boldsymbol{P}_{\text{c}}\bar{\boldsymbol{A}}^T]\bar{\dot{\boldsymbol{R}}}^{-1}\begin{bmatrix}\bar{\boldsymbol{A}}\boldsymbol{P}\\ \bar{\boldsymbol{A}}^*\boldsymbol{P}_{\text{c}}^*\end{bmatrix}\right)^T\right],\\
\dot{\boldsymbol{D}} &= 4\text{Re}\Big[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}) \circ (\boldsymbol{Z}\bar{\boldsymbol{A}}\boldsymbol{P})^T + (\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}) \circ (\boldsymbol{Z}_{\text{c}}\bar{\boldsymbol{A}}^*\boldsymbol{P}_{\text{c}}^*)^T\Big],\\
\dot{\boldsymbol{F}} &= 2\left[\boldsymbol{Z}^T \circ \boldsymbol{Z} - (\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}\boldsymbol{Z})^T \circ (\boldsymbol{\Pi}_{\bar{\boldsymbol{A}}}\boldsymbol{Z})\right].
\end{aligned}$$

Result 10: Denote the DOA-related block of the noncircular stochastic CRB in the presence of AUN as $\boldsymbol{B}^{\text{aun}}_{\text{sto,nc}}$. If $K<M$, then the closed-form expression for $\boldsymbol{B}^{\text{aun}}_{\text{sto,nc}}$ is given by (88), with

$$\begin{aligned}
\dot{\boldsymbol{C}} &= 4\text{Re}\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}\bar{\boldsymbol{A}}') \circ \left([\boldsymbol{P}\bar{\boldsymbol{A}}^H, \boldsymbol{P}_{\text{c}}\bar{\boldsymbol{A}}^T]\bar{\dot{\boldsymbol{R}}}^{-1}\begin{bmatrix}\bar{\boldsymbol{A}}\boldsymbol{P}\\ \bar{\boldsymbol{A}}^*\boldsymbol{P}_{\text{c}}^*\end{bmatrix}\right)^T\right],\\
\dot{\boldsymbol{D}} &= 4\text{Re}\left\{\boldsymbol{\mathcal{E}}_K^T\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}) \otimes (\boldsymbol{Z}\bar{\boldsymbol{A}}\boldsymbol{P})^T\right]\bar{\boldsymbol{\mathcal{Q}}}^*\right\} + 4\text{Re}\left\{\boldsymbol{\mathcal{E}}_K^T\left[(\bar{\boldsymbol{A}}'^H\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}) \otimes (\boldsymbol{Z}_{\text{c}}\bar{\boldsymbol{A}}^*\boldsymbol{P}_{\text{c}}^*)^T\right]\bar{\boldsymbol{\mathcal{Q}}}^*\right\},\\
\dot{\boldsymbol{F}} &= 4\text{Re}\left[\bar{\boldsymbol{\mathcal{Q}}}^H(\bar{\boldsymbol{R}}^{-T} \otimes \boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}})\bar{\boldsymbol{\mathcal{Q}}}\right] - 2\bar{\boldsymbol{\mathcal{Q}}}^H\left[(\boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}})^T \otimes \boldsymbol{\Pi}^{\perp}_{\bar{\boldsymbol{A}}}\right]\bar{\boldsymbol{\mathcal{Q}}}.
\end{aligned}$$

Results 8-10 were originally derived in [52], [53] from the extended Slepian-Bangs formula. In particular, *Result* 8 was also obtained indirectly from the asymptotic covariance matrix of the noncircular stochastic ML estimator [52]. It was mentioned that the deterministic CRB expression for circular signals remains valid for complex noncircular Gaussian signals [53], which invokes *Result* 7.

D. CLOSED-FORM NONCIRCULAR STOCHASTIC CRB EXPRESSIONS WITH UNCORRELATED SOURCES

If the sources are known *a priori* to be spatially uncorrelated, then (59) and (60) are preserved. In addition, we introduce

$$\boldsymbol{P}_{\text{c}} = \text{diag}(\boldsymbol{p}_{\text{c}}), \quad \boldsymbol{p}_{\text{c}} = \left[p_{\text{c},1}, p_{\text{c},2}, \ldots, p_{\text{c},K}\right]^T, \tag{95}$$

where $\{p_{\mathrm{c},k}\}_{k=1}^{K} \in \mathbb{C}$. Therefore,

$$\boldsymbol{R}_{\mathrm{c}} = \sum_{k=1}^{K} p_{\mathrm{c},k}\boldsymbol{a}_k\boldsymbol{a}_k^T.$$

Applying the vectorization operator yields

$$\boldsymbol{r}_{\mathrm{c}} = \boldsymbol{T}_{\mathrm{c}}\boldsymbol{p}_{\mathrm{c}}, \tag{96}$$

where

$$\boldsymbol{T}_{\mathrm{c}} = \boldsymbol{A} \odot \boldsymbol{A}. \tag{97}$$

It should be emphasized that in this case, $\dot{\boldsymbol{P}}$ is not only determined by its elements on the main diagonal, but also those on the off-diagonal, so we cannot treat $\dot{\boldsymbol{P}}$ as an arbitrary diagonal Hermitian matrix. Instead,

$$\dot{\boldsymbol{p}} = \left[\boldsymbol{p}^T, \mathrm{Re}(\boldsymbol{p}_{\mathrm{c}}^T), \mathrm{Im}(\boldsymbol{p}_{\mathrm{c}}^T)\right]^T. \tag{98}$$

According to the block-wise vectorization concept [116], it follows from (61) and (96) that $\dot{\boldsymbol{r}}$ can be expressed as

$$\begin{aligned}\dot{\boldsymbol{r}} &= \dot{\boldsymbol{J}}_2\left[\boldsymbol{r}^T, \boldsymbol{r}_{\mathrm{c}}^H, \boldsymbol{r}_{\mathrm{c}}^T, \boldsymbol{r}^H\right]^T \\ &= \dot{\boldsymbol{J}}_2\begin{bmatrix}\boldsymbol{T}\boldsymbol{p}\\ \boldsymbol{T}_{\mathrm{c}}^*\boldsymbol{p}_{\mathrm{c}}^*\\ \boldsymbol{T}_{\mathrm{c}}\boldsymbol{p}_{\mathrm{c}}\\ \boldsymbol{T}^*\boldsymbol{p}\end{bmatrix} + \dot{\boldsymbol{J}}_2\begin{bmatrix}\mathrm{vec}(\boldsymbol{Q})\\ \boldsymbol{0}\\ \boldsymbol{0}\\ \mathrm{vec}(\boldsymbol{Q}^*)\end{bmatrix}\end{aligned} \tag{99}$$

where $\dot{\boldsymbol{J}}_2 = (\boldsymbol{I}_2 \otimes \boldsymbol{O}_3 \otimes \boldsymbol{I}_M)^{-1} \in \mathbb{R}^{4M^2\times 4M^2}$, and $\boldsymbol{O}_3 \in \mathbb{R}^{2M\times 2M}$ is referred to as the communication matrix [117].

By (96), we have

$$\frac{\partial \boldsymbol{r}_{\mathrm{c}}}{\partial \boldsymbol{\theta}^T} = \boldsymbol{T}_{\mathrm{c}}'\boldsymbol{P}_{\mathrm{c}}, \tag{100}$$

where

$$\boldsymbol{T}_{\mathrm{c}}' = \boldsymbol{A} \odot \boldsymbol{A}' + \boldsymbol{A}' \odot \boldsymbol{A}. \tag{101}$$

Combining (86), (63), (99), and (100), we obtain

$$\dot{\boldsymbol{G}}_{\mathrm{sto}} = \dot{\boldsymbol{W}}\dot{\boldsymbol{J}}_2\begin{bmatrix}\boldsymbol{T}'\boldsymbol{P}\\ \boldsymbol{T}_{\mathrm{c}}'^*\boldsymbol{P}_{\mathrm{c}}^*\\ \boldsymbol{T}_{\mathrm{c}}'\boldsymbol{P}_{\mathrm{c}}\\ \boldsymbol{T}'^*\boldsymbol{P}\end{bmatrix}, \quad \dot{\boldsymbol{V}} = \dot{\boldsymbol{W}}\dot{\boldsymbol{J}}_2\begin{bmatrix}\boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{T}_{\mathrm{c}}^* & -\mathrm{j}\boldsymbol{T}_{\mathrm{c}}^*\\ \boldsymbol{0} & \boldsymbol{T}_{\mathrm{c}} & \mathrm{j}\boldsymbol{T}_{\mathrm{c}}\\ \boldsymbol{T}^* & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}. \tag{102}$$

Given these ingredients, we can derive the closed-form noncircular stochastic CRB expressions with uncorrelated sources in the presence of different noise fields, which are shown below.

Result 11: Consider the *a priori* knowledge that the sources are spatially uncorrelated. Denote the DOA-related block of the noncircular stochastic CRB in the presence of UN, NUN, and AUN as $\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{un}}$, $\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{nun}}$, and $\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{aun}}$, respectively. The closed-form expressions for $\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{un}}$, $\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{nun}}$, and $\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{aun}}$ are given by (87), with the same $\dot{\boldsymbol{G}}_{\mathrm{sto}}$ and $\dot{\boldsymbol{V}}$ shown in (102), but different $\dot{\boldsymbol{U}}$ given by

$$\mathrm{UN}: \dot{\boldsymbol{U}}_{\mathrm{un}} = \dot{\boldsymbol{W}}\dot{\boldsymbol{J}}_2\left[\mathrm{vec}(\boldsymbol{I}_M)^T, \boldsymbol{0}^T, \boldsymbol{0}^T, \mathrm{vec}(\boldsymbol{I}_M)^T\right]^T,$$

$$\mathrm{NUN}: \dot{\boldsymbol{U}}_{\mathrm{nun}} = \dot{\boldsymbol{W}}\dot{\boldsymbol{J}}_2\left[\boldsymbol{\mathcal{E}}_M^T, \boldsymbol{0}^T, \boldsymbol{0}^T, \boldsymbol{\mathcal{E}}_M^T\right]^T,$$

$$\mathrm{AUN}: \dot{\boldsymbol{U}}_{\mathrm{aun}} = \dot{\boldsymbol{W}}\dot{\boldsymbol{J}}_2\left[\boldsymbol{\mathcal{Q}}^T, \boldsymbol{0}^T, \boldsymbol{0}^T, \boldsymbol{\mathcal{Q}}^H\right]^T.$$

To the best of our knowledge, this result has not been presented in the literature, but it is relevant for assessing the performance of some newly proposed algorithms developed for this case, such as [94], [118]. In [94], an approximation to the true noncircular CRB was derived, which is written as (in our notations)

$$\boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{un}} = \frac{1}{N}(\dot{\boldsymbol{G}}_{\mathrm{sto}}^H\dot{\boldsymbol{G}}_{\mathrm{sto}})^{-1}, \tag{103}$$

where $\dot{\boldsymbol{G}}_{\mathrm{sto}}$ was further rewritten by substituting (128) and (135) into (102), as will be illustrated in Section V.

In fact, (103) is incorrect since it was derived based on the Slepian-Bangs formula in (8), which does not account for noncircularity. According to (79), the coefficient in the front of (103) should be $2/N$. If this mistake was corrected, (103) would be an approximation of the true CRB for the signal model in [94]. Let $\dot{\boldsymbol{\mathcal{F}}}$ denote the FIM in this case, and then the true CRB is calculated from $[\dot{\boldsymbol{\mathcal{F}}}^{-1}]_{\boldsymbol{\theta\theta}}$, whereas (103) actually results from $\boldsymbol{\mathcal{F}}_{\boldsymbol{\theta\theta}}^{-1}$. In general, $[\dot{\boldsymbol{\mathcal{F}}}^{-1}]_{\boldsymbol{\theta\theta}} \geq \boldsymbol{\mathcal{F}}_{\boldsymbol{\theta\theta}}^{-1}$ [119, p. 65], indicating that the latter is usually too optimistic, thus not attainable. In particular, $\boldsymbol{\mathcal{F}}_{\boldsymbol{\theta\theta}}^{-1}$ can be the true CRB, provided that the nuisance parameters are assumed to be known.

E. COMPARISON AND DISCUSSION

Properties 1-4 can be easily extended to the noncircular scenario, and some of these extended properties involving the stochastic model without *a priori* knowledge have been proved in [53]. Since the circular and noncircular deterministic CRB expressions are identical, we shall focus on comparisons between the stochastic ones in these two scenarios.

In the following *Theorem* 2, we directly examine the circular and noncircular stochastic CRBs for all unknown parameters, i.e., $\boldsymbol{B}(\boldsymbol{\alpha}_{\mathrm{sto}})$ and $\boldsymbol{B}(\dot{\boldsymbol{\alpha}}_{\mathrm{sto}})$, and then the relationship between their DOA-related blocks can be obtained naturally.

Theorem 2: Consider the case where the signals are actually circular, but this information is not known *a priori*. Then,

$$\boldsymbol{B}(\dot{\boldsymbol{\alpha}}_{\mathrm{sto}}) = \begin{bmatrix}\boldsymbol{B}(\boldsymbol{\alpha}_{\mathrm{sto}}) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{B}(\boldsymbol{\beta})\end{bmatrix}, \tag{104}$$

where $\boldsymbol{\beta}$ holds the unknown parameters associated with $\boldsymbol{P}_{\mathrm{c}}$.

Proof: See Appendix B. ∎

Equation (104) shows that in the asserted case, the DOA-related blocks of the circular and noncircular stochastic CRBs are identical, leading to the following *Property* 5.

Property 5: Assume that the signals are circular, but this information is not known *a priori*. Then, we have

$$\begin{aligned}&\boldsymbol{B}_{\mathrm{sto,nc}}^{\mathrm{un}}\Big|_{\substack{\boldsymbol{P}_{\mathrm{c}}=\boldsymbol{0}\\ \boldsymbol{R}_{\mathrm{c}}=\boldsymbol{0}}} = \boldsymbol{B}_{\mathrm{sto}}^{\mathrm{un}}, \quad \boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{un}}\Big|_{\substack{\boldsymbol{P}_{\mathrm{c}}=\boldsymbol{0}\\ \boldsymbol{R}_{\mathrm{c}}=\boldsymbol{0}}} = \boldsymbol{B}_{\mathrm{unc}}^{\mathrm{un}},\\ &\boldsymbol{B}_{\mathrm{sto,nc}}^{\mathrm{nun}}\Big|_{\substack{\boldsymbol{P}_{\mathrm{c}}=\boldsymbol{0}\\ \boldsymbol{R}_{\mathrm{c}}=\boldsymbol{0}}} = \boldsymbol{B}_{\mathrm{sto}}^{\mathrm{nun}}, \quad \boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{nun}}\Big|_{\substack{\boldsymbol{P}_{\mathrm{c}}=\boldsymbol{0}\\ \boldsymbol{R}_{\mathrm{c}}=\boldsymbol{0}}} = \boldsymbol{B}_{\mathrm{unc}}^{\mathrm{nun}},\\ &\boldsymbol{B}_{\mathrm{sto,nc}}^{\mathrm{aun}}\Big|_{\substack{\boldsymbol{P}_{\mathrm{c}}=\boldsymbol{0}\\ \boldsymbol{R}_{\mathrm{c}}=\boldsymbol{0}}} = \boldsymbol{B}_{\mathrm{sto}}^{\mathrm{aun}}, \quad \boldsymbol{B}_{\mathrm{unc,nc}}^{\mathrm{aun}}\Big|_{\substack{\boldsymbol{P}_{\mathrm{c}}=\boldsymbol{0}\\ \boldsymbol{R}_{\mathrm{c}}=\boldsymbol{0}}} = \boldsymbol{B}_{\mathrm{unc}}^{\mathrm{aun}}.\end{aligned} \tag{105}$$

This property can also be verified by substituting $\boldsymbol{P}_{\rm c} = \boldsymbol{0}$ and $\boldsymbol{R}_{\rm c} = \boldsymbol{0}$ into the CRB expressions given in *Results* 8-11. The three equations on the left of (105) have been checked by this approach in [52], [53], whereas the other three on the right have not been reported in the literature. Verify them by substituting $\boldsymbol{P}_{\rm c} = \boldsymbol{0}$ and $\boldsymbol{R}_{\rm c} = \boldsymbol{0}$ into *Result* 11 would take much more efforts due to the absence of more explicit expressions. However, *Theorem* 2 demonstrates all these six equations as an entirety.

Furthermore, another property was given in [52], [53] as shown below.

Property 6: Under the UN, NUN, and AUN models, the corresponding noncircular CRBs are upper bounded by the circular ones.

$$\boldsymbol{B}^{\rm un}_{\rm sto} \succeq \boldsymbol{B}^{\rm un}_{\rm sto,nc}, \quad \boldsymbol{B}^{\rm nun}_{\rm sto} \succeq \boldsymbol{B}^{\rm nun}_{\rm sto,nc}, \ \boldsymbol{B}^{\rm aun}_{\rm sto} \succeq \boldsymbol{B}^{\rm aun}_{\rm sto,nc}.$$

From *Results* 3 and 8, the first inequality was proved in [52] by means similar to *Theorem* 1. On the other hand, the other two inequalities were demonstrated based on the fact that the asymptotic covariance matrix of the ML estimation errors for circular Gaussian signals is preserved in the noncircular scenario [53]. To the best of our knowledge, when the sources are known *a priori* to be uncorrelated, there is no evidence that the noncircular CRB is upper bounded by the circular one.

V. EXTENSION TO WIDEBAND SIGNALS

A. WIDEBAND MODEL BASED ON FREQUENCY DECOMPOSITION

As mentioned in Section I-D, the temporal samples for wideband signals cannot be modeled as in (1). Conventionally, the observation interval is divided into N nonoverlapping subintervals with the same duration ξ. Then, a Φ-point DFT is applied to each subinterval. Thus, the processing bandwidth is decomposed into Φ narrow frequency bins.

Assume that ξ is much larger than the signal propagation time delay across sensors and also the correlation time of the source signals and the noise. Then, the Fourier coefficients at different frequencies are asymptotically uncorrelated [66], [78], which can be modeled as

$$\boldsymbol{x}_\phi(t) = \boldsymbol{A}_\phi(\boldsymbol{\theta})\boldsymbol{s}_\phi(t) + \boldsymbol{n}_\phi(t), \tag{106}$$

where $\phi = 1, 2, \ldots, \Phi$ is the frequency index. It should be noted that $t = 1, 2, \ldots, N$ herein denotes the frequency domain snapshot index, which is different from the temporal snapshot index in the narrowband scenario. We do not replace it with another symbol in order to highlight the connection between the narrowband CRB and the wideband one. $\boldsymbol{x}_\phi(t)$, $\boldsymbol{s}_\phi(t)$ and $\boldsymbol{n}_\phi(t)$ contain all Fourier coefficients of the array output signals, the sources signals, and the additive noise, respectively, at the ϕ-th bin:

$$\begin{aligned}
\boldsymbol{x}_\phi(t) &= \left[x_{1,\phi}(t), x_{2,\phi}(t), \ldots, x_{M,\phi}(t)\right]^T \in \mathbb{C}^{M\times 1},\\
\boldsymbol{s}_\phi(t) &= \left[s_{1,\phi}(t), s_{2,\phi}(t), \ldots, s_{K,\phi}(t)\right]^T \in \mathbb{C}^{K\times 1},\\
\boldsymbol{n}_\phi(t) &= \left[n_{1,\phi}(t), n_{2,\phi}(t), \ldots, n_{M,\phi}(t)\right]^T \in \mathbb{C}^{M\times 1}.
\end{aligned}$$

The array manifold matrix at the ϕ-th bin is given by

$$\boldsymbol{A}_\phi(\boldsymbol{\theta}) = \left[\boldsymbol{a}_\phi(\theta_1), \boldsymbol{a}_\phi(\theta_2), \ldots, \boldsymbol{a}_\phi(\theta_K)\right].$$

According to the central limit theorem, if ξ is sufficiently long, the Fourier coefficients will be asymptotically Gaussian distributed [71, p. 94]. Hence, the Gaussian assumption made on the temporal signals can be relaxed. For the deterministic model and stochastic model, respectively, we have

$$\boldsymbol{x}_\phi(t) \sim \mathcal{CN}[\boldsymbol{A}_\phi(\boldsymbol{\theta})\boldsymbol{s}_\phi(t), \boldsymbol{Q}_\phi], \tag{107}$$

$$\boldsymbol{x}_\phi(t) \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{R}_\phi), \tag{108}$$

where

$$\begin{aligned}
&\boldsymbol{Q}_\phi \triangleq E[\boldsymbol{n}_\phi(t)\boldsymbol{n}_\phi^H(t)], \quad \boldsymbol{P}_\phi \triangleq E[\boldsymbol{s}_\phi(t)\boldsymbol{s}_\phi^H(t)],\\
&\boldsymbol{R}_\phi \triangleq E[\boldsymbol{x}_\phi(t)\boldsymbol{x}_\phi^H(t)] = \boldsymbol{A}_\phi(\boldsymbol{\theta})\boldsymbol{P}_\phi\boldsymbol{A}_\phi^H(\boldsymbol{\theta}) + \boldsymbol{Q}_\phi.
\end{aligned}$$

Under the assumptions given above, the frequency domain sample covariance matrix, $\boldsymbol{R}_\phi$, approximately equals the cross-spectral density matrix of the array output signals. In practice, $\boldsymbol{R}_\phi$ is estimated from $\hat{\boldsymbol{R}}_\phi = 1/N\sum_{t=1}^{N}\boldsymbol{x}_\phi(t)\boldsymbol{x}_\phi^H(t)$, which is a sufficient statistic for the wideband Gaussian problem [78]. Note that the spectral leakage inherent in the DFT might break the consistency of the established frequency bin model [57], so that the wideband CRB derived here is an approximation to the truth, but is of practical value since most algorithms regard $\hat{\boldsymbol{R}}_\phi$ as the actual measured data [78].

B. WIDEBAND SLEPIAN-BANGS FORMULA

Practically, it is not desirable to process all frequency bins [79]. The uncorrelatedness of different frequencies actually alludes to a spectra smoothness condition [78], and thus processing a large number of frequency bins may result in an overparameterized signal spectra. When this happens, the FIM might be very close to singular, and the CRB should be calculated from the Moore–Penrose pseudo inverse of the FIM [120]. To circumvent this difficulty and derive a bound with more practical value, we redefine Φ as the number of frequency bins that will be used by a practical algorithm and ϕ their indices. To avoid repeated definition, we use those notations in Section III with an additional subscript ϕ to represent the variable of the same definition at the ϕ-th frequency bin.

Let the overall data vector be $\tilde{\boldsymbol{x}} = \left[\bar{\boldsymbol{x}}_1^T, \bar{\boldsymbol{x}}_2^T, \ldots, \bar{\boldsymbol{x}}_\Phi^T\right]^T$ with $\bar{\boldsymbol{x}}_\phi = [\boldsymbol{x}_\phi^T(1), \boldsymbol{x}_\phi^T(2), \ldots, \boldsymbol{x}_\phi^T(N)]^T$. Its mean and covariance are denoted by $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Gamma}}$, respectively, which are functions of the extended unknown parameter vector $\tilde{\boldsymbol{\alpha}}$. Since uncorrelatedness is equivalent to independence under the joint Gaussian distribution, the p.d.f. of $\tilde{\boldsymbol{x}}$ can be expressed as

$$f(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{\alpha}}) = \prod_{\phi=1}^{\Phi} f(\bar{\boldsymbol{x}}_\phi; \boldsymbol{\alpha}_\phi). \tag{109}$$

Moreover,

$$\tilde{\boldsymbol{\mu}} = \left[\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T, \ldots, \boldsymbol{\mu}_\Phi^T\right]^T,$$

$$\tilde{\boldsymbol{\Gamma}} = \text{blkdiag}\left(\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \ldots, \boldsymbol{\Gamma}_\Phi\right), \tag{110}$$

where blkdiag$(\cdot)$ is the block diagonalization operation of matrices in the bracket. Hence, the wideband Slepian-Bangs formula can be derived as

$$\langle \tilde{\boldsymbol{B}}^{-1}(\tilde{\boldsymbol{\alpha}})\rangle_{i,j} = \sum_{\phi=1}^{\Phi}\left[\text{tr}\left(\boldsymbol{\Gamma}_\phi^{-1}\frac{\partial \boldsymbol{\Gamma}_\phi}{\partial\langle\tilde{\boldsymbol{\alpha}}\rangle_i}\boldsymbol{\Gamma}_\phi^{-1}\frac{\partial \boldsymbol{\Gamma}_\phi}{\partial\langle\tilde{\boldsymbol{\alpha}}\rangle_j}\right) + 2\text{Re}\left(\frac{\partial \boldsymbol{\mu}_\phi^H}{\partial\langle\tilde{\boldsymbol{\alpha}}\rangle_i}\boldsymbol{\Gamma}_\phi^{-1}\frac{\partial \boldsymbol{\mu}_\phi}{\partial\langle\tilde{\boldsymbol{\alpha}}\rangle_j}\right)\right]. \tag{111}$$

Equation (111) implies that the whole wideband CRB matrix can be calculated through the summation of FIMs at all frequencies, and then take the inverse. It should be noted that the derivative of $\boldsymbol{r}_\phi$ is w.r.t. $\tilde{\boldsymbol{\alpha}}^T$ instead of $\boldsymbol{\alpha}_\phi^T$. In general, $\tilde{\boldsymbol{\alpha}}$ incorporates all the elements in $\{\boldsymbol{\alpha}_\phi\}_{\phi=1}^{\Phi}$, some of which may be common. For example, if no *a priori* knowledge is available, the unknowns in $\boldsymbol{P}_\phi$ and $\boldsymbol{Q}_\phi$ vary with ϕ. In contrast, the DOAs, $\boldsymbol{\theta}$, are invariant across frequencies.

C. UNIFIED WIDEBAND CRB FRAMEWORK

1) DETERMINISTIC MODEL

Let $\tilde{\boldsymbol{\mu}}_{\text{det}}$ and $\tilde{\boldsymbol{\Gamma}}_{\text{det}}$ denote the mean and the covariance matrix of $\tilde{\boldsymbol{x}}$ under the deterministic model. Then, $\tilde{\boldsymbol{\mu}}_{\text{det}}$ and $\tilde{\boldsymbol{\Gamma}}_{\text{det}}$ are given by (110), with each frequency component taking the same form as in (9). The unknown parameter vector is expressed as

$$\tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{\alpha}}_{\text{det}} = \left[\boldsymbol{\theta}^T, \text{Re}\left(\tilde{\boldsymbol{s}}^T\right), \text{Im}\left(\tilde{\boldsymbol{s}}^T\right), \tilde{\boldsymbol{\sigma}}^T\right]^T,$$

where

$$\tilde{\boldsymbol{s}} = \left[\bar{\boldsymbol{s}}_1^T, \bar{\boldsymbol{s}}_2^T, \ldots, \bar{\boldsymbol{s}}_\Phi^T\right]^T, \quad \tilde{\boldsymbol{\sigma}} = \left[\boldsymbol{\sigma}_1^T, \boldsymbol{\sigma}_2^T, \ldots, \boldsymbol{\sigma}_\Phi^T\right]^T.$$

Starting from (111) and following the derivation steps in Section II-C, we can obtain the submatrix of the wideband FIM, which is associated with $\boldsymbol{\theta}$, Re$(\tilde{\boldsymbol{s}})$, and Im$\left(\tilde{\boldsymbol{s}}^T\right)$:

$$\tilde{\mathcal{F}} = 2\text{Re}\left\{\begin{bmatrix}\tilde{\boldsymbol{G}}_{\text{det}}^H \\ \tilde{\boldsymbol{\Delta}}_{\text{det}}^H \\ -\text{j}\tilde{\boldsymbol{\Delta}}_{\text{det}}^H\end{bmatrix}\left[\tilde{\boldsymbol{G}}_{\text{det}}, \tilde{\boldsymbol{\Delta}}_{\text{det}}, \text{j}\tilde{\boldsymbol{\Delta}}_{\text{det}}\right]\right\}, \tag{112}$$

where

$$\begin{aligned}\tilde{\boldsymbol{G}}_{\text{det}} &= \left[\bar{\boldsymbol{G}}_{\text{det},1}^T, \bar{\boldsymbol{G}}_{\text{det},2}^T, \ldots, \bar{\boldsymbol{G}}_{\text{det},\Phi}^T\right]^T, \\ \tilde{\boldsymbol{\Delta}}_{\text{det}} &= \text{blkdiag}(\bar{\boldsymbol{\Delta}}_{\text{det},1}, \bar{\boldsymbol{\Delta}}_{\text{det},2}, \ldots, \bar{\boldsymbol{\Delta}}_{\text{det},\Phi}).\end{aligned} \tag{113}$$

Assume that $K<M$, and then the unified framework for the DOA-related block of the wideband deterministic CRB is given by

$$\tilde{\boldsymbol{B}}_{\text{det}} = \frac{1}{2}\text{Re}\left(\tilde{\boldsymbol{G}}_{\text{det}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{\Delta}}_{\text{det}}}^\perp\tilde{\boldsymbol{G}}_{\text{det}}\right)^{-1}. \tag{114}$$

2) STOCHASTIC MODEL

Let $\tilde{\boldsymbol{\mu}}_{\text{sto}}$ and $\tilde{\boldsymbol{\Gamma}}_{\text{sto}}$ denote the mean and the covariance matrix of $\tilde{\boldsymbol{x}}$ under the stochastic model. Combining (110) and (17) gives $\tilde{\boldsymbol{\mu}}_{\text{sto}}$ and $\tilde{\boldsymbol{\Gamma}}_{\text{sto}}$, and the unknown parameter vector is expressed as

$$\tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{\alpha}}_{\text{sto}} = \left[\boldsymbol{\theta}^T, \tilde{\boldsymbol{p}}^T, \tilde{\boldsymbol{\sigma}}^T\right]^T,$$

where

$$\tilde{\boldsymbol{p}} = \left[\boldsymbol{p}_1^T, \boldsymbol{p}_2^T, \ldots, \boldsymbol{p}_\Phi^T\right]^T.$$

First, the nuisance parameters in $\tilde{\boldsymbol{\alpha}}_{\text{sto}}$ are permuted to make $\boldsymbol{p}_\phi^T$ adjacent to $\boldsymbol{\sigma}_\phi^T$. According to *Theorem* 1, this does not change the DOA-related block of the stochastic CRB. Then, applying the derivation steps in Section II-D yields

$$\tilde{\boldsymbol{B}}^{-1}(\tilde{\boldsymbol{\alpha}}_{\text{sto}}) = N\begin{bmatrix}\tilde{\boldsymbol{G}}_{\text{sto}}^H \\ \tilde{\boldsymbol{\Delta}}_{\text{sto}}^H\end{bmatrix}\left[\tilde{\boldsymbol{G}}_{\text{sto}}, \tilde{\boldsymbol{\Delta}}_{\text{sto}}\right], \tag{115}$$

where

$$\begin{aligned}\tilde{\boldsymbol{G}}_{\text{sto}} &= \left[\boldsymbol{G}_{\text{sto},1}^T, \boldsymbol{G}_{\text{sto},2}^T, \ldots, \boldsymbol{G}_{\text{sto},\Phi}^T\right]^T, \\ \tilde{\boldsymbol{\Delta}}_{\text{sto}} &= \text{blkdiag}(\boldsymbol{\Delta}_{\text{det},1}, \boldsymbol{\Delta}_{\text{det},2}, \ldots, \boldsymbol{\Delta}_{\text{det},\Phi}).\end{aligned} \tag{116}$$

According to the partitioned matrix inversion formula, the unified framework for the DOA-related block of the wideband stochastic CRB can be expressed as

$$\tilde{\boldsymbol{B}}_{\text{sto}} = \frac{1}{N}\left(\tilde{\boldsymbol{G}}_{\text{sto}}^H\boldsymbol{\Pi}_{\tilde{\boldsymbol{\Delta}}_{\text{sto}}}^\perp\tilde{\boldsymbol{G}}_{\text{sto}}\right)^{-1}. \tag{117}$$

Remark 5: The DOA-related block of the wideband FIM is a summation of all DOA-related blocks across frequencies. To express the summation as matrix multiplication, $\tilde{\boldsymbol{G}}_{\text{det}}$ and $\tilde{\boldsymbol{G}}_{\text{sto}}$ stack all frequency components following the column direction. On the contrary, those nuisance blocks are affected by each frequency component separately, so that $\tilde{\boldsymbol{\Delta}}_{\text{det}}$ and $\tilde{\boldsymbol{\Delta}}_{\text{sto}}$ are block-diagonal. Consequently, $\boldsymbol{\Pi}_{\tilde{\boldsymbol{\Delta}}_{\text{det}}}^\perp$ and $\boldsymbol{\Pi}_{\tilde{\boldsymbol{\Delta}}_{\text{sto}}}^\perp$ are block-diagonal. If the DOA-related block of the CRB at each frequency is well-defined, then (114) and (117) can be rewritten as

$$\tilde{\boldsymbol{B}}_{\text{det}} = \left(\sum_{\phi=1}^{\Phi}\boldsymbol{B}_{\text{det},\phi}^{-1}\right)^{-1}, \quad \tilde{\boldsymbol{B}}_{\text{sto}} = \left(\sum_{\phi=1}^{\Phi}\boldsymbol{B}_{\text{sto},\phi}^{-1}\right)^{-1}. \tag{118}$$

Equation (118) shows that the wideband CRB for DOAs depends on the CRBs for DOAs rather than for nuisance parameters at all frequencies of interest, both in the deterministic and stochastic cases.

D. CLOSED-FORM WIDEBAND CRB EXPRESSIONS WITH UNIFORM AND NONUNIFORM NOISES

Based on (118), the narrowband results in Section III can be naturally extended to the wideband scenario. Most wideband algorithms model the covariance matrix of the noise Fourier coefficients as either UN or NUN rather than AUN, because many realizations of the AUN model are developed for the temporal noise. Existing closed-form expressions for the wideband deterministic and stochastic CRBs are given below.

1) DETERMINISTIC MODEL

Result 12: Consider the case where $\{\boldsymbol{Q}_\phi\}_{\phi=1}^{\Phi}$ are expressed as in (27). Denote the DOA-related block of the wideband deterministic CRB as $\tilde{\boldsymbol{B}}_{\text{det}}^{\text{un}}$. If $K<M$, then the closed-form expression for $\tilde{\boldsymbol{B}}_{\text{det}}^{\text{un}}$ is given by

$$\tilde{\boldsymbol{B}}_{\text{det}}^{\text{un}} = \frac{1}{2N}\left\{\sum_{\phi=1}^{\Phi}\frac{1}{\sigma_\phi}\text{Re}\left[(\boldsymbol{A}'^H_\phi\boldsymbol{\Pi}^\perp_{\boldsymbol{A}_\phi}\boldsymbol{A}'_\phi)\circ\hat{\boldsymbol{P}}_\phi{}^T\right]\right\}^{-1}. \quad (119)$$

Result 13: Consider the case where $\{\boldsymbol{Q}_\phi\}_{\phi=1}^{\Phi}$ are expressed as in (30). Denote the DOA-related block of the wideband deterministic CRB as $\tilde{\boldsymbol{B}}_{\text{det}}^{\text{nun}}$. If $K<M$, then the closed-form expression for $\tilde{\boldsymbol{B}}_{\text{det}}^{\text{nun}}$ is given by

$$\tilde{\boldsymbol{B}}_{\text{det}}^{\text{nun}} = \frac{1}{2N}\left\{\sum_{\phi=1}^{\Phi}\text{Re}\left[(\bar{\boldsymbol{A}}'^H_\phi\boldsymbol{\Pi}^\perp_{\bar{\boldsymbol{A}}_\phi}\bar{\boldsymbol{A}}'_\phi)\circ\hat{\boldsymbol{P}}_\phi^T\right]\right\}^{-1}. \quad (120)$$

In many wideband algorithms developed for the deterministic model, the DFT is applied to the whole observation interval instead of the divided subintervals [5], [68], [76], so that (119) and (120) are modified by $N = 1$. *Result* 13 was first reported in [5]. The original derivation started from (111). All submatrices of the partitioned FIM were calculated and then the partitioned inversion formula was applied to reach (120). In particular, if the variances of the noise Fourier coefficients are uniform across sensors and frequencies, (119) and (120) will degenerate to the same result [5]. Another expression that resembles (119) was presented in [76], where the Fourier coefficients of the temporal colored noise are modeled as the frequency domain UN. The power spectrum density of the noise was denoted by $Q(f_\phi)\Phi f_s$ with f_s being the sampling frequency. The covariance matrix of the noise Fourier coefficients at the ϕ-th frequency was written as in (27), with $\sigma_\phi = Q(f_\phi)\Phi f_s$. The authors used the general CRB formula in (5) to obtain *Result* 12. However, detailed derivation was not given in the published paper, and we find that there was a missing coefficient $1/2$ in the CRB expression derived therein.

2) STOCHASTIC MODEL

As for the stochastic model, the covariance matrix of the Fourier coefficients of the source signals takes different forms according to whether the sources are known *a priori* to be spatially uncorrelated.

Result 14: Consider the case where $\{\boldsymbol{Q}_\phi\}_{\phi=1}^{\Phi}$ are expressed as in (27). Denote the DOA-related block of the wideband stochastic CRB as $\tilde{\boldsymbol{B}}_{\text{sto}}^{\text{un}}$. If $K<M$, then the closed-form expression for $\tilde{\boldsymbol{B}}_{\text{sto}}^{\text{un}}$ is given by

$$\tilde{\boldsymbol{B}}_{\text{sto}}^{\text{un}} = \frac{1}{2N}\left\{\sum_{\phi=1}^{\Phi}\frac{1}{\sigma_\phi}\text{Re}\left[(\boldsymbol{A}'^H_\phi\boldsymbol{\Pi}^\perp_{\boldsymbol{A}_\phi}\boldsymbol{A}'_\phi)\circ(\boldsymbol{P}_\phi\boldsymbol{A}_\phi^H\boldsymbol{R}_\phi^{-1}\boldsymbol{A}_\phi\boldsymbol{P}_\phi)^T\right]\right\}^{-1}. \quad (121)$$

Result 15: Consider the case where $\{\boldsymbol{Q}_\phi\}_{\phi=1}^{\Phi}$ are expressed as in (30). Denote the DOA-related block of the wideband stochastic CRB as $\tilde{\boldsymbol{B}}_{\text{sto}}^{\text{nun}}$. If $K<M$, then the closed-form expression for $\tilde{\boldsymbol{B}}_{\text{sto}}^{\text{nun}}$ is given by

$$\tilde{\boldsymbol{B}}_{\text{sto}}^{\text{nun}} = \frac{1}{N}\left(\sum_{\phi=1}^{\Phi}\boldsymbol{C}_\phi - \boldsymbol{D}_\phi\boldsymbol{F}_\phi^{-1}\boldsymbol{D}_\phi^H\right)^{-1}, \quad (122)$$

where $\boldsymbol{C}_\phi$, $\boldsymbol{D}_\phi$, and $\boldsymbol{F}_\phi$ are shown in (49), (50), and (51), respectively.

Result 16: Consider the case where the sources are known *a priori* to be spatially uncorrelated. Denote the DOA-related block of the wideband stochastic CRBs with $\{\boldsymbol{Q}_\phi\}_{\phi=1}^{\Phi}$ given by (27) and (30) as $\tilde{\boldsymbol{B}}_{\text{unc}}^{\text{un}}$ and $\tilde{\boldsymbol{B}}_{\text{unc}}^{\text{nun}}$, respectively. The closed-form expressions for $\tilde{\boldsymbol{B}}_{\text{unc}}^{\text{un}}$ and $\tilde{\boldsymbol{B}}_{\text{unc}}^{\text{nun}}$ are given by

$$\tilde{\boldsymbol{B}}_{\text{unc}}^{\text{un}} = \tilde{\boldsymbol{B}}_{\text{unc}}^{\text{nun}} = \frac{1}{N}\left(\sum_{\phi=1}^{\Phi}\boldsymbol{G}_{\text{sto},\phi}^H\boldsymbol{\Pi}^\perp_{\boldsymbol{\Delta}_{\text{sto},\phi}}\boldsymbol{G}_{\text{sto},\phi}\right)^{-1}, \quad (123)$$

with the same $\boldsymbol{G}_{\text{sto},\phi}$ and $\boldsymbol{V}_\phi$ shown in (64) and (66) but different $\boldsymbol{U}_\phi$ given by

$$\text{UN}: \boldsymbol{U}_\phi = \text{vec}(\boldsymbol{R}_\phi^{-1}),$$
$$\text{NUN}: \boldsymbol{U}_\phi = \boldsymbol{W}_\phi\boldsymbol{\mathcal{E}}_M.$$

Result 14 was first presented in [79] with $\boldsymbol{P}_\phi\boldsymbol{A}_\phi^H\boldsymbol{R}_\phi^{-1}\boldsymbol{A}_\phi\boldsymbol{P}_\phi$ written equivalently as $\boldsymbol{P}_\phi - (\boldsymbol{P}_\phi^{-1} + \frac{1}{\sigma_\phi}\boldsymbol{A}'^H_\phi\boldsymbol{A}_\phi)^{-1}$. However, the derivation was not given in the published paper. The expression in the UN case in *Result* 16 was derived in our previous work [81]. Note that *Result* 15 and the expression with NUN in *Result* 16 are new and not available in literature.

VI. FURTHER RESULTS BASED ON LINEAR ARRAYS

It is well-known that under the Gaussian distribution, the nonsingularity of the FIM, or the existence of the CRB, implies local identifiability of the unknown parameters [121]. On the other hand, a singular FIM indicates nonexistence of an unbiased estimator with finite variance [120]. As emphasized in the reviewed results, all the deterministic CRBs and the stochastic ones without *a priori* knowledge exist only if $K<M$, but the stochastic CRBs with the *a priori* knowledge of uncorrelated sources in *Results* 6, 11, and 16 may exist even if $K \geq M$.

The results above are applicable to various array geometries, as long as there is only one angular parameter to be estimated for each source. In practice, one of the most popular array geometries is the linear array located in the same plane with the sources. By specifying the array manifold matrix of a linear array, we can examine the explicit condition under which a particular CRB exists, and then discuss the number of resolvable Gaussian sources. In this section, we shall concentrate on the case of uncorrelated sources. We first review some existing results in the narrowband circular and wideband scenarios, and then present some supplementary results for the narrowband noncircular scenario, which have not been fully investigated before.

A. REVIEW OF THE CO-ARRAY CONCEPT

Consider a linear array consisting of M sensors and let d denote the unit inter-sensor spacing. Setting $0d$ as the reference, we can express the position of the m-th sensor as $z_m d$, $z_m \in \mathbb{R}$. Then, the array structure can be represented by a real set $\mathbb{A} = \{z_1, z_2, \ldots, z_M\}$.

Introduce the difference set of $\mathbb{A}$:

$$\mathbb{A}_{\text{diff}} = \{z_{m_1} - z_{m_2} | z_{m_1}, z_{m_2} \in \mathbb{A};\ m_1, m_2 = 1, 2, \ldots, M\}.$$

Let $\mathbb{D}$ collect all unique elements of $\mathbb{A}_{\text{diff}}$ in ascending order, and then $\mathbb{D}$ represents the difference co-array associated with $\mathbb{A}$. Denote the array manifold matrix of $\mathbb{D}$ as $\boldsymbol{A}_{\mathbb{D}} = [\boldsymbol{a}_{\mathbb{D}}(\theta_1), \boldsymbol{a}_{\mathbb{D}}(\theta_2), \ldots, \boldsymbol{a}_{\mathbb{D}}(\theta_K)] \in \mathbb{C}^{|\mathbb{D}|\times K}$, where $\boldsymbol{a}_{\mathbb{D}}(\theta_k)$ is the steering vector, and $|\cdot|$ is the cardinality of a set.

Similarly, the sum set of $\mathbb{A}$ is given by

$$\mathbb{A}_{\text{sum}} = \{z_{m_1} + z_{m_2} | z_{m_1}, z_{m_2} \in \mathbb{A};\ m_1, m_2 = 1, 2, \ldots, M\}.$$

Let $\mathbb{S}$ collect all unique elements of $\mathbb{A}_{\text{sum}}$ in ascending order, and then $\mathbb{S}$ represents the sum co-array associated with $\mathbb{A}$. The array manifold matrix of $\mathbb{S}$ is denoted by $\boldsymbol{A}_{\mathbb{S}} = [\boldsymbol{a}_{\mathbb{S}}(\theta_1), \boldsymbol{a}_{\mathbb{S}}(\theta_2), \ldots, \boldsymbol{a}_{\mathbb{S}}(\theta_K)] \in \mathbb{C}^{|\mathbb{S}|\times K}$, with $\boldsymbol{a}_{\mathbb{S}}(\theta_k)$ denoting the steering vector.

For narrowband signals with a central frequency f_0, the (m, k)-th element of the array manifold matrix for the linear array $\mathbb{A}$ can be explicitly written as

$$\langle \boldsymbol{A} \rangle_{m,k} = \mathrm{e}^{-\mathrm{j}2\pi \frac{d}{\lambda} z_m \sin(\theta_k)}, \tag{124}$$

where $\lambda = c/f_0$ is the signal wavelength, and $\mathrm{j} = \sqrt{-1}$ is the imaginary unit.

Substituting (124) into (62) and (97), we can write the $(\bar{m}, k)$-th element of $\boldsymbol{T}$ and $\boldsymbol{T}_{\text{c}}$, respectively, as

$$\begin{aligned} \langle \boldsymbol{T} \rangle_{\bar{m},k} &= \mathrm{e}^{-\mathrm{j}2\pi \frac{d}{\lambda}(z_{m_1} - z_{m_2})\sin(\theta_k)}, \\ \langle \boldsymbol{T}_{\text{c}} \rangle_{\bar{m},k} &= \mathrm{e}^{-\mathrm{j}2\pi \frac{d}{\lambda}(z_{m_1} + z_{m_2})\sin(\theta_k)}, \\ \bar{m} &= (m_2 - 1)M + m_1. \end{aligned} \tag{125}$$

Clearly, $\boldsymbol{T}$ and $\boldsymbol{T}_{\text{c}}$ are respectively associated with $\mathbb{A}_{\text{diff}}$ and $\mathbb{A}_{\text{sum}}$. The number of unique rows in $\boldsymbol{T}$ equals $|\mathbb{D}|$, whereas that for $\boldsymbol{T}_{\text{c}}$ equals $|\mathbb{S}|$.

Based on the co-array concept, the existence of the CRB can be interpreted as a rank condition for a particular matrix, as will be illustrated below.

B. RESULTS ON UNCORRELATED NARROWBAND CIRCULAR SIGNALS

The relationship between $\boldsymbol{A}_{\mathbb{D}}$ and $\boldsymbol{T}$ is given by [42]

$$\boldsymbol{A}_{\mathbb{D}} = \boldsymbol{J}_3 \boldsymbol{T}, \tag{126}$$

where $\boldsymbol{J}_3 \in \mathbb{R}^{|\mathbb{D}|\times M^2}$ is called the co-array selection matrix. Conversely, another useful relationship is given by [41]

$$\boldsymbol{T} = \boldsymbol{J}_4 \boldsymbol{A}_{\mathbb{D}}, \tag{127}$$

where $\boldsymbol{J}_4 \in \mathbb{R}^{M^2\times|\mathbb{D}|}$ is a binary matrix of full column rank. The function of $\boldsymbol{J}_4$ is twofold: it permutes the rows in $\boldsymbol{A}_{\mathbb{D}}$ and augments the row dimension from $|\mathbb{D}|$ to M^2 with $M^2 - |\mathbb{D}|$ duplicates of certain rows. By (127), we can rewrite $\boldsymbol{T}'$ in (65) as

$$\boldsymbol{T}' = \boldsymbol{J}_4 \boldsymbol{A}'_{\mathbb{D}}, \tag{128}$$

where

$$\boldsymbol{A}'_{\mathbb{D}} = \left[\frac{\partial \boldsymbol{a}_{\mathbb{D}}(\theta_1)}{\partial \theta_1}, \frac{\partial \boldsymbol{a}_{\mathbb{D}}(\theta_2)}{\partial \theta_2}, \ldots, \frac{\partial \boldsymbol{a}_{\mathbb{D}}(\theta_K)}{\partial \theta_K}\right].$$

In the presence of UN, it was shown in [41] that

$$\mathrm{vec}(\boldsymbol{I}_M) = \boldsymbol{J}_4 \boldsymbol{h}, \tag{129}$$

where $\boldsymbol{h} \in \mathbb{R}^{|\mathbb{D}|\times 1}$ satisfies $\langle \boldsymbol{h} \rangle_i = \delta_{\langle \mathbb{D} \rangle_i, 0}$, $\forall i = 1, 2, \ldots, |\mathbb{D}|$, with $\delta_{\langle \mathbb{D} \rangle_i, 0}$ denoting the Kronecker function. Substituting (128), (127), and (129) into (64), (66), and (67), respectively, yields

$$\begin{aligned} \boldsymbol{G}_{\text{sto}} &= \boldsymbol{W}\boldsymbol{J}_4 \boldsymbol{A}'_{\mathbb{D}} \boldsymbol{P}, \\ \boldsymbol{\Delta}_{\text{sto}} &= \boldsymbol{W}\boldsymbol{J}_4 \left[\boldsymbol{A}_{\mathbb{D}}, \boldsymbol{h}\right]. \end{aligned} \tag{130}$$

Using (130), we can rewrite (23) as the closed-form CRB expression derived in [41]. Since $\boldsymbol{\theta}$ is replaced by the normalized DOAs $\bar{\boldsymbol{\theta}} = \sin(\boldsymbol{\theta})d/\lambda$ therein, $\partial \boldsymbol{r}/\partial \boldsymbol{\theta}^T$ is updated to $\partial \boldsymbol{r}/\bar{\boldsymbol{\theta}}^T$.

Define the augmented co-array manifold (ACM) matrix as

$$\bar{\boldsymbol{A}}_{\text{ACM}} \triangleq \left[\boldsymbol{A}'_{\mathbb{D}}, \boldsymbol{A}_{\mathbb{D}}, \boldsymbol{h}\right] \in \mathbb{C}^{|\mathbb{D}|\times(2K+1)}. \tag{131}$$

It has been proved in [41] that $\boldsymbol{B}^{\text{un}}_{\text{sto}}$ exists if and only if $\bar{\boldsymbol{A}}_{\text{ACM}}$ has full column rank. This rank condition is necessarily true when the number of columns in $\bar{\boldsymbol{A}}_{\text{ACM}}$ is no larger than that of rows, which leads to an upper bound on the number of resolvable Gaussian sources by a given difference co-array:

$$\text{UN}:\ K \leq \frac{|\mathbb{D}| - 1}{2}. \tag{132}$$

In particular, assume that $\{z_m\}_{m=1}^{M} \in \mathbb{Z}$ with $\mathbb{Z}$ denoting the integer set, and then the central segment of $\mathbb{D}$, which contains consecutive integers, is linked to a uniform linear array (ULA) represented by $\mathbb{U}$. If $K \leq (|\mathbb{U}| - 1)/2$, then $\boldsymbol{B}^{\text{un}}_{\text{sto}}$ is guaranteed to exist for an arbitrary set of distinct DOAs. Meanwhile, in the region $(|\mathbb{U}| - 1)/2 < K \leq (|\mathbb{D}| - 1)/2$, whether $\boldsymbol{B}^{\text{un}}_{\text{sto}}$ exists or not depends on specific DOAs [41].

Remark 6: For a ULA whose first sensor is located at $0d$, $|\mathbb{D}| = |\mathbb{U}| = 2M - 1$, and then (132) becomes $K \leq M - 1$. This indicates that even if the sources are known *a priori* to be uncorrelated, it is infeasible to resolve more Gaussian sources than sensors based on a ULA. Therefore, a NULA structure is indispensable to underdetermined DOA estimation for narrowband circular signals [39], [40].

To the best of our knowledge, the rank condition in the presence of NUN or AUN has not been studied yet, but we can carry out a similar discussion. In general, the FIM is nonsingular if and only if $[\partial \boldsymbol{r}/\partial \boldsymbol{\theta}^T, \partial \boldsymbol{r}/\partial \boldsymbol{p}^T, \partial \boldsymbol{r}/\partial \boldsymbol{\sigma}^T]$ has full column rank, which requires the submatrix $[\partial \boldsymbol{r}/\partial \boldsymbol{\theta}^T, \partial \boldsymbol{r}/\partial \boldsymbol{p}^T] =$

$[\boldsymbol{T}'\boldsymbol{P}, \boldsymbol{T}] = \boldsymbol{J}_4[\boldsymbol{A}'_{\mathbb{D}}\boldsymbol{P}, \boldsymbol{A}_{\mathbb{D}}]$ to have full column rank. Since $\boldsymbol{J}_4$ is of full-column rank and $\boldsymbol{P}$ is nonsingular, this leads to a restriction on the column rank of the reduced ACM matrix defined as $\boldsymbol{A}_{\mathrm{ACM}} \triangleq \left[\boldsymbol{A}'_{\mathbb{D}}, \boldsymbol{A}_{\mathbb{D}}\right] \in \mathbb{C}^{|\mathbb{D}|\times 2K}$, which has full column rank only if

$$\text{Noiseless: } K \le \frac{|\mathbb{D}|}{2}. \tag{133}$$

Note that this coincides with the result in [40], where the power of UN is assumed to be known.

In the presence of NUN, it can be verified that augmenting $\boldsymbol{J}_4[\boldsymbol{A}'_{\mathbb{D}}\boldsymbol{P}, \boldsymbol{A}_{\mathbb{D}}]$ with $\partial\boldsymbol{r}/\partial\boldsymbol{\sigma}^T = \boldsymbol{\mathcal{E}}_M$ will generate $M-1$ and M linearly independent rows and columns, respectively. A detailed examination in [44] shows that the rank condition holds if and only if

$$\text{NUN}: \; K \le \frac{|\mathbb{D}|-1}{2}, \tag{134}$$

which is the same as the case of UN.

In the presence of AUN, the rank condition depends on $\partial\boldsymbol{r}/\partial\boldsymbol{\sigma}^T = \boldsymbol{\mathcal{Q}}$, which can lead to different upper bounds on the number of resolvable Gaussian sources based on different noise models. In particular, in the presence of partially unknown noise, the noise covariance matrix can be constructed by truncating the Fourier series [7], [8], whose basis matrices are Toeplitz ones. Therefore, some columns in $\boldsymbol{\mathcal{Q}}$ may be linearly dependent, leading to a tighter upper bound than those in (132) and (134).

C. RESULTS ON UNCORRELATED NARROWBAND NONCIRCULAR SIGNALS

For noncircular signals, $\boldsymbol{T}_{\mathrm{c}}$ is related to $\boldsymbol{A}_{\mathbb{S}}$ through the following relationships [94]:

$$\begin{aligned}\boldsymbol{T}_{\mathrm{c}} &= \boldsymbol{J}_5\boldsymbol{A}_{\mathbb{S}},\\ \boldsymbol{T}'_{\mathrm{c}} &= \boldsymbol{J}_5\boldsymbol{A}'_{\mathbb{S}},\end{aligned} \tag{135}$$

where $\boldsymbol{J}_5 \in \mathbb{R}^{M^2\times|\mathbb{S}|}$ is a binary matrix of full column rank constructed similarly as $\boldsymbol{J}_4$, and

$$\boldsymbol{A}'_{\mathbb{S}} = \left[\frac{\partial\boldsymbol{a}_{\mathbb{S}}(\theta_1)}{\partial\theta_1}, \frac{\partial\boldsymbol{a}_{\mathbb{S}}(\theta_2)}{\partial\theta_2}, \ldots, \frac{\partial\boldsymbol{a}_{\mathbb{S}}(\theta_K)}{\partial\theta_K}\right].$$

Substitute (127), (128), and (135) into (102), and then define the reduced ACM matrix for complex noncircular Gaussian signals as

$$\dot{\boldsymbol{A}}_{\mathrm{ACM}} \triangleq \begin{bmatrix} \boldsymbol{J}_4\boldsymbol{A}'_{\mathbb{D}}\boldsymbol{P} & \boldsymbol{J}_4\boldsymbol{A}_{\mathbb{D}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{J}_5{\boldsymbol{A}'_{\mathbb{S}}}^*\boldsymbol{P}_{\mathrm{c}}^* & \boldsymbol{0} & \boldsymbol{J}_5\boldsymbol{A}_{\mathbb{S}}^* & -\mathrm{j}\boldsymbol{J}_5\boldsymbol{A}_{\mathbb{S}}^* \\ \boldsymbol{J}_5\boldsymbol{A}'_{\mathbb{S}}\boldsymbol{P}_{\mathrm{c}} & \boldsymbol{0} & \boldsymbol{J}_5\boldsymbol{A}_{\mathbb{S}} & \mathrm{j}\boldsymbol{J}_5\boldsymbol{A}_{\mathbb{S}} \\ \boldsymbol{J}_4{\boldsymbol{A}'_{\mathbb{D}}}^*\boldsymbol{P} & \boldsymbol{J}_4\boldsymbol{A}_{\mathbb{D}}^* & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Following the proof in [41], it can be proved that if the noise covariance matrix is known, then the CRB exists if and only if $\dot{\boldsymbol{A}}_{\mathrm{ACM}}$ has full column rank.

Notice that the elements in $\mathbb{D}$ are symmetric w.r.t. zero, so that $\boldsymbol{A}_{\mathbb{D}}$ and $\boldsymbol{A}_{\mathbb{D}}^*$ contain the same $|\mathbb{D}|$ rows in reverse order. Left-multiplying $\boldsymbol{J}_4$ and $\boldsymbol{J}_5$ produces $M^2-|\mathbb{D}|$ and $M^2-|\mathbb{S}|$ repeated rows, respectively. Therefore, $\dot{\boldsymbol{A}}_{\mathrm{ACM}}$ has $|\mathbb{D}|+2|\mathbb{S}|$ linearly independent rows in total, while the number of columns is $4K$. Thus, $\dot{\boldsymbol{A}}_{\mathrm{ACM}}$ has full column rank only if

$$\text{Noiseless}: K \le \frac{|\mathbb{D}|+2|\mathbb{S}|}{4}, \tag{136}$$

which provides an upper bound on the number of resolvable noncircular Gaussian sources by a given linear array.

In particular, if the sources are all circular, $\dot{\boldsymbol{A}}_{\mathrm{ACM}}$ will degenerate to

$$\dot{\boldsymbol{A}}_{\mathrm{ACM}}|_{\mathrm{circular}} = \begin{bmatrix} \boldsymbol{J}_4\boldsymbol{A}'_{\mathbb{D}}\boldsymbol{P} & \boldsymbol{J}_4\boldsymbol{A}_{\mathbb{D}} \\ \boldsymbol{J}_4{\boldsymbol{A}'_{\mathbb{D}}}^*\boldsymbol{P} & \boldsymbol{J}_4\boldsymbol{A}_{\mathbb{D}}^* \end{bmatrix},$$

which contains $|\mathbb{D}|$ linearly independent rows and $2K$ columns. In this case, (136) is transformed into (133).

Comparing (133) with (136), we can see that more noncircular sources than circular ones can possibly be resolved based on the same linear array, due to the additional sensor positions in the virtual sum co-array. In other words, since noncircular signals carry extra information, a nonuniform linear array is theoretically no longer indispensable for resolving more noncircular Gaussian sources than the number of physical sensors.

In the presence of UN, NUN, and AUN, the corresponding ACM matrices are constructed by augmenting $\dot{\boldsymbol{A}}_{\mathrm{ACM}}$ with $[\mathrm{vec}(\boldsymbol{I}_M)^T, \boldsymbol{0}, \boldsymbol{0}, \mathrm{vec}(\boldsymbol{I}_M)^T]^T$, $[\boldsymbol{\mathcal{E}}_M^T, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\mathcal{E}}_M^T]^T$, and $[\boldsymbol{\mathcal{Q}}^T, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\mathcal{Q}}^H]^T$, respectively. Therefore, (136) should be modified accordingly. Extending the results in (132) and (134), we find that in the presence of UN or NUN, the upper bound is

$$\text{UN or NUN:} \quad K \le \frac{|\mathbb{D}|+2|\mathbb{S}|-1}{4}. \tag{137}$$

However, the upper bound in the AUN case cannot be obtained accurately. The results in this subsection are original in this paper.

D. RESULTS ON UNCORRELATED WIDEBAND SIGNALS

In the wideband scenario, the (m,k)-th element of the array manifold matrix at the ϕ-th frequency bin is expressed as

$$\langle\boldsymbol{A}_\phi\rangle_{m,k} = \mathrm{e}^{-\mathrm{j}2\pi\frac{d}{\lambda_\phi}z_m\sin(\theta_k)},$$

where the signal wavelength is $\lambda_\phi = c/f_\phi$ with f_ϕ denoting the central frequency for the ϕ-th frequency bin.

All the results on the difference co-array in Section VI-B can be directly extended to the ϕ-th frequency bin. In the following, we present the condition under which $\tilde{\boldsymbol{B}}_{\mathrm{unc}}^{\mathrm{un}}$ exists.

Theorem 3: Define the wideband ACM matrix as

$$\tilde{\boldsymbol{A}}_{\mathrm{ACM}} \triangleq \left[\tilde{\boldsymbol{A}}'_{\mathbb{D}}\bar{\boldsymbol{P}}, \tilde{\boldsymbol{A}}_{\mathbb{D}}, \tilde{\boldsymbol{h}}\right] \in \mathbb{C}^{|\mathbb{D}|\Phi\times(K+K\Phi+\Phi)}, \tag{138}$$

where

$$\begin{aligned}\tilde{\boldsymbol{A}}'_{\mathbb{D}} &= \mathrm{blkdiag}(\boldsymbol{A}'_{\mathbb{D},1}, \boldsymbol{A}'_{\mathbb{D},2}, \ldots, \boldsymbol{A}'_{\mathbb{D},\Phi}),\\ \tilde{\boldsymbol{A}}_{\mathbb{D}} &= \mathrm{blkdiag}(\boldsymbol{A}_{\mathbb{D},1}, \boldsymbol{A}_{\mathbb{D},2}, \ldots, \boldsymbol{A}_{\mathbb{D},\Phi}),\\ \bar{\boldsymbol{P}} &= [\boldsymbol{P}_1^T, \boldsymbol{P}_2^T, \ldots, \boldsymbol{P}_\Phi^T]^T, \quad \tilde{\boldsymbol{h}} = \boldsymbol{I}_\Phi\otimes\boldsymbol{h}.\end{aligned} \tag{139}$$

Then, the $\tilde{\boldsymbol{B}}_{\text{unc}}^{\text{un}}$ exists if and only if $\tilde{\boldsymbol{A}}_{\text{ACM}}$ has full column rank, i.e., if and only if

$$\text{rank}(\tilde{\boldsymbol{A}}_{\text{ACM}}) = K + K\Phi + \Phi. \tag{140}$$

Proof: See Appendix C. ■

Obviously, (140) holds true only if $K + K\Phi + \Phi \leq |\mathbb{D}|\,\Phi$, yielding

$$\text{UN: } K \leq \frac{\Phi}{\Phi + 1}(|\mathbb{D}| - 1). \tag{141}$$

The upper bound on the number of resolvable Gaussian sources in (141) is more relaxed than that in (132), since $\Phi/(\Phi+1) \geq 1/2,\ \forall \Phi \geq 1$. It can be inferred that more wideband Gaussian sources than narrowband ones can be resolved based on a given linear array, including both the uniform and nonuniform ones. Moreover, in contrast to the narrowband circular scenario mentioned in *Remark* 6, it is possible to conduct underdetermined DOA estimation for wideband sources without the assistance of a nonuniform linear array [81].

VII. CONCLUSION

A number of closed-form Gaussian CRB expressions for DOA estimation under various model assumptions were reviewed under a unified framework, with some new supplementary results reported. The reviewed results cover three scenarios: narrowband complex circular signals, narrowband complex noncircular signals, and wideband signals. In each scenario, three source signal models (the deterministic model, the stochastic model, and the stochastic model with the *a priori* knowledge of uncorrelated sources), and three Gaussian noise models (UN, NUN, and AUN) were considered. Starting from the Slepian-Bangs formula, a closed-form deterministic/stochastic CRB framework was developed according to the simplest derivation in the literature, based on which a class of closed-form deterministic CRB expressions and two classes of stochastic ones were directly derived under different noise models.

Comparisons were conducted among these CRB expressions, leading to a series of equalities and order relationships which show that: 1) The deterministic CRB under different noise models can be identical in some special cases, whereas more unknown parameters always lead to a larger stochastic CRB. 2) Under the same noise model, the circular and noncircular deterministic CRBs are always identical, whereas the noncircular stochastic CRB is upper bounded by the circular one, but they can be identical when the signals are actually circular. 3) The wideband deterministic/stochastic CRB for DOAs depends on the narrowband CRBs for DOAs rather than for nuisance parameters at all frequency components.

The deterministic CRBs and the stochastic ones without *a priori* knowledge exist only in the overdetermined case, regardless of the array geometry. However, those stochastic CRBs employing the *a priori* knowledge of uncorrelated sources can exist in the underdetermined case. In each scenario, the rank condition under which this kind of stochastic CRB exits was examined based on a linear array, with the upper bound on the number of resolvable Gaussian sources deduced. For narrowband circular signals, the virtual difference co-array plays an important role in resolving more sources than the number of physical sensors. In addition to the difference co-array, the sum co-array, which is associated with the covariance matrix of the conjugate part of noncircular signals, is able to further improve the source resolvability. Similar improvement can be offered by the information within multiple frequency components for wideband signals.

APPENDIX A
PROOF OF THEOREM 1

We introduce the following lemma [20, Lemma A4] to carry out the proof.

Lemma 1: Consider a positive definite matrix $\boldsymbol{\mathcal{A}} \in \mathbb{C}^{w\times w}$, which is partitioned as $\boldsymbol{\mathcal{A}} = \begin{bmatrix} \boldsymbol{\mathcal{A}}_1 & \boldsymbol{\mathcal{A}}_2 \\ \boldsymbol{\mathcal{A}}_2^H & \boldsymbol{\mathcal{A}}_3 \end{bmatrix}$, where $\boldsymbol{\mathcal{A}}_1 \in \mathbb{C}^{v\times v}$. Let $\boldsymbol{\mathcal{B}} \in \mathbb{C}^{w\times u}$ be another partitioned matrix such that $\boldsymbol{\mathcal{B}} = [\boldsymbol{\mathcal{B}}_1^T, \boldsymbol{\mathcal{B}}_2^T]^T$, where $\boldsymbol{\mathcal{B}}_1 \in \mathbb{C}^{v\times u}$. We have

$$\boldsymbol{\mathcal{B}}^H \boldsymbol{\mathcal{A}}^{-1} \boldsymbol{\mathcal{B}} \succeq \boldsymbol{\mathcal{B}}_1^H \boldsymbol{\mathcal{A}}_1^{-1} \boldsymbol{\mathcal{B}}_1,$$

The equality holds if and only if $\boldsymbol{\mathcal{B}}_2 - \boldsymbol{\mathcal{A}}_2^H \boldsymbol{\mathcal{A}}_1^{-1} \boldsymbol{\mathcal{B}}_1 = \boldsymbol{0}$.

Adding $\boldsymbol{\omega}$ to $\boldsymbol{p}$ or $\boldsymbol{\sigma}$ produces an extended matrix $\boldsymbol{\Delta}_{\text{ext}}$ satisfying $\boldsymbol{\Delta}_{\text{ext}}\boldsymbol{O}_1 = [\boldsymbol{\Delta}_{\text{sto}}, \boldsymbol{\Omega}]$, where $\boldsymbol{\Omega} = \boldsymbol{W}\partial \boldsymbol{r}/\partial \boldsymbol{\omega}^T$, and $\boldsymbol{O}_1$ is a permutation matrix satisfying $\boldsymbol{O}_1^{-1} = \boldsymbol{O}_1^T$ [122, p. 32]. Note that this property of a permutation matrix will be used elsewhere. By *Lemma* 1, we have

$$\begin{aligned}
\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{ext}}\boldsymbol{O}_1} &= \boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{ext}}} \\
&= \boldsymbol{I} - [\boldsymbol{\Delta}_{\text{sto}}, \boldsymbol{\Omega}] \begin{bmatrix} \boldsymbol{\Delta}_{\text{sto}}^H \boldsymbol{\Delta}_{\text{sto}} & \boldsymbol{\Delta}_{\text{sto}}^H \boldsymbol{\Omega} \\ \boldsymbol{\Omega}^H \boldsymbol{\Delta}_{\text{sto}} & \boldsymbol{\Omega}^H \boldsymbol{\Omega} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\Delta}_{\text{sto}}^H \\ \boldsymbol{\Omega}^H \end{bmatrix} \\
&\preceq \boldsymbol{I} - \boldsymbol{\Delta}_{\text{sto}} \left(\boldsymbol{\Delta}_{\text{sto}}^H \boldsymbol{\Delta}_{\text{sto}}\right)^{-1} \boldsymbol{\Delta}_{\text{sto}}^H = \boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{sto}}}.
\end{aligned} \tag{A.1}$$

The equality holds true if and only if $\boldsymbol{\Omega}^H \boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{sto}}} = \boldsymbol{0}$. Since $\boldsymbol{\Omega} \neq \boldsymbol{0}$ and $\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{sto}}} \neq \boldsymbol{0}$, the equality holds true if and only if $\boldsymbol{\Omega}$ lies in the range space of $\boldsymbol{\Delta}_{\text{sto}}$. The linear dependence among $\boldsymbol{\omega}, \boldsymbol{p}$, and $\boldsymbol{\sigma}$ indeed satisfies this condition. As a result, we have $\boldsymbol{G}_{\text{sto}}^H(\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{ext}}} - \boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{sto}}})\boldsymbol{G}_{\text{sto}} \preceq \boldsymbol{0}$. Therefore, *Theorem* 1 (a) and (b) follows from (23) directly. Furthermore, the first equation in (A.1) also implies that permuting nuisance parameters does not change $\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Delta}_{\text{sto}}}$, which proves *Theorem* 1 (c). This completes the whole proof.

APPENDIX B
PROOF OF THEOREM 2

Recall the FIM for noncircular signals in (85). According to [123], the following relationship holds:

$$\begin{aligned}
&\dot{\boldsymbol{R}}^T \otimes \dot{\boldsymbol{R}} \\
&= \dot{\boldsymbol{J}}_2 \left(\dot{\boldsymbol{R}}^T \circledast \dot{\boldsymbol{R}}\right) \dot{\boldsymbol{J}}_2 \\
&= \dot{\boldsymbol{J}}_2 \begin{bmatrix} \boldsymbol{R}^T \otimes \boldsymbol{R} & \boldsymbol{R}^T \otimes \boldsymbol{R}_{\text{c}} & \boldsymbol{R}_{\text{c}}^* \otimes \boldsymbol{R} & \boldsymbol{R}_{\text{c}}^* \otimes \boldsymbol{R}_{\text{c}} \\ \boldsymbol{R}^T \otimes \boldsymbol{R}_{\text{c}}^* & \boldsymbol{R}^T \otimes \boldsymbol{R}^T & \boldsymbol{R}_{\text{c}}^* \otimes \boldsymbol{R}_{\text{c}}^* & \boldsymbol{R}_{\text{c}}^* \otimes \boldsymbol{R}^T \\ \boldsymbol{R}_{\text{c}} \otimes \boldsymbol{R} & \boldsymbol{R}_{\text{c}} \otimes \boldsymbol{R}_{\text{c}} & \boldsymbol{R} \otimes \boldsymbol{R} & \boldsymbol{R} \otimes \boldsymbol{R}_{\text{c}} \\ \boldsymbol{R}_{\text{c}} \otimes \boldsymbol{R}_{\text{c}}^* & \boldsymbol{R}_{\text{c}} \otimes \boldsymbol{R}^T & \boldsymbol{R} \otimes \boldsymbol{R}_{\text{c}}^* & \boldsymbol{R} \otimes \boldsymbol{R}^T \end{bmatrix} \dot{\boldsymbol{J}}_2,
\end{aligned} \tag{B.1}$$

where Ⓣ denotes the block Kronecker product. From (99), we can get

$$\frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{\alpha}}_{\text{sto}}^T} = \dot{\boldsymbol{J}}_2 \boldsymbol{O}_4 \left[\frac{\partial \boldsymbol{r}^T}{\partial \dot{\boldsymbol{\alpha}}_{\text{sto}}^T}, \frac{\partial \boldsymbol{r}^H}{\partial \dot{\boldsymbol{\alpha}}_{\text{sto}}^T}, \frac{\partial \boldsymbol{r}_c^T}{\partial \dot{\boldsymbol{\alpha}}_{\text{sto}}^T}, \frac{\partial \boldsymbol{r}_c^H}{\partial \dot{\boldsymbol{\alpha}}_{\text{sto}}^T} \right]^T = \dot{\boldsymbol{J}}_2 \boldsymbol{O}_4 \left[\boldsymbol{r}'^T, \boldsymbol{r}'^H, \boldsymbol{r}_c'^T, \boldsymbol{r}_c'^H \right]^T, \tag{B.2}$$

where $\boldsymbol{O}_4 \in \mathbb{R}^{4M^2 \times 4M^2}$ denotes a permutation matrix which interchanges the second block with the fourth one. Using (B.1), (B.2), we can rewrite (85) as

$$\boldsymbol{B}^{-1}(\dot{\boldsymbol{\alpha}}_{\text{sto}}) = \frac{N}{2} \begin{bmatrix} \boldsymbol{r}' \\ \boldsymbol{r}'^* \\ \boldsymbol{r}_c' \\ \boldsymbol{r}_c'^* \end{bmatrix}^H \dot{\boldsymbol{R}}_{\textcircled{\tiny T}}^{-1} \begin{bmatrix} \boldsymbol{r}' \\ \boldsymbol{r}'^* \\ \boldsymbol{r}_c' \\ \boldsymbol{r}_c'^* \end{bmatrix}, \tag{B.3}$$

where

$$\dot{\boldsymbol{R}}_{\textcircled{\tiny T}} = \begin{bmatrix} \boldsymbol{R}^T \otimes \boldsymbol{R} & \boldsymbol{R}_c^* \otimes \boldsymbol{R}_c & \boldsymbol{R}_c^* \otimes \boldsymbol{R} & \boldsymbol{R}^T \otimes \boldsymbol{R}_c \\ \boldsymbol{R}_c \otimes \boldsymbol{R}_c^* & \boldsymbol{R} \otimes \boldsymbol{R}^T & \boldsymbol{R} \otimes \boldsymbol{R}_c^* & \boldsymbol{R}_c \otimes \boldsymbol{R}^T \\ \boldsymbol{R}_c \otimes \boldsymbol{R} & \boldsymbol{R} \otimes \boldsymbol{R}_c & \boldsymbol{R} \otimes \boldsymbol{R} & \boldsymbol{R}_c \otimes \boldsymbol{R}_c \\ \boldsymbol{R}^T \otimes \boldsymbol{R}_c^* & \boldsymbol{R}_c^* \otimes \boldsymbol{R}^T & \boldsymbol{R}_c^* \otimes \boldsymbol{R}_c^* & \boldsymbol{R}^T \otimes \boldsymbol{R}^T \end{bmatrix}.$$

For the asserted case, we have $\boldsymbol{P}_c = \boldsymbol{0}$ and $\boldsymbol{R}_c = \boldsymbol{0}$. Consequently, (B.3) becomes

$$\boldsymbol{B}^{-1}(\dot{\boldsymbol{\alpha}}_{\text{sto}}) = N\boldsymbol{r}'^H \left(\boldsymbol{R}^{-T} \otimes \boldsymbol{R}^{-1} \right) \boldsymbol{r}' + N\boldsymbol{r}_c'^H \left(\boldsymbol{R}^{-1} \otimes \boldsymbol{R}^{-1} \right) \boldsymbol{r}_c'. \tag{B.4}$$

We can always permute the nuisance parameters to partition the overall unknown parameter vector as $\dot{\boldsymbol{\alpha}}_{\text{sto}} = [\boldsymbol{\alpha}_{\text{sto}}^T, \boldsymbol{\beta}^T]^T$. By *Theorem* 1, this does not affect the DOA-related block of the stochastic CRB. Therefore, we have

$$\boldsymbol{r}' = \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\alpha}_{\text{sto}}^T}, \boldsymbol{0} \right], \quad \boldsymbol{r}_c' = \left[\boldsymbol{0}, \frac{\partial \boldsymbol{r}_c}{\partial \boldsymbol{\beta}^T} \right]. \tag{B.5}$$

Substituting (B.5) into (B.4) and taking the inverse yields (104). The proof is complete.

APPENDIX C
PROOF OF THEOREM 3

A very similar rank condition was proved in [81, Theorem 1]. It differs from *Theorem* 3 in this paper only in the definition of the wideband ACM matrix. The definition in [81] is

$$\tilde{\boldsymbol{A}}_{\text{acm}} \triangleq \left[\tilde{\boldsymbol{T}}' \bar{\boldsymbol{P}}, \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{i}} \right] \in \mathbb{C}^{M^2\Phi \times (K+K\Phi+\Phi)}, \tag{C.1}$$

where

$$\tilde{\boldsymbol{T}}' = \text{blkdiag}(\boldsymbol{T}_1', \boldsymbol{T}_2', \ldots, \boldsymbol{T}_\Phi'),$$
$$\tilde{\boldsymbol{T}} = \text{blkdiag}(\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_\Phi),$$
$$\tilde{\boldsymbol{i}} = \boldsymbol{I}_\Phi \otimes \text{vec}(\boldsymbol{I}_M).$$

Using (127), (128), and (129), we can rewrite $\tilde{\boldsymbol{T}}'$, $\tilde{\boldsymbol{T}}$, and $\tilde{\boldsymbol{i}}$ as

$$\tilde{\boldsymbol{T}}' = \tilde{\boldsymbol{J}}_4 \tilde{\boldsymbol{A}}_{\mathbb{D}}', \quad \tilde{\boldsymbol{T}} = \tilde{\boldsymbol{J}}_4 \tilde{\boldsymbol{A}}_{\mathbb{D}}, \ \tilde{\boldsymbol{i}} = \tilde{\boldsymbol{J}}_4 \tilde{\boldsymbol{h}},$$

where $\tilde{\boldsymbol{J}}_4 = \boldsymbol{I}_\Phi \otimes \boldsymbol{J}_4$.

The relationship between the two ACM matrices defined in [81] and in this paper is given by

$$\tilde{\boldsymbol{A}}_{\text{acm}} = \tilde{\boldsymbol{J}}_4 \tilde{\boldsymbol{A}}_{\text{ACM}}. \tag{C.2}$$

Since $\boldsymbol{J}_4$ has full column rank, so does $\tilde{\boldsymbol{J}}_4$. When $\tilde{\boldsymbol{A}}_{\text{ACM}}$ is left-multiplied by $\tilde{\boldsymbol{J}}_4$, its rank is unchanged. Therefore, the proof of the rank condition in [81] is preserved in this paper. The proof is complete.

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