Proceedings Paper:

Baldivieso Monasterios, P.R. orcid.org/0000-0003-1518-1177 and Konstantopoulos, G.C. (2021) Constrained control for microgrids with constant power loads. In: Proceedings of 2020 59th IEEE Conference on Decision and Control (CDC). 2020 59th IEEE Conference on Decision and Control (CDC), 14-18 Dec 2020, Jeju Island, Korea (South), Online conference. IEEE , pp. 3341-3346. ISBN 9781728174488

https://doi.org/10.1109/CDC42340.2020.9304345

Constrained control for microgrids with constant power loads

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***Abstract*— This paper analyses the reachability properties of Microgrids connected to constant power loads subject to input and state constraints. Constraint requirements in microgrids are either inherent, *i.e.* inverter modulation indices limitations, or imposed, such as current and voltage limitation which are safety critical. In this paper, we propose an analysis of the controllability properties of a microgrid using set theoretic notions; this analysis sheds light on the constraint admissibility properties of a microgrid with constant power loads in terms of constraint satisfaction and robustness to changes in power demands. Lastly, we provide a method of recasting the original nonlinear microgrid control problem into controlling a linear system subject to bounded additive disturbances and output constraints.**

I. Introduction

This paper is concerned with with the reachability properties for Constant Power Load (CPL) in ac systems using set invariance. This type of loads can be found, given the latest advances in Microgrid (MG) technology, in devices that are tightly regulated such as the ones controlled using power inverters as mentioned in [1]. In general, inverter based MGs have acquired a widespread use and are the key enablers of the smart-grid concept. In this setting, constraints arise naturally in the form of actuator limits, *i.e.* modulation indices and current capacities, see for example [2]. Among the most successful tools developed that excels in the analysis of constrained systems is set invariance, see the excellent survey [3]. Different concepts of invariance can be found in the literature for different classes of systems; these concepts have a close link to control Lyapunov functions and stability notions [4].

On the other hand, CPLs have been thoroughly studied within the context of dc systems; voltage dynamics of such load when connected to a current source and in parallel to a capacitance and admittance are

$$C\frac{dV}{dt} = -gV - \frac{P}{V} + i.$$

A relevant question associated to these dynamics is the existence of long term equilibrium which have been studied in [5] and [6]. It is easy to see, when linearising, that a CPL introduces negative impedance in the resulting circuit which may cause a voltage collapse in the system. The voltage stability properties of networks with nonlinear loads have been studied in [2], and dates back to the works of [7]. An efficient method for controlling and counteracting this negative impedance has been proposed in [8] using passivity based control. These existing approaches, however, do not take constraints into consideration. The concepts of set invariance, then, offer avenues of research to study the effect of constraints on a MG. Set invariance for both deterministic and uncertain cases, have been widely studied for linear systems. In recent years, several attempts have been made to understand the nonlinear case: [9] and [10] proposed methods for computing maximal invariant sets for nonlinear dynamics emphasising the potential non-convexities that a nonlinear approach may have. In particular, the ideas of [4] regarding the boundary structure of invariant sets of nonlinear systems characterise the structure of these sets. Several new advances have applied set invariance concepts, such as the maximal positive invariant set, to power system transients, see [11]. These set invariance methods provide the necessary tools to obtain certificates of stability and constraint satisfaction that are safety critical. These ideas of set invariance, to the best of the author's knowledge, have not been applied in the context of MGs with CPLs. The presence of this type of loads require tight regulation to keep voltages within pre-specified regions, in terms of magnitude and angles. The problem of CPL stabilisation is further exacerbated with the presence of input constraints, in form of current limitations. The contributions and structure of this paper are the following:

- In section III, we propose a novel invariance inducing control law for a node with a $3-$phase ac CPL. We exploit the structure of the load dynamics and its reachable sets to guarantee that the latter is always contained within the original constraint set. The proposed control law is set-valued in nature, and we explore its properties in terms of robustness to load demand changes, available selection properties. Section II contains the MG model together with the necessary concepts of continuous and discrete reachable sets.
- In section IV, we leverage on the properties of the proposed set-valued control law to recast the MG control problem. We also show the multi-stable nature of a MG when connected to CPLs. Using set invariance on the load side allows us to bound the voltages at the load side, provided the line currents are chosen according the proposed control law. Therefore the remaining MG dynamics can be understood as, *i.e.* inverter filters and lines, as a linear system subject to bounded disturbances and output constraints. Section V illustrates the proposed controller properties when applied to several CPLs.

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Notation: For an index set $\mathcal{I}$, x_i denotes a quantity indexed by an element $i \in \mathcal{I}$, whereas upper case subscripts $x_{\mathcal{I}}$ denote concatenation of elements $\{x_i\}_{i\in\mathcal{I}}$. The $2-$norm is denoted $|x| = \|x\|_2$. A MG can be seen as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the set of nodes $\mathcal{V}$ represent a collection of inverters $\mathcal{V}_I$ and loads $\mathcal{V}_L$ such that $\mathcal{V} = \mathcal{V}_I \cup \mathcal{V}_L$ and $\mathcal{V}_I \cap \mathcal{V}_L = \emptyset$; the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ defines the MG topology, characterised by the node-edge matrix $\mathcal{B} \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{V}|}$ which for edge $e = (i, j) \in \mathcal{E}$ involving nodes i and j can be defined as $[\mathcal{B}]_{ei} = 1$ if node i is the source of $e \in \mathcal{E}$, and $[\mathcal{B}]_{ej} = -1$ if node j is its sink, and zero otherwise. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted $\mathbb{I}_n$, the complex structure in $\mathbb{R}^{2n\times 2n}$ is

$$\mathbb{J}_{2n} = \begin{bmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix}.$$

A $C-$set is a compact and convex set, a $PC-$set is a C-set set with the origin in its nonempty interior.

II. Preliminaries

A. Microgrid Model

In this section, we aim to describe the model of a MG composed of $|\mathcal{V}_I|$ $3-$phase power inverters connected to $|\mathcal{V}_L|$ loads via inductive lines. Each inverter, which is considered to be a controllable voltage source, is interfaced with the rest of the network via LC-filters; the lines are predominantly inductive and are represented as L filters with parasitic resistances; while loads can be seen as CPLs in parallel to an admittance. We consider a dq model in the global reference frame rotating at a constant frequency ω^{c} such that

$$L_i \frac{di_i}{dt} = -Z_i i_i - v_i + \bar{v}_i \tag{1a}$$

$$C_i \frac{dv_i}{dt} = -Y_i v_i + i_i - \mathcal{B}_i^\top i_E \tag{1b}$$

$$L_e \frac{di_e}{dt} = -Z_e i_e + \mathcal{B}_I v_I + \mathcal{B}_L v_L \tag{1c}$$

$$C_l \frac{dv_l}{dt} = -Y_l v_l - h(S_l, v_l) - \mathcal{B}_l^\top i_E. \tag{1d}$$

These different MG components can be characterised as a feedback interconnection. In this model, $v_i \in \mathbb{R}^2$ and $i_i \in \mathbb{R}^2$ represent the filter voltage and current for all $i \in \mathcal{V}_I$; similarly, $i_e \in \mathbb{R}^2$ and $v_l \in \mathbb{R}^2$ represent line currents and load voltages for all $e \in \mathcal{E}$ and $l \in \mathcal{V}_L$ respectively. The passive components, *i.e.* impedances and admittances, can be defined as $Z_\mu = r_\mu \mathbb{I}_2 + \omega^{\mathrm{c}} L_\mu \mathbb{J}_2$ and $Y_\mu = g_\mu \mathbb{I}_2 + \omega^{\mathrm{c}} C_\mu \mathbb{J}_2$ for all $\mu \in \mathcal{V}_I \cup \mathcal{V}_L \cup \mathcal{E}$ with resistances, inductances, conductances and capacitances given by r_μ, L_μ, g_μ, and C_μ respectively. The interconnection between inverters and loads with lines can be characterised using the node-edge incidence matrix $\mathcal{B}$; $\mathcal{B}_I$ and $\mathcal{B}_L$ are columns associated with inverter and load nodes. The nonlinearity introduced by the load characteristics $S_l = (P_l, Q_l)$ is

$$h(S_l, v_l) = \frac{2}{3}(P_l \mathbb{I}_2 - Q_l \mathbb{J}_2)\frac{v_l}{v_l^\top v_l}$$

with P_l and Q_l are the active and reactive power demands for load $l \in \mathcal{V}_L$. The actuation point of the MG corresponds to the inverter output voltage $\bar{v}_i$ for all $i \in \mathcal{V}_I$. Implicitly, we consider in (1d) that the CPL current dynamics operate at a slower time-scale for example if considering thermostatically controlled loads. The overall state, control, and disturbances are $x = (i_I, v_I, i_E, v_L)$, $u = \bar{v}_I$, and $d = S_L$, a short hand notation for (1) is $\dot{x} = f_c(x, u, d)$. We invoke the following assumption with respect to the MG topology:

Assumption 1: For a MG defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,

i) $\mathcal{G}$ has a single connected component.
ii) $[\mathcal{B}]_{el} = -1$ for each $l \in \mathcal{V}_L$, $e = (i, l) \in \mathcal{E}$, and $i \in \mathcal{V}_I$.

The above assumption implies that no two loads are connected to each other with a distribution line; the only connections in the network are between inverters and loads. The MG is subject to hard actuation and operational constraints constraints, *i.e.* inverter modulation indices, regions of operation in terms of load voltages, and current capacity for distribution lines, *i.e.* $u \in \mathbb{U}$, $x \in \mathbb{X}$. Furthermore, the load demand $d\colon \mathbb{R} \to \mathbb{D}$ is a priori unknown but with values in a given set $d(t) \in \mathbb{D}$.

Assumption 2 (System constraints): The MG constraint sets satisfy

i) The constraint sets $\mathbb{U}_i \subset \mathbb{R}^2$ for all $i \in \mathcal{V}_I$, $\mathbb{X}_\mu \subset \mathbb{R}^2$ for all $\mu \in \mathcal{V} \cup \mathcal{E}$ are polytopic $C-$sets.
ii) For each $l \in \mathcal{V}_L$, the load demand satisfies $d_l \in \mathcal{D}_l = \{d_l \colon \mathbb{R} \to \mathbb{D}_l \colon d \text{ measurable}\}$ with $\mathbb{D}_l$ is a $PC-$set.

Remark 1: The overall state constraint set $\mathbb{X} = \prod_{\mu\in\mathcal{V}\cup\mathcal{E}} \mathbb{X}_\mu$ can be computed using a cartesian product which implies that there are no coupled constraints present in the MG. Following Assumption 1, the set of measurable disturbances is also the cartesian product of the sets corresponding to the individual loads, *i.e.* $\mathcal{D} = \prod_{l\in\mathcal{V}_L} \mathcal{D}_l$.

B. Reachability sets

Control sets are an important tool in the analysis and study of controllability in nonlinear systems. In particular, we are interested in the region of attraction of given target sets. The first concept is that of reachable sets as defined in [12].

Definition 1: For a set $\mathcal{X} \subset \mathbb{R}^n$ and a control system $\dot{x} = f_c(x, u, d)$ with $\varphi(t, x, u, d)$ denoting its solution, or flow, at time t for a control input $u \in \mathbb{U}$ and load demand $d \in \mathcal{D}$; the *Reachable set* from any $x \in \mathcal{X}$ up to time $T > 0$ is $\mathcal{O}^+_{\leq T}(x) = \{\varphi(t, x, u, d) \in \mathbb{R}^n \colon u \in \mathbb{U},\ t \in [0, T]\}$. Furthermore, the reachable set from $x \in \mathbb{R}^n$ is $\mathcal{O}^+(x) = \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x)$. These sets are hard to compute in practice, because the flow corresponding to (1) can rarely be obtained in explicit form. A way to circumvent these issues is to use a discretisation[1] of (1) to then obtain discrete versions of its reachable sets. The resulting nonlinear discrete time dynamics

$$x^+ = f(x, u, d) \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^p$ represent the state, input, and disturbances for the discretised MG; x^+ is the successor state. The state, control and disturbance dimensions are $n = 4|\mathcal{V}_I| + 2|\mathcal{E}| + 2|\mathcal{V}_L|$, $m = 2|\mathcal{V}_I|$, and $p = 2|\mathcal{V}_L|$ respectively. The function $f\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is an absolutely

[1] In the context of our paper, any method for discretising the ODE may be applied. The simplest case for this being an Euler discretisation, for further detail we direct the reader to [13, Chapter 11].

continuous function. Using this discrete approximation we can define the reachable operator from a target set $\mathcal{X} \subset \mathbb{R}^n$ for a given disturbance instantiation $d \in \mathbb{D}$:

$$\mathfrak{R}(\mathcal{X}, d) = \{f(x, u, d) \in \mathbb{R}^n : x \in \mathcal{X},\ u \in \mathbb{U}\}. \tag{3}$$

These operators allow us to define the $k-$step $d-$reachable sets via the following recursions:

$$\mathcal{R}_d^{k+1} = \mathfrak{R}(\mathcal{R}_e^k, d), \quad \mathcal{R}_\cdot^0 = \mathcal{X}, \tag{4}$$

for any $d, e \in \mathbb{D}$. In the above relations, we have considered, with a slight abuse of notation, different values of disturbances in $\mathbb{D}$. In other words, by fixing a $\tilde{d} \in \mathcal{D}$, for (4) and step $k+1$, we have $d = \tilde{d}(k) \in \mathbb{D}$ and for step k, the disturbance is $e = \tilde{d}(k-1) \in \mathbb{D}$. This procedure allows us to define a correspondence between disturbances and sequences of sets $\tilde{d} \mapsto \{\mathcal{R}^k_{\tilde{d}(k)}\}_{k=0}^{\infty}$. The initial set of this sequence is not affected by the disturbance which justifies the notation $\mathcal{R}_\cdot^0$. We note that these $k-$step reachable sets (4) define discrete positive orbits passing through the set $\mathcal{X} \subset \mathbb{R}^n$. Furthermore, the discretised dynamics satisfy in general $f(x, u, d) = \varphi(t + T_s, t, x, u, d)$ with constant controls in the interval $[t, t + T_s]$. These discrete orbits and its continuous counterparts are related via the sampling time $T_s > 0$.

Lemma 1: For each $x \in \mathcal{X}$ and $\epsilon > 0$ there exists $T_s > 0$ such that $\text{dist}_H(\mathcal{O}^+(x), \mathcal{R}_d^\infty) < \epsilon$.[2]

We refer the reader to [12, Appendix C] for a proof. Before proceeding, we recall pertinent definitions from the literature regarding set invariance for discrete time systems, see [3] for further details:

Definition 2: A set $\Omega \subset \mathbb{R}^n$ is Robust Control invariant set if for the controlled system $x^+ = f(x, u, d)$ with $d \in \mathbb{D}$ if for any $x \in \Omega$, there exists a control action $u \in \mathbb{U}$ such that for all $d \in \mathbb{D}$, the successor state satisfies $x^+ \in \Omega$.

III. Reachability analysis

In this section, we propose a controller that uses set invariance and reachability concepts to guarantee constraint satisfaction for each load $l \in \mathcal{V}_L$. From a practical standpoint, one of the objectives of controlling a MG is to supply the load with adequate levels of voltage and frequency, while coping with the corresponding power demands.

A. Load controller

From Assumption 2 each load is constrained to a C-set which can be represented by n_l inequalities, *i.e.* $\mathbb{X}_l = \{v_l \in \mathbb{R}^2 : \Xi_{xl} v_l \leq \xi_{xl}\}$ with $\Xi_{xl} \in \mathbb{R}^{n_{xl} \times 2}$ and $\xi_{xl} \in \mathbb{R}^{n_{xl}}$. On the other hand, from Assumption 1, the load dynamics have a diagonal structure, which allow a separate analysis of (1d) considering the inputs $i_l = -\mathcal{B}_l^\top i_E$ such that

$$C_l \frac{dv_l}{dt} = -Y_l v_l - h(d_l, v_l) + i_l = f_l(v_l, i_l, d_l).$$

We are interested in finding a control law $\kappa(\cdot)$ such that $\mathbb{X}_l$ becomes an invariant set. To this aim, consider a constraint admissible steady state pair $(v_l^{\text{ss}}, i_l^{\text{ss}}) \in \mathbb{X}_l \times \mathbb{U}_l$ such that the translated state, $\mathbb{X}_l^t = \mathbb{X}_l - v_l^{\text{ss}}$, and control $\mathbb{U}_l^t = \mathbb{U}_l - i_l^{\text{ss}}$ sets are PC-sets; the input constraint set arises from the constraints imposed on the the line currents connected to the l^{th} load, *i.e.* $\mathbb{U}_l = \bigoplus_{e \in \mathcal{E}_l} \mathbb{X}_e$ with $\mathcal{E}_l = \{e : (\mu, l) \in \mathcal{E}, \forall \mu \in \mathcal{V}\}$. Similarly, using the coordinate transformation $x_l = v_l - v_l^{ss}$ and $u_l = v_l - v_l^{ss}$ results in

$$C_l \dot{x} = f_l^t(x_l, u_l, d_l) = f_l(x_l + v_l^{ss}, u_l + i_l^{ss}, d_l) \tag{5}$$

with the origin as an equilibrium point. The desired control law $\kappa(\cdot)$ forces the closed loop vector field orbits $f_l^t(x_l, \kappa(x_l, d_l), d_l)$ to be contained within $\mathbb{X}_l$ at all times $t \geq 0$ and for all load demands $d_l(t) \in \mathcal{D}_l$.

As mentioned in Section II-B, the computation of orbits for nonlinear systems is problematic, in this case the source of this problem is $h(\cdot, \cdot)$; however, leveraging on Lemma 1, we can compute reachable sets for the discrete approximation of (5). Given a sampling time $\tilde{T}_s > 0$, an Euler discretisation yields $x_l^+ = x_l + \tilde{T}_s C_l^{-1} f_l^t(x_l, u_l, d_l)$. Any such discretisation induces computation errors that are linked with the choice of the sampling time. To avoid any such error to incur in the computation of the orbits, we further impose the following condition on the desired control law $x_l + T_s f_l^t(x_l, \kappa(x_l, d_l), d_l) \in \lambda \mathbb{X}_l$ where[3] $\lambda \in [0, 1]$ and $T_s = \tilde{T}_s C_l^{-1}$. For a given state $x \in \mathbb{X}_l$ and a $d_l \in \mathbb{D}_l$, the control action forcing the one-step reachable set to satisfy $\mathfrak{R}(x_l, d_l) \subset \lambda \mathbb{X}_l$ can be characterised by

$$\left.\begin{array}{ll} \Xi_{xl}(x_l + T_s f(x_l, u_l, d_l)) & \leq \lambda \xi_{xl} \\ \Xi_{ul} u & \leq \xi_{ul} \end{array}\right\} u_l \in \mathcal{U}_l(x_l, d_l) \tag{6}$$

yielding a set valued control law $\mathcal{U}_l : \mathbb{X}_l \times \mathbb{D}_l \rightrightarrows \mathbb{U}_l$. Before proceeding, we state a property of the function $g : \mathbb{X}_l \times \mathbb{D}_l \to \mathbb{R}^n$ which will be useful in subsequent developments.

Lemma 2: Suppose Assumption 2 holds, then for $d \in \mathbb{D}_l$

$$g(x_l, d_l) = (\mathbb{I}_2 - T_s Y_l) x_l + T_s(w_l - h(x_l, d_l))$$

with $w_l = Y_l v_l^{\text{ss}} + i_l^{\text{ss}}$ is locally Lipschitz in $\mathbb{X}_l$.

We, now, explore the properties of our proposed control law:

Theorem 3: For a fixed $d_l \in \mathbb{D}_l$, $\mathcal{U}_l(x_l, d_l)$ is nonempty and a $PC-$set for all $x \in \mathbb{X}_l$ if

$$|u| \geq \max_{y \in \mathbb{X}_l} \min_{z \in \lambda \mathbb{X}_l} |g(y, d_l) - z|^2.$$

Proof: The proof is constructive, for an $x \in \mathbb{X}_l$, its associated one-step reachable set, which is a convex set by construction, is $\mathfrak{R}(x_l, d_l) = g(x_l, d_l) \oplus T_s \mathbb{U}_l$. We can distinguish three cases:

i) $g(x_l, d_l) \in \text{int}\, \mathbb{X}_l$: Since $\text{int}\, \mathbb{X}_l$ is an open set, it contains a neighbourhood $\mathcal{N}(g(x_l, d_l))$. In particular the largest ball contained in $\lambda \mathbb{X}_l$ centred at $g(x_l, d_l)$ has a radius $\rho_l = d(g(x_l, d_l), \lambda \partial \mathbb{X}_l) > 0$. As a result, a set of controls belonging to the closure of $u_l \in \overline{\mathcal{N}}(0, \rho_l) \subseteq T_s \mathbb{U}_l$ ensures $x_l^+ \in \lambda \mathbb{X}_l$.

[2]The Hausdorff distance between any two sets $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ is $\text{dist}_H(\mathcal{C}, \mathcal{D}) = \max\{\sup_{x \in \mathcal{C}} \inf_{y \in \mathcal{D}} |x - y|, \sup_{y \in \mathcal{D}} \inf_{x \in \mathcal{C}} |x - y|\}$

[3]The goal of λ is to ensure a degree of "robustness" with respect to approximation and discretisation errors.

ii) $g(x_l, d_l) \in \lambda\partial\mathbb{X}_l$: the recession cone $\mathcal{R}_C(g(x_l, d_l))$ of $\lambda\mathbb{X}_l$ contains the set of directions $\eta \in \mathbb{R}^2$ for which $g(x_l, d_l) + t\eta \in \lambda\mathbb{X}_l$, and its closure satisfies

$$\overline{\mathcal{R}_{\lambda\mathbb{X}_l}}(g(x_l, d_l)) = \liminf_{t\to 0} \frac{\lambda\mathbb{X}_l - g(x_l, d_l)}{t}.$$

Since the state constraint set is a $PC-$set, then it can be arbitrarily approximated by a polyhedral set composed of n_l inequalities with normal vectors Ξ_{xl}. Since $g(x_l, d_l)$ lies in the boundary, then there are $\tilde{n}_{lx} > 0$ inequalities that are active, *i.e.* $\tilde{\Xi}_{xl} g(x_l, d_l) = \tilde{\xi}_{xl}$ with $\tilde{\Xi}_{xl} \in \mathbb{R}^{\tilde{n}_l \times 2}$ and $\tilde{\xi}_{xl} \in \mathbb{R}^{\tilde{n}_l}$. By definition of boundary, for any point $z \in \lambda\partial\mathbb{X}_l$ there exists a neighbourhood $\mathcal{N}$ of z such that $\mathcal{N} \cap \lambda\mathbb{X}_l \neq \emptyset$. Similarly to the previous case, we leverage on the fact that $\mathbb{U}_l$ is a $PC-$set; the intersection $T_s\mathbb{U}_l \cap \overline{\mathcal{R}_C}(g(x_l, d_l))$ is nonempty which yields a nonempty control law $\mathcal{U}_l(x_l) \neq \emptyset$.

iii) $g(x_l, d_l) \notin \lambda\partial\mathbb{X}_l$. By the second geometric form of the Hahn-Banach theorem, given $\lambda\mathbb{X}_l$ is a $PC-$set, there exists a linear functional $\Gamma\colon \mathbb{R}^2 \to \mathbb{R}$ separating $\lambda\mathbb{X}_l$ and $g(x_l, d_l)$, *i.e.* $\Gamma(g(x_l, d_l)) > 0$ and $\Gamma(y) \leq 0$ for all $y \in \lambda\mathbb{X}_l$. On the other hand, $s \in \lambda\mathbb{X}_l$ is the projection of $g(x_l, d_l)$ onto $\lambda\mathbb{X}_l$, such that $\gamma = g(x_l, d_l) - s$ is a normal vector to the hyperplane generated by $\Gamma(\cdot)$. If the hypothesis on the size of controls is satisfied, then $|T_s u_l| \geq |g(x_l, d_l)| - |s|$ such that $g(x_l, d_l) + T_s u_l \in \lambda\mathbb{X}_l$. ∎

The control invariance of $\mathbb{X}_l$ follows as a corollary.

Corollary 1: Given a constant $d_l \in \mathbb{D}_l$. The constraint set $\mathbb{X}_l$ is a control invariant set for $\dot{x} = f_l^t(x_l, u_l, d_l)$ with $u_l \in \mathcal{U}_l(x_l) \subseteq \mathbb{U}_l$.

Proof: For a state $x_l \in \mathbb{X}_l$, by Theorem 3, its associated control law satisfies $\mathcal{U}_l(x_l) \neq \emptyset$. The successor state satisfies $x_l^+ \in \lambda\mathbb{X}_l$ which implies $\mathfrak{R}(x_l, d_l) \subset \mathbb{X}_l$. Assuming $\mathcal{R}_d^k(\mathbb{X}_l) \subset \mathbb{X}_l$, then it is clear that $\mathcal{R}_d^{k+1}(\mathbb{X}_l) \subset \mathbb{X}_l$ is also contained within $\mathbb{X}_l$. As a result, $\mathcal{R}_d^\infty(\mathbb{X}_l) \subset \mathbb{X}_l$ holds by induction; the discrete orbit is therefore contained within the constraint set. Our claim follows from a direct application of Lemma 1. ∎

B. Variation in the power demand

Theorem 3 ensures the proposed control law is nonempty as long as the control effort available is larger than the drift caused by the CPL dynamics. This raises an interesting question, how much of a change in power demand can the controller handle before $\mathcal{U}(x_l) = \emptyset$?

Consider a state $x_l \in \mathbb{X}_l$ and a constant power demand $d_l \in \mathbb{D}_l$ for which a control action u_l exists within $\mathcal{U}_l(x_l, d_l)$. When the power demand is different from the one estimated, *i.e.* $x_l^+ = x_l + T_s f_l^t(x_l, \kappa(x_l, d_l), d_l + \delta_l)$. Given that the variation in load demand belongs to a set $\delta_l \in \Delta_l$ such that $d_l \oplus \Delta_l \subset \mathbb{D}_l$. Then, the action of the power demand on the load dynamics can be represented as

$$h(x_l, d_l + \delta_l) = h(x_l, d_l) + \underbrace{\frac{2}{3}\frac{1}{v_l^\top v_l}\begin{bmatrix} v_{d,l} & v_{q,l} \\ v_{q,l} & -v_{d,l}\end{bmatrix}}_{\Lambda(x_l + v_l^{\mathrm{ss}})} \delta_l.$$

The state dependent transformation $\Lambda(x_l)$ depends only on x_l since $v_l = x_l + v_l^{\mathrm{ss}}$. As a result, the successor state lies in

$$x_l^+ \in \mathfrak{R}(x_l, d_l) \oplus T_s\Lambda(x_l)\Delta_l. \quad (7)$$

One of our aims is to characterise the largest $\Delta_l \subset \mathbb{D}_l$ such that $\mathfrak{R}(x_l, d_l) \oplus \Lambda(x_l)\Delta_l \subseteq \mathbb{X}_l$. Following set manipulations, we can readily obtain a condition that guarantees robustness against variations in d_l, $\Delta_l \subset \big(T_s\Lambda(x_l)\big)^{-1}\big(\mathbb{X}_l \sim \mathfrak{R}(x_l, d_l)\big)$.[4]

The above relation is well defined since $\mathfrak{R}(x_l, d_l) \subseteq \mathbb{X}_l$ by construction. The amount variation in load power demand is clearly state dependant, in fact the amount of variation depends proportionally on $T_s^{-1}|x_l|$. We are interested in finding the largest possible $\Delta_l \subset \mathbb{D}_l$ such that (7) holds for all $x_l \in \mathbb{X}_l$.

Proposition 1: Suppose Assumption 2 holds, then the set $\mathbb{X}_l$ is robust control invariant for $x_l^+ = x_l + T_s f_l^t(x_l, u_l, d_l + \delta_l)$ if the variation in power demand satisfies

$$\Delta_l = \frac{1-\lambda}{T_s} \bigcap_{x_l \in \mathbb{X}_l} \Lambda(x_l)\mathbb{X}_l \quad (8)$$

The variations defined in Proposition (1) are conservative, since the set Δ_l considers only a worst-case scenario. As mentioned previously, the control law can handle different variation magnitudes depending on the state. If, on the other hand, a bounded variation is known *a priori*; then it is possible to find a subset of the constraint set $\mathbb{X}_l$ that is robust to the given set of variations.

C. Further properties of $\mathcal{U}_l(\cdot, \cdot)$

In this section, we explore further properties of the set-valued feedback control law $\mathcal{U}(\cdot)$. As a first step, we investigate the continuity properties of such map which ensures the existence of a selection map $\kappa\colon \mathbb{X}_l \times \mathbb{D}_l \to \mathbb{U}_l$ with $\kappa_l(x_l, d_l) \in \mathcal{U}(x_l, d_l)$. Central to this developments is the concept of lower semicontinuous maps:

Definition 3 (Lower semicontinuity [14]): A set-valued map $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous at $x \in \mathbb{R}^n$ iff for any $y \in F(x)$ and for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$ such that $x_n \to x$, there exists a sequence of elements $y_n \in F(x_n) \to y$.

Theorem 4: Suppose Assumption 2 holds. For a fixed power demand $d_l \in \mathbb{D}_l$, the set valued map $\mathcal{U}_l(\cdot, d_l)$ is lower semicontinuous.

Proof: Consider a sequence of states $\{x_{l,n}\}_{n\in\mathbb{N}}$ that converges to x_l, and a control action $u_l \in \mathcal{U}_l(x_l, d_l)$. The images $\mathcal{U}_l(x_{l,n}, d_l)$ are not empty by virtue of Theorem 3; these images are characterised by a set of inequalities, see (6), and is a $PC-set$. On the other hand, by definition of a converging sequence, for any open set $\mathcal{A} \subset \mathbb{R}^n$ with $x_l \in \mathcal{A}$, then $\exists N_x > 0$ such that $x_n \in \mathcal{A}$ for all $n \geq N_x$. The sequence of states $\{x_{l,n}\}_{n\in\mathbb{N}}$ generate a sequence of sets $\{\mathcal{U}_l(x_{l,n}, d_l)\}_{n\in\mathbb{N}}$; the limit inferior of which, see [14, Proposition 1.1.2], contains the limits of all sequences $\{u_{n,l}\}_{n\in\mathbb{N}} \subseteq \mathbb{U}_l$ such that $u_{n,l} \in \mathcal{U}_l(x_{n,l} d_l)$. What remains to be proven is[5] $\liminf_{n\to\infty} \mathcal{U}_l(x_{l,n}, d_l) = \mathcal{U}_l(x_l, d_l)$.

[4] Given two $C-$sets $\mathcal{A}$ and $\mathcal{B}$, the Pontryagin difference of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \sim \mathcal{B} = \{x \in \mathcal{A} \colon x + b \in \mathcal{A}, \forall b \in \mathcal{B}\}$.

[5] The limit inferior of a sequence of sets $\{K_n\}_{n\in\mathbb{N}}$ can be characterised by $\liminf_{n\to\infty} K_n = \bigcap_{\epsilon>0} \bigcup_{N>0} \bigcap_{n\geq N} (K_n \oplus \epsilon\mathbb{B})$ with $\mathbb{B}$ the unit ball [14].

Any $u \in \mathcal{U}_l(x,d)$ satisfies for each $i \in \{1,\dots,n_{xl}\}$ the inequality $|T_s\Xi_{xl,i}^\top u_l| \leq \lambda|\xi_{xl,i}| + |\Xi_{xl,i}||g(x_l,d_l)|$ holds. From Lemma 2, $g(x_l,d_l)$ is Lipschitz with constant $L \geq 0$, this implies that $|T_s\Xi_{xl,i}^\top u_{n,l}| \leq \lambda|\xi_{xl,i}|+|\Xi_{xl,i}||g(x_{l,n},d_l)|+|\Xi_{xl,i}|L\varepsilon$ with $|x - x_{l,n}| \leq \varepsilon$. As a result $u \in \mathcal{U}_l(x_{l,n},d_l) \oplus \varepsilon\mathbb{B}$.

For the other inclusion, for any $u \in \liminf_{n\to\infty} \mathcal{U}_l(x_{l,n},d_l)$, then $|u - u_{l,n}| \leq \varepsilon_u$ for some $N_u > 0$ or $u \in \mathcal{U}_l(x_{l,n},d_l) \oplus \mathbb{B}\varepsilon_u$. The assertion follows from, *mutatis mutandis*, the previous argument. From Definition 3, we conclude that the map $\mathcal{U}(\cdot,d_l)$ is lower semicontinuous. ■

The existence of a continuous selection follows:

Corollary 2: Suppose Assumption 2 holds. There exists a continuous control law $\kappa_l \colon \mathbb{X}_l \to \mathbb{U}_l$ such that $\kappa_l(x_l) \in \mathcal{U}_l(x_l,d_l)$.

The proof follows from applying the celebrated Michael selection theorem, see [15], to the set-valued map $\mathcal{U}_l(\cdot,d_l)$ Leveraging on the fact that $\mathcal{U}(\cdot,d_l)$ is lower semicontinuous, we propose the selection based on minimising the distance of the successor state x^+ to the origin. To this aim, the following optimisation problem defines the desired control law

$$\mathbb{P}_l\colon\ \min\{\frac{1}{2}|x_l + T_s f_l^t(x_l,u_l,d_l)|^2 \colon u_l \in \mathcal{U}(x,d)\} \tag{9}$$

The solution of this problem yields a control action $u_l^* \in \mathcal{U}(x,d_l)$ such that the continuous selection is $\kappa_l(x_l,d_l) = u_l^*$. This controller can be expressed as a nonlinear complementary problem of the form

$$\kappa_l(x_l,d_l) = \frac{1}{T_s^2}\tilde{\Xi}_{ul}^\top\eta - \frac{1}{T_s}g(x_l,d_l)$$

$$0 \leq \eta \perp \tilde{\xi}_{ul}(x_l,d_l) - \tilde{\Xi}_{ul}\kappa_l(x_l,d_l) \geq 0$$

where η is a Lagrange multiplier that is related on the cost of operating the CPL, the matrix $\tilde{\Xi}_{ul} = [T_s\Xi_{xl}^\top \ \ \Xi_{ul}^\top]^\top$ defines the normal vectors of $\mathcal{U}_l(\cdot,\cdot)$, and $\tilde{\xi}_{ul}(x_l,d_l) = [(\lambda\xi_{xl} - \Xi_{xl}g(x_l,d_l))^\top \ \ \xi_{ul}^\top]^\top$ define the corresponding half spaces.

IV. Interconnection with a MG

In this section we leverage on the results of Section III-A to propose a new control setting for a MG that guarantees constraint satisfaction at the loads and lines. We first proceed to the analysis of the equilibria and controllability of a MG.

A. Analysis of Equilibria

One of our objectives is to analyse the equilibrium points of the interconnection of (1a)–(1c) and (1d). The following proposition characterises these points.

Proposition 2 (MG Equilibria): Suppose Assumption 1 holds. Then, the MG has $2^{|\mathcal{V}_L|}$ equilibrium points.

Proof: We begin noting that system (1) can be partitioned into a linear part given by (i_I,v_I,i_E) and a nonlinear part corresponding to the loads v_L such that the steady state of the latter can be rewritten as:

$$0 = (Y_l + \mathcal{B}_l^\top Z_E^{-1}\mathcal{B}_l)v_l + \underbrace{(\mathcal{B}_l^\top Z_E^{-1}\mathcal{B}_I v_I)}_{i_l} + h(S_l,v_l) \tag{10}$$

where $i_E = Z_E^{-1}(\mathcal{B}_I v_I + \mathcal{B}_L v_L)$, and i_l can be interpreted as input current to the l^{th} load. Using the complex representation of admittances, voltages, currents and powers[6], the equilibrium pairs $(v_l^{\text{ss}}, i_l^{\text{ss}})$ for (10) can be characterised for each $l \in \mathcal{V}_L$ by

$$i_l^{\text{ss}}(S_l) = \left(Y_l + \frac{2S_l^*}{3|v_l^{\text{ss}}|^2}\right)v_l^{\text{ss}}.$$

where the complex representation of the nonlinearity is $h(S_l,v_l) = \frac{2}{3}\frac{S_l^*}{v_l^*}$. However, for a given i_l as computed above, there exist an additional equilibrium point given by the solution of the following complex quadratic equation:

$$v_l v_l^* + \frac{i_l}{\tilde{a}_l}v_l^* + \frac{2}{3}\frac{S_l^*}{\tilde{a}_l} = 0,$$

where $\tilde{a} = Y_l^* + \mathcal{B}_l^\top Z_E^{-1}\mathcal{B}_l \in \mathbb{C}$. As a result, for each $l \in \mathcal{V}_L$ there exist $v_{1,l}^{\text{eq}}(S_l,i_l) \in \mathbb{C}$ and $v_{2,l}^{\text{eq}}(S_l,i_l) \in \mathbb{C}$.

For the linear part of (1), the corresponding equilibrium pairs are $i_I^{\text{eq}} = Z_I^{-1}(\bar{v}_I^{\text{eq}} - v_I^{\text{eq}})$ and $0 = -(Y_I + Z_I^{-1} + \mathcal{B}_I^\top Y_E\mathcal{B}_I)v_I^{\text{eq}} - \mathcal{B}_I^\top Y_E\mathcal{B}_L v_L^{\text{eq}} + Z_I^{-1}\bar{v}_I^{\text{eq}}$. From the previous discussion, the load control equilibrium input is $u_L = -\mathcal{B}_L^\top Y_E\mathcal{B}_I x_I^{\text{eq}}$ which, via Assumption 1, has full row rank. The equilibrium inputs $\bar{v}_I^{\text{eq}}$ depend on existing solutions for each load $l \in \mathcal{V}_L$ in a one-to-one way. Since we have 2 solutions per load, and $|\mathcal{V}_L|$ loads in the MG which yield $2^{|\mathcal{V}_L|}$ possible combinations of points. ■

This characterisation of the equilibrium points relies on Assumption 1 which allows to consider $Y_L + \mathcal{B}_L^\top Z_E^{-1}\mathcal{B}_L$ in (10) to be diagonal. A further observation shows that the equilibrium points are directly affected by the impedance of the part of the network connected to the loads. The study of these equilibrium points sheds light on the global behaviour of these CPLs, and provides motivation on the developments of Section III-A.

B. Reformulation of the MG control problem

Following a similar argument from the proof of Proposition 2, the MG dynamics can be partitioned into two parts, the loads v_L and the rest of the network $x_n = (i_I,v_I,i_E)$. A direct consequence of Corollary 1 and Assumption 1 is that $v_L \in \prod_{l\in\mathcal{V}_L}\mathbb{X}_l$ as long as the closed loop load currents satisfy $i_L \in \prod_{l\in\mathcal{V}_L}\mathcal{U}_l(v_l,d_l)$. Therefore, the interconnection dynamics between networks and loads can be described as

$$M_n\frac{dx_n}{dt} = -A_n x_n + B_n u_n + e_l \tag{11a}$$

$$C_n x_n \in \prod_{l\in\mathcal{V}_L}\mathcal{U}_l(x_l,d_l) \tag{11b}$$

where $e_l = \mathcal{B}_L v_L \in \mathcal{B}_L\prod_{l\in\mathcal{V}_L}\mathbb{X}_l$ is a bounded disturbance; $C_n x_n = -\mathcal{B}_L i_E$ is the output matrix mapping the state x_n to the corresponding load currents. Theorem 3 and its corollary are instrumental in transforming the problem of controlling load voltages to that of controlling a linear system subject to additive disturbances and output constraints. The benefits of such transformation are twofold: the problem focuses now

[6] $a + jb \iff \begin{bmatrix} a & -b \\ b & a\end{bmatrix} \iff \begin{bmatrix} a \\ b\end{bmatrix}$.

on the actuation point, *i.e.* the inverter side of the MG; the load voltages are contained within a prescribed range.

Remark 2: The partitioning of the state in two is not limited to the loads and the rest of the network, in fact, another possible partition would involve $x_n = (v_L, i_E)$ which are the states that describe loads and line currents, and $x_i = (i_I, v_I)$ corresponding to only inverter related variables. The set-valued control law $\mathcal{U}_n(\cdot, d_L)$ retains the same structure and goal, *i.e.* to keep the state x_n within a λ scaled version of the constraint set $\mathbb{X}_L \times \mathbb{X}_E$.

V. Simulations and Examples

In this section, we present simulation results pertinent to Section III, where we apply the set-valued control law to a CPL with an uncertain load demand with values oscillating around a mean. The network is composed of two loads $\mathcal{V}_L = \{l_1, l_2\}$, four generation nodes $\mathcal{V}_I = \{i_1, i_2, i_3, i_4\}$, the set of edges are $\mathcal{E} = \{(i_1, i_2), (i_1, l_2), (i_2, l_1), (i_3, i_1), (i_3, i_4), (i_3, l_2), (i_4, l_1)\}$. The active and reactive load power demand has on average $P_{l_1} = 30$ [kW], $P_{l_2} = 20$ [kW], and $Q_{l_1} = 2$ [kV AR], $Q_{l_2} = 0.3$ [kV AR]. The power rating for each of the generators is $S_I = \{84, 74, 89, 90\}$ [kW]. The constraints imposed on the line currents are $\mathbb{I}_e = \{i_e \in \mathbb{R}^2 \colon |i_e| \leq 65[\text{A}]\}$ for all $e \in \mathcal{E}$. The constraints on the node voltages are: $\mathbb{V}_l = \{v_l \in \mathbb{R}^2 \colon 205 \leq |v_l| \leq 231, \ |\arctan \frac{v_{ql}}{v_{dl}}| \leq 0.5\}$. Applying the proposed set-valued control law, using the selection proposed in (9), renders the point $v_l^{\text{ss}} = (220\sqrt{2}, 0)$ stable. However, the assumption that exact knowledge of the power demand is not practical, for the simulations we have introduced noise in the power demand, *i.e.* $d_l = \bar{d}_l + \delta_l$, and the controller uses only nominal values. To test the robustness limits of the proposed control law, we introduce changes in the power demand at time instants $t = 3, 5, 7, 10, 15$. In Figure 1, we illustrate these load demand changes; the response of the controller is to vary the current accordingly while keeping voltages confined within the desired values.

VI. Conclusions and Future Work

In this paper, we presented a set theoretic approach to guarantee constraint satisfaction for a MG with CPLs. This method proposes a set-valued control law that renders the load constraint set to be control invariant. The set valued map is lower semicontinuous which guarantees the existence of continuous selections; one of these selections is used to control a CPL with an uncertain power demand. Leveraging on set invariance, the proposed method reduces the MG to a linear system subject to a bounded additive disturbance and output constraints; this exploits the feedback interconnection structure between the inverters and the rest of the MG.

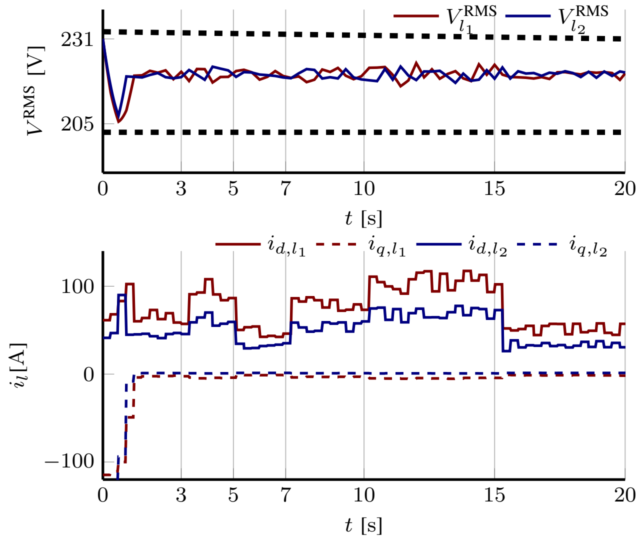

Fig. 1. (*top*) Voltage RMS evolution in time for both loads. The constraints are satisfied at all times despite the variations in the load power demand. In addition, sudden demand changes occur at times $t = 3, 5, 7, 10, 15$[s], the proposed control law is able to handle these variations without disturbing the voltage levels. (*bottom*) The corresponding input currents for each load. The load variations are more evident in the currents. The input current constraints are satisfied at all times since both $\mathbb{I}_{l_1} = \mathbb{I}_{(i_2,l_1)} \oplus \mathbb{I}_{(i_4,l_1)}$ and $\mathbb{I}_{l_2} = \mathbb{I}_{(i_1,l_2)} \oplus \mathbb{I}_{(i_3,l_2)}$ admit up to 130[A] in magnitude.

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