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Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations

In memory of A.V. Skorokhod (10.09.1930 – 03.01.2011)

Yu. S. Mishura[*] & A. Yu. Veretennikov[†]

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Abstract

New weak and strong existence and weak and strong uniqueness results for multi-dimensional stochastic McKean–Vlasov equation are established under relaxed regularity conditions. Weak existence requires a non-degeneracy of diffusion and no more than a linear growth of both coefficients in the state variable. Weak and strong uniqueness are established under the restricted assumption of diffusion, yet without any regularity of the drift; this part is based on the analysis of the total variation metric.

1 Introduction

Our subject is solutions of the stochastic Itô-McKean-Vlasov equation in $\mathbb{R}^d$

$$dX_t = B[t, X_t, \mu_t]dt + \Sigma[t, X_t, \mu_t]dW_t, \ \ t \geq 0, \qquad X_0 = x_0, \tag{1}$$

[*]Taras Shevchenko National University of Kyiv; myus@univ.kiev.ua

[†]University of Leeds, UK & National Research University Higher School of Economics, Russian Federation, and Institute for Information Transmission Problems, Moscow, Russia; a.veretennikov@leeds.ac.uk

in a particular situation called "true McKean-Vlasov case" under the convention

$$B[t,x,\mu] = \int b(t,x,y)\mu(dy), \quad \Sigma[t,x,\mu] = \int \sigma(t,x,y)\mu(dy), \tag{2}$$

and under certain non-degeneracy assumptions. Here W is a standard d_1-dimensional Wiener process, b and σ are vector and matrix Borel functions of corresponding dimensions d and $d \times d_1$, μ_t is the distribution of the process X at time t. The initial data x_0 may be random but independent of W; a non-random value is also allowed. Historically, Vlasov's idea, proposed originally in 1938 and contained in the reprinted paper [31], called mean field interaction in mathematical physics and stochastic analysis, assumes that for a large multi-particle ensemble with "weak interaction" between particles, this interaction for one particle with others may be effectively replaced by an averaged field. A class of equations of type (1) was proposed by M. Kac [15] as a stochastic "toy model" for the Vlasov kinetic equation of plasma. The systematic study of such equations was started by McKean [22]. The reference [25] provides an introduction to the whole area with links to the paper [6] as the most important preceding background deterministic paper.

McKean–Vlasov's equations, being clearly more involved than Itô's SDEs, arise in multi-agent systems (see [2, 3]), as well as in some other areas of high interest such as filtering (see [5]). These processes also very closely relate to so called self-stabilizing processes (diffusions, in particular), which is, actually, another name for non-linear diffusions in the "ergodic" situation, (see [11]). In what concerns "propagation of chaos" for the equation (1), we refer the reader to [25] and [4, Theorem 4.3]. In the authors' view, it may be fruitful to separate different aspects, including time discretization and "propagation of chaos" for multi-particle case, and to consider approximations differently from the basic existence and uniqueness issues; only the latter two are the main subjects of the present paper. Many control problems lead to discontinuous coefficients. This is one of motivations for looking for existence and uniqueness under minimal regularity of the coefficients.

As to earlier works in this area, one of the most important papers is [9] where the martingale problem for a similar McKean-Vlasov SDE is tackled. It is not very easy to compare our regularity assumptions with those in [9] because the latter are given not directly in terms of coefficients (please compare with (2.1) in the Assumption I from [9]). We do not assume continuity with respect to the state variable x replacing it by the non-degeneracy of the diffusion matrix. Neither our linear growth bound is comparable directly with the Lyapunov type conditions in [9]. More general growth conditions were studied in [4]; however, our regularity conditions admit just measurable coefficients in x, especially, for weak existence, and, hence, overall, our results are not covered by [4] either.

Our goal is to establish weak existence analogues to Krylov's weak existence for Itô's equations which is more general than in earlier papers. A more general equation is tackled with a possibly non-square matrix σ, which may be useful in applications and which case was not covered in [4]. Further, we propose a different method which could be of interest in some other settings. In the homogeneous case and under less general conditions, using a different technique, weak existence and weak uniqueness were established in [13] and [14]. In [29] there is a result on strong existence for the equation similar to (1) only with a unit matrix diffusion; however, strong and weak uniqueness, along with "propagation of chaos", i.e., with convergence of particle approximations, are established there under restrictive additional assumptions on the drift which include Lipschitz and some other conditions. In the present paper, weak and strong uniqueness are established for linear growing in the state variable and measurable drifts under additional assumptions on the (variable) diffusion coefficient.

In applications where some additional regularization by white noise is often required it may be useful to have a result for references with dimensions $d_1 \geq d$ rather than just for $d_1 = d$. This case is rarely tackled in the literature and it is not easy to find a suitable reference; this was the main reason why we included this extension. Despite the widespread intuitive belief that for weak solutions or weak uniqueness everything which may be desired only depends on the matrix $\sigma\sigma^*$, in fact, conditions in the McKean–Vlasov case usually do require certain properties of σ, not $\sigma\sigma^*$ (please compare, for example, with [9]). Hence, even if some results for $d_1 = d$ can be extended to $d_1 \geq d$, yet it is not automatic. Unlike the setting in the paper [4], we allow non-homogeneous coefficients depending on time; a formal reduction to a homogeneous case by considering a couple (t, X_t) would require unnecessary additional conditions due to the degeneracy. Our method of proof is also different from that used in [4]: we use explicitly Skorokhod's single probability space approach combined with Krylov's integral estimates for Itô's processes.

Strong existence for McKean–Vlasov equations in our paper is derived from strong existence for "ordinary" or "linearized" Itô's equations with a fixed flow (μ_t). The famous Yamada–Watanabe principle (see [12], [21], [32], [33]) concerning weak existence and pathwise uniqueness here has a remote analogue in terms of the equivalence of weak and strong uniqueness, yet, under additional assumptions. In all main results of the paper (but not in Propositions 1 and 2) it is assumed that the drift, and in the Theorem 1 diffusion as well, satisfies a linear growth bound condition. The linear growth is useful because of numerous applications where, at least, the drift is often unbounded; further extensions on a faster non-linear growth usually require Lyapunov type conditions, which are not considered in this paper.

The structure of the paper is as follows. In the Section 2 weak existence is established. Theorem 1 there mimics Krylov's weak existence result for Itô's SDEs from [17] for a homogeneous case, and from [19] for a non-homogeneous case; see also [30]. No regularity of the coefficients is assumed with respect to the state variable x. The proof is split into three parts. The first two parts, given in the Proposition 1 and Proposition 2, are devoted to the case under a bit restrictive additional assumptions on the diffusion; the third part extends the consideration to the general situation, i.e. to a not necessarily quadratic and symmetric diffusion matrix. Section 3 is devoted to strong solutions and to weak and strong uniqueness. Weak uniqueness and strong uniqueness are established simultaneously under identical (for weak and for strong uniqueness) sets of conditions. The latter do involve some restriction on the diffusion coefficient which should not depend on the measure in the Theorem 3. For a completeness of the paper, two important classical Skorokhod lemmas (Lemma 4 and Lemma 5) are provided in Appendix (Section 4), along with a "localized" version of certain Krylov bounds (Lemma 3). We shall use the following abbreviations for inequalities: by CBS we encrypt Cauchy-Buniakovsky-Schwarz inequality and BCM denotes Bienaymé-Chebyshev-Markov inequality.

2 Weak existence

2.1 Main results

Before we turn to the main results, let us recall the definitions and a fact from functional analysis.

Definition 1. *The triple (X_t, μ_t, W_t) is called solution of the equation (1) iff (W_t) is a d_1-dimensional Wiener process with a filtration $(\mathcal{F}_t)$ such that for each t, X_t is $\mathcal{F}_t$-measurable, X_t is continuous in t, and*

$$\mathbb{P}\left(X_t - x_0 - \int_0^t B[s, X_s, \mu_s]ds - \int_0^t \Sigma[s, X_s, \mu_s]dW_s = 0,\ t \geq 0\right) = 1,$$

with all the integrals under the sign of probability being correctly defined, and with μ_t being a marginal distribution of X_t for each $t \geq 0$. This solution is called strong iff for each t the random variable X_t is measurable with respect to the sigma-algebra $\mathcal{F}_t^W$ (sigma-algebra generated by Wiener process W); all other solutions are called weak.

Note that in the case of strong solution, it exists on any probability space with a d_1-dimensional Wiener process W. Following a tradition of Itô SDE theory and slightly abusing a rigorous wording in the definition above, we will usually call solution just the first component X_t of the triple (X_t, μ_t, W_t) yet with a compulsory property that μ_t is a marginal distribution of X_t for each t. The next lemma is probably a common knowledge since it is never mentioned in the papers on the topic. Yet, it seems that this result does require some integrability conditions; so it is stated here with a brief sketch of the proof so as to make sure that the linear growth conditions suffice. Let $|\cdot|$ stand for the Euclidean norm for any vector in $\mathbb{R}^d$, and $\|\cdot\|$ for the standard matrix norm, namely, $\|\sigma\| = \left(\sum_{i,j} \sigma_{ij}^2\right)^{1/2}$.

Lemma 1. *In terms of notation (2), let the Borel coefficients $b(t, x, y)$ and $\sigma(t, x, y)$ for each x satisfy*

$$\sup_{t,y}(|b(t, x, y)| + \|\sigma(t, x, y)\|) \le C(x)$$

with some locally bounded Borel function $C(x)$, $x \in \mathbb{R}^d$, and let $\mu_t(dy)$ be a marginal distribution of any solution X_t of the equation (1). Then the functions $\widetilde{b}(t, x) := B[t, x, \mu_t]$ and $\widetilde{\sigma}(t, x) := \Sigma[t, x, \mu_t]$ are Borel measurable in (t, x).

Proof. Let (X_t, μ_t, W_t) be a solution of (1) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a d_1-dimensional Wiener process W, and consider another *independent* solution (ξ_t, μ_t, W'_t) *with the same marginal distribution μ_t of ξ_t*, say, on another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ with a d_1-dimensional Wiener process W':

$$d\xi_t = B[t, \xi_t, \mu_t]dt + \Sigma[t, \xi_t, \mu_t]dW'_t, \quad t \ge 0, \qquad \mathcal{L}(\xi_0) = \mathcal{L}(x_0). \tag{3}$$

Then the coefficient $B[t, x, \mu_t]$ can be written as

$$B[t, x, \mu_t] = \mathbb{E}' b(t, x, \xi_t),$$

where $\mathbb{E}'$ stands for expectation with respect to the probability measure $\mathbb{P}'$. (Later on we will be using the notation $\mathbb{E}^3$ instead of $\mathbb{E}'$.) Now, the function $b(t, x, y)$ is Borel measurable in (t, x, y) by the assumption, and the function $\xi_t(\omega')$ is $\mathcal{B}[0, \infty) \otimes \mathcal{F}$-measurable in (t, ω') due to continuity of solution ξ_t in t and its measurability in ω' (see, e.g., [20, Lemma 1.5.7]). Hence, the function $\widehat{b}(t, x, \omega') := b(t, x, \xi_t(\omega'))$ is $\mathcal{B}[0, \infty) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$- measurable in (t, x, ω'). Further, one of the statements of Fubini theorem (see [20, Theorem 1.5.5]) claims that in this case the function

$$\mathbb{E}' b(t, x, \xi_t) = \int b(t, x, \xi_t(\omega'))\mathbb{P}'(d\omega') = \int \widehat{b}(t, x, \omega')\mathbb{P}'(d\omega')$$

is $\mathcal{B}[0,\infty)\otimes\mathcal{B}(\mathbb{R}^d)$-measurable, as required. Here we used the condition of boundedness of b in y for each x, which implies integrability

$$\iint\limits_D \int |b(t,x,\xi_t(\omega'))|\mathbb{P}'(d\omega')dtdx \le \iint\limits_D C(x)dtdx < \infty,$$

over any *bounded* Borel subset $D\in\mathcal{B}[0,\infty)\otimes\mathcal{B}(\mathbb{R}^d)$. Lemma 1 is proved. •

In this way, under the condition of the at most linear growth in x assumed in the sequel (see (4) a few lines below), the coefficients B and Σ in (1) are Borel measurable in (t,x); so, the equation (1) does make sense under this condition.

The next theorem is the main result of the paper about weak existence.

Theorem 1. *Let the initial value x_0 have a finite 4th moment. For the problem (1)–(2), suppose that the following two conditions are satisfied.*

(*i*) *The functions b and σ admit linear growth condition in x, i.e., there exists $C>0$ such that for any s,x,y,*

$$|b(s,x,y)| + \|\sigma(s,x,y)\| \le C(1+|x|), \tag{4}$$

(*ii*) *The diffusion matrix σ is uniformly non-degenerate in the following sense: there is a value $\nu>0$ such that for any probability measure μ,*

$$\inf_{s,x}\inf_{|\lambda|=1}\lambda^*\left(\int\sigma(s,x,y)\mu(dy)\right)\left(\int\sigma^*(s,x,y)\mu(dy)\right)\lambda \ge \nu. \tag{5}$$

Then the equation (1) has a weak solution, that is, a solution on some probability space with a standard d_1-dimensional Wiener process with respect to some filtration $(\mathcal{F}_t,\, t\ge 0)$.

Remark 1. *If $d_1=d$ and σ is* symmetric and positive definite, *then the assumption (5) can be replaced by an equivalent but much easier one, which is frequently in use,*

$$\inf_{s,x,y}\inf_{|\lambda|=1}\lambda^*\sigma(s,x,y)\lambda \ge \nu > 0. \tag{6}$$

The intuitive meaning of the condition (5) in the simplest 1D (that is, with $d_1=d=1$) situation is that the diffusion coefficient is non-degenerate and cannot change sign for any fixed (s,x) and varying y. It is plausible that any moment of order $2+\epsilon$ for x_0 suffices for all the statements (except for Theorem 3 under the linear growth condition on the drift where an exponential moment will be required), but this goal is not pursued here. Under the additional assumption of boundedness of b and σ, the 4th moment of the initial value x_0 is not necessary and can be further relaxed.

The structure of the proof of Theorem 1 is such that first the case of symmetric non-degenerate σ is tackled, i.e., with $d_1 = d$ and under the assumption (6). A motivation for this approach is that under the relaxed assumption (5) and under the symmetry of σ it is easy to find a smoothing of this matrix function which would keep the non-degeneracy of the diffusion coefficient. Because of this we formulate some provisional statement under more restrictive conditions as Proposition 1. Its proof will be simultaneously the beginning of the proof of Theorem 1.

In the following intermediate simplified version of the Theorem 1, we allow both coefficients to grow, but σ is assumed to be symmetric and positive definite. In the first part of the proof the coefficients are assumed to be bounded. We will treat now a more general structure of the coefficients B and Σ, namely, we assume that instead of B and Σ we have in (1) coefficients of the form

$$\overline{\Sigma}[t, X, \mu] = \phi\left(\int \sigma(t, x, y)\mu(dy)\right), \quad \overline{B}[t, X, \mu] = \psi\left(\int b(t, x, y)\mu(dy)\right),$$

where conditions on the matrix-valued functions $\phi : \mathbb{R}^{d\times d_1} \mapsto \mathbb{R}^{d\times d_1}$ (this includes the case of $d_1 = d$ as in the Proposition 1) and vector-valued function $\psi : \mathbb{R}^d \mapsto \mathbb{R}^d$ will be specified. In fact, for the proof of Theorem 1 we only need ϕ; however, ψ is added just by analogy since it does not bring any new difficulty. We use notations

$$\Sigma[t, X, \mu] = \int \sigma(t, x, y)\mu(dy) = \langle \sigma, \mu\rangle_{t,x}, \quad B[t, X, \mu] = \int b(t, x, y)\mu(dy) = \langle b, \mu\rangle_{t,x},$$

and denote also

$$A[t, x, \mu] := \Sigma\Sigma^*[t, x, \mu].$$

Actually, the conditions on the functions ϕ and ψ imposed in the Proposition 1 below might be further relaxed, but the authors do not have a motivation for that at the moment, as the main goal in this section is Theorem 1. Correspondingly, we now consider the equation (1) with coefficients $\overline{B}$ instead of B and with $\overline{\Sigma}$ instead of Σ.

Proposition 1. *Assume that $d_1 = d$, let σ and b satisfy the linear growth condition (4), and for functions ϕ and ψ there exist $m_1, m_2 > 0$ such that for any $\Sigma', \Sigma'' \in \mathbb{R}^{d\times d}$ and any $B', B'' \in \mathbb{R}^d$*

$$\|\phi(\Sigma') - \phi(\Sigma'')\| \le C\|\Sigma' - \Sigma''\|(1 + (\|\Sigma'\| \vee \|\Sigma''\|)^{m_1}), \tag{7}$$

$$|\psi(B') - \psi(B'')| \le C|B' - B''|(1 + (|B'| \vee |B''|)^{m_2}), \tag{8}$$

as well as

$$\|\phi(\Sigma')\| \le C(1 + \|\Sigma'\|),$$

$$|\psi(B')| \le C(1 + |B'|),$$

Also, $\overline{\Sigma}[t, X, \mu]$ *is assumed to be a symmetric positive definite uniformly non-degenerate for any* μ *and uniformly w.r.t.* t, x, μ.

Let the initial value x_0 *have a finite fourth moment, matrix* $\sigma(t, x, y)$ *be symmetric for each triple* (t, x, y)*, and the inequality (6) hold. Then equation (1) with coefficients* $\overline{B}$ *and* $\overline{\Sigma}$ *has a weak solution, that is, a solution on some probability space with a standard d-dimensional Wiener process with respect to some filtration* $(\mathcal{F}_t,\, t \ge 0)$.

Remark 2. *In the case of bounded coefficients, we use in the proof of Proposition 1 the fact that the class of functions* ϕ *and* ψ *satisfying conditions* (7) *and* (8) *must include identical* $\psi = Id$ *and* $\phi(\langle\sigma, \mu\rangle_{t,x}) = \sqrt{A[t, x, \mu]}$ *defined via the Cauchy-Riesz-Dunford formula for a function of a positive self-adjoint square root of the matrix* A *(see, e.g., [7, VII.3.9]),*

$$\phi(\langle\sigma, \mu\rangle_{t,x}) = (2\pi i)^{-1} \oint_{\Gamma} \lambda^{1/2}(\lambda I - A[t, x, \mu])^{-1} d\lambda \tag{9}$$

with $\Gamma = \Gamma(x) = \{\lambda : |\lambda| = r(x)\}$*, whose radius* $r(x)$ *is such that this contour contains all eigenvalues of* $A[t, x, \mu]$*. For example, it can be* $r(x) = \sup_{t,\mu} \|A[t, x, \mu]\| + 1$*. Under the boundedness condition on all coefficients which will be accepted temporary for the Proposition 1 in the first part of the proof, it suffices to choose a unique contour with* $\bar{r} = \sup_{t,x,\mu} \|A[t, x, \mu]\| + 1$*, since this value is finite.*

Now, let us check condition (7) *in this example with bounded coefficients. Indeed, we have for any probability measures* μ, μ' *and any* t, x,

$$\|\phi(\langle\sigma, \mu\rangle_{t,x}) - \phi(\langle\sigma, \mu'\rangle_{t,x})\|$$

$$= \frac{1}{2\pi} \left\| \oint_{\Gamma} \lambda^{1/2}(\lambda I - A[t, x, \mu])^{-1} d\lambda - \oint_{\Gamma} \lambda^{1/2}(\lambda I - A[t, x, \mu'])^{-1} d\lambda \right\|$$

$$\le C\bar{r}^{3/2} \sup_{|\lambda| = \bar{r}} \sup_{\nu} \|(\lambda I - A[t, x, \nu])^{-2}\| \, \|A[t, x, \mu] - A[t, x, \mu']\|$$

$$\leq C\bar{r}^2 \,\|\Sigma[t,x,\mu] - \Sigma[t,x,\mu']\|,$$

Recall that here $\phi(\langle \sigma, \mu\rangle_{t,x}) = \phi(\mathbb{E}^3\sigma(t,x,\xi))$ is assumed to be symmetric positive definite non-degenerate uniformly with respect to t, x and μ, where $\mu = \mathcal{L}(\xi)$.

Under the linear growth assumptions of Theorem 1, we will need $m_1 = 2$ and $m_2 = 0$, as we need to include $\psi = Id$ and $\phi(\langle \sigma, \mu\rangle)$ of the form (9) with $\Gamma = \Gamma(x) = \{\lambda : |\lambda| = r(x)\}$, whose radius $r(x)$ is such that this contour contains all eigenvalues of $A[t,x,\mu]$. It can be $r(x) = C|x|^2 + 1$ with some C large enough. In this case we have

$$\|\phi(\langle \sigma, \mu\rangle_{t,x}) - \phi(\langle \sigma, \mu'\rangle_{t,x})\|$$

$$= \frac{1}{2\pi}\left\| \oint_\Gamma \lambda^{1/2}(\lambda I - A[t,x,\mu])^{-1} d\lambda - \oint_\Gamma \lambda^{1/2}(\lambda I - A[t,x,\mu'])^{-1} d\lambda \right\|$$

$$\leq C r(x)^{3/2} \sup_{|\lambda| = r(x)} \sup_\nu \|(\lambda I - A[t,x,\nu])^{-2}\| \, \|A[t,x,\mu] - A[t,x,\mu']\|$$

$$= C r(x)^{3/2} \sup_{|\lambda| = r(x)} \sup_\nu \|(\lambda I - A[t,x,\nu])^{-2}\| \, \|\Sigma\Sigma^*[t,x,\mu] - \Sigma\Sigma^*[t,x,\mu']\|$$

$$\leq C r(x)^2 \|\Sigma[t,x,\mu] - \Sigma[t,x,\mu']\|.$$

Note that another slightly different option to define $\sqrt{A[t,x,\mu]}$ using the same idea in the unbounded case is as follows:

$$\sqrt{A[t,x,\mu]} = \frac{1}{2\pi i} \sum_{i=1}^{\infty} \mathbf{1}(i-1 \leq |x| < i) \oint_{\Gamma_i} \lambda^{1/2} (\lambda I - A[t,x,\mu])^{-1} \, d\lambda, \tag{10}$$

where

$$\Gamma_i = \{\lambda \in \mathbb{C} : \, |\lambda| = \sup_{t,x,\mu:\, |x| \leq i} \|A[t,x,\mu]\| + 1\},$$

where the contour $\Gamma_i \subset \mathbb{C}$ in the complex plane is chosen in a way so that its interior contains all the eigenvalues of the elliptic matrix $A[s,x,\cdot]$ for $|x| \leq i$.

2.2 Proof of Proposition 1

1. Assume temporarily that σ and b are bounded, and instead of (7)–(8) suppose for the first several steps that

$$\|\phi(\Sigma') - \phi(\Sigma'')\| \ \le C\|\Sigma' - \Sigma''\|, \tag{11}$$

and

$$|\psi(B') - \psi(B'')| \le C|B'' - B''|. \tag{12}$$

These restrictions will be waived at the step 6 of the proof. Let us smooth out both coefficients with respect to all variables by convolutions in such a way that they become globally Lipschitz in x and y. Namely, let

$$b^n(t,x,y) = b(t,x,y) * \varphi_n(x) * \varphi_n(y),$$

and

$$\sigma^n(t,x,y) = \sigma(t,x,y) * \varphi_n(x) * \varphi_n(y),$$

where the sequence $\varphi_n(\cdot)$ is defined in a standard way, i.e., as non-negative C^∞ functions with a compact support integrated to one, and so that this compact support squeezes to the origin of the corresponding variable as $n \to \infty$; or, in other words, that they are delta-sequences in the corresponding variables. Note that, of course, we may assume that for every n the smoothed coefficient of the drift remains to be under the linear growth condition (4) with the same constant for each n (in reality this constant may increase a little bit in comparison to the constant C from (4), but still remain uniformly bounded); also, under the assumption (6) the smoothed diffusion remains uniformly non-degenerate with ellipticity constants independent of n.

Now we shall explain why the equation with smoothed coefficients has a (strong) solution. We use successive approximations. For any fixed n, let

$$X^n(0)_t := x_0, \quad \mu^n(0)_t = \mathcal{L}(X^n(0)_t) = \mu_0, \quad t \ge 0;$$

further, if $X(m)_t$ and $\mu(m)_t$ are already determined, let us define

$$X^n(m+1)_t := x_0 + \int_0^t \bar{B}^n[s, X^n(m)_s, \mu^n(m)_s]ds + \int_0^t \bar{\Sigma}^n[s, X^n(m)_s, \mu^n(m)_s]dW_s,$$

where

$$\bar{B}^n[t,x,\mu] = \psi\left(\int b^n(t,x,y)\mu(dy)\right), \ \ \bar{\Sigma}^n[t,x,\mu] = \phi\left(\int \sigma^n(t,x,y)\mu(dy)\right).$$

We can say that $\bar{B}^n[s, X^n(m)_s, \mu^n(m)_s] = \psi(\mathbb{E}^3 b^n(s, X^n(m)_s, \xi^n(m)_s))$, where $\xi^n(m)_s$ is a random variable equivalent to $X^n(m)_s$ on some independent probability space, and, moreover, the sequence $(\xi^n(m).), m \geq 1$ can be chosen independent on $(X^n(m).), m \geq 1$, and so that the whole sequence $(\xi^n(m).), m \geq 1$ has the same distribution as the sequence $(X^n(m).), m \geq 1$. Then by induction the second moments of any $X^n(m)_t$ are finite and uniformly bounded for $t \leq T$, and by Itô's isometry and CBS inequality,

$$
\begin{aligned}
&\mathbb{E}|X^n(m+1)_t - X^n(m)_t|^2 \\
&\leq C_T \mathbb{E} \int_0^t |\psi(\mathbb{E}^3 b^n(s, X^n(m)_s, \xi^n(m)_s)) - \psi(\mathbb{E}^3 b^n(s, X^n(m-1)_s, \xi^n(m-1)_s))|^2 ds \\
&+ C\mathbb{E} \int_0^t \|\phi(\mathbb{E}^3 \sigma^n(s, X^n(m)_s, \xi^n(m)_s)) - \phi(\mathbb{E}^3 \sigma^n(s, X^n(m-1)_s, \xi^n(m-1)_s))\|^2 ds \\
&\leq C_T \mathbb{E} \int_0^t |\mathbb{E}^3 b^n(s, X^n(m)_s, \xi^n(m)_s) - \mathbb{E}^3 b^n(s, X^n(m-1)_s, \xi^n(m-1)_s)|^2 ds \\
&+ C\mathbb{E} \int_0^t \|\mathbb{E}^3 \sigma^n(s, X^n(m)_s, \xi^n(m)_s) - \mathbb{E}^3 \sigma^n(s, X^n(m-1)_s, \xi^n(m-1)_s)\|^2 ds \\
&\leq C_{T,n} \mathbb{E} \int_0^t |X^n(m)_s - X^n(m-1)_s|^2 ds + C\mathbb{E} \int_0^t \mathbb{E}^3 |\xi^n(m)_s - \xi^n(m-1)_s|^2 ds \\
&\leq C_{T,n} \mathbb{E} \int_0^t |X^n(m)_s - X^n(m-1)_s|^2 ds.
\end{aligned}
$$

Since all terms here are finite, we obtain by induction

$$
\mathbb{E}|X^n(m+1)_t - X^n(m)_t|^2 \leq C_{T,n} \frac{T^m}{m!}, \quad t \leq T.
$$

Due to the Doob inequality we also get

$$
\mathbb{E} \sup_{t \leq T} |X^n(m+1)_t - X^n(m)_t|^2 \leq C_{T,n} \frac{T^m}{m!}.
$$

From here by standard methods it follows easily convergence in probability of the sequence $X^n(m)_t$ to a solution X^n_t as $m \to \infty$, uniformly with respect to $t \leq T$, as required.

2. In a standard way (see, e.g., [19], [24]), absolutely similar to the inequalities of Lemma 2, we get the estimates uniform in n,

$$
\mathbb{E} \sup_{0 \leq t \leq T} |X^n_t|^4 \leq C_T (1 + \mathbb{E}|x_0|^4), \tag{13}
$$

and

$$\sup_{0\le s\le t\le T;\, t-s\le h} \mathbb{E}|X^n_t - X^n_s|^4 \le C_T h^2, \tag{14}$$

with some constants C_T which may be different for different inequalities but do not depend on n. In fact, similar a priori bounds hold true for any power function assuming the appropriate initial moment, although, this will not be used in this paper. The proof can be done following the lines in [10, Theorem 1.6.4]. Why do we need the fourth degree will be clear in the next step: it is useful for verifying continuity for the processes with equivalent finite-dimensional distributions. Note that all these bounds are valid under the linear growth assumptions in x.

3. Let us introduce new processes ξ^n, the copies of X^n, on some other independent probability space (i.e., we will consider both on the direct product of the two probability spaces); it also satisfies a similar SDE. In the sequel by $\mathbb{E}^3\sigma^n(s, X^n_s, \xi^n_s)$ or $\mathbb{E}^3\sigma(s, X_s, \xi_s)$ we denote expectation with respect to the third variable ξ^n_s, or ξ_s i.e., *conditional* expectation given the second variable X^n_s or X_s; in other words,

$$\mathbb{E}^3\sigma^n(s, X^n_s, \xi^n_s) = \int \sigma^n(s, X^n_s, y)\mu^{\xi^n}_s(dy),$$

where $\mu^{\xi^n}_s$ stands for the marginal distribution of ξ^n_s; likewise,

$$\mathbb{E}^3(\sigma^n(s, X^n_s, \xi^n_s) - \sigma^n(s, X_s, \xi_s))$$

means simply

$$\int \sigma^n(s, X^n_s, y)\mu^{\xi^n}_s(dy) - \int \sigma^n(s, X_s, y)\mu^{\xi}_s(dy),$$

where μ^{ξ}_s is the marginal distribution of ξ_s, and, finally,

$$\mathbb{E}^3\|\sigma^n(s, X^n_s, \xi^n_s) - \sigma(s, X_s, \xi_s)\|^2$$

is understood as

$$\int \|\sigma^n(s, X^n_s, y) - \sigma^n(s, X_s, y')\|^2\mu^{\xi^n,\xi}_s(dy, dy'),$$

where $\mu^{\xi^n,\xi}_s(dy, dy')$ denotes the marginal distribution of the couple (ξ^n_s, ξ_s).

Now, due to the estimates (13)–(14) and by virtue of Skorokhod's Lemma about a single probability space and convergence in probability (see Lemma 4 in the Appendix, or [24, §6, ch. 1], or [19, Lemma 2.6.2], without loss of generality we assume that not only $\mu^n \Longrightarrow \mu$, but also on some probability space

$$(\widetilde{X}^n_t, \widetilde{\xi}^n_t, \widetilde{W}^n_t) \stackrel{\mathbb{P}}{\to} (\widetilde{X}_t, \widetilde{\xi}_t, \widetilde{W}_t), \quad n\to\infty,$$

for any t and for some equivalent random processes $(\widetilde{X}^n, \widetilde{\xi}^n, \widetilde{W}^n)$, generally speaking, over a sub-sequence. Slightly abusing notations, we denote initial values still by x_0 without tilde. Also, without loss of generality we assume that each process $(\widetilde{\xi}^n_t,\, t\ge 0)$ for any $n\ge 1$ is independent of $(\widetilde{X}^n, \widetilde{W}^n)$, as well as their limit $\widetilde{\xi}_t$ may be chosen independent of the limits $(\widetilde{X}, \widetilde{W})$ (this follows from the fact that on the original probability space ξ^n is independent of (X^n, W^n) and on the new probability space their joint distribution remains the same; hence, independence of $\widetilde{\xi}^n$ is also valid and in the limit this is still true). See the details in the proof of the Theorem 2.6.1 in [19]. We may also introduce Wiener processes for ξ^n_t and $\widetilde{\xi}^n_t$, and will do it because it will be useful at one of the steps of the proof of the Proposition 2 (for the Proposition 1 it is not necessary). Namely, on an independent probability spaces we have,

$$d\xi^n_t = \overline{B}^n[t, \xi^n_t, \mu_t]dt + \overline{\Sigma}^n[t, \xi^n_t, \mu_t]dW'^{,n}_t, \ \ t\ge 0, \qquad \mathcal{L}(\xi^n_0) = \mathcal{L}(x_0), \tag{15}$$

and

$$d\widetilde{\xi}^n_t = \overline{B}^n[t, \widetilde{\xi}^n_t, \mu_t]dt + \overline{\Sigma}^n[t, \widetilde{\xi}^n_t, \mu_t]d\widetilde{W}'^{,n}_t, \ \ t\ge 0, \qquad \mathcal{L}(\widetilde{\xi}^n_0) = \mathcal{L}(x_0).$$

In what follows, let us fix some arbitrary $T>0$ and consider t in the interval $[0,T]$. Due to the inequality (14), the same inequality holds for $\widetilde{X}^n$ and $\widetilde{W}^n$, in particular,

$$\sup_{0\le s\le t\le T;\, t-s\le h} \mathbb{E}|\widetilde{X}^n_t - \widetilde{X}^n_s|^4 \le C_T h^2. \tag{16}$$

Due to Kolmogorov's continuity theorem, it means that all processes $\widetilde{X}^n$ may be regarded as continuous, and $\widetilde{W}^n$ can be assumed also continuous by the same reason. Further, due to the independence of the increments of W^n after time t of the sigma-algebra $\sigma(X^n_s, W^n_s, s\le t)$, the same property holds true for $\widetilde{W}^n$ and $\sigma(\widetilde{X}^n_s, \widetilde{W}^n_s, s\le t)$, as well as for $\widetilde{W}^n$ and for the completions of the sigma-algebras $\sigma(\widetilde{X}^n_s, \widetilde{W}^n_s, s\le t)$ which we denote by $\mathcal{F}^{(n)}_t$. Also, the processes $\widetilde{X}^n$ are adapted to the filtration $(\mathcal{F}^{(n)}_t)$. So, all stochastic integrals which involve $\widetilde{X}^n$ and $\widetilde{W}^n$ are well defined. The same relates to the processes $\widetilde{\xi}^n$.

Hence, again by using Skorokhod's Lemma (Lemma 4), we may choose a subsequence $n'\to\infty$ so as to pass to the limit in the equation

$$\widetilde{X}^{n'}_t = x_0 + \int_0^t \psi(\mathbb{E}^3 b^{n'}(s, \widetilde{X}^{n'}_s, \widetilde{\xi}^{n'}_s))\, ds + \int_0^t \phi(\mathbb{E}^3 \sigma^{n'}(s, \widetilde{X}^{n'}_s, \widetilde{\xi}^{n'}_s)) d\widetilde{W}^{n'}_s,$$

in order to get

$$\widetilde{X}_t = x_0 + \int_0^t \psi(\mathbb{E}^3 b(s, \widetilde{X}_s, \widetilde{\xi}_s)) ds + \int_0^t \phi(\mathbb{E}^3 \sigma(s, \widetilde{X}_s, \widetilde{\xi}_s))\, d\widetilde{W}_s,$$

or, equivalently,

$$\widetilde{X}_t = x_0 + \int_0^t \bar{B}[s, \widetilde{X}_s, \mu_s]\, ds + \int_0^t \bar{\Sigma}[s, \widetilde{X}_s, \mu_s]\, d\widetilde{W}_s,$$

with

$$\mu_s = \mathcal{L}(\widetilde{X}_s).$$

First of all, recall that a priori bounds (13) – (14) and (16) hold true with constants not depending on n. Now, by Skorokhod's Lemma 4, on some probability space we have a sequence of equivalent processes $(\widetilde{X}_t^{n'}, \widetilde{\xi}_t^{n'}, \widetilde{W}_t^{n'})$ and a limiting triple $(\widetilde{X}_t, \widetilde{\xi}_t, \widetilde{W}_t)$ such that for any t,

$$(\widetilde{X}_t^{n'}, \widetilde{\xi}_t^{n'}, \widetilde{W}_t^{n'}) \stackrel{\mathbb{P}}{\to} (\widetilde{X}_t, \widetilde{\xi}_t, \widetilde{W}_t).$$

By virtue of the above a priori estimates for $\widetilde{W}^n$, the process $\widetilde{W}$ is continuous and it is a Wiener process. Also, the limits are adapted to the corresponding filtration $\widetilde{\mathcal{F}}_t := \bigvee_n \mathcal{F}_t^{(n)}$ and $\widetilde{W}$ is continuous and it is a Wiener process with respect to this filtration. In particular, related Lebesgue and stochastic integrals are all well defined. Moreover, by virtue of the uniform estimates (14), the limit $(\widetilde{X}_t, \widetilde{\xi}_t)$ may be also regarded as continuous due to Kolmogorov's continuity theorem because a priori bounds (13) – (14) remain valid for the limiting processes $\widetilde{X}, \widetilde{\xi}$. In particular, note because it will come in handy later, that

$$\sup_{0\le t\le T} \mathbb{E}|\widetilde{X}_t|^2 \le C_T(1 + \mathbb{E}|x_0|^2).$$

4. Still in the case of bounded coefficients b and σ, let us now show that

$$\int_0^t \psi(\mathbb{E}^3 b^{n'}(s, \widetilde{X}_s^{n'}, \widetilde{\xi}_s^{n'}))ds \stackrel{\mathbb{P}}{\to} \int_0^t \psi(\mathbb{E}^3 b(s, \widetilde{X}_s, \widetilde{\xi}_s))ds, \tag{17}$$

and

$$\int_0^t \phi(\mathbb{E}^3 \sigma^{n'}(s, \widetilde{X}_s^{n'}, \widetilde{\xi}_s^{n'}))d\widetilde{W}_s^{n'} \stackrel{\mathbb{P}}{\to} \int_0^t \phi(\mathbb{E}^3 \sigma(s, \widetilde{X}_s, \widetilde{\xi}_s))d\widetilde{W}_s, \quad n' \to \infty. \tag{18}$$

We start with the drift term. Let us fix some n_0 and let $n > n_0$. Due to the assumption (12), we have for any $t \le T$,

$$\mathbb{P}\left(\left|\int_0^t \left(\psi(\mathbb{E}^3 b^n(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n)) - \psi(\mathbb{E}^3 b(s, \widetilde{X}_s, \widetilde{\xi}_s))\right) ds\right| > c\right)$$

$$
\leq \mathbb{P}\left(\left|\int_0^t \left(\psi(\mathbb{E}^3 b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)) - \psi(\mathbb{E}^3 b^{n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))\right) ds\right| > \frac{c}{3}\right)
$$

$$
+ \mathbb{P}\left(\left|\int_0^t \left(\psi(\mathbb{E}^3 b^{n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)) - \psi(\mathbb{E}^3 b^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s))\right) ds\right| > \frac{c}{3}\right)
$$

$$
+ \mathbb{P}\left(\left|\int_0^t \left(\psi(\mathbb{E}^3 b^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s)) - \psi(\mathbb{E}^3 b(s,\widetilde{X}_s,\widetilde{\xi}_s))\right) ds\right| > \frac{c}{3}\right)
$$

$$
=: I^1 + I^2 + I^3.
$$

Let

$$
\gamma_{n,R} := \inf(t \geq 0 : \sup_{0\leq s\leq t} (|\widetilde{X}^n_s| \vee |\widetilde{\xi}^n_s|) \geq R), \ \gamma_R := \inf(t \geq 0 : \sup_{0\leq s\leq t} (|\widetilde{X}_s| \vee |\widetilde{\xi}_s|) \geq R),
$$
$$
\gamma^X_R := \inf(t \geq 0 : \sup_{0\leq s\leq t} |\widetilde{X}_s| \geq R), \ \gamma^\xi_R := \inf(t \geq 0 : \sup_{0\leq s\leq t} |\widetilde{\xi}_s| \geq R),
$$
$$
\gamma^X_{n,R} := \inf(t \geq 0 : \sup_{0\leq s\leq t} |\widetilde{X}^n_s| \geq R), \ \gamma^\xi_{n,R} := \inf(t \geq 0 : \sup_{0\leq s\leq t} |\widetilde{\xi}^n_s| \geq R).
$$

We have that for any $\epsilon > 0$ there exists $R > 0$ such that (to have $R-1$ instead of R will be convenient shortly)

$$
\mathbb{P}(\sup_{0\leq t\leq T} (|\widetilde{X}_t| \vee |\widetilde{\xi}_t|) \geq R-1) < \epsilon,
$$

or, equivalently,

$$
\mathbb{P}(\gamma_{R-1} \leq T) < \epsilon,
$$

and similarly,

$$
\sup_n \mathbb{P}(\sup_{0\leq t\leq T} (|\widetilde{X}^n_t| \vee |\widetilde{\xi}^n_t|) \geq R-1) < \epsilon,
$$

or, equivalently,

$$
\sup_n \mathbb{P}(\gamma_{n,R-1} \leq T) < \epsilon.
$$

Denote

$$
g^{n,n_0}(s,x,\xi) := b^n(s,x,\xi) - b^{n_0}(s,x,\xi), \quad g^{n_0}(s,x,\xi) := b^{n_0}(s,x,\xi) - b(s,x,\xi).
$$

Then the first summand I^1 may be estimated by BCM inequality due to the condition (12) as follows without ψ,

$$I^1 \le \frac{3}{c}\,\mathbb{E}\int_0^T C\,\mathbb{E}^3|b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b^{n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds$$

$$= \frac{3C}{c}\,\mathbb{E}\mathbb{E}^3\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds = \frac{3C}{c}\,\mathbb{E}\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds$$

$$= C\mathbb{E}\mathbf{1}(\gamma_{n,R}\le T)\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds + C\mathbb{E}\mathbf{1}(\gamma_{n,R}> T)\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds.$$

Here the first term $\mathbb{E}\mathbf{1}(\gamma_{n,R}\le T)\displaystyle\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds$ admits the bound (recall that on each line constants C can be different)

$$\mathbb{E}\mathbf{1}(\gamma_{n,R}\le T)\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds \le CT\mathbb{P}(\gamma_{n,R}\le T),$$

and the last expression is small uniformly in n if R is large enough.

The second term $\mathbb{E}\mathbf{1}(\gamma_{n,R} > T)\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds$ admits a bound via Krylov's estimate (see the Theorems 2.4.1 or 2.3.4 in [19]) as follows: there exists a constant N depending on the dimension d and on R through the ellipticity constants of the diffusion matrix and the sup-norm of the drift on the set $B_R\times B_R$, where $B_R = \{x\in\mathbb{R}^d : |x|\le R\}$, so that

$$\mathbb{E}\mathbf{1}(\gamma_{n,R} > T)\int_0^T |g^{n,n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds$$

$$\le N_R\left(\int_0^T\int_{|x|\le\widetilde{R}}\int_{|\xi|\le\widetilde{R}} |g^{n,n_0}(s,x,\xi)|^{2d+1}\,dxd\xi ds\right)^{\frac{1}{2d+1}}$$

$$\le N_R\left(\int_0^T\int_{|x|\le\widetilde{R}}\int_{|\xi|\le\widetilde{R}} |b^n(s,x,\xi)-b(s,x,\xi)|^{2d+1}\,dxd\xi ds\right)^{\frac{1}{2d+1}}$$

$$+N_R\left(\int_0^T\int_{|x|\le\widetilde{R}}\int_{|\xi|\le\widetilde{R}} |b^{n_0}(s,x,\xi)-b(s,x,\xi)|^{2d+1}\,dxd\xi ds\right)^{\frac{1}{2d+1}} \to 0, \quad n,n_0\to\infty,$$

for each R, by virtue of the well-known property of mollified functions. Hence, overall, we obtain that

$$I^1\to 0, \quad n,n_0\to\infty.$$

Further, under the assumptions of Proposition 1, the second term I^2 admits for any $0 \le t \le T$ the estimate, again without ψ in the right hand side,

$$I^2 = \mathbb{P}\left(\left|\int_0^t \left(\psi(\mathbb{E}^3 b^{n_0}(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s)) - \psi(\mathbb{E}^3 b^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s))\right) ds\right| > \frac{c}{3}\right)$$
$$\le C\, \mathbb{E}\mathbb{E}^3 \int_0^T |b^{n_0}(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s) - b^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds,$$

which tends to zero as $n \to \infty$ under the assumptions of the Proposition 1, due to the Lebesgue bounded convergence theorem because of the convergence in probability $(\widetilde{X}^n_s, \widetilde{\xi}^n_s) \to (\widetilde{X}_s, \widetilde{\xi}_s)$. So, for each n_0

$$\lim_{n\to\infty} I^2 = 0.$$

As for the term I^3, it admits the following bounds (again without ψ in the right hand side):

$$I^3 = \mathbb{P}\left(\left|\int_0^t \left(\psi(\mathbb{E}^3 b^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)) - \psi(\mathbb{E}^3 b(s, \widetilde{X}_s, \widetilde{\xi}_s))\right) ds\right| > \frac{c}{3}\right)$$
$$\le C\, \mathbb{E}\mathbb{E}^3 \int_0^T |b^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s) - b(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds$$
$$= C\, \mathbb{E}(\mathbf{1}(\gamma_{n,R} \wedge \gamma_R \le T) + \mathbf{1}(\gamma_{n,R} \wedge \gamma_R > T)) \int_0^T |g^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds.$$

Furthermore, the term $\mathbb{E}\mathbf{1}(\gamma_{n,R} \wedge \gamma_R \le T) \displaystyle\int_0^T |g^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds$ admits the bound

$$\mathbb{E}\mathbf{1}(\gamma_{n,R} \wedge \gamma_R \le T) \int_0^T |g^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds \le C\mathbb{P}(\gamma_{n,R} \wedge \gamma_R \le T) \to 0, \quad R \to \infty,$$

uniformly in n. Hence, it follows that

$$\lim_{n_0\to\infty} \lim_{R\to\infty} \mathbb{E}\mathbf{1}(\gamma_{n,R} \wedge \gamma_R \le T) \int_0^T |g^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds = 0.$$

In order to evaluate the term $\mathbb{E}\mathbf{1}(\gamma_{n,R} \wedge \gamma_R > T) \int_0^T |g^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)|\, ds$, we note that the values of the function $g^{n_0}(s, x, \xi)$ outside the set $\{(x, \xi) : (|x| \vee |\xi|) \le R\}$ with large R whose exact value is not relevant. So, for evaluating this term, without losing of generality we may assume that $g^{n_0}(s,x,\xi)$ vanishes outside the ball $B_{R+1}\times B_{R+1}$: if not, we just truncate accepting that $g^{n_0}=0$ outside $B_{R+1}\times B_{R+1}$. Then the desired convergence follows from Krylov's bound of Lemma 3. Indeed, provided that we denote

$$g^{n_0}_R(s,x,\xi) := g^{n_0}(s,x,\xi)\mathbf{1}(|x|\le R, |\xi|\le R),$$

the following bounds follow immediately:

$$\mathbb{E}\mathbf{1}(\gamma_{n,R}\wedge\gamma_R > T)\int_0^T |g^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds \le \mathbb{E}\int_0^T |g^{n_0}_R(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds.$$

Now, for continuous g^{n_0}, in the limit as $n\to\infty$ we get

$$\begin{aligned}\mathbb{E}\mathbf{1}(\gamma_R > T)\int_0^T |g^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds &\le \mathbb{E}\int_0^T |g^{n_0}_R(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds \\ &= \lim_{n\to\infty}\mathbb{E}\int_0^T |g^{n_0}_R(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s)|\,ds \\ \le CN_R\|g^{n_0}_R\|_{L_{2d+1}((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)} &= CN_R\|g^{n_0}_R\|_{L_{2d+1}((0,\infty)\times B_R\times B_R)}.\end{aligned}$$

The latter bound extends to all Borel measurable functions g_R as in [19, Section II].

$$\mathbb{E}\int_0^T |g_R(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds \le N_R\|g_R\|_{L_{2d+1}([0,T]\times B_R\times B_R)}. \tag{19}$$

Let us show that this bound remains true for any Borel function g_R vanishing outside $B_R\times B_R$. This is analogous to the justification of a similar inequality in [19, Section II.6], but we want to add some details for the convenience of the reader. Firstly, it clearly suffices to prove it for non-negative functions only, $g_R\ge 0$. Secondly, since indicator of any compact in a finite-dimensional Euclidean space may be approximated monotonically (from above) pointwise by continuous bounded functions, again by Fatou lemma (19) remains valid for all non-negaitive functions represented by finite sums

$$g(s,x,\xi) = \sum_{i=1}^K c_i 1_{\Gamma_i}(s,x,\xi),$$

with any compacts $\Gamma_i\subset[0,T]\times B_R\times B_R$ and with all $c_i>0$ by the monotonic convergence theorem. Thirdly, the expression $\mathbb{E}\int_0^T g(s,\widetilde{X}_s,\widetilde{\xi}_s)\,ds$ may be rewritten via a "Green measure"

$$\mathbb{E}\int_0^T g(s,\widetilde{X}_s,\widetilde{\xi}_s)\,ds = \int g(s,x,\xi)\,\nu(ds,dx,d\xi)$$

with a positive sigma-additive measure ν:

$$\nu(ds,dx,d\xi) = ds\,\mathbb{P}(\tilde X_s \in dx)\mathbb{P}(\tilde\xi_s \in d\xi).$$

Fourthly, since on finite-dimensional Euclidean spaces any such measure is regular, it is possible to approximate monotonically any expression of the form

$$\int g(s,x,\xi)\,\nu(ds,dx,d\xi) = \int (\sum_{i=1}^K c_i 1_{D_i}(s,x,\xi))\,\nu(ds,dx,d\xi)$$

with any Borel $D_i \subset [0,T]\times B_R\times B_R$ and $c_i > 0$ by $\int(\sum_{i=1}^K c_i 1_{\Gamma_i}(s,x,\xi))\,\nu(ds,dx,d\xi)$ with compact Γ_i. Hence, for such functions $\sum_{i=1}^K c_i 1_{D_i}(s,x,\xi)$ with Borel D_i the estimate (19) also remains valid. Finally, by such sums $\sum_{i=1}^K c_i 1_{D_i}(s,x,\xi)$ any bounded Borel measurable $g_R \ge 0$ can be approximated uniformly and monotonically; so, the bound (19) is valid for any bounded Borel measurable $g_R \ge 0$, as required. For what follows, let us note that it also suffices for extension of the estimate to all Borel measurable from $L_{2d+1}([0,T]\times B_R\times B_R)$ again by monotonic approximations.

Hence, by the properties of the mollified functions it follows that

$$\lim_{n_0\to\infty} I^3 = 0.$$

The convergence (17) is, thus, proved.

5. Now still for bounded coefficients let us consider convergence of stochastic integrals in (18). Our goal is an estimate similar to that for the drift and Lebesgue integrals above:

$$\mathbb{P}\left(\left\|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde X^n_s,\widetilde\xi^n_s))d\widetilde W^n_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde X_s,\widetilde\xi_s))d\widetilde W_s\right\| > c\right) < C\epsilon, \qquad (20)$$

for any $c,\epsilon > 0$ and n large enough. In principle, the task is similar to the convergence of Lebesgue integrals studied in the previous steps. Hence, we mainly show how to overcome the additional obstacle due to different Wiener processes $\widetilde W$ and $\widetilde W^n$ in the stochastic integrals. We have a tool for this which is Skorokhod's Lemma 5.

By virtue of [20, Theorem 6.2.1(v)] and similarly to the calculus for the drift with Lebesgue integrals in the previous steps, yet using second moments instead of the first ones for the evident reason, we estimate

$$\mathbb{P}\left(\left\|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde X^n_s,\widetilde\xi^n_s))d\widetilde W^n_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde X_s,\widetilde\xi_s))d\widetilde W_s\right\| > c\right)$$

$$
\begin{aligned}
&\leq \mathbb{P}\left(\left\|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n))d\widetilde{W}_s^n - \int_0^t \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n))d\widetilde{W}_s^n\right\| > c/3\right)\\
&+ \mathbb{P}\left(\left\|\int_0^t \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n))d\widetilde{W}_s^n - \int_0^t \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right\| > c/3\right)\\
&+ \mathbb{P}\left(\left\|\int_0^t \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right\| > c/3\right)\\
&=: J^1 + J^2 + J^3.
\end{aligned}
$$

Using the assumptions, we apply Itô-Skorokhod inequality with any $\delta > 0$ and get that

$$
\begin{aligned}
J^1 &= \mathbb{P}\left(\left\|\int_0^t (\phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)) - \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)))d\widetilde{W}_s^n\right\| > c/3\right)\\
&\leq \mathbb{P}\left(\int_0^t \left\|\phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)) - \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n))\right\|^2 ds > \delta\right)\\
&+ \frac{9}{c^2}\mathbb{E}\left(\delta \wedge \int_0^T \left\|\phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)) - \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n))\right\|^2 ds\right).
\end{aligned}
$$

Here the second term is small if we choose $\delta > 0$ small. Let us consider the first term given $\delta > 0$. We have that

$$
\begin{aligned}
&\mathbb{P}\left(\int_0^t \left\|\phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)) - \phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n))\right\|^2 ds > \delta\right)\\
&\quad\leq \mathbb{P}\left(C\int_0^t \left\|\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n) - \mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)\right\|^2 ds > \delta\right)
\end{aligned}
$$

By BCM inequality,

$$
\begin{aligned}
&\mathbb{P}\left(C\int_0^t \left\|\mathbb{E}^3\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n) - \mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)\right\|^2 ds > \delta\right)\\
&\leq (\delta/C)^{-1}\mathbb{E}\int_0^t \left\|\mathbb{E}^3(\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n) - \sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)\right\|^2 ds\\
&\leq (\delta/C)^{-1}\mathbb{E}\int_0^t \mathbb{E}^3\left\|\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n) - \sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)\right\|^2 ds\\
&= (\delta/C)^{-1}\mathbb{E}\mathbb{E}^3\int_0^t \left\|\sigma^n(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n) - \sigma^{n_0}(s,\widetilde{X}_s^n,\widetilde{\xi}_s^n)\right\|^2 ds.
\end{aligned}
$$

Convergence of the latter term to zero as $n, n_0 \to \infty$ follows from the same considerations as for the drift in the previous step of the proof for the analogous term I^1 via Krylov's bound. So, we have

$$0 \le \lim_{n,n_0\to\infty} J^1 \le \lim_{n,n_0\to\infty} \mathbb{E}\mathbb{E}^3 \int_0^t \left\|\sigma^n(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) - \sigma^{n_0}(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n)\right\|^2 ds = 0.$$

The term J^2 converges to zero by Skorokhod's Lemma 5:

$$\int_0^t \phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n)) d\widetilde{W}_s^n \overset{\mathbb{P}}{\to} \int_0^t \phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)) d\widetilde{W}_s, \quad n \to \infty,$$

with $f_s^n := \phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n)), \quad f_s^0 := \phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s))$ in this lemma.

Consider J^3:

$$\begin{aligned} J^3 &= \mathbb{P}\left(\left\|\int_0^t (\phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)) - \phi(\mathbb{E}^3\sigma(s, \widetilde{X}_s, \widetilde{\xi}_s))) d\widetilde{W}_s\right\| > c/3\right) \\ &\le \mathbb{P}\left(\int_0^t \left\|\phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)) - \phi(\mathbb{E}^3\sigma(s, \widetilde{X}_s, \widetilde{\xi}_s))\right\|^2 ds > \delta\right) \\ &+ \frac{9}{c^2}\mathbb{E}\left(\delta \wedge \int_0^T \left\|\phi(\mathbb{E}^3\sigma^{n}(s, \widetilde{X}_s, \widetilde{\xi}_s)) - \phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s))\right\|^2 ds\right). \end{aligned}$$

Similarly to J^1, the second term in the last sum is small for small δ. For the first one we have, similarly to J^1,

$$\begin{aligned} &\mathbb{P}\left(\int_0^t \left\|\phi(\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s)) - \phi(\mathbb{E}^3\sigma(s, \widetilde{X}_s, \widetilde{\xi}_s))\right\|^2 ds > \delta\right) \\ &\le \mathbb{P}\left(C\int_0^t \left\|\mathbb{E}^3\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s) - \mathbb{E}^3\sigma(s, \widetilde{X}_s, \widetilde{\xi}_s)\right\|^2 ds > \delta\right) \\ &\le (\delta/C)^{-1}\mathbb{E}\mathbb{E}^3 \int_0^t \left\|\sigma^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s) - \sigma(s, \widetilde{X}_s, \widetilde{\xi}_s)\right\|^2 ds. \end{aligned}$$

Convergence of this term to zero as $n_0 \to \infty$ follows from Lemma 3, similarly to the analogous convergence of I^3 in the previous step. So,

$$\lim_{n_0\to\infty} J^3 = 0.$$

This finishes the proof of the desired bound (20). Thus, a weak solution of the equation (1)–(2) exists in the case of $d_1 = d$ and under the assumption (6) instead of (5). For bounded coefficients and under (11)–(12) the Proposition 1 is proved.

6. Now consider the general case of unbounded coefficients satisfying the linear growth condition in x along with (7)–(8). Let

$$\sigma^n(t,x,y) = (\sigma(t,x,y)\mathbf{1}(|x|\le n) + \sigma(0,0,0)\mathbf{1}(|x|>n)) * \phi_n(x),$$
$$b^n(t,x,y) = b(t,x,y)\mathbf{1}(|x|\le n).$$

Note that the function σ^n is Borel measurable, bounded, uniformly non-degenerate, and smooth (at least, Lipschitz) in x. Denote by X^n a solution of the equation

$$X^n_t = X_0 + \int_0^t \overline{B}^n[s, X^n_s, \mu^n_s]\,ds + \int_0^t \overline{\Sigma}^n[s, X^n_s, \mu^n_s]\,dW^n_s,$$

or

$$X^n_t = X_0 + \int_0^t \psi(\mathbb{E}^3 b^n[s, X^n_s, \xi^n_s])\,ds + \int_0^t \phi(\mathbb{E}^3\sigma^n[s, X^n_s, \xi^n_s])\,dW^n_s,$$

with

$$\overline{\Sigma}^n[s,x,\mu] = \phi\left(\int \sigma^n(s,x,y)\,\mu(dy)\right) = \phi(\Sigma^n[s,x,\mu]),$$
$$\overline{B}^n[s,x,\mu] = \psi\left(\int b^n(s,x,y)\,\mu(dy)\right) = \phi(B^n[s,x,\mu]),$$

and where (ξ^n) are independent of (X^n, W^n) processes with the same distributions as X^n on some independent probability space. (Note that we use the same notations as in the step 1; however, the approximations b^n and σ^n here are different; so we define them as the new ones.) This weak (in fact, strong) solution exists for each n due to the Proposition 1 for bounded coefficients. We will use again Skorokhod's technique; note that we could not apply it in one step because in our method of justifying the previous steps the boundedness of both coefficients is essential.

A priori moment inequalities of Lemma 2 hold true uniformly with respect to n. Hence, by Skorokhod's Lemma 4, choose a subsequence of equivalently distributed triples $(\widetilde{X}^{n_k}, \widetilde{\xi}^{n_k}, \widetilde{W}^{n_k})$ converging in probability for any s to some limiting triple $(\widetilde{X}, \widetilde{\xi}, \widetilde{W})$. Here $\widetilde{W}$ is a Wiener process of dimension d_1 (cf. with [19, Chapter 2], where, however, dimensions are equal, but it does not affect the conclusion). Convergence in the equation

$$\widetilde{X}^{n_k}_t = \widetilde{X}_0 + \int_0^t \psi(\mathbb{E}^3 b^{n_k}[s, \widetilde{X}^{n_k}_s, \widetilde{\xi}^{n_k}_s])\,ds + \int_0^t \phi(\mathbb{E}^3\sigma^{n_k}[s, \widetilde{X}^{n_k}_s, \widetilde{\xi}^{n_k}_s])\,d\widetilde{W}^{n_k}_s$$

towards the limiting equation

$$\widetilde{X}_t = \widetilde{x}_0 + \int_0^t \psi(\mathbb{E}^3 B[s, \widetilde{X}_s, \widetilde{\xi}_s])\,ds + \int_0^t \phi(\mathbb{E}^3\sigma[s, \widetilde{X}_s, \widetilde{\xi}_s])\,d\widetilde{W}_s$$

follows from the same calculus as in the proof of the previous steps of Proposition 1 with the only difference that now σ may also be unbounded; however, this does require some additional care, in particular, because we want to use Krylov's bounds stated for unbounded coefficients (and do not forget about ϕ and ψ). Yet, in fact, we can use Krylov's bound from Lemma 3.

First of all, recall that a priori bounds (13) and (14) hold true with constants not depending on n. Now, by Skorokhod's Lemma 4, on some probability space we have some equivalent processes $(\widetilde{X}_t^{n'}, \widetilde{\xi}_t^{n'}, \widetilde{W}_t^{n'})$ and a limiting triple $(\widetilde{X}_t, \widetilde{\xi}_t, \widetilde{W}_t)$ such that for any t,

$$(\widetilde{X}_t^{n'}, \widetilde{\xi}_t^{n'}, \widetilde{W}_t^{n'}) \stackrel{\mathbb{P}}{\to} (\widetilde{X}_t, \widetilde{\xi}_t, \widetilde{W}_t).$$

By virtue of a priori bounds for $\widetilde{W}^n$, the process $\widetilde{W}$ is continuous and it is a Wiener process. Also, the limits are adapted to the corresponding filtration $\widetilde{\mathcal{F}}_t := \bigvee_n \mathcal{F}_t^{(n)}$, and $\widetilde{W}$ is a Wiener process with respect to this filtration. In particular, related Lebesgue and stochastic integrals are all well defined. Moreover, by virtue of the uniform estimates (14), the limit $(\widetilde{X}_t, \widetilde{\xi}_t)$ may be also regarded as continuous due to Kolmogorov's continuity theorem because a priori bounds (13) – (14) remain valid for the limiting processes $\widetilde{X}, \widetilde{\xi}$. In particular, it is useful to note for the sequel that

$$\sup_{0\le t\le T} \mathbb{E}|\widetilde{X}_t|^2 \le C_T(1+\mathbb{E}|x_0|^2) \text{ and } \mathbb{E}\sup_{0\le t\le T} |\widetilde{X}_t|^2 \le C_T(1+\mathbb{E}|x_0|^2).$$

7. Let us now show that

$$\int_0^t \psi(\mathbb{E}^3 b^{n'}(s, \widetilde{X}_s^{n'}, \widetilde{\xi}_s^{n'}))ds \stackrel{\mathbb{P}}{\to} \int_0^t \psi(\mathbb{E}^3 b(s, \widetilde{X}_s, \widetilde{\xi}_s))ds, \tag{21}$$

and

$$\int_0^t \phi(\mathbb{E}^3 \sigma^{n'}(s, \widetilde{X}_s^{n'}, \widetilde{\xi}_s^{n'}))d\widetilde{W}_s^{n'} \stackrel{\mathbb{P}}{\to} \int_0^t \phi(\mathbb{E}^3 \sigma(s, \widetilde{X}_s, \widetilde{\xi}_s))d\widetilde{W}_s, \quad n'\to\infty. \tag{22}$$

Denote $\widetilde{R} = R-1$. Given any $\epsilon>0$, and simplifying notation by renaming $n'\to n$, for any $t\le T$ by virtue of the BCM inequality we conclude that for any $\epsilon>0$ there exists R such that

$$\mathbb{E}\mathbf{1}(\gamma_{n,\widetilde{R}} \wedge \gamma_{\widetilde{R}} \le T) < \epsilon.$$

Further at one place we will need more precise estimates:

$$\mathbb{P}(\gamma^X_{n,\widetilde{R}} \le T) \le (\widetilde{R})^{-2}\mathbb{E}\sup_{0\le t\le T}|\widetilde{X}_t^n|^2 \le (\widetilde{R})^{-2}C(1+\mathbb{E}|x_0|^2),$$

and

$$\mathbb{P}(\gamma^X_{\widetilde{R}} \le T) \le (\widetilde{R})^{-2}\mathbb{E}\sup_{0\le t\le T}|\widetilde{X}_t|^2 \le (\widetilde{R})^{-2}C(1+\mathbb{E}|x_0|^2),$$

by virtue of the BCM inequality. Hence,

$$\mathbb{P}(\gamma^X_{n,\widetilde{R}} \wedge \gamma^X_{\widetilde{R}} \le T) \le 2C(\widetilde{R})^{-2}(1+\mathbb{E}|x_0|^2).$$

Now, we estimate, with $\widetilde{c} = c/C$, $m = m_2$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\int_0^t \psi(\mathbb{E}^3 b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))\,ds - \int_0^t \psi(\mathbb{E}^3 b(s,\widetilde{X}_s,\widetilde{\xi}_s))\,ds\right| > c\right)\\
&\le \mathbb{P}\left(C\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\left|\mathbb{E}^3\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > c\right)\\
&= \mathbb{P}\left(\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\left|\left(\mathbf{1}(\gamma_{n,\widetilde{R}}\wedge\gamma_{\widetilde{R}} \le T) + \mathbf{1}(\gamma_{n,\widetilde{R}}\wedge\gamma_{\widetilde{R}} > T)\right)\right.\right.\\
&\qquad\qquad\left.\left.\times\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}\right)\\
&\le \mathbb{P}\left(\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma_{n,\widetilde{R}}\wedge\gamma_{\widetilde{R}} \le T)\left|\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}/2\right)\\
&+ \mathbb{P}\left(\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma_{n,\widetilde{R}}\wedge\gamma_{\widetilde{R}} > T)\left|\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}/2\right)\\
&\le \mathbb{P}\left(\mathbf{1}(\gamma^X_{n,\widetilde{R}}\wedge\gamma^X_{\widetilde{R}} \le T)\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\left|\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}/2\right)\\
&+ \mathbb{P}\left(\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}}\wedge\gamma^\xi_{\widetilde{R}} \le T)\left|\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}/2\right)\\
&+ \mathbb{P}\left(\mathbf{1}(\gamma^X_{n,\widetilde{R}}\wedge\gamma^X_{\widetilde{R}} > T)\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}}\wedge\gamma^\xi_{\widetilde{R}} > T)\right.\\
&\qquad\qquad\left.\times\left|\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}/2\right)\\
&\le \mathbb{P}(\gamma^X_{n,\widetilde{R}}\wedge\gamma^X_{\widetilde{R}} \le T)\\
&+ \mathbb{P}\left(\int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}}\wedge\gamma^\xi_{\widetilde{R}} \le T)\left|\left(b^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s) - b(s,\widetilde{X}_s,\widetilde{\xi}_s)\right)\right|\,ds > \widetilde{c}/2\right)\\
&+ \mathbb{P}\left(\gamma^X_{n,\widetilde{R}}\wedge\gamma^X_{\widetilde{R}} > T; \int_0^t (1+(|\widetilde{X}^n_s|\vee|\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}}\wedge\gamma^\xi_{\widetilde{R}} > T)\right.
\end{aligned}$$

$$\times \left| \left(b^n(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) - b(s, \widetilde{X}_s, \widetilde{\xi}_s) \right) \right| ds > \frac{\widetilde{c}}{2} \Bigg) =: L^1 + L^2 + L^3.$$

Here the first term L^1 does not exceed ϵ if R is large enough, uniformly with respect to n.

Consider the term L^2. We estimate it using the linear growth in x assumption:

$$\mathbb{P}\left(\int_0^t (1 + (|\widetilde{X}_s^n| \vee |\widetilde{X}_s|)^m) \mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} \le T) \left| \left(b^n(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) - b(s, \widetilde{X}_s, \widetilde{\xi}_s) \right) \right| ds > \widetilde{c}/2 \right)$$
$$\le \mathbb{P}\left(\mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} \le T) \int_0^t (1 + (|\widetilde{X}_s^n| \vee |\widetilde{X}_s|)^{m+1}) ds > \widetilde{c}/2 \right).$$

Since the processes $\widetilde{X}_s^n$ together with their limit $\widetilde{X}_s$ are bounded in probability uniformly in n, and because

$$\mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} \le T) \to 0, \quad R \to \infty,$$

we conclude that

$$\lim_{R\to\infty} \sup_n L^2 = 0.$$

Further, consider L^3. Let us fix some n_0 and let $n > n_0$. Replacing $\widetilde{c}/2$ by c for simplicity, we have for any $t \le T$,

$$\mathbb{P}\Bigg(\gamma^X_{n,\widetilde{R}} \wedge \gamma^X_{\widetilde{R}} > T; \int_0^t (1 + (|\widetilde{X}_s^n| \vee |\widetilde{X}_s|)^m) \mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)$$
$$\times \left| \left(b^n(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) - b(s, \widetilde{X}_s, \widetilde{\xi}_s) \right) \right| ds > c \Bigg)$$

$$\le \mathbb{P}\Bigg(\gamma^X_{n,\widetilde{R}} \wedge \gamma^X_{\widetilde{R}} > T; \int_0^t (1 + (|\widetilde{X}_s^n| \vee |\widetilde{X}_s|)^m) \mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)$$
$$\times \left| \left(b^n(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) - b^{n_0}(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) \right) \right| ds > \frac{c}{3} \Bigg)$$

$$+\mathbb{P}\Bigg(\gamma^X_{n,\widetilde{R}} \wedge \gamma^X_{\widetilde{R}} > T; \int_0^t (1 + (|\widetilde{X}_s^n| \vee |\widetilde{X}_s|)^m) \mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)$$
$$\times \left| \left(b^{n_0}(s, \widetilde{X}_s^n, \widetilde{\xi}_s^n) - b^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s) \right) \right| ds > \frac{c}{3} \Bigg)$$

$$+\mathbb{P}\left(\gamma^X_{n,\widetilde{R}} \wedge \gamma^X_{\widetilde{R}} > T; \int_0^t (1 + (|\widetilde{X}^n_s| \vee |\widetilde{X}_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)\right.$$

$$\left. \times \left|\left(b^{n_0}(s, \widetilde{X}_s, \widetilde{\xi}_s) - b(s, \widetilde{X}_s, \widetilde{\xi}_s)\right)\right| ds > \frac{c}{3}\right)$$

$$=: M^1 + M^2 + M^3.$$

Denote as earlier

$$g^{n,n_0}(s,x,\xi) := b^n(s,x,\xi) - b^{n_0}(s,x,\xi), \quad g^{n_0}(s,x,\xi) := b^{n_0}(s,x,\xi) - b(s,x,\xi).$$

Let $\alpha \le 1/(2m+2)$. Note that

$$\mathbb{E}(1 + (\sup_s |\widetilde{X}^n_s| \vee \sup_s |\widetilde{X}_s|)^{2m\alpha}) \le C < \infty,$$

and that this constant C is uniform in n and does not depend on R. The first summand M^1 may be estimated by the BCM inequality as

$$M^1 \le C\,\mathbb{E}\left(\mathbf{1}(\gamma^X_{n,\widetilde{R}} > T, \gamma^X_{\widetilde{R}} > T) \int_0^T (1 + (|\widetilde{X}^n_s| \vee |\widetilde{X}_s|)^m)\right.$$

$$\left. \times \mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)|b^n(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s) - b^{n_0}(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s)|\, ds\right)^\alpha$$

$$\le C\,\mathbb{E}\mathbf{1}(\gamma^X_{n,\widetilde{R}} > T, \gamma^X_{\widetilde{R}} > T)(1 + (\sup_s |\widetilde{X}^n_s| \vee \sup_s |\widetilde{X}_s|)^{m\alpha})$$

$$\times \left(\int_0^T \mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)|b^n(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s) - b^{n_0}(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s)|\, ds\right)^\alpha$$

$$\stackrel{\text{CBS}}{\le} C\left(\mathbb{E}\mathbf{1}(\gamma^X_{n,\widetilde{R}} > T, \gamma^X_{\widetilde{R}} > T)(1 + (\sup_s |\widetilde{X}^n_s| \vee \sup_s |\widetilde{X}_s|)^{2m\alpha})\right)^{1/2}$$

$$\times \left(\mathbb{E}\left(\int_0^T \mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)|b^n(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s) - b^{n_0}(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s)|\, ds\right)^{2\alpha}\right)^{1/2}$$

$$\stackrel{\text{Hölder}}{\le} C\left(\int_0^T \mathbb{E}\mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)|b^n(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s) - b^{n_0}(s, \widetilde{X}^n_s, \widetilde{\xi}^n_s)|\, ds\right)^\alpha \tag{23}$$

$$= C\left(\int_0^T \mathbb{E}\mathbf{1}(\gamma^\xi_{n,\widetilde{R}} \wedge \gamma^\xi_{\widetilde{R}} > T)|g^{n,n_0}(s, \widetilde{X}^n_{s\wedge\gamma_{n,\widetilde{R}}}, \widetilde{\xi}^n_{s\wedge\gamma_{n,\widetilde{R}}})|\, ds\right)^\alpha.$$

Here the couple $(\widetilde X^n_{s\wedge\gamma_{n,\widetilde R}}, \widetilde \xi^n_{s\wedge\gamma_{n,\widetilde R}})$ is a stopped diffusion with coefficients bounded by norm in state variables (x,ξ) by the value $C\widetilde R$ uniformly with respect to n, and with the diffusion coefficient uniformly non-degenerate.

Denote by $\widehat B^{n,\widetilde R}[s,x,\mu]$ and $\widehat \Sigma^{n,\widetilde R}[s,x,\mu]$ bounded vector and matrix functions in x respectively, with $\widehat \Sigma^{n,\widetilde R}[s,x,\mu]$ uniformly non-degenerate and smooth (e.g., C^1), such that

$$\widehat B^{n,\widetilde R}[s,x,\mu] = B^n[s,x,\mu], \ \ \widehat \Sigma^{n,\widetilde R}[s,x,\mu] = \Sigma^n[s,x,\mu], \quad |x|\le \widetilde R.$$

Let $(\widehat X^n_s) = (\widehat X^{n,\widetilde R}_s)$ be a strong solution of the Itô equation,

$$d\widehat X^n_t = \psi(\widehat B^{n,\widetilde R}[t,\widehat X^n_t,\mu^n_t])\,dt + \phi(\widehat \Sigma^{n,\widetilde R}[t,\widehat X^n_t,\mu^n_t])\,d\widetilde W^n_t, \quad \widehat X^n_0 = x_0,$$

where μ^n_t is still the marginal distribution of X^n_t and $\widetilde X^n_t$. Let also $\widehat \xi^n$ be an equivalent independent copy of the process $\widehat X^n_t$ satisfying the equation

$$d\widehat \xi^n_t = \psi(\widehat B^{n,\widetilde R}[t,\widehat \xi^n_t,\mu_t])dt + \phi(\widehat \Sigma^{n,\widetilde R}[t,\widehat \xi^n_t,\mu_t]d\widetilde W'^{,n}_t), \ \ t\ge 0, \quad \mathcal L(\xi^n_0) = \mathcal L(x_0).$$

We may assume that the Wiener processes $\widetilde W'^{,\,n}$ here are the same as in the equation (15) for $\widetilde \xi^n$. (Emphasize that solutions $\widehat \xi^n$ are strong ones; this is why we have mollified σ in the variable x). Then it follows that on $[0,t\wedge\gamma_{n,\widetilde R}]$ the processes $\widetilde X^n$ and $\widehat X^n$ coincide, see [20, Theorem 6.2.1(v)], as well as $\widetilde \xi^n$ coincide with $\widehat \xi^n$. We highlight that the same stopping times $\gamma_{n,\widetilde R}$ can be used. Then the bound for I^1 in the second line of (23) may be rewritten as

$$\begin{aligned}
M^1 &\le C\left(\mathbb E \mathbf 1(\gamma_{n,\widetilde R} > T, \gamma_{\widetilde R} > T)\int_0^T |g^{n,n_0}(s,\widehat X^n_{s\wedge\gamma_{n,\widetilde R}}, \widehat \xi^n_{s\wedge\gamma_{n,\widetilde R}})|\,ds\right)^\alpha \\
&\le C\left(\mathbb E \int_0^T |g^{n,n_0}(s,\widehat X^n_{s\wedge\gamma_{n,\widetilde R}}, \widehat \xi^n_{s\wedge\gamma_{n,\widetilde R}})|\,ds\right)^\alpha .
\end{aligned}$$

The values of the function $g^{n,n_0}(s,x,\xi)$ outside the set $\{(x,\xi):\ (|x|\vee|\xi|)\le \widetilde R\}$ are not relevant to the evaluation of the expression in the second line of (24). So, without loss of generality we may assume for our goal that $g^{n,n_0}(s,x,\xi)$ vanishes outside of this ball. Then, by Krylov's estimate from Lemma 3 we obtain with some constant N depending on the dimension d and on R through the ellipticity constants of the diffusion matrix and the sup-norm of the drift with some N_R,

$$M^1 \le C\left(\mathbb E \int_0^T |g^{n,n_0}(s,\widehat X^n_{s\wedge\gamma_{n,\widetilde R}}, \widehat \xi^n_{s\wedge\gamma_{n,\widetilde R}})|\,ds\right)^\alpha$$

$$= N_R \left(\int_0^T \int_{|x|\le \widetilde R}\int_{|\xi|\le \widetilde R} |b^n(s,x,\xi) - b^{n_0}(s,x,\xi)|^{2d+1}\, dx d\xi ds\right)^{\frac{\alpha}{2d+1}}$$

$$\le N_R \left(\int_0^T \int_{|x|\le \widetilde R}\int_{|\xi|\le \widetilde R} |b^n(s,x,\xi) - b(s,x,\xi)|^{2d+1}\, dx d\xi ds\right)^{\frac{\alpha}{2d+1}}$$

$$+N_R \left(\int_0^T \int_{|x|\le \widetilde R}\int_{|\xi|\le \widetilde R} |b^{n_0}(s,x,\xi) - b(s,x,\xi)|^{2d+1}\, dx d\xi ds\right)^{\frac{\alpha}{2d+1}} \to 0, \quad n, n_0 \to \infty,$$

for each R, by virtue of the properties of mollified functions. Hence, for any R,

$$\lim_{n,n_0\to\infty} M^1 = 0.$$

Let $0 \le w(x,\xi) \le 1$ be any continuous function which equals 1 for every $|x|\vee|\xi| \le \widetilde R$ and zero for every $|x|\vee|\xi| > R$, and $g^{n_0}_R(s,x,\xi) := g^{n_0}(s,x,\xi)w(x,\xi)$.

Further, the second term M^2 for any $0 \le t \le T$ admits the following upper bound:

$$M^2 \le \mathbb{P}\biggl(\gamma^X_{n,\widetilde R}\wedge\gamma^X_{\widetilde R} > T; \int_0^t (1 + (|\widehat X^n_{s\wedge\gamma_{n,\widetilde R}}| \vee |\widehat X_{s\wedge\gamma_{n,\widetilde R}}|)^m)$$

$$\times \mathbb{E}^3 \mathbf{1}(\gamma^\xi_{n,\widetilde R}\wedge\gamma^\xi_{\widetilde R} > T)\Bigl|\Bigl(b^{n_0}(s,\widetilde X^n_s,\widetilde\xi^n_s)ds - b^{n_0}(s,\widetilde X_s,\widetilde\xi_s)\Bigr)\Bigr| ds > \tfrac{c}{3}\biggr)$$

$$\le \frac{3}{c}(1+\widetilde R^m)\mathbb{E}\mathbb{E}^3\mathbf{1}(\gamma_{n,\widetilde R} > T, \gamma_{\widetilde R} > T)\int_0^T |b^{n_0}(s,\widetilde X^n_s,\widetilde\xi^n_s) - b^{n_0}(s,\widetilde X_s,\widetilde\xi_s)|\, ds$$

$$\le \frac{3}{c}(1+\widetilde R^m)\mathbb{E}\mathbb{E}^3\mathbf{1}(\gamma_{n,\widetilde R} > T, \gamma_{\widetilde R} > T)$$

$$\times \int_0^T |b^{n_0}(s,\widetilde X^n_s,\widetilde\xi^n_s)w(\widetilde X^n_s,\widetilde\xi^n_s) - b^{n_0}(s,\widetilde X_s,\widetilde\xi_s)w(\widetilde X_s,\widetilde\xi_s)|\, ds.$$

Let us show that the latter expression tends to zero as $n \to \infty$.

The random variable $\int_0^T |b^{n_0}(s,\widetilde X^n_s,\widetilde\xi^n_s)w(\widetilde X^n_s,\widetilde\xi^n_s) - b^{n_0}(s,\widetilde X_s,\widetilde\xi_s)w(\widetilde X_s,\widetilde\xi_s)|\, ds$ is bounded uniformly in n.

So, it suffices to show that

$$\int_0^T |b^{n_0}(s,\widetilde X^n_s,\widetilde\xi^n_s)w(\widetilde X^n_s,\widetilde\xi^n_s) - b^{n_0}(s,\widetilde X_s,\widetilde\xi_s)w(\widetilde X_s,\widetilde\xi_s)|\, ds \to 0, \quad n\to\infty, \qquad (24)$$

in probability. Indeed, (24) follows from the standard trick in Krylov's estimates: firstly we approximate again the bounded function $b^{n_0}(s,x,\xi)w(x,\xi)$ by smooth (continuous suffices) in (x,ξ) function $\bar b^{n_0}(s,x,\xi)$ also with a compact support, so that their difference in the L_{2d+1} norm is small. Then

$$\int_0^T |\bar b^{n_0}(s,\widetilde X^n_s,\widetilde \xi^n_s) - \bar b^{n_0}(s,\widetilde X_s,\widetilde \xi_s)|\, ds \to 0, \quad n\to\infty,$$

due to the continuity of $\bar b^{n_0}$. Also, the difference

$$\mathbb{E}\int_0^T |b^{n_0}(s,\widetilde X^n_s,\widetilde \xi^n_s) w(\widetilde X^n_s,\widetilde \xi^n_s) - \bar b^{n_0}(s,\widetilde X^n_s,\widetilde \xi^n_s)|\, ds$$

is small due to Krylov's estimate (Lemma 3); it does not exceed some constant N_R multiplied by $\|b^{n_0}w - \bar b^{n_0}\|_{[0,T]\times B_R\times B_R}$.

For any Borel function g_R Krylov's bound (Lemma 3) implies the inequality

$$\sup_n \mathbb{E}\int_0^T |g_R(s,\widetilde X^n_s,\widetilde \xi^n_s)|\, ds \le N_R \|g_R\|_{L_{2d+1}([0,T]\times B_R\times B_R)},$$

where, as before, $B_R = \{z\in\mathbb{R}^d : |z|\le R\}$. Assume that g_R is continuous in (x,y) and vanishing outside $B_R\times B_R$. Then in the limit we obtain by Fatou lemma that

$$\mathbb{E}\int_0^T |g_R(s,\widetilde X_s,\widetilde \xi_s)|\, ds \le N_R \|g_R\|_{L_{2d+1}([0,T]\times B_R\times B_R)}. \tag{25}$$

This inequality for unbounded coefficients was already justified in the comments after (19).

Therefore,

$$\lim_{n,n_0\to\infty} M^2 = 0.$$

Let us consider the third term M^3. We have that

$$\begin{aligned} M^3 &= \mathbb{P}\Big(\gamma^X_{n,\widetilde R}\wedge\gamma^X_{\widetilde R} > T; \int_0^T (1+(|\widetilde X^n_s|\vee|\widetilde X_s|)^m)\mathbb{E}^3\mathbf{1}(\gamma^\xi_{n,\widetilde R}\wedge\gamma^\xi_{\widetilde R} > T) \\ &\quad \times \Big|\Big(b^{n_0}(s,\widetilde X_s,\widetilde \xi_s) - b(s,\widetilde X_s,\widetilde \xi_s)\Big)\Big| ds > \frac{c}{3}\Big) \\ &\le C(1+\widetilde R^m)\,\mathbb{E}\mathbf{1}(\gamma_{\widetilde R}\wedge\gamma_{n,\widetilde R} > T)\int_0^T |g^{n_0}(s,\widetilde X_s,\widetilde \xi_s)|\, ds \\ &\le C(1+\widetilde R^m)\,\mathbb{E}\int_0^T |g^{n_0}_R(s,\widetilde X_s,\widetilde \xi_s)|\, ds. \end{aligned}$$

Now, by the same reasons as for the term M^2, we obtain

$$
\begin{aligned}
M^3 &\le C(1+\widetilde{R}^m)\,\mathbb{E}\mathbf{1}(\gamma_{\widetilde{R}}\wedge\gamma_{n,\widetilde{R}}>T)\int_0^T |g^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds \\
&\le C(1+\widetilde{R}^m)\,\mathbb{E}\int_0^T |g^{n_0}_R(s,\widetilde{X}_s,\widetilde{\xi}_s)|\,ds \\
&\le C(1+\widetilde{R}^m)\,N_R\|g^{n_0}_R\|_{L_{2d+1}([0,T]\times B_R\times B_R)}
\end{aligned}
$$

due to (25). Hence,

$$
\lim_{n_0\to\infty} M^3 = 0.
$$

The convergence (21) is, thus, proved.

8. Now let us consider convergence of stochastic integrals in (22). Our goal is an estimate the next probability similarly to the drift and Lebesgue integrals above:

$$
\mathbb{P}\left(\left|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))d\widetilde{W}^n_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right| > c\right) < C\epsilon, \tag{26}
$$

for any $c,\epsilon>0$ and n large enough. As we know from the step 5 in the proof of Proposition 1, the task is similar to the convergence of Lebesgue integrals, and we mainly show how to tackle the additional obstacle due to different Wiener processes $\widetilde{W}$ and $\widetilde{W}^n$ in the stochastic integrals. A tool for this is Skorokhod's Lemma 5, as before. However, it is not applicable directly because our processes may be unbounded, so we should overcome this with the help of a truncation which reduces the problem to bounded processes.

By virtue of [20, Theorem 6.2.1(v)] and similarly to the calculus for Lebesgue integrals in the previous steps, yet using second moments instead of the first ones by the evident reason we estimate,

$$
\begin{aligned}
&\mathbb{P}\left(\left|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))d\widetilde{W}^n_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right| > c\right) \\
&= \mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}\le T; \left|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))d\widetilde{W}^n_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right| > c\right) \\
&+\mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}> T; \left|\int_0^t \phi(\mathbb{E}^3\sigma^n(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))d\widetilde{W}^n_s - \int_0^t \phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right| > c\right) \\
&\le \mathbb{P}(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}\le T)
\end{aligned}
$$

$$+\mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}>T;\left|\int_0^t\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widehat{\xi}^n_s))d\widetilde{W}^n_s-\int_0^t\phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right|>c\right).$$

The first term here $\mathbb{P}(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}\le T)$ is small if R is large enough, as we have seen in the earlier steps. It remains to consider the second term. We will evaluate it as follows:

$$\mathbb{P}(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}>T;\left|\int_0^t\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widehat{\xi}^n_s))d\widetilde{W}^n_s-\int_0^t\phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right|>c)$$
$$\le\mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}>T;\left|\int_0^T\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widehat{\xi}^n_s))d\widetilde{W}^n_s-\int_0^t\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widehat{\xi}^n_s))d\widetilde{W}^n_s\right|>c/3\right)$$
$$+\mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}>T;\left|\int_0^T\phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}^n_s,\widetilde{\xi}^n_s))d\widetilde{W}^n_s-\int_0^t\phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right|>c/3\right)$$
$$+\mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}>T;\left|\int_0^T\phi(\mathbb{E}^3\sigma^{n_0}(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s-\int_0^t\phi(\mathbb{E}^3\sigma(s,\widetilde{X}_s,\widetilde{\xi}_s))d\widetilde{W}_s\right|>c/3\right)$$
$$=:J^1+J^2+J^3.$$

Now for all three terms the evaluation is similar to that for the drift term, except for one difference. We start with J^1:

$$J^1=\mathbb{P}\left(\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}>T;\left|\int_0^T(\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))d\widetilde{W}^n_s-\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widetilde{\xi}^n_s)))d\widetilde{W}^n_s\right|>c/3\right)$$
$$\le\mathbb{P}\left(\left|\int_0^{T\wedge\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}}}(\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))-\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widetilde{\xi}^n_s)))d\widetilde{W}^n_s\right|>c/3\right)$$
$$=\mathbb{P}\left(\left|\int_0^T\mathbf{1}(s<\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}})(\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))-\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widetilde{\xi}^n_s)))d\widetilde{W}^n_s\right|>c/3\right).$$

By virtue of the simplified version of the Itô-Skorokhod's inequality (see [20, Theorem 6.3.5], we get that for any $\delta>0$

$$\mathbb{P}\left(\left|\int_0^T\mathbf{1}(s<\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}})(\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))-\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widetilde{\xi}^n_s)))d\widetilde{W}^n_s\right|>c/3\right)$$
$$\le\mathbb{P}\left(\int_0^T\mathbf{1}(s<\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}})\left\|\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))-\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))\right\|^2ds>\delta\right)$$
$$+\frac{9}{c^2}\mathbb{E}\left(\delta\wedge\int_0^T\mathbf{1}(s<\gamma^X_{\widetilde{R}}\wedge\gamma^X_{n,\widetilde{R}})\left\|\phi(\mathbb{E}^3\sigma^n(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))-\phi(\mathbb{E}^3\sigma^{n_0}(s,\widehat{X}^n_s,\widetilde{\xi}^n_s))\right\|^2ds\right)$$

$=: F^1 + F^2.$

The term F^2 is small if we choose δ small enough. Let us consider F^1. It can be bounded as follows:

$$\mathbb{P}\left(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}}) \left\|\phi(\mathbb{E}^3\sigma^n(s, \widehat{X}^n_s, \widetilde{\xi}^n_s)) - \phi(\mathbb{E}^3\sigma^{n_0}(s, \widehat{X}^n_s, \widetilde{\xi}^n_s))\right\|^2 ds > \delta\right)$$

$$\leq \mathbb{P}\left(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})(1 + |\widehat{X}^n_s|^m)^2 \left\|\mathbb{E}^3(\sigma^n(s, \widehat{X}^n_s, \widetilde{\xi}^n_s) - \sigma^{n_0}(s, \widehat{X}^n_s, \widetilde{\xi}^n_s))\right\|^2 ds > \delta\right)$$

$$= \mathbb{P}\bigg(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})(1 + |\widehat{X}^n_s|^m)^2 \Big\|\mathbb{E}^3(\mathbf{1}(T \geq \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}})$$

$$+\mathbf{1}(T < \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}})) \times (\sigma^n(s, \widehat{X}^n_s, \widetilde{\xi}^n_s) - \sigma^{n_0}(s, \widehat{X}^n_s, \widetilde{\xi}^n_s))\Big\|^2 ds > \delta\bigg)$$

$$\leq \mathbb{P}\bigg(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})\mathbb{E}^3(1 + |\widehat{X}^n_s|^m)^2 \mathbf{1}(T \geq \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}})$$

$$\times\|\sigma^n(s, \widehat{X}^n_s, \widetilde{\xi}^n_s) - \sigma^{n_0}(s, \widehat{X}^n_s, \widetilde{\xi}^n_s)\|^2 ds > \delta/2\bigg)$$

$$+\mathbb{P}\bigg(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})(1 + |\widehat{X}^n_s|^m)^2 \mathbb{E}^3 \mathbf{1}(T < \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}})$$

$$\times\|\sigma^n(s, \widehat{X}^n_s, \widetilde{\xi}^n_s) - \sigma^{n_0}(s, \widehat{X}^n_s, \widetilde{\xi}^n_s)\|^2 ds > \delta/2\bigg) =: S^1 + S^2.$$

For S^2 we can replace $\widetilde{\xi}$ by $\widehat{\xi}$ and then use BCM inequality as in the drift evaluation, to obtain an L_{4d+1}-bound (not L_{2d+1} because of $\|\sigma^n - \sigma^{n_0}\|^2$) by virtue of Krylov's bound (see Lemma 3):

$$S^2 = \mathbb{P}\bigg(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})(1 + |\widehat{X}^n_s|^m)^2 \mathbb{E}^3 \mathbf{1}(T < \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}})$$

$$\times\|\sigma^n(s, \widehat{X}^n_s, \widehat{\xi}^n_s) - \sigma^{n_0}(s, \widehat{X}^n_s, \widehat{\xi}^n_s)\|^2 ds > \delta/2\bigg)$$

$$\leq C_R \|g^{n,n_0}\|^2_{L_{4d+1}([0,T]\times B_R \times B_R)} \to 0, \quad n, n_0 \to \infty.$$

For S^1 let $\beta = 1/(4m+4)$ and also use the BCM inequality:

$$\mathbb{P}\bigg(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})\mathbb{E}^3(1 + |\widehat{X}^n_s|^m)^2 \mathbf{1}(T \geq \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}}) \times \|\sigma^n(s, \widehat{X}^n_s, \widetilde{\xi}^n_s)$$

$$-\sigma^{n_0}(s, \widehat{X}^n_s, \widetilde{\xi}^n_s)\|^2 ds > \delta/2\bigg) \leq C\mathbb{E}\bigg(\int_0^T \mathbf{1}(s < \gamma^X_{\widetilde{R}} \wedge \gamma^X_{n,\widetilde{R}})\mathbb{E}^3(1$$

$$+|\widehat{X}_s^n|^m)^2\mathbf{1}(T \ge \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}}) \times \|\sigma^n(s,\widehat{X}_s^n,\widetilde{\xi}_s^n) - \sigma^{n_0}(s,\widehat{X}_s^n,\widetilde{\xi}_s^n)||^2 ds\Big)^\beta$$

$$\overset{\text{CBS}}{\le} C(T\mathbb{E}\sup_s(1+|\widehat{X}_s^n|))^{1/2}(\mathbb{P}(T \ge \gamma^\xi_{\widetilde{R}} \wedge \gamma^\xi_{n,\widetilde{R}}))^{\beta/2}.$$

Due to a priori bounds from Lemma 2, the first multiplier here is bounded, while the second is small if R is large enough, all uniformly in n. Hence $\lim_{R\to\infty} S^1 = 0$. Therefore,

$$\lim_{n,n_0\to\infty} J^1 = 0.$$

Both other two terms J^2 and J^3 are tackled similarly to the drift terms in the previous step with the use of (25) and Itô's isometry.

This proves the desired bound (26). So, (21) and (22) hold true, and thus, the proposition is proved.

2.3 Proof of Theorem 1

We will use a hint from [30, section 4] in order to reduce the statement to the case considered earlier in the Proposition 1. However, for the reader's convenience we repeat the details.

Denote $\widetilde{\Sigma}[t,x,\mu] := \sqrt{A[t,x,\mu]}$, where $A[t,x,\mu] := \Sigma[t,x,\mu]\Sigma^*[t,x,\mu]$, and suppose that there exists a weak solution $\widetilde{X}$ of the equation

$$\widetilde{X}_t = x_0 + \int_0^t B[s,\widetilde{X}_s,\mu_s]ds + \int_0^t \widetilde{\Sigma}[s,\widetilde{X}_s,\mu_s]d\widetilde{W}_s,$$

with some d-dimensional Wiener process $(\widetilde{W}_t,\, t \ge 0)$ on some probability space and where μ_s stands for the distribution of $\widetilde{X}_s$.

Existence of this weak solution is already justified in the Proposition 1. Here the positive-definite matrix-function $\sqrt{A[t,x,\mu]}$ is defined via the Cauchy-Riesz-Dunford formula (9), or, equivalently, (10).

Further, without loss of generality we may and will assume that on the same probability space there exists another *independent* d_1-dimensional Wiener process $(\overline{W}_t,\, t \ge 0)$. Let I denote a $d_1 \times d_1$-dimensional unit matrix and let

$$p[s,x,\mu] := \widetilde{\Sigma}[s,x,\mu]^{-1}\,\Sigma[s,x,\mu].$$

Note that the matrix $\widetilde{\Sigma}[s,x,\mu]$ is symmetric and that

$$p^*p[s,x,\mu] = \Sigma^*[s,x,\mu](\widetilde{\Sigma}^*[s,x,\mu])^{-1}\widetilde{\Sigma}[s,x,\mu]^{-1}\,\Sigma[s,x,\mu]$$

$$
\begin{aligned}
&= \Sigma^*[s,x,\mu](A)^{-1}[s,x,\mu]\Sigma[s,x,\mu],\\
&p^*[s,x,\mu]p[s,x,\mu]p^*[s,x,\mu]p[s,x,\mu]\\
&= \Sigma^*[s,x,\mu](A)^{-1}[s,x,\mu]\Sigma[s,x,\mu]\Sigma^*[s,x,\mu](A)^{-1}[s,x,\mu]\Sigma[s,x,\mu]\\
&= \Sigma^*(A)^{-1}(A)(A)^{-1}\Sigma[s,x,\mu] = \Sigma^*(A)^{-1}\Sigma[s,x,\mu],
\end{aligned}
$$

and let

$$
W^0_t := \int_0^t p^*[s,\widetilde{X}_s,\mu_s]\, d\widetilde{W}_s + \int_0^t (I - p^*[s,\widetilde{X}_s,\mu_s]p[s,\widetilde{X}_s,\mu_s])\, d\overline{W}_s.
$$

Notice that

$$
\begin{aligned}
\Sigma[s,x,\mu]p^*[s,x,\mu] &= A[s,x,\mu](A[s,x,\mu])^{-1/2} = (A[s,x,\mu])^{1/2},\\
\Sigma[s,x,\mu]p^*[s,x,\mu]p[s,x,\mu] &= (A[s,x,\mu])^{1/2}p[s,x,\mu]\\
&= (A[s,x,\mu])^{1/2}(a[s,x,\mu])^{-1/2}\Sigma[s,x,\mu] = \Sigma[s,x,\mu].
\end{aligned}
$$

Due to the multivariate Lévy characterization theorem this implies that W^0 is a d_1-dimensional Wiener process, since its matrix angle characteristic (also known as a matrix angle bracket) equals

$$
\begin{aligned}
\langle W^0, W^0\rangle_t &= \int_0^t p^*p[s,\widetilde{X}_s,\mu_s]\, ds + \int_0^t (I - p^*p[s,\widetilde{X}_s,\mu_s])^*(I - p^*p[s,\widetilde{X}_s,\mu_s])\, ds\\
&= \int_0^t (p^*p[s,\widetilde{X}_s,\mu_s] + I - 2p^*p[s,\widetilde{X}_s,\mu_s] + p^*pp^*p[s,\widetilde{X}_s,\mu_s])\, ds\\
&= \int_0^t (I - p^*p[s,\widetilde{X}_s,\mu_s] + p^*pp^*p[s,\widetilde{X}_s,\mu_s])\, ds\\
&= \int_0^t (I - \Sigma^*(A)^{-1}\Sigma[s,\widetilde{X}_s,\mu_s] + \Sigma^*(A)^{-1}(A)(A)^{-1}\Sigma[s,\widetilde{X}_s,\mu_s])\, ds = \int_0^t I\, ds = t\, I.
\end{aligned}
$$

Next, due to the stochastic integration rules (see [12]),

$$
\begin{aligned}
&\int_0^t \Sigma[s,\widetilde{X}_s,\mu_s]\, dW^0_s = \int_0^t \Sigma p^*[s,\widetilde{X}_s,\mu_s]\, d\widetilde{W}_s + \int_0^t \Sigma(I - p^*p)[s,\widetilde{X}_s,\mu_s]\, d\overline{W}_s\\
&= \int_0^t (A)^{1/2}[s,\widetilde{X}_s,\mu_s]\, d\widetilde{W}_s = \int_0^t \widetilde{\Sigma}[s,\widetilde{X}_s,\mu_s]\, d\widetilde{W}_s = \widetilde{X}_t - x_0 - \int_0^t B[s,\widetilde{X}_s,\mu_s]\, ds.
\end{aligned}
$$

In other words, $(\widetilde{X}, W^0)$ is a weak solution of the equation (1). It remains to notice that since we did not change measures, μ_s is still the distribution of $\widetilde{X}_s$ by the assumption. The proof of the Theorem 1 is thus completed.

Remark 3. *There is a non-rigorous view that for SDE solutions everything related to weak solutions and weak uniqueness depends only on the matrix $\sigma^*\sigma$ and not on σ itself. This is not precise. Firstly, for strong solutions this is not true because regularity such as Lipschitz condition or even a simple continuity may fail for a badly chosen square root, let us forget about non-Borel square roots. Secondly, even for weak solutions in the absence of non-degeneracy and if the square root is not continuous, there is no guarantee that weak solution exists for any square root. Also, existing results about weak solutions and weak uniqueness, see [4, 9], impose conditions on σ and not on $\sigma\sigma^*$. Hence, a vague "common knowledge" is not sufficient and had to show the calculus.*

3 Strong solutions; strong and weak uniqueness

3.1 On strong existence

In this section it is shown that strong solution of the equation (1)–(2) exists under appropriate conditions. Emphasize that we do not claim strong uniqueness in this section, but only strong existence in the sense of the Definition 1. We also notice for interested readers that in [28] the assumption of continuity in time was dropped in comparison to [27]; so, just a certain local Lipschitz condition suffices for our aim.

Proposition 2. *Let $\mathbb{E}|x_0|^4 < \infty$. Also, let the coefficients b and σ satisfy all conditions of the Theorem 1 and the non-degeneracy assumption (6), and let just σ be Lipschitz in x uniformly with respect to s and locally with respect to y,*

$$\|\sigma(t,x,y) - \sigma(t,x',y)\| \le C(1+|y|^2)|x-x'|. \tag{27}$$

Then the equation (1)–(2) has a strong solution and, moreover, every solution is strong and, in particular, solution may be constructed on any probability space equipped with a d_1-dimensional Wiener process.

This result is likely to be a common knowledge. A brief sketch of the proof is presented below for completeness and because the authors were unable to find an exact reference.

1. First of all, note that weak solutions exist and a priori bounds (13)–(14) are valid. Considerations are based on the results from [27] and [28] about strong solutions for SDEs for a Borel measurable drift which is assumed bounded or with a linear growth in both papers. Since weak solution does exist, whatever is its distribution μ, the process X may be considered as an ordinary SDE with coefficients depending on time,

$$\widetilde{b}(t,x) = B[t,x,\mu_t], \qquad \widetilde{\sigma}(t,x) = \Sigma[t,x,\mu_t],$$

and, hence,

$$dX_t = \widetilde{b}(t,X_t)dt + \widetilde{\sigma}(t,X_t)dW_t, \qquad X_0 = x. \tag{28}$$

Recall that according to Lemma 1, the new coefficients $\widetilde{b}(t,x)$ and $\widetilde{\sigma}(t,x)$ are Borel measurable.

2. Now in order to establish strong existence it suffices to verify that the new coefficients $\widetilde{b}$ and $\widetilde{\sigma}$ both satisfy linear growth in x condition uniform in time, and $\widetilde{\sigma}$ is Lipschitz continuous in x and remains uniformly non-degenerate.

We have, for any $T > 0$ and $0 \le t \le T$,

$$|\widetilde{b}(t,x)| = \left|\int b(t,x,y)\,\mu_t(dy)\right| \le C\int (1+|x|)\,\mu_t(dy) = C\,(1+|x|).$$

Similarly, it also follows that

$$\|\widetilde{\sigma}(t,x)\| \le C\int (1+|x|)\,\mu_t(dy) = C\,(1+|x|).$$

Further, by virtue of (27),

$$\begin{aligned}
&\|\widetilde{\sigma}(t,x) - \widetilde{\sigma}(t,x')\| = \|\Sigma[t,x,\mu_t] - \Sigma[t,x',\mu_t]\| \\
&= \left\|\int \sigma(t,x,y)\,\mu_t(dy) - \int \sigma(t,x',y)\,\mu_t(dy)\right\| \\
&\le C\,|x-x'|\int (1+|y|^2)\,\mu_t(dy) \le C_T\,|x-x'|.
\end{aligned}$$

The uniform non-degeneracy of σ, hence, also of $\sigma\sigma^*$, follows from the inequality (6) by integration with respect to μ_t.

These properties suffice for the local strong uniqueness of solution of the equation (1)–(2) by virtue of the results in [27]. However, because weak solution is well-defined for all values of time, strong uniqueness is global. According to the Yamada-Watanabe principle ([32]), any solution of the equation (2) is strong. So, any solution of the original equation (1) is also strong. This completes the proof of the Proposition 2.

Remark 4. *Notice that as a solution of the "linearized" equation (28), X is pathwise unique, but so far it is not known if this implies the same property for X as a solution of (1), unless* weak uniqueness *for the equation (1) has been established. In a restricted framework this will be done in Theorem 3 below.*

Remark 5. *In the case of* $d = 1$*, Lipschitz condition may be relaxed to Hölder of order* $1/2$ *and, actually, a little bit further by using techniques from [32] and [26].*

3.2 Weak uniqueness

In this section weak uniqueness will be shown for the equation (1) – (2) under appropriate conditions. This result requires only a Borel measurability of the drift with respect to the state variable x, although, it assumes that diffusion σ does not depend on y along with some additional continuity condition in x and non-degeneracy. The drift may be unbounded in the state variable x.

Theorem 2. *Let* $\mathbb{E}\exp(r|x_0|^2) < \infty$ *for some* $r > 0$*, and let the functions* b *and* σ *be Borel measurable, and*

$$\sigma(s, x, y) \equiv \sigma(s, x),$$

that is, σ *does not depend on the variable* y*; let* σ *satisfy the non-degeneracy assumption (6); let* $d_1 = d$*, the matrix* σ *be bounded, symmetric and invertible, and let there exist* $C > 0$ *such that the function*

$$\widetilde{B}[s, x, \mu] := \sigma^{-1}(s, x)\, B[s, x, \mu]$$

satisfies the linear growth condition: there is $C > 0$ *such that for all* $x \in \mathbb{R}^d$*,*

$$\sup_{s,\mu} |\widetilde{B}[s, x, \mu]| \le C(1 + |x|). \tag{29}$$

Also assume that the matrix-function $\sigma(t, x)$ *satisfies the uniform continuity condition in* x *which guarantees that the equation*

$$dX_t^0 = \sigma(t, X_t^0)\, dW_t, \quad X_0^0 = \xi, \tag{30}$$

with an $\mathcal{F}_0$*-measurable initial data* ξ *possessing a given distribution* $\mu_0 = \mathcal{L}(x_0)$*, has a unique weak solution (see [27, 28]). Then under the assumptions of Theorem 1 solution of the equation (1)–(2) is weakly unique.*

Remark 6. *In case of* $d = 1$ *continuity of* σ *in* x *is not needed. Under the additional assumption of boundedness of* $\widetilde{b}$ *or* $\widetilde{B}$ *an exponential moment of the initial value* x_0 *is not necessary and can be replaced by the fourth moment, or even weaker.*

3.3 Proof of Theorem 2

Denote by X^0_t any (weakly unique) weak solution of the Itô equation (30). Note that

$$dW_t = \sigma^{-1}(t, X^0_t) dX^0_t.$$

1. Recall that under the assumptions of the theorem, any solution of the equation (1)–(2) is strong by virtue of the Proposition 2. Hence, it suffices to show weak uniqueness, after which strong uniqueness for this equation will follow from strong uniqueness for the "linearized" equation (28). We will show this weak uniqueness by contradiction. Suppose there are two solutions (X^1, W^1) and (X^2, W^2) of the equation (1) with distributions μ^1 and μ^2 respectively in the space of trajectories $C[0,\infty;\mathbb{R}^d]$:

$$dX^1_t = \sigma(t, X^1_t)\, dW^1_t + B[t, X^1_t, \mu^1_t]\, dt, \quad X^1_0 = \xi^1, \tag{31}$$

and

$$dX^2_t = \sigma(t, X^2_t)\, dW^2_t + B[t, X^2_t, \mu^2_t]\, dt, \quad X^2_0 = \xi^2, \tag{32}$$

respectively, with $\mathcal{L}(\xi^1) = \mathcal{L}(\xi^2) = \mathcal{L}(x_0)$. Yet, under the present setting it will be shown that firstly $\mu^1 = \mu^2$ and secondly $X^1 = X^2$ a.s. Note that both X^1 and X^2 are Markov processes ([18]).

Both solutions (X^i, μ^i) in the weak sense may be obtained from the Wiener process W and solution X^0 of the equation without the drift (30) via Girsanov's transformations using the following stochastic exponents:

$$\gamma^i_T = \exp\left(\int_0^T \widetilde{B}[s, X^0_s, \mu^i_s]\, dW_s - \frac{1}{2}\int_0^T |\widetilde{B}[s, X^0_s, \mu^i_s]|^2\, ds\right), \quad i = 1, 2,$$

where $\widetilde{b}(t,x,y) := \sigma^{-1}(t,x)\, b(t,x,y)$, $\widetilde{B}[t,x,\mu] := \sigma^{-1}(t,x)\, B[t,x,\mu]$, $|\widetilde{B}|$ stands for the modulus of the vector $\widetilde{B}$, and $\widetilde{B}[s, X^0_s, \mu^i_s]\, dW_s$ is understood as a scalar product, $\sum_{j=1}^d \widetilde{B}^j[s, X^0_s, \mu^i_s] d\widetilde{W}^j_s$.

It is well-known that in the case of bounded $\widetilde{B}$ the random variables γ^i_T, $i = 1, 2$, are probability densities due to Girsanov theorem (see, e.g., [20, Theorem 6.8.8]). So, till the step 4 we assume $\widetilde{B}$ bounded; note that in this case we have,

$$|\widetilde{B}[s,x,\mu] - \widetilde{B}[s,x,\nu]| \le C\|\mu - \nu\|_{TV}. \tag{33}$$

Indeed,

$$|\widetilde{B}[s,x,\mu] - \widetilde{B}[s,x,\nu]| = |\sigma^{-1}(s,x)\int (b(s,x,y)\mu(dy) - b(s,x,y)\nu(dy)])|$$

$$= |\sigma^{-1}(s,x)\int(b(s,x,y)(\mu(dy)-\nu(dy))])| \le \int |\sigma^{-1}(s,x)b(s,x,y)|\,|\mu(dy)-\nu(dy)|$$

$$\le \|\sigma^{-1}(s,x)b(s,x,\cdot)\|_B\|\mu-\nu\|_{TV}.$$

The calculus with a bounded $\widetilde{B}$ is needed so as to explain the idea which will be further expanded to the case without this restriction. Also this will justify the statement in the Remark 6.

Denote

$$\widetilde{W}_t^1 := W_t - \int_0^t \widetilde{B}[s, X_s^0, \mu_s^1]\,ds, \quad 0 \le t \le T.$$

This is a new Wiener process on $[0,T]$ under the probability measure P^{γ^1} defined by its density as $(dP^{\gamma^1}/dP)(\omega) = \gamma_T^1$. Then, on the same interval $[0,T]$, on the probability space with a Wiener process $(\Omega, \mathcal{F}, (\widetilde{W}_t^1, F_t), \mathbb{P}^{\gamma^1})$, the process $(X_t^0,\, 0 \le t \le T)$ satisfies the equation,

$$\begin{aligned} dX_t^0 &= \sigma(t,X_t^0)d\widetilde{W}_t^1 + \sigma(t,X_t^0)\widetilde{B}[t,X_t^0,\mu_t^1]dt \\ &= \sigma(t,X_t^0)d\widetilde{W}_t^1 + B[t,X_t^0,\mu_t^1]\,dt, \end{aligned} \tag{34}$$

with the initial condition $X_0^0 = x_0$. In other words, the process X^0 on $[0,T]$ satisfies the equation (31), just with another Wiener process and under another probability measure. However, given μ_t^1, $0 \le t \le T$, this solution considered as a solution of Itô's or "linearized" equation is weakly unique. This is a well-known fact for bounded coefficients due to the results on uniqueness for solutions of parabolic equations, see [23]. For unbounded coefficients under the linear growth conditions this follows by truncation and via stopping times in a standard way. Further, this uniqueness for X^0 implies weak uniqueness for the pair (X^0, W), see [1] et al.

So, the pair $(X_t^0, \widetilde{W}_t^1,\, 0 \le t \le T)$ has the same distribution under the measure $\mathbb{P}^{\gamma^1}$ as the pair $(X_t^1, W_t,\, 0 \le t \le T)$ under the measure $\mathbb{P}$. Therefore, the marginal distribution of X_t^0 under the measure $\mathbb{P}^{\gamma^1}$ equals μ_t^1, i.e., the couple (X_t^0, μ_t^1) under $\mathbb{P}^{\gamma^1}$ solves the McKean-Vlasov equation (1), that is, it is equivalent to the pair $(X_t^1, \mu_t^1,\, 0 \le t \le T)$ under the measure $\mathbb{P}$.

Note for the sequel that $d\widetilde{W}_t^1$ admits a representation

$$d\widetilde{W}_t^1 = \sigma^{-1}(t,X_t^0)\,dX_t^0 - \sigma^{-1}(t,X_t^0)B[t,X_t^0,\mu_t^1]\,dt = \sigma^{-1}(t,X_t^0)dX_t^0 - \widetilde{B}[t,X_t^0,\mu_t^1]\,dt,$$

or, equivalently,

$$\sigma^{-1}(t, X_t^0) dX_t^0 = d\widetilde{W}_t^1 + \widetilde{B}[t, X_t^0, \mu_t^1]\, dt.$$

Similarly, let

$$\widetilde{W}_t^2 := W_t - \int_0^t \widetilde{B}[s, X_s^0, \mu_s^2]\, ds, \quad 0 \le t \le T.$$

This is a new Wiener process on $[0, T]$ under the probability measure P^{γ^2} defined by its density as $(dP^{\gamma^2}/dP)(\omega) = \gamma^2$. Then, on the interval $[0, T]$, on the probability space with a Wiener process $(\Omega, \mathcal{F}, (\widetilde{W}_t^2, F_t), \mathbb{P}^{\gamma^2})$, the process $(X_t^0,\, 0 \le t \le T)$ satisfies the equation,

$$dX_t^0 = \sigma(t, X_t^0) d\widetilde{W}_t^2 + B[t, X_t^0, \mu_t^2]\, dt,$$

with the initial condition $X_0^0 = x_0$. In other words, the process X^0 on $[0, T]$ satisfies the equation (32), just with another Wiener process and under another measure. However, given μ_t^2, $0 \le t \le T$, this solution considered as a solution of Itô's equation is weakly unique. Therefore, the couple (X_t^0, μ_t^2) under the probability measure $\mathbb{P}^{\gamma^2}$ solves the McKean-Vlasov equation (1), that is, it is equivalent to the pair $(X_t^2, \mu_t^2,\, 0 \le t \le T)$ under the measure $\mathbb{P}$.

2. This provides us a way to write down the density of the distribution of X^1 on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the distribution of X^2 on $(\Omega, \mathcal{F}, \mathbb{P})$ on the interval of time $[0, T]$. We have, for any measurable $A \subset C[0, T; \mathbb{R}^d]$,

$$\mu_{0,T}^{\mathbf{1}}(A) := \mathbb{P}(X^1 \in A) = \mathbb{P}^{\gamma^1}(X^0 \in A) = \mathbb{E}^{\gamma^1} \mathbf{1}(X^0 \in A) = \mathbb{E}\gamma_T^1 \mathbf{1}(X^0 \in A), \quad (35)$$

and

$$\mu_{0,T}^2(A) := \mathbb{P}(X^2 \in A) = \mathbb{P}^{\gamma^2}(X^0 \in A) = \mathbb{E}^{\gamma^2} \mathbf{1}(X^0 \in A) = \mathbb{E}\gamma_T^2 \mathbf{1}(X^0 \in A). \quad (36)$$

So, on the sigma-algebra $\mathcal{F}_T^W$ we obtain that

$$\frac{\mu_{[0,T]}^2(dX)}{\mu_{[0,T]}^1(dX)}(X^0) = \frac{\gamma_T^2}{\gamma_T^1}(X^0) = \exp\left(\int_0^T \widetilde{B}[s, X_s^0, \mu_s^2] dW_s - \frac{1}{2}\int_0^T |\widetilde{B}[s, X_s^0, \mu_s^2]|^2 ds\right)$$

$$\times \exp\left(-\int_0^T \widetilde{B}[s, X_s^0, \mu_s^1] dW_s + \frac{1}{2}\int_0^T |\widetilde{B}[s, X_s^0, \mu_s^1]|^2 ds\right)$$

$$= \exp\left(\int_0^T (\widetilde{B}[s, X_s^0, \mu_s^2] - \widetilde{B}[s, X_s^0, \mu_s^1]) dW_s - \frac{1}{2}\int_0^T (|\widetilde{B}[s, X_s^0, \mu_s^2]|^2 - |\widetilde{B}[s, X_s^0, \mu_s^1]|^2) ds\right)$$

$$= \exp\left(\int_0^T (\widetilde{B}[s, X_s^0, \mu_s^2] - \widetilde{B}[s, X_s^0, \mu_s^1]) \sigma^{-1}(s, X_s^0) dX_s^0\right)$$

$$\times \exp\left(-\frac{1}{2}\int_0^T (|\widetilde{B}[s,X_s^0,\mu_s^2]|^2 - |\widetilde{B}[s,X_s^0,\mu_s^1]|^2)ds\right)$$

$$= \exp\left(\int_0^T (\widetilde{B}[s,X_s^0,\mu_s^2] - \widetilde{B}[s,X_s^0,\mu_s^1])(d\widetilde{W}_s^1 + \widetilde{B}[s,X_s^0,\mu_s^1]\,ds)\right)$$

$$\times \exp\left(-\frac{1}{2}\int_0^T (|\widetilde{B}[s,X_s^0,\mu_s^2]|^2 - |\widetilde{B}[s,X_s^0,\mu_s^1]|^2)ds)\right)$$

$$= \exp\left(\int_0^T (\widetilde{B}[s,X_s^0,\mu_s^2] - \widetilde{B}[s,X_s^0,\mu_s^1])\,d\widetilde{W}_s^1 - \frac{1}{2}\int_0^T |\widetilde{B}[s,X_s^0,\mu_s^2] - \widetilde{B}[s,X_s^0,\mu_s^1]|^2 ds\right).$$

Further, due to (35) and (36) the measure μ^i is an image of $\mathbb{P}^{\gamma^i}$ under the mapping X^0 for $i = 1, 2$. So,

$$v(t) := \|\mu^1_{[0,t]} - \mu^2_{[0,t]}\|_{TV} \le \|P^{\gamma^1}|_{\mathcal{F}_t^W} - P^{\gamma^2}|_{\mathcal{F}_t^W}\|_{TV}. \tag{37}$$

Since the two measures P^{γ^1} and P^{γ^1} on $\mathcal{F}_t^W$ are equivalent with the density

$$\frac{dP^{\gamma^2}}{dP^{\gamma^1}}\Big|_{\mathcal{F}_t^W}(\omega) = \frac{\gamma_t^2}{\gamma_t^1}(\omega),$$

the total variation distance between them equals (denoting $\rho_t = \gamma_t^2/\gamma_t^1$),

$$\frac{1}{2}\|P^{\gamma^2}|_{\mathcal{F}_t^W} - P^{\gamma^1}|_{\mathcal{F}_t^W}\|_{TV} = \int_\Omega \left(1 - \frac{\gamma_t^2}{\gamma_t^1}(\omega)\wedge 1\right)\mathbb{P}^{\gamma^1}(d\omega) = 1 - \mathbb{E}^{\gamma^1}\rho_t \wedge 1 \le \sqrt{E^{\gamma^1}\rho_t^2 - 1}.$$

Let us justify the last inequality for completeness, dropping the sub-index t:

$$\begin{aligned} 1 - \mathbb{E}^{\gamma^1}(\rho\wedge 1) &= \mathbb{E}^{\gamma^1}(1-\rho\wedge 1) \\ \le \sqrt{\mathbb{E}^{\gamma^1}(1-\rho\wedge 1)^2} &= \sqrt{E^{\gamma^1}(1-\rho\mathbf{1}(\rho\le 1) - \mathbf{1}(\rho>1))^2} \\ = \sqrt{\mathbb{E}^{\gamma^1}(\mathbf{1}(\rho\le 1) - \rho\mathbf{1}(\rho\le 1))^2} &= \sqrt{\mathbb{E}^{\gamma^1}\mathbf{1}(\rho\le 1)(\rho-1)^2} \\ \le \sqrt{\mathbb{E}^{\gamma^1}(\rho-1)^2} &= \sqrt{\mathbb{E}^{\gamma^1}\rho^2 - 1}, \end{aligned}$$

as required. We used the CBS inequality. So, due to (37),

$$v(t) \le 2\sqrt{\mathbb{E}^{\gamma^1}\rho_t^2 - 1}. \tag{38}$$

Now, again by virtue of the CBS inequality,

$$\mathbb{E}^{\gamma^1}\rho_T^2 = \mathbb{E}^{\gamma^1}\exp\Big(-2\int_0^T (\widetilde{B}[s,X_s^0,\mu_s^2] - \widetilde{B}[s,X_s^0,\mu_s^1])d\widetilde{W}_s^1$$

$$
\begin{aligned}
& - \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds\Big) \\
&= \mathbb{E}^{\gamma^1} \exp(-2 \int_0^T (\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]) d\widetilde{W}^1_s \\
& -4 \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds) \\
& \times \exp\Big(3 \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds\Big) \\
&\le \Bigg(\mathbb{E}^{\gamma^1} \exp\Big(- 4 \int_0^T (\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]) d\widetilde{W}^1_s \\
& -8 \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds\Big)\Bigg)^{1/2} \\
& \times \left(\mathbb{E}^{\gamma^1} \exp\left(6 \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds\right)\right)^{1/2} \\
&\le (=) \sqrt{\mathbb{E}^{\gamma^1} \exp\left(6 \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds\right)}.
\end{aligned} \tag{39}
$$

The last inequality is always true; for a bounded $\widetilde{B}$ it is, apparently, an equality.

3. If $\widetilde{B}$ is bounded with the norm $\|\widetilde{B}\|_B := \sup_{s,\mu} |\widetilde{B}[s, x, \mu]|$, then

$$
\begin{aligned}
& \mathbb{E}^{\gamma^1} \exp\left(6 \int_0^T |\widetilde{B}[s, X^0_s, \mu^2_s] - \widetilde{B}[s, X^0_s, \mu^1_s]|^2 ds\right) \\
& \le \mathbb{E}^{\gamma^1} \exp\left(6 \|\widetilde{B}\|^2_B \int_0^T \|\mu^1_s - \mu^2_s\|^2_{TV}\, ds\right).
\end{aligned} \tag{40}
$$

Here the value under the expectation is non-random; hence, the symbol of this expectation may be dropped. Therefore, we have with $C = 6\|\widetilde{B}\|^2_B$,

$$
v(T) \le 2 \sqrt{\exp\left(C \int_0^T v(s)^2 ds\right) - 1}. \tag{41}
$$

Recall that $v(t) \leq 2$, and the function v increases in t. Let us choose $\alpha_0 > 0$ small so that for any $0 \leq \alpha \leq \alpha_0$,

$$\exp(\alpha) - 1 \leq 2\alpha, \tag{42}$$

and take $T \leq \alpha_0/(4C)$. Then $C \int_0^T v(s)^2\, ds \leq CTv(T)^2 \leq 4CT \leq \alpha_0$. So,

$$v(T) \leq 2\sqrt{\exp\left(C\int_0^T v(s)^2\, ds\right) - 1} \leq 2\sqrt{2CTv(T)^2} = 2\sqrt{2CT}v(T).$$

If we choose T so small that $2\sqrt{2CT} < 1$, that is, $T < 1/(8C)$, then it follows that $v(T) = 0$. Hence, $v(T) = 0$ for any $T < \min(1/(8C), \alpha_0/(4C))$. Let us fix some $T > 0$ satisfying this inequality.

Further, we conclude by induction that

$$v(2T) = v(3T) = \ldots = 0. \tag{43}$$

Indeed, assume that $v(kT) = 0$ is already established for some integer $k > 0$. Redefine the stochastic exponents:

$$\gamma^i_{kT,(k+1)T} = \exp\left(\int_{kT}^{(k+1)T} \widetilde{B}[s, X^0_s, \mu^i_s]\, dW_s - \frac{1}{2}\int_{kT}^{(k+1)T} |\widetilde{B}[s, X^0_s, \mu^i_s]|^2\, ds\right), \quad i = 1, 2,$$

and re-denote

$$\widetilde{W}^1_t := W_t - \int_{kT\wedge t}^{t} \widetilde{B}[s, X^0_s, \mu^1_s]\, ds, \quad 0 \leq t \leq (k+1)T.$$

Then $\widetilde{W}^1_t$ is a new Wiener process on $[kT, (k+1)T]$ starting at W_{kT} under the probability measure P^{γ^1} defined by its density as $(dP^{\gamma^1}/dP)(\omega) = \gamma^1_{kT,(k+1)T}$. Repeating the calculus leading to (39), (40) and (41), and having in mind the induction assumption $v(kT) = 0$, we obtain with the same constant C that

$$v((k+1)T) \leq \sqrt{\exp\left(C\int_{kT}^{(k+1)T} v(s)^2 ds\right) - 1},$$

which straightforwardly implies that

$$v((k+1)T) \leq \sqrt{2CTv((k+1)T)^2} = \sqrt{2CT}v((k+1)T).$$

As earlier, the condition $T < \min(1/(2C), 1/(\alpha C))$ (see (42)) guarantees that

$$v((k+1)T) = 0,$$

as required. This completes the induction (43).

Hence, solution is weakly unique on the whole $\mathbb{R}_+$. As noticed above, strong uniqueness also follows. For bounded $\widetilde{B}$ the statements of Theorem 3 as well as of the Remark 6 are justified.

4. Now let us return to the inequality (39) and explain how to drop the additional assumption of boundedness of $\widetilde{B}$, and also how to deal with a localized version of (33). First of all, prior to (39) we have to show that γ^i, $i = 1, 2$, are, indeed, probability densities for which it suffices to show uniform integrability for $T > 0$ small enough: for example, it suffices to check that

$$\mathbb{E}(\gamma_T^i)^2 < \infty, \quad i = 1, 2.$$

Via the estimates similar to (39) by virtue of CBS inequality, this problem is reduced to the question whether or not the following expression is finite:

$$\begin{aligned}\mathbb{E}(\gamma_T^i)^2 \le \left(\mathbb{E}\exp\left(4\int_0^T \widetilde{B}[s, X_s^0, \mu_s^i]\, dW_s - 8\int_0^T |\widetilde{B}[s, X_s^0, \mu_s^i]|^2\, ds\right)\right)^{1/2} \\ \times \left(\mathbb{E}\exp\left(6\int_0^T |\widetilde{B}[s, X_s^0, \mu_s^i]|^2\, ds\right)\right)^{1/2} \\ \le \left(\mathbb{E}\exp\left(6\int_0^T |\widetilde{B}[s, X_s^0, \mu_s^i]|^2\, ds\right)\right)^{1/2} \\ \le \left(\mathbb{E}\exp\left(C\int_0^T (1 + |X_s^0|^2)\, ds\right)\right)^{1/2}. \end{aligned} \tag{44}$$

In the last inequality the assumption on the linear growth of $\widetilde{B}$ was used.

Suppose for instant that the finiteness of the last expectation in the last line of (44) has been shown; then, by standard induction arguments with conditional expectations it follows that both γ_T^i are, indeed, probability densities for *any* $T > 0$. Hence, the calculus leading to (38) and (39) is valid and we have that

$$v(t) = \|\mu^1_{[0,t]} - \mu^2_{[0,t]}\|_{TV} \le 2\sqrt{\mathbb{E}^{\gamma^1}\rho^2 - 1}, \tag{45}$$

and

$$\mathbb{E}^{\gamma^1}\rho^2 \le \sqrt{\mathbb{E}^{\gamma^1}\exp\left(6\int_0^T \left|\widetilde{B}[s, X_s^0, \mu_s^2] - \widetilde{B}[s, X_s^0, \mu_s^1]\right|^2 ds\right)}. \tag{46}$$

It is a general fact which does not use any boundedness of $\widetilde{B}$ in any variable but only in the last variable is,

$$\left|\widetilde{B}[s, X_s^0, \mu_s^2] - \widetilde{B}[s, X_s^0, \mu_s^1]\right| \le \sup_y \left|\widetilde{B}(s, X_s^0, y)\right| \|\mu_s^2 - \mu_s^1\|_{TV}. \tag{47}$$

Due to the linear growth assumption (29), the inequality (47) implies

$$\left|\widetilde{B}[s, X_s^0, \mu_s^2] - \widetilde{B}[s, X_s^0, \mu_s^1]\right| \le C(1 + |X_s^0|)\|\mu_s^1 - \mu_s^2\|_{TV}. \tag{48}$$

Hence, by virtue of (48) we obtain

$$\begin{aligned}\mathbb{E}^{\gamma^1}\rho^2 &\le \mathbb{E}^{\gamma^1}\exp\left(6\int_0^T [C(1 + |X_s^0|)\|\mu_s^1 - \mu_s^2\|_{TV}]^2 ds\right) \\ &\le \mathbb{E}^{\gamma^1}\exp\left(6C^2 v(T)^2 \int_0^T (1 + |X_s^0|^2)\, ds\right). \end{aligned} \tag{49}$$

Recall that the process X^0 satisfies the equation (34) on $[0, T]$ with respect to the measure $\mathbb{P}^{\gamma^1}$. We want to show that given C, the right hand side in (44) is finite for any T small enough. For this end, denote $6C^2 v(T)^2 := r \ge 0$. Recall that in any case $v(T) \le 2$. We would like to show that for any fixed constant $K > 0$ ($K = 24C^2 + 1$ suffices), the value

$$\mathbb{E}^{\gamma^1}\exp\left(r\int_0^T (1 + |X_s^0|^2)\, ds\right)$$

is finite for $0 \le r < K$, and differentiable with respect to r, and that this derivative is non-negative and small uniformly in $r \in [0, K)$, if $T > 0$ is small enough.

It suffices to show the same properties, still for small enough T, for the function

$$\psi(r, T) = \mathbb{E}\exp\left(r\int_0^T (1 + |X_s^1|^2)\, ds\right), \tag{50}$$

where X^1 solves the equation (34) on $[0, T]$ with respect to the original measure $\mathbb{P}$, because X^1 solves the same equation with respect to the measure $\mathbb{P}$ as the process X^0 with respect to the measure $\mathbb{P}^{\gamma^1}$ on $[0, T]$.

Denote $\bar{X}_t^1 := \sup_{0\le s\le t}|X_s^1|$. From the equation (31) and due to the assumption (29) and boundedness of σ, we have that

$$|X_t^1| \le |x_0| + C\left(\int_0^t (1 + |X_s^1|)\, ds + \left|\int_0^t \sigma(s, X_s^1)\, dW_s^1\right|\right).$$

Now, by virtue of Gronwall's inequality and since both sides in this inequality are finite, we obtain with some $C > 0$,

$$\bar{X}_t^1 \le C \exp(Ct) \left(|x_0| + \sup_{0 \le t' \le t} \left| \int_0^{t'} \sigma(s, X_s^1)\, dW_s^1 \right| \right).$$

Therefore, for each $r > 0$

$$\exp(rT(\bar{X}_T^1)^2) \le C_T \exp(2rT|x_0|^2) \exp\left(2rT \sup_{0 \le t' \le T} \left| \int_0^{t'} \sigma(s, X_s^1)\, dW_s^1 \right|^2 \right),$$

with some $C_T < \infty$. Recall that the matrix-function σ is bounded. Denote by $\sigma^i(r,x)$ the ith row of the matrix σ, $1 \le i \le d$. Due to the exponential martingale inequalities following from Girsanov theorem along with Doob inequality for martingales (see, e.g., [20, Sections 6.8 & 3.4]) there exist constants $C_1, C_2 > 0$ such that for any $a > 0$, and for any $t > 0$, and for any i,

$$\mathbb{P}(\sup_{0 \le s \le t} \left| \int_0^s \sigma^i(r, X_r^1) dW_r^1 \right| > a) \le C_1 \exp(-a^2/(C_2 t)).$$

This implies that (with new $C_1, C_2 > 0$)

$$\begin{aligned} &\mathbb{P}\left(\sup_{0 \le s \le T} \left| \int_0^s \sigma(r, X_r^1) dW_r^1 \right| > a \right) \\ &\le \sum_{i=1}^d \mathbb{P}\left(\sup_{0 \le s \le T} \left| \int_0^s \sigma^i(r, X_r^1) dW_r^1 \right| > \frac{a}{d} \right) \le C_1 \exp(-a^2/(C_2 T)). \end{aligned}$$

By integration, it follows that for $2rT < 1/(C_2 T)$ with C_2 from the last line,

$$\mathbb{E} \exp\left(2rT \sup_{0 \le s \le T} \left| \int_0^s \sigma(r, X_r^1) dW_r^1 \right|^2 \right) < \infty,$$

and hence we also have that

$$\psi(r,T) \le \exp(rT)\, C_T \mathbb{E} \exp(2rT(1 + |x_0|^2))\, \mathbb{E} \exp(2rT \sup_{0 \le s \le T} |X_s^1|^2) < \infty,$$

as required, in particular, due to the exponential moment assumed for x_0.

Thus, the function $\psi(r,T)$ (see (50)) is finite for r from some (any) finite range $0 \le r < K$, if T is small enough. It follows that all expressions in (44) for small enough $T>0$ are finite. So, in particular, both γ^i_T are, indeed, probability densities for small $T>0$ under the linear growth condition (29), too. Hence, we can return to the inequalities (38) earlier established for bounded $\widetilde{B}$, and by virtue of (45), (46), and (49) we get,

$$v(T) \le 2\sqrt{E^{\gamma^1}\rho_T^2 - 1} \le \sqrt{CTv(T)^2},$$

with some constant C which constant may depend on the initial distribution (or value). Therefore, $v(T)=0$ for $T>0$ small enough.

7. Denote

$$\mathcal{N} := \{t \ge 0 : \, v(t) = 0\}.$$

The previous steps show that $\sup(\mathcal{N}) > 0$ and that $0 \in \mathcal{N}$. Note that $t \in \mathcal{N} \Longrightarrow s \in \mathcal{N}$ for any $0 \le s \le t$. Recall that $v(t) \le \sqrt{\mathbb{E}^{\gamma^1}\rho_t^2 - 1}$ (see (38)) where the right hand side is clearly continuous in t. Moreover, as it follows from (45), (46), and (49),

$$v(t)^2 \le 4(\mathbb{E}^{\gamma^1}\rho_t^2 - 1) \le 4\left(\mathbb{E}^{\gamma^1}\exp\left(6\int_0^t [C(1+|X_s^0|)\|\mu_s^1 - \mu_s^2\|_{TV}]^2 ds\right) - 1\right),$$

which implies that the set $\mathcal{N}$ is closed, since $\|\mu_s^1 - \mu_s^2\|_{TV} = 0$ for all $s<t$ if $[0,t) \subset \mathcal{N}$.

On the other hand, consider any $N \in (0, \sup(\mathcal{N}))$. Recall that $\mathbb{E}\exp(c\sup_{s\le N}|X_s^0|^2) < \infty$ with some positive c. Hence, the same calculus as above shows that $v(t)=0$ in some small right neighbourhood of N. In other words, the set on the positive half-line $\mathbb{R}_+$ where $v(t)=0$ is non-empty, closed and open in $\mathbb{R}_+$. Thus, it coincides with $\mathbb{R}_+$ itself. In other words, for all $t \ge 0$,

$$v(t) = 0,$$

which finishes the proof of Theorem 2.

3.4 Strong uniqueness

In this section it will be shown that in certain cases weak uniqueness implies strong uniqueness for the equation (1) – (2), and both properties will be established under appropriate conditions. This result, Theorem 3 below, requires only a Borel measurability of the drift with respect to the state variable x, although, it assumes that diffusion σ does not depend on y along with Lipschitz condition in x and non-degeneracy. The drift may be unbounded in the state variable x.

Theorem 3. *Let* $\mathbb{E}\exp(r|x_0|^2) < \infty$ *for some* $r > 0$*, and let the functions* b *and* σ *be Borel measurable, and*

$$\sigma(s,x,y) \equiv \sigma(s,x),$$

that is, σ *does not depend on the variable* y*; let* σ *satisfy the non-degeneracy assumption (6); let* $\mathrm{d}_1 = d$*, the matrix* σ *be bounded, symmetric and invertible, and let there exist* $C > 0$ *such that the function*

$$\widetilde{B}[s,x,\mu] := \sigma^{-1}(s,x)\, B[s,x,\mu]$$

satisfies the linear growth condition: there is $C > 0$ *such that for all* $x \in R^d$*,*

$$\sup_{s,\mu} |\widetilde{B}[s,x,\mu]| \le C(1+|x|).$$

Also assume that the matrix-function $\sigma(t,x)$ *satisfies the following Lipschitz condition (for simplicity) which guarantees that the equation*

$$dX^0_t = \sigma(t,X^0_t)\,dW_t, \quad X^0_0 = x_0,$$

has a unique strong solution for any x *(see [27, 28]):*

$$\sup_{t\ge 0}\ \sup_{x,x':\, x'\neq x} \frac{\|\sigma(t,x)-\sigma(t,x')\|}{|x-x'|} < \infty. \tag{51}$$

Then solution of the equation (1)–(2) is weakly and strongly unique; this solution is strong.

Proof. It follows straightforwardly from the Theorem 2 and from the fact that with a given μ_t any solution is strong [27, 28] (note that linear growth of the drift is allowed in both [27, 28]). •

Remark 7. *Note that under the condition (51), not only the equation (30) but any equation with the same diffusion and a Borel measurable drift with a linear growth assumption in* x *will have a strong solution. It concerns both solutions of the equation (1) and its "linearized" version (28).*

Emphasize that no regularity on the function b *is needed in either variable. Also, a linear growth condition on the drift in* x *is equivalent to the condition (29); the latter was assumed in order to make the presentation more explicit. The price for the no regularity and linear growth is a special form of* σ *which may not depend on the "measure variable"* y*; in particular, in such a case* $\Sigma(t,x) = \sigma(t,x)$*, and we will use the lower case to denote the diffusion coefficient in the remaining sections.*

Remark 8. *Instead of Lipschitz condition (51), it suffices if diffusion coefficient* σ *belongs to the Sobolev class* $\sigma(t,x) \in W^{0,1}_{2d+2,loc}$. *More general conditions on Sobolev derivatives for* σ *can be found in [27, Theorem 1] and [28], and any of them can be used in our Theorem 3 above. Note that in the latter reference* σ *is assumed Lipschitz but it is shown that continuity is necessary only with respect to the state variable* x, *which is also applied to the conditions from [27]. As usual, a more relaxed conditions on* σ *can be stated in the case of dimension one as in [32], or in [27, Theorem 2], the simplest version of both being just Hölder 1/2.*

4 Appendix

To prove Theorem 1, we need two auxiliary lemmas. Lemma 2, is about very standard a priori moment bounds for solutions; another one, Lemma 3, is a localized version of Krylov's bounds where coefficients are locally bounded and the obtained upper bound also relates to a bounded domain.

Lemma 2. *Under assumption (4) and the standard measurability, a priori estimates*

$$\mathbb{E} \sup_{0\le t\le T} |X_t|^4 \le C_T(1+\mathbb{E}|x_0|^4), \tag{52}$$

and

$$\sup_{0\le s\le t\le T;\, t-s\le h} \mathbb{E}|X_t - X_s|^4 \le C_T h^2, \tag{53}$$

hold true with some constants C_T *(generally, different) that do not depend on* n. *In (52) the constant* C_T, *generally speaking, also depends on the value of the moment* $\mathbb{E}|x_0|^4$.

Remark 9. *In the proof of the Proposition 1 similar a priori bounds are stated for the successive approximations of solutions. Recall that* $x_0 \in \mathbb{R}^d$ *is the initial value of the process* X *and that it may be random with a certain finite moment. In fact, similar a priori bounds hold true for any power function assuming the appropriate initial moment, although, this will not be used in this paper. The proof of (52) is very standard and can be done following the lines in [10, Theorem 1.6.4], or [19, Corollary 2.5.6], or [24] (and many other places) combined with Doob's inequalities. The bounds (53) follow from a similar calculus starting from* s *instead of* 0.

Why do we need the 4th moment, is clarified in the proofs of the Proposition 1 and Theorem 1: it is useful for verifying continuity for the processes with equivalent finite-dimensional distributions; for this purpose the 2nd moment is not sufficient, although $2+\epsilon$ *should probably work.*

The next lemma is a version of Krylov's bounds. Let Z_t be a non-explosive strong Markov process in $\mathbb{R}^d$ satisfying an SDE

$$dZ_t = b_t(Z_t)dt + \sigma_t(Z_t)dW_t, \quad Z_0 = z_0,$$

where (non-random) functions $b_t(z)$ and $\sigma_t(z)$ are d-vector and matrix $d \times d$ respectively, σ_t is uniformly non-degenerate, and locally in z bounded uniformly in t, that is,

$$d(R) := \sup_{|z|\le R} \sup_t (|b_t(z)| + \|\sigma_t(z)\|) < \infty, \quad \forall \ R > 0, \tag{54}$$

and the random variable z_0 is independent of the filtration of the Wiener process $W = (W_t, \mathcal{F}_t)$. Let D be a bounded domain in $\mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^1$, and $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^1$ be Borel measurable functions. Then we state the versions of Krylov's bounds in which only local boundedness of the coefficients is assumed, and the statements are also local.

Lemma 3. *Let $D \subset B_R = \{z \in R^d : |z| \le R\}$. Under assumption (54), for any $p \ge d$ there exists a constant $N = N_R$ which also depends on d, on the constants of ellipticity of $\sigma\sigma^*$ and on the upper bounds for the norms of b and σ for $|x|, |y| \le R$, such that for any g satisfying $g(z) \equiv 0$, $z \notin D$,*

$$\mathbb{E} \int_0^T g(Z_t)dt \le N_R \|g\|_{L_p(D)}, \tag{55}$$

and for any f satisfying $f(t, z) \equiv 0$, $z \notin D$,

$$\mathbb{E} \int_0^T f(t, Z_t)dt \le N_R \|f\|_{L_{p+1}([0,T]\times D)}. \tag{56}$$

Notice, that in the proof of Theorem 1 the role of Z was played by the pairs $(\widetilde{X}^n, \widetilde{\xi}^n)$. These are yet not full Krylov's estimates for general Itô processes; however, they suffice for our goals in the present paper.

Proof. The proof is based on Krylov's bounds for Itô processes with bounded coefficients and from the hint similar to the one in the proof of the Lemma 4.3.1 [16]. We only establish bound (56) and the bound (55) follows similarly. Let D' be another bounded domain containing the closure of D: $\overline{D} \subset D'$. Without loss of generality, we may assume that $\inf_{y\in\overline{D}, z\in\overline{\partial D'}} |y - z| > 0$ and $D' \subset B_{R+1} = \{z \in R^d : |z| \le R+1\}$. Denote

$$\tau^0 = 0, \quad T^1 := \inf(t \ge \tau_0 : Z_t \notin \bar{D}'),$$

$$\tau^k := \inf(t \ge T^k : Z_t \in D),\ T^{k+1} := \inf(t \ge \tau^k : Z_t \not\in \bar D'),\ k \ge 1,$$

and let

$$\widehat Z^0_t := Z_0 + \int_0^t \mathbf{1}(s < T^1) b_s(Z_s) ds + \int_0^t (\mathbf{1}(s < T^1)\sigma_s(Z_s) + \mathbf{1}(s \ge T^1)) dW_s,$$

for $t \ge 0$, and by induction,

$$\widehat Z^k_t := Z_{\tau^k} + \int_{\tau^k}^t \mathbf{1}(s < T^{k+1}) b_s(Z_s) ds + \int_{\tau^k}^t (\mathbf{1}(s < T^{k+1})\sigma_t(Z_s) + \mathbf{1}(s \ge T^{k+1})) dW_s,$$

for $t \ge \tau^k$, $k \ge 0$. (That is, each process $\widehat Z^k$ starts in τ^k at state Z_{τ^k} and follows the trajectory of Z until T^{k+1}, after which continues as a Wiener process.) Rigorously, recall that $Z_t = \widehat Z^0_t$ on the set $(t < T^1)$, and $\widehat Z^k_t = Z_t$ on $(\tau^k \le t < T^{k+1})$ (see [20, Theorem 6.3.7(iii)]). Since $f(s, Z_s) = 0$ on any interval $T^k \le s \le \tau^k$, we have, according to Krylov's bound [19, Theorems 2.2.2, 2.2.4], that for $p \ge d$

$$\begin{aligned}
\mathbb{E}\int_0^T |f(s,Z_s)|\,ds &= \sum_{k=0}^\infty \mathbb{E}\int_{\tau^k\wedge T}^{T^{k+1}\wedge T} |f(s,Z_s)|\,ds = \sum_{k=0}^\infty \mathbb{E}\mathbf{1}(\tau^k \le T)\int_{\tau^k\wedge T}^{T^{k+1}\wedge T} |f(s,Z_s)|\,ds \\
&= \sum_{k=0}^\infty \mathbb{E}\mathbf{1}(\tau^k \le T)\int_{\tau^k}^{T^{k+1}\wedge T} |f(s,Z_s)|\,ds \le \sum_{k=0}^\infty \mathbb{E}\mathbf{1}(\tau^k \le T)\int_{\tau^k}^{T^{k+1}} |f(s,Z_s)|\,ds \\
&= \sum_{k=0}^\infty \mathbb{E}\mathbf{1}(\tau^k \le T)\mathbb{E}\left(\int_{\tau^k}^{T^{k+1}} |f(s,\widehat Z^k_s)|\,ds|\mathcal{F}_{\tau^k}\right) \le N_R\|f\|_{L_{p+1}([0,T]\times R^d)}\sum_{k=0}^\infty \mathbb{P}(\tau^k \le T).
\end{aligned}$$

Recall that $\mathbb{P}(\tau^k \le T) \le \mathbb{P}(T^k \le T)$. For a strong Markov process Z with a positive probability to exit from $\bar D'$ on any finite interval of time due to the non-degeneracy of its diffusion coefficient and boundedness of both coefficients in $\bar D'$, the probabilities $\mathbb{P}(T^k \le T)$ admit exponential bounds

$$\mathbb{P}(T^k \le T) \le Cq^{k-1},\ k \ge 1,$$

with some $C < \infty$ and $q < 1$. So, with a new constant C and since $(1 - \mathbf{1}(D))f \equiv 0$,

$$\mathbb{E}\int_0^T |f(s,Z_s)|\,ds \le N_R\|f\|_{L_{p+1}([0,T]\times\mathbb{R}^d)}\sum_{k=0}^\infty \mathbb{P}(T^k \le T) \le CN_R\|f\|_{L_{p+1}([0,T]\times D)},$$

as required. •

Lemma 4 (Skorokhod (on unique probability space and convergence))**.** *Let* $\{\xi^n_t,\ t \ge 0,\ n \ge 0\}$ *be some d-dimensional stochastic processes defined on some probability space (the spaces can be different for different n) and let for any* $T > 0$, $\varepsilon > 0$ *the following hold true:*

$$\lim_{c\to\infty} \sup_n \sup_{t\le T} \mathbb{P}(|\xi^n_t| > c) = 0,$$

and

$$\lim_{h\downarrow\infty} \sup_n \sup_{t,s\le T;\,|t-s|\le h} \mathbb{P}(|\xi^n_t - \xi^n_s| > \varepsilon) = 0,$$

Then for any sequence $n' \to \infty$ *a new probability space* $(\Omega', \mathcal{F}', \mathbb{P}')$ *can be constructed that supports the processes* $\widetilde{\xi}^{n'}_t$, $t \ge 0$ *and* $\widetilde{\xi}_t$, $t \ge 0$, *such that all finite-dimensional distributions of* $\widetilde{\xi}^{n'}_\cdot$ *coincide with those of* $\xi^{n'}_\cdot$ *and there exists a subsequence* $n \to \infty$ *such that for any* $t \ge 0$

$$\widetilde{\xi}^n_t \to \widetilde{\xi}_t, \quad n \to \infty,$$

in probability $\mathbb{P}'$.

See [24, Ch.1, §6].

Lemma 5 (Skorokhod)**.** *Let* $f^n : \mathbb{R} \times \Omega \to \mathbb{R}$, $n \ge 0$ *be uniformly bounded random processes on some probability space; let* $(W^n,\ n \ge 0)$ *be a sequence of one-dimensional Wiener processes on the same probability space, and let all Itô's stochastic integrals* $\displaystyle\int_0^T f^n_s\, dW^n_s, n \ge 0$ *be well-defined. Assume that for any* $\varepsilon > 0$,

$$\lim_{h\to 0} \sup_n \sup_{|s-t|\le h} \mathbb{P}\{|f^n_s - f^n_t| > \varepsilon\} = 0,$$

and let for each $s \in [0, T]$

$$(f^n_s, W^n_s) \stackrel{\mathbb{P}}{\to} (f^0_s, W^0_s).$$

Then

$$\int_0^T f^n_s\, dW^n_s \stackrel{\mathbb{P}}{\to} \int_0^T f^0_s\, dW^0_s.$$

See [24, Ch.2, §3, Theorem], where W^n are allowed to be more general martingales with brackets converging to that of a Wiener process.

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