Quantum choice models: a flexible new approach for understanding moral decision-making

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ABSTRACT
Quantum probability, first developed in theoretical physics, has recently been successfully used in cognitive psychology to model data from experiments that previously resisted effective modelling by classical methods. This has led to the development of choice models based on quantum probability, which have greater flexibility than standard models due to the implementation of complex numbers through, for example, complex phases or 'quantum rotations'. This paper tests whether these new models can also capture choice modification under *implicit* 'changing perspectives' in choice contexts with salient moral attributes. We apply these models to two distinctly different case-studies. In the first, respondents have to make choices between route alternatives with variable 'concrete' and 'moral' attributes - Chorus et al. (2018)'s 'taboo trade-off' between time-cost and deaths-injuries. The second study investigates how an individual weighs wages and commuting times for themselves relative to the wages and commuting times for their partner. Under both scenarios, we find that the flexibility provided by quantum choice models allows them to accurately capture and formally explain choices across the differing contexts.

Keywords: Quantum probability; moral choice; travel behaviour

1. INTRODUCTION

Moral choice scenarios can be summarised as those where the choices or actions a decision-maker takes could negatively impact other individuals. Thus, to the decision-maker, the choice alternatives may to some extent be categorised as 'right' or 'wrong', depending on how serious (and possibly how likely) the consequences are. As a result, the associated choices can perhaps be more complex as they do not involve straightforward trade-offs between rather concrete attributes of alternatives. For example, a decision-maker may not choose the alternative that they would choose based on more attractive concrete features as they believe it to be an overall morally contentious option. Alternatively, a set of options may all have negative features, where different schools of moral thought suggest different actions should be taken (for example, Awad et al. (2020) discuss country-level variations in decision-making in 'moral machine' choice tasks).

While moral choice behaviour has received much attention in economics and psychology, it is rarely considered in the choice modelling literature (see Chorus 2015 for a detailed discussion). This is despite the fact that many typical experiments conducted to understand or interpret an individual's preferences in moral choice scenarios use paradigms such as variations of the well-known trolley problem (where a 'runaway trolley' has two possible paths, both of which will result in the death of some individual(s), and the decision-maker must choose who to save), for which a precise understanding of the trade-offs that are being made could be obtained using choice models. This is perhaps due to the fact that an individual's moral preferences are difficult to investigate outside of the laboratory, with typical experimental methods for examining moral choice scenarios often suffering from low external validity (Bauman et al., 2014). However, more recently, moral choice behaviour has become more prominent in the travel behaviour modelling community through, for example, the reinvention of the trolley problem as a self-driving car problem (Awad et al., 2018). Thus far, there has not been much consideration given to the types of choice models used for the modelling of such scenarios, despite the wide range of theoretical explanations for moral behaviour that have been proposed (Chorus, 2015). However, some steps towards the development of choice models specifically for moral choice contexts have been made (Chorus et al., 2018).

In this paper, we specifically look at models based on quantum probability theory. These have not yet been applied to moral choice scenarios, despite the adoption of such methods 'allowing for a re-examination of the challenge of formalising psychological concepts of conflict, ambiguity, and uncertainty' (Wang et al., 2013). Quantum probability theory has recently made a significant impact in cognitive psychology (Bruza et al., 2015). This impact is in part due to the underlying logic of quantum probability theory which revealed a fundamental lack of distributivity of propositions concerning non-compatible features of an observed system (Birkhoff and Von Neumann, 1936). This key difference between classical and quantum logic reveals that under quantum theory, the law of probability following the distributivity of 'and' and 'or' of propositions – $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ – may fail to hold (for a detailed example, see Hancock et al. 2020). Another essential difference follows from the description of a system by using state vectors with complex-valued components which entail the occurrence of interference effects when such states are superposed, famously leading to the paradoxical state of Schrödinger's cat being both dead and alive at the same time in a historical thought experiment devised to point out the consequences of the entanglement of the quantum system and its observer (Schrödinger, 1935). In effect, the measurement of a property of a system occurs differently, namely by applying projection operators on the state vector of a system which inherently 'changes the system by making an observation' - as opposed to simply reading of the value of a pre-existent property of the system. Crucially, these features mean that the adoption of quantum probability theory allows for a powerful and elegant framework for modelling and understanding many 'paradoxical' findings which become 'intuitive' (Wang et al., 2013), such as probability judgement errors (Busemeyer et al., 2011), question ordering effects (Trueblood and Busemeyer, 2011) and violations of the 'sure thing principle' (Pothos and Busemeyer, 2009; Broekaert et al., 2020). A classic example of a probability judgement error is given by Tversky and Kahneman (1983), who found that participants, after reading 'Linda was a philosophy major. She is bright and concerned with issues of discrimination and social justice', were more likely to agree with the statement 'Linda is a feminist bank teller' than the statement 'Linda is a bank teller'. This subjective assessment clearly contradicts logical set theory in which the category "feminist bank teller" is a subset of the category "bank teller", and hence on probabilistic grounds of set membership, this should lead to a lower association of Linda with the former category.

With, for example, ordering effects also frequently observed in choice modelling applications, it is unsurprising that quantum models have also since made the transition into choice modelling (Lipovetsky, 2018). Furthermore, quantum models can be used to accurately capture the 'change of decision context and mental state' when moving between choices made under revealed preference and stated preference settings (Yu and Jayakrishnan, 2018). Additionally, it has been demonstrated that quantum probability theory can be implemented into choice models to accurately understand route choice problems as well as best-worst choice behaviour in the context of alternative routes (Hancock et al., 2020). Thus there appears to be ample scope for further developments of quantum choice models, with our previous development of the notion of a 'quantum rotation' within a choice model providing useful transitions across choice contexts. The aim of this paper is to build on work presented in Hancock et al. (2020), which focussed solely on typical travel behaviour data, by testing these models on more complex choice scenarios. We specifically test whether these rotations and other quantum choice model features can equivalently be used to accurately capture changes in choice context within moral choice scenarios.[1]

[1] A prior version of a formal model using rotations in choice scenarios with moral trade-off was developed by

We apply the models to two very different datasets. The first allows us to test whether quantum choice models can be used to capture the impact of the presence of a 'taboo trade-off' (Chorus et al., 2018) involving trade-offs between 'moral' and 'concrete' features.[2] The moral attributes of the route alternatives appeal to the personal sense of *right* versus *wrong* grounded in the decision-maker's socio-cultural and philosophical or religious association - like the personal answerability or blame for opting for a route alternative with a higher expected number of deaths or severely injured travellers. The concrete attributes on the other hand call for a more pragmatic material utility which a priori does not ponder rightness or wrongness of the choice - like for instance the additional time on a route alternative. It goes without saying that these categories may well be perceived as intertwined; a faster route alternative with implicit detrimental environmental effects can appeal to the decision-maker's ethical principles. Vice versa, a utilitarian based ethical approach held by a decision-maker could result in equating moral attributes with pragmatic features of the alternative.

The second dataset tests whether quantum choice models can be used to capture differences between how an individual weighs wage and commuting times for themselves relative to considering the wages and commuting times for both their partner and themselves, developed by Swärdh and Algers (2009), with descriptions also in Beck and Hess (2016). An aspect of morality is again appealed to in this experimental paradigm. The consideration of the partner's situation may appeal to the decision-maker's empathy or selfishness with respect to the partner, or, a particular balanced choice may result from an evaluation of the pragmatic joint utility for the couple.

The remainder of this paper is organised as follows. Section 2 gives an introduction to quantum probability, discusses how it has provided useful explanations for choices with a moral component in cognitive psychology, and shows how we mathematically build our quantum choice models. Section 3 shows the empirical application to our two moral choice datasets. We finish with some conclusions and directions for future research.

2. THE QUANTUM PROBABILITY APPROACH

In this section, we first give a basic overview of the quantum probability approach; we refer the reader to Khrennikov (2010); Busemeyer and Bruza (2012); Broekaert et al. (2016); Yearsley and Busemeyer (2016); Yearsley (2017) for a more extensive coverage on the application of quantum theory in decision-making. Next, we in turn look at how quantum probability can be used to capture a change in perspective, and how it has been used to explain a number of 'paradoxical' phenomena in cognitive psychology, some of which have moral components. Finally, we demonstrate how we mathematically operationalise the quantum probability approach into the models utilised in this paper.

2.1. Basic features of the approach

Under quantum models, each choice scenario is represented in a n-dimensional 'Hilbert' space, which is spanned by a set of n orthonormal (possibly complex) vectors, with one vector for each possible choice alternative. In essence, the cognitive process corresponding to the experimental paradigm is implemented by performing operations on specifically constructed vectors of the

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Hancock (2019).

[2] The authors Chorus et al. (2018) have coined the types of attributes as 'sacred' and 'secular'. We have opted to denominate the attributes by more culturally neutral terms in comparison to Chorus et al. (2018).

Hilbert space.[3] This vector represents the decision-maker's behavioural belief-action state at a given moment in the experimental paradigm, in particular for the present datasets, expressing their preferences.

A basic example of this is given in Fig. (1), where we adapt the Hilbert space to the paradigm corresponding to the first dataset (Chorus et al., 2018). A similar example is described in more detail in the introduction of Hancock et al. (2020), where an individual is choosing whether to commute to work by car or by train.

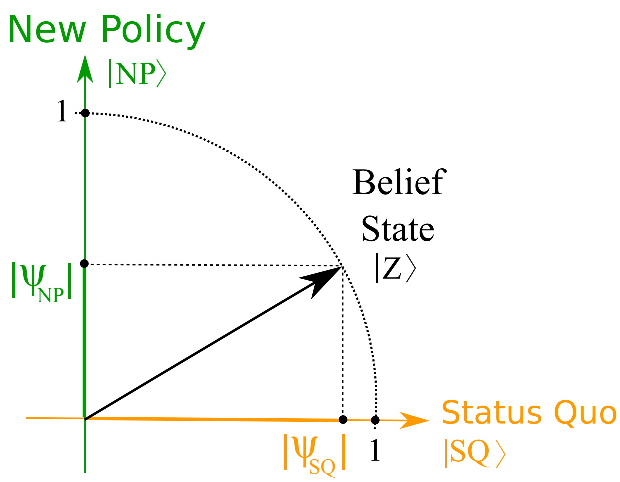

FIGURE 1 : A schematic representation of the belief state in the geometric quantum-like model for a binary choice between the 'New Policy' or the 'Status Quo'. The belief state $|Z\rangle$ is a superposition of $|NP\rangle$ and $|SQ\rangle$, meaning that the decision-maker has a propensity to choose both the 'New Policy' or their 'Status Quo'. The numerical probabilities of choosing each alternative are obtained from the complex-valued amplitudes of the projections on the respective axes by squaring the moduli $|\psi_{NP}|$ and $|\psi_{SQ}|$.

The preference of an individual decision-maker is represented by a (normalised) belief state vector which is denoted $|Z\rangle$. The action of making a choice is represented by a projection from the belief state vector onto the vector representing the chosen alternative, i.e. $|NP\rangle$ for the 'New Policy' or $|SQ\rangle$ for the 'Status Quo', in Fig. (1).

The projection operations are represented by the dotted lines connecting the belief state (on the arc) to the axes orthogonally spanned by the two choice alternatives, resulting in the two respective component moduli $|\psi_{NP}|$ and $|\psi_{SQ}|$. To be used as probabilities, the outputs of these projection processes need to fulfil two properties; a) they need to all be between 0 and 1, and b) they need to sum to 1. With the quantum approach, this is achieved by using the *squared* 'length' of each projection as the probability for that alternative. Since the state vector is normalised - i.e. of unit 'length' (or modulus) - these two requirements are fulfilled. One can easily verify that with the two choices represented by a set of orthonormal vectors, the set of squared 'length' projections will sum to one according to Pythagoras' Theorem (see Fig. 1).

[3] A Hilbert space is a regular real, or complex-valued, vector space with an inner product and a completeness property that assures converging limits will exist within the space itself.

2.2. A change in choice perspective

In quantum mechanics, two observables of a system are considered *incompatible* if a measurement of one of them influences the outcome of the other. Conversely, two observables are compatible if they do not influence each other.[4] Thus in the context of our model, if two choices have no relation to each other and their respective answers do not impact each other, they are compatible. In such cases, the *same* belief state vector - albeit with *different* components dedicated to each of the choice tasks - can invariably be used for the two tasks. However, if we had two tasks which were related to each other by simple variation of some concrete attributes and hence require the *same* components of the belief state vector - then the belief state needs to be updated as well. For instance, in repeated choice tasks with only modified attributes of the alternatives, the belief state of the decision-maker is updated in line with the cognitive process associated with each new choice. Mathematically, the adaptation of the belief state to the different values of the attributes is determined by immediate implementation[5] in the vector components, Eq. (3), and effectively corresponds to a rotation between the two state vectors (Hancock et al., 2020).

However, 'incompatible' choice tasks at a deeper level - when a pair of choices impact each other on different components of the belief state - require different belief vectors for each of the tasks. To give a more detailed example of such task 'incompatibility', consider a scenario where the decision-maker has to choose their *favourite* and *least favourite* alternative from a set. The sensitivities for what constitutes the best alternative may not be equivalent to what constitutes the worst. This can be represented in quantum models through different vectors for an alternative being the best compared to the same alternative being the worst. To capture the change of perspective (considering the best, to considering the worst), a 'quantum rotation' is required, which maps the belief state vector representing the choice of alternatives as being the best, to the belief state vector representing the choice of alternatives as being the worst - and where the projection on the respective axes remain with their interpretation of providing the amplitudes for the respective alternatives. One can equivalently describe this rotation from a passive perspective in which the belief state remains invariant but the basis is rotated in the opposite direction. Hancock et al. (2020) have shown that such rotations (in Hilbert space) can capture the difference in the representation (value) of an alternative when evaluated as best compared to when evaluated as worst. In this paper, we use the same concept of a quantum rotation to capture changes of perspective in moral choice scenarios. We also introduce a supplementary method based on inserting complex phases at the level of attribute value functions in the belief state vectors to implement an alternative perspective operation. We thus assume that choices under moral contexts involve more of a dilemma within the deliberation process, with these model extensions capturing this additional process.

In a given choice context, we assume that an individual would evaluate the scenario differently if they were first asked explicitly about the 'ethical answerability' of their choice. The presence of a salient moral component may lead the decision-maker to a similar implicit intermediate assessment and result in the decision-maker considering their choice from a different perspective. In the event of such an intermediate assessment of the moral attributes - from an effective change of perspective

[4]In case of incompatibility, a Heisenberg uncertainty relation can be derived which states that the product of the standard deviations of both observables should always be larger or equal to half the expectation value of their commutator. Compatible observables will hence be represented by commuting operators, see e.g. Griffiths (1994) section 3.4.

[5]Note that at this point, in stated preference settings, we make the assumption that previous choices do not impact the current choice.

on the choice - the choice proportions for the alternatives will have changed depending on the acceptance or dismissal of the moral components (see Fig. 2).

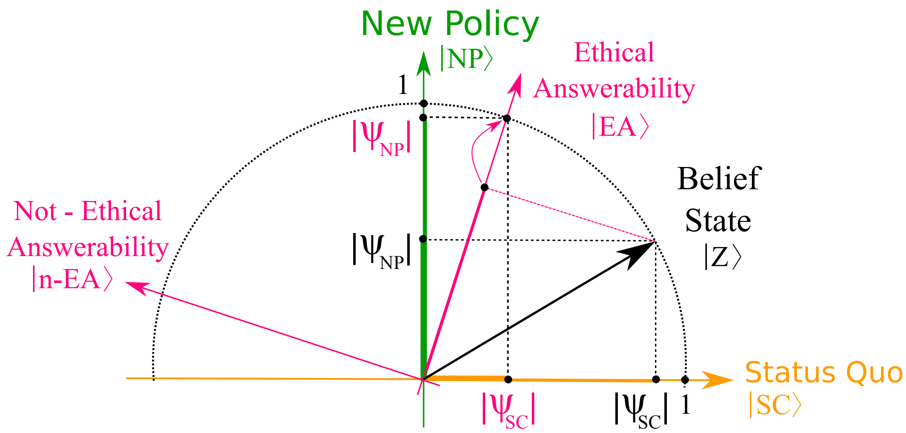

FIGURE 2 : Schematic representation of making explicit consecutive binary choices under quantum probability theory in the geometric quantum-like model; first the 'Ethical Answerability' or 'Not - Ethical Answerability' question, followed by 'New Policy' or 'Status Quo' question. In this particular illustration, the change of the belief state is shown following a *positive* outcome for the 'Ethical Answerability' question. While the initial belief state $|Z\rangle$ only had some latent tendency for responding 'Ethical Answerability', after the positive outcome, the updated belief state coincides with the ethical answerability belief state $|EA\rangle$ (the curved pink arrow shows the renormalisation of the belief state after the collapse of $|Z\rangle$ onto $|EA\rangle$). Note that in this particular case, the intermediate question results in an increase of the belief support for the choice 'New Policy' on a *positive* outcome for 'Ethical Answerability' since the amplitude norm $|\psi_{NP}|$, in pink, is larger in our case than the amplitude norm $|\psi_{NP}|$, in black, and the reverse is true for the 'Status Quo' scenario. In the present rotation-based model, we make the assumption that the salient moral attribute(s) of the alternatives can elicit an *implicit* questioning that does not entail a collapse of the belief state but leads to a rotation approaching either towards the $|EA\rangle$ or $|n\text{-}EA\rangle$ belief state. The rotation induced by implicit questioning thus causes a change of the belief support for both alternatives SQ and NP compared to the initial belief state $|Z\rangle$.

Hence in general, under a quantum model, when a decision-maker makes a choice - albeit implicit - this will update their belief state. If, for example, they implicitly decide that a particular alternative is ethically answerable, their state vector would converge more closely to the 'Ethical Answerability' vector itself. This results in a change in the 'lengths' of projection onto the vectors representing the choice of the New Policy or the Status Quo alternative. We follow Chorus et al. (2018) by defining a new policy as involving *a 'taboo trade-off' if a decision-maker could choose to decrease tax or travel time (a concrete attribute) at the cost of increasing the number of injuries or deaths (a moral attribute).* Thus, in this example, under choice tasks that feature taboo trade-offs, the decision-maker is more likely to choose the new policy if they first decide that it is ethically answerable, and is less likely to choose it otherwise. Both tendencies are present in the decision-maker's belief state, hence the implemented rotation for the decision-maker's implicit change of perspective results effectively from a weighted combination of the two possible positions. The result of this is that quantum models can capture a change in perspective through a quantum rotation, which can be mathematically represented simply by estimating the impact a change of basis (passive) - or change of the belief state (active) - has on the 'lengths' of projections (Sections 3.1.3, 3.2.3). Besides implementing the change of perspective through a quantum rotation of the belief state vector, we also implement such a change of projection 'lengths' by the insertion of a complex phase on the attribute values in the belief states, (Sections 3.1.4, 3.2.4). In the latter case, the change of projection 'length' results from a constructive or destructive interference between the complex summands within each belief state component itself. This complex phase method is similar to the more encompassing quantum rotation method in its effect of summing complex components but differs in that this interference occurs at the more basal level of each attribute itself. Since specific complex phasing can augment the effect of moral attributes in the moral choice scenarios, this approach implements a perspective operation by a more detailed process than the encompassing effect of a quantum rotation.

2.3. Quantum theory and formal modelling of moral choices in psychology

Whilst choice models with a quantum logic framework have not yet been tested on moral choice data, there have been a number of applications of quantum logic to experiments for paradigms with a moral component (where, for example, decision-makers may make choices that impact a number of other individuals) in cognitive psychology. In particular, quantum probability theory has been used to explain 'interference' effects where an additional decision task impacts the probability of a subsequent decision for an action. For example, Busemeyer et al. (2009) tested the impact of additionally asking decision-makers to categorise a digitally modified face - according to pre-learned ad-hoc criteria - as 'good' or 'bad', before choosing how to respond by either a 'withdraw' or 'attack' action, and in which a bonus was provided for responding with the action 'attack' after categorisation 'bad', or the action 'withdraw' after the category 'good', and a penalty otherwise. Their study found that the quantum approach could be used to accurately capture the difference in action responses with and without the categorisation task. Furthermore, in simulated jury decision-making experiments, where participants read strong or weak defences and prosecutions, quantum probability theory provided a better account of the ordering effects that were observed relative to models based on classical probability (Trueblood and Busemeyer, 2010). Ordering effects observed when participants state opinions about political figures can also be explained by quantum models (Pothos and Busemeyer, 2013). In the context of a 'taboo trade-off', where an individual can sacrifice 'moral' features in favour of 'concrete' features, a similar interference may take place in that a decision-maker may not wish to appear 'unethical' or expose socially undesirable choices. Similarly, an individual may consider their own welfare differently if they are also required to consider the welfare of their partner.[6] For this reason, we use quantum models to test for interference effects in both of the choice datasets considered in this paper.

2.4. Mathematical outline for basic quantum choice models

Whilst the quantum approach provides a convenient structure for capturing phenomena in cognitive psychology, its operationalisation into a choice model is less simple. The key component (as discussed in detail by Hancock et al. 2020) is that a decision-maker has some 'belief state' $|Z\rangle$

[6] A recent theoretical model by Yilmaz (2019) proposes unitary transformations of the decision-maker's belief state based on first-person perspectives on imagined belief states of third-person agents to produce an effective ethical choice outcome. In contrast, our model implements the decision-maker's implicit intermediate belief state rotation to potentially consider their choice from their ethically concerned perspective, or not.

regarding their preferences over J alternatives presented in the experimental paradigm. When a decision-maker makes a choice, their state goes from ‘indefinite’ to ‘definite’, by projecting their belief state onto the vector representing the chosen alternative, where we further assume that the presented alternatives are mutually exclusive and exhaust all choice possibilities. This means that the choice probability, $Pr[\text{Alt}_j]$, for a specific alternative Alt_j, is given by the modulus square of the amplitude for that alternative appearing in the decision-maker’s belief state

$$Pr[\text{Alt}_j] \quad = \quad |\psi_j|^2, \tag{1}$$

where $|Z\rangle$ is a column vector, with $|Z\rangle = (\psi_1 \dots \psi_j \dots \psi_J)^\tau$. Since the belief state vector is normalised, the probabilities for the alternatives add up to 1:

$$\sum_{j=1}^{J} |\psi_j|^2 = 1. \tag{2}$$

Consequently, we must build quantum choice models by developing methods for defining a belief state vector based on functions of the attributes of the alternatives. For the applications in this paper, we consider an approach based on the ‘quantum amplitude model’, as developed in Hancock et al. (2020). The key feature of the quantum amplitude model (QA) is that the amplitudes of each alternative are explicitly implemented with the use of some value function. This allows us to directly estimate the probabilities with which each alternative is chosen. Whilst a number of different value functions can be used, we focus on the use of regret-like functions (Chorus, 2010) for the applications in this paper. The amplitude for an alternative i for individual n in choice task t is thus defined as:

$$\psi_{nti} = \left(\delta_{QA,i} + \sum_{j \neq i}^{J} \sum_{k=1}^{K} \ln(1 + \mathrm{e}^{\beta_k (x_{ntik} - x_{ntjk})}) \right) / \sqrt{\mathcal{N}_{nt}}, \tag{3}$$

where $j = 1, ..., J$ is an index across alternatives, $k = 1, ..., K$ is an index across attributes, $\delta_{QA,i}$ are alternative specific constants, β_k are attribute-specific weights and $\mathcal{N}$ is a normalisation factor. This factor, which ensures that the probabilities with which each alternative is chosen sum to one, is obtained from the sum of the squared moduli numerators:

$$\mathcal{N}_{nt} = \sum_{i}^{J} \left| \left(\delta_{QA,i} + \sum_{j \neq i}^{J} \sum_{k=1}^{K} \ln(1 + \mathrm{e}^{\beta_k (x_{ntik} - x_{ntjk})}) \right) \right|^2. \tag{4}$$

Note that given the probabilities with which the different alternatives are chosen are based on squaring these amplitudes, the same probabilities will be generated if all amplitudes are multiplied by the same factor - as opposed to the *addition* of the same term. Thus the number of additive constants that are identifiable is equal to the number of alternatives. Additionally, the equations as presented here imply that the amplitudes are in real-valued space, with operations moving the amplitudes into complex space introduced in the following section. Our approach thus makes explicit use of complex-valued operations and hence enacts interference effects which cannot be obtained in the real-valued trigonometric approach of Lipovetsky (2018). Alternative methods for capturing moral features can still use the above implementations, with for example, the use of additional constants or separate β-coefficients depending on the choice context.

2.5. Quantum model perspective operations for a change in choice context

We consider two possible extensions (which we label 'perspective operations') to the basic quantum choice models described above, with each extension attempting to capture a 'change of perspective' for the taboo or moral trade-off in a different way.

1. A **'quantum rotation'** (models QAR-1 and QAR-2). We follow Hancock et al. (2020) in using Pauli matrices to implement a rotation operation on the belief state itself. For scenarios involving two alternatives, this rotation occurs in a 2-dimensional Hilbert space, with the rotation matrix generated by Pauli matrices;

   $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

   The rotation operator, R, itself - about axis $\mathbf{n} = (n_x, n_y, n_z)$ and over angle ϑ - is then given by;

   $$R = \mathrm{e}^{-i\vartheta \mathbf{n}\cdot\sigma}, \tag{6}$$

   where $\mathbf{n}\cdot\sigma$ gives some combination of the Pauli factors, with the restriction that $|\mathbf{n}| = 1$.

   For choice scenarios involving three alternatives, as in the second dataset, we apply two consecutive quantum rotations on different pairs of alternatives. Given the importance of ordering within quantum models, the choice of pairs and order in which the two rotations are made will impact the outcome.

   To apply the rotation, we use the initial belief state, Ψ_0, which is simply the vector with amplitudes for each alternative using Eq. (3) - i.e. model QA without additional features. We then obtain the belief state for the changed perspective, Ψ_f, by applying the rotation matrix:

   $$\Psi_f = R\Psi_0. \tag{7}$$

   The rotation thus appropriately adjusts the amplitudes for the different alternatives depending on the impact that the change of perspective has on the choice being made. A matrix R with zero off-diagonal elements (i.e. if we have $\mathbf{n}\cdot\sigma = \sigma_z$) would result in no change in the probabilities with which each alternative is chosen. As a contrast, $\vartheta = \pi/2$ and $\mathbf{n}\cdot\sigma = \sigma_y$ results in the probabilities for a pair of alternatives perfectly swapping.

   We consider two different options for constructing R:

   (a) A rotation, R_1, that uses a single estimated parameter. It is based on previous work where we used a Hamiltonian approach to quantum modelling (see Hancock et al. 2020). We implement the rotation with a fixed constant ($\vartheta = \pi/2$) and modulate it with a single parameter (h) that also weights the Pauli matrices:

   $$R_1 = \mathrm{e}^{-i\frac{\pi}{2}\sqrt{1+h^2}\mathbf{n}\cdot\sigma}, \tag{8}$$

   with $\mathbf{n} = \left(0, \frac{h}{\sqrt{1+h^2}}, \frac{1}{\sqrt{1+h^2}}\right)$.

(b) A rotation, R_2, that uses up to three estimated parameters. The angle ϑ is estimated directly and ω_1 and ω_2 are axis parameters used to weight the Pauli matrices, with $\mathbf{n} = (\sin(\omega_1), \cos(\omega_1) \cdot \cos(\omega_2), \cos(\omega_1) \cdot \sin(\omega_2))$.

2. The introduction of **'complex phases'** (model QAP), such that the amplitudes for the choice alternatives contain real and imaginary parts. We implement these complex phases in the QA model by multiplying $\ln(1 + \mathrm{e}^{\beta_k(x_{ntik} - x_{ntjk})})$ in Eqs. (3, 4), with $\mathrm{e}^{i\varphi_k}$. Note that we could estimate a different φ_k for each attribute k, or alternatively have a simpler structure such that a single additional parameter is estimated ($\varphi_k = \varphi$, $\forall k$).

To apply a model with a rotation, we first estimate amplitudes for each alternative based on the basic QA model. Then, under choice scenarios in which there is a 'change of perspective', we apply either of two implementations of the quantum rotation (QAR-1 and QAR-2). The second feature of the complex phases is instead implemented directly into the basic QA model. It thus assumes that moral attributes are 'treated differently' to others. If these attributes are very different, then the estimates for their respective phases, φ_k, will be different, allowing interference interactions.

3. EMPIRICAL APPLICATION

In this section, we present the results of two case studies. In each case, we first detail the dataset that is used for testing our quantum choice models. We then apply our quantum model under basic settings before introducing quantum rotations for specific choice contexts or alternatively adding complex phases in the specification. We conclude by providing combined models with both rotations and complex phases.

3.1. Quantum modelling for taboo trade-offs

3.1.1. Description of data

The first dataset we use involves 'taboo trade-offs' and comes from Chorus et al. (2018) (and is thus henceforth labelled the 'taboo trade-off dataset'). Decision-makers choose between the introduction of a new transport policy or keeping the status quo. To simplify the choice scenarios, each new policy simply offered an increase or decrease compared to the status quo for four attributes, with shifts by ±300 EUR vehicle ownership tax, ±20 minutes travel time for each car commuter per day, ±100 serious injuries in traffic accidents and ±5 deaths in traffic accidents. This results in a total of 16 possible new policies, which are offered in turn to each of 99 decision-makers, resulting in a dataset with a total of 1,584 choices. For consistency, we follow Chorus et al. (2018) by defining a choice as involving a 'taboo trade-off' if a decision-maker could choose a policy that involves decreasing tax or travel time (a concrete attribute) at the cost of increasing the number of injuries or deaths (a moral attribute). One could of course argue that a scenario that increases injuries and reduces time or cost is not a taboo trade-off if deaths are also reduced at the same time. We also follow Chorus et al. (2018) in including all 16 choice scenarios in our dataset to aid a direct comparison with their 'Taboo Trade-off Aversion' model (TTOA). Two of these scenarios include dominated alternatives (Scenarios 1 and 5 in Table 3). A summary of the observed share of choices under the different scenarios is given in Table 1.

For all attributes, we observe that the new policy is more likely to be chosen if there is a decrease in the attribute, as expected. Additionally, we observe that individuals are less likely to pick the new policy if it falls into the category of a taboo trade-off. The observed shares for each

TABLE 1 : Observed shares for choosing the new policy or the status quo depending on the attribute change

		Chosen alternative	
		New Policy	Status Quo
Tax	Decrease	50.88%	49.12%
	Increase	20.32%	79.68%
Time	Decrease	43.44%	56.56%
	Increase	27.78%	72.22%
Injuries	Decrease	53.28%	46.72%
	Increase	17.92%	82.08%
Deaths	Decrease	47.72%	52.28%
	Increase	23.48%	76.52%
Taboo Trade-Off	Yes	30.08%	69.92%
	No	42.72%	57.28%

specific choice scenario are given together with model results in Table 3. This dataset is suitable for quantum choice modelling as decision-makers may not process the different attributes or the different choice tasks in the same way. Thus, quantum rotations and complex phases may both provide a method for capturing these differences.

3.1.2. Basic models for the taboo trade-off dataset

For the first set of models tested, we do not include either quantum rotations or complex phases in the specifications, so as to test the basic structure of the quantum models.

We test models without any parameters to control for the presence of a taboo trade-off, as well as models with an additional constant added to represent the presence of a taboo-trade off. We have a parameter to capture the relative importance of each attributes, and test the quantum model as specified by Eq. (3). We now look in turn at the amplitudes for the status quo (SQ) and new policy (NP) options, where we do not show an index for individuals, n, as all participants complete all 16 choice tasks and there is no variation in the attributes at the individual level:

$$\psi_{t,SQ} = \Big(\delta_{SQ} + \delta_{base} + \ln(1+\mathrm{e}^{-\beta_{TT}\cdot\Delta_{t,TT}}) + \ln(1+\mathrm{e}^{-\beta_{Tax}\cdot\Delta_{t,Tax}}) + \ln(1+\mathrm{e}^{-\beta_{DE}\cdot\Delta_{t,DE}}) + \ln(1+\mathrm{e}^{-\beta_{IN}\cdot\Delta_{t,IN}})\Big)/\mathcal{N}_t, \quad (9)$$

and for the new policy:

$$\psi_{t,NP} = \Big(\delta_{Taboo}\cdot z_{t,taboo} + \delta_{base} + \ln(1+\mathrm{e}^{\beta_{TT}\cdot\Delta_{t,TT}}) + \ln(1+\mathrm{e}^{\beta_{Tax}\cdot\Delta_{t,Tax}}) + \ln(1+\mathrm{e}^{\beta_{DE}\cdot\Delta_{t,DE}}) + \ln(1+\mathrm{e}^{\beta_{IN}\cdot\Delta_{t,IN}})\Big)/\mathcal{N}_t, \quad (10)$$

where t is an index across choice tasks, $t = 1..16$, $\Delta_{t,x} = 1$ if attribute x increases under the new policy or $\Delta_{t,x} = -1$ if the attribute decreases under the new policy, and where the normalisation factor

$\mathcal{N}_t$ satisfies Eq. (4). We include relative importance parameters for the four different attributes, β_{TT} for travel time, β_{Tax} for travel tax, β_{DE} for number of deaths and β_{IN} for number of injuries. As all of these attributes are unfavourable - less is better - we expect negative estimates for these coefficients. This leads to a decrease in the amplitude of the new policy if $\Delta_{t,x} = 1$ (i.e. there is an increase in the attribute), resulting in a smaller probability. We add a constant, δ_{base}, to both numerators and a constant, δ_{SQ}, that allows us to statistically test the underlying bias towards the status quo. As a contrast to random utility models, the additional constant here does not result in an overspecification, with an increased value for δ_{base} corresponding to less deterministic choices. Under the basic model that accounts for the presence of a taboo trade off, we additionally estimate a taboo trade-off constant where appropriate, δ_{Taboo}, which is multiplied by $z_{t,taboo}$, an indicator that takes a value of one for choice tasks where there is the presence of a taboo trade-off, and a value of zero otherwise. In our model that does not account for taboo trade-offs, we fix δ_{Taboo} to a value of zero. Additionally, as there are no attribute levels in this dataset, multiplying all parameters by the same constant results in the same likelihood for the quantum amplitude model. We consequently fix the first β-coefficient to a value of -1 to avoid an overspecification. This will result in the QA model having the same number of free parameters as the logit model.

We compare our quantum models to logit models that are equivalent to those specified by Chorus et al. (2018). The utility for the two alternatives is defined as:

$$U_{t,SQ} = (\delta_{SQ} - \beta_{TT} \cdot \Delta_{t,TT} - \beta_{Tax} \cdot \Delta_{t,Tax} - \beta_{DE} \cdot \Delta_{t,DE} - \beta_{IN} \cdot \Delta_{t,IN}) + \varepsilon_{t,SQ}, \tag{11}$$

and:

$$U_{t,NP} = \left(\delta_{taboo} \cdot z_{t,taboo} + \beta_{TT} \cdot \Delta_{t,TT} + \beta_{Tax} \cdot \Delta_{t,Tax} + \beta_{DE} \cdot \Delta_{t,DE} + \beta_{IN} \cdot \Delta_{t,IN}\right) + \varepsilon_{t,NP}, \tag{12}$$

where ε is a type I extreme value error. The addition of δ_{taboo} to the utility for the new policy in the presence of a taboo trade-off gives us the 'Taboo Trade-off Aversion' model (TTOA) as specified by Chorus et al. (2018).

The results of our quantum and logit models are given in Table 2, where we first report basic logit and basic quantum models, before reporting models that include an additional constant for choices including a taboo trade-off (Logit-t and QA-t, respectively). For all of the model estimation in this paper, we use R packages maxLik (Henningsen and Toomet, 2011) and Apollo (Hess and Palma, 2019).

Without any parameter for a taboo trade-off, the quantum amplitude (QA) model is outperformed by the logit model. Notably, the model appears to find very similar relative ratios for the different β-attribute coefficients as logit. The addition of a taboo parameter (which is significant in each model) results in a reduction in the estimates for injuries and deaths for both models. It improves the fit of the QA model slightly more than the logit model, but the logit model still has the best log-likelihood and adjusted ρ^2 value at this point.[7] For the basic models, the best overall BIC value is obtained by a logit model without a taboo parameter.

3.1.3. Models with quantum rotations for the taboo trade-off dataset

We next turn to models with additional rotations implemented in the presence of a taboo trade-off, which attempt to capture the 'change of perspective', as described in Section 2.5. Thus, if the

[7] Note that 'Logit-t' is identical to the 'Generic Taboo Trade-Off Aversion' (TTOA) model as described by Chorus et al. (2018).

decision-maker can decrease travel time or tax at the cost of increasing the number of fatalities or serious injuries, a rotation is applied.

TABLE 2 : Results of all models applied to the taboo trade-off dataset, together with all parameter estimates, where ○ and ⋆ indicate attribute pairs which have the same phase.

Type		Basic models				Quantum rotations		Models with complex phases					Combined
Specification		**Logit**	**QA**	**Logit-t**	**QA-t**	**QAR-1**	**QAR-2**	**QAPh-1**	**QAPh-2a**	**QAPh-2b**	**QAPh-2c**	**QAPh-3**	**QAC**
Free parameters		5	5	6	6	6	7	6	7	7	7	9	9
Log-likelihood		-721.23	-725.40	-719.47	-722.61	-717.94	-717.77	-719.17	-712.94	-719.16	-718.83	-712.79	**-712.35**
BIC		1479.29	1487.63	1483.15	1489.43	1480.09	1487.12	1482.54	**1477.45**	1489.90	1489.24	1491.89	1491.02
B-S test p-value (vs. RRM-t)						0.0402	0.2655	0.2181	0.0008			0.0184	0.0111
Adj. ρ^2		0.3386	0.3348	0.3392	0.3364	0.3406	0.3399	0.3395	**0.3443**	0.3386	0.3389	0.3426	0.3430
Average probability of choosing NP	No taboo	41.57%	41.22%	42.71%	42.12%	41.86%	42.01%	42.06%	43.07%	42.08%	42.23%	43.04%	42.87%
	Taboo (before rotation)	30.97%	32.39%	30.08%	31.18%	28.82%	30.61%	30.92%	30.13%	30.90%	30.80%	30.07%	31.49%
	Taboo (after rotation)	-	-	-	-	30.69%	30.36%	-	-	-	-	-	30.16%
β_{Tax}	est.	-0.4888	-1.0000	-0.5232	-1.0000	-1.0000	-1.0000	-1.3989	-1.4536	-1.3963	-1.3679	-1.4293	-1.6455
	rob.t-rat.	-10.31	**fixed**	-10.05	**fixed**	**fixed**	**fixed**	-9.05	-6.88	-9.31	-9.45	-5.22	-5.70
β_{TT}	est.	-0.2598	-0.5271	-0.2834	-0.5675	-0.5225	-0.5304	-0.7442	-0.8368	-0.7476	-0.7792	-0.8517	-0.9638
	rob.t-rat.	-6.54	-5.08	-7.01	-5.83	-5.38	-5.40	-6.56	-5.23	-6.79	-6.59	-5.12	-4.53
β_{IN}	est.	-0.5548	-1.1239	-0.5052	-0.9660	-1.1731	-1.1201	-1.5786	-1.8668	-1.5764	-1.6693	-1.7748	-1.9672
	rob.t-rat.	-10.31	-7.26	-8.99	-6.75	-7.49	-6.84	-8.84	-9.21	-8.90	-7.05	-8.11	-7.74
β_{DE}	est.	-0.3957	-0.7870	-0.3444	-0.6627	-0.8458	-0.7988	-1.1301	-1.4356	-1.1364	-1.1128	-1.5081	-1.5517
	rob.t-rat.	-9.43	-6.95	-7.80	-6.32	-7.15	-6.15	-8.22	-7.93	-7.13	-8.24	-5.22	-6.61
δ_{SQ}	est.	-0.9269	-0.9208	-0.6293	-0.6244	-0.9704	-0.8747	-1.7840	-1.2720	-1.7945	-1.8456	-1.2543	-1.2672
	rob.t-rat.	-8.17	-6.95	-4.12	-4.35	-7.02	-5.10	-6.80	-9.52	-6.28	-6.53	-8.35	-7.90
δ_{base}	est.		-0.2995		-0.5226	-0.7632	-0.8383	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	rob.t-rat.		-1.37		-2.65	-4.14	-4.38	**fixed**	**fixed**	**fixed**	**fixed**	**fixed**	**fixed**
δ_{taboo}	est.			-0.4409	-0.3506								
	rob.t-rat.			-2.30	-2.75								
h	est.					-0.2779							
	rob.t-rat.					-6.07							
ω	est.						-1.85						3.0729
	rob.t-rat. (vs $\pi/2$)						-94.29						28.79
ϑ	est.					1.5708	1.7257						2.0577
	rob.t-rat. (vs $\pi/2$)					**fixed**	0.91						0.43
φ_{Tax}	est.							-0.5644°	-0.9913°	-0.5797°	-0.7790°	-0.9584	-1.1296°
	rob.t-rat.							-6.23	-6.27	-3.23	-4.25	-4.77	-7.56
φ_{TT}	est.							-0.5644°	-0.9913°	-0.5540*	-0.4428*	-1.0188	-1.1296°
	rob.t-rat.							-6.23	-6.27	-3.60	-3.40	-4.21	-7.56
φ_{IN}	est.							-0.5644°	0.3904*	-0.5797°	-0.4428*	0.3367	0.4453*
	rob.t-rat.							-6.23	2.03	-3.23	-3.40	1.77	2.56
φ_{DE}	est.							-0.5644°	0.3904*	-0.5540*	-0.7790°	0.4794	0.4453*
	rob.t-rat.							-6.23	2.03	-3.60	-4.25	2.23	2.56
Rotation matrix R elements	R[1,1]					-0.15-0.95i	-0.15+0.95i						-0.47-0.88i
	R[1,2]					0.27+0.00i	0.27+0.00i						-0.06+0.00i
	R[2,1]					-0.27+0.00i	-0.27+0.00i						0.06+0.00i
	R[2,2]					-0.15+0.95i	-0.15-0.95i						-0.47+0.88i

The models in this section are based on those in Section 3.1.2, with the amplitudes being estimated equivalently using Eqs. (9) and (10). However, instead of adding a constant to capture the presence of a taboo trade-off, an additional rotation is applied to the estimated amplitudes using Eq. (7). We test two different rotations, with each based on the Pauli matrices, as described in Section 2.5.

1. We first use a rotation matrix, R_1, with one free parameter, h, for weighting the Pauli matrices $n_y = \frac{h}{\sqrt{1+h^2}}$, $n_z = \frac{1}{\sqrt{1+h^2}}$ and set the rotation angle $\vartheta = \frac{\pi}{2} \cdot \sqrt{1+h^2}$. Notice that an estimate of $h = 0$ would indicate no change in probability. Positive estimates indicate a shift towards alternative 1, whereas negative estimates indicate a shift towards alternative 2.

2. We next use a rotation matrix, R_2, based on our second method using trigonometric functions to define the weights for $|\mathbf{n}|$. We find that fixing $n_x = 0$ results in no loss of model fit, leaving us with two free rotation parameters: one for the angle, ϑ, and another ω, where we set $n_y = \cos(\omega)$ and $n_z = \sin(\omega)$ (which guarantees $|\mathbf{n}| = 1$).

The results of models with quantum rotations are again given in Table 2. This table also reports p-values from Ben-Akiva and Swait tests (Ben-Akiva and Swait, 1986) for non-nested models, assessing whether the quantum models have a statistically better fit than the TTOA (Logit-t) model. This table further reports the average probability of choosing the new policy (NP) before and after the quantum rotation is applied to a choice scenario which contains a taboo trade-off (the observed choice proportions appear in Table 1).

For model QAR-1, which has the quantum rotation implemented, we see a significant improvement in log-likelihood from the addition of 1 parameter. This quantum model now has a better BIC (1,483.15) than the TTOA model (1,480.09) and is statistically better at the 5% level, with a p-value of 0.040 from the Ben-Akiva and Swait test. The addition of a second free parameter in the model (QAR-2) does not result in a significant improvement in the log-likelihood compared to the QAR-1 model.

In line with the results of Chorus et al. (2018) and Table 1, we observe that across all models, the presence of a taboo trade-off results in the decision-maker being less likely to choose the New Policy alternative. As the number of model parameters increases, the average probabilities under the model become increasingly closer to matching the observed share of new policy choices (30.08% when there is a taboo trade-off, 42.72% when there is not, see Table 1). For QAR-2, we observe that the rotation, on average, reduces the probability of choosing the new policy. Under QAR-1, the opposite is true. Whilst this result may appear counterintuitive, we observe a greater estimate for β_{DE} (see Table 2) in QAR-1 in comparison to QA, demonstrating that the additional flexibility of including a rotation allows for more extreme estimates to help capture choices in general. We will return to the impact of quantum rotations on the probabilities of choosing the alternatives in more detail later (see Section 3.1.5).

3.1.4. Models with complex phases for the taboo trade-off dataset

An alternative mechanism for capturing different 'processes' with a quantum choice model is the implementation of complex phases, as described in Section 2.5. As with models implementing quantum rotations, we have a number of options for how many free parameters to use in the specifications for the quantum choice models. We consider the following three possibilities:

1. A single complex phase, φ, for all of the attributes, such that all $\ln(1+e^{\beta_x \cdot \Delta_x})$ are replaced with $e^{-i\varphi} \cdot \ln(1+e^{\beta_x \cdot \Delta_x})$ in Eqs. (9) and (10). This implementation of phases tests whether the introduction of complex phases improves the model performance in general.

2. Two complex phases, φ_1 and φ_2, each respectively applied to two attributes. This gives us three distinct configurations. Our focus of interest is on the configuration with one phase applied to the two moral attributes and the other phase applied to the two concrete attributes. If this model is significantly better than the other configurations, it would suggest that the moral and concrete attributes are indeed 'different' and can be categorised as such.

3. Four complex phases with four free parameters, $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, with a different phase for each attribute. This configuration allows us to test the performance of the introduction of relative complex phases overall.

The results of models with complex phases are given in Table 2. As with quantum rotations, we again observe that there is a significant improvement obtained by including complex phases. The first model offers a good improvement over the base QA model, but is not statistically better than the TTOA model. Further additional free parameters result in improvements in model fit. As these parameters are significantly different from zero (if $\varphi = 0$, then we have $e^{i0} = 1$, which corresponds to real-only amplitudes), we have evidence to reject models without complex phases in favour of models with complex phases. Note that the introduction of complex phases into the specification of the amplitudes (Eq. 3) means that we no longer have an overspecification by not fixing one of the β-coefficients. This is a direct result of having real-valued constants in the amplitudes. However, we still have five base parameters as the estimate for δ_{base} becomes insignificant, and is thus fixed to a value of zero. Crucially, the model results suggest that concrete and moral attributes are treated 'differently' in the cognitive choice process, as a substantial gain is found through the use of separate phases for the concrete and moral attributes, but not for other combinations of uses of two complex phases. Furthermore, we obtain insignificantly different estimates for the tax and time phases and the deaths and injuries phases when each parameter has a separate phase (model QAPh-3). This suggests that the concrete attributes are treated 'equivalently' in the cognitive choice process and similarly so for the moral attributes. In comparison to models with a quantum rotation, the models with complex phases record better BIC values, with the best adjusted ρ^2 value of 0.3443 for a model with complex phases compared to 0.3406 for a model with a rotation. This implies that the moral aspect in the choice tasks is better captured by a perspective operation that implements separate complex phases for moral and concrete attributes, as opposed to the inclusion of a rotation for particular choice tasks.

3.1.5. Combined model for the taboo trade-off dataset

For our final model, we test the use of a model that incorporates both a quantum rotation and complex phases simultaneously. Our final model for the taboo trade-off dataset is based on the best performing model thus far (QAPh-2a) combined with the use of a rotation based on Pauli matrices. It thus has two complex phase parameters (one for concrete attributes, and one for moral attributes), as well as two rotation parameters. This results in the model having an additional four parameters to capture the moral components in the choice tasks, on top of the five parameters of the basic QA model.

The results of the combined model (QAC) are shown in Table 2. For this model, we observe significant estimates for the parameters for the complex phases, where these are not significantly different from the estimates for these parameters under a model without additional rotation parameters (QAPh-2a). The combined model does not offer a significant improvement over the version with complex phases, with the estimates for the rotation parameter ϑ not being significantly different from $\pi/2$, indicating that the rotation has minimal impact. This implies that for this dataset, complex phases and additional rotations are approximately equivalent.

However, there is evidence that there is still an effect by including the rotation, through consideration of the results in Table 3. This table gives the probability of supporting the new policy under each of the different choice scenarios before and after the quantum rotation is applied. Crucially, our final model has smaller mean absolute deviations from the true share of support than the TTOA model (which is unsurprising given that this model has more free parameters and records a statistically significant improvement in the model fit). This is only the case for the 'taboo tasks' after the implementation of the rotation, suggesting that the combined model still benefits from the inclusion of the rotations. Note that a rotation is not like an additive constant, which would always result in a bias towards one alternative. Instead, the combination of real and imaginary numbers (the interference effect) results in a shift that may swing the probabilities in either direction, which is hence unlikely to result in a direct bias towards one alternative. In this case, the rotation almost always reduces the probability of choosing the new alternative. This is in line with our expectations: the presence of a taboo trade-off reduces the likelihood of choosing the new policy.

3.2. Quantum modelling for moral trade-offs involving a couple's respective commutes

3.2.1. Description of data

The second dataset we test involves decision-makers completing two distinct sets of choice tasks based on an individual's willingness to accept longer commutes for better salaries (see Beck and Hess, 2016, for a detailed description of the survey). The first set of tasks involved trade-offs between the individual's current travel time and salary or an increased salary (of 500 or 1000 SEK in net wage per month) at a cost of an increase in one-way travel time (of either 10 or 25 minutes). The second set additionally included attributes for increased travel time and salaries for the partner of the decision-maker (under the assumption that both the decision-maker and their partner both commute to work), meaning that the decision-maker has to make choices about who to prioritise (the dataset is thus henceforth referred to as the 'couple commuter dataset'). All choice tasks included a status quo alternative, a new location and an 'I am indifferent' option. A sample of 1,179 households (with both partners in each household, resulting in 2,358 individuals) completed 4 tasks for the first set involving only attributes affecting themselves, and 4 or 5 tasks for the second set with attributes impacting both members of the household. This resulted in a total of 20,041 choice observations.

While the first set of choice tasks involves typical time-cost trade-offs that can potentially be captured well with traditional choice models, the latter involves a more complex decision context without any 'crisp' trade-off element in that there may not be a clear ethical protocol for how to make the decision. This dataset thus provides another test for our quantum model features that capture changes in choice context. The observed choice shares for the alternatives are given in Table 4, where we see that a decision-maker is more likely to pick the status quo (SQ) over the new location (NL) if the choice task also includes attributes concerning their partner.

At the outset, it should already be noted that the presence of an 'indifference' option in a SC

TABLE 3 : Observed and theoretical choice probabilities in the taboo trade-off dataset. The 'taboo trade-off' occurs if a decision-maker chooses to decrease tax or travel time at the cost of increasing the number of injuries or deaths. The impact of the quantum rotations is rendered explicit; 'Before' is the probability of choosing the new policy without applying a quantum rotation, 'After' is the probability following the application of the quantum rotation. TTOA gives theoretical probabilities from the 'Generic Taboo Trade-Off Aversion' model (Chorus et al., 2018).

Scenario	Attributes				Taboo Trade-Off?	Share of support for New Policy			
	Tax	Time	Injuries	Deaths		Observed	TTOA (Logit-t)	QAC before	QAC after
1	-	-	-	-	No	98.0%	93.6%	97.5%	
2	-	-	-	+	Yes	68.7%	70.4%	67.6%	65.3%
3	-	-	+	+	Yes	29.3%	23.9%	32.6%	27.3%
4	-	+	+	+	Yes	11.1%	9.2%	16.1%	12.2%
5	+	+	+	+	No	2.0%	1.9%	2.7%	
6	+	-	-	-	No	62.6%	64.3%	63.3%	
7	+	+	-	-	No	44.4%	36.7%	42.7%	
8	+	+	+	-	No	4.0%	7.1%	3.5%	
9	-	+	-	+	Yes	42.4%	43.3%	40.6%	41.9%
10	+	-	+	-	Yes	15.2%	13.3%	12.2%	13.8%
11	-	-	+	-	Yes	46.5%	55.5%	56.4%	53.0%
12	-	+	-	-	No	80.8%	82.5%	81.4%	
13	-	+	+	-	Yes	30.3%	28.7%	29.5%	29.2%
14	+	-	-	+	Yes	22.2%	22.6%	21.0%	24.2%
15	+	-	+	+	Yes	5.1%	3.7%	7.3%	4.6%
16	+	+	-	+	No	7.1%	12.8%	9.1%	
Mean absolute deviation from true share of support (%; all choice tasks)							3.03	2.19	**1.57**
Mean absolute deviation from true share of support (%; taboo tasks only)							2.68	3.15	**2.05**

survey calls for special attention in model specification. Indeed, as discussed by Hess et al. (2014), the inclusion of an 'indifference' option means that non context-dependent models are likely not suitable. To understand this point, note that making both the status quo and the alternative option worse or better by the same amount, be this through changes in time, salary, or both, should not affect the degree to which a decision-maker is indifferent between them. However, in structures based on random utility maximisation (RUM), changes to time or salary for the non-indifference options would change their utilities and hence their probabilities relative to the indifference option, whose utility is unchanged. Hess et al. (2014) shows that on the contrary, as a result of regret models using a value function that is reliant on pairwise comparisons of alternatives, the same change in all non-indifference alternatives does not impact the probability of choosing the indifference option. Within a quantum choice model framework, there are numerous possibilities for capturing indifference. For the work on this dataset, we implement the simplest solution. This is to assume that the indifference choice is a separate component of the belief state, using a 3-dimensional Hilbert space. The indifferent alternative thus appears in the model equivalently to any of the other alternatives, except that it does not depend directly on the attributes of the other

TABLE 4 : Observed shares of alternatives under each choice scenario in the couple commuter dataset.

Scenario	Attribute changes Travel time (TT, mins). Salary (SEK/month)				Observed		
	Own TT	Own Salary	Partner TT	Partner Salary	SQ	NL	Indifferent
1	+10	+500	0	0	70.3%	24.5%	5.2%
2	+25	+500	0	0	70.0%	23.9%	6.1%
3	+10	+1000	0	0	71.1%	24.1%	4.9%
4	+25	+1000	0	0	69.5%	25.3%	5.2%
5	+10	+500	+10	+500	74.4%	20.6%	5.0%
6	+10	+500	+25	+500	73.6%	21.2%	5.2%
7	+10	+500	+10	+1000	73.8%	20.3%	5.9%
8	+10	+500	+25	+1000	76.0%	19.2%	4.7%
9	+25	+500	+10	+500	74.4%	21.3%	4.3%
10	+25	+500	+10	+1000	73.4%	20.4%	6.2%
11	+10	+1000	+10	+500	74.8%	20.6%	4.6%
12	+10	+1000	+25	+500	74.0%	20.9%	5.1%
13	+25	+1000	+10	+500	72.4%	22.5%	5.1%

alternatives, c.f. Eq. (3). It does however depend on the attributes indirectly, by the normalisation of the components (see Eq. 4). However, by implementing the RRM value functions, our quantum model inherits the property that choice probabilities will be invariant to uniform increases or decreases of the attribute values. If, for example, we observe an increase of Δ_x across attribute x for all alternatives, then there is no change in the amplitudes: $\mathrm{e}^{\beta_k((x_{ntik}+\Delta_x)-(x_{ntjk}+\Delta_x))} = \mathrm{e}^{\beta_k(x_{ntik}-x_{ntjk})}$. This means that for the choice scenarios detailed in Table 4, the absolute values for a decision-maker's travel time and salary do not have an impact: it is only the relative differences between the status quo and new location that impact the choice probabilities, both in RRM and our quantum amplitude model. This discussion explains the use of RRM as the base model against which we compare our quantum model.

3.2.2. Basic models for the couple commuter dataset

For this dataset, there are two distinct choice sets: the first only includes factors impacting the decision-maker alone (CT1) while the second additionally includes impacts on the partner (CT2).

For the QA model on the CT1 data, the amplitudes for the status quo (SQ), new location (NL) and indifference (Ind) alternatives are then:

$$\psi_{nt,SQ} = \frac{\delta_{base} + \delta_{SQ} + \ln(1 + \mathrm{e}^{-\beta_{O_{TT}} \cdot \Delta_{O_{nt,TT}}}) + \ln(1 + \mathrm{e}^{-\beta_{O_{Sal}} \cdot \Delta_{O_{nt,Sal}}})}{\mathcal{N}_{nt}}, \tag{13}$$

$$\psi_{nt,NL} = \frac{\delta_{base} + \ln(1 + \mathrm{e}^{\beta_{O_{TT}} \cdot \Delta_{O_{nt,TT}}}) + \ln(1 + \mathrm{e}^{\beta_{O_{Sal}} \cdot \Delta_{O_{nt,Sal}}})}{\mathcal{N}_{nt}}, \tag{14}$$

and

$$\psi_{nt,Ind} = \frac{\delta_{Ind} + \delta_{base}}{\mathcal{N}_{nt}}. \tag{15}$$

We again estimate a constant that is added to the amplitude for all alternatives, δ_{base}, and $\mathscr{N}_{nt}$ is a normalisation factor calculated using the numerators from each amplitude, Eqs. (13, 14, 15), in line with Eq. (4). For the remaining terms, we have that $\Delta_{O_{nt,TT}}$ is the change in the decision-maker's travel time, $\Delta_{O_{nt,Sal}}$ is the change in their salary and the β-coefficients estimate the relative importance of these attributes ('O'for 'own'). The amplitude for the new location alternative replaces $-\beta$ with β and obviously drops δ_{SQ}, where we do not show δ_{NL}, which is normalised to zero, while the constant δ_{Ind} is included for the indifference alternative.

For the random regret minimisation models, the random regret functions for the CT1 data are given by:

$$RR_{nt,SQ} = \delta_{SQ} + \ln(1 + \mathrm{e}^{\beta_{O_{TT}} \cdot \Delta_{O_{nt,TT}}}) + \ln(1 + \mathrm{e}^{\beta_{O_{Sal}} \cdot \Delta_{O_{nt,Sal}}}) + \varepsilon_{nt,SQ}, \tag{16}$$

$$RR_{nt,NL} = \delta_{NL} + \ln(1 + \mathrm{e}^{-\beta_{O_{TT}} \cdot \Delta_{O_{nt,TT}}}) + \ln(1 + \mathrm{e}^{-\beta_{O_{Sal}} \cdot \Delta_{O_{nt,Sal}}}) + \varepsilon_{nt,NL}, \tag{17}$$

and

$$RR_{nt,Ind} = \delta_{Ind} + \varepsilon_{nt,Ind}. \tag{18}$$

Note that the direct comparison of the equations for amplitudes and regret allows for a clear mathematical interpretation of the difference between the models. Whereas the quantum models additionally have a normalisation factor such that the probabilities can be calculated directly from these amplitudes, using Eq. (2), the regret model implements uncertainty in which alternative is chosen through use of type I extreme value distributed error terms, ε. A further difference arises in that Δ_x and $-\Delta_x$ are interchanged when moving between amplitudes and regret. This is simply to ensure the correct sign for the directionality of the attributes in the respective models, with the negative of the regret used to calculate probabilities.

For CT2, additional terms are added for the attributes impacting the partner. An additional layer of flexibility is possible (and explored below), by allowing the parameters for own time and salary to be different in CT1 and CT2, i.e. not just allowing for differences between the evaluation of the impact on the respondent themselves (vs on the partner), but allowing that impact to be different when the impact on the partner is also considered.

The results of the basic quantum choice models together with the equivalent RRM models are given in Table 5. The first models (RRM-1 and QA-1) keep the parameters for the importance of a decision-maker's own salary and travel time constant between CT1 and CT2, labelling them as $\beta_{O_{Sal}}$ and $\beta_{O_{TT}}$. The second set (RRM-2 and QA-2) have separate parameters for CT1 and CT2 for the importance of own salary and cost, as well as separate constants for CT1 and CT2.

Regardless of whether RRM and QA are compared with or without the use of separate parameters, the results indicate a substantial advantage for the quantum models. Additionally, both QA and RRM find clear evidence that the use of separate CT1 and CT2 parameters lead to further gains in fit, demonstrating that there is an inconsistency in how a decision-maker considers factors impacting themselves in the absence (CT1) or presence (CT2) of factors impacting their partner. These differences are clearly visible in the second set of models, where we see a reduction in both the own salary and own time parameters when going from CT1 to CT2, where this is an indication of differences in noise (lower parameters mean a less deterministic choice process), but also differences in relative valuations as the reduction is larger for the salary coefficient than for the time coefficient. The models imposing homogeneity between CT1 and CT2 are biased as a result and show that individuals give higher importance to their own salary and their own travel time, while the second set of models shows that this is only the case for salary.

TABLE 5 : Results of all basic models and models with quantum rotations for the couple commuter dataset, together with all parameter estimates.

Type		Basic Models						Models with a quantum rotation					
Specification		**RRM-1**	**QA-1**	**RRM-2**		**QA-2**		**QAR-2a**	**QAR-2b**	**QAR-2c**	**QAR-2d**	**QAR-2e**	**QAR-2f**
Free parameters		6	7	10		12		11	11	11	11	11	11
Log-likelihood		-12,784.21	-12,624.13	-12,426.71		-12,289.38		-12,430.74	-12,463.49	-12,283.33	-12,426.12	-12,461.48	-12,436.75
Adj. ρ^2		0.41908	0.42631	0.43514		0.44129		0.43491	0.43342	0.44161	0.43512	0.43351	0.43464
BIC		25,612.62	25,299.83	24,927.10		24,667.17		24,942.52	25,008.01	24,647.70	24,933.28	25,004.00	24,954.55
Average probability of chosen alternative (by type of alternative and by experiment)													
CT1	Status Quo	77.66%	75.61%	76.97%		77.73%		77.84%	77.74%	77.63%	77.90%	77.70%	78.07%
	New Location	44.47%	40.97%	42.39%		43.10%		44.04%	44.00%	43.04%	44.24%	44.22%	43.97%
	Indifferent	1.91%	9.28%	5.01%		6.22%		6.76%	6.76%	6.24%	6.60%	6.67%	6.70%
CT2	Status Quo	76.62%	78.70%	77.86%		78.19%		77.44%	77.75%	78.23%	77.62%	77.93%	77.07%
	New Location	29.38%	29.41%	33.47%		33.54%		28.78%	27.73%	34.08%	28.40%	27.57%	28.72%
	Indifferent	8.04%	3.54%	5.00%		5.37%		5.22%	5.29%	5.25%	5.35%	5.26%	5.35%
Overall		64.38%	64.18%	64.84%		65.35%		64.69%	64.66%	65.39%	64.76%	64.72%	64.61%
Parameter Estimates													
Choice Set		both	both	CT1	CT2	CT1	CT2	both	both	both	both	both	both
$\beta_{O_{TT}}$	est.	-0.1434	-3.0200	-0.1609	-0.1290	-0.0747	-0.0315	-0.1875	-0.1554	-0.0692	-0.1617	-0.1445	-0.1960
	rob.t-rat.	-39.89	-16.95	-37.19	-30.43	-7.16	-13.04	-8.80	-10.91	-9.96	-10.33	-11.63	-6.79
$\beta_{O_{Sal}}$	est.	2.1004	26.3856	2.4834	1.4860	0.9225	0.3381	1.8225	1.6141	0.8541	1.6326	1.5364	1.8534
	rob.t-rat.	35.69	17.99	31.93	15.80	9.18	6.95	11.38	13.77	11.80	13.55	14.72	9.05
$\beta_{P_{TT}}$	est.	-0.0956	-1.5624		-0.1315		-0.0332	-0.1580	-0.1370	-0.0754	-0.1381	-0.1234	-0.1773
	rob.t-rat.	-35.77	-26.04		-28.78		-14.74	-13.88	-17.37	-15.31	-15.74	-17.57	-10.46
$\beta_{P_{Sal}}$	est.	1.3103	15.1615		0.8569		0.2052	1.7746	1.5900	0.5142	1.6317	1.6599	1.5995
	rob.t-rat.	21.77	35.19		5.40		3.74	8.42	8.79	4.67	10.41	9.56	7.37
δ_{SQ}	est.	-0.3498	-8.7204	-0.5274	0.8100	0.0056	-0.2434	-0.4447	-0.2817	0.0055	-0.3365	-0.2380	-0.5032
	rob.t-rat.	-5.43	-8.03	-6.88	4.33	0.15	-7.28	-4.03	-3.78	0.20	-4.02	-3.60	-3.44
δ_{IND}	est.	5.1402	10.2110	4.2313	6.1702	1.1199	2.4040	1.1086	1.1086	1.1221	1.0972	1.1075	1.1066
	rob.t-rat.	64.77	12.01	50.96	52.16	45.65	352.05	35.11	40.10	55.29	39.97	42.23	33.92
δ_{base}	est.		1.2753			-0.8342	-2.2278	-0.5590	-0.6243	-0.8522	-0.6151	-0.6524	-0.5510
	rob.t-rat.		2.06			-16.93	-221.78	-11.29	-15.27	-23.14	-14.55	-17.10	-9.56
First Rotation								1-2	1-2	1-3	1-3	2-3	2-3
Second Rotation								1-3	2-3	1-2	2-3	1-3	1-2
ω_{1-2}	est.							1.3914	1.0363	0.8699		1.3074	
	rob.t-rat. (vs $\pi/2$)							-3.34	-0.79	-15.70		-2.05	
ϑ_{1-2}	est.							2.8047	3.0650	1.9568		0.2188	
	rob.t-rat. (vs $\pi/2$)							44.56	23.58	9.33		-26.57	
ω_{1-3}	est.							0.3423		1.6702	2.7346		1.3994
	rob.t-rat. (vs $\pi/2$)							13.75		7.19	2.86		-9.03
ϑ_{1-3}	est.							0.5017		1.6371	2.8437		0.8244
	rob.t-rat. (vs $\pi/2$)							-22.57		0.74	28.91		-5.16
ω_{2-3}	est.								1.2713		1.2225	0.5071	1.6737
	rob.t-rat. (vs $\pi/2$)								-16.68		-16.78	-20.56	2.03
ϑ_{2-3}	est.								1.4465		1.5556	2.8602	1.2224
	rob.t-rat. (vs $\pi/2$)								-2.13		-0.46	108.97	-0.75

3.2.3. Models with quantum rotations for the couple commuter dataset.

This dataset also provides opportunities for the use of additional features from quantum choice models to test for an inconsistency or 'change of perspective' incurred through changing from thinking about just yourself compared to yourself and your partner. We test the change of perspective to the choice task with attributes impacting the partner through two consecutive quantum rotations over the alternatives (status quo, new location and indifferent). For our models implementing quantum rotations, we use Eqs. (13, 14, 15) to define the amplitudes for the alternatives within the first set of choice tasks. For the amplitudes under the second set of choice tasks, we initially use these equations to estimate the amplitudes before then applying quantum rotations. Thus, for these models, we estimate a single set of coefficients that apply to choices made in both choice sets. We then require a product of two rotation matrices for adjusting the amplitudes appropriately when additionally considering travel time and salary changes for the partner. The new amplitudes after the rotation are then given by:

$$\Psi_f = R_B R_A \Psi_0, \tag{19}$$

where R_A and R_B are both estimated using Eq. (6) and rotation matrices based on R_2 with the use of axis and angle parameters. We again find that fixing $n_x = 0$ results in no loss of model fit, leaving us with four free parameters, ω_A, ω_B, ϑ_A and ϑ_B, where we again set $n_y = \cos(\omega)$ and $n_z = \sin(\omega)$. Given that we implement two consecutive rotations on pairs of alternatives, there are six combinations of pairwise rotations. The results of these six models are given in Table 5. In all six cases, we see an improvement in fit over the basic model (QA-1). The third option (QAR-2c) which first rotates between the status quo and the indifference option (alternatives 1 and 3), before rotating between the status quo and the new option (alternatives 1 and 2), offers the most substantial improvement in fit. It also outperforms the second basic model (QA-2), suggesting that the rotations can better account for the differences between tasks completed in the different choice sets than is the case for using separate parameters.

3.2.4. Models with complex phases for the couple commuter dataset

We next turn to the incorporation of complex phases, where we again test a number of possible specifications, given that there are six different attributes and many possibilities for how many phases to implement. We consider the following possibilities for how to introduce complex phases.

1. A model with a single complex phase, φ, that is applied to all of the attributes.
2. A model with two complex phases, φ_1 and φ_2, with the first applied to attributes impacting the decision-maker and the second applied to attributes impacting the partner.
3. A model with two complex phases, with the first applied to travel time attributes and the second applied to salaries.
4. A model with two complex phases, with the first applied to attributes in the first set of choice tasks, and the second applied to attributes in the second set of choice tasks.
5. A model with six complex phases with six free parameters, $\{\varphi_1, \ldots, \varphi_6\}$, with a different phase for each attribute in each set of choice tasks.

TABLE 6 : Results of all models with complex phases and the combined models for the couple commuter dataset, together with all parameter estimates, where ∘ and ⋆ indicate attribute pairs which have the same phase.

Type		**Models with complex phases**					**Combined**	
Specification		**QAPh-1**	**QAPh-2**	**QAPh-3**	**QAPh-4**	**QAPh-5**	**QAC-1**	**QAC-2**
	Free parameters	8	9	9	9	13	17	15
	Log-likelihood	-12,536.66	-12,366.51	-12,536.65	-12,487.07	-12,319.10	**-12,249.70**	-12,250.21
	Adj. ρ^2	0.43024	0.43792	0.43019	0.43244	0.43989	0.44286	**0.44293**
	BIC	25,132.26	24,799.33	25,139.60	25,040.45	24,733.98	24,624.65	**24,610.94**
Average probability of chosen alternative (by type of alternative and by experiment)								
CT1	Status Quo	75.57%	76.99%	75.57%	76.21%	77.79%	78.15%	78.18%
	New Location	39.84%	42.51%	39.84%	39.36%	42.96%	43.92%	43.84%
	Indifferent	8.77%	6.83%	8.76%	8.34%	6.65%	6.13%	6.15%
CT2	Status Quo	79.57%	78.57%	79.58%	78.99%	78.32%	78.21%	78.22%
	New Location	33.13%	31.82%	33.13%	33.51%	32.81%	34.15%	34.13%
	Indifferent	4.01%	5.40%	4.01%	4.37%	5.40%	5.26%	5.27%
	Overall	64.79%	65.02%	64.79%	64.76%	65.34%	65.65%	65.66%
Parameter Estimates								
$\beta_{O_{TT}}$	est.	-3.1325	-0.1441	-3.1557	-3.3643	-0.0709	-0.0385	-0.0383
	rob.t-rat.	-12.99	-28.52	-8.63	-14.15	-18.58	-9.75	-9.96
$\beta_{O_{Sal}}$	est.	17.9584	1.4095	18.0740	26.0038	0.8198	0.6026	0.5430
	rob.t-rat.	13.83	22.79	8.92	14.09	17.04	7.28	11.10
$\beta_{P_{TT}}$	est.	-2.0343	-0.0992	-2.0487	-2.6486	-0.0908	-0.0513	-0.0543
	rob.t-rat.	-14.07	-18.67	-8.96	-14.95	-10.33	-7.12	-9.03
$\beta_{P_{Sal}}$	est.	9.0215	0.6876	9.0752	10.1627	0.5504	0.2528	0.2530
	rob.t-rat.	1.20	7.43	8.38	18.89	6.44	5.07	5.00
δ_{SQ}	est.	-41.2320	-0.5687	-41.4785	-33.3201	-0.0046	0.0992	0.0784
	rob.t-rat.	-12.40	-6.16	-8.57	-11.00	-0.15	4.85	5.04
δ_{IND}	est.	6.6360	1.2861	6.7424	8.9254	0.7372	1.0810	1.0512
	rob.t-rat.	10.09	36.59	7.70	10.35	15.23	81.06	32.53
δ_{base}	est.	1.1958	-0.8424	1.1482	0.7655	-0.9828	-1.2130	-1.1817
	rob.t-rat.	2.59	-25.32	2.75	17.27	-27.24	-99.16	-55.73
ω_{1-2}	est.						0.7219	0.3527
	rob.t-rat. (vs $\pi/2$)						-7.57	-9.96
ϑ_{1-2}	est.						2.3048	2.5050
	rob.t-rat. (vs $\pi/2$)						5.86	18.47
ω_{1-3}	est.						1.6817	1.7523
	rob.t-rat. (vs $\pi/2$)						3.55	3.94
ϑ_{1-3}	est.						2.3981	2.7212
	rob.t-rat. (vs $\pi/2$)						3.49	9.95
$\varphi_{O_{TT_1}}$	est.	0.3958∘	0.5282∘	0.3960∘	0.7655∘	0.1119	0.1653	0.0619∘
	rob.t-rat.	17.62	31.39	17.60	17.27	2.12	2.55	1.76
$\varphi_{O_{TT_2}}$	est.	0.3958∘	0.5282∘	0.3958∘	-3.04E-06⋆	-1.0690	0.2035	0.0619∘
	rob.t-rat.	17.62	31.39	17.60	-4.13	-15.75	1.41	1.76
$\varphi_{P_{TT}}$	est.	0.3958∘	-1.2984⋆	0.3958∘	-3.04E-06⋆	1.2073	0.3628	0.2338
	rob.t-rat.	17.62	-29.61	17.60	-4.13	25.92	2.69	3.78
$\varphi_{O_{Sal_1}}$	est.	0.3958∘	0.5282∘	0.1602⋆	0.7655∘	0.2671	0.1045	0.1502⋆
	rob.t-rat.	17.62	31.39	0.41	17.27	11.21	4.69	6.75
$\varphi_{O_{Sal_2}}$	est.	0.3958∘	0.5282∘	0.1602⋆	-3.04E-06⋆	0.7928	0.5541	0.1502⋆
	rob.t-rat.	17.62	31.39	0.41	-4.13	12.23	3.77	6.75
$\varphi_{P_{Sal}}$	est.	0.3958∘	-1.2984⋆	0.1602⋆	-3.04E-06⋆	-1.0220	-0.5316	-0.7645
	rob.t-rat.	17.62	-29.61	0.41	-4.13	-27.09	-2.99	-3.29

The results of these different possibilities are given in Table 6. The incorporation of a larger number of complex phases opens up the potential for large gains in fit, but the actual specification of the phases is important. For example, a significant gain is found by moving from a single phase to two phases for each combination except for the model with different phases for travel time as opposed to salaries (QAPh-3). The most substantial of these gains is found by QAPh-2, which has separate phases for attributes impacting the decision-maker and attributes impacting their partner. The best performing model overall is a model with a different phase for each of the different attributes, suggesting that, as with the first dataset, the attributes are considered differently. However, this model does not perform as well as a basic quantum amplitude model with separate parameters for the different choice sets (see Table 5). Consequently, as opposed to the results of the first dataset, it is the addition of quantum rotations rather than complex phases that better captures the implicit change in choice context.

3.2.5. Combined model for the couple commuter dataset

For our combined model, we again utilise a model with both quantum rotations and complex phases to capture the change of perspective when commute attributes concerning the partner are also present. Given the good performance of QAPh-5, which has six phases, and of QAR-2c, we opt to combine these models for our final model. This means that it has 7 parameters in common with the basic model (QA-1), 6 complex phases (with one for each attribute) and 4 rotation parameters, as before for the models with quantum rotations. Note that we implement rotations based on the best performing rotation model, thus first rotating between the status quo and the indifference option (axis parameter ω_{13}, angle ϑ_{13}) before rotating between the status quo and the new location alternative (axis parameter ω_{12}, angle ϑ_{12}).

Table 6 gives the results of our combined model. This time, in contrast with the results for the taboo trade-off dataset, we see that the model (QAC-1) combining quantum rotations and complex phases does offer a significant improvement over a model offering only one of these additional features to capture the change of perspective. We also see that the model with six phases has rather different estimates for the phases for the same attributes across the different choice sets, suggesting that these attributes cannot be treated equivalently (Fig. 3). For the combined models, we find significant estimates for both the phase and rotation parameters. However, we note that there is not a significant difference between the phase parameters across choice sets ($\varphi_{O_{Sal_1}}$ and $\varphi_{O_{Sal_2}}$) and ($\varphi_{O_{TT_1}}$ and $\varphi_{O_{TT_2}}$). It thus appears that the quantum rotation, which is used for the 'change of perspective' from the first choice task to the second choice task, already captures the difference between choice sets. We consequently include a second combined model (QAC-2) with just four complex phases. This final model does not result in a significant loss of model fit, as expected, and achieves the best adjusted ρ^2 and BIC.

This effect is particularly clear through closer consideration of the estimates for the complex phase parameters, φ. These are displayed graphically in Fig. (3). For the model with complex phases only (QAPh-5), we observe very different estimates across attributes, choice task set and whether an attribute affects the decision-maker or the partner. This illustrates why the QAPh-5 model can outperform the models with only two complex phases, with the flexibility of interactions across all attributes clearly helping to improve performance. The addition of quantum rotations, (moving from QAPh-5 to QAC-1) results in phases regressing closer to smaller values (modulo 2π). This results in a weaker interference interaction across the attributes, with the real parts growing in magnitude whilst the imaginary parts shrink. The only exception is the phase to attribute OTT-2 which appears to vary strongly. However, the phase to OTT-2 in QAC-1 is not reliable (t-value = 1.41), hence its value shift from QAPh-5 to QAC-1 should not necessarily be understood as a significant adaptation but rather a shift to a spurious local optimisation value in QAC-1. Notably, the estimates for OTT-1 and OTT-2 are similar in QAC-1, and the use of only one phase for the decision-maker's travel times (and one phase for salary) gives us the result in QAC-2, which records an insignificant loss of model fit.

Table 6 also compares the average probability for the chosen alternative under each of these models. The combined models do better than other models for choices where the decision-maker opts to change to the new location, with the combination of quantum rotations and complex phases evidently helping capture these choices. The coefficients associated with attributes (β) change substantially across the different models, with the ratios of parameters also changing. A decision-maker's own salary ranges from being approximately equivalently as important as their partner's salary, to more than double the importance in the combined model. The converse is true for travel times, with the combined model indicating a partner's travel time is of greater importance than that of the decision-maker. The opposite is true in the basic model.

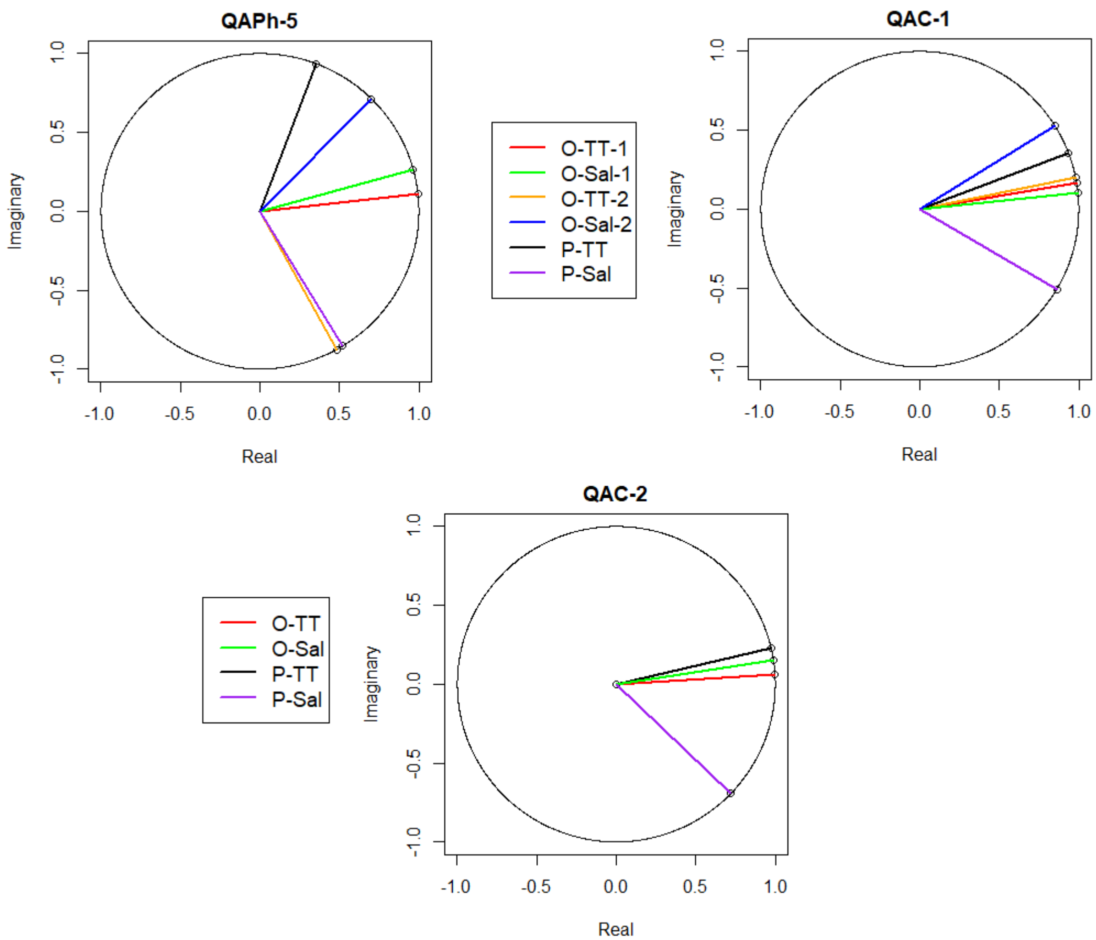

FIGURE 3 : Illustration of the estimated complex phase coefficients for the travel time (TT), salary increase (Sal) of the decision-maker (O) and partner (P) in the quantum models QAPh-5 (only complex phases), QAC-1 (rotations and 6 phases) and QAC-2 (rotations and 4 phases). Model QAC-2 keeps the same phases for the choice set CT1 (self) and CT2 (self and partner).

4. CONCLUSIONS

The growing interest in moral decision-making in choice modelling calls for the development of appropriate model specifications. The present paper has focussed on quantum probability and demonstrated that quantum rotations, as well as complex phases, accurately capture an implicit change in decision context when a more salient moral element enters the dimension of choice.

For our first choice paradigm with a 'Taboo trade-off', we find that models containing a perspective operation implemented by both rotations and complex phases do not significantly improve upon models with just one of these features. This however is not the case for the second choice paradigm with a moral component due to choices impacting the partner, in which our combined model with a perspective operation that has both features outperforms simpler specifications. Further work is thus required to establish the relative strengths and merits of these different features for the decision-maker's implicit perspective change, with it being possible that the first choice scenario is too simple (in having only two alternatives) to merit further model features. Importantly, quantum models have the potential to offer better performance than more conventional approaches in both datasets, as shown by our empirical results.

Overall, whilst the results for the quantum choice models in this paper are promising, it is not clear that they are distinctly *better* than those of Hancock et al. (2020). This implies that we cannot necessarily attribute the success of the quantum rotations and complex phases to the fact that there are moral components in the choices modelled. This is particularly clear from the result that our quantum model already has better model fit than the random regret model for the second dataset tested in this paper *before* the moral component was captured through the additional quantum model features. Further tests of quantum probability theory based models could shed light on whether they are models that are particularly suited to moral decision-making with salient attribute scenarios, or whether they are suitable for decision-making in general.

Further models should consider different sorts of moral choice data and scenarios. For example, quantum models may be well suited for modelling choices made in 'moral machine' choice tasks. The application of these models to choice scenarios where multiple individuals disagree, communicate and reassess on what is their 'most ethical' choice would be particularly interesting. On such socially sensitive matters moreover, results may also differ significantly for *revealed* preference datasets due to continuing concerns about external validity of *stated* moral choices (Bauman et al., 2014).

Closer to our present study, given that quantum models explicitly assume that a decision-maker is uncertain about their choice, an 'indifferent' (or equivalently a 'neither') alternative may be better modelled not as a separate alternative, but based on the superposition principle. The indifferent belief state would be expressed through a superposition of the belief states for each of the two choices. Such a superposition could contain a relative complex phase, which could depend on specific attributes of the alternatives. Such an approach would then resort to an interference effect between the belief states supporting the respective choices. The representation space would be smaller again - a 2-dimensional Hilbert space, and its choice probabilities renormalised to include the third - indifference - option. Other extensions to explore scenarios with an indifference option could make the indifference-component of the belief state explicitly dependent on the attributes of the choices (as a contrast to Eq. 15). Such a function could, for example, express an additional effect at play in the balance of the two other choices; the indifference between two very favourable alternatives may be less prominent than the indifference between two pale alternatives - due to a lack of interest. A further theoretical development of the quantum model could incorporate the impact of previous choices through a carry-over parameter that modifies the amplitude of the present quantum perspective operation. Finally, another next step could be to test the decision-maker for explicit ethical answerability of the choice alternatives first. For example, in the taboo trade-off paradigm, a decision-maker's explicit change of perspective could then be further analysed from the tensorial belief state (Status Quo, New Policy) $\otimes$ (Ethical, Not-Ethical), and compared to a test where only an *implicit* change of perspective is assumed.

Whilst we include a discussion on various parameter ratios, one clear weakness of the new quantum choice models developed here is that, by not being grounded in microeconomic theory, they cannot be used to compute context independent welfare measures. This is a common limitation of all models that include departures from a random utility framework. With our previous work (Hancock et al., 2020) demonstrating that quantum choice models can produce forecasts and elasticities, further research is needed to establish how the outputs can be used in an appraisal context.

Overall, however, our results indicate that choice models with a quantum probability framework have vast potential, both within moral choice scenarios and more generally.

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