Friezes satisfying higher SL$_k$-determinants

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With an appendix by Michael Cuntz and Pierre-Guy Plamondon

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In this article, we construct SL_k-friezes using Plücker coordinates, making use of the cluster structure on the homogeneous coordinate ring of the Grassmannian of k-spaces in n-space via the Plücker embedding. When this cluster algebra is of finite type, the SL_k-friezes are in bijection with the so-called mesh friezes of the corresponding Grassmannian cluster category. These are collections of positive integers on the AR-quiver of the category with relations inherited from the mesh relations on the category. In these finite type cases, many of the SL_k-friezes arise from specializing a cluster to 1. These are called unitary. We use Iyama–Yoshino reduction to analyze the nonunitary friezes. With this, we provide an explanation for all known friezes of this kind. An appendix by Cuntz and Plamondon proves that there are 868 friezes of type E_6.

1. Introduction

In this paper, we establish an explicit connection between SL_k-friezes and Grassmannian cluster categories and algebras.

Integral SL_k-friezes are certain arrays of integers consisting of finitely many rows of infinite length, see Example 2.4. Moreover, entries in an SL_k-frieze satisfy the so-called diamond rule, where for every $k \times k$-diamond formed by the neighboring entries, the determinant of the corresponding matrix equals 1.

For example, when $k = 2$ each diamond $b \; {}^{a}_{\;\;d}\; c$ satisfies the relation $\left| \begin{smallmatrix} b & a \\ d & c \end{smallmatrix} \right| = 1$. Moreover, we consider tame SL_k-frieze, that is friezes where the determinant of every $(k+1) \times (k+1)$-diamond is 0. Such friezes have horizontal period n, where n is determined by k and the number of rows.

SL_k-friezes were introduced in the seventies by Coxeter [1971] in the case $k = 2$, while higher SL_k-frieze patterns first appeared in work of Cordes and Roselle [1972]. Conway and Coxeter [1973a; 1973b] further studied SL_2-friezes, showing that there exists a bijection between SL_2-friezes and triangulations of polygons. Interest in friezes and their various generalizations renewed after the introduction of cluster algebras in 2001, as cluster algebras coming from the Grassmannian of 2 planes in an n-dimensional space are also in bijection with triangulations of polygons. Moreover, it was shown later that SL_2-friezes can be obtained by specializing all cluster variables in a given cluster to one [Caldero and Chapoton 2006].

In this way, SL_2-friezes are well-understood and they are closely related to the combinatorics of cluster algebras. On the other hand, the classification of integral SL_k-friezes remains elusive. Our paper makes a step in the direction of a complete classification. We show that in the finite type cases, all integral SL_k-friezes can be obtained from the combinatorics of the cluster algebras on coordinate rings of Grassmannians, and Grassmannian cluster categories. We make use of the combinatorial tools that we call *Plücker friezes*, which arise from the cluster algebras, and *mesh friezes*, which arise from Grassmannian cluster categories.

Plücker friezes play a crucial role in our construction of SL_k-friezes. Their entries are given by a constellation of Plücker coordinates in the homogeneous coordinate ring $\mathcal{A}(k, n)$ of the Grassmannian $\mathrm{Gr}(k, n)$ via the Plücker embedding. We show that the (specialized) Plücker frieze of type (k, n) deserves its name: it is indeed an SL_k-frieze — all its $k \times k$-diamonds have determinant 1 (see Theorem 3.1). In particular, each map from $\mathcal{A}(k, n)$ to $\mathbb{Z}$, given by a specialization of a cluster to one, yields a tame integral SL_k-frieze (see Corollary 3.12). We call friezes obtained in this way *unitary*. This generalizes the results about friezes from the case $k = 2$, which have been well-known and studied for some time: Indeed, all SL_2-friezes arise from the (specialized) Plücker frieze of type $(2, n)$ in this way. However, for $k > 2$ not all friezes are unitary.

Next, we use categorification of cluster algebras to obtain all SL_k-friezes whenever $\mathcal{A}(k, n)$ is of finite type. A mesh frieze is an integral frieze on the Auslander–Reiten quiver of a Grassmannian cluster category such that entries coming from a mesh satisfy a certain frieze-like relation. In the case $k = 2$, a mesh frieze of the cluster category exactly agrees with the SL_2-frieze. Thus, in light of the connection that we found, it makes sense to consider relationship between SL_k-friezes and mesh friezes of Grassmannian cluster categories for $k > 2$. In finite type we show that there exists a bijection between the two. In particular, this implies that in finite type every SL_k-frieze is obtained by specializing the collection of cluster variables in a given cluster to some set of positive integers.

This also allows us to use tools from representation theory, such as Iyama–Yoshino reduction, to study SL_k-friezes. It enables us to pass from a mesh frieze of a given "rank" (that is, a mesh frieze for a cluster category whose rank is given by the number of indecomposables in a cluster tilting object), to one of lower rank. Making use of known restrictions for smaller rank cases (in particular, of type A cases) we

can then draw conclusions about the nature of our original mesh frieze. The natural bijection, in finite type cases, between mesh friezes and SL_k-friezes means that Iyama–Yoshino reduction helps us better our understanding of integral SL_k-friezes. In particular, we obtain new results on the number of such friezes and their possible entries.

An appendix by Cuntz and Plamondon determines the number of friezes in type $(3, 7)$ ([Appendix B](#)).

2. Background and notation

2A. SL_k-friezes.

Definition 2.1. Let R be an integral domain.

An SL_k-*frieze (over R)* is a grid in the plane, consisting of a finite number of rows:

$$
\begin{array}{ccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdots \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & * & & * & & * & & m_{0,w-1} & & m_{1,w} & & m_{2,w+1} & & * & \\
 & & & & & & & \cdot^{\cdot^{\cdot}} & & & & & & & \\
\cdots & & \cdots & & m_{01} & & m_{12} & & m_{23} & & m_{34} & & \cdots & \\
 & & \cdots & & m_{00} & & m_{11} & & m_{22} & & m_{33} & & \cdots & & \cdots \\
 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
 & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & & \\
\cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

The frieze is formed by $k-1$ rows of zeroes followed by a row of 1s from top and from bottom respectively and by $w \geq 1$ rows of elements $m_{ij} \in R$ in between, such that every $k \times k$-diamond of entries of the frieze has determinant 1, i.e., whenever we consider a matrix having k successive entries of a row of an SL_k-frieze on its diagonals and all other entries above and below the diagonal accordingly, the determinant of this matrix is 1. The integer w is called the *width* of the frieze.

Definition 2.2. Let F be an SL_k-frieze over R.

(1) F is *tame* if all $(k+1) \times (k+1)$-diamonds in F have determinant 0. In general, we define an $s \times s$-diamond of the frieze, where $1 \leq s \leq k+1$, as the matrix having s successive entries of the frieze on its diagonals and all other entries above and below the diagonal accordingly.

(2) If all nontrivial entries m_{ij} with $0 \leq i, j \leq w$ of F are positive integers, we call F an *integral* SL_k-frieze.

Remark 2.3. One can prove that every tame SL_k-frieze of width w over a field K has horizontal period $n = w + k + 1$, which was first proven for certain SL_k-friezes in [Cordes and Roselle 1972] and in general in [Morier-Genoud et al. 2014, Corollary 7.1.1].

SL_2-friezes were first studied by Coxeter and Conway and Coxeter in the early 1970s, see Example 2.4. Higher SL_k-frieze patterns made their first appearance 1972 in work of Cordes and Roselle [1972] (with extra conditions on the minors of the first $k \times k$-diamonds) and seem only to have re-emerged with the introduction of cluster algebras: as 2-friezes [Propp 2020], SL_k-tilings [Bergeron and Reutenauer 2010] and more systematically [Morier-Genoud et al. 2012; 2014; Morier-Genoud 2012; Cuntz 2017]. See also [Morier-Genoud 2015] for a survey on different types of frieze patterns.

Example 2.4. An SL_2-frieze is simply called a *frieze pattern*. Frieze patterns have first been studied by Coxeter [1971] and by Conway and Coxeter [1973a; 1973b]. They showed that integral frieze patterns of width w are in bijection with triangulations of $w+3$-gons. Their horizontal period is $w + 3$ or a divisor of $w + 3$. The following example:
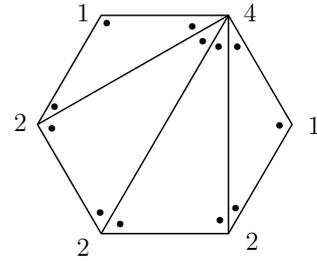
arises from a fan triangulation of a hexagon: The first nontrivial row of the frieze pattern (in bold face) is given by the number of triangles of the triangulation meeting at the vertices of the hexagon (counting these while going around the hexagon).

2B. *Plücker relations.* Throughout, we fix $k, n \in \mathbb{Z}_{>0}$ with $2 \leq k \leq \frac{n}{2}$ and consider the Grassmannian $Gr(k, n)$ as a projective variety via the Plücker embedding, with homogeneous coordinate ring

$$\mathcal{A}(k, n) = \mathbb{C}[Gr(k, n)].$$

This coordinate ring can be equipped with the structure of a cluster algebra as we briefly recall here. For details, we refer to [Scott 2006].

We write $[1, n] = \{1, 2, \ldots, n\}$ for the closed interval of integers between 1 and n. A k-subset of $[1, n]$ is a k-element subset of $[1, n]$. Every k-subset $I = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$ of $[1, n]$ gives rise to a *Plücker coordinate* $p_{i_1, i_2, \ldots, i_k}$. We extend this definition to allow different ordering on the indices and repetition of indices: for an arbitrary k-tuple $I = (i_1, \ldots, i_k)$ we set $p_{i_1, \ldots, i_k} = 0$ if there exists $\ell \neq m$ such that $i_\ell = i_m$ and $p_{i_1, \ldots, i_k} = \text{sgn}(\pi) p_{j_1, \ldots, j_k}$ if $(i_1, \ldots, i_k) = (j_1, \ldots, j_k)$, with $j_1 < j_2 < \cdots < j_k$ and π is the permutation sending j_m to i_m for $1 \leq m \leq k$. Sometimes, by abuse of notation, we will also call any tuple $(i_1, \ldots, i_k)$ a k-subset. Scott [2006] proved that $\mathcal{A}(k, n)$ is a cluster algebra, where all the Plücker

coordinates are cluster variables. For each (k, n), there exists a cluster consisting of Plücker coordinates and the exchange relations arise from the Plücker relations (see below).

Note that throughout the article we will consider sets I up to reduction modulo n, where we choose the representatives $1, \ldots, n$ for the equivalence classes modulo n. If I consists of k consecutive elements, up to reducing modulo n, we call I a *consecutive k-subset* and p_I a *consecutive Plücker coordinate*. These are frozen cluster variables or coefficients, and we will later set all of them equal to 1. Of importance for our purposes are the Plücker coordinates which arise from k-subsets of the form $I = I_0 \cup \{m\}$, where I_0 is a consecutive $(k-1)$-subset of $[1, n]$ and m is any entry of $[1, n] \setminus I_0$. We call such k-subsets *almost consecutive*. Note that each consecutive k-subset is almost consecutive. In particular the frozen cluster variables are consecutive and thus almost consecutive.

We follow the exposition of [Marsh 2013, Section 9.2] for the presentation of the Plücker relations. In the cluster algebra $\mathcal{A}(k, n)$, the relations

$$\sum_{\ell=0}^{k} (-1)^l p_{i_1, \ldots, i_{k-1}, j_\ell} \, p_{j_0, \ldots, \widehat{j_\ell}, \ldots, j_k} = 0 \tag{1}$$

hold for arbitrary $1 \le i_1 < i_2 < \cdots < i_{k-1} \le n$ and $1 \le j_0 < j_1 < \cdots < j_k \le n$, where $\widehat{}$ signifies omission. These relations are called the *Plücker relations*. For $k = 2$, the nontrivial Plücker relations are of the form

$$p_{a,c} \, p_{b,d} = p_{a,b} \, p_{c,d} + p_{a,d} \, p_{b,c} \tag{2}$$

for arbitrary $1 \le a < b < c < d \le n$. They are often called the *three-term Plücker relations*.

Obviously, the order of the elements in a tuple plays a significant role. We will often need to use ordered tuples, or partially ordered tuples. For this, we introduce the following notation. For a tuple $J = \{i_1, \ldots, i_\ell\}$ of $[1, n]$ and $\{a_1, \ldots, a_\ell\} = \{i_1, \ldots, i_\ell\}$ with $1 \le a_1 \le \cdots \le a_\ell \le n$ we set

$$o(J) = o(i_1, \ldots, i_\ell) = (a_1, \ldots, a_\ell).$$

With this notation, if $0 \le s, \ell \le k$, we will write

$$p_{b_1, \ldots, b_s, o(i_1, \ldots, i_\ell), c_1, \ldots, c_{k-\ell+a}} \quad \text{for} \quad p_{b_1, \ldots, b_s, a_1, \ldots, a_\ell, c_1, \ldots, c_{k-\ell+a}}.$$

For arbitrary $I = \{i_1, \ldots, i_{k-1}\}$ and $J = \{j_0, \ldots, j_k\}$, the Plücker relations (1) then take the form

$$\sum_{l=0}^{k} (-1)^\ell p_{o(I) j_\ell} \cdot p_{o(J \setminus j_\ell)} = 0.$$

2C. *Plücker friezes, and special Plücker friezes.* To the homogeneous coordinate ring $\mathcal{A}(k, n)$ we now associate a certain frieze pattern which we will use later to obtain SL_k-friezes. It uses the almost consecutive Plücker coordinates. For brevity, it is convenient to write $[r]^\ell$ for the set $\{r, r+1, \ldots, r+\ell-1\}$. For example, if $k = 4$ and $n = 9$, $p_{[1]^3, 6}$ is the short notation for p_{1236} and $p_{o([8]^3, 4)}$ is short for p_{1489}.

Definition 2.5. The *Plücker frieze* of type (k, n) is a $\mathbb{Z} \times \{1, 2, \ldots, n + k - 1\}$-grid with entries given by the map

$$\varphi_{(k,n)} \colon \mathbb{Z} \times \{1, 2, \ldots, n + k - 1\} \to \mathcal{A}(k, n), \quad (r, m) \mapsto p_{o([r']^{k-1}, m')},$$

where $r' \in [1, n]$ is the reduction of r modulo n and $m' \in [1, n]$ is the reduction of $m + r' - 1$ modulo n. We denote the Plücker frieze by $\mathcal{P}_{(k,n)}$.

Example 2.6 and Figure 1 illustrate the map $\varphi_{(k,n)}$.

Example 2.6. We draw $\mathcal{P}_{(2,5)}$ as grid of Plücker coordinates with the positions in $\mathbb{Z} \times \{1, 2, \ldots, 6\}$ written in gray above them:

Remark 2.7. Notice that if $m' \notin \{1, 2, \ldots, k - 1\}$ then the image $\varphi_{(k,n)}(r, m)$ corresponds to a Plücker coordinate $p_{o([r']^{k-1}, m')} \neq 0$ and for $m' \in \{1, 2, \ldots, k - 1\}$, the image is 0. In particular, the top and bottom $k - 1$ rows of $\mathcal{P}_{(k,n)}$ consist of solely 0 entries.

Remark 2.8 (specializing coefficients). In what follows, we will replace all the frozen variables by 1. For that, let J be the ideal of $\mathcal{A}(k, n)$ generated by

$$\{x - 1 \mid x \text{ is a consecutive Plücker coordinate}\}.$$

The consecutive Plücker coordinates are precisely the frozen variables in the cluster algebra $\mathcal{A}(k, n)$. The quotient $\mathcal{A}(k, n)/J$ is a coefficient-free cluster algebra, and therefore — as a subring of a field of rational functions over $\mathbb{C}$ — an integral domain. Let $s : \mathcal{A}(k, n) \to \mathcal{A}(k, n)/J$ be the map induced from the identity on nonconsecutive Plücker coordinates and from replacing the consecutive Plücker coordinates by 1. This map is an algebra homomorphism $\mathcal{A}(k, n) \to \mathcal{A}(k, n)/J$. We write $s\mathcal{A}(k, n)$ for $\mathcal{A}(k, n)/J$.

Definition 2.9. The *special Plücker frieze of type* (k, n), denoted by $s\mathcal{P}_{(k,n)}$, is the grid we get from $\mathcal{P}_{(k,n)}$ by substituting the consecutive Plücker coordinates with 1s. In other words, $s\mathcal{P}_{(k,n)}$ is the grid with entries given by the map $s \circ \varphi_{(k,n)}$ with entries in $s\mathcal{A}(k, n)$.
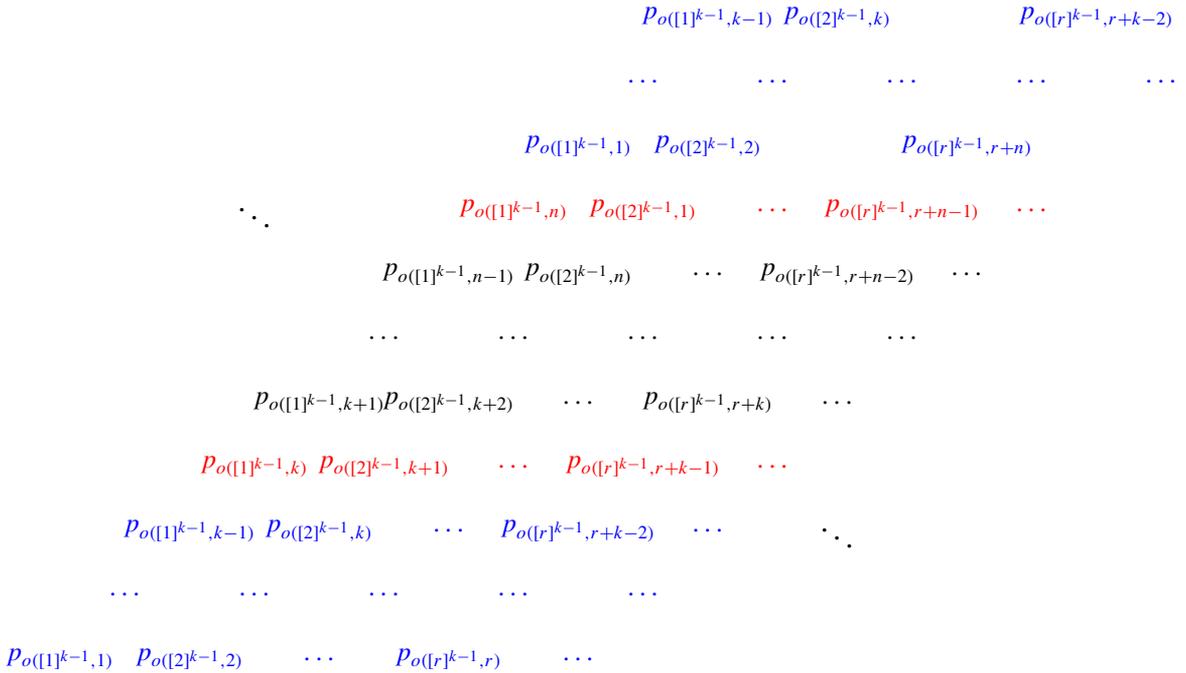
$$\begin{array}{ccccccccc}
& & & & p_{o([1]^{k-1},k-1)} & p_{o([2]^{k-1},k)} & & p_{o([r]^{k-1},r+k-2)} \\
& & & \cdots & \cdots & \cdots & \cdots & & \cdots \\
& & & p_{o([1]^{k-1},1)} & p_{o([2]^{k-1},2)} & & p_{o([r]^{k-1},r+n)} \\
& \ddots & & {\color{red}p_{o([1]^{k-1},n)}} & {\color{red}p_{o([2]^{k-1},1)}} & \cdots & {\color{red}p_{o([r]^{k-1},r+n-1)}} & \cdots \\
& & p_{o([1]^{k-1},n-1)} & p_{o([2]^{k-1},n)} & \cdots & p_{o([r]^{k-1},r+n-2)} & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
& p_{o([1]^{k-1},k+1)} & p_{o([2]^{k-1},k+2)} & \cdots & p_{o([r]^{k-1},r+k)} & \cdots \\
{\color{red}p_{o([1]^{k-1},k)}} & {\color{red}p_{o([2]^{k-1},k+1)}} & \cdots & {\color{red}p_{o([r]^{k-1},r+k-1)}} & \cdots \\
p_{o([1]^{k-1},k-1)} & p_{o([2]^{k-1},k)} & \cdots & p_{o([r]^{k-1},r+k-2)} & \cdots & \ddots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
p_{o([1]^{k-1},1)} & p_{o([2]^{k-1},2)} & \cdots & p_{o([r]^{k-1},r)} & \cdots
\end{array}$$

Figure 1. The grid defined by $\varphi_{(k,n)}$, reducing indices modulo n. The top and bottom $k-1$ rows (blue) are 0, the k-th row from top/bottom (red) consist of consecutive k-tuples, i.e., frozen Plücker coordinates.

Example 2.10. The special Plücker frieze of type $(2, 5)$, $s\mathcal{P}_{(2,5)}$, (see Example 2.6) is as follows (writing 0 for the entries p_{ii}):

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
p_{13} & p_{24} & p_{35} & p_{14} & p_{25} & p_{13} & p_{24} & \cdots \\
\cdots & p_{14} & p_{25} & p_{13} & p_{24} & p_{35} & p_{14} & p_{25} \\
& 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

And $s\mathcal{P}_{(3,6)}$:

$$\begin{array}{ccccccccc}
& 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
p_{146} & p_{125} & p_{236} & p_{134} & p_{245} & p_{356} & p_{146} & \cdots \\
\cdots & p_{124} & p_{235} & p_{346} & p_{145} & p_{256} & p_{136} & p_{124} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Remark 2.11. For $k = 3$, [Morier-Genoud et al. 2012, Proposition 3.3] describes an explicit correspondence between SL_3-friezes and 2-friezes (for a definition of 2-friezes see Appendix B). It is stated

in [Morier-Genoud et al. 2012, Remark 5.8] that it can be checked that all the $n(n-4)$ entries in a 2-frieze are Plücker coordinates for the Grassmannian Gr(3, n). Performing this check and combining the two observations yields an alternative way to obtain the Plücker frieze of type (k, n) for the special case $k = 3$. More generally, in [Morier-Genoud 2015, Remark 3.19] it is observed that, for $k \geq 3$, cluster variables for $\mathcal{A}(k, n)$ appear as entries in the derived arrays of an SL_k-frieze.

3. Integral tame SL_k-friezes from Plücker friezes

3A. *The special Plücker frieze $s\mathcal{P}_{(k,n)}$ is a tame SL_k-frieze.* In this section, we prove the following result.

Theorem 3.1. *The frieze $s\mathcal{P}_{(k,n)}$ is a tame SL_k-frieze over $s\mathcal{A}(k, n)$.*

Before we provide a proof for Theorem 3.1, we make some observations on notation and set-up.

Notation 3.2. Throughout we calculate modulo n. More precisely, we will always reduce integers modulo n, and identify integers with their representatives in $[1, n]$. In addition, the following notation will be useful: For any $a, b \in [1, n]$ we denote by $[a, b]$ the closed interval between a and b modulo n, defined as

$$[a, b] = \begin{cases} \{p \in [1, n] \mid a \leq p \leq b\} & \text{if } a \leq b, \\ \{p \in [1, n] \mid a \leq p \leq n \text{ or } 1 \leq p \leq b\} & \text{if } b < a, \end{cases}$$

and by (a, b) the open interval between a and b modulo n, defined as

$$(a, b) = \begin{cases} \{p \in [1, n] \mid a < p < b\} & \text{if } a \leq b, \\ \{p \in [1, n] \mid a < p \leq n \text{ or } 1 \leq p < b\} & \text{if } b < a. \end{cases}$$

Analogously, we define the half-open intervals $[a, b)$ and $(a, b]$.

Recall that we obtain the frieze $s\mathcal{P}_{(k,n)}$ from the frieze $\mathcal{P}_{(k,n)}$ by specializing the consecutive Plücker coordinates to 1. We will compute the determinants of a diamond in the frieze $s\mathcal{P}_{(k,n)}$ via the corresponding diamond in $\mathcal{P}_{(k,n)}$. The $k \times k$-diamonds in the Plücker frieze $\mathcal{P}_{(k,n)}$ are matrices of the form

$$A_{[m]^k;r} := (p_{o([r+i-1]^{k-1}, m+j-1)})_{1 \leq i, j \leq k},$$

with

$$r \in [1, n] \quad \text{and} \quad m \in [r+k-1, r+n-1] = [r+k-1, r-1],$$

where as before we calculate modulo n, with representatives $1, \ldots, n$.

To inductively compute the determinants of the matrices $A_{[m]^k;r}$, we need to consider other matrices of the following form: For $r \in [1, n]$, $1 \leq s \leq n$ and $\underline{m} = (m_1, \ldots, m_s)$ with $m_i \in [1, n]$ we define

$$A_{\underline{m};r} = (a_{ij})_{1 \leq i, j \leq s}$$

to be the $(s \times s)$-matrix with entries $a_{ij} = p_{o([r+i-1]^{k-1}, m_j)}$ for $1 \leq i, j \leq s$:

$$
A_{\underline{m};r} := \begin{bmatrix}
p_{o([r]^{k-1}, m_1)} & p_{o([r]^{k-1}, m_2)} & \cdots & p_{o([r]^{k-1}, m_s)} \\
p_{o([r+1]^{k-1}, m_1)} & p_{o([r+1]^{k-1}, m_2)} & \cdots & p_{o([r+1]^{k-1}, m_s)} \\
\vdots & \vdots & \vdots & \vdots \\
p_{o([r+s-1]^{k-1}, m_1)} & p_{o([r+s-1]^{k-1}, m_2)} & \cdots & p_{o([r+s-1]^{k-1}, m_s)}
\end{bmatrix}. \tag{3}
$$

Example 3.3. As an example, let $k = 3$ and $n = 6$ and consider the matrix $A_{\underline{m};r}$ in $\mathcal{P}_{(3,6)}$ for $r = 1$, $s = 3$ and $\underline{m} = (3, 4, 5)$. It is given by

$$
A_{(3,4,5);1} := \begin{bmatrix}
p_{123} & p_{124} & p_{125} \\
p_{233} & p_{234} & p_{235} \\
p_{334} & p_{344} & p_{345}
\end{bmatrix} = \begin{bmatrix}
p_{123} & p_{124} & p_{125} \\
0 & p_{234} & p_{235} \\
0 & 0 & p_{345}
\end{bmatrix}
$$

We set

$$
b_{\underline{m};r} := \det A_{\underline{m};r}.
$$

In particular, for $\underline{m} = [m]^k = (m, m+1, \ldots, m+k-1)$, the matrix $A_{\underline{m};r}$ is a $k \times k$-diamond in the frieze $\mathcal{P}_{(k,n)}$ whose determinant it is our first goal to compute.

We can compute the determinants $b_{\underline{m};r}$ provided the two conditions (c1) and (c2) are satisfied for the tuple $\underline{m}$:

Notation 3.4. Fix $r \in [1, n]$ and let $1 \leq s \leq k$. Choose elements $m_j \in [1, n]$ for each $1 \leq j \leq s$. We can impose the following conditions on the ordered tuple $(m_1, \ldots, m_s)$:

(c1) It is ordered cyclically modulo n, that is there exists a number $b \in \{1, \ldots, s\}$ such that

$$
m_b < m_{b+1} < \cdots < m_s < m_1 < m_2 < \cdots < m_{b-1},
$$

if $b \in \{2, \ldots, s\}$ or

$$
m_1 < m_2 < \cdots < m_s
$$

if $b = 1$.

(c2) We have $r + k - 2 \notin [m_1, m_s)$.

Conditions (c1) and (c2) are technical conditions needed to ensure that we get the correct signs in our computations (see the proof of Proposition 3.5).

We now provide a formula for computing the determinants of the matrices $A_{\underline{m};r}$ with entries in $\mathcal{A}(k, n)$.

Proposition 3.5. *Let $r \in [1, n]$ and let $1 \leq s \leq k$. Let $\underline{m} = (m_1, \ldots, m_s)$ with $m_i \in [1, n]$ for all i, and assume that $\underline{m}$ satisfies conditions (c1) and (c2). Let $b_{\underline{m};r}$ be the determinant of the matrix $A_{\underline{m};r}$ from (3). Then we have*

$$
b_{\underline{m};r} = \left[\prod_{l=0}^{s-2} p_{o([r+l]^k)} \right] \cdot p_{o([r+s-1]^{k-s}, m_1, \ldots, m_s)}. \tag{4}
$$

The proof of this proposition can be found in Appendix A.

Remark 3.6. In particular, if $k = s$ and $\underline{m} = [m]^k$ in the statement of Proposition 3.5, then we have

$$b_{\underline{m};r} = b_{m_1, m_2, \ldots, m_k; r} = \left[\prod_{l=0}^{k-2} p_{o([r+l]^k)} \right] \cdot p_{o([m]^k)};$$

a product of consecutive Plücker coordinates.

Proof of Theorem 3.1. All entries in the bottom $k - 1$ rows and in the top $k - 1$ rows of $s\mathcal{P}_{(k,n)}$ are zero by definition and the entries in the k-th row from top and from bottom are 1 since the consecutive Plücker coordinates are set to be 1, see Definition 2.9.

It remains to show that the $k \times k$-determinants in $s\mathcal{P}_{(k,n)}$ are all 1 and that the $(k+1) \times (k+1)$-determinants all vanish. We will use Proposition 3.5 for both claims.

Observe that each $k \times k$-diamond in the special Plücker frieze $s\mathcal{P}_{(k,n)}$ is of the form $s(A_{\underline{m};r}) = (s(a_{ij}))_{1 \le i,j \le k}$, where $A_{\underline{m};r} = (a_{ij})_{1 \le i,j \le k}$ is a $k \times k$-diamond in $\mathcal{P}_{(k,n)}$ (the integer s from (3) is equal to k here) and the map s is the specialization of consecutive Plücker coordinates to 1 (see Definition 2.9). Recall that $A_{\underline{m};r}$ is of the form as in (3), where

$$r \in [1, n] \quad \text{and} \quad m \in [r+k-1, r-1]$$

and $\underline{m} = (m, m+1, \ldots, m+k-1)$. Then the k-tuple $\underline{m}$ clearly satisfies condition (c1). We are going to show that it also satisfies condition (c2). Indeed, if we had $r + k - 2 \in [m, m+k-1)$, then $r + k - 2 = m + j - 1$ for some $1 \le j < k$. However, since $m \in [r+k-1, r-1]$, this would imply

$$r + k - 2 = m + j - 1 \in [r+k+j-2, r+j-2].$$

But for $1 \le j < k$ we have

$$r + k - 2 \in (r+j-2, r+k+j-2), \quad \text{and} \quad [r+k+j-2, r+j-2] \cap (r+j-2, r+k+j-2) = \varnothing;$$

a contradiction. So we must have $r + k - 2 \notin [m, m+k-1)$.

Therefore, both (c1) and (c2) are satisfied by $\underline{m}$ and we can apply Proposition 3.5 to obtain

$$\det(A_{\underline{m};r}) = b_{\underline{m};r} = \prod_{l=0}^{k-2} p_{o([r+l]^k)} \cdot p_{o([m]^k)}.$$

This yields

$$\det(s(A_{\underline{m};r}) = s(b_{\underline{m};r}) = \prod_{l=0}^{k-2} s(p_{o([r+l]^k)}) \cdot s(p_{o([m]^k)}) = 1,$$

for the $k \times k$-diamond $s(A_{\underline{m};r})$ of the special Plücker frieze $s\mathcal{P}_{(k,n)}$. Since this holds for every $k \times k$-diamond, $s\mathcal{P}_{(k,n)}$ is indeed a SL$_k$-frieze.

To show that it is tame, consider an arbitrary $(k+1) \times (k+1)$-diamond of $s\mathcal{P}_{(k,n)}$. It must be of the form $s(A_{[m]^{k+1};r}) = (s(a_{ij}))_{1 \le i,j \le k+1}$, where $A_{[m]^{k+1};r}$ is the corresponding diamond in the Plücker frieze $\mathcal{P}_{(k,n)}$ given by (3) for $[m]^{k+1} = (m, m+1, \ldots, m+k)$ and $m \in [r+k, r-2]$. Similarly to the first part of the proof (for $k \times k$-diamonds), one can show that conditions (c1) and (c2) are satisfied for $[m]^{k+1}$ and

for any order-inheriting subtuple thereof. So we can apply Proposition 3.5 to compute the determinants $b_{\widetilde{m}_l;r}$ for $\widetilde{m}_l = (m, \ldots, \widehat{m+l}, \ldots, m+k)$, for $0 \leq l \leq k$, and Laplace expansion for the determinant of $A_{[m]^{k+1};r}$ by the last row yields

$$
\begin{aligned}
\det(A_{[m]^{k+1};r}) &= \sum_{l=1}^{k+1} (-1)^{k+1+l} p_{o([r+k]^{k-1}, m+l-1)} \cdot b_{\widetilde{m}_l;r} \\
&= \sum_{l=1}^{k+1} (-1)^{k+1+l} p_{o([r+k]^{k-1}, m_l)} \cdot \prod_{j=0}^{k-2} p_{o([r+j]^k)} p_{o(m_1, \ldots, \widehat{m_l}, \ldots, m_{k+1})} \\
&= \prod_{j=0}^{k-2} p_{o([r+j]^k)} \cdot \underbrace{\sum_{l=1}^{k+1} (-1)^{k+1+l} p_{o([r+k]^{k-1}, m_l)} p_{o(m_1, \ldots, \widehat{m_l}, \ldots, m_{k+1})}}_{(*)} \cdot
\end{aligned}
$$

Now since $(*)$ is precisely the Plücker relation on $I = [r+k]^{k-1}$ and $J = [m_1, m_{k+1}]$, this term equals zero and so $\det(A_{[m]^{k+1};r}) = 0$.

It follows that the determinant of the $(k+1) \times (k+1)$-diamond $s(A_{[m]^{k+1};r})$ vanishes,

$$
\det(s(A_{[m]^{k+1};r})) = s(\det(A_{[m]^{k+1};r})) = 0,
$$

and thus $s\mathcal{P}_{(k,n)}$ is tame. $\qquad\square$

Remark 3.7. In [Morier-Genoud et al. 2014, Proposition 3.2.1], it is shown that the variety of SL_k-friezes of width $w = n - k - 1$, say over $\mathbb{C}$, can be embedded into $\mathrm{Gr}(k, n)$ as the subvariety of points which can be represented by matrices whose consecutive $k \times k$-minors all coincide (and are nonvanishing). In the strategy from [Morier-Genoud et al. 2014, Proposition 3.2.1], we could choose the $(k \times n)$-matrix P with entries in $s\mathcal{A}(k, n)$ given by

$$
\begin{bmatrix}
1 & p_{o([n-k+2]^{k-1}2)} & \cdots & & p_{o([n-k+2]^{k-1}n-k)} & 1 & 0 & \cdots & 0 \\
0 & 1 & p_{o([n-k+3]^{k-1}3)} & \cdots & p_{o([n-k+3]^{k-1}n-k)} & p_{o([k-1]^{k-1}n-k+1)} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & & & & & \ddots & \vdots \\
0 & 0 & & \cdots & 1 & p_{o([1]^{k-1}k+1)} & p_{o([1]^{k-1}k+2)} & \cdots & p_{o([1]n-1)} \quad 1
\end{bmatrix}.
$$

Each consecutive $(k \times k)$-minor of the above matrix is 1 (as follows, e.g., from Proposition 3.5). We can complete this to a unique tame SL_k-frieze over $s\mathcal{A}(k, n)$. By Theorem 3.1 this must be precisely the specialized Plücker frieze of type (k, n).

Let now $\mathcal{F}$ be any tame SL_k-frieze over $\mathbb{C}$. As explained in [Morier-Genoud et al. 2014], it is uniquely determined by a $k \times n$ slice of the form

$$
M = \begin{bmatrix}
1 & m_{12} & m_{13} & \cdots & m_{1,n-k} & 1 & 0 & \cdots & 0 \\
0 & 1 & m_{23} & \cdots & m_{2,n-k} & m_{2,n-k+1} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & & & & & \ddots & \vdots \\
0 & 0 & \cdots & 1 & m_{k,k+1} & & \cdots & & m_{k,n-1} \quad 1
\end{bmatrix},
$$

representing a point in the cone over $\mathrm{Gr}(k, n)$ with respect to the Plücker embedding. The minors $M_{[r]^k l}$ of M, for $1 \leq r \leq k$ and $l \in [r + k + 1, r - 2]$ (with the interval taken cyclically modulo n), are precisely the entries of the matrix M, and we can view the matrix M as a pointwise evaluation of the matrix P at a point in the cone over the Grassmannian, and consequently, the frieze $\mathcal{F}$ as a pointwise evaluation of the specialized Plücker frieze $s\mathcal{P}_{(k,n)}$ at a point in the cone over the Grassmannian $\mathrm{Gr}(k, n)$. This might justify considering the specialized Plücker frieze $s\mathcal{A}(k, n)$ the "universal" SL_k-frieze of width $w = n - k - 1$.

3B. Integral tame SL_k-friezes from Plücker friezes. As an application of Theorem 3.1 we obtain that specializing a cluster to 1 yields an integral tame SL_k-frieze as we will show now.

Definition 3.8. Let $\mathcal{A}$ be a cluster algebra of rank m, i.e., its clusters have cardinality $m \in \mathbb{Z}_{>0}$. The specialization of a cluster $\underline{x} = (x_1, \ldots, x_m)$ in $\mathcal{A}$ to a tuple $(a_1, \ldots, a_m) \in \mathbb{C}^m$ is the algebra homomorphism $\mathcal{A} \to \mathbb{C}$ determined by sending x_i to a_i for all $1 \leq i \leq m$. If $(a_1, \ldots, a_m) = (1, \ldots, 1)$, we call this the specialization of $\underline{x}$ to 1.

Remark 3.9. Note that we consider clusters in a cluster algebra to be ordered tuples, rather than sets.

We observe here that specializing a cluster to a tuple $(a_1, \ldots, a_m)$ determines values for all cluster variables, since the cluster algebra is generated by the cluster variables, each of which can be expressed as a Laurent polynomial in any given cluster. Since we are interested in integral SL_k-friezes we consider specializations with respect to tuples in $\mathbb{Z}_{>0}^m$.

Remark 3.10. The image of the cluster variables in $\mathcal{A}$ under a specialization of a cluster to a tuple in $(\mathbb{Z}_{>0})^m$ (or in $(\mathbb{Q}_{>0})^m$) lies in $\mathbb{Q}_{>0}$. This is due to the Laurent phenomenon [Fomin and Zelevinsky 2002b] and positivity (see [Gross et al. 2018; Lee and Schiffler 2015; Musiker et al. 2011]): Every noninitial cluster variable is a Laurent polynomial whose denominator is a monomial in the x_i and whose numerator is a polynomial in the x_i with positive coefficients. In particular, specializing a cluster to 1 sends every cluster variable in $\mathcal{A}$ to a positive integer.

Remark 3.11 (tameness). Let F be a tame SL_k-frieze over an integral domain R, and let $\varphi : R \to S$ be a unitary ring homomorphism from R to an integral domain S. Assume that the images of the entries of F lie in S' for some subring S' of S. Then the grid $\varphi(F)$ we obtain by evaluating φ entry-wise is a tame SL_k-frieze over S'.

Corollary 3.12. Let $\underline{x}$ be a cluster in $\mathcal{A}(k, n)$ and let $\varphi_{\underline{x}} : \mathcal{A}(k, n) \to \mathbb{C}$ be the specialization of $\underline{x}$ to 1. Then $\varphi_{\underline{x}}(s\mathcal{P}_{(k,n)})$ is a tame integral SL_k-frieze of width $w = n - k - 1$.

Proof. By Theorem 3.1 and Remark 3.10, $\varphi_{\underline{x}}(s\mathcal{P}_{(k,n)})$ is an integral SL_k-frieze; the tameness follows from Remark 3.11, since $\varphi_{\underline{x}}$ is unitary. Its width follows from the definition of $\mathcal{P}_{(k,n)}$.   □

Remark 3.13. We see later (Lemma 4.8) that for $k \leq 3$ and arbitrary n, as well as for $k = 4$ and $n = 6$ two different clusters $\underline{x} \neq \underline{x}'$ of a cluster algebra $\mathcal{A}$ produce different images $\varphi_{\underline{x}}(s\mathcal{P}_{(k,n)})$ and $\varphi_{\underline{x}'}(s\mathcal{P}_{(k,n)})$ if and only if $\underline{x}$ is not a permutation of $\underline{x}'$.

4. Connection between the categories $\mathcal{C}(k, n)$ and SL_k-friezes

In this section, we use indecomposable modules of the Grassmannian cluster categories $\mathcal{C}(k, n)$ to form friezes of the same shape as the (special) Plücker friezes. We describe the Ext-hammocks for entries in these friezes and characterize the cases where such a frieze of modules (see Definition 4.2) gives rise to cluster-tilting objects. With that, we will then give a bijection between friezes on Auslander–Reiten quivers of Grassmannian cluster categories which we will call *mesh friezes* (see Definition 4.9), and integral tame SL_3-friezes.

4A. *The Grassmannian cluster categories.* We recall the definition of and results about the Grassmannian cluster categories from [Jensen et al. 2016]. Note that an alternative method for constructing an additive categorification of Grassmannian cluster algebras with coefficients has been provided by Demonet and Luo [2016]; in the special case $k = 2$ they applied Amiot's construction [2009] of the generalized cluster category to an ice quiver with potential coming from a triangulation of a polygon. Let $Q(n)$ be the cyclic quiver with vertices $1, \ldots, n$ and $2n$ arrows $x_i : i - 1 \to i$, $y_i : i \to i - 1$. Let $B = B_{k,n}$ be the completion of the path algebra $\mathbb{C}Q(n)/\langle xy - yx, x^k - y^{n-k}\rangle$, where $xy - yx$ stands for the n relations $x_i y_i - y_{i+1}x_{i+1}$, $i = 1, \ldots, n$ and $x^k - y^{n-k}$ stands for the n relations $x_{i+k} \ldots x_{i+1} - y_{i+n-k+1} \ldots y_i$ (reducing indices modulo n). Let $t = \sum x_i y_i$. The center $Z = Z(B)$ is isomorphic to $\mathbb{C}[\![t]\!]$. Then we define the *Grassmannian cluster category* of type (k, n), denoted by $\mathcal{C}(k, n)$, as the category of maximal Cohen Macaulay modules for B. In particular, the objects of $\mathcal{C}(k, n)$ are (left) B-modules M, such that M is free over Z. This category is a Frobenius category, and it is stably 2-CY. It is an additive categorification of the cluster algebra structure of $\mathcal{A}(k, n)$, as proved in [Jensen et al. 2016]. Note that the stable category $\underline{\mathcal{C}(k, n)}$ is triangulated, which follows from [Buchweitz 1986, Theorem 4.4.1], since B is Iwanaga–Gorenstein, or from [Happel 1988, I, Theorem 2.6], since $\mathcal{C}(k, n)$ is a Frobenius category. Both $\mathcal{C}(k, n)$ and its stable version $\underline{\mathcal{C}(k, n)}$ will be called Grassmannian cluster categories.

We recall from [Jensen et al. 2016] that the number of indecomposable not projective-injective summands in any cluster-tilting object of $\mathcal{C}(k, n)$ is $(k - 1)(n - k - 1)$. This is called the *rank* of the cluster category $\mathcal{C}(k, n)$. We say that $\mathcal{C}(k, n)$ is of *finite type*, if it has finitely many isomorphism classes of indecomposable modules. Note also that (for $k \leq n/2$) the category $\mathcal{C}(k, n)$ is of finite type if and only if either $k = 2$ and n arbitrary, or $k = 3$ and $n \in \{6, 7, 8\}$. These categories are of Dynkin type A_{n-3}, D_4, E_6 and E_8 respectively. This means that one can find a cluster-tilting object in $\underline{\mathcal{C}(k, n)}$ whose quiver is of the corresponding Dynkin type.

Remark 4.1 (indecomposable objects of $\mathcal{C}(k, n)$). The following results are from [Jensen et al. 2016]. The rank of an object $M \in \mathcal{C}(k, n)$ is defined to be the length of $M \otimes_Z K$ for K the field of fractions of the center Z [Jensen et al. 2016, Definition 3.5]. There is a bijection between the rank one indecomposable objects of $\mathcal{C}(k, n)$ and k-subsets of $[1, n]$. We may thus write any rank one module as M_I, where I is a k-subset of $[1, n]$. In particular, the indecomposable projective-injective objects are indexed by the n consecutive k-subsets of $[1, n]$ (note here that we consider cyclic consecutive k-subsets modulo n. The

rank one indecomposables are thus in bijection with the cluster variables of $\mathcal{A}(k, n)$ which are Plücker coordinates, see Section 2B. We denote the bijective association $p_I \mapsto M_I$ by $\psi_{k,n}$.

Definition 4.2. Using the bijection between rank one modules and Plücker variables, we can form a frieze of rank 1 indecomposable modules through the composition $\psi_{k,n} \circ \varphi_{(k,n)}$. For the remainder of this section, we will write $\mathcal{M}_{(k,n)}$ to denote the image of $\psi_{k,n} \circ \varphi_{(k,n)}$. We write 0s for the images of the p_I where I has a repeated entry.

As an example, we take $(k, n) = (2, 5)$.

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M_{23} & M_{34} & M_{45} & M_{15} & M_{12} & M_{23} & M_{34} \\
M_{13} & M_{24} & M_{35} & M_{14} & M_{25} & M_{13} & M_{24} & \cdots \\
\cdots & M_{14} & M_{25} & M_{13} & M_{24} & M_{35} & M_{14} & M_{25} \\
M_{45} & M_{15} & M_{12} & M_{23} & M_{34} & M_{45} & M_{15} & M_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Remark 4.3. In the finite types, the indecomposable modules are well-known. For $k = 2$, they are exactly the rank 1 indecomposables. The Auslander–Reiten quivers of the categories $\mathcal{C}(3, n)$, for $n = 6, 7, 8$ are described in [Jensen et al. 2016]. We recall some of this information here.

(i) For $(3, 6)$, there are 22 indecomposable objects. Among them, 20 are rank one modules. The additional two are rank two modules filtered by M_{135} and M_{246} (in both ways). Altogether, there are 6 projective-injective indecomposables and 16 other indecomposable objects. The Dynkin type of $\mathcal{C}(3, 6)$ is D_4.

(ii) For $(3, 7)$, there are 14 rank 2 modules filtered by two rank 1 modules in addition to the 35 rank one modules. The category $\mathcal{C}(3, 7)$ has 7 projective-injectives objects and is a cluster category of type E_6.

(iii) In addition to the 56 rank one modules, $\mathcal{C}(3, 8)$ has 56 rank two modules and 24 rank 3 modules, all filtered by rank 1 modules. It has 8 projective-injective objects. The Dynkin type of $\mathcal{C}(3, 8)$ is E_8.

The categories $\mathcal{C}(3, 9)$ and $\mathcal{C}(4, 8)$ are tame, these categories are known to be of tubular type, see [Baur et al. 2020], where the nonhomogeneous tubes are described.

4B. *Description of* **Ext***-hammocks.* We determine the shape of the Ext-*hammocks in* $\mathcal{M}_{(k,n)}$, that is, given I we describe the set of all J appearing in the frieze $\mathcal{M}_{(k,n)}$ such that $\mathrm{Ext}^1_{\mathcal{C}(k,n)}(M_I, M_J)$ is nonzero. We prove that the Ext-hammocks consist of two maximal rectangles determined by I. We use the result in [Jensen et al. 2016], where the authors determine a precise combinatorial condition for two rank 1 modules to have a nonzero extension. For this section, we allow arbitrary $1 < k < n - 1$.

Two k-subsets I, J of $[1, n]$ are said to be *noncrossing* if there are no cyclically ordered a, b, c, d with $a, c \in I \setminus J$ and $b, d \in J \setminus I$. If such a, b, c, d do occur, then I and J are *crossing*.

Proposition 4.4 [Jensen et al. 2016, Proposition 5.6]. *Let I, J be k-subsets of $[1, n]$. Then*

$$\text{Ext}^1_{\mathcal{C}(k,n)}(M_I, M_J) = 0$$

if and only if I and J are noncrossing.

Next, we define a maximal rectangle in the Plücker frieze starting (resp. ending) at I. Let $I = o([r]^{k-1}, m)$ be a nonconsecutive k-subset. Then we say that J belongs to the *maximal rectangle starting at I* if $J = o([s]^{k-1}, p)$ where $s \in \{r, r + 1, \ldots, m - k\}$ and $p \in \{m, m + 1, \ldots, r - 2\}$. Similarly, we say that J belongs to the *maximal rectangle ending at I* whenever $s \in \{r, r - 1, \ldots, m + 2\}$ and $p \in \{m, m - 1, \ldots, r + k\}$. Note that the collection of such elements indeed forms a maximal rectangle in $\mathcal{M}_{(k,n)}$.

Proposition 4.5. *Let $I = o([r]^{k-1}, m)$ be a nonconsecutive k-subset and let J be another element of $\mathcal{P}_{(k,n)}$. Then $\text{Ext}^1_{\mathcal{C}(k,n)}(M_I, M_J)$ is nonzero if and only if J belongs to the maximal rectangle starting at $o([r + 1]^{k-1}, m + 1)$ or the maximal rectangle ending at $o([r - 1]^{k-1}, m - 1)$.*

Proof. By Proposition 4.4 it suffices to show that I, J are crossing if and only if J belongs to the maximal rectangle starting at $o([r + 1]^{k-1}, m + 1)$ or the maximal rectangle ending at $o([r - 1]^{k-1}, m - 1)$. Recall that
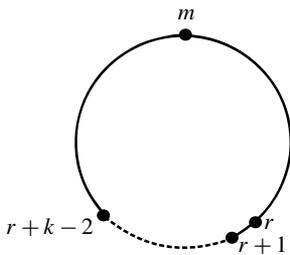
$$I = o([r]^{k-1}, m) = \{r, r + 1, \ldots, r + k - 2, m\}$$

and let

$$J = o([s]^{k-1}, p) = \{s, s + 1, \ldots, s + k - 2, p\}$$

such that I and J cross. In the figure below we depict elements in I on a circle, and observe that I and J cross if and only if the following three conditions are satisfied:

$$J \cap (r + k - 2, m) \neq \varnothing,$$
$$J \cap (m, r) \neq \varnothing,$$
$$m \notin J.$$

Note that $p \notin [r, r + k - 2]$, because otherwise for the first two conditions above to be satisfied we must have $m \in [s]^{k-1} \subset J$. This clearly contradicts the third condition. Therefore, we have two cases $p \in (m, r)$ or $p \in (r + k - 2, m)$.

Suppose $p \in (m, r)$. Then I and J cross if and only if $s + j \in (r + k - 2, m)$ for some $j \in \{0, 1, \ldots, k - 2\}$, and also $s + j \neq m$ for any such j. This implies that $s + k - 2 < m$. On the other hand, we must also have that $r < s$. All together, we obtain $r < s < m - k + 2$. Thus, we see from Figure 2 that J lies in the maximal rectangle starting at $o([r + 1]^{k-1}, m + 1)$.
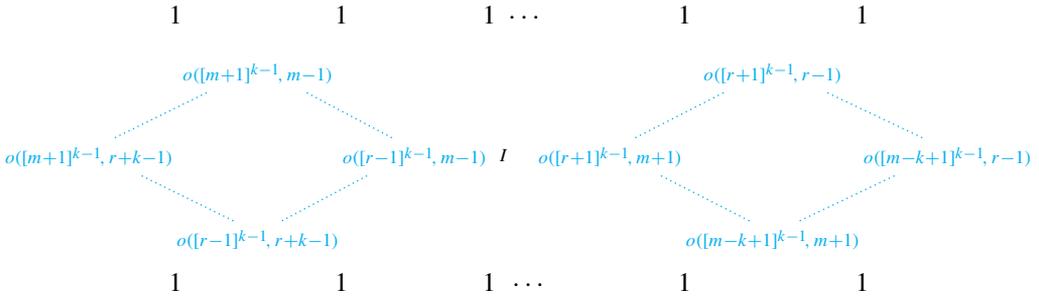
Figure 2. Ext-hammock of M_I, where $I = o([r]^{k-1}, m)$.

Similarly, suppose $p \in (r + k - 2, m)$. Then I and J cross if and only if $s \in (m, r)$. From Figure 2 we see that this mean J lies in the maximal rectangle ending at $o([r-1]^{k-1}, m-1)$. This shows the desired claim. □

For arbitrary k and n, the category $\mathcal{C}(k, n)$ contains a cluster-tilting object consisting entirely of modules indexed by Plücker coordinates. This follows from the results in [Scott 2006]. Next, we describe for which k and n there exists a cluster-tilting objected indexed by the almost consecutive Plücker coordinates.

Proposition 4.6. *A Plücker frieze of type (k, n) contains a collection $\mathcal{I}$ of almost consecutive k-element subsets such that $\bigoplus_{I \in \mathcal{I}} M_I$ is a cluster-tilting object of $\mathcal{C}(k, n)$ if and only if $k = 2, 3$ with n arbitrary, or $k = 4$ and $n = 6$.*

Proof. In general, the cluster category $\mathcal{C}(k, n)$ contains a cluster-tilting object consisting of $(k-1)(n-k-1)$ indecomposable summands. Therefore, we want to show that a Plücker frieze contains $(k-1)(n-k-1)$ distinct k-element subsets that are pairwise noncrossing if and only if $k = 2, 3$ or $k = 4$ and $n = 6$.

If $k = 2$ then every object in $\mathcal{C}(k, n)$ is of the form M_I where I is a 2 element subset of $[1, n]$. In particular, every such I lies in the associated Plücker frieze. This shows the claim for $k = 2$.

If $k = 3$ we want to find $2(n - 4)$ pairwise noncrossing elements in the Plücker frieze. Consider the following collection of 3-element subsets of $[1, n]$ consisting of two disjoint families of size $n - 4$:

$$\{1, 2, m\}_{m=4,5,\ldots,n-1} \quad \text{and} \quad \{s, s+1, 1\}_{s=3,4,\ldots,n-2}.$$

Each of these corresponds to a diagonal in the Plücker frieze, thus by Proposition 4.5 we see that no two elements in the same family cross. Moreover, $\{1, 2, m\}$ does not cross $\{s, s + 1, 1\}$ by definition. This shows that the collection of 3-element subsets above are pairwise noncrossing.

If $k \geq 4$ we want to show that the associated Plücker frieze does not contain $(k-1)(n-k-1)$ pairwise noncrossing elements unless $k = 4$ and $n = 6$. We make use of the following key observation here: the Ext-hammocks in a frieze of width w have the same shape as the Ext-hammocks in the cluster category of type A with rank w. In particular, any region in the frieze that has the same shape as the fundamental region in the associated cluster category of type A, as depicted on the left in Figure 3, contains at most w pairwise noncrossing objects.
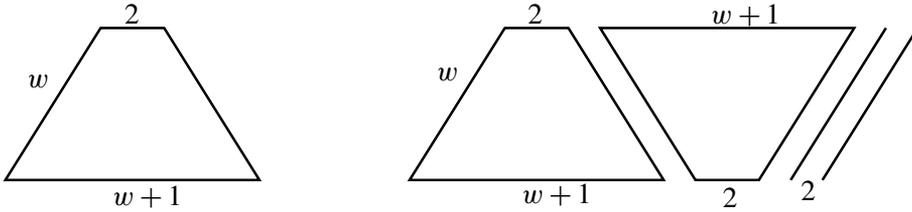
Figure 3. Fundamental region (left) and Plücker frieze where $k = 4$ (right).

First we consider the case $k = 4$ separately. We see that the Plücker frieze admits two copies of the fundamental region followed by two columns, because we have $w = n - k - 1 = n - 5$ and the period of the frieze is n. See Figure 3 on the right. Now suppose that such frieze admits a collection $\mathcal{I}$ of $3w$ pairwise noncrossing objects. Because the frieze contains two such fundamental regions, each of which admits at most w noncrossing object, we conclude that the two remaining columns contain at least w element of $\mathcal{I}$. However, a frieze is n-periodic, and the same argument as above implies that any two neighboring columns contain at least w elements of $\mathcal{I}$. Because there are $w + 5$ columns we conclude that $\mathcal{I}$ contains at least

$$\left\lfloor \frac{w+5}{2} \right\rfloor w$$

elements. We also know that the size of $\mathcal{I}$ is $3w$ which implies that $\lfloor (w + 5)/2 \rfloor \leq 3$. Therefore, there are two possibilities, $w = 1$, $k = 4$, $n = 6$ or $w = 2$, $k = 4$, $n = 7$. In the first case, we see from this diagram that we can take $\mathcal{I} = \{1235, 1345, 1356\}$:

1		1		1		1		1		1		1		1
	1235		2346		1345		2456		1356		1246		1235	
1		1		1		1		1		1		1		1
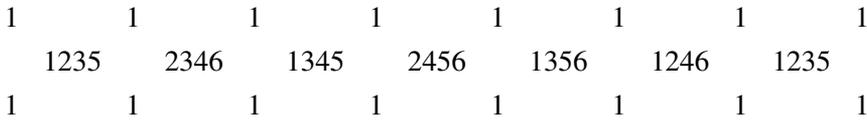
In the second case, one can check that it is not possible to find $\mathcal{I}$ of size 6. This completes the proof for $k = 4$.

Finally, suppose $k > 4$ and let $\mathcal{I}$ be a collection of pairwise noncrossing elements in the Plücker frieze of size $(k - 1)(n - k - 1) = (k - 1)w$. By the same reasoning as in the previous case we think of subdividing the frieze into fundamental regions. We begin by taking two copies of the frieze, so we have a rectangular array of elements of width w and length $2n$. Next, we want to subdivide it into rectangles of width w and length $w + 3$, see Figure 3 on the right. Moreover, each such rectangular region contains at most $2w$ elements of $\mathcal{I}$. But in the beginning we took two copies of the frieze, so dividing by two, we see that a Plücker frieze contains at most

$$\left\lceil \frac{2n}{w+3} \right\rceil w$$

pairwise noncrossing elements. To prove the claim it suffices to show that

$$(k-1)w > \left\lceil \frac{2n}{w+3} \right\rceil w.$$

Below, we will show that

$$k - 1 > \frac{2n}{w+3} + 1 \tag{5}$$

holds, which in turn implies the equation above. Multiplying Equation (5) by $w + 3$ and substituting $w + 3 = n - k + 2$ we obtain

$$(k - 4)n > (k - 2)^2 \tag{6}$$

Since $k > 4$ we can write

$$n > k - 2 > \frac{(k - 2)(k - 2)}{k - 4}$$

which in turn implies Equations (6) and (5). This completes the proof in the case $k > 4$.   □

In terms of cluster algebras we get the following:

Corollary 4.7. *A Plücker frieze of type (k, n) contains a collection $\mathcal{I}$ of almost consecutive k-element subsets such that the collection $\{p_I \mid I \in \mathcal{I}\}$ is a cluster in $s\mathcal{A}(k, n)$ if and only if $k = 2, 3$ with n arbitrary or $k = 4$ and $n = 6$.*

We can apply this to determine when the specialization $\varphi_{\underline{x}}$ of a cluster $\underline{x}$ to $(1, \ldots, 1)$ uniquely determines a specialized Plücker frieze.

Lemma 4.8. *Let $k \leq 3$ and n arbitrary or let $k = 4$ and $n = 6$. Let $\underline{x}$ and $\underline{x}'$ be two clusters of Plücker variables. Then $\varphi_{\underline{x}}(s\mathcal{P}_{(k,n)}) = \varphi_{\underline{x}'}(s\mathcal{P}_{(k,n)})$ if and only if $\underline{x}$ is a permutation of $\underline{x}'$.*

Proof. The assumptions on k and n guarantee that the Plücker frieze $\mathcal{P}_{(k,n)}$ contains a cluster $\underline{a}$ of almost consecutive Plücker coordinates, see Proposition 4.6 and Corollary 4.7.

If $\underline{x}$ is a permutation of $\underline{x}'$ then $\varphi_{\underline{x}} = \varphi_{\underline{x}'}$, which shows the reverse implication. On the other hand, assume that there are two clusters $\underline{x} = (x_1, \ldots, x_N)$ and $\underline{x}' = (x'_1, \ldots, x'_N)$ of Plücker coordinates such that $\varphi_{\underline{x}}(s\mathcal{P}_{(k,n)}) = \varphi_{\underline{x}'}(s\mathcal{P}_{(k,n)})$, where $N = (k-1)(n-k-1) + n$ is the cardinality of a cluster in $\mathcal{A}(k, n)$. Since $\underline{x}$ and $\underline{a}$ are clusters, there exist Laurent polynomials $Q_i^{\underline{x}, \underline{a}}$ such that $a_i = Q_i^{\underline{x}, \underline{a}}(x_1, \ldots, x_N)$. On the other hand, there also exist exchange Laurent polynomials $Q_i^{\underline{a}, \underline{x}}$ such that $x_i = Q_i^{\underline{a}, \underline{x}}(a_1, \ldots, a_N)$ and $Q^{\underline{a}, \underline{x}}(Q^{\underline{x}, \underline{a}}(\underline{x})) = \underline{x}$ (where we understand $Q^{\underline{x}, \underline{a}}$ as the vector $(Q_1^{\underline{x}, \underline{a}}, \ldots, Q_N^{\underline{x}, \underline{a}})$). Similarly we can find $Q^{\underline{x}', \underline{x}}$ and $Q^{\underline{x}, \underline{x}'}$.

By definition $\varphi_{\underline{x}}(\underline{x}) = (1, \ldots, 1)$, and $\varphi_{\underline{x}'}(\underline{x}') = (1, \ldots, 1)$, and in particular, the entries of both evaluations agree on the cluster $\underline{a}$ in $s\mathcal{P}_{(k,n)}$. This means

$$\varphi_{\underline{x}}(\underline{a}) = \varphi_{\underline{x}}(Q^{\underline{x}, \underline{a}}(\underline{x})) = Q^{\underline{x}, \underline{a}}(1, \ldots, 1). \tag{7}$$

On the other hand

$$\varphi_{\underline{x}'}(\underline{a}) = \varphi_{\underline{x}'}\big(Q^{\underline{x}, \underline{a}}(Q^{\underline{x}', \underline{x}}(\underline{x}'))\big) = Q^{\underline{x}, \underline{a}}(Q^{\underline{x}', \underline{x}}(1, \ldots, 1)). \tag{8}$$

Since $Q^{\underline{a}, \underline{x}}(Q^{\underline{x}, \underline{a}}(1, \ldots, 1)) = (1, \ldots, 1)$, applying $Q^{\underline{a}, \underline{x}}$ to (7) and (8) yields

$$(1, \ldots, 1) = Q^{\underline{a}, \underline{x}}\big(Q^{\underline{x}, \underline{a}}(Q^{\underline{x}', \underline{x}}(1, \ldots, 1))\big) = Q^{\underline{x}', \underline{x}}(1, \ldots, 1).$$

Note that, while we have defined the cluster algebra $\mathcal{A}(k, n)$ as a $\mathbb{C}$-algebra, we have in fact

$$\mathcal{A}(k, n) = \mathbb{C} \otimes_{\mathbb{C}} \mathcal{A}_{\mathbb{Z}}(k, n),$$

where $\mathcal{A}_{\mathbb{Z}}(k, n)$ is a cluster algebra of geometric type, defined as the subring of $\mathbb{Q}(x_1, \ldots, x_N)$ generated by the cluster variables. Assume now as a contradiction that $\underline{x}$ is not a permutation of $\underline{x}'$. Then there is an $1 \leq i \leq N$ such that $x_i \notin \{x'_1, \ldots, x'_N\}$ and

$$x_i = Q_i^{\underline{x}', \underline{x}}(x'_1, \ldots, x'_N) = \frac{f(x'_1, \ldots, x'_N)}{x'^{d_1}_1 \cdots x'^{d_N}_N},$$

with $x'_j \nmid f(\underline{x}')$ for all $1 \leq j \leq N$. By positivity [Gross et al. 2018; Lee and Schiffler 2015] this is a Laurent polynomial over $\mathbb{Z}_{\geq 0}$. Therefore, $Q_i^{\underline{x}', \underline{x}}(x'_1, \ldots, x'_N) = 1$ implies that f is the constant polynomial $f = 1$ and

$$x_i = \frac{1}{x'^{d_1}_1 \cdots x'^{d_N}_N}.$$

However, by positivity of d-vectors, which was shown for skew-symmetric cluster algebras by [Cao and Li 2018, Theorem 1.2], we have $d_j \geq 0$ for all $1 \leq j \leq N$ and so x_i is a unit in $\mathcal{A}_{\mathbb{Z}}(k, n)$; contradicting [Geiss et al. 2013, Theorem 1.3]. $\square$

4C. *Mesh friezes.* Assume $k \leq n/2$. We recall that the Grassmannian cluster category $\mathcal{C}(k, n)$ is of finite type if and only if either $k = 2$, or $k = 3$ and $n \in \{6, 7, 8\}$ (Section 4A).

Definition 4.9. Let $\mathcal{C}(k, n)$ be of finite type. A *mesh frieze* $\mathcal{F}_{k,n}$ for the Grassmannian cluster category $\mathcal{C}(k, n)$ is a collection of positive integers, one for each indecomposable object in $\mathcal{C}(k, n)$ such the $\mathcal{F}_{k,n}(P) = 1$ for every indecomposable projective-injective P and such that all mesh relations on the AR-quiver of $\mathcal{C}(k, n)$ evaluate to 1 thus, whenever we have an AR-sequence

$$0 \to A \to \bigoplus_{i=1}^{I} B_i \to C \to 0, \tag{9}$$

where the B_i are indecomposable, and $I \geq 1$, we get

$$\mathcal{F}_{k,n}(A) \mathcal{F}_{k,n}(C) = \prod_{i=1}^{I} \mathcal{F}_{k,n}(B_i) + 1.$$

(Note that $\mathcal{C}(k, n)$ has Auslander–Reiten sequences and an Auslander–Reiten quiver, see for example [Jensen et al. 2016, Remark 3.3].)

Example 4.10. In case $k = 2$, a mesh frieze $\mathcal{F}_{2,n}$ for $\mathcal{C}(2, n)$ is an integral frieze pattern (or an integral SL_2-frieze). In that case, the notion of mesh friezes and of integral SL_2-friezes coincide. Such friezes are in bijection with triangulations of an n-gon and hence arise from specializing a cluster to 1, see Example 2.4.

Remark 4.11. Note that [Assem et al. 2010] introduce the notion of "friezes of Dynkin type." It coincides with certain mesh friezes (on the stable version $\underline{\mathcal{C}(k, n)}$) in types A_n, D_4, E_6, E_8.

Remark 4.12. Recall that $\mathcal{C}(k, n)$ is a categorification of the cluster algebra $\mathcal{A}(k, n)$. Furthermore, the mesh frieze has 1 at the projective-injective objects, i.e., the objects for consecutive k-subsets. In that sense, any mesh frieze $\mathcal{F}_{k,n}$ for $\mathcal{C}(k, n)$ is also a mesh frieze associated to the cluster algebra $s\mathcal{A}(k, n) = \mathcal{A}(k, n)/J$ with coefficients specialized to 1, see Remark 2.8. The requirement that mesh relations evaluate to 1 translates to requiring that certain exchange relations in the cluster algebra hold. We will use both points of view. The approach to friezes via $\mathcal{C}(k, n)$ is particularly useful when we want to check for the existence of cluster-tilting objects in the SL_k-frieze. It will also be of importance in Section 5 as we want to use Iyama–Yoshino reductions on these categories to analyze frieze patterns.

Remark 4.13. Note that the entries in a mesh frieze satisfy the exchange relations of the associated cluster algebra $\mathcal{A}(k, n)$. That is, denoting by x_N the cluster variable in $\mathcal{A}(k, n)$ associated to an indecomposable object N in $\mathcal{C}(k, n)$ the following holds: Whenever we have an exchange relation

$$x_A x_C = \prod_{i=1}^{\widetilde{m}} x_{\widetilde{B}_i} + \prod_{j=1}^{\widetilde{m}'} x_{\widetilde{B}'_j}$$

in $\mathcal{A}(k, n)$, then

$$\mathcal{F}(A)\mathcal{F}(B) = \prod_{i=1}^{\widetilde{m}} \mathcal{F}(\widetilde{B}_i) + \prod_{j=1}^{\widetilde{m}'} \mathcal{F}(\widetilde{B}'_j).$$

Indeed, this follows from the fact that a mesh frieze is determined uniquely by the values of one cluster tilting object. Thus, we may take an acyclic cluster that forms a slice because $k = 2$ and n is arbitrary or $k = 3$ and $n = 6, 7, 8$, and then use sink/source mutations to determine all entries in the mesh frieze uniquely. Now this exact same cluster also determines values on the mesh frieze uniquely using arbitrary mutations of the cluster algebra. Thus, these two approaches should yield the exact same mesh frieze, which implies that the entries in the mesh frieze satisfy all relations of the cluster algebra, not just the ones coming from sink/source mutations.

4D. *Mesh friezes for $\mathcal{C}(k, n)$ yield SL_k-friezes.* In this section, we show the correspondence between integral tame SL_k-friezes and mesh friezes in the finite type cases. For $k = 2$, these two notions coincide (see Example 4.10). So for the rest of this section, we assume $k = 3$ and $n \in \{6, 7, 8\}$. The width of such an SL_3-frieze is $n - 4$, see Remark 2.3.

The following lemma shows that the entries in a tame SL_3-frieze are determined by a cluster of $\mathcal{A}(k, n)$, which consists of consecutive Plücker coordinates.

Lemma 4.14. *Let $F_{3,n}$ with $n \in \{6, 7, 8\}$ be a tame integral SL_3-frieze. Then the entries of $F_{3,n}$ are uniquely determined by the values on the $2w$ entries $x_1 = m_{00}$, $x_2 = m_{01}$, ..., $x_w = m_{0,n-5}$ and $x_{w+1} = m_{2,n-3}$, ..., $x_{2w} = m_{n-3,n-3}$.*

See the picture for $(3, 7)$:

	0		0		0		0		0		0		0	
0		0		0		0		0		0		0		
	1		1		1		1		1		1		1	
m_{22}		m_{33}		x_6		m_{55}		m_{66}		x_1		m_{11}		
	m_{23}		x_5		m_{45}		m_{56}		m_{60}		x_2		m_{12}	
m_{13}		x_4		m_{35}		m_{46}		m_{50}		m_{61}		x_3		
	1		1		1		1		1		1		1	
0		0		0		0		0		0		0		
	0		0		0		0		0		0		0	

Proof. We show the assertion for $F_{3,7}$. Here $w = 3$. Assume that the values of the entries $(x_1, \ldots, x_6)$ are given. We first complete the frieze by repeating the entries along the diagonals (according to the (anti)periodicity rule given in [Morier-Genoud et al. 2014; Cuntz 2017]), that is, the frieze becomes infinite also in the vertical directions. See the picture for $F_{3,7}$:

	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		
$\ldots$	1		1		1		1		1		1		1 $\ldots$
m_{22}		m_{33}		x_6		m_{55}		m_{66}		x_1		m_{11}		
	m_{23}		x_5		m_{45}		m_{56}		m_{60}		x_2		m_{12}	
m_{13}		x_4		m_{35}		m_{46}		m_{50}		m_{61}		x_3		
	1		1		1		1		1		1		1	
0		0		0		0		0		0		0		
	0		0		0		0		0		0		0	
1		1		1		1		1		1		1		
	m_{66}		x_1		m_{11}		m_{22}		m_{33}		x_6		m_{55}	
m_{56}		m_{60}		x_2		m_{12}		m_{23}		x_5		m_{45}		
	m_{50}		m_{61}		x_3		m_{13}		x_4		m_{35}		m_{46}	
$\ldots$	1		1		1		1		1		1		$\ldots$
	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$

Then we can successively compute the remaining entries m_{ij} of the frieze from the x_i using the tameness condition, in particular the two rows of 0s for calculating the 4×4 determinants. First compute m_{35} from the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & m_{35} \\ 1 & 0 & 0 & 1 \\ x_1 & 1 & 0 & 0 \end{pmatrix}.$$

Since by tameness $\det(A) = 0$, this yields $m_{35} = (x_1 + x_5)/x_4$. Moreover from the determinant of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & m_{35} \\ 1 & 0 & 0 & 1 \\ x_1 & 1 & 0 & 0 \\ x_2 & m_{11} & 1 & 0 \end{pmatrix}$$

we obtain $m_{11} = (x_2 + m_{35})/x_1$. Similarly, we obtain $m_{61} = (x_2 + x_6)/x_3$ and $m_{33} = (x_5 + m_{61})/x_6$. Next we obtain m_{45}, m_{46} from the determinants of

$$\begin{pmatrix} 0 & 1 & m_{35} & m_{45} \\ 0 & 0 & 1 & m_{46} \\ 1 & 0 & 0 & 1 \\ m_{11} & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_4 & x_5 & x_6 \\ 1 & m_{35} & m_{45} \\ 0 & 1 & m_{46} \end{pmatrix}.$$

These two determinants yield the system of linear equations

$$\underbrace{\begin{pmatrix} 1 & -m_{35} \\ -x_4 & (x_4 m_{35} - x_5) \end{pmatrix}}_{B} \begin{pmatrix} m_{45} \\ m_{46} \end{pmatrix} = \begin{pmatrix} -m_{11} \\ 1 - x_6 \end{pmatrix}.$$

This system has a unique solution (m_{45}, m_{46}), since the determinant of the matrix B equals $-x_5 \neq 0$. Similarly, we may also uniquely determine m_{50} and m_{60}. The remaining entries may be calculated entirely by using determinants of the 3×3-diamonds.

The assertion for the cases $(3, 6)$ and $(3, 8)$ are proven by analogous tedious calculations. $\qquad\square$

Proposition 4.15. *Let $w \in \{2, 3, 4\}$ and $n = w + 4$. For every tame integral SL_3-frieze of width w there exists a mesh frieze on $\mathcal{C}(3, n)$.*

Proof. Let $F = F_{3,n}$ be a SL_3-frieze of horizontal period $n = w + 4$, for $w \in \{2, 3, 4\}$. Consider the Plücker frieze $\mathcal{P}(k, n)$ of type $(3, n)$. It has the same width and period as F, and we can overlay $\mathcal{P}(k, n)$ on top of F such that every p_I for an almost consecutive 3-subset I of $[1, n]$ comes to lie on top of an entry in F. (Note that a choice is involved here, the way to overlay F by $\mathcal{P}(k, n)$ is not unique.) This assigns to each entry p_I in the Plücker frieze $\mathcal{P}(k, n)$ an integer $f(p_I)$ lying underneath p_I. By Proposition 4.6, the Plücker frieze $\mathcal{P}(k, n)$ contains a collection of entries $\{p_I \mid I \in \mathcal{I}\}$ such that $\bigoplus_{I \in \mathcal{I}} M_I$ is a cluster-tilting object of $\mathcal{C}(3, n)$. Its image under $\psi_{3,n}^{-1}$ is a cluster in $\mathcal{A}(3, n)$ (see Corollary 4.7). Specializing the indecomposable object M_I for $I \in \mathcal{I}$ to $f(p_I)$ and calculating the remaining entries using the mesh relations yields a mesh frieze on the AR-quiver of $\mathcal{C}(3, n)$: By construction, the mesh relations are satisfied and as explained in Remark 3.10 all its entries are positive rationals. It remains to show that the entries are all integers. For this, we associate positive integers from F to their positions in the corresponding Auslander–Reiten quiver/mesh frieze and use the mesh relations to see that all the other entries are sums and differences of products of these given positive integers. That proves the claim. We include pictures of the Auslander–Reiten quivers of these three categories along the way.
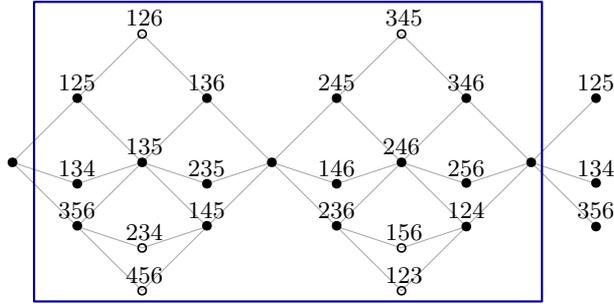
Figure 4. Auslander–Reiten quiver of $\mathcal{C}(3, 6)$. Projective-injectives are drawn as white circles.
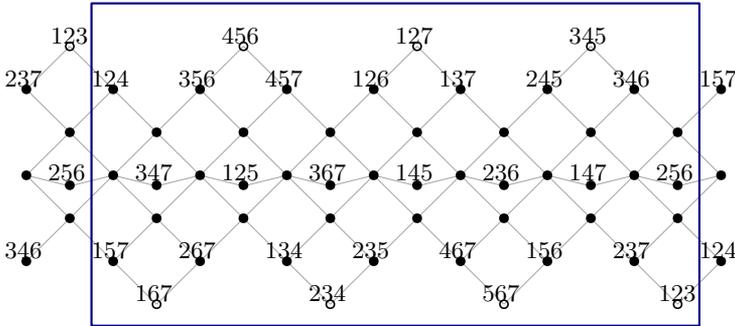

Figure 5. Auslander–Reiten quiver of $\mathcal{C}(3, 7)$.

(i) Let $w = 2$ and $n = 6$. The Auslander–Reiten quiver of $\mathcal{C}(3, 6)$ is shown in Figure 4. The indecomposables modules are indicated by black circles. Among them, the one corresponding to indecomposable rank 1 modules (or Plücker coordinates) appear with their label. The remaining two nodes correspond to rank 2 modules.

The entries of $F_{3,6}$ yield positive integers in the τ-orbits of the nodes 1, 3 and 4 (see Figure 7) of the Auslander–Reiten quiver of $\mathcal{C}(3, 6)$, as this is where the objects indexed by almost consecutive 3-subsets sit. To compute the entries of the τ-orbit of node 2, we use the mesh relations. Here, they are SL$_2$-relations. They involve the τ-orbit of node 1 and the 1s above, if present. In particular, every entry in the τ-orbit of node 2 is of the form $d = \frac{ab-1}{1} = ab - 1$, for a and b in the τ-orbit of node 1. So $d \in \mathbb{Z}$ and we have a mesh frieze. The uniqueness follows since the entries in $F_{3,6}$ are uniquely determined by the choice of the entries corresponding to a cluster.

(ii) The Auslander–Reiten quiver of $\mathcal{C}(3, 7)$ is in Figure 5. The unlabeled nodes correspond to rank 2 modules. Any integer SL$_3$-frieze $F = F_{3,7}$ of width 3 yields positive integers in the τ-orbits of the nodes 1 (or 6) and 2, as the almost consecutive labels are exactly in these τ-orbits. (Note that the τ-orbits of node 1 and node 6 coincide). Using the frieze relation from Conway and Coxeter (see Equation (2) in [Baur et al. 2016]) one can write all entries of the mesh frieze as subtractions of multiples of the entries in the τ-orbit of node 1. So as in (i), all entries are integers and we have a mesh frieze.
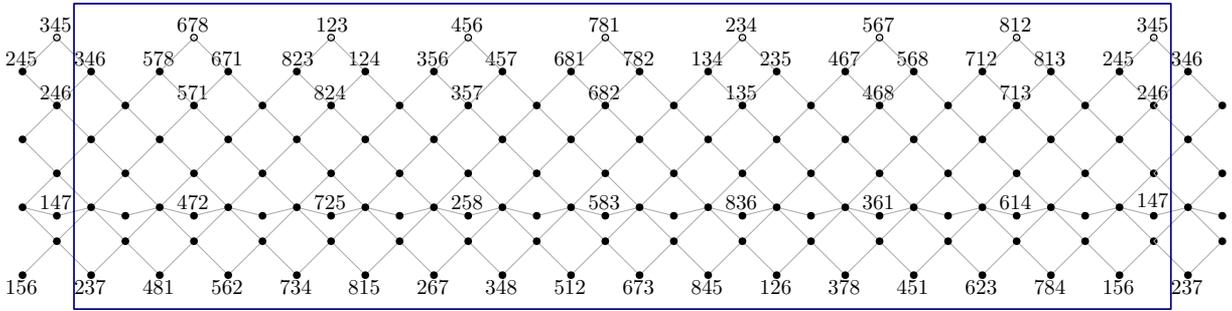
Figure 6. Auslander–Reiten quiver of $\mathcal{C}(3, 8)$.

(iii) Let $F = F_{3,8}$ be an integral SL_3-frieze of width 4. Its entries yield positive integers in the τ-orbits of the nodes 1 and 8 of the Auslander–Reiten quiver of $\mathcal{C}(3, 8)$ (see Figure 6). The unlabeled nodes are rank 2 and rank 3 modules. Observe that only 32 of the entries in the mesh frieze arise from the SL_3-frieze in this case: The entries in the first and fourth row of F form the τ-orbit of 8, the entries of the second and third row the τ-orbit of 1. Using the frieze relation from Conway and Coxeter one can write all entries in the τ-orbits of the nodes 3, 4, 5, 6, 7 of the mesh frieze of $\mathcal{C}(3, 8)$ as subtractions of multiples of entries in the τ-orbits above/below them (see Equation (2) in [Baur et al. 2016]). So again, all these entries are integers. To see that the entries in the τ-orbit of node 2 are integers, use the Plücker relations. For example, to see that the entry for $I = \{2, 5, 7\}$ is an integer, we use the Plücker relations (1) for $(i_1, i_2) = (5, 6)$ and $(j_0, \ldots, j_3) = (2, 4, 5, 7)$: $p_{562}p_{457} - p_{564}p_{257} + p_{565}p_{247} - p_{567}p_{245} = 0$. This yields $p_{257} = p_{256}p_{457} - p_{567}p_{245} = p_{256}p_{457} - 1$ with $p_{256} \in \mathbb{Z}$, $p_{457} \in \mathbb{Z}$ (as they are Plücker coordinates of almost consecutive 3-subsets). $\qquad\square$

Corollary 4.16. *Let $n \in \{6, 7, 8\}$. There is a bijection between mesh friezes for $\mathcal{C}(3, n)$ and integral tame SL_3-friezes of width $n - 4$.*

Proof. For $n = 6$, this can be found in [Morier-Genoud 2015]. We use Proposition 4.15 to see that there is a surjective map from the set of mesh friezes for $\mathcal{C}(3, n)$ to the set of integral tame SL_3-friezes. This map is injective since by Lemma 4.14, in these cases, the SL_3-frieze contains a cluster which uniquely determines it. $\qquad\square$

5. Mesh friezes via IY-reduction

In this section, we explain the effect of Iyama–Yoshino reduction on mesh friezes of cluster categories. We will consider cluster categories of not necessarily connected Dynkin type, i.e., the case where the Auslander–Reiten quiver consists of (possibly more than one) connected components, each of which is isomorphic to the Auslander–Reiten quiver of a cluster category $D^b \pmod{H}/\tau^{-1}\Sigma$, where H is a hereditary algebra with quiver given by a disjoint union of Dynkin diagrams.
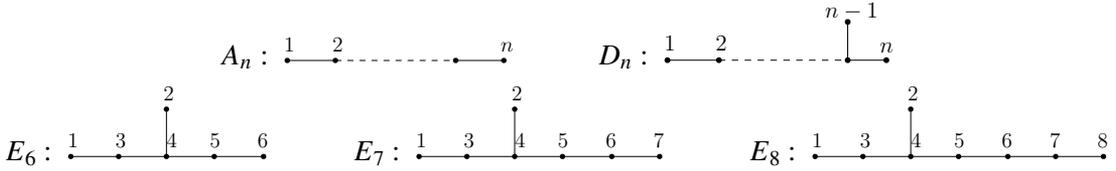
Figure 7. The Dynkin diagrams $\Gamma_{\mathcal{C}}$ of the finite type cluster categories.

We say that such a cluster category is of finite type if it has only finitely many indecomposable objects. In this case, the *rank* is the number of summands in any cluster-tilting object.

Note that when the stable category of $\mathcal{C}(k, n)$ is of finite type (i.e., if $k = 2$ and n is arbitrary, or if $k = 3$ and $n \in \{6, 7, 8\}$), then its Auslander–Reiten quiver is isomorphic to the Auslander–Reiten quiver of the cluster category of the same type in the sense of [Buan et al. 2006].

Recall that for any $\mathcal{A}(k, n)$, specializing a cluster to 1 yields an integral tame SL_k-frieze of width $w = n - k - 1$ (Corollary 3.12). Mesh friezes for $\mathcal{C}(2, n)$ all arise from specializing a cluster to 1. It is known that there exist mesh friezes on finite type cluster categories which do not arise from specializing a cluster, the first example is a mesh frieze in type D_4 containing only 2s and 3s, see Example 5.9 below. It has appeared in [Baur and Marsh 2009, Appendix A], see also [Fontaine and Plamondon 2016, Section 3], where all mesh friezes of type D are characterized. For $\mathcal{C}(3, 6)$ there are thus 51 different mesh friezes: 50 arising from specializing a cluster to 1 and the example above.

5A. *Mesh friezes and their reductions.* Let M be a rigid indecomposable object of a triangulated 2-Calabi–Yau category $\mathcal{C}$. Let $M^{\perp} := \{Y \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(M, Y[1]) = 0\} = \{Y \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(Y, M[1]) = 0\}$ and let $\mathcal{C}(M) = M^{\perp}/(M)$ where (M) denotes the ideal of the category $M^{\perp}$ consisting of morphisms that factor through $\mathrm{add}(M)$. Then by [Iyama and Yoshino 2008, Theorems 4.2 and 4.7], $\mathcal{C}(M)$ is triangulated 2-Calabi–Yau. In particular, it has a Serre functor, and thus Auslander–Reiten triangles.

If $\mathcal{C}$ is a (stable) cluster category of finite type, every indecomposable object is rigid. Moreover, the reduction $\mathcal{C}(M)$ is also of finite type. Let $\Gamma_{\mathcal{C}}$ be the Dynkin diagram associated to $\mathcal{C}$. See Figure 7. Then the Auslander–Reiten quiver of $\mathcal{C}$ has the shape of a quotient of the AR-quiver of the bounded derived category of this type, i.e., of $\Gamma_{\mathcal{C}} \times \mathbb{Z}$ (see [Happel 1988]). For M indecomposable, consider the indecomposable objects reached by sectional paths from M. The arrows of these paths form an oriented Dynkin diagram (see Figure 7) whose vertices are M and the indecomposable objects on the sectional paths. This is a slice in the Auslander–Reiten quiver. The vertex of M in the Dynkin diagram will be called the *node of M*.

By considering the Auslander–Reiten quiver of $\mathcal{C}(M)$, it is straightforward to prove the following statement (note that $\mathcal{C}(M)$ may be a product of cluster categories of finite type):

Proposition 5.1. *Let $\mathcal{C}$ be a cluster category of type A, D or E. Then $\mathcal{C}(M)$ is a cluster category. Its type is obtained by deleting the node of M of the underlying Dynkin diagram.*

Notation 5.2. Let $\mathcal{F}$ be a mesh frieze on a finite type cluster category $\mathcal{C}$. Let M be an indecomposable object in $\mathcal{C}$. We write $\mathcal{F}(N)$ for the (integer) value of an indecomposable N in $\mathcal{F}$. Then $\mathcal{F}|_{\mathcal{C}(M)}$ denotes the collection of numbers of $\mathcal{F}$ on the AR-quiver of $\mathcal{C}(M)$.

Proposition 5.3. *Let $\mathcal{F}$ be a mesh frieze on a finite type cluster category $\mathcal{C}$, containing an entry* 1. *Let $M \in \operatorname{ind} \mathcal{C}$ such that the frieze entry at M is* 1. *Then $\mathcal{F}|_{\mathcal{C}(M)}$ is a mesh frieze for $\mathcal{C}(M)$.*

Proof. Denote by $\langle 1 \rangle$ suspension in $\mathcal{C}(M)$ and let

$$A \to B \to C \to A\langle 1 \rangle \tag{10}$$

be an AR-triangle in $\mathcal{C}(M)$. We need to show that the corresponding values of the mesh frieze satisfy the equation $\mathcal{F}(A)\mathcal{F}(C) = \mathcal{F}(B) + 1$, where $\mathcal{F}(B) = \mathcal{F}(B_1) \cdots \mathcal{F}(B_t)$ whenever $B = \bigoplus_{i=1}^{t} B_i$.

Let T be a cluster tilting object in $\mathcal{C}(M)$ consisting of all indecomposable objects N that admit a sectional path starting at A and ending at N. Note that T forms a slice in the AR-quiver of $\mathcal{C}(M)$ with A being a source. Moreover, A, $B \in \operatorname{add} T$ while $C \notin \operatorname{add} T$, and the triangle (10) corresponds to a minimal left $\operatorname{add}(T/A)$-approximation of A. By construction, the minimal right $\operatorname{add}(T/A)$-approximation of A is zero.

$$A\langle -1 \rangle \to C \to 0 \to A \tag{11}$$

Next consider the object $\widetilde{T} = T \oplus M$ of $\mathcal{C}$. Since, $T \in \mathcal{C}(M)$ it follows that $\operatorname{Ext}^1_{\mathcal{C}}(T, M) = 0$ Also, $\operatorname{Ext}^1_{\mathcal{C}}(T, T) = 0$, because T is a cluster-tilting object in $\mathcal{C}(M)$. Finally, M is an indecomposable object of $\mathcal{C}$, which means that it is rigid. This implies that $\widetilde{T}$ is rigid in $\mathcal{C}$. Thus, $\widetilde{T}$ is a cluster-tilting object in $\mathcal{C}$ as it has the correct number of indecomposable summands.

The AR-triangles (10) and (11) in $\mathcal{C}(M)$ lift to the corresponding triangles in $\mathcal{C}$:

$$A \to \widetilde{B} \to C \to A[1], \quad A[-1] \to C \to \widetilde{B}' \to A, \tag{12}$$

where $\widetilde{B} \cong B \oplus M^i$ and $\widetilde{B}' \cong M^j$ for some nonnegative integers i, j and $[1]$ denotes suspension in $\mathcal{C}$. Here, M^0 denotes the zero module. Moreover, we see that the two triangles in (12) are the left and right add $\widetilde{T}/A$-approximations of A respectively.

Let $\mathcal{A}$ be the cluster algebra associated to $\mathcal{C}$. For an object $N \in \mathcal{C}$ let x_N denote the associated (product of) cluster variable(s) in $\mathcal{A}$. Because $\mathcal{C}$ is the categorification of $\mathcal{A}$ it follows that the two triangles (12) give rise to the following exchange relation in the cluster algebra:

$$x_A x_C = x_{\widetilde{B}} + x_{\widetilde{B}'}.$$

Note that the entries of the associated mesh frieze $\mathcal{F}$ also satisfy the relations of the cluster algebra $\mathcal{A}$. By Remark 4.13 we obtain the desired equation

$$\mathcal{F}(A)\mathcal{F}(C) = \mathcal{F}(\widetilde{B}) + \mathcal{F}(\widetilde{B}') = \mathcal{F}(B)\mathcal{F}(M^i) + \mathcal{F}(M^j) = \mathcal{F}(B) + 1$$

because $\mathcal{F}(M^1) = \mathcal{F}(M) = 1$ and $\mathcal{F}(M^0) = \mathcal{F}(0) = 1$.   □

Remark 5.4. We can use Proposition 5.3 to argue that a mesh frieze for $\mathcal{C}$ always has $\leq m$ entries 1, if m is the rank of $\mathcal{C}$: Let $\mathcal{F}$ be a mesh frieze for $\mathcal{C}$, a cluster category of rank m. Assume that $\mathcal{F}$ has $> m$ entries 1. Take M_0 indecomposable such that its entry in the frieze is 1. Let $\mathcal{C}_1 := \mathcal{C}(M_0)$. This has rank $m - 1$. Let $\mathcal{F}_1 = \mathcal{F}|_{\mathcal{C}_1}$. Take $M_1 \in \text{ind}\,\mathcal{C}_1$ such that the entry of M_1 in $\mathcal{F}_1$ is 1. Let $\mathcal{C}_2 := \mathcal{C}_1(M_1)$. This is a cluster category of rank $m - 2$ and $\mathcal{F}_2 := \mathcal{F}_1|_{\mathcal{C}_2}$ has $\geq m - 1$ entries equal to 1. Iterate until the resulting cluster category is a product of cluster categories of type A, whose rank is m' (and $m' \geq 3$) and where the associated frieze $\mathcal{F}'$ has $\geq m' + 1$ entries equal to 1s. Contradiction. Here, we also might need to justify that passing from $\mathcal{F}$ to $\mathcal{F}_1$ we loose exactly one entry in the frieze that equals 1. This holds, because suppose on the contrary that $\mathcal{F}(M_0) = \mathcal{F}(M_0') = 1$, where M_0' is some other indecomposable object of $\mathcal{C}$, and $M_0' \notin M_0^\perp$. This implies that $\text{Ext}^1_{\mathcal{C}}(M_0, M_0') \neq 0$. In particular, there exist two triangles in $\mathcal{C}$,

$$M_0 \to B \to M_0' \to M_0[1], \quad M_0' \to B' \to M_0 \to M_0'[1].$$

And by the same reasoning as in the proof of the above proposition, we obtain

$$\mathcal{F}(M_0)\mathcal{F}(M_0') = \mathcal{F}(B) + \mathcal{F}(B').$$

The right hand side is clearly ≥ 2, because $\mathcal{F}$ is an integral mesh frieze. This is a contradiction with $\mathcal{F}(M_0) = \mathcal{F}(M_0') = 1$.

5B. *Counting friezes.* In this section, we study the number of mesh friezes for the Grassmannian cluster categories $\mathcal{C}(3, n)$ in the finite types (types D_4, E_6 and E_8) and for the cluster category of type E_7. We recover the known number of mesh friezes for $\mathcal{C}(3, 7)$ and find all the known mesh friezes for $\mathcal{C}(3, 8)$ and for cluster categories of type E_7. Recall that specializing a cluster to 1 gives a unique mesh frieze on $\mathcal{C}(k, n)$; see Corollary 3.12.

Definition 5.5. A *unitary mesh frieze* is a mesh frieze which arises from specializing a cluster to 1.

Not every frieze pattern arises this way. Some mesh friezes have less 1s than the rank of the cluster category. These are called *nonunitary friezes*. We will focus on these in what follows.

Remark 5.6. For $k = 2$, the notion of mesh frieze coincides with the notion of SL$_2$-frieze and the numbers of them are well-known, they are counted by the Catalan numbers. In particular, they all arise from specializing a cluster to 1. From now on, we thus concentrate on $k = 3$ and $n \in \{6, 7, 8\}$. Recall first that the case $(3, 6)$ corresponds to a cluster category of type D_4, the case $(3, 7)$ to a cluster category of type E_6 and the case $(3,8)$ to a cluster category of type E_8. The number of clusters in these types are 50, 833 and 25080 respectively, see, e.g., [Fomin and Zelevinsky 2003, Table 3].

It is known that the number of SL$_3$-friezes of width 2 or, equivalently, the number of mesh friezes for $\mathcal{C}(3, 6)$ is 51 (this is proven via 2-friezes in [Morier-Genoud et al. 2012, Proposition 6.2], and can be checked using a computer algebra system or counting the mesh friezes on a cluster category of type D_4, see Example 5.9 below). The number of SL$_3$-friezes (equivalently, the number of mesh friezes for $\mathcal{C}(3, 7)$) of width 3 is 868 by Theorem B.1. Note that this number was already stated, citing a private communication by Cuntz, in [Morier-Genoud 2015]. Appendix B now provides a proof for this statement.

Fontaine and Plamondon found 26952 mesh friezes for $\mathcal{C}(3, 8)$, see the lists on [Fontaine and Plamondon 2008], see also [Cuntz 2017, Conjecture 2.1; Morier-Genoud 2015, Section 4.4].

We also recall that the number of clusters in a cluster category of type E_7 is 4160. Conjecturally, there are 4400 mesh friezes in type E_7 [Fontaine and Plamondon 2008].

Corollary 5.7. *Let $\mathcal{F}$ be a mesh frieze on a finite type cluster category $\mathcal{C}$ and $M \in \operatorname{ind} \mathcal{C}$ an object whose frieze entry is* 1. *Then the following holds*:

(1) *If $\mathcal{F}' := \mathcal{F}|_{\mathcal{C}(M)}$ is unitary, then $\mathcal{F}$ is unitary.*

(2) *If the Dynkin type of $\mathcal{C}(M)$ is A or a product of A-types, then $\mathcal{F}$ is unitary.*

Proof. Let n be the rank of $\mathcal{C}$. By Proposition 5.3, $\mathcal{F}' := \mathcal{F}|_{\mathcal{C}(M)}$ is a mesh frieze for $\mathcal{C}(M)$. If $\mathcal{C}(M)$ is unitary, then $\mathcal{F}'$ contains $n - 1$ entries equal to 1. But then $\mathcal{F}$ contains n 1s and is unitary. The second statement then follows, since in type A, all mesh friezes are unitary. $\square$

An immediate consequence of Corollary 5.7 is the following.

Corollary 5.8. *Let $\mathcal{F}$ be a mesh frieze on a cluster category of type D or E. If $\mathcal{F}$ contains an entry* 1 *at the branch node or at a node which is an immediate neighbor of the branch node, then $\mathcal{F}$ is unitary.*

By Corollary 5.8, nonunitary friezes cannot exist in ranks ≤ 3: In terms of ranks, the smallest possible cluster category with nonunitary mesh friezes is of type D_4. Indeed, we have one in this case, see Example 5.9.

Example 5.9. There exists a nonunitary mesh frieze in type D_4 [Baur and Marsh 2009, Appendix A]. It is the only nonunitary mesh frieze in type D_4; see [Fontaine and Plamondon 2016, Section 3]:

$$
\begin{array}{ccccccccc}
 & 2 & & 2 & & 2 & & 2 & \\
 & & 3 & & 3 & & 3 & & 3 \\
\cdots & & 2 & & 2 & & 2 & & 2 & \cdots \\
 & & 2 & & 2 & & 2 & & 2 \\
\end{array}
$$

5B1. *Nonunitary mesh friezes in type E_6.* Let $\mathcal{C}$ be a cluster category of type E_6. Using the result of Cuntz and Plamondon (Theorem B.1), one can deduce that there are 35 nonunitary mesh friezes for $\mathcal{C}$. One can extract these from the lists available on [Fontaine and Plamondon 2008]. One can check that each of them contains two 1s.

Here, we explain how nonunitary mesh friezes arise in type E_6 and how any frieze with a 1 must contain at least two 1s. Observe that the AR-quiver of a cluster category of type E_6 consists of 7 slices of the Dynkin diagram E_6:

(Note that the last slice with circles involves a twist, the top node is identified with the bottom node in the first slice, etc.)

By Corollary 5.7, if a nonunitary (mesh) frieze in type E_6 contains an entry 1, the node of this entry has to be 1 (or 6) (see Figure 7 for the labeling of nodes). So let $\mathcal{F}$ be a nonunitary frieze for $\mathcal{C}$ and M an indecomposable corresponding to an entry 1 in the top row. Then $\mathcal{C}(M)$ is a cluster category of type D_5. Furthermore, $\mathcal{F}|_{\mathcal{C}(M)}$ is a nonunitary mesh frieze on $\mathcal{C}(M)$ (since otherwise, $\mathcal{F}$ is unitary, by Corollary 5.7). There are exactly 5 nonunitary frieze on a cluster category of type D_5, the five τ-translates of the following mesh frieze:

$$
\begin{array}{ccccccccc}
2 & & 3 & & 1 & & 3 & & 2 \\
& 5 & & 2 & & 2 & & 5 & & 3 \\
& & 3 & & 3 & & 3 & & 7 & & 7 \\
& & & 2 & & 2 & & 2 & & 4 & & 2 \\
& & & 2 & & 2 & & 2 & & 4 & & 2
\end{array}
$$

Since there are 7 positions for the first entry 1 we picked (respectively for M), all in all there are exactly 35 different nonunitary friezes for $\mathcal{C}$.

5B2. *Nonunitary mesh friezes in type E_7.* Let $\mathcal{C}$ be a cluster category of type E_7. Conjecturally, there are 240 nonunitary mesh friezes for $\mathcal{C}$. We now show how 240 nonunitary mesh friezes arise from nonunitary friezes for a cluster category of type D_6 or of type D_4. First recall that the AR-quiver of $\mathcal{C}$ consists of 10 slices of a Dynkin diagram of type E_7. If $\mathcal{F}$ is a nonunitary for $\mathcal{C}$, it cannot contain entries equal to 1 in the nodes 2,3,4 or 5 (see Figure 7) for the labels of the nodes in the Dynkin diagram).

Remark 5.10. Let $\mathcal{F}$ be a nonunitary frieze for $\mathcal{C}$.

 (i) Assume that $\mathcal{F}$ contains an entry 1 in the τ-orbit of node 1, let M be the corresponding indecomposable object. Then $\mathcal{C}(M)$ is of type D_6.

   (a) There are 2 mesh friezes of type D_6 without any 1s, see Example 5.12. Hence we get 20 mesh friezes of type E_7 having an entry 1 in the τ-orbit of 1 and all other entries ≥ 2.

   (b) So assume that $\mathcal{F}' = \mathcal{F}_{\mathcal{C}(M)}$ contains an additional entry 1, with corresponding module N. Then doing a further reduction $(\mathcal{C}(M))(N)$ yields a cluster category of type D_5 or of type $A_1 \times D_4$. In both cases, $\mathcal{F}'$ has two entries equal to 1. Analyzing the AR-quiver of $\mathcal{C}(M)$ and the possible positions of these 1s yields another 220 nonunitary mesh friezes, as one can check.

 (ii) The above mesh friezes cover are all known nonunitary mesh friezes in type E_7 having at least one entry 1, according to the lists of Fontaine and Plamondon.

Conjecture 5.11. *There are no mesh friezes for $\mathcal{C}$ of type E_7 where all entries are ≥ 2.*

Example 5.12. In type D_6 there are exactly two mesh friezes without any 1s, see [Fontaine and Plamondon 2016, Section 3]. One of them is here, the other one is the translate of it by τ:

$$
\begin{array}{ccccccc}
2 & 2 & 2 & 2 & 2 & 2 & \\
& 3 & 3 & 3 & 3 & 3 & 3 \\
& & 4 & 4 & 4 & 4 & 4 & 4 \\
\cdots & & & 5 & 5 & 5 & 5 & 5 & 5 & \cdots \\
& & & & 2 & 3 & 2 & 3 & 2 & 3 \\
& & & & 3 & 2 & 3 & 2 & 3 & 2
\end{array}
$$
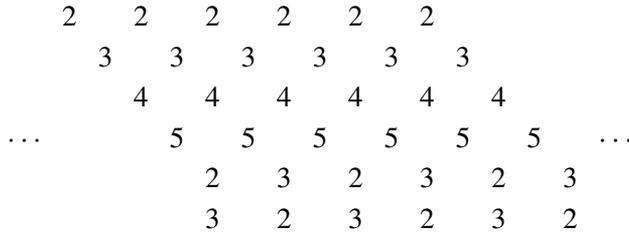
5B3. *Nonunitary mesh friezes in type E_8.* Let $\mathcal{C}$ be a cluster category of type E_8. Conjecturally, there are 1872 nonunitary mesh friezes for $\mathcal{C}$. Recall the labeling of the nodes in the Dynkin diagram from Figure 7. Nonunitary (mesh) friezes do not contain 1s in the τ-orbits of the nodes 2, 3, 4 or 5, see Corollary 5.8.

So in order to study nonunitary mesh friezes for $\mathcal{C}$ we can start considering friezes containing an entry 1 in the τ-orbit of node 6. There are 16 choices for this, as the Auslander–Reiten quiver of $\mathcal{C}$ has 16 slices. Let M be the corresponding indecomposable. Then $\mathcal{C}(M)$ is a cluster category of type $D_5 \times A_2$. There must be two more 1s in the frieze in the factor of type A_2 and one in the factor of type D_5. There are 5 choices of a cluster in type A_2 and there are five choices for an entry 1 in the type D_5 factor (see Section 5B1. All in all there are $16 \times 5 \times 5 = 400$ possibilities for a nonunitary frieze with an entry 1 in the τ-orbit of node 6.

Continuing with similar arguments, one finds 1852 nonunitary friezes with four 1s and 16 friezes with two 1s. There are no known mesh friezes for $\mathcal{C}$ with only one entry $= 1$, giving additional evidence for Conjecture 5.11:

Lemma 5.13. *If Conjecture 5.11 is true, then every mesh frieze for $\mathcal{C}$ of type E_8 contains $0, 2, 4$ or 8 entries equal to* 1.

Proof. We only need to discuss nonunitary mesh friezes.

If there is an entry 1 in the τ-orbit of node 8, the corresponding category $\mathcal{C}(M)$ is of type E_7 and the claim follows by assumption (Conjecture 5.11) and by Remark 5.10(i)(a) and (b).

If there is an entry 1 in the τ-orbit of node 1, $\mathcal{C}(M)$ is a cluster category of type D_7. By the results in [Fontaine and Plamondon 2016, Section 3], every nonunitary frieze of type D_7 contains either one or three entries $= 1$. All the other cases arise from two entries equal to 1 and a nonunitary mesh frieze for a cluster category of type E_6, these are known to have two entries $= 1$. $\qquad\square$

Conjecture 5.14. *If $\mathcal{F}$ is a mesh frieze for $\mathcal{C}$ of type E_8 and if all entries of $\mathcal{F}$ are ≥ 2, then $\mathcal{F}$ is one of the four friezes from Example 5.15.*

Example 5.15. In type E_8 there are 4 known mesh friezes without any 1s, appearing in the lists by Fontaine and Plamondon [2008]. They are the 4 translates of the mesh frieze of Figure 8. These mesh friezes arise from specializing the following cluster to $(3, 3, 3, 3, 3, 3, 3, 3)$:

$$
\begin{array}{ccccccc}
p_{235} & \longrightarrow & p_{236} & \longleftarrow & p_{126} & \longrightarrow & p_{127} \\
\uparrow & & \downarrow & & \uparrow & & \downarrow \\
p_{356} & \longleftarrow & p_{256} & \longrightarrow & p_{267} & \longleftarrow & p_{167}
\end{array}
$$
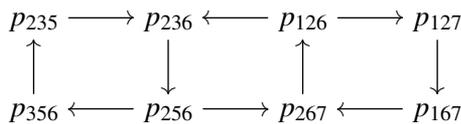
3		2		2		3		3		2		2		3		
	5		3		5		8		5		3		5		8	
13		7		7		13		13		7		7		13		$\cdots$
$\cdots$	18		16		18		21		18		16		18		21	
29	6	41	7	41	6	29	5	29	6	41	7	41	6	29	5
	11		15		11		8		11		15		11		8	
3		4		4		3		3		4		4		3		

Figure 8. Example 5.15: A 4-periodic frieze, here, 8 slices are shown.

Appendix A: Proof of Proposition 3.5

In the following, we are mainly interested in Plücker coordinates $p_{i_1,\ldots,i_k}$ where a portion of the i_l is consecutive. For brevity we will write $[r]^l$ for the set $\{r, r+1, \ldots, r+l-1\}$. We reduce integers modulo n. So if $l = k - 1$, such a tuple forms an almost consecutive k-subset. As before, we will use the map o to indicate reordering by size, e.g., if $k = 4$, $n = 9$, $p_{o([8]^3),4}$ stands for the Plücker coordinate p_{1894} and $p_{(o([8]^3,4)}$ for p_{1489}. We recall the proposition we are to prove here.

Proposition 3.5. *Let $r \in [1, n]$ and let $1 \leq s \leq k$. Let $\underline{m} = (m_1, \ldots, m_s)$ with $m_i \in [1, n]$ for all i, and assume that $\underline{m}$ satisfies conditions (c1) and (c2). Let $b_{\underline{m};r}$ be the determinant of the matrix $A_{\underline{m};r}$ from (3). Then we have*

$$b_{\underline{m};r} = \left[\prod_{l=0}^{s-2} p_{o([r+l]^k)}\right] \cdot p_{o([r+s-1]^{k-s}, m_1, \ldots, m_s)}. \tag{13}$$

Proof of Proposition 3.5. If we have $m_i = m_j$ for some $1 \leq i < j \leq s$, then

$$b_{\underline{m};r} = 0 = \left[\prod_{l=0}^{s-2} p_{o([r+l]^k)}\right] \cdot p_{o([r+s-1]^{k-s}, m_1, \ldots, m_s)},$$

as the last term is 0. We can thus assume that $m_1, \ldots, m_s$ are mutually distinct.

We prove the claim by induction over s. For $s = 1$ the claim holds, since $b_{\underline{m};r} = \det(p_{o([r]^{k-1}, m_1)}) = p_{o([r]^{k-1}, m_1)}$. Let now $2 \leq s \leq k$ and assume the statement is true for $1 \leq l < s$, that is for all $b_{(\widetilde{m}_1, \ldots, \widetilde{m}_l);s}$, where the $\widetilde{m}_j$ satisfy conditions (c1) and (c2). We show that it also holds for $b_{\underline{m};r}$.

We have

$$\begin{aligned} b_{\underline{m};r} &= \sum_{j=1}^{s} (-1)^{s+j} p_{o([r+s-1]^{k-1}, m_j)} \cdot b_{(m_1, \ldots, \widehat{m_j}, \ldots, m_s);r} \\ &= (-1)^s \cdot \left[\prod_{l=0}^{s-3} p_{o([r+l]^k)}\right] \cdot \left[\sum_{j=1}^{s} (-1)^j p_{o([r+s-1]^{k-1}, m_j)} \cdot p_{o([r+s-2]^{k-(s-1)}, m_1, \ldots, \widehat{m_j} \ldots, m_s)}\right], \end{aligned} \tag{14}$$

where the first equality is by Laplace expansion along the last row and the second equality is by induction assumption and pulling the term $X := (-1)^s \cdot \prod_{l=0}^{s-3} p_{o([r+l]^k)}$ out of the sum. Notice that the conditions

(c1) and (c2) pass down from the set $\{m_1, \ldots, m_s\}$ to any subset $\{m_1, \ldots, \widehat{m_j}, \ldots, m_s\}$, so we are justified in using the induction assumption on the minors of the form $b_{(m_1,\ldots,\widehat{m_j},\ldots,m_s);r}$.

We distinguish the following cases.

<u>Case 1</u>. Assume $s \neq k$ and there exists a $1 \leq l \leq s$ with $m_l \in [r+s-2]^{k-(s-1)}$.

<u>Case 1a</u>. Assume first that $m_l \neq r+s-2$. Then $m_l \in [r+s-1]^{k-s}$ and the following expressions all vanish:

$$p_{o([r+s-1]^{k-1}, m_l)} = p_{o([r+s-1]^{k-s}, m_1,\ldots,m_s)} = 0 \quad \text{and} \quad p_{[r+s-2]^{k-(s-1)}, m_1,\ldots,\widehat{m_j}\ldots,m_s} = 0 \quad \text{for all } j \neq l.$$

Combining this with (14) it follows that

$$b_{\underline{m};r} = 0 = \prod_{l=0}^{s-2} p_{o([r+l]^k)} \cdot p_{o([r+s-1]^{k-s}, m_1,\ldots,m_s)}.$$

This proves the claim in this case.

<u>Case 1b</u>. Assume now that $m_l = r+s-2$ and $l < s$. If we had $m_{l+1} \in (r+k-2, m_l]$, then we would have $r+k-2 \in (m_l, m_{l+1}) \subseteq (m_1, m_s)$ contradicting condition (c2). Therefore, we must have $m_{l+1} \in (m_l, r+k-2] = [r+s-1]^{k-s}$ and we are in case 1a with m_{l+1} taking on the role of m_l.

<u>Case 1c</u>. As a last subcase for case 1 assume that $l = s$ and $m_s = r+s-2$. Then all terms in (14) with factors $p_{o([r+s-2]^{k-(s-1)}, m_1,\ldots,\widehat{m_j},\ldots,m_s)}$ with $j \neq s$ vanish and we obtain

$$b_{\underline{m};r} = (-1)^s \cdot \left[\prod_{l=0}^{s-3} p_{o([r+l]^k)}\right] \cdot (-1)^s p_{o([r+s-1]^{k-1}, m_s)} \cdot p_{o([r+s-2]^{k-(s-1)}, m_1,\ldots,m_{s-1})},$$

$$= \left[\prod_{l=0}^{s-2} p_{o([r+l]^k)}\right] \cdot p_{o([r+s-1]^{k-s}, m_1,\ldots,m_s)},$$

where the second equality follows directly from the equality $m_s = r+s-2$. This proves the claim in this case.

<u>Case 2</u>. Assume now $s = k$ and there exists a $1 \leq l \leq k$ such that $m_l = r+k-2$. This cannot happen if $l \neq k$: By condition (c2) we have $r+k-2 \notin [m_1, m_k)$, but by condition (c1), we have $m_l \in [m_1, m_k)$ for $1 \leq l < k$. Therefore, $m_l = r+k-2$ implies $l = k$, and thus in Case 2 we have $m_k = r+k-2$. Then the same argument as in Case 1c, replacing s by k, proves the claim.

<u>Case 3</u>. We are now left with the case where $m_l \notin [r+s-2]^{k-(s-1)}$ for all $1 \leq l \leq s$. We are going to show that

$$\sum_{j=1}^{s}(-1)^j p_{o([r+s-1]^{k-1},m_j)} \cdot p_{o([r+s-2]^{k-(s-1)}, m_1,\ldots\widehat{m_j}\ldots,m_s)} = (-1)^s \cdot p_{o([r+s-2]^k)} \cdot p_{o([r+s-1]^{k-s}, m_1,\ldots,m_s)}, \quad (15)$$

which, when substituting into (14) yields the desired equality (13). We use the Plücker relations on $I = [r+s-1]^{k-1}$ of cardinality $|I| = k-1$ and $J = [r+s-2]^{k-(s-1)} \cup \{m_1, m_2, \ldots, m_s\}$ of cardinality $|J| = k+1$.

<u>Case 3a</u>. Assume that $1 = r + a \in [r + s - 1]^{k-1} \cap [r + s - 2]^{k-(s-1)}$. Then I is ordered as follows:

$$1 = r + a < r + a + 1 < \cdots < r + s + k - 3 < r + s - 1 < \cdots < r + a - 1$$

and J is ordered as follows:

$$1 = r + a < r + a + 1 < \cdots < r + k - 2 < m_1 < m_2 < \cdots < m_s < r + s - 2 < r + s - 1 < \cdots < r + a - 1 = n.$$

Notice that we must have $m_1 < m_l$ for all $l \neq 1$. Otherwise, if there exists a $1 < l \leq s$ with $m_l < m_1$, then if we have $m_l \in [1, r + k - 2] \subseteq [r + s - 2, r + k - 2]$ this contradicts the assumptions of Case 3, and if we have $m_l \in (r + k - 2, n]$, then it follows from $m_l < m_1$ that $r + k - 2 \in [m_1, m_l) \subseteq [m_1, m_s)$, which contradicts condition (c2). The Plücker relation on I and J (see Equation (1)) thus reads as

$$0 = \sum_{l=0}^{k-a-2} (-1)^l \cdot p_{o([r+s-1]^{k-1}), r+a+l} \cdot p_{o(J \setminus \{r+a+l\})} + \sum_{l=1}^{s} (-1)^{k-a-2+l} \cdot p_{o([r+s-1]^{k-1}), m_l} \cdot p_{o(J \setminus m_l)}$$
$$+ \sum_{l=0}^{a-s+1} (-1)^{k-a+s+l-1} \cdot p_{o([r+s-1]^{k-1}), r+s-2+l} \cdot p_{o(J \setminus \{r+s-2+l\})}.$$

Notice that $p_{o([r+s-1]^{k-1}), r+j} = 0$ for all $j \in [s-1, k-2]$ and thus the whole first sum among those three vanishes, and all the summands, except for the first one ($l = 0$) in the last sum vanish as well. The Plücker relation thus reduces to

$$0 = \sum_{l=1}^{s} (-1)^{k-a-2+l} \cdot p_{o([r+s-1]^{k-1}), m_l} \cdot p_{o(J \setminus m_l)} + (-1)^{k-a+s-1} \cdot p_{o([r+s-1]^{k-1}), r+s-2} \cdot p_{o([r+s-1]^{k-s}, m_1, \ldots, m_s)}.$$

We need to get the remaining k-tuples in linear order. Observe that

$$p_{o([r+s-1]^{k-1}, m_l)} = p_{r+a, \ldots, r+s+k-3, m_l, r+s-1, \ldots, r+a-1}$$
$$= \text{sgn}(\pi) \, p_{o([r+s-1]^{k-1}), \, m_l}$$

and

$$p_{o([r+s-2]^k)} = \big(r + a, \ldots, r + s + k - 3, r + s - 2, r + s - 1, \ldots, r + a - 1 \big)$$
$$= \text{sgn}(\pi) \, p_{o([r+s-1]^{k-1}), r+s-2},$$

where $\pi = (s + k - a - 1, s + k - a, \ldots, k - 1, k) \in S_k$. Substituting this into the Plücker relation on I and J (and multiplying by $\text{sign}(\pi) \cdot (-1)^{k-a}$) yields

$$0 = \sum_{l=1}^{s} (-1)^l \cdot p_{o([r+s-1]^{k-1}, m_l)} \cdot p_{o(J \setminus m_l)} - (-1)^s \cdot p_{o([r+s-2]^k)} \cdot p_{o([r+s-1]^{k-s}, m_1, \ldots, m_s)},$$

which, observing that $p_{o(J \setminus m_l)} = p_{o([r+s-2]^{k-(s-1)}, m_1, \ldots, \widehat{m_l}, \ldots, m_s)}$ and pulling the second term to the left hand side yields the desired Equation (15).

<u>Case 3b</u>. For the final case, assume that $1 \notin [r + s - 1]^{k-1} \cap [r + s - 2]^{k-(s-1)}$. We first observe what the linear order on J looks like. Notice that we have $r + s - 2 < r + s - 1 < \cdots < r + k - 2$. Since

$m_j \notin [r+s-2, r+k-2]$, the set J is ordered in one of the following three ways (16)–(18):

$$m_b < m_{b+1} < \cdots < m_s < r+s-2 < \cdots < r+k-2 < m_1 < m_2 < \cdots < m_{b-1}, \tag{16}$$

$$\text{for some } b \in \{2, \ldots, s\},$$

$$m_1 < m_2 < \cdots < m_s < r+s-2 < \cdots < r+k-2, \quad \text{or} \tag{17}$$

$$r+s-2 < \cdots < r+k-2 < m_1 < \cdots < m_s. \tag{18}$$

If it is ordered as in (17), we set $b = 1$, and if it is ordered as in (18), we set $b = s+1$. Then the Plücker relation on I and J reads as follows:

$$0 = \sum_{l=b}^{s} (-1)^{l-b} p_{o(\langle[r+s-1]^{k-1}),m_l} \cdot p_{o(J\backslash m_l)} + \sum_{l=0}^{k-s} (-1)^{s-b+1+l} \cdot p_{o([r+s-1]^{k-1}),r+s-2+l} \cdot p_{o(J\backslash\{r+s-2+l\})}$$

$$+ \sum_{l=1}^{b-1} (-1)^{k-b+1+l} \cdot p_{o([r+s-1]^{k-1}),m_l} \cdot p_{o(J\backslash m_l)}.$$

Notice again that $p_{o([r+s-1]^{k-1}),r+j} = 0$ for all $j \in [s-1, k-2]$, and therefore the corresponding terms in the Plücker relation vanish. Again, we need to get the remaining k-tuples in linear order. Observe that

$$p_{o([r+s-1]^{k-1}),m_l} = \begin{cases} p_{o([r+s-1]^{k-1},m_l)} & \text{if } 1 \leq l \leq b-1, \\ \operatorname{sgn}(\sigma) p_{o([r+s-1]^{k-1},m_l)} & \text{if } b \leq l \leq s, \end{cases}$$

and that $p_{o([r+s-1]^{k-1}),r+s-2} = \operatorname{sgn}(\sigma) p_{o([r+s-2]^k)}$, where $\sigma = (1, 2, \ldots, k) \in S_k$ with $\operatorname{sgn}(\sigma) = (-1)^{k-1}$. Therefore, if J is ordered as in (17) or (18), the Plücker relation on I and J can be rewritten as

$$0 = \sum_{l=b}^{s} (-1)^{l-b} \cdot (-1)^{k-1} p_{o([r+s-1]^{k-1},m_l)} \cdot p_{o(J\backslash m_l)} + (-1)^{s-b+1} \cdot (-1)^{k-1} p_{o([r+s-2]^k)} \cdot p_{o(J\backslash\{r+s-2\})}$$

$$+ \sum_{l=1}^{b-1} (-1)^{k-b+1+l} \cdot p_{o([r+s-1]^{k-1},m_l)} \cdot p_{o(J\backslash m_l)}.$$

Juggling the signs in a straight forward way, and pulling together the first and last sum we obtain

$$0 = (-1)^s \cdot (-1)^{k-b} \cdot p_{o([r+s-2]^k)} \cdot p_{o(J\backslash\{r+s-2\})} + (-1)^{k-b+1} \sum_{l=1}^{s} (-1)^l \cdot p_{o([r+s-1]^{k-1},m_l)} \cdot p_{o(J\backslash m_l)}.$$

Pulling the second term on the left hand side and multiplying by $(-1)^{k-b}$ yields as desired Equation (15), observing that $p_{o(J\backslash m_l)} = p_{o([r+s-2]^{k-(s-1)}, m_1,\ldots,\widehat{m_l},\ldots,m_s)}$. This concludes the proof. □

Appendix B: Counting friezes in type E_6
by Michael Cuntz and Pierre-Guy Plamondon

Friezes (also called "friezes of type A" and "SL_2-friezes") were introduced by H. Coxeter [1971] and later studied by Coxeter and J. Conway [1973a; 1973b]. It was observed by P. Caldero that the theory of cluster algebras of S. Fomin and A. Zelevinsky [2002a] allows for a far-reaching generalization of the original notion of frieze; this generalization was first studied in [Assem et al. 2010]. Recall that in

[Assem et al. 2010], a frieze (of Dynkin type) is a mesh frieze on an associated Auslander–Reiten quiver, see Remark 4.11. Since then, many generalizations and variations on the notion of friezes have been introduced, as can be seen in the survey paper [Morier-Genoud 2015].

It is known that, for a given non-Dynkin type, there are infinitely many friezes of that type. However, it follows from [Conway and Coxeter 1973a; 1973b] that friezes of Dynkin type A come in a finite number (given by Catalan numbers), and it was proved in [Fontaine and Plamondon 2016] that friezes of Dynkin type B, D and G also come in a finite (explicit) number (the result for type D_4 was also proved in [Morier-Genoud et al. 2012]). It was conjectured in [Propp 2020; Morier-Genoud et al. 2012; Fontaine and Plamondon 2016] that the number of friezes in type E_6 is 868; in [Fontaine and Plamondon 2016] the precise number of friezes in any Dynkin type is also conjectured.

In this appendix, we settle the case of type E_6, and obtain the result for type F_4 as a corollary.

Theorem B.1. *The number of friezes of type E_6 is exactly* 868.

Since Dynkin type F_4 is a folding of type E_6, it follows from the work of [Fontaine and Plamondon 2016] that we then have:

Corollary B.2. *The number of friezes of type F_4 is exactly* 112.

Our proof relies on a reduction to 2-friezes (we recall the definition below); our strategy is to show that the entries in a 2-frieze of height 3 are bounded.

We have attempted to apply the methods used in this appendix to types E_7 and E_8, without success.

2-friezes. We shall not be using the definition of a frieze from [Assem et al. 2010], but rather a slightly different notion, that of a 2-frieze as defined in [Morier-Genoud et al. 2012].

Definition B.3. A 2-*frieze of height h* is an array of positive integers $(a_{i,j})$, where

- $i \in \mathbb{Z}$ and $j \in \{0, 1, \ldots, h+1\}$;
- for all $i \in \mathbb{Z}$, we have that $a_{i,0} = a_{i,h+1} = 1$;
- for all $i \in \mathbb{Z}$ and all $j \in \{1, \ldots, h\}$, we have that $a_{i,j} = a_{i-1,j}a_{i+1,j} - a_{i,j-1}a_{i,j+1}$.

The reason we are interested in 2-friezes is the following result, which is a consequence of [Morier-Genoud et al. 2012, Proposition 5.4] and of the fact that the quiver depicted in [Morier-Genoud et al. 2012, (2.3)] is mutation-equivalent to a quiver of type E_6.

Theorem B.4 [Morier-Genoud et al. 2012]. *Entries in a 2-frieze of height* 3 *are entries in a frieze of type E_6 which determine the frieze. This defines an injection from the set of friezes of type E_6 to the set of 2-friezes of height* 3.

Our strategy to prove Theorem B.1 is thus to prove that the number of 2-friezes of height 3 is finite. We will do this by showing that there is a bound on the possible values appearing in a 2-frieze of height 3.

1	s	t	u	1
1	v	w	x	1
1	$\frac{v+w}{s}$	$\frac{vx+w}{t}$	$\frac{w+x}{u}$	1
1	$\frac{svx+sw+tv+tw}{stv}$	X	$\frac{tw+tx+uvx+uw}{tux}$	1
1	$\frac{suvx+suw+tuw+tvw+tvx}{tuvw}$	Y	$\frac{stw+suvx+suw+tvx+twx}{stwx}$	1
1	$\frac{sux+tw+tx+uw}{uwx}$	Z	$\frac{suv+sw+tv+tw}{svw}$	1
1	$\frac{t+u}{x}$	$\frac{su+t}{w}$	$\frac{s+t}{v}$	1
1	u	t	s	1
1	x	w	v	1

Figure 9. Entries for 2-friezes of height 3 with the first two rows described by s, t, u, v, w and x, and where

$$X = \frac{suvx+suw+tvw+tvx+tw^2+twx}{stuw},$$

$$Y = \frac{stuvx+stw^2+stwx+suvwx+suw^2+t^2vw+t^2vx+t^2w^2+t^2wx+tuvw+tuw^2}{stuvwx},$$

$$Z = \frac{stw+suvx+suw+t^2w+tuw+tvx}{tvwx}.$$

Two choices of initial variables for 2-friezes of height 3. If we fix the entries of the first two rows of a 2-frieze of height 3 to be s, t, u, v, w and x, then we get the expressions shown in Figure 9 for all its entries.

If, instead, we fix the six first entries of the leftmost nontrivial column, then we get the expressions shown in Figure 10.

Finiteness of the number of 2-friezes of height 3. We use the second choice of initial variables of the previous section. We can assume, without loss of generality, that the greatest entry in the first and third columns is u. Then

$$u \geq suw - sv - tw + 1$$
$$\geq suw - su - wu + 1 \quad \text{(since } u \geq t, v\text{)}$$
$$= u(s-1)(w-1) - u + 1.$$

Therefore

$$1 \geq (s-1)(w-1) - 1 + \frac{1}{u} > (s-1)(w-1) - 1.$$

1	s	$\frac{sux-s-tx+t}{tvx-tw-ux+1}$	B	1
1	t	$su-t$	$\frac{suw-sv+tv-tw-u+1}{tvx-tw-ux+1}$	1
1	u	$tv-u$	$suw-sv-tw+1$	1
1	v	$uw-v$	$tvx-tw-ux+1$	1
1	w	$vx-w$	$\frac{tvx-tw+uw-ux-v+1}{suw-sv-tw+1}$	1
1	x	$\frac{svw-sw+w-x}{suw-sv-tw+1}$	C	1
1	$\frac{sux-s-tx+1}{suw-sv-tw+1}$	A	$\frac{svx-sw-x+1}{tvx-tw-ux+1}$	1
1	B	$\frac{sux-s-tx+t}{tvx-tw-ux+1}$	s	1
1	$\frac{suw-sv+tv-tw-u+1}{tvx-tw-ux+1}$	$su-t$	t	1
1	$suv-sv-tw+1$	$tv-u$	u	1

Figure 10. Entries for 2-friezes of height 3 with the six first entries of the leftmost nontrivial column fixed, and where

$$A = \frac{stvx-stw+suwx-sux-svx+s-twx+tw+x-1}{stuvwx-stuw^2-stv^2x+stvw-su^2wx+suvx+suw-sv-t^2vwx+t^2w^2+tuwx+tvx-2tw-ux+1},$$

$$B = \frac{stuvx-stuw-sr^2x+suw+su-sv-t^2vx+t^2w+tux+tv-tw-t-u+1}{stuvwx-stuw^2-stv^2x+stvw-su^2wx+suvx+suw-sv-t^2vwx+t^2w^2+tuwx+tvx-2tw-ux+1},$$

$$C = \frac{suvwx-suw^2-sv-2x+svw-tvwx+tvx+tw^2-tw+uw-ux+vx-v-w+1}{stuvwx-stuw^2-stv^2x+stvw-su^2wx+suvx+suw-sv-t^2vwx+t^2w^2+tuwx+tvx-2tw-ux+1}.$$

Hence $(s-1)(w-1) < 2$. This implies that $s = 1$ or $w = 1$ or $s = w = 2$.

The case $s = 1$ or $w = 1$. In this case, the associated frieze of type E_6 contains a frieze of type D_5, and it is known [Fontaine and Plamondon 2016] that there are only 187 of these. Thus there is a finite number of cases where $s = 1$ or $w = 1$.

The case $s = w = 2$. Consider the following inequalities:

$$u \geq suw - sv - tw + 1 = 4u - 2t - 2v + 1$$

and $u \geq v$, which implies that $3u \geq 4u - 2t + 1$, so

$$u \leq 2t - 1.$$

But together with $u \geq tvx - tw - ux + 1 = tvx - 2t - ux + 1$ this yields

$$(2t - 1)(x + 1) \geq tvx - 2t + 1,$$

and hence

$$4t \geq tvx + 2 - 2tx + x = x + 2 + (v - 2)tx > (v - 2)tx.$$

Thus we obtain $(v - 2)x < 4$ which gives $v < 6$. For symmetry reasons, the same argument produces $t < 6$. But then $u \leq 9$ since $u \leq 2t - 1$. Hence we have reduced the problem to a finite number of cases. In fact, an easy computation shows that the only solution is

$$(s, t, u, v, w, x) = (2, 4, 5, 4, 2, 1),$$

in which case the (transposed) 2-frieze is:

```
1  1  1   1  1  1  1  1  1  1   1   1  1  1
1  2  4   5  4  2  1  1  2  4   5   4  2  1
1  2  6  11  6  2  1  1  2  6  11   6  2  1
1  2  4   5  4  2  1  1  2  4   5   4  2  1
1  1  1   1  1  1  1  1  1  1   1   1  1  1
```

Thus the number of 2-friezes of height 3 is finite, and we know a bound on the values appearing in such a 2-frieze. A computer check then allows us to show that there are only 868 such 2-friezes. Moreover, by [Morier-Genoud et al. 2012], the number of friezes of type E_6 is at most the number of 2-friezes of height 3. In fact, the two numbers are equal; this follows from Corollary 4.16, since 2-friezes are in bijection with SL$_3$-friezes [Morier-Genoud et al. 2012, Section 3]. Since we know from [Propp 2020; Morier-Genoud et al. 2012; Fontaine and Plamondon 2016] that this number is at least 868, we have thus proved that the number is exactly 868. This finishes the proof of Theorem B.1.

Acknowledgements

References

[Amiot 2009] C. Amiot, "Cluster categories for algebras of global dimension 2 and quivers with potential", *Ann. Inst. Fourier (Grenoble)* **59**:6 (2009), 2525–2590. MR Zbl

[Assem et al. 2010] I. Assem, C. Reutenauer, and D. Smith, "Friezes", *Adv. Math.* **225**:6 (2010), 3134–3165. MR Zbl

[Baur and Marsh 2009] K. Baur and B. R. Marsh, "Frieze patterns for punctured discs", *J. Algebraic Combin.* **30**:3 (2009), 349–379. MR Zbl

[Baur et al. 2016] K. Baur, M. J. Parsons, and M. Tschabold, "Infinite friezes", *European J. Combin.* **54** (2016), 220–237. MR Zbl

[Baur et al. 2020] K. Baur, D. Bogdanic, and A. Garcia Elsener, "Cluster categories from Grassmannians and root combinatorics", *Nagoya Math. J.* **240** (2020), 322–354. MR Zbl

[Bergeron and Reutenauer 2010] F. Bergeron and C. Reutenauer, "SL$_k$-tilings of the plane", *Illinois J. Math.* **54**:1 (2010), 263–300. MR Zbl

[Buan et al. 2006] A. B. Buan, B. Marsh, M. Reineke, I. Reiten, and G. Todorov, "Tilting theory and cluster combinatorics", *Adv. Math.* **204**:2 (2006), 572–618. MR Zbl

[Buchweitz 1986] R.-O. Buchweitz, "Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings", preprint, 1986, available at http://hdl.handle.net/1807/16682.

[Caldero and Chapoton 2006] P. Caldero and F. Chapoton, "Cluster algebras as Hall algebras of quiver representations", *Comment. Math. Helv.* **81**:3 (2006), 595–616. MR Zbl

[Cao and Li 2018] P. Cao and F. Li, "Positivity of denominator vectors of skew-symmetric cluster algebras", *J. Algebra* **515** (2018), 448–455. MR Zbl

[Conway and Coxeter 1973a] J. H. Conway and H. S. M. Coxeter, "Triangulated polygons and frieze patterns", *Math. Gaz.* **57**:400 (1973), 87–94. MR Zbl

[Conway and Coxeter 1973b] J. H. Conway and H. S. M. Coxeter, "Triangulated polygons and frieze patterns", *Math. Gaz.* **57**:401 (1973), 175–183. MR Zbl

[Cordes and Roselle 1972] C. M. Cordes and D. P. Roselle, "Generalized frieze patterns", *Duke Math. J.* **39** (1972), 637–648. MR Zbl

[Coxeter 1971] H. S. M. Coxeter, "Frieze patterns", *Acta Arith.* **18** (1971), 297–310. MR Zbl

[Cuntz 2017] M. Cuntz, "On wild frieze patterns", *Exp. Math.* **26**:3 (2017), 342–348. MR Zbl

[Demonet and Luo 2016] L. Demonet and X. Luo, "Ice quivers with potential associated with triangulations and Cohen–Macaulay modules over orders", *Trans. Amer. Math. Soc.* **368**:6 (2016), 4257–4293. MR Zbl

[Fomin and Zelevinsky 2002a] S. Fomin and A. Zelevinsky, "Cluster algebras, I: Foundations", *J. Amer. Math. Soc.* **15**:2 (2002), 497–529. MR Zbl

[Fomin and Zelevinsky 2002b] S. Fomin and A. Zelevinsky, "The Laurent phenomenon", *Adv. Appl. Math.* **28**:2 (2002), 119–144. MR Zbl

[Fomin and Zelevinsky 2003] S. Fomin and A. Zelevinsky, "Y-systems and generalized associahedra", *Ann. of Math.* (2) **158**:3 (2003), 977–1018. MR Zbl

[Fontaine and Plamondon 2008] B. Fontaine and P.-G. Plamondon, Lists of friezes of finite type, 2008, available at https://www.math.u-psud.fr/~plamondon/friezes.

[Fontaine and Plamondon 2016] B. Fontaine and P.-G. Plamondon, "Counting friezes in type D_n", *J. Algebraic Combin.* **44**:2 (2016), 433–445. MR Zbl

[Geiss et al. 2013] C. Geiss, B. Leclerc, and J. Schröer, "Factorial cluster algebras", *Doc. Math.* **18** (2013), 249–274. MR Zbl

[Gross et al. 2018] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, "Canonical bases for cluster algebras", *J. Amer. Math. Soc.* **31**:2 (2018), 497–608. MR Zbl

[Happel 1988] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, Lond. Math. Soc. Lect. Note Ser. **119**, Cambridge Univ. Press, 1988. MR Zbl

[Iyama and Yoshino 2008] O. Iyama and Y. Yoshino, "Mutation in triangulated categories and rigid Cohen–Macaulay modules", *Invent. Math.* **172**:1 (2008), 117–168. MR Zbl

[Jensen et al. 2016] B. T. Jensen, A. D. King, and X. Su, "A categorification of Grassmannian cluster algebras", *Proc. Lond. Math. Soc.* (3) **113**:2 (2016), 185–212. MR Zbl

[Lee and Schiffler 2015] K. Lee and R. Schiffler, "Positivity for cluster algebras", *Ann. of Math.* (2) **182**:1 (2015), 73–125. MR Zbl

[Marsh 2013] B. R. Marsh, *Lecture notes on cluster algebras*, Eur. Math. Soc., Zürich, 2013. MR Zbl

[Morier-Genoud 2012] S. Morier-Genoud, "Arithmetics of 2-friezes", *J. Algebraic Combin.* **36**:4 (2012), 515–539. MR Zbl

[Morier-Genoud 2015] S. Morier-Genoud, "Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics", *Bull. Lond. Math. Soc.* **47**:6 (2015), 895–938. MR Zbl

[Morier-Genoud et al. 2012] S. Morier-Genoud, V. Ovsienko, and S. Tabachnikov, "2-frieze patterns and the cluster structure of the space of polygons", *Ann. Inst. Fourier* (*Grenoble*) **62**:3 (2012), 937–987. MR Zbl

[Morier-Genoud et al. 2014] S. Morier-Genoud, V. Ovsienko, R. E. Schwartz, and S. Tabachnikov, "Linear difference equations, frieze patterns, and the combinatorial Gale transform", *Forum Math. Sigma* **2** (2014), art. id. e22. MR Zbl

[Musiker et al. 2011] G. Musiker, R. Schiffler, and L. Williams, "Positivity for cluster algebras from surfaces", *Adv. Math.* **227**:6 (2011), 2241–2308. MR Zbl

[Propp 2020] J. Propp, "The combinatorics of frieze patterns and Markoff numbers", *Integers* **20** (2020), art. id. A12. MR Zbl

[Scott 2006] J. Scott, "Grassmannians and cluster algebras", *Proc. Lond. Math. Soc.* (3) **92**:2 (2006), 345–380. MR Zbl

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