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SEMISIMPLIFICATION FOR SUBGROUPS OF REDUCTIVE ALGEBRAIC GROUPS

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Abstract. Let G be a reductive algebraic group—possibly non-connected—over a field k and let H be a subgroup of G. If $G = \mathrm{GL}_n$ then there is a degeneration process for obtaining from H a completely reducible subgroup H' of G; one takes a limit of H along a cocharacter of G in an appropriate sense. We generalise this idea to arbitrary reductive G using the notion of G-complete reducibility and results from geometric invariant theory over non-algebraically closed fields due to the authors and Herpel. Our construction produces a G-completely reducible subgroup H' of G, unique up to $G(k)$-conjugacy, which we call a *k-semisimplification* of H. This gives a single unifying construction which extends various special cases in the literature (in particular, it agrees with the usual notion for $G = \mathrm{GL}_n$ and with Serre's "G-analogue" of semisimplification for subgroups of $G(k)$ from [19]). We also show that under some extra hypotheses, one can pick H' in a more canonical way using the Tits Centre Conjecture for spherical buildings and/or the theory of optimal destabilising cocharacters introduced by Hesselink, Kempf and Rousseau.

1. Introduction

The aim of this paper is to present a construction of the *semisimplification* of a subgroup H of a (possibly non-connected) reductive linear algebraic group G over an arbitrary field k. This construction unifies and generalizes many concepts already in the literature within a single framework. For example, the semisimplification of a module for a group is a well-known construction in representation theory, corresponding in our case to the situation where $H \subseteq \mathrm{GL}_n(k)$. Building on this idea, for G a connected reductive linear algebraic group over a field k and H a subgroup of $G(k)$, Serre introduced the concept of a "G-analogue" of semisimplification from representation theory in [19, §3.2.4]. This notion is also used for representations of various kinds of algebras, e.g., see [12], [8], [16], [23], and [24]. It is also an ingredient in work of Lawrence-Sawin on the Shafarevich Conjecture for abelian varieties [13] and work of Lawrence-Venkatesh on Mordell's Conjecture [14], which involve Galois representations taking values in possibly non-connected reductive p-adic groups.

We begin by recalling how the most basic case works. Let $n \in \mathbb{N}$ and let H be a subgroup of $\mathrm{GL}_n(k)$. There is an H-module filtration of k^n such that the successive quotients are irreducible, by the Jordan-Hölder Theorem. In terms of matrices, this implies that, by changing basis if necessary, we may assume that H is in upper block-triangular form, with the action of H on each quotient being represented by the corresponding block on the diagonal. Letting H' be the subgroup of $\mathrm{GL}_n(k)$ consisting of the block diagonal matrices obtained by taking each element of H and replacing the entries above the block diagonal with 0s, we obtain a subgroup which acts semisimply on k^n—that is, H' is completely reducible. Since

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this construction is independent of the choice of basis up to $\mathrm{GL}_n(k)$-conjugacy, again by the Jordan-Hölder Theorem, it is therefore reasonable to call H' the *semisimplification* of H.

We now explain some of the ingredients of our construction in the case that k is algebraically closed, which removes some technicalities. Recall [2], [19] that if G is connected and H is a subgroup of G then H is *G-completely reducible* (G-cr for short) if for any parabolic subgroup P of G such that P contains H, there is a Levi subgroup L of P such that L contains H. If $G = \mathrm{GL}_n$ then H is G-cr if and only if k^n is completely reducible as an H-module; this follows from the usual characterisation of parabolic subgroups of GL_n as stabilizers of flags of subspaces. We make the same definition for arbitrary reductive G, replacing parabolic subgroups and Levi subgroups with R-parabolic subgroups and R-Levi subgroups instead (see Section 2 for details).

To perform our construction, we apply a characterisation of G-complete reducibility in terms of geometric invariant theory (GIT). We see this idea already in our original example: we can view H' as a degeneration of H in the following sense. Let the sizes of the blocks down the diagonal be $n_1, \ldots, n_r$, and define a cocharacter $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ by

$$\lambda(a) = \mathrm{diag}(a^r, \ldots, a^r, \ldots, a^1, \ldots, a^1), \text{ with } n_i \text{ occurrences of } a^{r-i+1}, 1 \leq i \leq r.$$

For each $a \in k^*$, define $H_a = \lambda(a)H\lambda(a)^{-1}$ for $a \in k^*$. Then $H' = \lim_{a\to 0} H_a$ in an appropriate sense.

Our definition of k-semisimplification (Definition 4.1) for arbitrary k is new, generalizes the one given by Serre in [19, §3.2.4], and is closely related to the definition given in [6] using optimal destabilising cocharacters; the two notions agree whenever the latter makes sense (cf. also [15, Sec. 4] for the algebraically closed case). We prove that the k-semisimplification of a subgroup H of G is unique up to conjugacy (Theorem 4.5), generalizing [19, Prop. 3.3(b)]. In Theorem 5.4 we show that a normal subgroup of a G-completely reducible subgroup H is G-completely reducible and that the process of k-semisimplification behaves well under passing to normal subgroups of H, if k is perfect or G is connected. The proof rests on deep results from the theory of spherical buildings and the Hesselink-Kempf-Rousseau theory of optimal destabilising cocharacters. We give a short and self-contained exposition, bringing together some results (such as Corollary 3.5) that follow from previous work but are not easily extracted from earlier papers.

2. Cocharacter-closed orbits

Following [7] and our earlier work [6], [1], we regard an affine variety over a field k as a variety X over the algebraic closure $\overline{k}$ together with a choice of k-structure. We denote the separable closure of k by k_s. We write $X(k)$ for the set of k-points of X and $X(\overline{k})$ (or just X) for the set of $\overline{k}$-points of X. By a subvariety of X we mean a closed $\overline{k}$-subvariety of X; a k-subvariety is a subvariety that is defined over k. We denote by M_n the associative algebra of $n\times n$ matrices over k. Below G denotes a possibly non-connected reductive linear algebraic group over k. By a subgroup of G we mean a closed $\overline{k}$-subgroup and by a k-subgroup we mean a subgroup that is defined over k. (We note here that much of what follows works for non-closed subgroups—most of the important conditions hold for H if and only if they hold for the Zariski closure $\overline{H}$; the details are left to the reader.) By G^0 we denote the identity component of G, and likewise for subgroups of G.

We define $Y_k(G)$ to be the set of k-defined cocharacters of G and $Y(G) := Y_{\overline{k}}(G)$ to be the set of all cocharacters of G.

Let H be a subgroup of G. Even if H is k-defined, the (set-theoretic) centralizer $C_G(H)$ need not be k-defined in general. It is useful to have criteria to ensure that $C_G(H)$ is k-defined (see Proposition 3.4 and Section 5). For instance, if k is perfect and H is k-defined then $C_G(H)$ is k-defined. We say that H is *separable* if the scheme-theoretic centralizer $\mathscr{C}_G(H)$ is smooth [2, Def. 3.27]; for instance, any subgroup of GL_n is separable [2, Ex. 3.28] (see [5] for more examples of separable subgroups). If H is k-defined and separable then $C_G(H)$ is k-defined (see [1, Prop. 7.4]).

Next we recall some basic notation and facts concerning parabolic subgroups in (non-connected) reductive groups G from [2, §6] and [6]. Given $\lambda \in Y(G)$, we define

$$P_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$$

and $L_\lambda = C_G(\mathrm{Im}(\lambda))$ (for the definition of a limit, see [20, Sec. 3.2.13]). We call P_λ an *R-parabolic subgroup* of G and L_λ an *R-Levi subgroup* of P_λ; they are subgroups of G. We have $P_\lambda = L_\lambda = G$ if $\mathrm{Im}(\lambda)$ is contained in the centre of G. For ease of reference, we record without proof some basic facts about these subgroups.

Lemma 2.1. (i) *If P is a k-defined R-parabolic subgroup then $R_u(P)$ is k-defined.*
(ii) *If P is a parabolic subgroup of G^0 then the normalizer $N_G(P)$ is an R-parabolic subgroup of G, and $N_G(P)$ is k-defined if P is.*

If G is connected then every pair (P, L) consisting of a parabolic k-subgroup P of G and a Levi k-subgroup L of P is of the form $(P, L) = (P_\lambda, L_\lambda)$ for some $\lambda \in Y_k(G)$, and vice versa [20, Lem. 15.1.2(ii)]. In general, if $\lambda \in Y_k(G)$ then P_λ and L_λ are k-defined [6, Lem. 2.5], but the converse is not so straightforward. If P is an R-parabolic k-subgroup and L is an R-Levi k-subgroup of P then for any maximal k-torus T of L, there exists $\lambda \in Y_{k_s}(T)$ such that $P = P_\lambda$ and $L = L_\lambda$. However, it is possible that P is a k-defined R-parabolic subgroup and yet there does not exist any $\mu \in Y_k(G)$ such that $P = P_\mu$, and similarly for R-Levi subgroups—see [6, Rem. 2.4]. This complicates some of the arguments below.

Lemma 2.2. *Let P be an R-parabolic subgroup of G and L an R-Levi subgroup of P.*
(i) *We have $P \cong L \ltimes R_u(P)$, and this is a k-isomorphism if P and L are k-defined.*
(ii) *Any two R-Levi k-subgroups of an R-parabolic k-subgroup P are $R_u(P)(k)$-conjugate.*

We denote the canonical projection from P to L by c_L; this is k-defined if P and L are. If we are given $\lambda \in Y(G)$ such that $P = P_\lambda$ and $L = L_\lambda$ then we often write c_λ instead of c_L. We have $c_\lambda(g) = \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1}$ for $g \in P_\lambda$; the kernel of c_λ is the unipotent radical $R_u(P_\lambda)$ and the set of fixed points of c_λ is L_λ.

Let $m \in \mathbb{N}$. Below we consider the action of G on G^m by simultaneous conjugation: $g \cdot (g_1, \ldots, g_m) = (gg_1g^{-1}, \ldots, gg_mg^{-1})$. Given $\lambda \in Y(G)$, we have a map $P_\lambda^m \to L_\lambda^m$ given by $\mathbf{g} \mapsto \lim_{a\to 0} \lambda(a) \cdot \mathbf{g}$; we abuse notation slightly and also call this map c_λ. For any $\mathbf{g} \in P_\lambda^m$, there exists an R-Levi k-subgroup L of P_λ with $\mathbf{g} \in L^n$ if and only if $c_\lambda(\mathbf{g}) = u \cdot \mathbf{g}$ for some $u \in R_u(P_\lambda)(k)$.

Our main tool from GIT is the notion of cocharacter-closure, introduced in [6] and [1].

Definition 2.3. Let X be an affine G-variety and let $x \in X$ (we do not require x to be a k-point). We say that the orbit $G(k) \cdot x$ is *cocharacter-closed over k* if for all $\lambda \in Y_k(G)$ such that $x' := \lim_{a\to 0} \lambda(a)\cdot x$ exists, x' belongs to $G(k)\cdot x$. If $k = \overline{k}$ then it follows from the Hilbert-Mumford Theorem that $G(k)\cdot x$ is cocharacter-closed over k if and only if $G(k)\cdot x$ is closed [11, Thm. 1.4]. If $\mathcal{O}$ is a $G(k)$-orbit in X then we say that $\mathcal{O}$ is *accessible from x over k* if there exists $\lambda \in Y_k(G)$ such that $x' := \lim_{a\to 0} \lambda(a)\cdot x$ belongs to $\mathcal{O}$.

Example 2.4. If $X = G^m$, $\lambda \in Y_k(G)$ and $\mathbf{g} \in P_\lambda^m$ then $G(k)\cdot c_\lambda(\mathbf{g})$ is accessible from $\mathbf{g}$ over k.

The following result is [1, Thm. 1.3].

Theorem 2.5 (Rational Hilbert-Mumford Theorem)**.** *Let G, X, x be as above. Then there is a unique $G(k)$-orbit $\mathcal{O}$ such that $\mathcal{O}$ is cocharacter-closed over k and accessible from x over k.*

3. G-complete reducibility

Definition 3.1. Let H be a subgroup of G. We say that H is *G-completely reducible over k* (G-cr over k) if for any R-parabolic k-subgroup P of G such that P contains H, there is an R-Levi k-subgroup L of P such that L contains H. We say that H is *G-irreducible over k* (G-ir over k) if H is not contained in any proper R-parabolic k-subgroup of G at all.

Remark 3.2. We say that H is G-cr if H is G-cr over $\overline{k}$—cf. Section 1. More generally, if k'/k is an algebraic field extension then we may regard G as a k'-group and it makes sense to ask whether H is G-cr over k'.

For more on G-complete reducibility, see [18], [19], [2].

Note that the definitions make sense even if H is not k-defined. It is immediate that G-irreducibility over k implies G-complete reducibility over k. We have $P_{g\cdot\lambda} = gP_\lambda g^{-1}$ and $L_{g\cdot\lambda} = gL_\lambda g^{-1}$ for any $\lambda \in Y(G)$ and any $g \in G$ (see, e.g., [2, §6]). It follows that if H is G-cr over k (resp., G-ir over k) then so is any $G(k)$-conjugate of H. More generally, one can show that if H is G-cr over k (resp., G-ir over k) then so is $\phi(H)$, for any k-defined automorphism ϕ of G. If $k = \overline{k}$ and H is G-cr then H is reductive [19, Prop. 4.1], [2, §2.4, §6.2]. It follows from Proposition 3.4 below that if H is k-defined, k is perfect and H is G-cr over k then H is reductive. We see below (Corollary 3.5) that the converse holds in characteristic 0. On the other hand, the converse is false in general, as is shown by the example in [22, Proof of Prop. 1.10].

We now explain the link between G-complete reducibility and GIT. Fix a k-embedding $\iota\colon G \to \mathrm{GL}_n$ for some $n \in \mathbb{N}$. Let H be a subgroup of G. Let $m \in \mathbb{N}$ and let $\mathbf{h} = (h_1, \ldots, h_m) \in H^m$. We call $\mathbf{h}$ a *generic tuple for H with respect to ι* if $h_1, \ldots, h_m$ generate the subalgebra of M_n generated by H [6, Def. 5.4]. Note that we don't insist that $\mathbf{h}$ is a k-point. Our constructions below do not depend on the choice of ι, so we suppress the words "with respect to ι". It is immediate that if $\mathbf{h} \in H^m$ is a generic tuple for H and $g \in G$ then $g\cdot\mathbf{h}$ is a generic tuple for gHg^{-1}.

Theorem 3.3 ([1, Thm. 9.3])**.** *Let H be a subgroup of G and let $\mathbf{h} \in H^m$ be a generic tuple for H. Then H is G-completely reducible over k if and only if $G(k)\cdot\mathbf{h}$ is cocharacter-closed over k.*

Using this result one can derive many results on G-complete reducibility: for instance, see [2] for the algebraically closed case and [6], [1] for arbitrary k. Note that if $\mathbf{h} \in H^m$ is a generic tuple for H then the centralizer $C_G(H)$ coincides with the stabilizer $G_{\mathbf{h}}$.

Proposition 3.4. *Let H be a k-subgroup of G. Suppose k is perfect. Then H is G-completely reducible over k if and only if H is G-completely reducible.*

Proof. If k is perfect then $\overline{k}/k$ is separable and $C_G(H)$ is k-defined. The result now follows from [1, Cor. 9.7(i)]. ☐

Corollary 3.5. *Suppose* $\mathrm{char}(k) = 0$*. Let H be a k-subgroup of G. Then H is G-completely reducible over k if and only if H is reductive.*

Proof. If $k = \overline{k}$ then this is well known (see [19, Prop. 4.2], [2, §2.2, §6.3], for example). The result for arbitrary k now follows from Proposition 3.4. ☐

Recall that if S is a k-split torus of G, then $C_G(S)$ is an R-Levi k-subgroup of G [1, Lem. 2.5]. Part (i) of the next result gives the converse, and part (ii) strengthens [1, Cor. 9.7(ii)]: we do not need the hypotheses that H and $C_G(H)$ are k-defined. See also [19, Prop. 3.2].

Proposition 3.6. *Let L be an R-Levi k-subgroup of G and let H be a subgroup of L.*

(a) *There exists a k-split torus S in G such that $L = C_G(S)$.*
(b) *H is G-completely reducible over k if and only if H is L-completely reducible over k.*

Proof. (a). We can choose $\lambda \in Y_{k_s}(G)$ such that $L = C_G(\mathrm{Im}(\lambda))$. Let $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_r \in Y_{k_s}(G)$ be the $\mathrm{Gal}(k_s/k)$-conjugates of λ and let S be the subtorus of $Z(L)^0$ generated by the subtori $\mathrm{Im}(\lambda_i)$. Then S is k-defined and $L = C_G(S)$. The product map $\lambda_1 \times \cdots \times \lambda_r$ gives an epimorphism from $\overline{k}^* \times \cdots \times \overline{k}^*$ onto S. But a quotient of a split k-torus is k-split [7, III.8.4 Cor.], so S is split.

(b). Given (a), the result now follows from Theorem 3.3 together with [1, Thm. 5.4(ii)]. ☐

We finish the section with some results involving non-connected reductive groups which are needed in the sequel. Note that if Q is an R-parabolic k-subgroup of G and M is an R-Levi k-subgroup of Q then Q^0 is a parabolic k-subgroup of G^0 and M^0 is a Levi k-subgroup of Q^0; see [2, Sec. 6].

Lemma 3.7. *Let P be an R-parabolic subgroup of G and let T be a maximal torus of P. Then there is a unique R-Levi subgroup L of P such that $T \subseteq L$. If P and T are k-defined then L is k-defined.*

Proof. The first assertion is [2, Cor. 6.5]. For the second, suppose P and T are k-defined. Then the unique R-Levi subgroup L of P containing T must be Galois-stable and hence k-defined also. ☐

Lemma 3.8. *(a) Let Q be an R-parabolic k-subgroup of G and set $P = Q^0$. Then the R-Levi k-subgroups of Q are precisely the subgroups of the form $N_Q(L)$ for L a Levi k-subgroup of P.*

(b) Let Q, P be as in (a) and let H be a subgroup of P. Then H is contained in an R-Levi k-subgroup of Q if and only if H is contained in a Levi k-subgroup of P. Moreover, if L is a Levi k-subgroup of P then $c_{N_Q(L)}(H)$ is $N_Q(L)$-completely reducible over k if and only if $c_L(H)$ is L-completely reducible over k.

(c) Let H be a subgroup of G^0. Then H is G-completely reducible over k if and only if H is G^0-completely reducible over k.

Proof. (a) As observed above, if M is an R-Levi subgroup of Q then M^0 is a Levi subgroup of P, and $N_Q(M^0)^0 = N_P(M^0)^0 = M^0$. Let L be a Levi subgroup of P and let T be a maximal torus of L. By Lemma 3.7 there is a unique R-Levi subgroup M of Q such that $T \subseteq M$. The Levi subgroups M^0 and L of P both contain T, so by Lemma 3.7 they are equal; in particular, M normalizes L. Now $N_Q(T)$ normalizes L by Lemma 3.7, so $N_Q(L)$ meets every component of Q. Since $Q = M \ltimes R_u(Q)$, M also meets every component of Q. It follows that $M = N_Q(L)$. Finally, L contains a maximal k-torus of P if and only if $N_Q(L)$ does, so L is k-defined if and only if $N_Q(L)$ is, by Lemma 3.7.

(b) The first assertion follows immediately from (a), and part (c) now follows. For the second assertion of (b), note that the restriction of $c_{N_Q(L)}(H)$ to P is c_L; the desired result now follows from part (c) applied to the reductive k-group $N_Q(L)$. ☐

4. k-semisimplification

Now we come to our main definition.

Definition 4.1. Let H be a subgroup of G. We say that a subgroup H' of G is a *k-semisimplification of H (for G)* if there exist an R-parabolic k-subgroup P of G and an R-Levi k-subgroup L of P such that $H \subseteq P$, $H' = c_L(H)$ and H' is G-completely reducible (or equivalently by Proposition 3.6(ii), L-completely reducible) over k. We say *the pair* (P, L) *yields* H'.

Remarks 4.2.
(a) Let H be a subgroup of G. If H is G-cr over k then clearly H is a k-semisimplification of itself, yielded by the pair (G, G).
(b) Suppose (P, L) yields a k-semisimplification H' of H. Let L_1 be another R-Levi k-subgroup of P. Then $L_1 = uLu^{-1}$ for some $u \in R_u(P)(k)$, so $c_{L_1}(H) = uc_L(H)u^{-1}$. Hence (P, L_1) also yields a k-semisimplification of H. We say that P *yields a k-semisimplification of H*.
(c) It is straightforward to check that if ϕ is an automorphism of G (as a k-group), H is a subgroup of G and (P, L) yields a k-semisimplification H' of H then $\phi(H')$ is a k-semisimplification of $\phi(H)$, yielded by $(\phi(P), \phi(L))$.
(d) For G connected and H a subgroup of $G(k)$, Definition 4.1 recovers Serre's "G-analogue" of a semisimplification from [19, §3.2.4]. For $k = \overline{k}$, Definition 4.1 generalizes the definition of $\mathcal{D}(H)$ following [15, Lem. 4.1].

Remark 4.3. Let $\mathbf{h} = (h_1, \ldots, h_m) \in H^m$ be a generic tuple for H. Note that c_λ extends in the obvious way to a homomorphism from a parabolic subalgebra $\mathcal{P}_\lambda$ of M_n onto a Levi subalgebra $\mathcal{L}_\lambda$ of $\mathcal{P}_\lambda$, and $\mathcal{P}_\lambda$ contains the subalgebra $\mathcal{A}$ generated by H. Since the elements h_i generate $\mathcal{A}$, the elements $c_\lambda(h_i)$ generate $c_\lambda(\mathcal{A})$. But $c_\lambda(\mathcal{A})$ is the subalgebra of $\mathcal{L}_\lambda$ generated by $c_\lambda(H)$, so we deduce that $c_\lambda(\mathbf{h}) = (c_\lambda(h_1), \ldots, c_\lambda(h_m))$ is a generic tuple for $c_\lambda(H)$. Hence by Theorem 3.3, $c_\lambda(H)$ is a k-semisimplification of H if and only if $G(k)\cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k. It follows from Theorem 2.5 that H admits at least one k-semisimplification: for we can choose $\lambda \in Y_k(G)$ such that $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k, so $c_\lambda(H)$ is a k-semisimplification of H, yielded by (P_λ, L_λ).

Lemma 4.4. *Suppose that H' is a k-semisimplification of H. Then there is $\lambda \in Y_k(G)$ such that H' is yielded by the pair (P_λ, L_λ).*

Proof. Suppose H' is yielded by the pair (P, L). By the discussion in Section 2, there exist a maximal k-torus T of L and $\mu \in Y_{k_s}(T)$ such that $P = P_\mu$ and $L = L_\mu$. Choose a finite Galois extension k'/k such that T splits over k', and let $\lambda = \sum_{\gamma \in \mathrm{Gal}(k'/k)} \gamma \cdot \mu \in Y_k(T)$. One checks easily that $H \subseteq P_\lambda$ and $c_\lambda|_H = c_\mu|_H$ (cf. the proof of [6, Lem. 2.5(ii)]). Hence (P_λ, L_λ) also yields H'. □

Here is our main result, which was proved in the special case $k = \overline{k}$ in [6, Prop. 5.14(i)], cf. [19, Prop. 3.3(b)]. The uniqueness asserted in Theorem 4.5 is akin to the theorem of Jordan–Hölder.

Theorem 4.5. *Let H be a subgroup of G. Then any two k-semisimplifications of H are $G(k)$-conjugate.*

Proof. Let H_1, H_2 be k-semisimplifications of H. By Lemma 4.4, there exist $\lambda_1, \lambda_2 \in Y_k(G)$ such that $(P_{\lambda_1}, L_{\lambda_1})$ realizes H_1 and $(P_{\lambda_2}, L_{\lambda_2})$ realizes H_2. Let $\mathbf{h} \in H^m$ be a generic tuple for H. Then $c_{\lambda_i}(\mathbf{h})$ is a generic tuple for H_i for $i = 1, 2$, and each orbit $G(k) \cdot c_{\lambda_i}(\mathbf{h})$ is cocharacter-closed over k and accessible from $\mathbf{h}$ over k (Example 2.4). It follows from the uniqueness result in Theorem 2.5 that the closed subset $C_{\mathbf{h}} := \{g \in G \mid g \cdot c_{\lambda_1}(\mathbf{h}) = c_{\lambda_2}(\mathbf{h})\}$ contains a k-point.

Pick $g \in C_{\mathbf{h}}$. If $H_2 = gH_1g^{-1}$ then we are done. Otherwise there exists $h \in H$ such that $gc_{\lambda_1}(h)g^{-1} \notin H_2$ or $g^{-1}c_{\lambda_2}(h)g \notin H_1$. Without loss assume the former. We can repeat the above argument, replacing $\mathbf{h}$ with the generic tuple $\mathbf{h}' := (\mathbf{h}, h) \in H^{m+1}$; note that $C_{\mathbf{h}'}$ is properly contained in $C_{\mathbf{h}}$. The result now follows by a descending chain condition argument. □

Definition 4.6. We define $\mathcal{D}_k(H)$ to be the set of $G(k)$-conjugates of any k-semisimplification of H (cf. the discussion preceding [15, Thm. 1.4]). This is well-defined by Theorem 4.5.

Example 4.7. Let H be a subgroup of G. As noted in Remark 4.2(a), if H is G-cr over k then H is a k-semisimplification of itself, yielded by the pair (G, G). If H is a G-ir subgroup of G, then H is the only k-semisimplification of H: this shows that not every element of $\mathcal{D}_k(H)$ need be a k-semisimplification of H. In a similar vein, if P and Q are arbitrary R-parabolic k-subgroups of G and $Q \supseteq P$ then it is easily seen that Q yields a k-semisimplification of P if and only if $P^0 = Q^0$.

Example 4.8. Let H be a subgroup of G and let P be minimal among the R-parabolic k-subgroups that contain H. Let L be an R-Levi k-subgroup of P. We claim that $c_L(H)$ is L-ir over k (cf. [19, Prop. 3.3(a)] and [2, Sec. 3]); it then follows from Proposition 3.6(ii) that $c_L(H)$ is a k-semisimplification of H. Suppose $c_L(H)$ is not L-ir: say, $c_L(H) \subseteq Q$, where Q is a proper R-parabolic k-subgroup of L. There exist a maximal k-torus T of Q and cocharacters $\lambda, \mu \in Y_{k_s}(T)$ such that $P = P_\lambda$, $L = L_\lambda$ and $Q = P_\mu$. Now $H \subseteq QR_u(P) \subsetneq P$, and clearly $QR_u(P)$ is k-defined. But it is easily checked that $QR_u(P) = P_{m\lambda+\mu}$ for suitably large $m \in \mathbb{N}$ (cf. [2, Lem. 6.2(i)]), so $QR_u(P)$ is an R-parabolic k-subgroup of G, contradicting the minimality of P. Conversely, if P is an R-parabolic k-subgroup with R-Levi k-subgroup L such that $P \supseteq H$ and $c_L(H)$ is L-ir over k then a similar argument shows that P is minimal among the R-parabolic k-subgroups containing H. This proves the claim.

In particular, let G, H, λ and H' be as in the GL_n example in Section 1. Let $P = P_\lambda$ be the parabolic subgroup of block upper triangular matrices with blocks of size $n_1, \ldots, n_r$ down the leading diagonal. Let $L = L_\lambda$ be the subgroup of block diagonal matrices with blocks of size $n_1, \dots, n_r$ down the leading diagonal. Since each $n_i \times n_i$ block yields an irreducible representation of $H' := c_\lambda(H)$, H' is L-ir over k, so P is minimal among the R-parabolic k-subgroups of G containing H; hence H' is the k-semisimplification of H yielded by (P, L).

Example 4.9. Suppose $\mathrm{char}(k) = 0$. Let H be a k-subgroup of G and let P be an R-parabolic subgroup of G with R-Levi subgroup L such that $P \supseteq H$. Then Corollary 3.5 implies that $c_L(H)$ is a k-semisimplification of H if and only if $R_u(H) \subseteq R_u(P)$.

Remark 4.10. Given a reductive k-group G and a subgroup H of G, we may (as in Remark 3.2) regard G as a $\overline{k}$-group by forgetting the k-structure, so it makes sense to consider the semisimplification (i.e., the $\overline{k}$-semisimplification) of H. The reader is warned that it can happen that H is G-cr over k but not G-cr, or vice versa (see [2, Ex. 5.11] and [5, Ex. 7.22]), so there is no direct relation between the notions of k-semisimplification and semisimplification.

5. Optimality and normal subgroups

In Example 4.7 we observed that not every element of $\mathcal{D}_k(H)$ need be a k-semisimplification of H. On the other hand, it can happen that H is contained in many different R-parabolic subgroups of G, and there may exist many conjugate, but different, k-semisimplifications. We now recall two constructions that give under some extra hypotheses a more canonical choice of R-parabolic subgroup yielding a k-semisimplification. They apply in particular when $G = \mathrm{GL}_n$ (see Example 5.6); this does not seem to be well known even when $k = \overline{k}$.

First construction: Suppose G is connected, H is a subgroup of G and H is not G-cr over k. We use the theory of spherical buildings (see [18], [19]) and the argument of [3, Proof of Thm. 1.1]. Recall that the spherical building $\Delta_k(G)$ of G is a simplicial complex whose simplices are the parabolic k-subgroups of G, ordered by reverse inclusion (the proper k-parabolic subgroups correspond to the non-empty simplices). The apartments of $\Delta_k(G)$ are the sets of all k-parabolic subgroups of G that contain a fixed maximal split k-torus S of G. The set Σ of parabolic k-subgroups P of G such that $P \supseteq H$ is a convex subcomplex of $\Delta_k(G)$, and Σ is not completely reducible in the sense of [19, §2.2] because H is not G-cr over k (see [19, §3.2.1]). By the Tits Centre Conjecture—see, e.g., [4, §2.6], and [19, §2.4] and the references therein—Σ has a so-called "centre": a proper parabolic k-subgroup $P_c \in \Sigma$ such that P_c is fixed by any building automorphism of $\Delta_k(G)$ that stabilizes Σ. In particular, P_c is stabilized by any k-automorphism of G that stabilizes H.

Lemma 5.1. *Let G, H and Σ be as above. Let P_c be a centre for Σ such that P_c is not properly contained in any other centre for Σ. Then P_c yields a k-semisimplification of H.*

Proof. Let Λ be the set of k-parabolic subgroups Q of G such that $Q \subseteq P_c$. Fix a Levi k-subgroup L of P_c. We have an inclusion-preserving bijection ψ from Λ to $\Delta_k(L)$ given by $Q \mapsto Q \cap L$, with inverse given by $R \mapsto RR_u(P_c)$. Let Σ_L be the subset of $\Delta_k(L)$ consisting of all the k-parabolic subgroups of L that contain $c_L(H)$. It is clear that $\psi(\Sigma \cap \Lambda) = \Sigma_L$. If ϕ is a building automorphism of $\Delta_k(G)$ that fixes P_c then ϕ stabilizes Λ, and we get an automorphism ϕ_L of $\Delta_k(L)$ (as a simplicial complex) given by $\phi_L(Q \cap L) = \phi(Q) \cap L$; moreover, if ϕ stabilizes Σ then ϕ_L stabilizes Σ_L.

We claim that ϕ_L is a building automorphism of $\Delta_k(L)$. It is enough to show that ϕ_L maps apartments to apartments. Let S be a maximal split k-torus of L (and hence of G).

Since ϕ is a building automorphism, there is a maximal split k-torus S' of G such that for every k-parabolic subgroup Q of G that contains S, $\phi(Q)$ contains S'. In particular, $S' \subseteq P_c$ since $\phi(P_c) = P_c$. By Lemma 3.7 there is a k-Levi subgroup L' of P_c such that $S' \subseteq L'$. By Lemma 2.2(ii) there exists $u \in R_u(P_c)(k)$ such that $uS'u^{-1} \subseteq L$. Let $R \in \Delta_k(L)$ such that $S \subseteq R$: say, $R = Q \cap L$ for $Q \in \Lambda$. Then $S' \subseteq \phi(Q)$. Since $\phi(Q) \subseteq P_c$, $R_u(\phi(Q))$ contains $R_u(P_c)$, so $uS'u^{-1} \subseteq \phi(Q)$. Hence $uS'u^{-1} \subseteq \phi(Q) \cap L = \phi_L(R)$. This proves the claim.

Now suppose P_c does not yield a k-semisimplification of H. Then $c_L(H)$ is not L-cr over k. By the discussion before the lemma, Σ_L has a centre $R \subsetneq L$. We have $R = Q \cap L$ for some $Q \in \Lambda$ with $Q \subsetneq P_c$. But the results in the previous paragraph imply that Q is a centre for Σ, contradicting the minimality of P_c. ☐

Second construction: We allow G to be non-connected again. Suppose the following property holds for a subgroup H of G:

($*$) there exists an R-parabolic k-subgroup P of G such that $H \subseteq P$ but H is not contained in any R-Levi subgroup—that is, any R-Levi $\overline{k}$-subgroup—of P.

This hypothesis implies in particular that H is not G-cr over k. The construction in [6, Sec. 5.2] then yields a canonical so-called 'optimal destabilising' R-parabolic k-subgroup $P_{\rm opt}$ of G such that $H \subseteq P_{\rm opt}$ but H is not contained in any R-Levi subgroup of $P_{\rm opt}$. If k is perfect then $P_{\rm opt}$ yields both a $\overline{k}$-semisimplification of H and a k-semisimplification of H by [11, Thm. 4.2], but both can fail for general k. Moreover, $P_{\rm opt}$ is stabilized by any k-automorphism of G that stabilizes H; in particular, if M is a k-subgroup of G that normalizes H then $M(k)$ normalizes $P_{\rm opt}$. See [6, Thm. 5.16] for details.

This construction rests on the notion of an "optimal destabilising cocharacter" due to work of Hesselink [10], Kempf [11] and Rousseau [17]. Roughly speaking, the idea is as follows. Take a generic tuple $\mathbf{h} \in H^m$ for H. Choose $\mathbf{g} \in G^m$ such that $G(k) \cdot \mathbf{g}$ is accessible from $\mathbf{h}$ over k and $G(k) \cdot \mathbf{g}$ is cocharacter-closed over k. Set $\mathcal{O}(\mathbf{h}) = G(\overline{k}) \cdot \mathbf{g}$; note that $\mathcal{O}(\mathbf{h})$ is uniquely defined by Theorem 2.5. Roughly speaking, we define $\lambda_{\rm opt} \in Y_k(G)$ to be the cocharacter that takes $\mathbf{h}$ into $\mathcal{O}(\mathbf{h})$ as quickly as possible (in an appropriate sense), and we define $P_{\rm opt}$ to be $P_{\lambda_{\rm opt}}$. (In fact, we need a slight variation—due to Hesselink—on this construction: rather than taking a single generic tuple $\mathbf{h}$, one considers the action of a cocharacter λ on all elements of H at once.) Note that $P_{\rm opt}$ is not uniquely determined (see [6, Rem. 5.22]).

Now suppose that H is a subgroup of G such that $C_G(H)$ is k-defined. One can show that if H is G-cr then H is G-cr over k (as previously noted, the converse is false). In fact, we prove a slightly stronger result: if H is not G-cr over k then hypothesis ($*$) holds. To see this, choose a generic tuple $\mathbf{h} \in H^m$. We can find $\lambda \in Y_k(G)$ such that (P_λ, L_λ) yields a k-semisimplification H' of H; so $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k but $G(k) \cdot \mathbf{h}$ is not. If H is contained in an R-Levi $\overline{k}$-subgroup L of P_λ then $c_\lambda(\mathbf{h}) = u \cdot \mathbf{h}$ for some $u \in R_u(P_\lambda)$. But then [1, Thm. 7.1] implies that $c_\lambda(\mathbf{h}) = u_1 \cdot \mathbf{h}$ for some $u_1 \in R_u(P_\lambda)(k)$, so $G(k) \cdot c_\lambda(\mathbf{h}) = G(k) \cdot \mathbf{h}$, a contradiction.

Remark 5.2. Let M be a k-subgroup of G such that M normalizes H, and let P be the R-parabolic subgroup of G obtained from one of the constructions above. Then it is automatic that $M(k)$ normalizes P. However, under the extra hypothesis that H is k-defined, we can in fact show that $M \subseteq N_G(P)$. To see this, one can first extend the field from k to k_s and then show that the R-parabolic subgroup obtained from either of the constructions is k-defined (cf. [3, Proof of Thm. 1.1] and [11, Sec. 4]), and hence coincides with P—this implies that $M(k_s)$, and hence M, normalizes P.

Remark 5.3. There are some limitations on the constructions given above. First, without the hypothesis that k is perfect, it can happen that the subgroup obtained from P_{opt} is not G-cr over k, and is therefore not a k-semisimplification of H. (It is, however, $G(\overline{k})$-conjugate to a k-semisimplification of H.) Second, as yet there is no theory of optimal destabilising subgroups that holds for arbitrary fields—this means that we do not know how to define a version of P_{opt} for a subgroup H that is not G-cr over k if $(*)$ does not hold. See [6, Sec. 1 and Ex. 5.21] for further discussion of this latter point.

By combining the two constructions above we obtain the following "Clifford theory" result, exploring the link between the semisimplification of a group and a normal subgroup. In the case k is algebraically closed, part (a) is [2, Thm. 3.10].

Theorem 5.4. *Let M be a k-subgroup of G and let H be a normal k-subgroup of M. Suppose at least one of the following holds:*

(i) k is perfect.
(ii) G is connected.

Then:

(a) If M is G-completely reducible over k then H is G-completely reducible over k.
(b) There is an R-parabolic subgroup P of G such that $M \subseteq P$ and P yields both a k-semisimplification of M and a k-semisimplification of H. In particular, there exist k-semisimplifications M' (resp., H') of M (resp., of H) such that H' is normal in M'.

Proof. Suppose H is not G-cr over k. Choose $P = P_{\text{opt}}$ in case (i) and $P = P_c$ in case (ii). Then $M \subseteq N_G(P)$ by Remark 5.2. Since H is not contained in any R-Levi k-subgroup of P, H is not contained in any R-Levi k-subgroup of $N_G(P)$ (Lemma 3.8). Hence M is not contained in any R-Levi k-subgroup of $N_G(P)$. It follows that M is not G-cr over k. This proves part (a).

For (b), pick $\lambda \in Y_k(G)$ such that (P_λ, L_λ) yields a semisimplification $M' := c_\lambda(M)$ of M. Then $c_\lambda(M)$ is G-cr over k and $c_\lambda(H)$ is normal in $c_\lambda(M)$. Now $c_\lambda(M)$ and $c_\lambda(H)$ satisfy the hypotheses of the theorem, so $c_\lambda(H)$ is G-cr over k by (a). Hence (P_λ, L_λ) yields a semisimplification $H' := c_\lambda(H)$ of H as well, and H' is normal in M'. □

Remark 5.5. The hypothesis in part (ii) can be weakened: one only needs to assume that $H \subseteq G^0$. In order to make the proof go through, one needs to verify that the first construction above extends to this situation.

Example 5.6. Let H be a k-subgroup of $G = \mathrm{GL}_n$ such that H is not completely reducible over k. Since H is separable, $C_G(H)$ is k-defined, so H is not G-completely reducible; we obtain a parabolic k-subgroup P_{opt} as above which yields a subgroup H'. We claim that H' is a k-semisimplification of H. For suppose H' is not G-cr over k. Choose $\mathbf{h}$, $\mathbf{g}$ as above, and let $\mathbf{h}' = c_{\lambda_{\text{opt}}}(\mathbf{h})$ (so that $\mathbf{h}'$ is a generic tuple for H'). Since $C_G(H')$ is k-defined, hypothesis $(*)$ holds, so we obtain an optimal cocharacter which takes $\mathbf{h}'$ out of $G \cdot \mathbf{h}' = \mathcal{O}(\mathbf{h})$ and into $\mathcal{O}(\mathbf{h}')$. But $\mathbf{g}$ is accessible from $\mathbf{h}'$ over k by [1, Thm. 4.3(ii)], so $\mathcal{O}(\mathbf{h}') = \mathcal{O}(\mathbf{h})$, a contradiction.

The parabolic subgroup $P_{\rm opt}$ is the stabilizer of some flag $\mathcal{F}$ of subspaces of k^n, and $\mathcal{F}$ does not admit a complementary H-stable flag of subspaces of k^n. By Remark 5.2, $C_G(H)$ is a subgroup of $P_{\rm opt}$—that is, $C_G(H)$ stabilizes $\mathcal{F}$—and likewise the normalizer $N_G(H)$ stabilizes $\mathcal{F}$ if $N_G(H)$ is k-defined. If k is perfect then $N_G(H)$ is automatically k-defined but it need not be k-defined in general; see [9] for further discussion.

Remark 5.7. Hesselink gives an example [10, (8.5) Ex.] of a subgroup H of an almost simple group G of type C_2 such that $P_{\rm opt}$ is not a minimal centre for Σ, the subcomplex of the building $\Delta_k(G)$ of G consisting of all parabolic subgroups of G that contain H. This shows that the two constructions above can yield different R-parabolic subgroups. Nevertheless, the corresponding k-semisimplifications of H are $G(k)$-conjugate, thanks to Theorem 4.5.

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