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NON-COERCIVE LYAPUNOV FUNCTIONS FOR INPUT-TO-STATE STABILITY OF INFINITE-DIMENSIONAL SYSTEMS *

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Abstract. We consider an abstract class of infinite-dimensional dynamical systems with inputs. For this class the significance of noncoercive Lyapunov functions is analyzed. It is shown that the existence of such Lyapunov functions implies norm-to-integral input-to-state stability. This property in turn is equivalent to ISS, if the system has some sort of regularity. For a particular class of linear systems with unbounded admissible input operators, explicit constructions of noncoercive Lyapunov functions are provided. The theory is applied to a heat equation with Dirichlet boundary conditions.

Key words. infinite-dimensional systems, input-to-state stability, Lyapunov functions, nonlinear systems, linear systems.

AMS subject classifications. 35Q93, 37B25, 37L15, 93C10, 93C25, 93D05, 93D09

1. Introduction.

The concept of input-to-state stability (ISS), introduced in [40] for ordinary differential equations (ODEs), unifies the classical Lyapunov and input-output stability theories and has broad applications in nonlinear control theory, in particular to robust stabilization of nonlinear systems [8], design of nonlinear observers [1], analysis of large-scale networks [6, 16], etc.

The influence of finite-dimensional ISS theory and a desire to develop powerful tools for robust control of linear and nonlinear distributed parameter systems resulted in extensions of ISS concepts to broad classes of infinite-dimensional systems, including partial differential equations (PDEs) with distributed and boundary controls, semilinear equations in Banach spaces, time-delay systems, etc. [5, 10, 14, 18, 19, 20], [26, 29, 43, 38].

Currently ISS of infinite-dimensional systems is an active research area at the intersection of nonlinear control, functional analysis, Lyapunov theory and PDE theory, which brings such important techniques for stability analysis as characterizations of ISS and ISS-like properties in terms of weaker stability concepts [10, 29, 37], constructions of ISS Lyapunov functions for PDEs with distributed and boundary controls [26, 36, 43, 48], efficient methods for study of boundary control systems [10, 12, 18, 20, 47], etc. For a survey on ISS of infinite-dimensional systems we refer to [28].

It is a basic result in input-to-state stability theory that the existence of an ISS Lyapunov function implies ISS. However, the construction of ISS Lyapunov functions for infinite-dimensional systems is a challenging task, especially for systems

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with boundary inputs and/or for nonlinear systems. Already for undisturbed linear systems over Hilbert spaces, "natural" Lyapunov function candidates constructed via solutions of Lyapunov equations are of the form $V(x) := \langle Px, x\rangle$, where $\langle\cdot,\cdot\rangle$ is a scalar product in X and P is a self-adjoint, bounded and positive linear operator, whose spectrum may contain 0. In this case V is not coercive and satisfies only the weaker property that $V(x) > 0$ for $x \neq 0$. Hence the question arises, whether such "non-coercive" Lyapunov functions can be used to conclude that a given system is ISS. A thorough study of a similar question related to characterizations of uniform global asymptotic stability has recently been performed in [32, 31].

In [29, Section III.B] it was shown for a class of semilinear equations in Banach spaces with Lipschitz continuous nonlinearities that the existence of a non-coercive Lyapunov function implies ISS provided the flow of the system has some continuity properties with respect to states and inputs at the origin and the finite-time reachability sets of the system are bounded. However, this class of systems does not include many important systems such as linear control systems with admissible inputs operators, which are crucially important for the study of partial differential equations with boundary inputs.

In this paper we extend the results from [29, Section III.B] to a broader class of systems, which includes at least some important classes of boundary control systems. The characterizations of ISS developed in [29] will play a central role in these developments.

We start by defining a general class of control systems in Section 2. This class covers a wide range of infinite-dimensional systems. For this class several stability concepts are defined which relate to the characterizations of ISS, in particular to the characterization with the help of noncoercive Lyapunov functions. We define also several further ISS-like properties, which we call norm-to-integral ISS and integral-to-integral ISS. Integral-to-integral ISS has been studied in [41] and it was shown that integral-to-integral ISS is equivalent to ISS for systems of ordinary differential equations with sufficiently regular right hand side f. Further relations of ISS and integral-to-integral ISS for ODE systems have been developed in [9, 21] and other works.

Although ISS is no longer equivalent to integral-to-integral ISS for infinite-dimensional systems, *we prove in Theorem 3.5 that ISS is equivalent to norm-to-integral ISS for a broad class of infinite-dimensional systems provided the flow of the system has some continuity properties w.r.t. states and inputs at the origin (CEP property) and the finite-time reachability sets of the system are bounded (BRS property).* The proof of this criterion is performed in 3 steps. First we show that integral-to-integral ISS implies a so-called uniform limit property. This result has been already obtained in [29, Section III.B]. Next we show that integral-to-integral ISS implies local stability of a control system provided the flow of the system is continuous w.r.t. state and inputs at the origin. This is done in Proposition 3.4. The third and final step in the proof of Theorem 3.5 is the application of the main result in [29].

In Section 4 we introduce non-coercive ISS Lyapunov functions for abstract control systems, and show that *existence of such a function for a forward-complete system implies norm-to-integral ISS (Proposition 4.3), and it implies ISS provided CEP and BRS properties are satisfied (Theorem Theorem 4.4).* Derivations of these results rely on the characterizations of ISS obtained in Section 3. In Section 4.1 we discuss the employed definition of the ISS Lyapunov function for various common choices of input spaces.

In Section 5 we derive a constructive converse ISS Lyapunov theorem (Theorem 5.3) for certain classes of linear systems with admissible input operators. In particular, our results can be applied for a broad class of analytic semigroups over Hilbert spaces generated by subnormal operators, as discussed in Section 6.2.

It is well-known that the classic heat equation with Dirichlet boundary inputs is ISS, which has been verified by means of several different methods: [10, 18, 27]. However, no constructions for ISS Lyapunov functions have been proposed. In Section 6 we show that using the constructions developed in Theorem 5.3 one can construct a non-coercive ISS Lyapunov function for this system. It is still an open question, whether a coercive ISS Lyapunov function for a heat equation with the Dirichlet boundary input exists (note, that for the system with Neumann boundary input a coercive quadratic ISS Lyapunov function can be constructed, see [48]).

Notation: We use the following notation. The nonnegative reals are denoted by $\mathbb{R}_+ := [0,\infty)$. The open ball of radius r around 0 in a normed vector space X is denoted by $B_r := B_{r,X} := \{x \in X : \|x\|_X < r\}$. Similarly, $B_{r,\mathcal{U}} := \{u \in \mathcal{U} : \|u\|_{\mathcal{U}} < r\}$. By $\overline{\lim}$ we denote the limit superior. For any normed linear space X, for any $S \subset X$ we denote the closure of S by $\overline{S}$. For a linear operator $A : X \to X$ (bounded or densely defined unbounded), we denote by A^* the adjoint of the operator A.

For a function $u : \mathbb{R}_+ \to U$, where U is any set, we denote by $u|_{[0,t]}$ the restriction of u to the interval $[0,t]$, that is $u|_{[0,t]} : [0,t] \to U$ and $u|_{[0,t]}(s) = u(s)$ for all $s \in [0,t]$.

Let U be a Banach space, I be a closed subset of $\mathbb{R}$ and $p \in [1,+\infty)$. We define the following spaces (see [13, Definition A.1.14] for details)

$$
\begin{aligned}
M(\mathbb{R}_+,U) &:= \{f : \mathbb{R}_+ \to U : f \text{ is strongly measurable}\},\\
L^p(\mathbb{R}_+,U) &:= \{f \in M(\mathbb{R}_+,U) : \|f\|_{L^p(\mathbb{R}_+,U)} := \Big(\int_0^\infty \|f(s)\|_U^p ds\Big)^{1/p} < \infty\},\\
L^\infty(\mathbb{R}_+,U) &:= \{f \in M(\mathbb{R}_+,U) : \|f\|_{L^\infty(\mathbb{R}_+,U)} := \operatorname*{ess\,sup}_{s\in\mathbb{R}_+} \|f(s)\|_U < \infty\}.
\end{aligned}
$$

Identifying the functions, which differ on a set of Lebesgue measure zero, the spaces $L^p(\mathbb{R}_+,U)$, $p \in [1,+\infty]$ are Banach spaces.

For the formulation of stability properties the following classes of comparison functions are useful:

$$
\begin{aligned}
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \ \gamma \text{ is continuous and strictly increasing, } \gamma(0) = 0\}\\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}\\
\mathcal{L} &:= \Big\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and decreasing with } \lim_{t\to\infty} \gamma(t) = 0\Big\}\\
\mathcal{KL} &:= \Big\{\beta : \mathbb{R}_+^2 \to \mathbb{R}_+ \mid \ \beta(\cdot,t) \in \mathcal{K},\ \forall t \ge 0,\ \beta(r,\cdot) \in \mathcal{L},\ \forall r > 0\Big\}
\end{aligned}
$$

2. Preliminaries. We begin by defining (time-invariant) forward complete control systems evolving on a Banach space X.

Definition 2.1. *Let $(X,\|\cdot\|_X)$, $(U,\|\cdot\|_U)$ be Banach spaces and $\mathcal{U} \subset \{f : \mathbb{R}_+ \to U\}$ be a normed vector space which satisfies the following two axioms:*

Axiom of shift invariance: For all $u \in \mathcal{U}$ and all $\tau \ge 0$ we have $u(\cdot+\tau) \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} \ge \|u(\cdot+\tau)\|_{\mathcal{U}}$.

Axiom of concatenation: For all $u_1, u_2 \in \mathcal{U}$ and for all $t > 0$ the concatenation of u_1 and u_2 at time t

$$
u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0,t],\\ u_2(\tau - t), & \text{otherwise,}\end{cases} \tag{2.1}
$$

belongs to $\mathcal{U}$. *Consider a map* $\phi : \mathbb{R}_+ \times X \times \mathcal{U} \to X$.

The triple $\Sigma = (X, \mathcal{U}, \phi)$ *is called a* forward complete control system, *if the following properties hold:*

(Σ1) Identity property: *for every* $(x, u) \in X \times \mathcal{U}$ *it holds that* $\phi(0, x, u) = x$.

(Σ2) Causality: *for every* $(t, x, u) \in \mathbb{R}_+ \times X \times \mathcal{U}$, *for every* $\tilde{u} \in \mathcal{U}$ *with* $u|_{[0,t]} = \tilde{u}|_{[0,t]}$ *it holds that* $\phi(t, x, u) = \phi(t, x, \tilde{u})$.

(Σ3) Continuity: *for each* $(x, u) \in X \times \mathcal{U}$ *the map* $t \mapsto \phi(t, x, u)$, $t \in [0, \infty)$ *is continuous.*

(Σ4) Cocycle property: *for all* $t, h \geq 0$, *for all* $x \in X$, $u \in \mathcal{U}$ *we have*

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

The space X *is called the* state space, $\mathcal{U}$ *the* input space *and* ϕ *the* transition map. This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, evolution partial differential equations (PDEs), abstract differential equations in Banach spaces and many others.

We single out two particular cases which will be of interest.

Example 2.2. (Semilinear systems with Lipschitz nonlinearities). *Let* A *be the generator of a strongly continuous semigroup (also called* C_0*-semigroup)* $(T(t))_{t\geq 0}$ *of bounded linear operators on* X *and let* $f : X \times U \to X$. *Consider the system*

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad u(t) \in U, \tag{2.2}$$

where $x(0) \in X$. *We study mild solutions of* (2.2), *i.e. solutions* $x : [0, \tau] \to X$ *of the integral equation*

$$x(t) = T(t)x(0) + \int_0^t T(t - s)f(x(s), u(s))ds, \quad t \in [0, \tau], \tag{2.3}$$

belonging to the space of continuous functions $C([0, \tau], X)$ *for some* $\tau > 0$.

For system (2.2), *we use the following assumption concerning the nonlinearity* f:

(i) $f : X \times U \to X$ *is Lipschitz continuous on bounded subsets of* X, *i.e. for all* $C > 0$, *there exists a* $L_f(C) > 0$, *such that for all* $x, y \in B_C$ *and for all* $v \in B_{C,U}$, *it holds that*

$$\|f(x, v) - f(y, v)\|_X \leq L_f(C)\|x - y\|_X. \tag{2.4}$$

(ii) $f(x, \cdot)$ *is continuous for all* $x \in X$ *and* $f(0, 0) = 0$.

Let $\mathcal{U} := PC_b(\mathbb{R}_+, U)$ *be the space of piecewise continuous functions, which are bounded and right-continuous, endowed with the supremum norm:* $\|u\|_{\mathcal{U}} := \sup_{t\geq 0} \|u(t)\|_U$. *Then our assumptions on* f *ensure that mild solutions of initial value problems of the form* (2.2) *exist and are unique, according to [2, Proposition 4.3.3]. For system* (2.2) *forward completeness is a further assumption. If these mild solutions exist on* $[0, \infty)$ *for every* $x(0) \in X$ *and* $u \in PC_b(\mathbb{R}_+, U)$, *then* $(X, PC_b(\mathbb{R}_+, U), \phi)$, *defines a forward complete control system, where* $\phi(t, x(0), u)$ *denotes the mild solution at time* t.

Example 2.3. (Linear systems with admissible control operators). *Consider linear systems of the form*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in X, \ t \geq 0, \tag{2.5}$$

where A *is the generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *and* $B \in L(U, X_{-1})$ *for some Banach space* U. *Here* X_{-1} *is the completion of* X *with respect to the norm* $\|x\|_{X_{-1}} = \|(\beta I - A)^{-1}x\|_X$ *for some* $\beta \in \mathbb{C}$ *in the resolvent set* $\rho(A)$ *of*

A. The semigroup $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} is an extension of A, see e.g. [7]. Thus we may consider equation (2.5) *on the Banach space X_{-1}. For every $x_0 \in X$ and every $u \in L^1_{\mathrm{loc}}([0,\infty),U)$, the function $x : [0,\infty) \to X_{-1}$,*

$$x(t) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geq 0,$$

is called a mild solution *of the system* (2.5).

*An operator $B \in L(U, X_{-1})$ is called a q-*admissible control operator *for $(T(t))_{t\geq 0}$, where $1 \leq q \leq \infty$, if*

$$\int_0^t T_{-1}(t-s)Bu(s)ds \in X$$

for every $t \geq 0$ and $u \in L^q([0,\infty),U)$ [46]. If the operator $B \in L(U, X_{-1})$ is an q-admissible control operator for $(T(t))_{t\geq 0}$, then there exists for any $t \geq 0$ a constant $\kappa(t) > 0$ such that

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,ds \right\|_X \leq \kappa(t)\|u\|_q, \quad u \in L^q([0,t),U), \tag{2.6}$$

see [46].

If B is ∞-admissible and for every initial condition $x_0 \in X$ and every input function $u \in L^\infty([0,\infty),U)$ the mild solution $x : [0,\infty) \to X$ is continuous, then $(X, L^\infty([0,\infty),U), \phi)$, where

$$\phi(t, x_0, u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds,$$

defines a forward-complete control system as defined in Definition 2.1.

Remark 2.4. *We note that, ∞-admissibility and continuity of all mild solutions $\phi(\cdot, x_0, u) : [0,\infty) \to X$, where $x_0 \in X$ and $u \in L^\infty([0,\infty),U)$ is implied by each of the following conditions:*

- *B is q-admissible for some $q \in [1,\infty)$, see [46, Proposition 2.3],*
- *$B \in L(U, X_{-1})$, $\dim(U) < \infty$, X is a Hilbert space and $A - \lambda I$ generates for a certain $\lambda \in \mathbb{R}$ an analytic semigroup which is similar to a contraction semigroup, see [12, Theorem 1].*

In this article various stability concepts are needed for forward complete control systems.

Definition 2.5. *Consider a forward complete control system $\Sigma = (X, \mathcal{U}, \phi)$.*

1. *We call $0 \in X$ an* equilibrium point (of the undisturbed system) *if $\phi(t, 0, 0) = 0$ for all $t \geq 0$.*
2. *We say that Σ is* continuous at the equilibrium point (CEP), *if 0 is an equilibrium and for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon, h) > 0$, so that*

$$t \in [0,h] \ \wedge \ \|x\|_X \leq \delta \ \wedge \ \|u\|_{\mathcal{U}} \leq \delta \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \varepsilon. \tag{2.7}$$

3. *We say that Σ has* bounded reachability sets (BRS), *if for any $C > 0$ and any $\tau > 0$ it holds that*

$$\sup\left\{ \|\phi(t,x,u)\|_X \,:\, \|x\|_X \leq C, \ \|u\|_{\mathcal{U}} \leq C, \ t \in [0,\tau] \right\} < \infty.$$

4. *System* Σ *is called* uniformly locally stable (ULS)*, if there exist* $\sigma \in \mathcal{K}_\infty$*,* $\gamma \in \mathcal{K}_\infty$ *and* $r > 0$ *such that for all* $x \in \overline{B_r}$ *and all* $u \in \overline{B_{r,\mathcal{U}}}$*:*

$$\|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}) \quad \forall t \geq 0. \tag{2.8}$$

5. *We say that* Σ *has the* uniform limit property (ULIM)*, if there exists* $\gamma \in \mathcal{K}$ *so that for every* $\varepsilon > 0$ *and for every* $r > 0$ *there exists a* $\tau = \tau(\varepsilon, r)$ *such that for all* x *with* $\|x\|_X \leq r$ *and all* $u \in \mathcal{U}$ *there is a* $t \leq \tau$ *such that*

$$\|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{2.9}$$

6. *System* Σ *is called* (uniformly) input-to-state stable (ISS)*, if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that for all* $x \in X$*,* $u \in \mathcal{U}$ *and* $t \geq 0$ *it holds that*

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{2.10}$$

7. *We call* Σ norm-to-integral ISS *if there are* $\alpha \in \mathcal{K}$ *and* $\psi \in \mathcal{K}_\infty$*,* $\sigma \in \mathcal{K}_\infty$ *so that for all* $x \in X$*,* $u \in \mathcal{U}$ *and* $t \geq 0$ *it holds that*

$$\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \leq \psi(\|x\|_X) + t\sigma(\|u\|_{\mathcal{U}}). \tag{2.11}$$

For the special case of L^∞ input spaces we introduce one more stability notion

Definition 2.6. *Consider a forward complete control system* $\Sigma := (X, \mathcal{U}, \phi)$ *with the input space* $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$*, where* U *is any normed linear space.*

We call Σ integral-to-integral ISS *if there are* $\alpha \in \mathcal{K}$ *and* $\psi \in \mathcal{K}_\infty$*,* $\sigma \in \mathcal{K}_\infty$ *so that for all* $x \in X$*,* $u \in \mathcal{U}$ *and* $t \geq 0$ *it holds that*

$$\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \leq \psi(\|x\|_X) + \int_0^t \sigma(\|u(s)\|_U)ds. \tag{2.12}$$

Remark 2.7. *The CEP and BRS properties are motivated by the notions of a robust equilibrium point and of robust forward completeness, which were widely employed in [17], see also [32] where these concepts were used in the context of non-coercive ISS Lyapunov theory.*

Example 2.8. (Linear systems with admissible control operators) *We continue with Example 2.3, that is, we consider again equation* (2.5) *and assume that* A *generates a* C_0*-semigroup,* $B \in L(U, X_{-1})$ *is* ∞*-admissible and for every initial condition* $x_0 \in X$ *and every input function* $u \in L^\infty([0,\infty), U)$ *the mild solution* $x : [0,\infty) \to X$ *is continuous. These assumptions guarantee that* $(X, L^\infty([0,\infty), U), \phi)$*, where*

$$\phi(t, x_0, u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds,$$

defines a forward-complete control system. The system has the following properties

1. $0 \in X$ *is an equilibrium point due to the linearity of the system,*
2. $(X, L^\infty([0,\infty), U), \phi)$ *has the CEP property, and bounded reachability sets (BRS), which follows easily from inequality* (2.6) *and linearity of the system.*
3. $(T(t))_{t\geq 0}$ *is exponentially stable if and only if* $(X, L^\infty([0,\infty), U), \phi)$ *is ISS [10, Proposition 2.10].*

4. *If $(T(t))_{t\geq 0}$ is exponentially stable, then $(X, L^\infty([0,\infty),U),\phi)$ has the uniform limit property (ULIM), and is uniformly locally stable (ULS), which follows from the previous item.*
5. *If $\Sigma := (X, L^\infty([0,\infty),U),\phi)$ has a so-called integral ISS property, then Σ is ISS [10]. To the best of the knowledge of the authors it is unknown, whether or not the converse implication holds for every linear system* (2.5).

3. Characterization of ISS in terms of norm-to-integral ISS. In this section, we characterize input-to-state stability in terms of the norm-to-integral ISS, which is interesting on its own right, but also it will be instrumental for the establishment of non-coercive ISS Lyapunov theorems in the next section. We start with

PROPOSITION 3.1. *If a forward complete control system is ISS, then it is norm-to-integral ISS.*

Proof. Let $\Sigma = (X,\mathcal{U},\phi)$ be a forward complete ISS control system and let $\beta \in \mathcal{KL}$ be as in Definition 2.5. By Sontag's $\mathcal{KL}$-lemma [41, Proposition 7], there are $\xi_1, \xi_2 \in \mathcal{K}_\infty$ so that $\beta(r,t) \leq \xi_1^{-1}(e^{-t}\xi_2(r))$ for all $r,t \in \mathbb{R}_+$. ISS of Σ now implies that there is $\gamma \in \mathcal{K}_\infty$ such that the following holds:

$$\|\phi(t,x,u)\|_X \leq \xi_1^{-1}(e^{-t}\xi_2(\|x\|_X)) + \gamma(\|u\|_\mathcal{U}).$$

Define $\bar{\xi}(r) := \xi_1(\frac{1}{2}r)$, $r \in \mathbb{R}_+$. Using the inequality $\bar{\xi}(a+b) \leq \bar{\xi}(2a) + \bar{\xi}(2b)$, which is valid for all $a,b \in \mathbb{R}_+$, we obtain that for all $x \in X$, $u \in \mathcal{U}$ and $t \geq 0$ it holds that

$$\bar{\xi}(\|\phi(t,x,u)\|_X) \leq e^{-t}\xi_2(\|x\|_X) + \xi_1(\gamma(\|u\|_\mathcal{U})). \tag{3.1}$$

Integrating (3.1), we see that

$$\int_0^t \bar{\xi}(\|\phi(s,x,u)\|_X)ds \leq \xi_2(\|x\|_X) + t\xi_1 \circ \gamma(\|u\|_\mathcal{U}),$$

which shows norm-to-integral ISS of Σ. □

Next we show that norm-to-integral ISS implies ISS for a class of forward-complete control systems satisfying the CEP and BRS properties. In order to prove this, we are going to use the following characterization of ISS, shown in [29]:

THEOREM 3.2. *Let $\Sigma = (X,\mathcal{U},\phi)$ be a forward complete control system. Then Σ is ISS if and only if Σ is ULIM, ULS, and BRS.*

In [29, Proposition 8] it was shown (with slightly different formulation, but the same proof) that

PROPOSITION 3.3. *Let $\Sigma = (X,\mathcal{U},\phi)$ be a forward complete control system. If Σ is norm-to-integral ISS, then Σ is ULIM.*

Next we provide a sufficient condition for the ULS property.

PROPOSITION 3.4. *Let $\Sigma = (X,\mathcal{U},\phi)$ be a forward complete control system satisfying the CEP property. If Σ is norm-to-integral ISS, then Σ is ULS.*

Proof. Let Σ be norm-to-integral ISS with the corresponding α,ψ,σ as in Definition 7.

By [29, Lemma 2] Σ is ULS if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|x\|_X \leq \delta \ \wedge \ \|u\|_\mathcal{U} \leq \delta \ \wedge \ t \geq 0 \ \Rightarrow \ \|\phi(t,x,u)\|_X \leq \varepsilon. \tag{3.2}$$

Seeking a contradiction, assume that Σ is not ULS. Then there is $\varepsilon > 0$ such that for any $\delta > 0$ there are $x \in B_\delta$, $u \in B_{\delta,\mathcal{U}}$ and $t \geq 0$ such that $\|\phi(t,x,u)\|_X = \varepsilon$. Then

there are sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{U}$, and $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that $x_k \to 0$ as $k \to \infty$, $u_k \to 0$ as $k \to \infty$ and

$$\|\phi(t_k, x_k, u_k)\|_X = \varepsilon \quad \forall k \geq 1.$$

Since Σ is CEP, for the above ε there is a $\delta_1 = \delta_1(\varepsilon, 1)$ so that

$$\|x\|_X \leq \delta_1 \ \wedge \ \|u\|_{\mathcal{U}} \leq \delta_1 \ \wedge \ t \in [0,1] \quad \Rightarrow \quad \|\phi(t,x,u)\|_X < \varepsilon. \tag{3.3}$$

Define for $k \in \mathbb{N}$ the following time sequence:

$$t_k^1 := \sup\{t \in [0, t_k] : \|\phi(t, x_k, u_k)\|_X \leq \delta_1\},$$

if the supremum is taken over a nonempty set, and set $t_k^1 := 0$ otherwise.

Again as Σ is CEP, for the above δ_1 there is a $\delta_2 > 0$ so that

$$\|x\|_X \leq \delta_2 \ \wedge \ \|u\|_{\mathcal{U}} \leq \delta_2 \ \wedge \ t \in [0,1] \quad \Rightarrow \quad \|\phi(t,x,u)\|_X < \delta_1. \tag{3.4}$$

Without loss of generality we assume that δ_2 is chosen small enough so that

$$\alpha(\delta_1) > \psi(\delta_2). \tag{3.5}$$

We now define

$$t_k^2 := \sup\{t \in [0, t_k] : \|\phi(t, x_k, u_k)\|_X \leq \delta_2\},$$

if the supremum is taken over a nonempty set, and set $t_k^2 := 0$ otherwise.

Since $u_k \to 0$ and $x_k \to 0$ as $k \to \infty$, there is $K > 0$ so that $\|u_k\|_{\mathcal{U}} \leq \delta_2$ and $\|x_k\|_X \leq \delta_2$ for $k \geq K$.

From now on, we always assume that $k \geq K$.

Using (3.3), (3.4) and the cocycle property, it is not hard to show that for $k \geq K$ it must hold that $t_k \geq 2$, as otherwise we arrive at a contradiction to $\|\phi(t_k, x_k, u_k)\|_X = \varepsilon$.

Assume that $t_k - t_k^1 < 1$. This implies (since $t_k \geq 2$), that $t_k^1 > 0$. By the cocycle property we have

$$\|\phi(t_k, x_k, u_k)\|_X = \|\phi(t_k - t_k^1, \phi(t_k^1, x_k, u_k), u_k(\cdot + t_k^1)\|_X.$$

The axiom of shift invariance justifies the inequalities

$$\|u_k(\cdot + t_k^1)\|_{\mathcal{U}} \leq \|u_k\|_{\mathcal{U}} \leq \delta_2 \leq \delta_1.$$

Since $\|\phi(t_k^1, x_k, u_k)\|_X = \delta_1$, and $t_k - t_k^1 < 1$, we have by (3.3) that $\|\phi(t_k, x_k, u_k)\|_X < \varepsilon$, a contradiction. Hence $t_k - t_k^1 \geq 1$ for all $k \geq K$.

Analogously, we obtain that $t_k^1 - t_k^2 \geq 1$ and $t_k - t_k^2 \geq 2$.

Define

$$x_k^2 := \phi(t_k^2, x_k, u_k), \quad u_k^2 := u_k(\cdot + t_k^2)$$

and

$$x_k^1 := \phi(t_k^1, x_k, u_k), \quad u_k^1 := u_k(\cdot + t_k^1).$$

Due to the axiom of shift invariance $u_k^1, u_k^2 \in \mathcal{U}$ and

$$\|u_k^1\|_{\mathcal{U}} \le \|u_k^2\|_{\mathcal{U}} \le \|u_k\|_{\mathcal{U}} \le \delta_2.$$

Also by the definition of t_k^2 we have $\|x_k^2\|_X = \delta_2$.

Applying (2.11), we obtain for $t := t_k - t_k^2$ that

$$\begin{aligned}\int_0^{t_k - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds &\le \psi(\|x_k^2\|_X) + (t_k - t_k^2)\sigma(\|u_k^2\|_{\mathcal{U}}) \\ &\le \psi(\delta_2) + (t_k - t_k^2)\sigma(\|u_k\|_{\mathcal{U}}). \end{aligned} \tag{3.6}$$

On the other hand, changing the integration variable and using the cocycle property we obtain that

$$\begin{aligned}\int_0^{t_k - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds &= \int_0^{t_k^1 - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds + \int_{t_k^1 - t_k^2}^{t_k - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds \\ &= \int_0^{t_k^1 - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds + \int_0^{t_k - t_k^1} \alpha(\|\phi(s + t_k^1 - t_k^2, x_k^2, u_k^2)\|_X) ds \\ &= \int_0^{t_k^1 - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds + \int_0^{t_k - t_k^1} \alpha(\|\phi(s, x_k^1, u_k^1)\|_X) ds.\end{aligned}$$

By definition of t_k^2 and t_k^1 we have that

$$\|\phi(s, x_k^2, u_k^2)\|_X \ge \delta_2, \quad s \in [t_k^2, t_k]$$

and

$$\|\phi(s, x_k^1, u_k^1)\|_X \ge \delta_1, \quad s \in [t_k^1, t_k].$$

Continuing the above estimates and using that $t_k - t_k^1 \ge 1$ and $\alpha(\delta_1) > \alpha(\delta_2)$, we arrive at

$$\begin{aligned}\int_0^{t_k - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds &\ge (t_k^1 - t_k^2)\alpha(\delta_2) + (t_k - t_k^1)\alpha(\delta_1) \\ &\ge (t_k - t_k^2 - 1)\alpha(\delta_2) + \alpha(\delta_1).\end{aligned}$$

Since $t_k - t_k^2 \ge 2$ and in view of (3.5), we derive

$$\int_0^{t_k - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X) ds > \frac{t_k - t_k^2}{2}\alpha(\delta_2) + \psi(\delta_2). \tag{3.7}$$

Combining inequalities (3.6) and (3.7), we obtain

$$\frac{t_k - t_k^2}{2}\alpha(\delta_2) < (t_k - t_k^2)\sigma(\|u_k\|_{\mathcal{U}}).$$

This leads to

$$\frac{1}{2}\alpha(\delta_2) < \sigma(\|u_k\|_{\mathcal{U}}), \quad k \ge K.$$

Finally, since $\lim_{k\to\infty} \|u_k\|_{\mathcal{U}} = 0$, letting $k \to \infty$ we come to a contradiction. This shows that Σ is ULS. □

Now we combine the derived results into a criterion of ISS in terms of norm-to-integral ISS.

THEOREM 3.5. *Let Σ be a forward complete control system. Then Σ is ISS if and only if Σ is norm-to-integral ISS and has CEP and BRS properties.*

Proof. "⇒". Clearly, ISS implies CEP and BRS properties. By Proposition 3.1 ISS implies norm-to-integral ISS.

"⇐". Propositions 3.3 and 3.4 imply that Σ is ULIM and ULS. Since Σ is BRS, Theorem 3.2 shows that Σ is ISS. □

Some forward complete systems have the BRS and CEP properties intrinsically, which leads to the equivalence between ISS and norm-to-integral ISS for such classes of systems. In particular, the following slight extension of [41, Theorem 1] holds:

COROLLARY 3.6. *Let $X := \mathbb{R}^n$ and $\mathcal{U} := L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. Consider ordinary differential equations of the form*

$$\dot{x} = f(x, u), \quad t > 0, \tag{3.8}$$

with f which is continuous in both variables and locally Lipschitz continuous in x. Define $\phi(\cdot, x, u)$ as the maximal solution of (3.8) with an initial condition $x \in \mathbb{R}^n$ and an input $u \in \mathcal{U}$.

Assume that (3.8) is forward complete. Then the following statements are equivalent:

(i) (3.8) is ISS

(ii) (3.8) is integral-to-integral ISS

(iii) (3.8) is norm-to-integral ISS

Proof. (i) ⇔ (ii). Was shown in [41, Theorem 1].

(i) ⇔ (iii). It is well-known that under made assumptions on f a forward complete system (3.8) satisfies the BRS property (see [22, Proposition 5.1]) and CEP property. The claim follows from Theorem 4.4 and Proposition 3.1. □

For linear systems with admissible control operators we have:

PROPOSITION 3.7. *Let X and U be Banach spaces and let $\mathcal{U} := L^\infty([0,\infty), U)$. Consider the system (2.5) and assume that A generates a C_0-semigroup, $B \in L(U, X_{-1})$ is ∞-admissible and for every initial condition $x_0 \in X$ and every input $u \in \mathcal{U}$ the mild solution $x : [0,\infty) \to X$ is continuous.*

Then (2.5) is ISS if and only if (2.5) is norm-to-integral ISS.

Proof. "⇒". Follows by Proposition 3.1.

"⇐". As mentioned in Example 2.8, under the assumptions made (2.5) is a well-posed control system satisfying CEP and BRS properties. Theorem 4.4 finishes the proof.

Alternative proof of "⇐". From norm-to-integral ISS of (2.5) for $u \equiv 0$ the exponential stability of the semigroup generated by A follows by means of a generalized Datko lemma [23, Theorem 2]. As we assume that B is ∞-admissible, ISS follows by [10, Proposition 2.10]. □

3.1. Remark on input-to-state practical stability. In some cases it is impossible (as in quantized control) or too costly to construct a feedback that results in an ISS closed-loop system. For these applications one defines the following relaxation of the ISS property:

DEFINITION 3.8. *A control system $\Sigma = (X, \mathcal{U}, \phi)$ is called* (uniformly) input-to-state practically stable (ISpS)*, if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and $c > 0$ such that for all $x \in X$, $u \in \mathcal{U}$ and $t \geq 0$ the following holds:*

$$\|\phi(t, x, u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_\mathcal{U}) + c. \tag{3.9}$$

The notion of ISpS has been proposed in [16] and has become very useful for control in the presence of quantization errors [39, 15], sample-data control [33] to name a few examples.

One of the requirements in Theorem 3.5 is that the CEP property holds. If this property is not available, we can still infer input-to-state practical stability of Σ, using the main result in [24].

THEOREM 3.9. *Let Σ be a forward complete control system, which is BRS. If Σ is norm-to-integral ISS, then Σ is ISpS.*

Proof. Proposition 3.3 implies that Σ is ULIM. Since Σ is also BRS, [24, Theorem III.1] shows that Σ is ISpS. □

4. Non-coercive ISS Lyapunov theorem. For a real-valued function $b : \mathbb{R}_+ \to \mathbb{R}$ define the *right-hand upper and lower Dini derivatives* at $t \in \mathbb{R}_+$ by

$$D^+b(t) := \overline{\lim_{h\to+0}} \frac{b(t+h)-b(t)}{h}, \qquad D_+b(t) := \underline{\lim_{h\to+0}} \frac{b(t+h)-b(t)}{h}$$

respectively. Note that for all $b : \mathbb{R}_+ \to \mathbb{R}$ and all $t \in \mathbb{R}_+$ it holds that

$$D^+b(t) = -D_+(-b(t)). \tag{4.1}$$

Let $x \in X$ and V be a real-valued function defined in a neighborhood of x. The *(right-hand upper) Dini derivative of V at x corresponding to the input $u \in \mathcal{U}$ along the trajectories of* Σ is defined by

$$\dot{V}_u(x) := D^+V\big(\phi(\cdot,x,u)\big)\Big|_{t=0} = \overline{\lim_{t\to+0}} \frac{1}{t}\Big(V\big(\phi(t,x,u)\big) - V(x)\Big). \tag{4.2}$$

We need a lemma on derivatives of monotone functions.

LEMMA 4.1. *Let $b : \mathbb{R}_+ \to \mathbb{R}$ be a nonincreasing function. Then for each $t \in \mathbb{R}_+$ it holds that*

$$b(t) \geq D^+ \int_0^t b(s)ds \geq D_+ \int_0^t b(s)ds \geq \lim_{h\to+0} b(t+h). \tag{4.3}$$

Proof. Pick any $t \geq 0$. By the definition of the Dini derivative and using monotonicity it holds that

$$\begin{aligned} D^+ \int_0^t b(s)ds = \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_0^{t+h} b(s)ds - \int_0^t b(s)ds\Big) &= \overline{\lim_{h\to+0}} \frac{1}{h}\int_t^{t+h} b(s)ds \\ &\leq \overline{\lim_{h\to+0}} \frac{1}{h}\int_t^{t+h} b(t)ds = b(t). \end{aligned}$$

On the other hand, we have that

$$D_+ \int_0^t b(s)ds \geq \underline{\lim_{h\to+0}} \frac{1}{h}\int_t^{t+h} b(t+h)ds = \underline{\lim_{h\to+0}}\, b(t+h).$$

The inequality $D^+ \int_0^t b(s)ds \geq D_+ \int_0^t b(s)ds$ is clear. □

For stability analysis of nonlinear control systems, Lyapunov functions are an essential tool.

DEFINITION 4.2. *Consider a control system $\Sigma = (X,\mathcal{U},\phi)$ with the input space $\mathcal{U} := L^\infty(\mathbb{R}_+,U)$. A continuous function $V : X \to \mathbb{R}_+$ is called a* non-coercive ISS Lyapunov function *for Σ, if there exist $\psi_2,\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that*

$$0 < V(x) \le \psi_2(\|x\|_X), \quad \forall x \in X \setminus \{0\}, \tag{4.4}$$

and the Dini derivative of V along the trajectories of Σ for all $x \in X$ and $u \in \mathcal{U}$ satisfies

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u\|_\mathcal{U}). \tag{4.5}$$

Moreover, if (4.5) holds just for $u = 0$, we call V a non-coercive Lyapunov function *for the undisturbed system Σ. If additionally there is $\psi_1 \in \mathcal{K}_\infty$ so that the following estimate holds:*

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \in X, \tag{4.6}$$

then V is called a coercive ISS Lyapunov function *for Σ.*

Note that continuity of V and the estimate (4.4) imply that $V(0) = 0$.

The next proposition shows that the norm-to-integral ISS property arises naturally in the theory of ISS Lyapunov functions:

PROPOSITION 4.3. *Let $\Sigma = (X,\mathcal{U},\phi)$ be a forward complete control system. Assume that there exists a non-coercive ISS Lyapunov function for Σ. Then Σ is norm-to-integral ISS.*

Proof. Assume that V is a non-coercive ISS Lyapunov function for Σ with the corresponding ψ_2,α,σ. Pick any $u \in \mathcal{U}$ and any $x \in X$. As we assume forward completeness of Σ, the trajectory $\phi(\cdot,x,u)$ exists for all $t \ge 0$ and due to (4.5), we have for any $t > 0$ that:

$$\dot{V}_{u(t+\cdot)}\big(\phi(t,x,u)\big) \le -\alpha(\|\phi(t,x,u)\|_X) + \sigma(\|u(t+\cdot)\|_\mathcal{U}). \tag{4.7}$$

By definition of $\dot{V}$, and using the cocycle property for Σ, we have that

$$\begin{aligned}\dot{V}_{u(t+\cdot)}\big(\phi(t,x,u)\big) &= \overline{\lim_{h\to+0}}\,\frac{1}{h}\Big(V\big(\phi(h,\phi(t,x,u),u(t+\cdot))\big) - V\big(\phi(t,x,u)\big)\Big)\\ &= \overline{\lim_{h\to+0}}\,\frac{1}{h}\Big(V\big(\phi(t+h,x,u)\big) - V\big(\phi(t,x,u)\big)\Big).\end{aligned}$$

Defining $y(t) := V\big(\phi(t,x,u)\big)$, we see that

$$\dot{V}_{u(t+\cdot)}\big(\phi(t,x,u)\big) = D^+y(t),$$

and $y(0) = V(x)$ due to the identity axiom of the system Σ.

In view of the continuity axiom of Σ, for fixed x,u the map $\phi(\cdot,x,u)$ is continuous, and thus $t \mapsto -\alpha(\|\phi(t,x,u)\|_X)$ is continuous as well.

For $t \ge 0$, define $G(t) := \int_0^t \alpha(\|\phi(s,x,u)\|_X)ds$ and $b(t) := \sigma(\|u(t+\cdot)\|_\mathcal{U})$. Note that by the axiom of shift invariance, b is non-increasing. As G is continuously differentiable, we can rewrite the inequality (4.7) as

$$D^+y(t) \le -\frac{d}{dt}G(t) + b(t). \tag{4.8}$$

Pick any $r > 0$ and define $b(s) = b(0)$ for $s \in [-r, 0]$. As b is a nonincreasing function on $[-r, \infty)$, it holds for any $t \geq 0$ that $b(t) \leq \lim_{h\to+0} b(t - r + h)$, and by the final inequality in Lemma 4.1 applied to $b(\cdot - r)$ we obtain

$$b(t) \leq D_+ \int_0^t b(s-r)ds = -D^+\Big(- \int_0^t b(s-r)ds\Big).$$

Thus, (4.8) implies that

$$D^+y(t) + \frac{d}{dt}G(t) + D^+\Big(- \int_0^t b(s-r)ds\Big) \leq 0. \tag{4.9}$$

Due to

$$D^+(f_1(t) + f_2(t)) \leq D^+(f_1(t)) + D^+(f_2(t)),$$

which holds for any functions f_1, f_2 on the real line, this implies that

$$D^+\Big(y(t) + G(t) - \int_0^t b(s-r)ds\Big) \leq 0.$$

It follows from [42, Theorem 2.1] that $t \mapsto y(t) + G(t) - \int_0^t b(s-r)ds$ is nonincreasing. As $G(0) = 0$, it follows that for all $r > 0$

$$y(t) + G(t) - \int_0^t b(s-r)ds \leq y(0) = V(x).$$

As b is bounded, we may pass to the limit $r \to 0$ and obtain

$$y(t) + G(t) - \int_0^t b(s)ds \leq y(0) = V(x).$$

Now $y(t) \geq 0$ for all $t \in \mathbb{R}_+$, and so

$$\begin{aligned}\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \leq& \psi_2(\|x\|_X) + \int_0^t \sigma(\|u(s+\cdot)\|_{\mathcal{U}})ds\\ \leq& \psi_2(\|x\|_X) + t\sigma(\|u\|_{\mathcal{U}}).\end{aligned} \tag{4.10}$$

This completes the proof. □

We can now state our main result on noncoercive ISS Lyapunov functions.

THEOREM 4.4. *Let $\Sigma = (X, \mathcal{U}, \phi)$ be a forward complete control system with the input space $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$, which is CEP and BRS. If there exists a (noncoercive) ISS Lyapunov function for Σ, then Σ is ISS.*

Proof. By Proposition 4.3, Σ is is norm-to-integral ISS. The application of Theorem 3.5 finishes the proof. □

4.1. Remarks on the definition of the ISS Lyapunov function. In this section we discuss the definition of the ISS Lyapunov function, which we adopted in this paper.

For any $u \in \mathcal{U}$ and any $\tau \geq 0$ define $u_\tau(s) := \begin{cases} u(s) & , \text{ if } s \in [0, \tau], \\ 0 & , \text{ if } s > \tau. \end{cases}$

We start with a restatement of the ISS Lyapunov function concept.

LEMMA 4.5. *Assume that for any $u \in \mathcal{U}$ it holds that* $\inf_{\tau>0} \|u_\tau\|_\mathcal{U} \le \|u\|_\mathcal{U}$.

Then a continuous function $V : X \to \mathbb{R}_+$ is a non-coercive ISS Lyapunov function *for the system $\Sigma = (X,\mathcal{U},\phi)$, if and only if there exist $\psi_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that* (4.4) *holds and the Dini derivative of V along the trajectories of Σ for all $x \in X$ and $u \in \mathcal{U}$ satisfies*

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\inf_{\tau>0} \|u_\tau\|_\mathcal{U}). \tag{4.11}$$

If additionally there is $\psi_1 \in \mathcal{K}_\infty$ so that (4.6) *holds, then V is a* coercive ISS Lyapunov function *for Σ.*

Proof. $\Leftarrow$. Follows from the assumption that $\inf_{\tau>0} \|u_\tau\|_\mathcal{U} \le \|u\|_\mathcal{U}$ for all $u \in \mathcal{U}$.

$\Rightarrow$. Let a continuous function $V : X \to \mathbb{R}_+$ be a non-coercive ISS Lyapunov function. Pick any $x \in X$ and any $u \in \mathcal{U}$. Since $\mathcal{U}$ is a linear space, $0 \in \mathcal{U}$ and the axiom of concatenation implies that $u_\tau \in \mathcal{U}$ for any $\tau > 0$. By definition of $\dot{V}_u$, for any $\tau > 0$ it holds that $\dot{V}_u(x) = \dot{V}_{u_\tau}(x)$, and thus

$$\dot{V}_u(x) = \dot{V}_{u_\tau}(x) \le -\alpha(\|x\|_X) + \sigma(\|u_\tau\|_\mathcal{U}).$$

Taking infimum over $\tau > 0$, we obtain (4.11). □

Our definition of an ISS Lyapunov function is defined for any normed linear space $\mathcal{U}$, that allows to develop the ISS Lyapunov theory for a very broad class of systems. However, for some input spaces this definition is far too restrictive, and for other systems simpler and more useful restatements of the ISS Lyapunov function concept may be more useful. The most important input spaces we consider next.

REMARK 4.6 (L^p input spaces). *For spaces $\mathcal{U} = L^p(\mathbb{R}_+, U)$, for a Banach space U and some $p \in [1, +\infty)$, Definition 4.2 is far too restrictive. Indeed, for any $u \in \mathcal{U} = L^p(\mathbb{R}_+, U)$ it holds that* $\inf_{\tau\ge 0} \|u_\tau\|_\mathcal{U} = 0$, *and thus the inequality* (4.11) *reduces to*

$$\dot{V}_u(x) \le -\alpha(\|x\|_X), \tag{4.12}$$

which ensures a much stronger property than ISS, namely the uniform global asymptotic stability (for a precise definition see, e.g., [32]).

Coercive and non-coercive ISS Lyapunov theory which is appropriate for systems with $\mathcal{U} = L^p(\mathbb{R}_+, U)$, $p \in [1, +\infty)$, has been developed in [25].

REMARK 4.7 (Piecewise continuous input functions). *If $\mathcal{U} = PC_b(\mathbb{R}_+, U)$, for a Banach space U, then the dissipation inequality* (4.11) *simplifies to*

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U), \tag{4.13}$$

which resembles the classical dissipation inequality, used in the ISS theory of ODE systems. Similarly to Proposition 4.3, one can show that for a forward-complete system $\Sigma = (X, PC_b(\mathbb{R}_+, U), \phi)$ the existence of a non-coercive ISS Lyapunov function implies integral-to-integral ISS.

In the rest of this section let $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$. In this case our definition of ISS Lyapunov function seems to be the most natural. For forward complete finite-dimensional systems with f as in the statement of Corollary 3.6, existence of a non-coercive ISS Lyapunov function implies not only ISS, but also integral-to-integral ISS, which follows from Proposition 4.3 and Corollary 3.6. However, for infinite-dimensional systems we were able to show only somewhat weaker property (which is still stronger than norm-to-integral ISS, but weaker than integral-to-integral ISS),

see (4.10). Thus, a question remains whether existence of an ISS Lyapunov function (coercive or non-coercive) as defined in Definition 4.2 implies integral-to-integral ISS for forward complete systems. Although we do not have an answer to this problem, in the following proposition we show that if an ISS Lyapunov function satisfies a somewhat stronger dissipative estimate, then integral-to-integral ISS can be verified.

PROPOSITION 4.8. *Let $\Sigma = (X, \mathcal{U}, \phi)$ be a forward complete control system with the input space $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$.*

Assume that there is a continuous function $V : X \to \mathbb{R}_+$, $\psi_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that (4.4) holds and the Dini derivative of V along the trajectories of Σ for all $x \in X$ and $u \in \mathcal{U}$ satisfies

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \Big[D_{+,\tau}\Big(\int_0^\tau \sigma(\|u(s)\|_U)ds\Big)\Big]_{\tau=0}, \tag{4.14}$$

where $D_{+,\tau}$ means that lower right-hand Dini derivative is taken with respect to the argument τ.

Then Σ is integral-to-integral ISS.

Before we prove this proposition note that the estimate (4.14) implies (4.5), since for any $x \in X$ and $u \in \mathcal{U}$ it holds that

$$\Big[D_{+,\tau}\Big(\int_0^\tau \sigma(\|u(s)\|_U)ds\Big)\Big]_{\tau=0} \le \Big[D_{+,\tau}\Big(\int_0^\tau \sigma(\|u\|_\mathcal{U})ds\Big)\Big]_{\tau=0} = \sigma(\|u\|_\mathcal{U}).$$

Thus, function V as in Proposition 4.8 is an ISS Lyapunov function for Σ, with a (potentially) stronger dissipative estimate.

Proof. Assume that V is a non-coercive ISS Lyapunov function for Σ with the corresponding ψ_2, α, σ. Pick any $u \in \mathcal{U}$ and any $x \in X$. As we assume forward completeness of Σ, the trajectory $\phi(\cdot, x, u)$ exists for all $t \ge 0$ and due to (4.5), we have for any $t > 0$ that:

$$\dot{V}_{u(t+\cdot)}\big(\phi(t,x,u)\big) \le -\alpha(\|\phi(t,x,u)\|_X) + \Big[D_{+,\tau}\Big(\int_0^\tau \sigma(\|u(t+s)\|_U)ds\Big)\Big]_{\tau=0}. \tag{4.15}$$

The last term can be rewritten in a simpler form:

$$\begin{aligned}\Big[D_{+,\tau}\Big(\int_0^\tau \sigma(\|u(t+s)\|_U)ds\Big)\Big]_{\tau=0} &= \Big[D_{+,\tau}\Big(\int_t^{t+\tau} \sigma(\|u(s)\|_U)ds\Big)\Big]_{\tau=0}\\ &= \Big[D_{+,\tau}\Big(\int_0^{t+\tau} \sigma(\|u(s)\|_U)ds\Big)\Big]_{\tau=0} = D_{+,t}\Big(\int_0^t \sigma(\|u(s)\|_U)ds\Big).\end{aligned}$$

By definition of $\dot{V}$, and using the cocycle property for Σ, we have that

$$\begin{aligned}\dot{V}_{u(t+\cdot)}\big(\phi(t,x,u)\big) &= \overline{\lim_{h\to+0}}\,\frac{1}{h}\Big(V\big(\phi(h,\phi(t,x,u),u(t+\cdot))\big) - V\big(\phi(t,x,u)\big)\Big)\\ &= \overline{\lim_{h\to+0}}\,\frac{1}{h}\Big(V\big(\phi(t+h,x,u)\big) - V\big(\phi(t,x,u)\big)\Big).\end{aligned}$$

Defining $y(t) := V\big(\phi(t,x,u)\big)$, we see that

$$\dot{V}_{u(t+\cdot)}\big(\phi(t,x,u)\big) = D^+ y(t),$$

and $y(0) = V(x)$ due to the identity axiom of the system Σ.

In view of the continuity axiom of Σ, for fixed x, u the map $\phi(\cdot, x, u)$ is continuous, and thus $t \mapsto -\alpha(\|\phi(t,x,u)\|_X)$ is continuous as well. Hence, we can rewrite the inequality (4.15) as

$$D^+y(t) \leq -\frac{d}{dt}\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds + D_+\Big(\int_0^t \sigma(\|u(s)\|_U)ds\Big).$$

As $-D_+\big(\int_0^t \sigma(\|u(s)\|_U)ds\big) = D^+\big(-\int_0^t \sigma(\|u(s)\|_U)ds\big)$, we proceed to

$$D^+y(t) + \frac{d}{dt}\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds + D^+\Big(-\int_0^t \sigma(\|u(s)\|_U)ds\Big) \leq 0.$$

Due to

$$D^+(f_1(t)+f_2(t)) \leq D^+(f_1(t)) + D^+(f_2(t)),$$

which holds for any functions f_1, f_2 on the real line, this implies that

$$D^+\Big(y(t) + \int_0^t \alpha(\|\phi(s,x,u)\|_X)ds - \int_0^t \sigma(\|u(s)\|_U)ds\Big) \leq 0.$$

It follows from [42, Theorem 2.1] that $t \mapsto y(t)+\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds-\int_0^t \sigma(\|u(s)\|_U)ds$ is nonincreasing. As $G(0) = 0$, for all $r > 0$ we have

$$y(t) + G(t) - \int_0^t \sigma(\|u(s)\|_U)ds \leq y(0) = V(x) \leq \psi_2(\|x\|_X).$$

Now $y(t) \geq 0$ for all $t \in \mathbb{R}_+$, and so

$$\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \leq \psi_2(\|x\|_X) + \int_0^t \sigma(\|u(s)\|_U)ds.$$

This completes the proof. □

5. Construction of ISS Lyapunov functions for linear systems with unbounded input operators. In the remainder of this paper we specialize to the case of complex Hilbert spaces X and to the input spaces $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$, where U is a Banach space.

A classical method for construction of Lyapunov functions for exponentially stable semigroups is the solution of the operator Lyapunov equation [4, Theorem 5.1.3]. This method can also be used for the construction of ISS Lyapunov functions for systems with bounded input operators [30].

The following result holds (this is a Hilbert-space version of [30, Proposition 6], and thus we omit the proof):

PROPOSITION 5.1. *Let X be a complex Hilbert space, $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$, where U is a Banach space, A generate a strongly continuous semigroup over X and let $B \in L(U,X)$. If (2.5) is ISS, then there is an operator $P = P^* \in L(X)$ so that $\langle Px, x\rangle > 0$ for $x \neq 0$ and P solves the Lyapunov equation*

$$\langle Px, Ax\rangle_X + \langle Ax, Px\rangle_X = -\|x\|_X^2, \quad x \in D(A). \tag{5.1}$$

Furthermore, $V : X \to \mathbb{R}_+$ defined by

$$V(x) = \langle Px, x\rangle \tag{5.2}$$

is a non-coercive ISS Lyapunov function for (2.5).

REMARK 5.2. *It is well-known that the existence of a positive solution to the Lyapunov equation* (5.1) *is equivalent to the exponential stability of the semigroup generated by A, see [4, Theorem 5.1.3], and thus to the ISS of the system* (2.5) *(for admissible B). The main contribution of Proposition 5.1 is that the quadratic function V, defined by* (5.2), *is also a non-coercive ISS Lyapunov function for* (2.5).

Note that the operator P and the Lyapunov function V can be chosen independently on the bounded input operator B (the function σ in the dissipative estimate (4.5) however does depend on B). In the next theorem we derive a counterpart of Proposition 5.1 for systems with merely admissible operators B. In contrast to systems with bounded input operators, we need further assumptions that relate the operators P and A.

THEOREM 5.3. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a complex Hilbert space X and let $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$, where U is a Banach space.*

Assume that there is an operator $P \in L(X)$ satisfying the following conditions:

(i) P satisfies

$$\operatorname{Re}\langle Px, x\rangle_X > 0, \quad x \in X\backslash\{0\}. \tag{5.3}$$

(ii) $\operatorname{Im}(P) \subset D(A^*)$.

(iii) PA has an extension to a bounded operator on X, that is, $PA \in L(X)$. (We also denote this extension by PA.)

(iv) P satisfies the Lyapunov inequality

$$\operatorname{Re}\langle (PA + A^*P)x, x\rangle_X \leq -\langle x, x\rangle_X, \quad x \in D(A), \tag{5.4}$$

Then for any ∞-admissible input operator $B \in L(U, X_{-1})$ the function

$$V(x) := \operatorname{Re}\langle Px, x\rangle_X \tag{5.5}$$

is a non-coercive ISS Lyapunov function for (2.5), *which satisfies for each $\varepsilon > 0$ the dissipation inequality*

$$\dot{V}_u(x_0) \leq (\varepsilon - 1)\|x_0\|_X^2 + c(\varepsilon)\|u\|_\infty^2, \tag{5.6}$$

where

$$\begin{aligned} c(\varepsilon) := \frac{1}{4\varepsilon}\big(\|A^*P\|_{L(X)} + \|PA\|_{L(X)}\big)^2\|A_{-1}^{-1}B\|_{L(U,X)}^2 M^2 \\ + M\|A^*P\|_{L(X)}\|A_{-1}^{-1}B\|_{L(U,X)}\kappa(0). \end{aligned} \tag{5.7}$$

and $\kappa(0) = \lim_{t\searrow 0}\kappa(t)$, where $\kappa(t) > 0$ is the smallest constant satisfying

$$\left\|\int_0^t T_{-1}(t-s)Bu(s)\,ds\right\|_X \leq \kappa(t)\|u\|_\infty, \tag{5.8}$$

for every $u \in L^\infty([0,t), U)$. (The existence of the constants $\kappa(t)$ is implied by the ∞-admissibility of B.)

In particular, existence of a non-coercive ISS Lyapunov function (5.5) *implies* (2.5) *is ISS for any ∞-admissible B.*

REMARK 5.4. *Note that we have to take the real parts of the expressions in* (5.5) *and* (5.4), *as we deal with complex Hilbert spaces and we do not assume that P is a positive operator on X.*

REMARK 5.5. *If in addition to the assumptions of Theorem 5.3 the operator P is self-adjoint, that is, if $P = P^*$, then equation* (5.1) *is equivalent to* (5.4).

Proof. Note that linear systems with admissible input operators satisfy both the CEP and the BRS property, which follows easily from inequality (2.6). Due to Theorem 4.4, Σ is ISS if V is a non-coercive ISS Lyapunov function for Σ.

By the assumptions

$$0 < V(x) \leq \|P\|_{L(X)}\|x\|_X^2, \quad x \in X\backslash\{0\},$$

and thus (4.4) holds. It remains to show the dissipation inequality (4.5) for V.

The operator $A : D(A) \subset X \to X$ is densely defined as an infinitesimal generator of a C_0-semigroup, and hence A^* is well-defined and again the generator of a C_0-semigroup, see [35, Corollary 10.6]. In particular, this implies that A^* is a closed operator. Since $P \in L(X)$, the operator $S := A^*P$ with the domain $D(S) := \{x \in X : Px \in D(A^*)\}$ is a closed operator, see [45, Exercise 5.6]. However, by our assumptions $\mathrm{Im}(P) \subset D(A^*)$, which implies $D(S) = X$, and thus $S = A^*P \in L(X)$ by the Closed Graph Theorem. In particular, the term $\|A^*P\|_{L(X)}$ in (5.7) makes sense.

For $x_0 \in X$ and $u \in L^\infty([0,\infty), U)$ we have

$$\begin{aligned}
(5.9)\quad & V(\phi(t,x_0,u)) - V(x_0) \\
&= \mathrm{Re}\left\langle P\left(T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds\right),\right. \\
&\qquad \left. T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X - \mathrm{Re}\,\langle Px_0, x_0\rangle_X \\
&= \mathrm{Re}\,\langle PT(t)x_0, x_0\rangle_X - \mathrm{Re}\,\langle Px_0, x_0\rangle_X \\
(5.10)\quad &\quad + \mathrm{Re}\,\langle PT(t)x_0, T(t)x_0\rangle_X - \mathrm{Re}\,\langle PT(t)x_0, x_0\rangle_X \\
(5.11)\quad &\quad + \mathrm{Re}\left\langle PT(t)x_0, \int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
(5.12)\quad &\quad + \mathrm{Re}\left\langle P\int_0^t T_{-1}(t-s)Bu(s)ds, T(t)x_0\right\rangle_X \\
(5.13)\quad &\quad + \mathrm{Re}\left\langle P\int_0^t T_{-1}(t-s)Bu(s)ds, \int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X .
\end{aligned}$$

The terms in line (5.10) of the previous expression can be transformed into:

$$(5.14)\quad \mathrm{Re}\,\langle PT(t)x_0, T(t)x_0\rangle_X - \mathrm{Re}\,\langle PT(t)x_0, x_0\rangle_X = \mathrm{Re}\,\langle PT(t)x_0, T(t)x_0 - x_0\rangle_X.$$

Applying [4, Theorem 5.1.3] to the operator $\frac{1}{2}(P+P^*)$, we see that the conditions (i), (ii) and (iv) imply that A generates an exponentially stable semigroup. This implies (see e.g. [13, Proposition 5.2.4]) that $0 \in \rho(A)$ and thus $A^{-1} \in L(X)$ exists. Further, the exponential stability of the C_0-semigroup $(T(t))_{t\geq 0}$ implies

$$\|T(t)\|_{L(X)} \leq Me^{-\omega t}, \quad t \geq 0,$$

for some constants M, $\omega > 0$. Thanks to $\rho(A) = \rho(A_{-1})$ the operator A_{-1}^{-1} exists as well.

By [7, Theorem II.5.5] the map $A : D(A) \to X$ can be continuously extended to the linear isometry A_{-1} which maps $(X, \|\cdot\|_X)$ onto $(X_{-1}, \|\cdot\|_{X_{-1}})$. Hence A_{-1}^{-1}, mapping $(X_{-1}, \|\cdot\|_{X_{-1}})$ onto $(X, \|\cdot\|_X)$, is again a linear isometry, and thus a bounded

operator. As $B \in L(U, X_{-1})$, we have that $A_{-1}^{-1}B \in L(U, X)$. In particular, $T_{-1}(t-s)A_{-1}^{-1}Bu(s) = T(t-s)A_{-1}^{-1}Bu(s) \in X$ for all $s \geq 0$. Due to the fact, that A_{-1}^{-1} and $T_{-1}(t-s)$ commute, we obtain

$$\begin{aligned}
\left\|A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\right\|_X &= \left\|\int_0^t A_{-1}^{-1}T_{-1}(t-s)Bu(s)ds\right\|_X \\
&= \left\|\int_0^t T_{-1}(t-s)A_{-1}^{-1}Bu(s)ds\right\|_X. \\
&\leq \int_0^t \|T(t-s)\|_{L(X)}\|A_{-1}^{-1}B\|_{L(U,X)}\|u(s)\|_U ds \\
&\leq \int_0^t M ds \|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty \\
&\leq Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty. \qquad (5.15)
\end{aligned}$$

Since $\mathrm{Im}(P) \subset D(A^*)$, we estimate the expression in (5.11) using the Cauchy-Schwarz inequality and (5.15)

$$\begin{aligned}
&\mathrm{Re}\left\langle PT(t)x_0, \int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
&\quad = \mathrm{Re}\left\langle PT(t)x_0, AA^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
&\quad = \mathrm{Re}\left\langle A^*PT(t)x_0, A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
&\quad \leq \|A^*PT(t)x_0\|_X \cdot \left\|A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\right\|_X \\
&\quad \leq \|A^*P\|_{L(X)}\|T(t)x_0\|_X \cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty. \qquad (5.16)
\end{aligned}$$

To bound the expression (5.12) we use again (5.15) to obtain

$$\begin{aligned}
&\mathrm{Re}\left\langle P\int_0^t T_{-1}(t-s)Bu(s)ds, T(t)x_0\right\rangle_X \\
&\quad = \mathrm{Re}\left\langle PAA^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds, T(t)x_0\right\rangle_X \\
&\quad \leq \|PA\|_{L(X)} \cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty\|T(t)x_0\|_X. \qquad (5.17)
\end{aligned}$$

Finally, we estimate the expression (5.13) using (5.15) and κ as defined in (5.8)

$$\begin{aligned}
&\mathrm{Re}\left\langle P\int_0^t T_{-1}(t-s)Bu(s)ds, \int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
&\quad = \mathrm{Re}\left\langle P\int_0^t T_{-1}(t-s)Bu(s)ds, AA_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
&\quad = \mathrm{Re}\left\langle A^*P\int_0^t T_{-1}(t-s)Bu(s)ds, A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X \\
&\quad \leq \|A^*P\|_{L(X)}\kappa(t)\|u\|_\infty \cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty. \qquad (5.18)
\end{aligned}$$

Substituting (5.14), (5.16), (5.17), and (5.18) into (5.9), we obtain:

$$\begin{aligned}V(\phi(t,x_0,u))-V(x_0)\leq{}&\operatorname{Re}\langle PT(t)x_0-Px_0,x_0\rangle_X+\operatorname{Re}\langle PT(t)x_0,T(t)x_0-x_0\rangle_X\\&+\|A^*P\|_{L(X)}\|T(t)x_0\|_X\cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty\\&+\|PA\|_{L(X)}\cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty\|T(t)x_0\|_X\\&+\|A^*P\|_{L(X)}\kappa(t)\|u\|_\infty\cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty.\end{aligned}$$

For $x_0\in X$, we have $A^{-1}x_0\in D(A)$ and we obtain from the definition of the generator A that

$$\begin{aligned}\overline{\lim_{t\searrow 0}}\operatorname{Re}\frac{1}{t}\langle PT(t)x_0-Px_0,x_0\rangle_X&=\overline{\lim_{t\searrow 0}}\operatorname{Re}\frac{1}{t}\langle PA[T(t)A^{-1}x_0-A^{-1}x_0],x_0\rangle_X\\&=\operatorname{Re}\langle PAx_0,x_0\rangle_X\end{aligned}$$

and similarly

$$\begin{aligned}\overline{\lim_{t\searrow 0}}\operatorname{Re}\langle PT(t)x_0,T(t)x_0-x_0\rangle_X&=\overline{\lim_{t\searrow 0}}\operatorname{Re}\frac{1}{t}\langle A^*PT(t)x_0,T(t)A^{-1}x_0-A^{-1}x_0\rangle_X\\&=\operatorname{Re}\langle A^*Px_0,x_0\rangle_X.\end{aligned}$$

This implies for every $\varepsilon>0$ that (recall the definition of $\kappa(0)$ before (5.8))

$$\begin{aligned}\dot V_u(x_0)&=\overline{\lim_{t\searrow 0}}\frac{1}{t}(V(\phi(t,x_0,u))-V(x_0))\\&\leq\operatorname{Re}\langle PAx_0,x_0\rangle_X+\operatorname{Re}\langle A^*Px_0,x_0\rangle_X\\&\quad+\|A^*P\|_{L(X)}\|x_0\|_X\|A_{-1}^{-1}B\|_{L(U,X)}M\|u\|_\infty\\&\quad+\|PA\|_{L(X)}\|A_{-1}^{-1}B\|_{L(U,X)}M\|u\|_\infty\|x_0\|_X\\&\quad+\|A^*P\|_{L(X)}\|A_{-1}^{-1}B\|_{L(U,X)}M\kappa(0)\|u\|_\infty^2\\&=\operatorname{Re}\langle PAx_0,x_0\rangle_X+\operatorname{Re}\langle A^*Px_0,x_0\rangle_X\\&\quad+\|x_0\|_X(\|A^*P\|_{L(X)}+\|PA\|_{L(X)})\|A_{-1}^{-1}B\|_{L(U,X)}M\|u\|_\infty\\&\quad+\|A^*P\|_{L(X)}\|A_{-1}^{-1}B\|_{L(U,X)}M\kappa(0)\|u\|_\infty^2.\end{aligned}$$

Using Young's inequality and the estimate (5.4) we proceed to

$$\begin{aligned}\dot V_u(x_0)\leq{}&-\|x_0\|_X^2+\varepsilon\|x_0\|_X^2+\frac{(\|A^*P\|_{L(X)}+\|PA\|_{L(X)})^2\|A_{-1}^{-1}B\|_{L(U,X)}^2M^2}{4\varepsilon}\|u\|_\infty^2\\&+\|A^*P\|_{L(X)}\|A_{-1}^{-1}B\|_{L(U,X)}M\kappa(0)\|u\|_\infty^2,\end{aligned}$$

which shows the dissipation inequality (5.6), and thus also (4.5). □

Remark 5.6. *Theorem 5.3 has been formulated as a direct Lyapunov theorem. However, the following reformulation as a partial converse result is also possible. Assume that* (2.5) *is ISS, and the solution P of the Lyapunov equation* (5.1) *satisfies* $\operatorname{Im}(P)\subset D(A^*)$ *and PA is bounded. Then* (5.5) *is an ISS Lyapunov function for* (2.5).

It is of virtue to compare the ISS Lyapunov theorem for bounded input operators (Proposition 5.1) and ISS Lyapunov theorem for admissible input operators (Theorem 5.3). The ISS Lyapunov function candidate considered in both these results, is the same. What differs is the assumptions and the set of input operators, for which

this function is indeed an ISS Lyapunov function. Proposition 5.1 states that if the semigroup, generated by A is exponentially stable, then there is an operator P, which satisfies the assumptions (i) and (iv) of Theorem 5.3 and the condition $P = P^*$, and furthermore (5.2) is an ISS Lyapunov function for (2.5) for any bounded input operator. *Thus, the key additional assumptions which we impose in order to tackle the unboundedness of an input operator, are the assumptions (ii) and (iii).* We note, that that with these assumptions (5.5) is an ISS Lyapunov function for any ∞-admissible operator B.

6. Applications of Theorem 5.3. In this section we show applicability of Theorem 5.3 for some important special cases. We start with sufficient conditions, guaranteeing that Theorem 5.3 can be applied with $P = -A^{-1}$. Then we show that these sufficient conditions are fulfilled for broad classes of systems, generated by subnormal operators. Finally, we proceed to diagonal semigroups (whose generators are self-adjoint operators) and finally we give a construction of a non-coercive ISS Lyapunov function for a heat equation with Dirichlet boundary inputs.

6.1. A special case: $P = -A^{-1}$. In this section we give sufficient conditions for the applicability of Theorem 5.3 with $P := -A^{-1}$.

PROPOSITION 6.1. *Let A be the generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on a (complex) Hilbert space X and let $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$, where U is a Banach space.*

Further, assume that

(a) $D(A) \subset D(A^*)$

(b) there is $\delta > 0$ such that for every $x \in X$ we have

$$\operatorname{Re}\langle A^*A^{-1}x, x\rangle_X + \delta\|x\|_X^2 \geq 0 \tag{6.1}$$

(c) $\operatorname{Re}\langle Ax, x\rangle_X < 0$ holds for every $x \in D(A)\backslash\{0\}$.

Then

$$V(x) := -\operatorname{Re}\langle A^{-1}x, x\rangle_X \tag{6.2}$$

is an ISS Lyapunov function for (2.5) for any ∞-admissible operator $B \in L(U, X_{-1})$.

Proof. As A generates an exponentially stable semigroup, $0 \in \rho(A)$ and thus $P := -A^{-1} \in L(X)$. We show step by step that this choice of P satisfies all the requirements (i)–(iv) of Theorem 5.3.

(i). For any $x \in X\backslash\{0\}$ there is $y \in D(A)\backslash\{0\}$ so that $x = Ay$. Then by the assumptions of the proposition it holds that

$$V(x) = -\operatorname{Re}\langle y, Ay\rangle = -\operatorname{Re}\langle Ay, y\rangle > 0.$$

(ii). We have $\operatorname{Im}(P) = \operatorname{Im}(A^{-1}) = D(A) \subset D(A^*)$, which holds by our assumptions.

(iii). Trivial as $PA = -I$.

(iv). By assumptions there is a $\delta < 1$ so that

$$\begin{aligned}\operatorname{Re}\langle (PA + A^*P)x, x\rangle_X = \operatorname{Re}\left\langle (-I - A^*A^{-1})x, x\right\rangle_X &= -\langle x, x\rangle_X - \operatorname{Re}\left\langle A^*A^{-1}x, x\right\rangle_X \\ &\leq -(1-\delta)\langle x, x\rangle_X ,\end{aligned}$$

and thus P satisfies the Lyapunov inequality up to a scaling coefficient (and $\tilde{P} := \frac{1}{1-\delta}P$ satisfies precisely (5.4)).

Hence all assumptions of Theorem 5.3 are satisfied, and application of Theorem 5.3 shows the claim. □

REMARK 6.2. *If $D(A) \subset D(A^*)$, then inequality* (6.1) *is equivalent to the existence of a constant $\delta' < 1$ satisfying*

$$\|(A + A^*)x\|_X^2 + \delta' \|Ax\|_X^2 \geq \|A^*x\|_X^2, \qquad x \in D(A).$$

If A generates a strongly continuous contraction semigroup, then (6.1) *implies that the semigroup $(T(t))_{t\geq 0}$ is 2-hypercontractive [11]. In particular, subnormal and normal operators whose spectrum lie in a sector, satisfy* (6.1)*, see Proposition 6.6.*

6.2. Analytic semigroups generated by subnormal operators. In this section we show that Theorem 5.3 can be applied to a broad class of analytic semigroups over Hilbert spaces generated by subnormal operators.

A closed, densely-defined operator A on a Hilbert space X is called *subnormal*, if there is a Hilbert space Z containing X as a subspace and a normal operator $(N, D(N)) : Z \to Z$ so that $A = N_{|X}$ (the restriction of N to X) and X is an invariant subspace for N, that is, $N(D(N) \cap X) \subset X$. We write P for the orthogonal projection from Z onto X. That is, up to unitary equivalence $N = M_\phi$, a multiplication operator on some $L^2(\mu)$ space, and $Ax = \phi x$, $A^*x = P(\overline{\phi} x)$. See, for example [3, Th. X.4.19].

We denote the spectrum of a linear closed operator A by $\sigma(A)$. It holds that:

LEMMA 6.3. *A subnormal operator A satisfies $D(A) \subset D(A^*)$. Further, there exists a minimal normal extension N satisfying $\sigma(N) \subset \sigma(A)$.*

Proof. The first assertion follows from $D(N) = D(N^*)$, see [3, Prop. X.4.3]. The second assertion is proved in [34, Theorem 2.3]. □

EXAMPLE 6.4.

1. *Clearly, every normal operator on a Hilbert space is subnormal.*
2. *Symmetric operators on Hilbert spaces and analytic Toeplitz operators T_g on the Hardy space $H^2(D)$ are subnormal, [34].*

For $\theta \in [0, \pi/2)$ we define

$$S_\theta := \{s \in \mathbb{C} \mid |\arg(-s)| \leq \theta\}.$$

PROPOSITION 6.5. *Let A be a subnormal operator on a Hilbert space X and assume $\sigma(A) \subset S_\theta$, for some $\theta \in [0, \pi/2)$, and $B \in L(\mathbb{C}^m, X_{-1})$. Then:*

(i) A generates a bounded analytic C_0-semigroup of contractions $(T(t))_{t\geq 0}$,

(ii) B is ∞-admissible for $(T(t))_{t\geq 0}$.

Moreover, if in addition $0 \notin \sigma(A)$, then

(iii) A generates an exponentially stable semigroup and the system (2.5) *is ISS.*

Proof. Assertion (i) follows from Lemma 6.3, the fact that normal operators N with $\sigma(N) \subset S_\theta$ generate bounded analytic C_0-semigroups, see [7, Corollary II.4.7], and from the observation that A is the restriction of N to an invariant subspace. The assertion (ii) has been proved in [12]. Finally, A generates an exponentially stable semigroup since for analytic semigroups the spectral bound equals the growth bound [7, Corollary IV.3.12], and ISS follows as B is assumed to be ∞-admissible, see Example 2.8. □

We have the following important inequality for the subnormal operators:

PROPOSITION 6.6. *Let A be a subnormal operator on a Hilbert space X satisfying $\sigma(A) \subset S_\theta$, for some $\theta \in [0, \pi/2)$. Then for $\delta \geq 1 - 2\cos^2\theta$ we have*

$$\operatorname{Re}\langle x, A^2x\rangle_X + \delta\|Ax\|_X^2 \geq 0, \qquad x \in D(A^2). \tag{6.3}$$

Proof. Expanding (6.3) we obtain the equivalent assertion

$$\operatorname{Re}\langle \phi x, P\overline{\phi} x\rangle + \delta\|\phi x\|^2 \geq 0, \tag{6.4}$$

and we note that $\langle \phi x, P\overline{\phi}x\rangle = \langle \phi x, \overline{\phi}x\rangle = \langle \phi^2 x, x\rangle$. The left hand side of (6.4) is

$$\langle (\operatorname{Re}\phi^2 + \delta|\phi|^2)x, x\rangle = \langle (2(\operatorname{Re}\phi)^2 + (\delta-1)|\phi|^2)x, x\rangle.$$

As the essential range of ϕ lies in $\sigma(A)$, we have by sectoriality

$$2(\operatorname{Re}\phi)^2 \geq 2\cos^2\theta|\phi|^2$$

and hence

$$\langle (2(\operatorname{Re}\phi)^2 + (\delta-1)|\phi|^2)x, x\rangle \geq 0,$$

for $\delta \geq 1 - 2\cos^2\theta$. □

Now we can derive a converse ISS Lyapunov theorem for a broad class of systems with subnormal generators:

COROLLARY 6.7. *Let A be a subnormal operator on a Hilbert space X satisfying $\sigma(A) \subset S_\theta\backslash\{0\}$, for some $\theta \in [0, \pi/2)$. Further, let $B \in L(\mathbb{C}^m, X_{-1})$ and let $\mathcal{U} := L^\infty(\mathbb{R}_+, \mathbb{C}^m)$.*

Then

$$V(x) := -\operatorname{Re}\langle A^{-1}x, x\rangle_X \tag{6.5}$$

is an ISS Lyapunov function for (2.5) satisfying

$$\dot{V}_u(x) \leq -c_1\|x_0\|_X^2 + c_2\|u\|_\infty^2$$

for some constants $c_1, c_2 > 0$ and all $x_0 \in X$ and $u \in \mathcal{U}$.

Proof. By Proposition 6.5 A generates an exponentially stable and analytic C_0-semigroup of contractions $(T(t))_{t\geq 0}$ and B is ∞-admissible for $(T(t))_{t\geq 0}$. Further, Lemma 6.3 guarantees that $D(A) \subset D(A^*)$. As A generates a contraction semigroup, the Lumer-Phillips theorem ensures that the operator A is dissipative (that is, $\operatorname{Re}\langle Ax, x\rangle \leq 0$ for $x \in D(A)$). This together with $0 \in \rho(A)$ implies $\operatorname{Re}\langle Ax, x\rangle < 0$ for $x \in D(A)\backslash\{0\}$.

Furthermore, as $0 \in \rho(A)$, for all $y \in D(A)$ there is $x \in D(A^2)$ so that $y = Ax$ and applying Proposition 6.6 we obtain

$$0 \leq \operatorname{Re}\langle x, A^2x\rangle_X + \delta\|Ax\|_X^2 = \operatorname{Re}\langle A^*A^{-1}y, y\rangle_X + \delta\|y\|_X^2.$$

This shows (6.1).

Hence all assumptions of Proposition 6.1 are satisfied, and application of Proposition 6.1 shows the claim. □

REMARK 6.8. *The above corollary also holds if we replace $B \in L(\mathbb{C}^m, X_{-1})$ by an ∞-admissible $B \in L(U, X_{-1})$, where U is a Hilbert space.*

6.3. ISS Lyapunov functions for input-to-state stable diagonal systems. Consider a linear system (2.5) with the state space

$$X = l_2(\mathbb{N}) := \big\{x = \{x_k\}_{k=1}^{\infty} : \|x\|_X = \Big(\sum_{k=1}^{\infty} |x_k|^2\Big)^{1/2} < \infty\big\}.$$

endowed in the usual way with the scalar product $\langle \cdot, \cdot \rangle$. Let $U := \mathbb{R}$.

Consider an operator $A : X \to X$, defined by $Ae_k = -\lambda_k e_k$, where e_k is the k-th unit vector of $l_2(\mathbb{N})$ and $\lambda_k \in \mathbb{R}$ with $\lambda_k < \lambda_{k+1}$ for all k, $\lambda_1 > \varepsilon > 0$ and $\lambda_k \to \infty$ as $k \to \infty$.

The operator A can be represented using the spectral decomposition

$$Ax := \sum_{k=1}^{\infty} -\lambda_k \langle x, e_k \rangle e_k, \quad x \in D(A), \tag{6.6}$$

with

$$D(A) = \{x \in l_2(\mathbb{N}) : \sum_{k=1}^{\infty} -\lambda_k \langle x, e_k \rangle e_k \text{ converges}\}. \tag{6.7}$$

We have the following result:

PROPOSITION 6.9. *Let A be given by* (6.6)-(6.7) *and $B \in L(\mathbb{C}^m, X_{-1})$. Then* (2.5) *is ISS and*

$$V(x) := \sum_{k=1}^{\infty} -\frac{1}{\lambda_k} \langle x, e_k \rangle^2 \tag{6.8}$$

is a non-coercive ISS Lyapunov function for (2.5).

Proof. By assumptions the operator A is self-adjoint with $\sigma(A) \subset (-\infty, 0)$. Thus the assumptions of Corollary 6.7 are satisfied. Furthermore, the inverse of A is given by

$$A^{-1}x := \sum_{k=1}^{\infty} -\frac{1}{\lambda_k} \langle x, e_k \rangle e_k, \tag{6.9}$$

and thus the Lyapunov function (6.5) has the form (6.8). □

It is easy to see that P (as well as the corresponding ISS Lyapunov function V) is not coercive since $\lambda_k \to \infty$ as $k \to \infty$.

6.4. ISS Lyapunov functions for a heat equation with Dirichlet boundary input. It is well-known that a classical heat equation with Dirichlet boundary inputs is ISS, which has been verified by means of several different methods: [10, 18, 27]. However, no constructions for ISS Lyapunov functions have been proposed. In the next example we show that using Theorem 5.3 one can construct a non-coercive ISS Lyapunov function for this system.

EXAMPLE 6.10. *Let us consider the following boundary control system given by the one-dimensional heat equation on the spatial domain* $[0, 1]$ *with Dirichlet boundary control at the point* 1*,*

$$\begin{aligned} x_t(\xi, t) &= a x_{\xi\xi}(\xi, t), \quad \xi \in (0, 1),\ t > 0, \\ x(0, t) &= 0, \quad x(1, t) = u(t), \quad t > 0, \\ x(\xi, 0) &= x_0(\xi), \end{aligned}$$

where $a > 0$. We refer the reader to [44, Chapter 10] for the definition and properties of boundary control systems. We choose $X = L^2(0,1)$, $U = \mathbb{C}$ and $\mathcal{U} := L^\infty(\mathbb{R}_+, \mathbb{C})$. Every boundary control system can be equivalently written in the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A generates a C_0-semigroup on X and $B \in L(U, X_{-1})$, see [44, Proposition 10.1.2 and Remark 10.1.4]. For the one-dimensional heat equation on the spatial domain $[0,1]$ with Dirichlet boundary conditions the operator A is given by

$$Af = af'', \quad f \in D(A) := \left\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \right\}.$$

Here $H^2(0,1)$ denotes the Sobolev space of functions $f \in L^2(0,1)$, which have weak derivatives of order ≤ 2, all of which belong to $L^2(0,1)$. It is well-known that A is a self-adjoint operator on X generating an exponentially stable analytic C_0-semigroup on X. By [12, Theorem 1 and Proposition 5], we get $B \in L(U, X_{-1})$ is ∞-admissible, for every $x_0 \in X$ and $u \in L^\infty(0,\infty)$ the corresponding mild solution is continuous with respect to time and $\kappa(0) = 0$. In [10] the following ISS-estimates have been shown:

$$\|x(t)\|_{L^2(0,1)} \leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + \frac{1}{\sqrt{3}}\|u\|_{L^\infty(0,t)},$$
$$\|x(t)\|_{L^2(0,1)} \leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + c\left(\int_0^t |u(s)|^p ds\right)^{1/p},$$

for every $x_0 \in X$, $u \in \mathcal{U}$, $p > 2$ and some constant $c = c(p) > 0$. Direct application of Corollary 6.7 shows that

$$V(x) = -\langle A^{-1}x, x\rangle_X = \int_0^1 \left(\int_\xi^1 (\xi - \tau)x(\tau)d\tau\right)\overline{x(\xi)}d\xi$$

is a non-coercive ISS Lyapunov function for the one-dimensional heat equation on the spatial domain $[0,1]$ with Dirichlet boundary control at the point 1. In turn, the constructed non-coercive ISS Lyapunov function implies ISS of the considered system.

7. Conclusions. In this paper, we have investigated the question to what extent the existence of a non-coercive ISS Lyapunov function implies that a forward complete system is input-to-state stable (ISS). It was shown that the property of norm-to-integral ISS follows from the existence of such Lyapunov functions for a large class of systems. Furthermore, we show that norm-to-integral ISS is equivalent to ISS for the systems possessing the continuity of the flow map near the equilibrium and boundedness of finite-time reachability sets. These assumptions are related to questions of the richness of the possible dynamics both close to the origin and in the large.

Non-coercive Lyapunov functions are to some extent natural in infinite dimensions. Already Datko's construction of quadratic Lyapunov functions $V(x) = \langle Px, x\rangle$ for exponentially stable linear systems on Hilbert space generally leads to non-coercive Lyapunov functions. In this work, we show that under some additional conditions, which relate the infinitesimal generator of a semigroup and an operator P, this function V is a non-coercive ISS Lyapunov function for a linear system with any ∞-admissible input operator. Furthermore, we have shown in this paper that for broad classes of linear systems with unbounded input operators (including analytic systems

with subnormal generators) the construction of Lyapunov functions using the resolvent at 0 as an operator P is a natural choice and one that leads to noncoercive Lyapunov functions. As an example we have constructed an ISS Lyapunov function for a heat equation with a Dirichlet boundary input, which seems to be the first construction of an ISS Lyapunov function for this system, which was widely studied by non-Lyapunov methods.

In a future work we plan to extend the class of systems for which explicit constructions are possible and to deepen our understanding of noncoercive ISS Lyapunov functions.

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