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Simplicity criteria for rings of differential operators

V. V. Bavula

Abstract

Let K be a field of arbitrary characteristic, $\mathcal{A}$ be a commutative K-algebra which is a domain of essentially finite type (eg, the algebra of functions on an irreducible affine algebraic variety), $\mathfrak{a}_r$ be its *Jacobian ideal*, $\mathcal{D}(\mathcal{A})$ be the algebra of differential operators on the algebra $\mathcal{A}$. The aim of the paper is to give a simplicity criterion for the algebra $\mathcal{D}(\mathcal{A})$: *The algebra $\mathcal{D}(\mathcal{A})$ is simple iff $\mathcal{D}(\mathcal{A})\mathfrak{a}_r^i\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})$ for all $i \geq 1$ provided the field K is a perfect field.* Furthermore, a simplicity criterion is given for the algebra $\mathcal{D}(R)$ of differential operators on an arbitrary commutative algebra R over an arbitrary field. This gives an answer to an old question to find a simplicity criterion for algebras of differential operators.

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1 Introduction

The following notation will remain fixed throughout the paper (if it is not stated otherwise): K is a field of arbitrary characteristic (not necessarily algebraically closed), module means a left module, $P_n = K[x_1, \ldots, x_n]$ is a polynomial algebra over K, $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n} \in \mathrm{Der}_K(P_n)$, $I := \sum_{i=1}^m P_n f_i$ is a **prime** but **not** a maximal ideal of the polynomial algebra P_n with a set of generators $f_1, \ldots, f_m$, the algebra $A := P_n/I$ which is a domain with the field of fractions $Q := \mathrm{Frac}(A)$, the epimorphism $\pi : P_n \to A$, $p \mapsto \overline{p} := p + I$, to make notation simpler we sometime write x_i for $\overline{x}_i$ (if it does not lead to confusion), the **Jacobian** $m \times n$ matrices $J = (\frac{\partial f_i}{\partial x_j}) \in M_{m,n}(P_n)$ and $\overline{J} = (\overline{J}_{ij}) \in M_{m,n}(A) \subseteq M_{m,n}(Q)$ where $\overline{J}_{ij} := \overline{\frac{\partial f_i}{\partial x_j}}$, $r := \mathrm{rk}_Q(\overline{J})$ is the **rank** of the Jacobian matrix $\overline{J}$ over the field Q, $\mathfrak{a}_r$ is the **Jacobian ideal** of the algebra A which is (by definition) generated by all the $r \times r$ minors of the Jacobian matrix $\overline{J}$ (Suppose that K is a perfect field. Then the algebra A is *regular* iff $\mathfrak{a}_r = A$, it is the **Jacobian criterion of regularity**, [5, Theorem 16.19]). For $\mathbf{i} = (i_1, \ldots, i_r)$ such that $1 \leq i_1 < \cdots < i_r \leq m$ and $\mathbf{j} = (j_1, \ldots, j_r)$ such that $1 \leq j_1 < \cdots < j_r \leq n$, $\Delta(\mathbf{i}, \mathbf{j})$ denotes the corresponding minor of the Jacobian matrix $\overline{J} = (\overline{J}_{ij})$, that is $\det(\overline{J}_{i_\nu, j_\mu})$, $\nu, \mu = 1, \ldots, r$, and the element $\mathbf{i}$ (resp., $\mathbf{j}$) is called **non-singular** if $\Delta(\mathbf{i}, \mathbf{j}') \neq 0$ (resp., $\Delta(\mathbf{i}', \mathbf{j}) \neq 0$) for some $\mathbf{j}'$ (resp., $\mathbf{i}'$). We denote by $\mathbf{I}_r$ (resp., $\mathbf{J}_r$) the set of all the non-singular r-tuples $\mathbf{i}$ (resp., $\mathbf{j}$).

Since r is the rank of the Jacobian matrix $\overline{J}$, it is easy to show that $\Delta(\mathbf{i}, \mathbf{j}) \neq 0$ iff $\mathbf{i} \in \mathbf{I}_r$ and $\mathbf{j} \in \mathbf{J}_r$, [3, Lemma 2.1].

A localization of an *affine* algebra is called an algebra of **essentially finite type**. Let $\mathcal{A} := S^{-1}A$ be a localization of the algebra $A = P_n/I$ at a multiplicatively closed subset S of A. Suppose that K is a perfect field. Then the algebra $\mathcal{A}$ is *regular* iff $\mathfrak{a}_r = \mathcal{A}$ where $\mathfrak{a}_r$ is the Jacobian ideal of $\mathcal{A}$, it is the **Jacobian criterion of regularity**, [5, Theorem 16.19]. For any regular algebra $\mathcal{A}$ over a perfect field, explicit sets of generators and defining relations for the algebra $\mathcal{D}(\mathcal{A})$ are given in [3] (char(K)=0) and [4] (char(K) > 0).

Let R be an arbitrary commutative K-algebra. We denote by $\mathcal{D}(R)$ the algebra of differential operators on the algebra R and by $\mathrm{Der}_K(R)$ the R-module of K-derivations of R. The action of a derivation δ on an element a is denoted by $\delta(a)$.

Simplicity criterion for the algebra $\mathcal{D}(\mathcal{A})$ where the algebra $\mathcal{A}$ is a domain of essentially finite type. Theorem 1.1 is a simplicity criterion for the algebra $\mathcal{D}(\mathcal{A})$ where the algebra $\mathcal{A}$ is a domain of essentially finite type.

Theorem 1.1 *Let the K-algebra $\mathcal{A}$ be a commutative domain of essentially finite type over a perfect field K, and $\mathfrak{a}_r$ be its Jacobian ideal. The following statements are equivalent:*

1. *The algebra $\mathcal{D}(\mathcal{A})$ of differential operators on $\mathcal{A}$ is a simple algebra.*
2. *For all $i \geq 1$, $\mathcal{D}(\mathcal{A})\mathfrak{a}_r^i\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})$.*
3. *For all $k \geq 1$, $\mathbf{i} \in \mathbf{I}_r$ and $\mathbf{j} \in \mathbf{J}_r$, $\mathcal{D}(\mathcal{A})\Delta(\mathbf{i},\mathbf{j})^k\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})$.*

As an application of Theorem 1.1 we show that the algebra of differential operators on the cusp is simple.

Simplicity criterion for the algebra $\mathcal{D}(R)$ where R is an arbitrary commutative algebra. An ideal $\mathfrak{a}$ of the algebra R is called $\mathrm{Der}_K(R)$-*stable* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ for all $\delta \in \mathrm{Der}_K(R)$. Theorem 1.2.(2) is a simplicity criterion for the algebra $\mathcal{D}(R)$ where R is an arbitrary commutative algebra. Theorem 1.2.(1) shows that every nonzero ideal of the algebra $\mathcal{D}(R)$ meets the subalgebra R of $\mathcal{D}(R)$. If, in addition, the algebra $R = \mathcal{A}$ is a domain of essentially finite type, Theorem 1.2.(3) shows that every nonzero ideal of the algebra $\mathcal{D}(R)$ contains a power of the Jacobian ideal of $\mathcal{A}$.

Theorem 1.2 *Let R be a commutative algebra over an arbitrary field K.*

1. *Let I be a nonzero ideal the algebra $\mathcal{D}(R)$. Then the ideal $I_0 := I \cap R$ is a nonzero $\mathrm{Der}_K(R)$-stable ideal of the algebra R such that $\mathcal{D}(R)I_0\mathcal{D}(R) \cap R = I_0$. In particular, every nonzero ideal of the algebra $\mathcal{D}(R)$ has nonzero intersection with R.*
2. *The ring $\mathcal{D}(R)$ is not simple iff there is a proper $\mathrm{Der}_K(R)$-stable ideal $\mathfrak{a}$ of R such that $\mathcal{D}(R)\mathfrak{a}\mathcal{D}(R) \cap R = \mathfrak{a}$.*
3. *Suppose, in addition, that K is a perfect field and the algebra $\mathcal{A} = R$ is a domain of essentially finite type, $\mathfrak{a}_r$ be its Jacobian ideal, I be a nonzero ideal of $\mathcal{D}(\mathcal{A})$ and $I_0 = I \cap \mathcal{A}$. Then $\mathfrak{a}_r^i \subseteq I_0$ for some $i \geq 1$.*

Theorem 1.3 is simplicity criterion for the ring of differential operators on an irreducible affine algebraic curve.

Theorem 1.3 *([10, Theorem B]) Let X be an irreducible affine algebraic curve over an algebraically closed field K of characteristic zero, $\tilde{X}$ be its normalization and $\pi : \tilde{X} \to X$ be the natural projection, $\mathcal{D}(X)$ and $\mathcal{D}(\tilde{X})$ be the rings of differential operators on X and $\tilde{X}$, respectively. The following are equivalent:*

1. *π is injective.*
2. *$\mathcal{D}(X)$ is a simple ring (and Morita equivalent to $\mathcal{D}(\tilde{X})$).*
3. *The global dimension of the ring $\mathcal{D}(X))$ is 1.*
4. *$gr(\mathcal{D}(x))$ is a finitely generated K-algebra (equivalently, $gr(\mathcal{D}(x))$ is Noetherian).*

Theorem 1.4 is simplicity criterion for the ring of differential operators on an irreducible affine algebraic surface with smooth normalization.

Theorem 1.4 *([9, Corollary 3.5]) Suppose that X is an irreducible affine algebraic variety of dimension 2, such that $\tilde{X}$ is non-singular. Then $\mathcal{D}(X)$ is a simple ring if and only if X is S_2 and $\pi : \tilde{X} \to X$ is injective. Furthermore, in this case $\mathcal{D}(X)$ is Noetherian and Morita equivalent to $\mathcal{D}(\tilde{X})$.*

2 Proofs of Theorem 1.1 and Theorem 1.2

In this section, proofs of Theorem 1.1 and Theorem 1.2 are given.

Let R be a commutative K-algebra. The ring of (K-linear) **differential operators** $\mathcal{D}(R)$ on R is defined as a union of R-modules $\mathcal{D}(R) = \bigcup_{i=0}^{\infty} \mathcal{D}_i(R)$ where

$$\mathcal{D}_i(R) = \{u \in \mathrm{End}_K(R) \mid [r,u] := ru - ur \in \mathcal{D}_{i-1}(R) \text{ for all } r \in R\}, \ \ i \geq 0, \ \ \mathcal{D}_{-1}(R) := 0.$$

In particular, $\mathcal{D}_0(R) = \mathrm{End}_R(R) \simeq R$, $(x \mapsto bx) \leftrightarrow b$. The set of R-bimodules $\{\mathcal{D}_i(R)\}_{i\geq 0}$ is the **order filtration** for the algebra $\mathcal{D}(R)$:

$$\mathcal{D}_0(R) \subseteq \mathcal{D}_1(R) \subseteq \cdots \subseteq \mathcal{D}_i(R) \subseteq \cdots \ \text{ and } \ \mathcal{D}_i(R)\mathcal{D}_j(R) \subseteq \mathcal{D}_{i+j}(R) \ \text{ for all } \ i,j \geq 0.$$

The subalgebra $\Delta(R)$ of $\mathcal{D}(R)$ which is generated by $R \equiv \mathrm{End}_R(R)$ and the set $\mathrm{Der}_K(R)$ of all K-derivations of R is called the **derivation ring** of R.

Suppose that R is a regular affine domain of Krull dimension $n \geq 1$ and char(K)=0. In geometric terms, R is the coordinate ring $\mathcal{O}(X)$ of a smooth irreducible affine algebraic variety X of dimension n. Then

- $\mathrm{Der}_K(R)$ *is a finitely generated projective* R*-module of rank* n,
- $\mathcal{D}(R) = \Delta(R)$,
- $\mathcal{D}(R)$ *is a simple (left and right) Noetherian domain of Gelfand-Kirillov dimension* $\mathrm{GK}\ \mathcal{D}(R) = 2n$ ($n = \mathrm{GK}\,(R) = \mathrm{Kdim}(R)$).

For the proofs of the statements above the reader is referred to [7], Chapter 15. So, the domain $\mathcal{D}(R)$ is a simple finitely generated infinite dimensional Noetherian algebra ([7], Chapter 15).

If char(K) > 0 then $\mathcal{D}(R) \neq \Delta(R)$ and the algebra $\mathcal{D}(R)$ is not finitely generated and neither left nor right Noetherian but analogues of the results above hold but the Gelfand-Kirillov dimension has to replaced by a new dimension introduced in [2].

Given a ring B and a non-nilpotent element $s \in B$. Suppose that the set $S_s := \{s^i \mid i \geq 0\}$ is a left denominator set of B. The localization $S_s^{-1}B$ of the ring B at S_s is also denoted by B_s.

Proof of Theorem 1.2. 1. (i) *The ideal* I_0 *of* R *is a* $\mathrm{Der}_K(R)$*-stable ideal*: For all $\delta \in \mathrm{Der}_K(R)$, $I_0 \supseteq [\delta, I_0] = \delta(I_0)$.

(ii) $\mathcal{D}(R)I_0\mathcal{D}(R) \cap R = I_0$: $I_0 \subseteq \mathcal{D}(R)I_0\mathcal{D}(R) \cap R \subseteq \mathcal{D}(R)I\mathcal{D}(R) \cap R = I \cap R = I_0$, and the statement (ii) follows.

(iii) $I_0 \neq 0$: Recall that the ring $\mathcal{D}(R)$ admits the order filtration $\{\mathcal{D}(R)_i\}_{i\geq 0}$. Therefore, $I = \bigcup_{i\geq 0} I_i$ where $I_i = I \cap \mathcal{D}(R)_i$. Let $s = \min\{i \geq 0 \mid I_i \neq 0\}$. Then $I_s \neq 0$ and

$$[r, I_s] \subseteq I_{s-1} = \{0\} = \mathcal{D}_{-1}(R) \ \text{ for all } \ r \in R,$$

i.e. $I_s \subseteq \mathcal{D}(R)_0 = R$, by the *definition of the order filtration* on $\mathcal{D}(R)$, and so $s = 0$, as required.

2. ($\Rightarrow$) If the ring $\mathcal{D}(R)$ is not simple then there is proper ideal, say I, of $\mathcal{D}(R)$. Then, by the statements (i) and (ii) in the proof of statement 1, it suffices to take $\mathfrak{a} = I_0$.

($\Leftarrow$) The implication is obvious.

3. Recall that the Jacobian ideal $\mathfrak{a}_r$ of the algebra $\mathcal{A}$ is generated by the *finite* set $\{\Delta(\mathbf{i},\mathbf{j}) \mid \mathbf{i} \in \mathbf{I}_r, \mathbf{j} \in \mathbf{J}_r\}$. For each element $\Delta(\mathbf{i},\mathbf{j})$, the algebra $\mathcal{A}_{\Delta(\mathbf{i},\mathbf{j})}$ is a regular domain of essentially finite type. So, the algebra $\mathcal{D}(\mathcal{A}_{\Delta(\mathbf{i},\mathbf{j})}) \simeq \mathcal{D}(\mathcal{A})_{\Delta(\mathbf{i},\mathbf{j})}$ is simple (the algebra $\mathcal{D}(\mathcal{A})_{\Delta(\mathbf{i},\mathbf{j})}$ is a left and right localization of $\mathcal{D}(\mathcal{A})$ at the powers of the element $\Delta(\mathbf{i},\mathbf{j})$). Therefore, $1 \in I_{\Delta(\mathbf{i},\mathbf{j})}$, and so $\Delta(\mathbf{i},\mathbf{j})^l \in I \cap \mathcal{A} = I_0$ for some $l \geq 1$. So, $\mathfrak{a}_r^i \subseteq I_0$ for some $i \geq 1$. $\Box$

Proof of Theorem 1.1. ($1 \Rightarrow 3$) The implication is trivial.

($3 \Rightarrow 2$) The implication follows from the fact that the Jacobian ideal $\mathfrak{a}_r$ of the algebra $\mathcal{A}$ is generated by the finite set $\{\Delta(\mathbf{i},\mathbf{j}) \mid \mathbf{i} \in \mathbf{I}_r, \mathbf{j} \in \mathbf{J}_r\}$. In particular, $\Delta(\mathbf{i},\mathbf{j})^k \subseteq \mathfrak{a}_r^k$ for all $k \geq 1$, and so $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})\Delta(\mathbf{i},\mathbf{j})^k\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})\mathfrak{a}_r^k\mathcal{D}(\mathcal{A})$.

$(2 \Rightarrow 1)$ Suppose that the algebra $\mathcal{D}(\mathcal{A})$ is not simple, we seek a contradiction. Fix a proper ideal, say I, of the algebra $\mathcal{D}(\mathcal{A})$. By Theorem 1.2.(3), $\mathfrak{a}_r^i \subseteq I_0$ for some natural number $i \geq 1$. Then

$$\mathcal{A} \neq I_0 = \mathcal{D}(\mathcal{A}) I_0 \mathcal{D}(\mathcal{A}) \cap \mathcal{A} \supseteq \mathcal{D}(\mathcal{A}) \mathfrak{a}_r^i \mathcal{D}(\mathcal{A}) \cap \mathcal{A}.$$

Therefore, $\mathcal{D}(\mathcal{A}) \mathfrak{a}_r^i \mathcal{D}(\mathcal{A}) \neq \mathcal{D}(\mathcal{A})$, a contradiction. $\square$

Given a commutative algebra R, we denote by $\mathcal{C}_R$ the set of *regular elements* of R (i.e. non-zero-divisors) and by $Q(R) := \mathcal{C}_R^{-1} R$ its *quotient algebra*.

Corollary 2.1 *Let $\mathcal{A}$ be a semiprime commutative algebra with finitely many minimal primes. Then the algebra $\mathcal{D}(\mathcal{A})$ is a simple algebra iff the algebra $\mathcal{A}$ is a domain and the algebra $\mathcal{D}(\mathcal{A})$ is a simple algebra.*

Proof. $(\Rightarrow)$ Suppose that the algebra $\mathcal{A}$ is not a domain. Then its quotient algebra $Q(\mathcal{A}) := \mathcal{C}_{\mathcal{A}}^{-1} \mathcal{A} \simeq \prod_{i=1}^s K_i$ is a direct product of fields K_i where $s \geq 2$ is the number of minimal primes of the algebra $\mathcal{A}$. Therefore,

$$\mathcal{C}_{\mathcal{A}}^{-1}(\mathcal{D}(\mathcal{A})) \simeq \mathcal{D}(\mathcal{C}_{\mathcal{A}}^{-1}\mathcal{A}) \simeq \mathcal{D}(Q(\mathcal{A})) \simeq \mathcal{D}(\prod_{i=1}^s K_i) \simeq \prod_{i=1}^s \mathcal{D}(K_i).$$

The algebra $\mathcal{D}(\mathcal{A})$ is an essential left $\mathcal{D}(\mathcal{A})$-submodule of $\mathcal{D}(Q(\mathcal{A}))$. Therefore, the intersection $\mathcal{D}(\mathcal{A}) \cap \mathcal{D}(K_1)$ is a proper ideal of the algebra $\mathcal{D}(\mathcal{A})$ since $s \geq 2$, a contradiction.

$(\Leftarrow)$ The implication is trivial. $\square$

Example. (The algebra of differential operators on the cusp) Let $A = K[x,y]/(y^2 - x^3)$, the algebra of regular functions on the cusp $y^2 = x^3$. The algebra A is isomorphic to the subalgebra $K + \sum_{i \geq 2} Kx^i$ of the polynomial algebra $K[x]$. Notice that $A \subseteq K[x] \subseteq A_x = K[x]_x = K[x, x^{-1}]$ and $\mathcal{D}(K[x,x^{-1}]) = \oplus_{i \in \mathbb{Z}} Dx^i = D[x, x^{-1}; \sigma]$ is a skew Laurent polynomial ring with coefficients in the polynomial algebra $D = K[h]$, where $h = x\partial$, and σ is a K-automorphism of D given by the rule $\sigma(h) = h - 1$. The algebra $A_1 = K\langle x, \partial \,|\, \partial x - x\partial = 1 \rangle$ is called the (first) *Weyl algebra*. Then $A_1 \simeq \mathcal{D}(K[x])$ and $A_{1,x} \simeq \mathcal{D}(K[x])_x \simeq \mathcal{D}(K[x]_x) \simeq D[x, x^{-1}; \sigma] \simeq \mathcal{D}(A)_x$. Notice that $\mathcal{D}(A) = \{\delta \in \mathcal{D}(A)_x \,|\, \delta(A) \subseteq A\}$. The algebra $\mathcal{D}(A)$ is simple, [8, 10], and explicit generators can be found in [6]. Below we give short proofs of these results.

Lemma 2.2 *Let $A = K + \sum_{i \geq 2} Kx^i (\simeq K[x,y]/(y^2 - x^3))$ and $D = K[h]$. Then*

1. *$\mathcal{D}(A) = \bigoplus_{i \in \mathbb{Z}} Dw_i \subseteq \mathcal{D}(A_x)$ where $w_0 = 1$, $w_1 = (h-1)x$ and $w_i = x^i$ for $i \geq 2$; $w_{-1} = (h+1)(h-1)x^{-1}$, $w_{-2} = (h+2)(h-1)x^{-2}$ and $w_{-i} = (h-1) \cdot (h+1) \cdots (h+i-2) \cdot (h+i) x^{-i}$ for $i \geq 3$.*

2. *The algebra $\mathcal{D}(A)$ is a simple finitely generated Noetherian domain. Furthermore, the elements h and w_i ($i = \pm 1, \pm 2, \pm 3$) are algebra generators of $\mathcal{D}(A)$. For all $i \geq 1$, $w_{-2i} = w_{-2}^i$ and $w_{-3-2i} = w_{-3} w_{-2}^i$.*

3. *$\mathrm{Der}_K(A) = K[x]h$, $\Delta(A) = K[h][x;\sigma]$ is a non-simple Noetherian algebra and $\Delta(A) \neq \mathcal{D}(A)$.*

4. *The Jacobian ideal $\mathfrak{a}_1 = \sum_{i \geq 2} Kx^i$ of A is $\Delta(R)$-stable but not $\mathcal{D}(R)$-stable.*

Proof. 1. Recall that $\mathcal{D}(A_x) = \bigoplus_{i \in \mathbb{Z}} Dx^i$ is a $\mathbb{Z}$-graded algebra ($Dx^i Dx^j \subseteq Dx^{i+j}$ for all $i, j \in \mathbb{Z}$), the $\mathcal{D}(A_x)$-module $A_x = K[x, x^{-1}]$ is a $\mathbb{Z}$-graded module and the algebra A is a homogeneous subalgebra of A_x. Now, statement 1 follows from obvious computations and the fact that $\mathcal{D}(A) = \{\delta \in \mathcal{D}(A)_x \,|\, \delta(A) \subseteq A\}$.

2. The Jacobian ideal $\mathfrak{a}_1$ of A is equal to $\sum_{i \geq 2} Kx^i$. Since $x^{2i} \in \mathfrak{a}_1^i$ for all $i \geq 1$, in order to prove simplicity of $\mathcal{D}(A)$ it suffices to show that $(x^i) = \mathcal{D}(A)$ for all $i \geq 2$, by Theorem 1.1. Notice that the polynomials of $D = K[h]$, $w_{-i}x^i$ and $x^i w_{-i}$, are coprime, hence $(x^i) = \mathcal{D}(A)$

for all $i \geq 2$. In more detail, $w_{-2}x^2 = (h+2)(h-1)$ and $x^2 w_{-2} = h(h-3)$; and for $i \geq 3$, $w_{-i}x^i = (h-1)\cdot(h+1)\cdots(h+i-2)\cdot(h+i)$ and $x^i w_{-i} = (h-i-1)\cdot(h-i+1)\cdots(h-2)\cdot h$.

The equalities in statement 2 are obvious. Then, by statement 1, the elements h and w_i ($i = \pm 1, \pm 2, \pm 3$) are algebra generators of $\mathcal{D}(A)$. The subalgebra $\Lambda := K\langle h, w_3, w_{-3}\rangle$ is a generalized Weyl algebra $D[w_3, w_{-3}; \sigma, a = (h+3)(h+1)(h-1)]$ which is a Noetherian algebra, [1]. The algebra $\mathcal{D}(A)$ is a finitely generated left and right Λ-module, hence $\mathcal{D}(A)$ is Noetherian.

3. By statement 2, $\mathrm{Der}_K(A) = K[x]h$ since $w_1 = xh$. The rest follows.

4. The Jacobian ideal $\mathfrak{a}_1$ of A is $\Delta(R)$-stable since $\mathrm{Der}_K(A) = K[x]h$ and $h(\mathfrak{a}_1) \subseteq \mathfrak{a}_1$. Since $x^2 \in \mathfrak{a}_1$ and $\omega_{-2}(x^2) = -2 \notin \mathfrak{a}_1$, so the ideal $\mathfrak{a}_1$ is not $\mathcal{D}(R)$-stable. $\square$

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Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk