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REPRESENTING KERNELS OF PERTURBATIONS OF TOEPLITZ OPERATORS BY BACKWARD SHIFT-INVARIANT SUBSPACES

YUXIA LIANG AND JONATHAN R. PARTINGTON

ABSTRACT. It is well known the kernel of a Toeplitz operator is nearly invariant under the backward shift S^* . This paper shows that kernels of finite rank perturbations of Toeplitz operators are nearly S^* -invariant with finite defect. This enables us to apply a recent theorem by Chalendar–Gallardo–Partington to represent the kernel in terms of backward shift-invariant subspaces, which we identify in several important cases.

1. INTRODUCTION

Let $H(\mathbb{D})$ be the space of all analytic functions on the open unit disc \mathbb{D} . The Hardy space $H^2 := H^2(\mathbb{D})$ is defined by

$$H^2 = \{f \in H(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 < +\infty\}.$$

The limit $\lim_{r \rightarrow 1^-} f(re^{it})$ exists almost everywhere, which gives the values of f on the unit circle \mathbb{T} . Since the H^2 norm of f and the $L^2(\mathbb{T})$ norm of its boundary function coincide, H^2 embeds isometrically as a closed subspace of $L^2(\mathbb{T})$ via

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n e^{int}.$$

This indicates a natural orthogonal decomposition $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$, where H^2 is identified with the subspace spanned by $\{e^{int} : n \geq 0\}$ and $\overline{H_0^2}$ is the subspace spanned by $\{e^{int} : n < 0\}$, respectively.

Let $L^\infty := L^\infty(\mathbb{T})$ be the space containing all essentially bounded functions on \mathbb{T} . And $H^\infty := H^\infty(\mathbb{D})$ is the Banach algebra of bounded

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analytic functions on \mathbb{D} with the norm defined

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

Similarly, the radial boundary function of an H^∞ function belongs to L^∞ , and then H^∞ can be viewed as a Banach subalgebra of L^∞ .

We recall an inner function is an H^∞ function that has unit modulus almost everywhere on \mathbb{T} . An outer function is a function $f \in H^1$ which can be written in the form

$$f(re^{i\eta}) = \alpha \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + re^{i\eta}}{e^{it} - re^{i\eta}} k(e^{it}) dt\right)$$

for $re^{i\eta} \in \mathbb{D}$, where k is a real-valued integrable function and $|\alpha| = 1$. It is known that each $f \in H^1 \setminus \{0\}$ has a factorization $f = \theta \cdot u$, where θ is inner and u is outer. This factorization is unique up to a constant of modulus 1 (cf. [9]).

Let $P : L^2(\mathbb{T}) \rightarrow H^2$ be the orthogonal projection on H^2 defined by a Cauchy integral

$$(Pf)(z) = \int \frac{f(\zeta)}{1 - \bar{\zeta}z} dm(\zeta), \quad |z| < 1.$$

Given $g \in L^\infty$, the Toeplitz operator $T_g : H^2 \rightarrow H^2$ is defined by

$$T_g f = P(gf)$$

for any $f \in H^2$. If θ is an inner function, then $\text{Ker } T_\theta$ is the model space $K_\theta = H^2 \ominus \theta H^2 = H^2 \cap \theta \overline{H^2}$ (cf. [10, 11]). It has also been proved that $\|T_g\| = \|g\|_\infty$ and $T_g^* = T_{\bar{g}}$ (cf. [3]). For more investigations into Toeplitz operators, the reader can refer to [7, 4, 14] and so on.

Beurling's theorem states that the subspaces θH^2 with inner function θ constitute the nontrivial invariant subspaces for the unilateral shift $S : H^2 \rightarrow H^2$ defined by $[Sf](z) = zf(z)$. Also the model space K_θ is invariant under the backward shift $S^* : H^2 \rightarrow H^2$ (cf. [10, Proposition 5.2]) defined by

$$S^* f(z) = \frac{f(z) - f(0)}{z} \quad (f \in H^2, z \in \mathbb{D}).$$

The invariant subspace problem is still an unresolved problem in operator theory and there are various related investigations (cf. [6, 5]). Moreover, the study of nearly S^* -invariant subspaces has attracted a lot of attention (cf. [12, 13, 5]).

Definition 1.1. A subspace $M \subset H^2$ is called nearly S^* -invariant if $S^* f \in M$ whenever $f \in M$ and $f(0) = 0$. Furthermore, a subspace $M \subset H^2$ is said to be nearly S^* -invariant with defect m if there is an

m -dimensional subspace F such that $S^*f \in M + F$ whenever $f \in M$ with $f(0) = 0$; we call F the defect space.

If $f \in \text{Ker } T_g$ with $f(0) = 0$, so $gf \in \overline{H_0^2}$ and then $g(\bar{z}f) \in \overline{H_0^2}$. Since $\bar{z}f \in H^2$, this implies $S^*f = \bar{z}f \in \text{Ker } T_g$, which shows the kernel of a Toeplitz operator is nearly S^* -invariant. Motivated by this well-known result, we continue to examine a question which has a close link with the invariant subspace problem:

Given a Toeplitz operator T_g acting on Hardy space H^2 , is the kernel of a rank n perturbation of T_g nearly S^ -invariant with finite defect?*

We recall that an operator $T : \mathcal{H} \rightarrow \mathcal{H}$ of rank n on a Hilbert space \mathcal{H} takes the form

$$Th = \sum_{i=1}^n \langle h, u_i \rangle v_i \text{ for all } h \in \mathcal{H},$$

where $\{u_i\}$ and $\{v_i\}$ are orthogonal sets in \mathcal{H} (we may also suppose that $\{u_i\}$ is orthonormal). For simplicity, write $A_n := \{1, 2, \dots, n\}$ and let $|\Lambda|$ stand for the number of integers in a set Λ .

A rank n perturbation of the Toeplitz operator $T_g : H^2 \rightarrow H^2$ denoted by $R_n : H^2 \rightarrow H^2$ is defined by

$$R_n(h) = T_g h + Th = T_g h + \sum_{i=1}^n \langle h, u_i \rangle v_i \tag{1.1}$$

with orthonormal set $\{u_i\}$ and orthogonal set $\{v_i\}$ in H^2 .

The rest of the paper is organized as follows. In Section 2, we discuss the nearly S^* -invariant subspace $\text{Ker } R_n$ with finite defect for several important classes of symbols and present the corresponding defect space in each case. Then we apply a recent theorem by Chalendar–Gallardo–Partington to represent the kernel of the operator R_1 in terms of backward shift-invariant subspaces in Section 3. The challenging task here is to identify the subspaces in question, which we do in various important cases. Note that even in the nearly S^* -invariant (defect 0) case, this is known to be a difficult question in general.

2. NEARLY S^* -INVARIANT $\text{Ker } R_n$ WITH FINITE DEFECT

In this section, we prove that the kernel of the operator R_n in (1.1) is nearly S^* -invariant with finite defect for various important cases, especially identify the finite-dimensional defect spaces. First of all, we recall a useful theorem for later use.

Theorem 2.1. [10, Theorem 4.22] *For $\psi, \varphi \in L^\infty$, the operator $T_\psi T_\varphi$ is a Toeplitz operator if and only if either $\bar{\psi} \in H^\infty$ or $\varphi \in H^\infty$. In both cases, $T_\psi T_\varphi = T_{\psi\varphi}$.*

So for all $g \in L^\infty$, it holds that

$$T_{\bar{z}}T_g = T_{\bar{z}g} = T_{g\bar{z}}. \quad (2.1)$$

For every $h \in \text{Ker } R_n$, it follows that

$$T_g h + \sum_{i=1}^n \langle h, u_i \rangle v_i = 0. \quad (2.2)$$

Letting $S^* = T_{\bar{z}}$ act on both sides of (2.2) and using (2.1), we have

$$T_{g\bar{z}} h + \sum_{i=1}^n \langle h, u_i \rangle S^* v_i = 0.$$

Now let $h \in \text{Ker } R_n$ satisfy $h(0) = 0$, and then the above equation implies the following equivalent expressions.

$$T_g\left(\frac{h}{z}\right) + \sum_{i=1}^n \langle h, u_i \rangle S^* v_i = 0 \quad (2.3)$$

$$\Leftrightarrow g\frac{h}{z} + \sum_{i=1}^n \langle h, u_i \rangle S^* v_i \in \overline{H_0^2}. \quad (2.4)$$

So the question of nearly S^* -invariant $\text{Ker } R_n$ with finite defect is that: *for each $h \in \text{Ker } R_n$ with $h(0) = 0$, find a vector w in some suitable finite-dimensional space F such that*

$$S^* h + w = \frac{h}{z} + w \in \text{Ker } R_n,$$

which is equivalent to the following equations.

$$T_g\left(\frac{h}{z} + w\right) + \sum_{i=1}^n \left\langle \frac{h}{z} + w, u_i \right\rangle v_i = 0 \quad (2.5)$$

$$\Leftrightarrow g\left(\frac{h}{z} + w\right) + \sum_{i=1}^n \left\langle \frac{h}{z} + w, u_i \right\rangle v_i \in \overline{H_0^2}. \quad (2.6)$$

Next we will construct the defect space F in several important cases.

2.1. $g = 0$ **a.e. on \mathbb{T} .** In this case, R_n is a rank- n operator and equation (2.5) with $g = 0$ implies

$$\text{Ker } R_n = \bigcap_{i=1}^n (\bigvee \{u_i\})^\perp = H^2 \ominus (\bigvee \{u_i, i \in A_n\}),$$

where \bigvee denotes the closed linear span in H^2 .

For any $h \in \text{Ker } R_n$ with $h(0) = 0$, it always holds that

$$S^*h \in \text{Ker } R_n \oplus (\bigvee \{u_i, i \in A_n\}) = H^2,$$

which gives the following elementary observation on the nearly S^* -invariant subspace $\text{Ker } R_n$ with finite defect.

Proposition 2.2. *Suppose $g = 0$ almost everywhere on \mathbb{T} . Then the subspace $\text{Ker } R_n$ is nearly S^* -invariant with defect n and defect space*

$$F = \bigvee \{u_i, i \in A_n\}.$$

2.2. $g = \theta$ **an inner function.** In this case $T_\theta f = \theta f$ is an isometric multiplication operator on H^2 . For each $h \in \text{Ker } R_n$ with $h(0) = 0$, the relation (2.4) becomes

$$\theta \frac{h}{z} + \sum_{i=1}^n \langle h, u_i \rangle S^* v_i = 0. \quad (2.7)$$

The required relation (2.6) turns into

$$\theta \left(\frac{h}{z} + w \right) + \sum_{i=1}^n \left\langle \frac{h}{z} + w, u_i \right\rangle v_i = 0.$$

Combining it with (2.7), the above equation is equivalent to

$$\left(\theta w - \sum_{k=1}^n \langle h, u_k \rangle S^* v_k \right) + \sum_{i=1}^n \left\langle \left(\theta w - \sum_{k=1}^n \langle h, u_k \rangle S^* v_k \right), \theta u_i \right\rangle v_i = 0. \quad (2.8)$$

Now choosing

$$w = \bar{\theta} \left(\sum_{k=1}^n \langle h, u_k \rangle S^* v_k \right) = \sum_{k=1}^n \langle h, u_k \rangle T_{\bar{\theta}}(S^* v_k) \in H^2,$$

the required equation (2.8) holds. So we can obtain a theorem on the nearly S^* -invariant $\text{Ker } R_n$ with finite defect.

Theorem 2.3. *Suppose $g = \theta$ an inner function. Then the subspace $\text{Ker } R_n$ is nearly S^* -invariant with defect at most n and defect space*

$$F = \bigvee \{T_{\bar{\theta}}(S^* v_i), i \in A_n\}.$$

Example 2.4. For $g(z) = z^m$ ($m \in \mathbb{N}$), $\text{Ker } R_n$ is nearly S^* -invariant with defect at most n and defect space $F = \bigvee \{(S^*)^{m+1}(v_i), i \in A_n\}$.

2.3. $g = f_1 \overline{f_2}$ with $f_j \in \mathcal{GH}^\infty$ for $j = 1, 2$. Here \mathcal{GH}^∞ denotes the set of all invertible elements in H^∞ . In [2], Bourgain proved: *If g is a bounded measurable function on \mathbb{T} , then the condition $\int_{\mathbb{T}} \log |g| dm > -\infty$ (m is the normalized invariant measure on \mathbb{T}) is the necessary and sufficient condition for $g \neq 0$ to be of the form $g = f_1 \cdot \overline{f_2}$ where $f_1, f_2 \in H^\infty$.* The interested reader can also refer to [1, Theorem 4.1] for a matricial version with norm estimates. In this subsection, we suppose $f_j \in \mathcal{GH}^\infty$ for $j = 1, 2$, and then Theorem 2.1 ensures that $T_{f_1 \overline{f_2}} = T_{\overline{f_2}} T_{f_1}$.

For each $h \in \text{Ker } R_n$ with $h(0) = 0$, (2.3) can be rewritten as

$$T_{\overline{f_2}} T_{f_1} \left(\frac{h}{z} \right) + \sum_{i=1}^n \langle h, u_i \rangle S^* v_i = 0, \quad (2.9)$$

which together with Theorem 2.1 imply

$$\frac{h}{z} + \sum_{i=1}^n \langle h, u_i \rangle T_{f_1^{-1}} T_{\overline{f_2}^{-1}} (S^* v_i) = 0. \quad (2.10)$$

The required equation (2.5) is changed into

$$T_{\overline{f_2}} T_{f_1} \left(\frac{h}{z} + w \right) + \sum_{i=1}^n \langle \frac{h}{z} + w, u_i \rangle v_i = 0,$$

which, by (2.9), is equivalent to

$$T_{\overline{f_2}} T_{f_1} w - \sum_{k=1}^n \langle h, u_k \rangle S^* v_k + \sum_{i=1}^n \langle \frac{h}{z} + w, u_i \rangle v_i = 0.$$

Now choosing

$$w = \sum_{k=1}^n \langle h, u_k \rangle T_{f_1^{-1}} T_{\overline{f_2}^{-1}} (S^* v_k)$$

and using (2.10), the result follows. Hence we can present a theorem on the nearly S^* -invariant $\text{Ker } R_n$ with finite defect.

Theorem 2.5. *Suppose $g = f_1 \overline{f_2}$ with $f_j \in \mathcal{GH}^\infty$ for $j = 1, 2$. Then the subspace $\text{Ker } R_n$ is nearly S^* -invariant with defect at most n and defect space*

$$F = \bigvee \{T_{f_1^{-1}} T_{\overline{f_2}^{-1}} (S^* v_i), i \in A_n\}.$$

The following is a remark on two special cases of Theorem 2.5.

Remark 2.6. (i) For the operator R_n in (1.1) with $\bar{g} \in \mathcal{GH}^\infty$, $\text{Ker } R_n$ is nearly S^* -invariant with defect at most n and defect space

$$F = \bigvee \{T_{g^{-1}}(S^*v_i), i \in A_n\}.$$

(ii) For the operator R_n in (1.1) with $g \in \mathcal{GH}^\infty$, $\text{Ker } R_n$ is nearly S^* -invariant with defect at most n and defect space

$$F = \bigvee \{T_{g^{-1}}(S^*v_i), i \in A_n\}.$$

2.4. $g(z) = \overline{\theta(z)}$ **with θ a nonconstant inner function.** In this case, $T_{\bar{\theta}}$ is a special conjugate analytic Toeplitz operator with kernel $K_{\bar{\theta}}$. And then the relation (2.4) becomes

$$\psi := \bar{\theta} \frac{h}{z} + \sum_{k=1}^n \langle h, u_k \rangle S^*v_k \in \overline{H_0^2},$$

with

$$\theta\psi = \frac{h}{z} + \sum_{k=1}^n \langle h, u_k \rangle \theta S^*v_k \in H^2. \quad (2.11)$$

The desired relation (2.6) now takes the form

$$\bar{\theta} \left(\frac{h}{z} + w \right) + \sum_{i=1}^n \left\langle \frac{h}{z} + w, u_i \right\rangle v_i \in \overline{H_0^2}, \quad (2.12)$$

which, by (2.11), is equivalent to

$$\begin{aligned} & \psi - \sum_{k=1}^n \langle h, u_k \rangle S^*v_k + \bar{\theta}w \\ & + \sum_{i=1}^n \left\langle \psi - \sum_{k=1}^n \langle h, u_k \rangle S^*v_k + \bar{\theta}w, \bar{\theta}u_i \right\rangle v_i \in \overline{H_0^2}. \end{aligned} \quad (2.13)$$

We denote the decompositions of u_i and ψ as below: $u_i = u_{i1} + \theta u_{i2}$ with $u_{i1} = P_{K_{\bar{\theta}}}u_i \in K_{\bar{\theta}}$, $u_{i2} \in H^2$, and $\psi = \psi_1 + \psi_2$ with $\psi_1 \in B := \bigvee \{\bar{\theta}u_{i1}, i \in A_n\} \subset \overline{H_0^2}$ and $\psi_2 \in \overline{H_0^2} \ominus B$. So it is clear that

$$\langle \psi_2, \bar{\theta}u_i \rangle = 0 \text{ and } \langle \psi_2, u_{i2} \rangle = 0 \text{ for all } i \in A_n.$$

The above indicates that (2.13) is equivalent to

$$\begin{aligned} & \left(\psi_1 - \sum_{k=1}^n \langle h, u_k \rangle S^*v_k + \bar{\theta}w \right) \\ & + \sum_{i=1}^n \left\langle \left(\psi_1 - \sum_{k=1}^n \langle h, u_k \rangle S^*v_k + \bar{\theta}w \right), \bar{\theta}u_{i1} + u_{i2} \right\rangle v_i \in \overline{H_0^2}. \end{aligned}$$

Choosing

$$w = \sum_{k=1}^n \langle h, u_k \rangle \theta S^* v_k - \theta \psi_1,$$

the above desired relation is true and the defect space F is

$$F = \bigvee \{ \theta S^* v_i, P_{K_\theta} u_i, i \in A_n \} = \bigvee \{ \theta S^* v_i, P_{K_\theta} u_k, i \in A_n, k \in \Lambda \},$$

where Λ denotes the subset of A_n consisting of all $k \in A_n$ such that $P_{K_\theta} u_k \neq 0$, i.e. $\theta \nmid u_k$. So in conclusion we have the following theorem.

Theorem 2.7. *Suppose $g(z) = \overline{\theta(z)}$ with θ an inner function. Then the subspace $\text{Ker } R_n$ is nearly S^* -invariant with defect at most $n + |\Lambda|$ and defect space*

$$F = \bigvee \{ \theta S^* v_i, P_{K_\theta} u_k, i \in A_n, k \in \Lambda \},$$

with $\Lambda \subset A_n$ consisting of all $k \in A_n$ such that $\theta \nmid u_k$.

3. THE APPLICATION OF THE C-G-P THEOREM

In this section, we apply a recent theorem (for short the C-G-P Theorem) by Chalendar–Gallardo–Partington to represent the kernels of rank one perturbations of Toeplitz operators in terms of backward shift-invariant subspaces. We shall take $n = 1$ and denote the operator

$$R_1 f = T_g f + \langle f, u \rangle v$$

with $\|u\| = 1$ and $S^* v \neq 0$. First we cite the C-G-P Theorem on nearly S^* -invariant subspaces with defect m from [8].

Theorem 3.1. [8, Theorem 3.2] *Let M be a closed subspace that is nearly S^* -invariant with defect m . Then*

(1) *in the case where there are functions in M that do not vanish at 0, then*

$$M = \{ f : f(z) = k_0(z) f_0(z) + z \sum_{j=1}^m k_j(z) e_j(z) : (k_0, \dots, k_m) \in K \},$$

where f_0 is the normalized reproducing kernel for M at 0, $\{e_1, \dots, e_m\}$ is any orthonormal basis for the defect space F , and K is a closed $S^* \oplus \dots \oplus S^*$ -invariant subspace of the vector-valued Hardy space $H^2(\mathbb{D}, \mathbb{C}^{m+1})$, and $\|f\|^2 = \sum_{j=0}^m \|k_j\|^2$.

(2) *in the case where all functions in M vanish at 0, then*

$$M = \{ f : f(z) = z \sum_{j=1}^m k_j(z) e_j(z) : (k_1, \dots, k_m) \in K \},$$

with the same notation as in (1), except that K is now a closed $S^* \oplus \dots \oplus S^*$ -invariant subspace of the vector-valued Hardy space $H^2(\mathbb{D}, \mathbb{C}^m)$, and $\|f\|^2 = \sum_{j=1}^m \|k_j\|^2$.

The following proposition asserts that the kernels of some Toeplitz operators with special symbols are model spaces.

Proposition 3.2. [10, Proposition 5.8] *Let $\varphi \in H^\infty \setminus \{0\}$ and let η be the inner factor of φ , then*

$$\text{Ker } T_{\overline{\varphi}} = K_\eta.$$

Now we apply the C-G-P Theorem to represent $\text{Ker } R_1$ by backward shift-invariant subspaces in several important cases. Note that we can find K as the largest S^* -invariant subspace such that

$$S^{*n}k_0f_0 + z \sum_{j=1}^m S^{*n}k_j e_j \in M \quad \text{or} \quad z \sum_{j=1}^m S^{*n}k_j e_j \in M$$

for all $n \in \mathbb{N}$.

3.1. $g = 0$ **a.e. on \mathbb{T} .** In this case $M = \text{Ker } R_1 = H^2 \ominus \bigvee \{u\}$, which is a vector hyperplane. It is clear that such a hyperplane is the solution of a single linear equation. Also Proposition 2.2 showed that $\text{Ker } R_1$ is nearly S^* -invariant with a 1-dimensional defect space $F = \bigvee \{u\}$. Using Theorem 3.1, we deduce a corollary on the representations of $\text{Ker } R_1$.

Corollary 3.3. *Given a nearly S^* -invariant subspace $M = H^2 \ominus \bigvee \{u\}$ with defect 1, let $f_0 = P_M 1 = 1 - \overline{u(0)}u$, $v_0 = P(u - u(0)|u|^2)$ and $v_1 = P(\overline{z}|u|^2)$. Then*

(1) *in the case $P_M 1 \neq 0$, we have*

$$M = \{f : f = k_0 f_0 + k_1 z u : (k_0, k_1) \in K\},$$

with an $S^ \oplus S^*$ -invariant subspace $K = \{(k_0, k_1) : \langle k_0, z^n v_0 \rangle + \langle k_1, z^n v_1 \rangle = 0 \text{ for } n \in \mathbb{N}\}$.*

(2) *in the case $P_M 1 = 0$, we have*

$$M = \{f : f = k_1 z u : k_1 \in K\},$$

with an S^ -invariant subspace $K = \{k_1 : \langle k_1, z^n v_1 \rangle = 0 \text{ for } n \in \mathbb{N}\}$.*

Here we show some examples illustrating the variety of subspaces K that can occur.

Example 3.4. (i) Suppose $u = 1$, then $M = zH^2$, $f_0 = 0$ and $v_0 = v_1 = 0$. So Corollary 3.3 (2) implies M has the representation

$$M = \{f : f = z k_1 : k_1 \in K\}$$

with $K = H^2$ a trivial S^* -invariant subspace.

(ii) Suppose u is a nonconstant inner function, then $M = K_u \oplus zuH^2$, $f_0 = 1 - \overline{u(0)}u \neq 0$ and $v_0 = u - u(0)$, $v_1 = 0$. So Corollary 3.3 (1) implies M has the representation

$$M = \{f : f = k_0(1 - \overline{u(0)}u) + k_1zu : (k_0, k_1) \in K\}$$

with an $S^* \oplus S^*$ -invariant subspace $K = K_\eta \times H^2$, where η is the inner factor of v_0 . Besides, Proposition 3.2 is used to show that K is backward shift-invariant.

(iii) Suppose u is a normalized reproducing kernel of H^2 , that is $u(z) = \sqrt{1 - |\alpha|^2}(1 - \bar{\alpha}z)^{-1}$, $\alpha \in \mathbb{D} \setminus \{0\}$, then $M = \{f : f(\alpha) = 0\}$, $f_0 = \bar{\alpha}(\alpha - z)(1 - \bar{\alpha}z)^{-1} \neq 0$ and $v_0 = 0$, $v_1 = \bar{\alpha}(1 - \bar{\alpha}z)^{-1}$. So Corollary 3.3 (1) implies M has the representation

$$M = \{f : f = \bar{\alpha}k_0 \frac{\alpha - z}{1 - \bar{\alpha}z} + zk_1 \frac{\sqrt{1 - |\alpha|^2}}{1 - \bar{\alpha}z} : (k_0, k_1) \in K\},$$

with an $S^* \oplus S^*$ -invariant subspace $K = H^2 \times \{0\}$.

(iv) Suppose $u(z) = (1 + z^k)/\sqrt{2}$ with $k \geq 1$, then $M = \bigvee\{1 - z^k, z, \dots, z^{k-1}, z^{k+1}, z^{k+2}, \dots\}$, $f_0 = 2^{-1}(1 - z^k) \neq 0$ and $v_0 = z^k/(2\sqrt{2})$, $v_1 = 2^{-1}z^{k-1}$. So Corollary 3.3 (1) implies M has the representation

$$M = \{f : f = k_0 \frac{1 - z^k}{2} + zk_1 \frac{1 + z^k}{\sqrt{2}} : (k_0, k_1) \in K\}$$

with an $S^* \oplus S^*$ -invariant subspace $K = \{(k_0, k_1) : \sqrt{2}(S^*)^{k-1}k_1 = -(S^*)^k k_0, k_0 \in H^2\}$.

3.2. $g = \theta$ an inner function. In this case, $M = \text{Ker } R_1 \subset \bigvee\{\bar{\theta}v\}$. Take any vector $f = \lambda\bar{\theta}v \in M$ satisfying $R_1 f = 0$, which is equivalent to $\lambda(1 + \langle \bar{\theta}v, u \rangle) = 0$. If $1 + \langle \bar{\theta}v, u \rangle \neq 0$, then $\lambda = 0$, meaning $M = \{0\}$ a trivial S^* -invariant subspace. So suppose $1 + \langle \bar{\theta}v, u \rangle = 0$, and then $M = \bigvee\{\bar{\theta}v\}$, which is nearly S^* -invariant with a 1-dimensional defect space $F = \bigvee\{S^*(\bar{\theta}v)\}$ from Theorem 2.3. So Theorem 3.1 implies a corollary on the representations of $\text{Ker } R_1$.

Corollary 3.5. *Given a nearly S^* -invariant subspace $M = \bigvee\{\bar{\theta}v\}$ with defect 1, then*

(1) *in the case $a_0 := \langle \bar{\theta}v, 1 \rangle \neq 0$, let $f_0 = P_M 1 = \bar{a}_0 \|v\|^{-2} \bar{\theta}v$, we have*

$$M = \{f : f = k_0 f_0 : (k_0, 0) \in K\},$$

with an $S^ \oplus S^*$ -invariant subspace $K = \mathbb{C} \times \{0\}$.*

(2) *in the case $a_0 := \langle \bar{\theta}v, 1 \rangle = 0$, we have*

$$M = \{f : f = \|S^*(\bar{\theta}v)\|^{-1} k_1 \bar{\theta}v : k_1 \in K\},$$

with an S^* -invariant subspace $K = \mathbb{C}$.

Proof. (1) in this case, using Theorem 3.1 (1), we represent M by

$$\begin{aligned} M &= \left\{ f : f = k_0 f_0 + z k_1 \frac{S^*(\bar{\theta}v)}{\|S^*(\bar{\theta}v)\|} : (k_0, k_1) \in K \right\} \\ &= \left\{ f : f = \frac{\bar{a}_0}{\|v\|^2} k_0 \bar{\theta}v + \|S^*(\bar{\theta}v)\|^{-1} k_1 (\bar{\theta}v - a_0) : (k_0, k_1) \in K \right\}. \end{aligned}$$

Since $M = \bigvee \{\bar{\theta}v\}$, it yields

$$k_0 \in \mathbb{C} \text{ and } \|S^*(\bar{\theta}v)\|^{-1} k_1 (\bar{\theta}v - a_0) = \mu \bar{\theta}v \text{ with } \mu \in \mathbb{C},$$

which is equivalent to $k_0 \in \mathbb{C}$ and $k_1 = 0$ due to $a_0 \neq 0$. So the statement (1) is true. The statement (2) can be similarly shown by Theorem 3.1 (2). \square

3.3. $g = f_1 \overline{f_2}$ with $f_j \in \mathcal{GH}^\infty$ for $j = 1, 2$. In this case, $M = \text{Ker } R_1 \subset \bigvee \{f_1^{-1}(T_{\overline{f_2^{-1}v}})\}$. Take any vector $f = \lambda f_1^{-1}(T_{\overline{f_2^{-1}v}}) \in M$ such that $R_1 f = 0$, which is equivalent to $\lambda(1 + \langle f_1^{-1}(T_{\overline{f_2^{-1}v}}), u \rangle) = 0$. It is clear $M = \{0\}$ is a trivial S^* -invariant subspace for $1 + \langle f_1^{-1}T_{\overline{f_2^{-1}v}}, u \rangle \neq 0$. Now we always assume $1 + \langle f_1^{-1}(T_{\overline{f_2^{-1}v}}), u \rangle = 0$, and obtain $M = \bigvee \{f_1^{-1}(T_{\overline{f_2^{-1}v}})\}$, which is nearly S^* -invariant with a 1-dimensional defect space $F = \bigvee \{f_1^{-1}T_{\overline{f_2^{-1}v}}(S^*v)\}$ from Theorem 2.5. Denote the Taylor coefficients of $T_{\overline{f_2^{-1}v}}$ and f_1^{-1} by $\{a_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$, respectively. So $\langle f_1^{-1}T_{\overline{f_2^{-1}v}}, 1 \rangle = a_0 b_0$, and using Theorem 3.1, we deduce a corollary on the representations of $\text{Ker } R_1$.

Corollary 3.6. *Given a nearly S^* -invariant subspace*

$M = \bigvee \{f_1^{-1}(T_{\overline{f_2^{-1}v}})\}$ *with defect 1, then*

(1) *in the case $a_0 b_0 \neq 0$, let $f_0 = P_M 1 = \overline{a_0 b_0} \|f_1^{-1}T_{\overline{f_2^{-1}v}}\|^{-2} f_1^{-1}T_{\overline{f_2^{-1}v}}$; then we have*

$$M = \{f : f = k_0 f_0 : (k_0, 0) \in K\},$$

with an $S^ \oplus S^*$ -invariant subspace $K = \mathbb{C} \times \{0\}$.*

(2) *in the case $a_0 b_0 = 0$, we have*

$$M = \left\{ f : f = k_1 \frac{f_1^{-1}T_{\overline{f_2^{-1}v}}}{\|f_1^{-1}T_{\overline{f_2^{-1}v}}(S^*v)\|} : k_1 \in K \right\}$$

with $K = \mathbb{C}$ an S^ -invariant subspace.*

Proof. (1) in this case, Theorem 3.1 (1) gives

$$M = \{f : f = k_0 f_0 + k_1 \frac{f_1^{-1}(T_{f_2^{-1}}v - a_0)}{\|f_1^{-1}T_{f_2^{-1}}(S^*v)\|} : (k_0, k_1) \in K\},$$

due to $z[f_1^{-1}T_{f_2^{-1}}(S^*v)] = f_1^{-1}z[S^*(T_{f_2^{-1}}v)] = f_1^{-1}(T_{f_2^{-1}}v - a_0)$. Further by $M = \bigvee\{f_1^{-1}(T_{f_2^{-1}}v)\}$, it follows that

$$k_0 \in \mathbb{C} \text{ and } k_1 \frac{f_1^{-1}(T_{f_2^{-1}}v - a_0)}{\|f_1^{-1}T_{f_2^{-1}}(S^*v)\|} = \mu f_1^{-1}(T_{f_2^{-1}}v) \text{ with } \mu \in \mathbb{C},$$

which is equivalent to $k_0 \in \mathbb{C}$ and $k_1 = 0$ by $a_0 \neq 0$.

(2) in this case, it follows either $a_0 = 0$ or $b_0 = 0$ and $f_0 = P_M 1 = 0$. If $b_0 = 0$, Theorem 3.1 (2) implies that

$$M = \{f : f = k_1 \frac{f_1^{-1}(T_{f_2^{-1}}v - a_0)}{\|f_1^{-1}T_{f_2^{-1}}(S^*v)\|} : k_1 \in K\},$$

which is valid if and only if $a_0 = 0$ and $k_1 \in \mathbb{C}$. □

3.4. $g = \bar{\theta}$ with θ nonconstant inner function. Because of its link with model spaces, this case is of particular interest. For every $h \in \text{Ker } R_1$, the equation (2.2) is equivalent to $h + \langle h, u \rangle \theta v \in \overline{\theta H_0^2}$. So

$$M = \text{Ker } R_1 \subset (H^2 \cap \overline{\theta H_0^2}) \oplus \bigvee\{\theta v\} = K_\theta \oplus \bigvee\{\theta v\}.$$

Take any vector $h = h_1 + \lambda \theta v \in M$ with $h_1 \in K_\theta$ and $\lambda \in \mathbb{C}$, such that $R_1 h = 0$, which is equivalent to

$$\lambda(1 + \langle \theta v, u \rangle) = -\langle h_1, u \rangle. \quad (3.1)$$

Now we divide this into two subsections to represent $M = \text{Ker } R_1$ in terms of backward shift-invariant subspaces.

3.4.1. $\theta|u$. In this case, the equation (3.1) now is changed into $\lambda(1 + \langle \theta v, u \rangle) = 0$. If $1 + \langle \theta v, u \rangle \neq 0$, then $\lambda = 0$ and $M = K_\theta$ a nearly S^* -invariant subspace. So we suppose $1 + \langle \theta v, u \rangle = 0$, and then $M = K_\theta \oplus \bigvee\{\theta v\}$ is nearly S^* -invariant with a 1-dimensional defect space $F = \bigvee\{\theta S^*v\}$ from Theorem 2.7. Using Theorem 3.1, we obtain a corollary on the representation of $\text{Ker } R_1$.

Corollary 3.7. *Given a nearly S^* -invariant subspace $M = K_\theta \oplus \bigvee\{\theta v\}$ with defect 1, and let $f_0 = P_M 1 = 1 - \overline{\theta(0)\theta} + \overline{\theta(0)v(0)}\|v\|^{-2}\theta v$, we have*

$$\begin{aligned} M = \{f : f = k_0 - (k_0 \overline{\theta(0)}) + k_1 \frac{v(0)}{\|S^*v\|} \theta + (k_0 \frac{\overline{\theta(0)v(0)}}{\|v\|^2} \\ + \|S^*v\|^{-1} k_1) \theta v : (k_0, k_1) \in K\}, \end{aligned} \quad (3.2)$$

with an $S^* \oplus S^*$ -invariant subspace $K = \{(k_0, k_1) : k_i \text{ satisfies (3.3) for } i = 0, 1\}$, where

$$k_0 - (k_0 \overline{\theta(0)} + k_1 \frac{v(0)}{\|S^*v\|})\theta \in K_\theta \text{ and } k_0 \frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \|S^*v\|^{-1}k_1 \in \mathbb{C}. \quad (3.3)$$

Proof. By Theorem 3.1 (1), we obtain

$$M = \{f : f = k_0 f_0 + k_1 \frac{\theta(v - v(0))}{\|S^*v\|} : (k_0, k_1) \in K\},$$

which equals $K_\theta \oplus \mathcal{V}\{\theta v\}$ implying the desired representation in (3.2). It is clear the second relation in (3.3) holds for S^*k_i , $i = 1, 2$. At the same time, the first relation in (3.3) together with the fact K_θ is an S^* -invariant subspace verify that

$$Y_\theta := S^*k_0 - S^*(k_0 \overline{\theta(0)})\theta + \frac{v(0)k_1}{\|S^*v\|}\theta \in K_\theta.$$

Then it turns out that

$$\begin{aligned} & S^*k_0 - (S^*k_0 \overline{\theta(0)} + \frac{v(0)}{\|S^*v\|}S^*k_1)\theta \\ &= Y_\theta + \overline{\theta(0)}k_0(0) \frac{\theta - \theta(0)}{z} + \frac{v(0)k_1(0)}{\|S^*v\|} \frac{\theta - \theta(0)}{z} \\ &= Y_\theta + (\overline{\theta(0)}k_0(0) + \frac{v(0)k_1(0)}{\|S^*v\|})T_{\bar{z}}\theta \in K_\theta, \end{aligned}$$

since $\langle T_{\bar{z}}\theta, \theta \rangle = \langle 1, z \rangle = 0$ holds. This means the first relation in (3.3) also holds for S^*k_i , $i = 1, 2$. So K is an $S^* \oplus S^*$ -invariant subspace. \square

3.4.2. $\theta \nmid u$. In this case, we decompose u into $u = u_1 + u_\theta$ with nonzero $u_1 \in K_\theta$ and $u_\theta \in \theta H^2$. Then the identity (3.1) becomes

$$\lambda(1 + \langle \theta v, u_\theta \rangle) = -\langle h_1, u_1 \rangle. \quad (3.4)$$

Especially Theorem 2.7 implies $\text{Ker } R_1$ is nearly S^* -invariant with a 2-dimensional defect space $F = \mathcal{V}\{\theta S^*v, u_1\}$. For later use we present a remark concerning the projection $P_M 1$.

Remark 3.8. Let $M = \text{Ker } R_1 \subset N := K_\theta \oplus \mathcal{V}\{\theta v\}$, and denote $N = M \oplus \mathcal{V}\{G\}$ with $G = g + \mu \theta v$, where $g \in K_\theta$ and $\mu \in \mathbb{C}$. Then

$$\begin{aligned} P_M 1 &= 1 - \overline{\theta(0)}\theta + \frac{\overline{\theta(0)v(0)}}{\|v\|^2}\theta v \\ &\quad - \frac{\langle 1 - \overline{\theta(0)}\theta, g \rangle + \overline{\theta(0)v(0)}\mu}{\|g\|^2 + |\mu|^2\|v\|^2}(g + \mu \theta v). \end{aligned} \quad (3.5)$$

For simplicity, we denote $w_\theta := 1 + \langle \theta v, u_\theta \rangle$ and

$$\rho_\theta := \frac{\overline{u_1(0)} + \overline{\theta(0)v(0)}w_\theta}{\|u_1\|^2 + |w_\theta|^2\|v\|^2}.$$

Applying Theorem 3.1, we present a corollary on $\text{Ker } R_1$.

Corollary 3.9. (1) *In the case $w_\theta \neq 0$, $M = N \ominus \bigvee\{u_1 + \overline{w_\theta}\theta v\}$ is nearly S^* -invariant with defect 2, and letting*

$$\begin{aligned} f_0 &= P_M 1 = 1 - \overline{\theta(0)}\theta + \overline{\theta(0)v(0)}\|v\|^{-2}\theta v - \rho_\theta(u_1 + \overline{w_\theta}\theta v), \\ v_0 &= P(u_1 + \overline{w_\theta}\theta v - \theta(0)\overline{w_\theta}v + \theta(0)v(0)\|v\|^{-2}\overline{w_\theta}|v|^2 - \overline{\rho_\theta}|u_1 + \overline{w_\theta}\theta v|^2), \\ v_1 &= \|S^*v\|^{-1}P(\overline{w_\theta}v(v - v(0))) \text{ and } v_2 = \|u_1\|^{-1}P(\overline{z}|u_1|^2), \end{aligned} \quad (3.6)$$

we have

$$\begin{aligned} M &= \{f : f = k_0 - (k_0\overline{\theta(0)} + k_1\frac{v(0)}{\|S^*v\|})\theta + (k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \frac{k_1}{\|S^*v\|} - k_2\frac{z\overline{w_\theta}}{\|u_1\|})\theta v \\ &\quad + (-k_0\rho_\theta + k_2\frac{z}{\|u_1\|})(u_1 + \overline{w_\theta}\theta v) : (k_0, k_1, k_2) \in K\}, \end{aligned} \quad (3.7)$$

with an $S^* \oplus S^* \oplus S^*$ -invariant subspace $K = \{(k_0, k_1, k_2) : k_i \text{ satisfies (3.8) for } i = 0, 1, 2\}$, where

$$\begin{cases} k_0 - (k_0\overline{\theta(0)} + k_1\frac{v(0)}{\|S^*v\|})\theta \in K_\theta, \\ k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \frac{k_1}{\|S^*v\|} - k_2\frac{z\overline{w_\theta}}{\|u_1\|} \in \mathbb{C}, \\ \langle k_0, z^n v_0 \rangle + \langle k_1, z^n v_1 \rangle + \langle k_2, z^n v_2 \rangle = 0 \text{ for all } n \in \mathbb{N}. \end{cases} \quad (3.8)$$

(2) *In the case $w_\theta = 0$, $M = N \ominus \bigvee\{u_1\}$ is nearly S^* -invariant with defect 2, and letting*

$$\begin{aligned} f_0 &= P_M 1 = 1 - \overline{\theta(0)}\theta - \overline{u_1(0)}\|u_1\|^{-2}u_1 + \overline{\theta(0)v(0)}\|v\|^{-2}\theta v, \\ v_0 &= P(u_1 - u_1(0)\|u_1\|^{-2}|u_1|^2) \text{ and } v_2 = P(\|u_1\|^{-1}\overline{z}|u_1|^2), \end{aligned}$$

we have

$$\begin{aligned} M &= \{f : f = k_0 - (k_0\overline{\theta(0)} + k_1\frac{v(0)}{\|S^*v\|})\theta + (k_0\frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \frac{k_1}{\|S^*v\|})\theta v \\ &\quad + (-k_0\frac{\overline{u_1(0)}}{\|u_1\|^2} + k_2\frac{z}{\|u_1\|})u_1 : (k_0, k_1, k_2) \in K\}, \end{aligned}$$

with an $S^* \oplus S^* \oplus S^*$ -invariant subspace $K = \{(k_0, k_1, k_2) : k_i \text{ satisfies (3.9) for } i = 0, 1, 2\}$, where

$$\begin{cases} k_0 - (k_0 \overline{\theta(0)} + k_1 \frac{v(0)}{\|S^*v\|})\theta \in K_\theta, \\ k_0 \frac{\overline{\theta(0)v(0)}}{\|v\|^2} + \|S^*v\|^{-1}k_1 \in \mathbb{C}, \\ \langle k_0, z^n v_0 \rangle + \langle k_2, z^n v_2 \rangle = 0 \text{ for } n \in \mathbb{N}. \end{cases} \quad (3.9)$$

Proof. For the case $w_\theta \neq 0$, the equation (3.4) implies $\lambda = -w_\theta^{-1} \langle h_1, u_1 \rangle$ and then $M = \text{Ker } R_1 = \{f : f = k - w_\theta^{-1} \langle k, u_1 \rangle \theta v, k \in K_\theta\}$. By some calculations, it follows

$$M = N \ominus \bigvee \{u_1 + \overline{w_\theta} \theta v\}. \quad (3.10)$$

Letting $g = u_1$ and $\mu = \overline{w_\theta}$ in (3.5), we obtain f_θ in (3.6). By Theorem 3.1 (1), it follows

$$M = \{f : f = k_0 f_0 + k_1 \frac{\theta(v - v(0))}{\|S^*v\|} + k_2 \frac{z u_1}{\|u_1\|} : (k_0, k_1, k_2) \in K\}$$

which together with (3.10) imply the representation of M in (3.7). Note the third formula in (3.8) holds for $S^*k_i, i = 0, 1, 2$. Following the similar lines for proving (3.3) is S^* -invariant, it is easy to check the first two relations of (3.8) are valid for $S^*k_i, i = 0, 1, 2$. So K is an $S^* \oplus S^* \oplus S^*$ -invariant subspace. In particular, if $w_\theta = 0$, the equation (3.4) implies

$$M = N \ominus \bigvee \{u_1\},$$

which is a special case of (3.10) with $w_\theta = 0$. Hence we can prove the statement (2) from the similar proof of statement (1) with $w_\theta = 0$. \square

In order to help understand the case $g = \bar{\theta}$ with θ an inner function, we present an example for Corollary 3.9.

Example 3.10. Let $\theta = z^m$ ($m \geq 1$) and $u = u_1 + u_2$ with $u_1 = z^{m-1}/4$ and $u_2 \in z^m H^2$. It easy to check the kernel of R_1 is

$$M = \bigvee \{1, z, \dots, z^{m-1}\} \oplus \bigvee \{z^m v\} \ominus \bigvee \{z^{m-1}/4 + \overline{w_\theta} z^m v\},$$

which is nearly S^* -invariant with 2-dimensional defect space $F = \bigvee \{z^m S^*v, z^{m-1}\}$ from Theorem 2.7. If $w_\theta = 1 + \langle z^m v, u_\theta \rangle \neq 0$, then Corollary 3.9 (2) indicates the following representation for M :

$$\begin{aligned} M = \{f : f = k_0 - k_1 \frac{v(0)}{\|S^*v\|} z^m + (k_1 \|S^*v\|^{-1} - 4k_2 z \overline{w_\theta}) z^m v \\ + k_2 z (z^{m-1} + 4\overline{w_\theta} z^m v) : (k_0, k_1, k_2) \in K\}, \end{aligned}$$

with an $S^* \oplus S^* \oplus S^*$ -invariant subspace $K = \{(k_0, k_1, k_2) : k_i \text{ satisfies (3.11) for } i = 0, 1, 2\}$ where

$$\begin{cases} k_0 - k_1 \frac{v(0)}{\|S^*v\|} z^m \in K_{z^m}, \\ k_1 \|S^*v\|^{-1} - 4k_2 z \overline{w_\theta} \in \mathbb{C}, \\ \langle k_0, z^n v_0 \rangle + \langle k_1, z^n v_1 \rangle = 0 \text{ for } n \in \mathbb{N}, \end{cases} \quad (3.11)$$

with

$$v_0 = 4^{-1} z^{m-1} + \overline{w_\theta} z^m v, \quad v_1 = \|S^*v\|^{-1} P(\overline{w_\theta} v (\overline{v - v(0)})).$$

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