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Inequality of ratios

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Abstract
Some socioeconomic indicators can be represented in alternative ways. For example, as either attainments (e.g. child survival rates) or shortfalls (e.g. child mortality rates) if the variables are bounded. The literature has long been concerned with the consistency of inequality comparisons across such alternative representations. The case of bounded variables and their two alternative representations (attainments versus shortfalls) has been largely settled. This paper addresses the extent of the consistency problem in inequality comparisons involving ratio indicators which also have two potential representations (e.g. number of people per room or number of rooms per capita in the case of overcrowding). First, we probe welfare comparisons based on the generalised Lorenz curve and find that consistency can be secured in the presence of rank dominance. Second, we show that robust inequality comparisons based on all possible Zoli partial orderings (which include all relative and absolute inequality partial orderings, among others) are inconsistent across alternative representations of ratios. Third, with the identification of a class of inequality indices based on the ratio of the harmonic to the arithmetic mean, we show that complete orderings consistent across alternative representations of ratios do exist. Then we consider and ponder the pros and cons of three alternative solutions: defending one representation, using inequality indices that combine both representations, and functional transformations of the ratio variable. We find that the costs of these alternatives render them inferior to the class of indices based on the harmonic and arithmetic means. Both the consistency problem and its preferred proposed solution are illustrated with an empirical study of intergenerational changes in overcrowding inequality in Mexico.

Keywords Inequality · Lorenz curves · Harmonic mean · Logarithmic transformations

1 Introduction

Some socioeconomic indicators can be represented in alternative ways. For instance, bounded variables can be represented as either attainments (e.g. child survival rates) or shortfalls (e.g. child mortality rates). The literature has long been concerned with the consistency

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of inequality comparisons across such alternative representations. In the case of bounded variables, Micklewright and Stewart [18] were among the first to notice that relative inequality comparisons were not consistent when switching from attainment to shortfall representations. In any given two years, it could happen that cross-country relative inequality in child mortality rates decreased, while inequality in child survival rates went up.

Several other contributions highlighted the problem and proposed solutions (e.g. see [1,4–6,8,13,14,16,17,19,20,22]). Most of these solutions involve defending the use of one particular representation (e.g. [13]), using absolute inequality measurement tools (absolute inequality indices, absolute Lorenz curves, etc.) which all happen to be consistent (e.g. [8,16]), inequality indices based on representations [1,17], and so-called normalised inequality indices [20]. Thus, it seems that this strand of the literature is largely settled.

Some other socioeconomic indicators come in the form of ratios; which means that, in theory, the numerator and the denominator could switch roles. For instance, several countries (e.g. the United States, the United Kingdom and Mexico [7,12]) measure overcrowding using the number of people per room, people per bedroom, or some alternative variation thereof in which the number of potential living quarters appears in the *denominator*. By contrast, the European Union uses a more complex measure that consists of counting the number of rooms available for particular groups of people under specific circumstances (e.g. there should be at least one room for each single person aged 18 or older, etc.) [9]. In essence, the EU's overcrowding measure has number of rooms in the *numerator*. Meanwhile, perhaps less conspicuously, other popular socioeconomic indicators like infection rates or household income per capita are also ratios.

This paper addresses the extent of the consistency problem in inequality comparisons involving ratio indicators. Are inequality comparisons consistent when we switch the numerator with the denominator (i.e. across the two alternative representations)? As a preamble assessment, we probe dispersion-sensitive welfare comparisons based on the generalised Lorenz curve and find that consistency can be secured in the presence of rank dominance. Then we move onto proper inequality assessments and show that comparisons based on all possible Zoli partial orderings (which includes all relative and absolute inequality partial orderings, plus a wide array of intermediate ones; see [16,25]) are inconsistent across alternative representations of ratios.

Having established these "impossibility" results, we ask whether consistent complete orderings are still possible with some inequality indices. We find that a class of scale-invariant indices based on the ratio of the harmonic mean to the arithmetic mean does yield inequality comparisons that are consistent across alternative representations of the ratio indicator. Moreover, this class satisfies all desirable properties for inequality measurement: chiefly the transfers principle, but also the population principle and suitable normalisation. Perhaps the most salient member of this class is the Atkinson index that ensues when the equally-distributed-equivalent "income" is computed using the harmonic mean.

Additionally we propose and ponder the pros and cons of three potential alternative solutions including: defending one representation, using inequality indices that combine both representations, and functional transformations of the ratio variable. More specifically, in the first proposed solution, we discuss the suitability for ratios of the arguments put forward by Kenny [13] to defend the use of attainments over shortfall representations. We add other potential criteria including popular intuition and the mean value of the socioeconomic indicator. Regarding the second solution we test the appropriateness of adapting the inequality indices combining both representations proposed by Lasso de la Vega and Aristondo [17] and Aristondo and Lasso de la Vega [1] originally for bounded variables. We find that these composite inequality indices do restore consistency, but at the expense of violating the transfer principle. Finally, we explore functional transformations of the ratio variable that restore consistency in inequality comparisons. We find that the logarithmic transformation converts ratios into bounded variables. Thence the solutions proposed for bounded variables apply to the logarithm of a ratio, but at the expense of losing some intuition since the principle of transfers can no longer be applied to the original variable (echoing Foster and Ok [10] regarding the variance of logarithms).

We explore the practical extent of the inconsistency problem in an empirical illustration on intergenerational inequality comparisons in overcrowding in Mexico. Focusing on eight birth cohorts of adults heads of households and spouses divided into four samples by urbanisation status (combining urban-rural residence at age 14 with present urban-rural residence in 2017), we first find plenty of cases of rank dominance among the cohort-urbanisation sample combinations. Hence a significant scope for ordering past and present overcrowding distributions by the generalised Lorenz criterion (favouring present distributions of rooms per capita). By contrast, when moving to the inequality assessment, we find that most intergenerational comparisons are inconsistent between the two representations of the overcrowding ratio, namely people per room and rooms per capita, when using two inequality indices which are not functions of the ratio of the harmonic to the arithmetic mean (the standard deviation and the coefficient of variation). When we apply the Atkinson inequality index based on the harmonic mean, thereby restoring consistency, we find consistent reductions in relative inequality *across* birth cohorts (from older to younger) in most samples, but without any discernible trend in structural mobility *within* cohorts.

The rest of the paper proceeds as follows. Section 2 starts assessing consistency in the context of first-order and second-order dominance comparisons. Then it moves onto presenting the key "impossibility" result showing the inconsistency of robust inequality comparisons based on Zoli partial orders applied to ratios. Finally, this section provides a solution ensuring consistent complete orderings based on a class of inequality indices based on the ratio of the harmonic mean to the arithmetic mean. Section 3 proposes and discusses the merits and limitations of three potential alternative solutions to the inconsistency problem in inequality comparisons with ratios. Section 4 provides the empirical illustration on intergenerational comparisons of inequality in overcrowding in Mexico. Finally Sect. 5 offers some concluding remarks.

2 Consistency of inequality measurement with ratios

2.1 Preliminaries and desirable properties

Let $X = \{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n_{++}$ be a vector of n strictly positive real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$.[1] Likewise let $X^{-1} = \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$. Therefore: $x_1^{-1} \geq x_2^{-1} \geq \cdots \geq x_n^{-1}$. $m(X) \equiv \frac{1}{n}\sum_{i=1}^{n} x_i$ is the arithmetic mean function.

We define an inequality index as a non-negative real-valued function such that $I : \mathbb{R}^n \to \mathbb{R}_+$. In general we will consider continuous at least twice-differentiable functions for I. Now we introduce a set of desirable axioms for I:

[1] We need strictly positive numbers, otherwise ratios become indeterminate as soon as we swap roles between numerator and denominator.

Axiom 2.1 *Anonymity: $I(X) = I(Y)$ where $Y = XP$ and P is a $n \times n$ permutation matrix.*

Anonymity requires I to pay attention only to the elements in X (or Y), but not to any other characteristics associated with them.

Axiom 2.2 *Respect of Zoli-λ-μ partial ordering: I respects the Zoli-λ-μ partial ordering* [25] *if, for some pair $\lambda \in [0, 1]$, $\mu \in [0, 1]$ and every pair of distributions X and Y (both elements of $\mathbb{R}^n_{++}$), $I(X) \leq I(Y)$ if $\sum_{i=1}^{k} \frac{x_i - m(X)}{(\mu m(X)+1-\mu)^{\lambda}} \geq \sum_{i=1}^{k} \frac{y_i - m(Y)}{(\mu m(Y)+1-\mu)^{\lambda}}$ for all $k = 1, 2, \ldots, n$.*

An inequality index respecting the Zoli-λ-μ partial ordering ranks a pair of distributions equally after some common transformations of their elements. For example, if Y is obtained from X such that $y_i = x_i \gamma$ for all $i = 1, \ldots, n$ with $\gamma > 0$ then $I(X) = I(Y)$ if I respects the Zoli-1-1 partial order. So, for instance, if I were the variance we would get $I(X) \neq I(Y)$ because the variance respects the Zoli-λ-μ partial ordering for any pair $\lambda \in [0, 1]$, $\mu \in [0, 1]$ as long as $\lambda\mu = 0$, which excludes the $\{1, 1\}$ case.

Next, we define a Pigou–Dalton (PD) transfer as a rank-preserving transfer of $\delta > 0$ from a richer individual (j) to a less affluent individual (i), such that: $x_i + \delta \leq x_j - \delta$. Then the transfer principle states that I should decrease after PD transfers:

Axiom 2.3 *Transfers principle: For any pair of distributions X and Y (both elements of $\mathbb{R}^n_{++}$), if distribution X is obtained from Y through a sequence of PD transfers then: $I(X) < I(Y)$.*

Alternatively one could consider a smoothing axiom as a more general criterion for ordering unequal distributions when the socioeconomic indicator is not "transferable"[2]:

Axiom 2.4 *Smoothing: $I(X) < I(Y)$ where $X = YB$ and B is a $n \times n$ bi-stochastic matrix.*[3]

The inequality indices considered below are all expected to fulfill the aforementioned axioms. For ease of exposition we perform the consistency analysis using fixed populations. However this paper's results can also be extended to comparisons of distributions with different population sizes as long as I also satisfies the population principle:

Axiom 2.5 *Population principle: If distribution X is obtained from Y through replicating each individual $c \in \mathbb{N}_{++}$ times, then: $I(X) = I(Y)$.*

Additionally, while not essential, most inequality indices in the literature also satisfy a normalisation axiom helpful to rank egalitarian distributions equally, regardless of their mean:

Axiom 2.6 *Normalisation: $I(X) = 0$ if and only if $x_1 = x_2 = \cdots = x_n$.*

Finally, we introduce the key consistency property based on the proposal by Lambert and Zheng [16, equation 1]:

Axiom 2.7 *Consistency: $I(X) \gtreqless I(Y)$ if and only if $I(X^{-1}) \gtreqless I(Y^{-1})$* for all possible X and Y (*both elements of* $\mathbb{R}^n_{++}$).

[2] PD transfers can also be represented using bi-stochastic matrices. Hence Axioms 2.3 and 2.4 are equivalent formalisations of the same principle.

[3] For the equivalence conditions linking bi-stochastic matrices and inequality measurement in the context of majorization orderings see Arnold and Sarabia [2].

Note that the consistency axiom does not rule out the occasional agreement of rankings provided by I for some particular pair of distributions. Instead, consistency requires that I ranks *every possible pair of distributions* consistently.[4]

Just to provide some motivation for the relevance of the consistency property, consider the following two distributions with $n = 8$: $X = (1, 2, 3, 4, 5, 6, 7, 8)$ and $Y = (1, 2.5, 2.5, 4, 5, 6, 7.5, 7.5)$. Both have the same mean (4.5) and actually Y can be obtained from X by performing two PD transfers (one involving the second and third lowest values and another one involving the two highest values). Therefore we would expect $I(X) > I(Y)$ for any I satisfying the transfers principle. Moreover we would expect Lorenz-style dominance of Y over X for any Zoli-respecting Lorenz-style curves (e.g. Lorenz, absolute Lorenz, etc.). Now consider the respective inverse distributions: $X^{-1} = (1, 0.5, 0.333, 0.25, 0.2, 0.166, 0.143, 0.125)$ and $Y^{-1} = (1, 0.4, 0.4, 0.25, 0.2, 0.166, 0.133, 0.133)$. Would it also be the case that $I(X^{-1}) > I(Y^{-1})$? If not, then the inequality comparison would depend on the choice of representation, which may or may not be arbitrary.

In order to answer the aforementioned question, it is helpful to introduce the Lambert-Zheng generalised Lorenz curve for the fixed-population case[5]:

$$L(X; k, \mu, \lambda) \equiv \sum_{i=1}^{k} \frac{x_i - m(X)}{(\mu m(X) + 1 - \mu)^{\lambda}}, \quad k = 1, 2, \ldots, n; \mu \in [0, 1], \lambda \in [0, 1] \quad (1)$$

Then (for the fixed-population case) we define the Zoli-λ-μ inequality partial ordering proposed by Lambert and Zheng [16][6] stating that, for a given pair $\mu \in [0, 1], \lambda \in [0, 1]$, a distribution X weakly dominates Y in the Zoli-λ-μ sense if $L(X; k, \mu, \lambda) \geq L(Y; k, \mu, \lambda)$ for all $k = 1, 2, \ldots, n$. Likewise, following Lambert and Zheng [16, pp. 215, 218] we state that X and Y are said to be inequality equivalent in the Zoli-λ-μ sense if $L(X; k, \mu, \lambda) = L(Y; k, \mu, \lambda)$ for all $k = 1, 2, \ldots, n$.

Finally, we state that, for a given pair $\mu \in [0, 1], \lambda \in [0, 1]$, the Zoli-$\lambda$-$\mu$ inequality partial ordering is consistent if X weakly dominates Y in the Zoli-λ-μ sense and X^{-1} weakly dominates Y^{-1} in the Zoli-λ-μ sense.

2.2 First-order dominance and generalised-Lorenz partial orderings

Before discussing the consistency of inequality comparisons, we start with welfare comparisons based on first-order and second-order dominance. Though these are not strictly inequality comparisons, they incorporate concerns for inequality into social welfare assessments.[7]

[4] A similar distinction is made in the case of bounded variables between consistent inequality indices, which are expected to rank every possible pair of distributions consistently; and inconsistent inequality indices, which can still rank some specific pairs of distributions consistently, but not all of them (e.g. see [16]).

[5] For the variable-population case we would use $L(X; n, k, \mu, \lambda) = \frac{1}{n} L(X; k, \mu, \lambda),\ k = 1, 2, \ldots, n; \mu \in [0, 1], \lambda \in [0, 1]$.

[6] We are following the definition and name (i.e. Zoli) of the partial orderings proposed by Lambert and Zheng [16] which, interestingly, is slightly different from the original proposal by Zoli [25].

[7] We could consider higher orders of dominance, but first and second order seem to be the most popular in the literature.

We can consider alternative definitions of first and second-order dominance by choosing different statements from the equivalence theorems on which these partial orders are based. We will use the following weak definitions for the fixed-population case:

Definition 1 For any pair of distributions $X, Y \in \mathbb{R}^n_{++}$ we say that X FOD ("first-order dominates") Y if and only if $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

X FOD Y is equivalent to stating that $W(X) \geq W(Y)$ for all $W : \mathbb{R}^n_{++} \to \mathbb{R}$ that are symmetric (i.e. satisfy anonymity) and increasing [21]. This implies that higher values of X and Y represent higher well-being (e.g. rooms per capita). If that were the case, then higher values of X^{-1} and Y^{-1} should represent higher deprivation or lower wellbeing, accordingly (e.g. people per room). Then following Saposnik [21], we can show that Y^{-1} FOD X^{-1} is equivalent to stating that $V(Y^{-1}) \leq V(X^{-1})$ for all $V : \mathbb{R}^n_{++} \to \mathbb{R}$ that are symmetric (i.e. satisfy anonymity) and *decreasing*. That is, if $W(X) \geq W(Y)$ means that wellbeing with X is at least as high as with Y, then $V(Y^{-1}) \leq V(X^{-1})$ must mean, equivalently, that wellbeing with Y^{-1} is not higher than with X^{-1}, and vice versa.

Definition 2 For any pair of distributions $X, Y \in \mathbb{R}^n_{++}$ we say that X SOD ("second-order dominates") Y if and only if $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n$.[8]

X SOD Y is equivalent to stating that $W(X) \geq W(Y)$ for all $W : \mathbb{R}^n_{++} \to \mathbb{R}$ that are symmetric, increasing and Schur-concave [23]. As discussed above, if higher values of X^{-1} and Y^{-1} represent lower wellbeing, then Y^{-1} SOD X^{-1} is equivalent to stating that $V(Y^{-1}) \leq V(X^{-1})$ for all $V : \mathbb{R}^n_{++} \to \mathbb{R}$ that are symmetric, *decreasing* and *Schur-convex*.

Our first result in Proposition 2.1 pertains to the consistency of first-order dominance with ratios:

Proposition 2.1 *Consistency of first-order stochastic dominance: for all possible X and Y (both elements of $\mathbb{R}^n_{++}$), X FOD Y if and only if Y^{-1} FOD X^{-1}.*

Proof By Definition 1, X FOD Y is tantamount to rank dominance, i.e. $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$ [21]. Inverting the values we get: $x_i^{-1} \leq y_i^{-1}$ for all $i = 1, 2, \ldots, n$. Hence Y^{-1} FOD X^{-1}. Clearly the reverse is also true. □

Proposition 2.1 sits well with intuition because if higher values within X and Y denote higher wellbeing, then it should be the case that higher values in X^{-1} and Y^{-1} denote deprivation. Therefore, Proposition 2.1 ascertains the consistency of first-order dominance analysis in the case of ratios: if we find first-order dominance with wellbeing representations then we must find first-order dominance with deprivation representations (in the form of inverted variables) and viceversa.[9]

An interesting implication of Proposition 2.1 is that if X FOD Y then X dominates Y in higher orders and Y^{-1} dominates X^{-1} in higher orders. Shown in Proposition 2.2, this result is a consequence of first-order dominance implying higher orders of dominance. Interestingly, when comparing distributions of ratios with the same population, consistent SOD also requires rank dominance of the two most extreme values:

[8] Analogously, for any pair of distributions $X^{-1}, Y^{-1} \in \mathbb{R}^n_{++}$ we say that X^{-1} SOD ("second-order dominates") Y^{-1} if and only if $\sum_{i=1}^{k} x_{n-i+1}^{-1} \geq \sum_{i=1}^{k} y_{n-i+1}^{-1}$ for all $k = 1, 2, \ldots, n$.

[9] It should be straightforward to show that this is also true in the case of bounded variables and attainment-versus-shortfall representations.

Proposition 2.2 *Consistency of second-order stochastic dominance: for all possible X and Y (both elements of* $\mathbb{R}^n_{++}$*), X SOD Y and* Y^{-1} *SOD* X^{-1} *if X rank-dominates Y and only if* $x_1 \geq y_1$ *and* $x_n \geq y_n$*.*

Proof If: Direct consequence of Proposition 2.1 and its proof.

Only if: By definition, SOD implies $x_1 \geq y_1$. Also by definition (see footnote in Definition 2), Y^{-1} SOD X^{-1} implies $y_n^{-1} \geq x_n^{-1}$, which is equivalent to $x_n \geq y_n$. □

2.3 The Zoli-λ-μ partial orderings

Lambert and Zheng [16, theorems 1 and 2] demonstrate that the Zoli-λ-μ partial orderings rank bounded variables consistently if and only when $\lambda\mu = 0$ (i.e. an absolute approach to inequality measurement). Hence *all* absolute inequality indices are also consistent between attainment and shortfall representations of the bounded variable. By contrast, the results of Lambert and Zheng [16] do not rule out consistent inequality indices among the classes of non-absolute inequality indices; rather their results imply that not all inequality indices among those classes are consistent. Applying their rationale to assess the consistency of inequality measurement with ratio variables, Proposition 2.3 provides a key "impossibility result" showing that all Zoli partial orderings (including the Lorenz and absolute Lorenz partial orderings as limiting cases) are inconsistent. That is, *robust* inequality measurement with ratios is inconsistent across alternative representations of the ratio variable:

Proposition 2.3 *For all possible pairs* $\lambda \in [0, 1]$, $\mu \in [0, 1]$*, any Zoli-*λ*-*μ *partial ordering is inconsistent.*

Proof We set out to prove that, for all possible pairs $\lambda \in [0, 1]$, $\mu \in [0, 1]$, the Zoli-λ-μ partial orderings are inconsistent for ratios. In particular, we prove that for all possible pairs $\lambda \in [0, 1]$, $\mu \in [0, 1]$ and any pair of distributions X and Y: inequality equivalence between X and Y in the Zoli-λ-μ sense does not guarantee inequality equivalence between X^{-1} and Y^{-1} in the same sense.

Let X and Y be two different distributions which are nonetheless inequality-equivalent in terms of the Zoli-λ-μ partial orderings for a pair $\lambda \in [0, 1]$, $\mu \in [0, 1]$. That is:

$$\sum_{i=1}^{k} \frac{x_i - m(X)}{(\mu m(X) + 1 - \mu)^{\lambda}} = \sum_{i=1}^{k} \frac{y_i - m(Y)}{(\mu m(Y) + 1 - \mu)^{\lambda}}, \quad k = 1, 2, \ldots, n. \tag{2}$$

Clearly, the equations in (2) hold if and only if:

$$\frac{x_i - m(X)}{(\mu m(X) + 1 - \mu)^{\lambda}} = \frac{y_i - m(Y)}{(\mu m(Y) + 1 - \mu)^{\lambda}}, \quad i = 1, 2, \ldots, n. \tag{3}$$

Solving for x_i in (3) yields:

$$x_i = (\mu m(X) + 1 - \mu)^{\lambda} \frac{y_i - m(Y)}{(\mu m(Y) + 1 - \mu)^{\lambda}} + m(X), \quad i = 1, 2, \ldots, n. \tag{4}$$

Then inverting x_i we get the same condition in terms of every element in X^{-1}, i.e. x_i^{-1}, in (5):

$$x_i^{-1} = \left[(\mu m(X) + 1 - \mu)^{\lambda} \frac{y_i - m(Y)}{(\mu m(Y) + 1 - \mu)^{\lambda}} + m(X) \right]^{-1}, \quad i = 1, 2, \ldots, n. \tag{5}$$

Would x_i^{-1} in (5) also necessarily and sufficiently guarantee the inequality-equivalence of X^{-1} and Y^{-1} in terms of the Zoli-λ-μ partial orderings for the same pair $\lambda \in [0,1]$, $\mu \in [0,1]$? For that to be the case we would need to obtain the same formula for x_i^{-1} in (5) from the relationships in (6) (which is both necessary and sufficient to ensure that inequality equivalence between X^{-1} and Y^{-1}):

$$\frac{x_i^{-1} - m(X^{-1})}{(\mu m(X^{-1}) + 1 - \mu)^{\lambda}} = \frac{y_i^{-1} - m(Y^{-1})}{(\mu m(Y^{-1}) + 1 - \mu)^{\lambda}}, \quad i = 1, 2, \ldots, n. \tag{6}$$

But solving for x_i^{-1} in (6) yields (7):

$$x_i^{-1} = (\mu m(X^{-1}) + 1 - \mu)^{\lambda} \frac{y_i^{-1} - m(Y^{-1})}{(\mu m(Y^{-1}) + 1 - \mu)^{\lambda}} + m(X^{-1}), \quad i = 1, 2, \ldots, n. \tag{7}$$

Clearly, x_i^{-1} from expression (7) will not generally be identical to x_i^{-1} from expression (5), whichever $\lambda \in [0,1]$, $\mu \in [0,1]$ pair is considered. Therefore, for all pairs $\lambda \in [0,1]$, $\mu \in [0,1]$ the Zoli-λ-*mu* partial orderings are inconsistent. □

However, there may be some specific pairs of distributions for which a *specific agreement of rankings* is attained (not the same as consistency, as mentioned before) . We can find some of these by looking into the $n = 2$ cases of expressions (5) and (7) in (8) and (9), respectively:

$$x_1^{-1} = \left[\left(\mu\frac{x_1 + x_2}{2} + 1 - \mu\right)^{\lambda} \frac{y_1 - \frac{y_1+y_2}{2}}{\left(\mu\frac{y_1+y_2}{2} + 1 - \mu\right)^{\lambda}} + \frac{x_1 + x_2}{2}\right]^{-1}. \tag{8}$$

$$x_1^{-1} = \left(\mu\frac{x_1^{-1} + x_2^{-1}}{2} + 1 - \mu\right)^{\lambda} \frac{y_1^{-1} - \frac{y_1^{-1}+y_2^{-1}}{2}}{\left(\mu\frac{y_1^{-1}+y_2^{-1}}{2} + 1 - \mu\right)^{\lambda}} + \frac{x_1^{-1} + x_2^{-1}}{2}. \tag{9}$$

Clearly, x_1^{-1} obtains the same value in (8) and (9) only if: (1) $m(X) = m(Y)$, in which case, trivially, $X = Y$ (because $\frac{x_i - m(X)}{(\mu m(X)+1-\mu)^{\lambda}} = \frac{y_i - m(Y)}{(\mu m(Y)+1-\mu)^{\lambda}}$ for all $i = 1, 2, \ldots, n$); or (2) the two distributions are egalitarian. In practice, most distributional comparisons involve non-egalitarian distributions with different means. Therefore the problem of inconsistency is bound to be pervasive.

Another instance where specific agreement of rankings may be secured is when one egalitarian distribution is compared against a non-egalitarian distribution. In such case, let X be egalitarian and Y be non-egalitarian. If we construct their respective Lambert-Zheng generalised Lorenz curves we will get:

$$\sum_{i=1}^{k} \frac{x_i - m(X)}{(\mu m(X) + 1 - \mu)^{\lambda}} = 0 \geq \sum_{i=1}^{k} \frac{y_i - m(Y)}{(\mu m(Y) + 1 - \mu)^{\lambda}} \quad \forall k = 1, 2, \ldots, n. \tag{10}$$

The statement in (10) is true because X is egalitarian (therefore $x_i = m(X)$ for all $i = 1, 2, \ldots, n$) and Y is non-egalitarian with $y_1 \leq y_2 \leq \cdots \leq y_n$ (therefore $\sum_{i=1}^{k} \frac{y_i - m(Y)}{(\mu m(Y)+1-\mu)^{\lambda}} < \sum_{i=1}^{n} \frac{y_i - m(Y)}{(\mu m(Y)+1-\mu)^{\lambda}} = 0$ for all $k = 1, 2, \ldots, n-1$). Then, one can easily note that X is egalitarian if and only if X^{-1} is also egalitarian. Therefore X^{-1} will be egalitarian whereas Y^{-1} will not be so, and we will have again:

$$\sum_{i=1}^{k} \frac{x_{n-i+1}^{-1} - m(X^{-1})}{(\mu m(X^{-1}) + 1 - \mu)^{\lambda}} = 0 \geq \sum_{i=1}^{k} \frac{y_{n-i+1}^{-1} - m(Y^{-1})}{(\mu m(Y^{-1}) + 1 - \mu)^{\lambda}} \quad \forall k = 1, 2, \ldots, n. \quad (11)$$

Likewise, we may be able to find other cases where $I(X) \gtreqless I(Y)$ if and only if $I(X^{-1}) \gtreqless I(Y^{-1})$ involving two non-egalitarian distributions. But, as mentioned, consistent inequality comparisons must hold *for every possible pair of distributions*. The consistency result for bounded variables works in exactly the same way.

2.4 Consistent inequality indices

As mentioned in Sect. 2.3, if a particular partial ordering were consistent then all inequality indices respecting it would also be consistent. In the absence of any consistent Zoli partial ordering (as proven in Proposition 2.3), we may still be able to find classes of consistent inequality indices, but not relying on the (in)consistency of their respective partial orderings. In this section we show one such class of consistent inequality indices based on harmonic means.[10]

Let $H(X)$ denote the harmonic mean of X; that is: $H(X) = [\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}]^{-1}$. As is well-known in the literature, the harmonic mean is a member of the subset of concave generalised means useful in inequality and inequality-sensitive welfare assessments (e.g. see [3,11]). Due to its concavity, the harmonic mean increases in the presence of Pigou–Dalton transfers (or smoothing with a bi-stochastic matrix). Moreover, $H(X) \leq m(X)$ and $H(X) = m(X)$ if and only if X is an egalitarian distribution. Then, Proposition 2.4 identifies a class of consistent relative inequality indices based on the ratio of the harmonic to the arithmetic mean:

Proposition 2.4 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a decreasing function with* $f(1) = 0$*; and define* $t(X) \equiv \frac{H(X)}{m(X)}$. *Then* $I(X) = (f \circ t)(X)$ *is an inequality index satisfying anonymity, transfer principle, normalisation, and consistency.*

Proof Satisfaction of anonymity is straightforward since both $H(X)$ and $m(X)$ are anonymous. Satisfaction of the transfer principle relies on the fact that a Pigou–Dalton transfer renders $m(X)$ unaltered but increases $H(X)$ due to its concavity. Then $t(X)$ increases accordingly. Since f is a decreasing function, it must be the case that $(f \circ t)(X)$ decreases in the event of a Pigou–Dalton transfer. Satisfaction of normalisation is also straightforward given that $t(X) = 1$ if and only if X is an egalitarian distribution, and by definition $f(1) = 0$.

Satisfaction of consistency requires, by definition: $(f \circ t)(X) \gtreqless (f \circ t)(Y)$ if and only if $(f \circ t)(X^{-1}) \gtreqless (f \circ t)(Y^{-1})$. Now note that: $(f \circ t)(X) \gtreqless (f \circ t)(Y)$ if and only if $t(X) \lesseqgtr t(Y)$. Therefore proving consistency is tantamount to demonstrating that $t(X) \lesseqgtr t(Y)$ if and only if $t(X^{-1}) \lesseqgtr (Y^{-1})$. Inserting the definition of $H(X)$ into $t(X)$ we get:

$$\begin{aligned} t(X) &= \frac{H(X)}{m(X)} = \frac{[m(X^{-1})]^{-1}}{m(X)} = \frac{1}{m(X^{-1})m(X)} \\ &= \frac{[m(X)]^{-1}}{m(X^{-1})} = \frac{H(X^{-1})}{m(X^{-1})} = t(X^{-1}) \end{aligned} \quad (12)$$

[10] We thank an anonymous referee for suggesting the harmonic mean as a potential foundation for consistent inequality indices.

Hence, clearly $t(X) \lesseqqgtr t(Y)$ if and only if $t(X^{-1}) \lesseqqgtr (Y^{-1})$ because $t(X) = t(X^{-1})$ and $t(Y) = t(Y^{-1})$, as per (12). □

A prominent example of $(f \circ t)$ is the Atkinson index (based on his notion of equally-distributed equivalent income) when $\epsilon = 2$ [3, p. 257]:

$$A(X; \epsilon = 2) = 1 - \frac{H(X)}{m(X)} \tag{13}$$

But there are other equally suitable indices in the identified class. For instance, if $X \in \mathbb{R}^n_{++}$ (as is assumed throughout this paper), then the following logarithmic form would also satisfy the properties listed in Proposition 2.4:

$$B(X) = -\ln\left(\frac{H(X)}{m(X)}\right) \tag{14}$$

Interestingly, all inequality indices of the form $(f \circ t)$ also satisfy the population principle, which conveniently, enables the comparison of distributions of ratios with different population sizes. Additionally, the indices described in Proposition 2.4 also satisfy scale invariance, which means that they only measure the relative notion of inequality respecting the Lorenz partial ordering (i.e. the Zoli-1-1 partial ordering).[11]

3 Alternative potential solutions

Solutions in the form of classes of indices such as that identified in Sect. 2.4 are arguably satisfactory. However, since we only found consistent inequality indices within the class respecting the Lorenz partial ordering, it may be worth considering alternative solutions to the problem of inconsistent inequality measurement for ratios. This section discusses three potential alternatives.

3.1 An ethical defense of one particular representation

In the context of bounded variables, Kenny [13] provides an interesting defence for choosing an attainment representation (e.g. child survival rates) over a shortfall representation (e.g. child mortality rates) in order to be able to use bound-inconsistent inequality measures (the coefficient of variation in his particular case). He gave two justifications for choosing attainment representations. Firstly, that measures of convergence, like the coefficient of variation across time, are more sensitive to small absolute changes near the lower bound (e.g. zero child mortality) than larger changes further away when the mean trends toward that lower bound (e.g. toward zero child mortality). This sensitivity is deemed "perverse" [13, p. 3]. Secondly, Kenny [13] argued that most of the literature on income trends (at that time) measured convergence toward "the upper, "positive" value". While the second reason may not be fully convincing (it resorts to the case of unbounded variables like income to defend a particular representation for bounded variables), the first reason is arguably better founded. Yet in theory, for the case of ratios we could invoke justifications analogous to the two aforementioned reasons put forward by Kenny [13].

[11] For the seminal discussion on relative ("rightist"), absolute ("leftist") and intermediate ("centrist") approaches to inequality measurement see Kolm [15].

For example, let us consider the case of between-country inequality measurement of income per capita. In theory we could invert the variable and express it in terms of people per monetary unit (e.g. some form of density measure like people per square kilometer). We would run into inconsistencies if we made inequality comparisons with both representations. But the concept of income per capita seems to be far more intuitive than people per currency. Moreover, to the best of our knowledge, nobody in the literature has ever attempted distributional analysis with the inverse of income per capita. Therefore this seems to be a case where just defending one particular representation solves the problem. Effectively, inconsistency is brushed aside as if it did not exist.

On the other hand, not all cases may be as clear-cut as that of income per capita. For instance, several countries (e.g. the United States, the United Kingdom and Mexico [7,12]) measure overcrowding using number of people per room, people per bedroom, or some alternative variation thereof in which the number of potential living quarters appears in the *denominator*. By contrast, the European Union uses a more complex measure that consists of counting the number of rooms available for particular groups of people under specific circumstances (e.g. there should be at least one room for each single person aged 18 or older) [9]. In essence, the EU's overcrowding measure has number of rooms in the *numerator*. In such cases inconsistency may be a more salient problem, in the absence of universally preferred representations.

We could follow the style of Kenny [13]'s arguments to defend a particular representation for variables like overcrowding. For instance, in countries or regions with relatively high overcrowding it may be easier to understand people per room as often these will be numbers higher than 1 (e.g. in Mexico the overcrowding deprivation line is 2.5 people per room [7]). Meanwhile, in developed countries with lower incidence of overcrowding it may be worth switching to rooms per people, again under the assumption that it is easier to relate to numbers higher than 1 as opposed to between 0 and 1. Similar arguments could be put forward for between-country inequality assessment involving variables like number of physicians or hospital beds per 1,000 people, infection rates and so forth.

3.2 A combination of inequality indices

The first proposed alternative solution effectively brushes aside inconsistency by allowing only one representation for inequality measurement. However, if someone values consistency in inequality measurement with ratios then one potential alternative could be adapting the solution of Lasso de la Vega and Aristondo [17] for the consistency problem with bounded variables. In the case of ratios, their approach would involve using a generalised mean of two *inconsistent* inequality indices with *the same functional form and parameters*, but one evaluated at X and the other one at X^{-1} as in Eq. 15:

$$I^r(X) = \begin{cases} \left(\frac{[I(X)]^r + [I(X^{-1})]^r}{2}\right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ (I(X)I(X^{-1}))^{\frac{1}{2}} & \text{if } r = 0 \end{cases} \tag{15}$$

This approach clearly solves the consistency problem, as indicated in Proposition 3.1:

Proposition 3.1 *For all $r \in \mathbb{R}$, $I^r(X)$ satisfies the consistency property.*[12]

[12] If I in (15) is consistent then, by definition, the problem is solved and there is no need to undertake the approach proposed in this subsection anymore; hence the focus on inconsistent indices. Note that, logically, if

Proof It is straightforward to note that for all $r \in \mathbb{R}$: $I^r(X) = I^r(X^{-1})$. Likewise: $I^r(Y) = I^r(Y^{-1})$. Therefore it must be the case that $I^r(X) \gtreqqless I^r(Y)$ if and only if $I^r(X^{-1}) \gtreqqless I^r(Y^{-1})$. □

In the case of bounded variables, the indices proposed by Lasso de la Vega and Aristondo [17] solve those variables' consistency problem while fulfilling the key desirable properties of inequality measurement, chiefly the principle of transfers. Moreover their proposal constitutes an elegant alternative to restricting the inequality assessment to the absolute approach (absolute inequality indices, absolute Lorenz curves, etc.). In the case of ratios, these indices also solve the consistency problem as shown in Proposition 3.1, but unfortunately, as stated in Proposition 3.2, at the cost of violating the transfers principle:

Proposition 3.2 *For any non-egalitarian distribution $X \in \mathbb{R}^n_{++}$ and pair $\lambda \in [0, 1]$, $\mu \in [0, 1]$, there is at least one combination of r and inequality index I respecting the Zoli-λ-μ partial ordering, such that $I^r(X)$ violates the transfer principle.*

Proof Let Y be obtained from X through a PD transfer involving individuals i and j and some $\delta > 0$ such that: $x_i + \delta \leq x_j - \delta$. For all $k \neq i, j$: $y_k = x_k$.

Then:

$$I^r(Y) - I^r(X) \approx \delta \left[\frac{\partial I^r(X)}{\partial x_i} - \frac{\partial I^r(X)}{\partial x_j} \right] = \delta \left[\frac{\partial I^r(X)}{\partial I(X)} \left(\frac{\partial I(X)}{\partial x_i} - \frac{\partial I(X)}{\partial x_j} \right) + \frac{\partial I^r(X)}{\partial I(X^{-1})} \left(\frac{\partial I(X^{-1})}{\partial x_i} - \frac{\partial I(X^{-1})}{\partial x_j} \right) \right] \quad (16)$$

Now, we know from (15) that $\frac{\partial I^r(X)}{\partial I(X)} > 0$ and $\frac{\partial I^r(X)}{\partial I(X^{-1})} > 0$. Likewise since $I(X)$ satisfies the transfer principle, it is also the case that: $\frac{\partial I(X)}{\partial x_i} - \frac{\partial I(X)}{\partial x_j} < 0$.

However it is not always the case that $\frac{\partial I(X^{-1})}{\partial x_i} - \frac{\partial I(X^{-1})}{\partial x_j} < 0$. For example, letting $z_i = \frac{x_i^{-1} - m(X^{-1})}{(\mu m(X^{-1}) + 1 - \mu)^{\lambda}}$, and recalling the Lambert-Zheng generalised Lorenz curves $L(X^{-1}; k, \mu, \lambda) = \sum_{t=1}^{k} z_t$ we get the following changes in the aftermath of the PD transfer:

$$\frac{\partial L(X^{-1}; k, \mu, \lambda)}{\partial x_i} - \frac{\partial L(X^{-1}; k, \mu, \lambda)}{\partial x_j} = \mathbb{I}(z_i \leq z_k) \left[\frac{\partial z_i}{\partial x_i} - \frac{\partial z_i}{\partial x_j} \right] + \mathbb{I}(z_j \leq z_k) \left[\frac{\partial z_j}{\partial x_i} - \frac{\partial z_j}{\partial x_j} \right] + \sum_{t \neq i,j}^{n} \mathbb{I}(z_t \leq z_k) \left[\frac{\partial z_t}{\partial x_i} - \frac{\partial z_t}{\partial x_j} \right], \quad (17)$$

where $\mathbb{I}(a) = 1$ if a is true, otherwise $\mathbb{I}(a) = 0$; and:

$$\frac{\partial z_i}{\partial x_i} - \frac{\partial z_i}{\partial x_j} = \frac{\lambda \mu (\mu m(X^{-1}) + 1 - \mu)^{-1} \frac{1}{n} (x_i^{-1} - m(X^{-1}))}{(\mu m(X^{-1}) + 1 - \mu)^{\lambda}} \left[x_i^{-2} - x_j^{-2} \right]$$

I is inconsistent, then $I(X) \neq I(X^{-1})$. Therefore the class of consistent indices identified in Proposition 2.4 does not overlap with the class I^r in (15) because the former cannot be represented as the generalised mean of two identical inconsistent inequality indices, one evaluated at X and the other one at X^{-1}.

$$-\frac{1}{(\mu m(X^{-1})+1-\mu)^{\lambda}}\left[\frac{n-1}{n}x_i^{-2}+\frac{1}{n}x_j^{-2}\right], \tag{18}$$

$$\frac{\partial z_j}{\partial x_i}-\frac{\partial z_j}{\partial x_j}=\frac{1}{(\mu m(X^{-1})+1-\mu)^{\lambda}}\left[\frac{n-1}{n}x_j^{-2}+\frac{1}{n}x_i^{-2}\right]$$
$$+\frac{\lambda\mu(\mu m(X^{-1})+1-\mu)^{-1}\frac{1}{n}(x_j^{-1}-m(X^{-1}))}{(\mu m(X^{-1})+1-\mu)^{\lambda}}\left[x_i^{-2}-x_j^{-2}\right], \tag{19}$$

and

$$\frac{\partial z_t}{\partial x_i}-\frac{\partial z_t}{\partial x_j}=\frac{\lambda\mu(\mu m(X^{-1})+1-\mu)^{-1}(x_t^{-1}-m(X^{-1}))+1}{n(\mu m(X^{-1})+1-\mu)^{\lambda}}\left[x_i^{-2}-x_j^{-2}\right] \tag{20}$$
$$\forall t\neq i,j.$$

Now for the PD transfer to yield from X^{-1} a new distribution unambiguously less unequal according to any Zoli-λ-μ criteria, we need (17) to be non-negative for all $k=1,2,\ldots,n$. Otherwise there could be curve-crossings. However there is no guarantee that (17) will always be non-negative, since some of the expressions in (18), (19), and (20) are bound to be negative (as their signs depend on $x_t^{-1}-m(X^{-1})$ for all $t=1,2,\ldots,n$ and these cannot have the same sign simultaneously, otherwise the expected value of deviations from the mean would not be zero), and there is no guarantee that the positive elements in (17) will always have higher absolute value than the negative ones. For example, let i and j (involved in the PD transfers) be the second-to-wealthiest and wealthiest individuals, respectively. Then, it could be the case that (17) for $k=1$ is negative (e.g. if $x_j^{-1}-m(X^{-1})$ is sufficiently low) but (17) with k taking values very close to n is positive (e.g. if $x_t^{-1}-m(X^{-1})$ is sufficiently high for several $t\neq i,j$).[13] Hence in the absence of a robust decrease in inequality there is no guarantee that $\frac{\partial I(X^{-1})}{\partial x_i}-\frac{\partial I(X^{-1})}{\partial x_j}<0$.

Therefore $I^r(X)$ can satisfy the transfer principle if $\frac{\partial I(X^{-1})}{\partial x_i}-\frac{\partial I(X^{-1})}{\partial x_j}<0$ for all X, which is not guaranteed. But what if $\frac{\partial I(X^{-1})}{\partial x_i}-\frac{\partial I(X^{-1})}{\partial x_j}>0$? In that case, if $I(X)>I(X^{-1})$ one can always find an r low enough (minus infinity in the limit if necessary) such that $\frac{\partial I^r(X)}{\partial I(X)}<\frac{\partial I^r(X^{-1})}{\partial I(X^{-1})}$. Otherwise, if $I(X)<I(X^{-1})$ one can always find an r high enough (plus infinity in the limit if necessary) such that $\frac{\partial I^r(X)}{\partial I(X)}<\frac{\partial I^r(X^{-1})}{\partial I(X^{-1})}$. In both cases expression (16) turns positive and there is at least one r with which $I^r(X)$ violates the transfer principle whichever the choice of I. □

In summary, the generalised-mean approach solves the consistency problem of inequality measurement with ratios, but only at a high price: the violation of the transfers principle.

3.3 A transformation of the variable

Another potential alternative solution to restore consistency is to apply a monotonically increasing transformation to the ratio variable such that:

$(I\circ G)(X)\gtreqless(I\circ G)(Y)$ if and only if $(I\circ G)(X^{-1})\gtreqless(I\circ G)(Y^{-1})$ *for all possible* X *and* Y,

where $G:\mathbb{R}^n_{++}\to\mathbb{R}^n$ and $G(X)=\{g(x_1),g(x_2),\ldots,g(x_n)\}$ with $g:\mathbb{R}_{++}\to\mathbb{R}$ being an increasing function. In principle, several choices of g could be considered. But it may be

[13] Numerical examples are available from the author upon request.

worth narrowing these down by focusing on increasing transformations that render $G(X)$ and $G(X^{-1})$ vectors of *bounded* variables. Then we can restore consistency relying on absolute inequality indices, i.e. applying the results of Lambert and Zheng [16].[14]

By definition, the attainment and shortfall representations of a bounded variable satisfy the following equation: for some pair of real numbers $a < b$ and an attainment variable x such that $a \leq x \leq b$, then $x + y = b$ (so that $y = b - x$ is the shortfall representation). Then the strategy relying on absolute inequality measurement requires finding a function g such that $g(x) + g(x^{-1}) = c$ where c is some real constant operating as an upper bound.

One suitable option is the logarithmic transformation based on the solution $g(x) = a\ln(bx)$, where $2a\ln(b) = c$ with $b > 0$ and $a \neq 0$ if both $b \neq 1$ and $c \neq 0$. Without loss of generality we can set $a = b = 1$, implying $c = 0$. Hence we apply an increasing logarithmic transformation $g = \ln(x)$.[15] Then Proposition 3.3 demonstrates that inequality comparisons with the logarithmic transformation of ratios are consistent if and only if an absolute approach to inequality is adopted:

Proposition 3.3 *Let I respect the Zoli-λ-μ partial ordering for some $\lambda \in [0, 1]$ and $\mu \in [0, 1]$, and $L(X) = \{\ln(x_1), \ln(x_2), \ldots, \ln(x_n)\}$. Then $(I \circ L) : \mathbb{R}^n_{++} \rightarrow \mathbb{R}^n_{+}$ is consistent if and only if $\lambda\mu = 0$.*

Proof First, in addition to $\ln(x) + \ln(\frac{1}{x}) = 0$, we note that: $(m \circ L)(X) = -(m \circ L)(X^{-1})$.

If: Let $\lambda\mu = 0$ and $X \neq Y$. Now assume:

$$\sum_{i=1}^{k}[\ln(x_i) - (m \circ L)(X)] \geq \sum_{i=1}^{k}[\ln(y_i) - (m \circ L)(Y)] \quad \forall k = 1, 2, \ldots, n. \tag{21}$$

Since $\ln(x_i) - (m \circ L)(X) = (m \circ L)(X^{-1}) - \ln(x_i^{-1})$ (and the same goes for $L(Y)$), then:

$$\sum_{i=1}^{k}[\ln(x_i^{-1}) - (m \circ L)(X^{-1})] \leq \sum_{i=1}^{k}[\ln(y_i^{-1}) - (m \circ L)(Y^{-1})] \quad \forall k = 1, 2, \ldots, n, \tag{22}$$

which is equivalent to:

$$\sum_{i=1}^{k}[\ln(x_{n-i+1}^{-1}) - (m \circ L)(X^{-1})] \geq \sum_{i=1}^{k}[\ln(y_{n-i+1}^{-1}) - (m \circ L)(Y^{-1})] \quad \forall k = 1, 2, \ldots, n. \tag{23}$$

That is, if I is an absolute inequality index (i.e. respecting the Zoli-λ-μ partial ordering with $\lambda\mu = 0$) then $(I \circ L)(X) \gtreqless (I \circ L)(Y)$ if and only if $(I \circ L)(X^{-1}) \gtreqless (I \circ L)(Y^{-1})$.

Only if:

Now let distributions $X \neq Y$ be inequality equivalent in terms of the Zoli-λ-μ partial ordering. Then $(I \circ L)(X) = (I \circ L)(Y)$ for any I respecting that same partial ordering.

[14] In principle we could also resort to the proposal by Lasso de la Vega and Aristondo [17]. However only absolute inequality indices could be admissible as choices for the combined index because $(m \circ G)(X)$ and $(m \circ G)(X^{-1})$ may not be constrained to be different from zero anymore. Hence one single absolute inequality index, as proposed by Lambert and Zheng [16], would suffice.

[15] In principle, alternative candidate functions for the increasing transformations may be feasible, but the other simple solutions to the functional equation, $g(x) + g(x^{-1}) = c$, are not as easy to come by as the logarithmic function.

Since $(I \circ L)$ is consistent it must be the case that: $(I \circ L)(X^{-1}) = (I \circ L)(Y^{-1})$ If so, then we would expect from the inequality equivalence both:

$$\frac{\ln(x_i) - (m \circ L)(X)}{(\mu(m \circ L)(X) + 1 - \mu)^{\lambda}} = \frac{\ln(y_i) - (m \circ L)(Y)}{(\mu(m \circ L)(Y) + 1 - \mu)^{\lambda}}, \quad i = 1, 2, \ldots, n, \tag{24}$$

and:

$$\frac{\ln(x_i^{-1}) - (m \circ L)(X^{-1})}{(\mu(m \circ L)(X^{-1}) + 1 - \mu)^{\lambda}} = \frac{\ln(y_i^{-1}) - (m \circ L)(Y^{-1})}{(\mu(m \circ L)(Y^{-1}) + 1 - \mu)^{\lambda}}, \quad i = 1, 2, \ldots, n. \tag{25}$$

However if we insert $\ln(x_i) - (m \circ L)(X) = (m \circ L)(X^{-1}) - \ln(x_i^{-1})$, $\ln(y_i) - (m \circ L)(Y) = (m \circ L)(Y^{-1}) - \ln(y_i^{-1})$, $(m \circ L)(X) = -(m \circ L)(X^{-1})$, and $(m \circ L)(Y) = -(m \circ L)(Y^{-1})$ into (24) we get, after some rearrangements:

$$\frac{\ln(x_i^{-1}) - (m \circ L)(X^{-1})}{(1 - \mu[1 + (m \circ L)(X^{-1})])^{\lambda}} = \frac{\ln(y_i^{-1}) - (m \circ L)(Y^{-1})}{(1 - \mu[1 + (m \circ L)(Y^{-1})])^{\lambda}}, \quad i = 1, 2, \ldots, n, \tag{26}$$

Given that $(m \circ L)(X) \neq (m \circ L)(Y)$, clearly expressions (25) and (26) do not coincide unless $\lambda\mu = 0$. Therefore $\lambda\mu = 0$ is necessary for $(I \circ L)$ to be consistent. □

Proposition 3.3 shows that a logarithmic transformation of the ratio enables consistent *absolute* inequality measurement. However this solution comes at a significant cost. Foster and Ok [10, and references therein] noted that the then popular variance of logarithms violated the transfers principle. More generally, we can show that any inequality measure $(I \circ L)$ where I is a Schur-convex function (thus satisfying the transfers principle) violates the transfers principle:

Proposition 3.4 $(I \circ L)$ *violates the transfers principle.*

Proof Let distribution Y be obtained from X through a PD transfer of $\delta > 0$ involving individuals i and j such that: $y_i = x_i + \delta \leq x_j - \delta = y_j$ and $y_k = x_k$ for all $k \neq i, j$. Then:

$$\begin{aligned}(I \circ L)(Y) - (I \circ L)(X) &\approx \delta \left[\frac{\partial (I \circ L)(X)}{\partial x_i} - \frac{\partial (I \circ L)(X)}{\partial x_j}\right] \\ &= \delta \left[\frac{\partial (I \circ L)(X)}{\partial \ln x_i} \frac{1}{x_i} - \frac{\partial (I \circ L)(X)}{\partial \ln x_j} \frac{1}{x_j}\right]\end{aligned} \tag{27}$$

Clearly, $(I \circ L)(Y) - (I \circ L)(X) < 0$, which would mean that $(I \circ L)$ satisfies the transfer principle, only if $\frac{\partial (I \circ L)(X)}{\partial \ln x_i} \frac{1}{x_i} - \frac{\partial (I \circ L)(X)}{\partial \ln x_j} \frac{1}{x_j} < 0$. However the latter is not guaranteed because even though $\frac{\partial (I \circ L)(X)}{\partial \ln x_i} < \frac{\partial (I \circ L)(X)}{\partial \ln x_j}$ (due to the Schur-convexity of I and $x_i < x_j$) it is also the case that $\frac{1}{x_i} > \frac{1}{x_j}$ (for the same second reason, i.e. $x_i < x_j$). Hence a priori the sign of $(I \circ L)(Y) - (I \circ L)(X)$ is ambiguous and depends on the relative magnitude of the values involved in the transfer (x_i and x_j). □

Hence this alternative potential solution based on measuring absolute inequality with logarithmic transformations of ratios also runs into the inconvenient trade-off whereby consistency is gained at the expense of the transfer principle. One possible way out of this dilemma is to consider PD transfers (or smoothing with bi-stochastic matrices) only applied to the transformed variable. In other words, we would not be considering transfers of the form $x_i + \delta \leq x_j - \delta$ (with some $\delta > 0$). Instead we would consider $\ln x_i + \delta \leq \ln x_j - \delta$ with respective "shortfall" counterparts $-[\ln x_i + \delta]$ and $-[\ln x_j - \delta]$. But even this patch comes with the cost of losing further intuition: one thing is to consider a transfer of one person-per-room from a very overcrowded dwelling to a less overcrowded dwelling; another thing is to interpret a transfer of a logarithm of one person-per-room.

Finally, even if we accept the aforementioned caveats, there would be an additional drawback from the way logarithmic transformations affect the dispersion of the original variable. As is well known, logarithmic transformations will compress the dispersion of any values of $X \in \mathbb{R}^n_{++}$ higher than 1 while doing the exact opposite to values lower than 1. For instance, if X has very similar values clustered somewhere between 0 and 1, while Y has more disperse values all higher than 1, it could happen that $I(X) < I(Y)$ but $(I \circ L)(X) > (I \circ L)(Y)$. Therefore even though the logarithmic transformation would enable a consistent comparison, the inequality ranking based on the original variable would be overturned.

4 Empirical illustration

We illustrate the problem of inconsistency in inequality measurement with ratios, as well as the implementation of the preferred proposed solution (inequality measures based on the ratio of the harmonic to the arithmetic mean), with a study of intergenerational change in overcrowding in Mexico.[16] Specifically, for different birth cohorts of the adult Mexican population in charge of households (heads and spouses) we ask whether inequality in dwelling overcrowding decreased between the time these adults were 14 years old (i.e. when they were adolescent and living with other adults heading the household) and the present in 2017 (when the survey we use was collected). A more detailed study of intergenerational mobility in overcrowding in Mexico can be found in Yalonetzky [24].

First, we implement simple tests of rank dominance to detect the presence of social-welfare comparisons that are both robust to alternative social welfare functions consistent with the generalised Lorenz curve *and* consistent across alternative ratio representations. Then we probe the inconsistency of intergenerational inequality comparisons. First, we provide numerous examples of inconsistent conclusions when we switch between ratio representations using two popular inequality indices. Then we show how consistency is restored by applying the preferred proposed solution.

4.1 Data and descriptive statistics

We use the ESRU Survey on Social Mobility in Mexico 2017 (ESRU-EMOVI 2017) collected by the Mexican Centro de Estudios Espinosa Yglesias (CEEY). We measure overcrowding in its two simplest ways. Firstly, dividing the number of people in the household by the number of rooms including the kitchen; i.e. the same indicator used by CONEVAL in the construction of Mexico's multidimensional poverty index. Secondly, inverting the ratio to yield number of rooms per person.[17]

We focus on adults aged 25–64 who are either heads of household or spouses and connect their present overcrowding condition with their homes' back when they were 14 years old.

16 Results for the alternative, less preferable solutions are available upon request.

17 Of course, overcrowding is only a partial indicator of dwelling conditions. It only captures the quantity of available rooms per family member (or the other way around), without accounting for those rooms' quality or any other relevant dwelling conditions (e.g. floor, walls, and ceiling material; heating; electricity; indoor plumbing; location in disaster-prone areas, etc.). CONEVAL [7] provides a detailed discussion of the plethora of indicators used to diagnose the fulfillment of the right to a dignified and decorous dwelling, ranging from access to housing subsidies to perceptions of insecurity in the neighbourhood and so forth.

Table 1 Sample sizes by degree of urbanisation

Cohort	Urban-to-urban	Rural-to-rural	Rural-to-urban	Urban-to-rural
1988–1992	1155	144	334	39
1983–1987	992	134	339	31
1978–1982	1407	178	432	57
1973–1977	1275	215	547	67
1968–1972	1108	201	498	40
1963–1967	1008	155	412	36
1958–1962	741	128	390	37
1953–1957	1325	218	730	48

We construct the following eight birth cohorts (with ages in 2017 in parenthesis): 1988–1992 (25–29), 1983–1987 (30–34), 1978–1982 (35–39), 1973–1977 (40–44), 1968–1972 (45–49), 1963–1967 (50–54), 1958–1962 (55–59), 1953–1957 (60–64). More specifically, we explore the interaction between birth cohort (of heads and spouses) and degree of urbanisation. Bearing in mind that the EMOVI 2017 deems rural any location with fewer than 2500 inhabitants, the degree of urbanisation is operationalised by dividing the sample into four groups: (1) urban-to-urban, comprising people who lived in urban areas (cities for short) when they were 14 years old and currently live in cities; (2) rural-to-rural, made of people who lived in rural areas when they were 14 years old and now also live in rural areas; (3) rural-to-urban, comprising people who grew up in rural areas when they were 14 years old and currently live in cities; and (4) urban-to-rural, made of people who grew up in cities when they were 14 years old and now live in rural areas.

The respective sizes for the urbanisation-cohort samples are in Table 1. The sizes for the urban-to-rural samples are rather small, reflecting the relative infrequency of de-urbanisation in countries like Mexico. Hence the estimates for this population group are bound to be less precise and reliable vis-a-vis the other urbanisation groups'.

As useful preliminary information, Fig. 1 presents the mean values of people per room and its inverse (rooms per capita) for the cohort-urbanisation combinations. Among several noteworthy features we note intergenerational reductions in mean overcrowding for all cohorts in all urbanisation groups (in the sense that mean overcrowding was higher at the parental home when the adults were 14 years old than at their present home where they are either the head or their spouse). Moreover, the older the cohort the greater the reduction in mean overcrowding. Coherently, mean rooms per capita increased for every cohort in every urbanisation group.

4.2 First- and second-order stochastic dominance

In order to test for the presence of rank dominance (i.e. FOD) in every cohort-urbanisation sample, first, we sort both distributions of current level of overcrowding measured by *people per room* and its past level (when adults were 14) in ascending order and match the values by ranking (e.g. the lowest current overcrowding level with the lowest past level, the second-to-lowest present and past levels, and so forth up to the highest values). Then we compute the difference between present and past values; hence a negative difference in a particular rank signifies a reduction in overcrowding from childhood to current age in the absence of

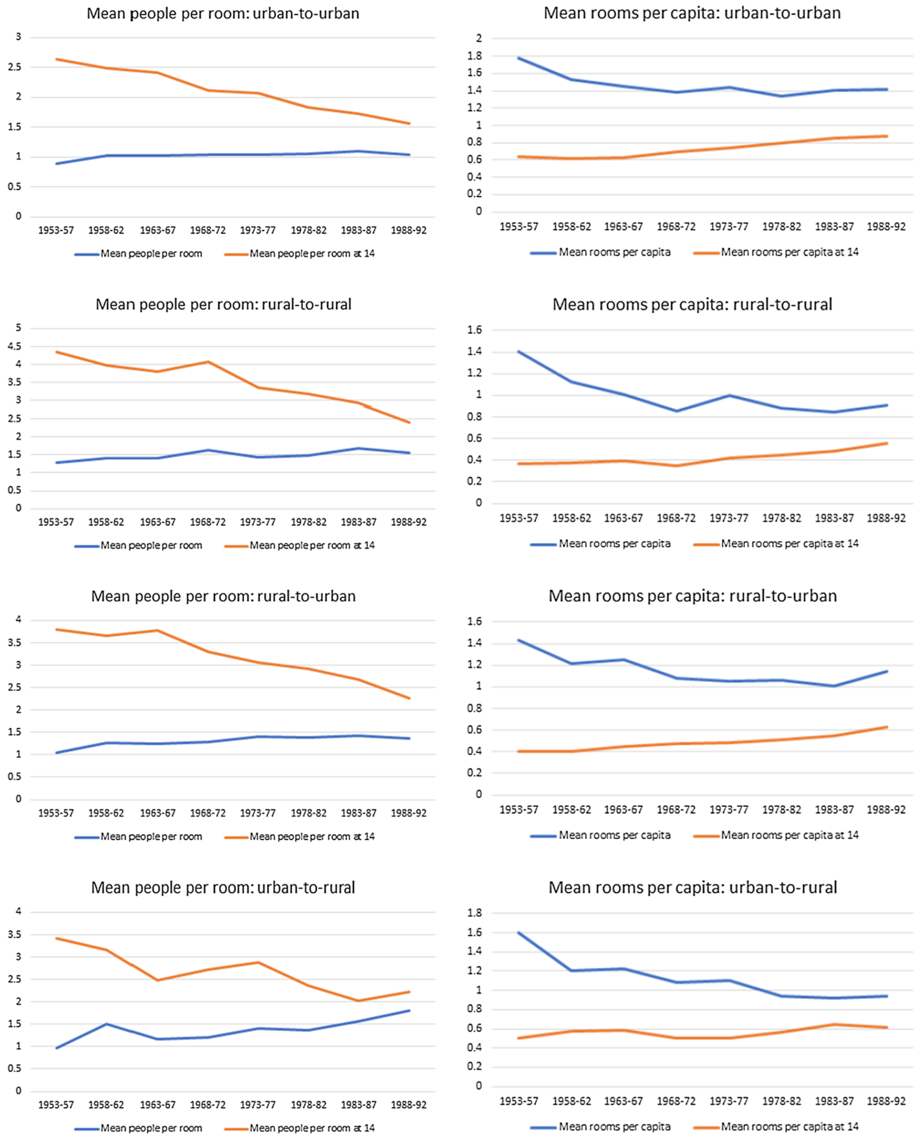

Fig. 1 Mean people per room and rooms per capita by degree of urbanisation

re-rankings. We conclude rank dominance of the past distribution of people per room over the present one *in the sample* (i.e. not in the population) if both the minimum and the maximum difference (both reported in Fig. 2) are negative, because in that case every other difference in between will also be negative.

Likewise, we conclude rank-dominance of the present distribution over its past counterpart if (and only if) the minimum and maximum differences are both positive. Finally, we conclude an absence of rank-dominance either way if and only if the minimum and maximum differences bear opposite signs. Then, as per Proposition 2.2, we conclude for a

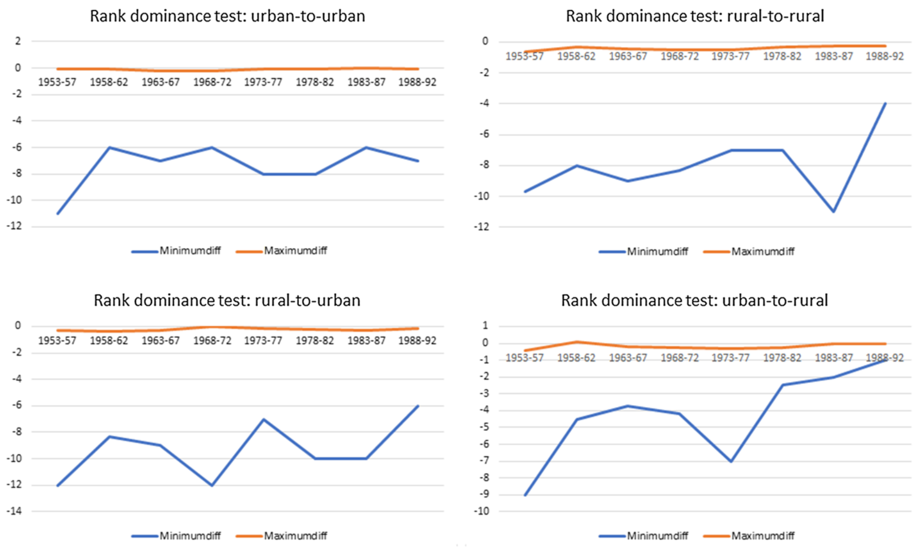

Fig. 2 Rank dominance tests for people per room by degree of urbanisation: minimum and maximum anonymously ranked differences between present and past values of people per room

particular sample that the current distribution of *rooms per capita* rank-dominates its past counterpart if the minimum and maximum differences in Fig. 2 are both negative; in turn, signalling second-order dominance by the current distribution of rooms per capita over its past counterpart.[18]

Figure 2 shows the rank dominance test results for the cohort-urbanisation combinations. As expected from the mean comparisons in Fig. 1, there is no evidence whatsoever of rank-dominance of past rooms per capita over the corresponding present distribution for any of the 32 cohort-urbanisation sample combinations. Meanwhile, we find rank-dominance of the present distribution of rooms per capita over its past counterpart in almost every sample, with the exception of the second-youngest cohort of the urban-to-urban sample and the second-to-oldest cohort of the urban-to-rural sample. In both cases, the signs of the difference statistics do not match, pointing to the absence of rank-dominance in any direction. Put differently, by and large the generalised Lorenz criterion can rank the pairs of overcrowding distributions across cohorts and urbanisation, whichever the ratio representation.

4.3 (In)consistency in the assessment of intergenerational changes in overcrowding inequality

Figure 3 shows the standard deviations of past and present overcrowding across urbanisation samples and for both ratio representations (people per room and rooms per capita). The inconsistency is quite stark. Applied to people per room, the standard deviation tells a story

[18] Note these are tests for rank-dominance *within the sample*, as opposed to tests of hypotheses about the *populations* relying on the evidence from samples. The latter is not a trivial pursuit: even if there was truly rank dominance in the unknown populations one could draw a large number of samples of different sizes featuring minimum and maximum differences with opposing signs and viceversa (i.e. one could draw samples featuring minimum and maximum differences with the same sign from populations that cannot be ordered in terms of rank dominance).

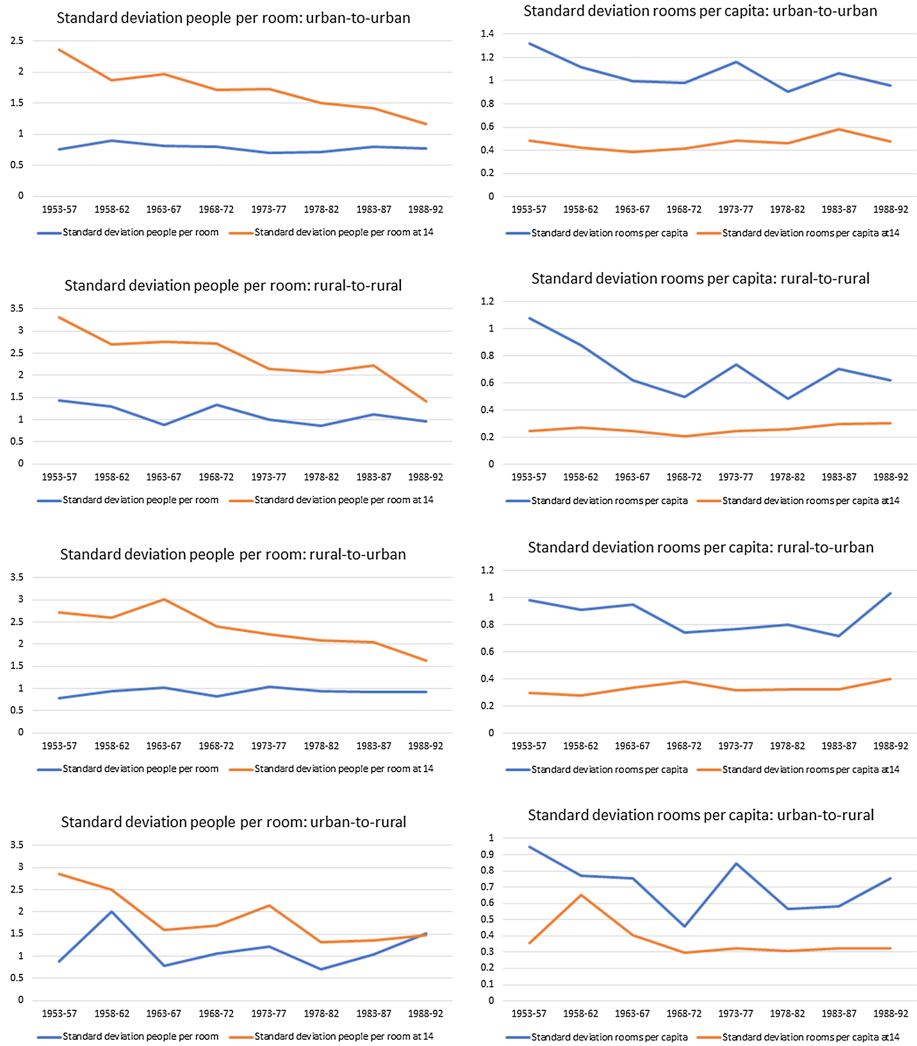

Fig. 3 Standard deviation by degree of urbanisation

of intergenerational decrease in absolute inequality across all cohorts and samples (except for the youngest cohort in the urban-to-rural sample, which suffers from relatively low size anyway). By contrast, applied to rooms per capita, the standard deviation describes the exact opposite picture without exceptions. Partly, this contrast could be explained by the fact that mean people per room decreased across all cohorts *toward* its lower bound (of 0, at least in principle), thereby reducing the scope for dispersion in the distribution; whereas mean rooms per capita has been simultaneously increasing *away* from its lower bound (of 0, again at least in principle), thereby increasing the scope for dispersion in the distribution. Whichever the explanation, the key point is the pervasiveness of inconsistency across comparisons.

Figure 4 corroborates the widespread presence of inconsistency across ratio representations, now considering relative inequality measurement. Albeit less clearly than in the case

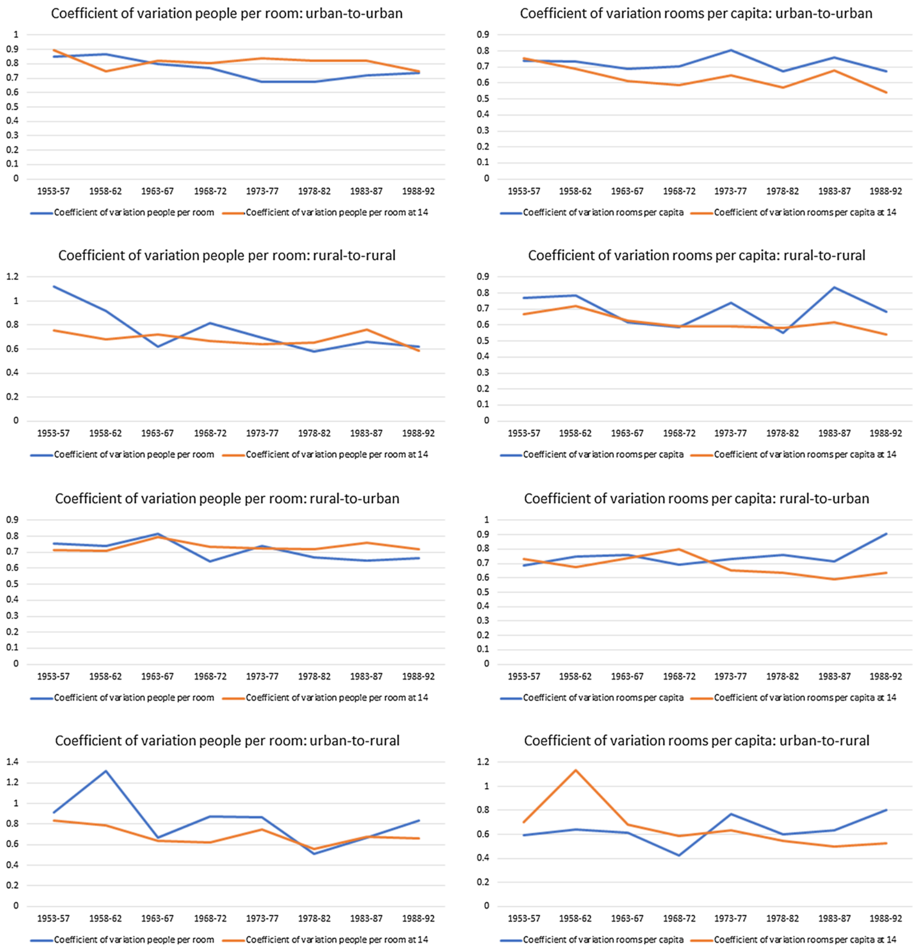

Fig. 4 Coefficient of variation by degree of urbanisation

of the standard deviation, plenty of inequality comparisons (among the 32 samples) are inconsistent across ratio representations with the coefficient of variation.

4.4 Consistent assessment of intergenerational changes in overcrowding inequality

Figure 5 shows the values of the consistent Atkinson index in Eq. 13. In general consistent relative inequality has decreased non-monotonically across cohorts for both past and present series, with the exceptio of present overcrowding in the small-sized urban-to-rural sample. Within each cohort, though, results vary widely. For instance, in the urban-to-urban sample, only three cohorts (including the two youngest ones) out of eight experienced an increase in inequality in overcrowding. Inequality increase is also observed in four out of eight cohorts in the rural-to-rural sample, two out of eight in the rural-to-urban sample, and four out of eight in the (small sized) urban-to-rural sample. Hence, it seems that, overall, the reduction

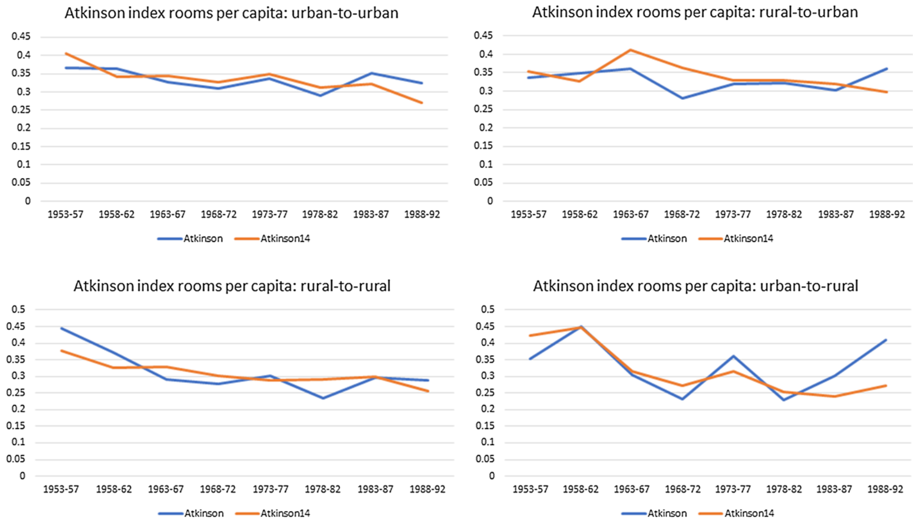

Fig. 5 Atkinson index based on harmonic mean by degree of urbanisation

in overcrowding inequality is more salient across cohorts than within them; that is, a cohort effect rather than structural mobility.

5 Concluding remarks

In the light of the recent interest in consistent inequality measurement when variables are bounded, we set out to study the existence of a similar problem when variables are ratios that can be represented in two alternative ways by swapping numerator for denominator. Rather than a mere academic curiosity, several important socioeconomic indicators come in the form of ratios, whether conspicuously (e.g. overcrowding) or less so (e.g. infection rates). Hence we are entitled to ask whether inequality comparisons of ratios are consistent across the two possible representations.

Before entering inequality measurement, we took some time to consider the consistency of dispersion-sensitive welfare comparisons with ratios. Concretely, we asked whether first and second-order dominance comparisons would be consistent to alternative representations and we found (i) that first-order dominance is always consistent; and (ii) that, while unnecessary, first-order dominance ensures the consistency of second-order dominance across the two alternative ratio representations. However, finding jointly necessary and sufficient conditions for consistent second-order dominance comparisons remains an open question for future research. For instance, future inquiries could ascertain whether consistency in higher orders of dominance may require a necessary full rank-dominance condition, i.e. first-order dominance. For second-order dominance, we found that only rank-dominance over the two extreme values was necessary in order to attain consistency.

Then came the first main finding of the paper in the form of an "impossibility" result: all the Zoli partial orderings (including a wide array of robust relative, absolute and intermediate approaches to inequality measurement) are inconsistent when the variables are ratios. At this point, we clarified that our definition of consistency requires inequality indices to satisfy the property *for all possible pairs of distributions*. Consequently, inconsistent inequality partial orderings can still rank *some pairs of distributions* consistently. We provided some examples. Perhaps a future line of inquiry could seek to characterise the set of distributional pairs which are always ranked consistently by all inequality indices respecting the Zoli partial orderings (or also by some subclass thereof, e.g. relative indices, etc.).

Inconsistent partial orderings rule out the joint consistency of entire classes of indices respecting a particular partial ordering; yet they do not rule out the consistency of specific subclasses of inequality indices. Hence our second main finding was the identification of a subclass of consistent inequality indices within the class of relative inequality indices. Remarkably, these consistent indices are functions of the ratio of the harmonic to the arithmetic mean. In addition to consistency, they satisfy all the key desirable properties of inequality measurement, and include among them the Atkinson index based on the harmonic mean. Since it is unlikely that this paper exhausted all possibilities, future research should prioritise an axiomatic characterisation of the complete class of consistent inequality indices for ratio variables.

We also explored three potential alternative solutions. The first one requires defending the use of a particular representation of the ratio at the expense of the other one. Effectively, this solution summarily dismisses the consistency problem. Implicitly, this is the approach normally taken in practice, e.g. nobody has yet proposed measuring people per currency as an inverse alternative representation of household income per capita. On the other hand, alternative representations of overcrowding have indeed been proposed. We discussed some of the criteria that might be brought to bear in order to adjudicate for or against a particular representation.

The second and third potential alternatives did attempt to take consistency seriously. One involves using generalised means of the same inequality index evaluated in both representations. The other one requires taking the logarithmic transformation of the ratio. Remarkably both solutions do succeed in restoring consistency, but at the expense of violating the transfers principle. Future research may inquire into whether the generalised-mean approach can be somewhat salvaged. As for the log-transformation approach, we found that if we consider transfers (or smoothing with bi-stochastic matrices) for the transformed variable then the transfer principle is restored alongside consistency. But even here there is a catch: the inequality comparison based on the transformed variable is consistent among alternative representations, but is not consistent with the comparison based on the original variable. Hence future research may consider alternative transformations of the variable.

In a nutshell, as a solution to the problem of consistent inequality measurement with ratio variables, the class of relative inequality indices based on the ratio of the harmonic to the arithmetic mean is arguably superior to the three proposed alternatives, given their respective non-trivial disadvantages. Still it is up to everyone to ponder them all and judge whether any is satisfactory enough. In the meantime, future research could explore further alternative solutions to the consistency problem in inequality measurement with ratios.

Though every empirical scenario is bound to be different, the case of intergenerational inequality comparisons in overcrowding in Mexico proved interesting, among other things, for featuring numerous instances of rank-dominance jointly with inconsistency in inequality comparisons across the board. The deployment of the consistent Atkinson index showed that overcrowding inequality in Mexico has largely decreased from older to younger cohorts, without being necessarily accompanied by structural mobility within cohorts. Future research could probe several other empirical situations and try out both the proposed solutions and any new alternatives for its own sake, but also to ascertain whether the case of intergenerational overcrowding in Mexico is typical or anomalous.

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