Article:

Nie, X., Coca, D., Luo, J. et al. (2020) Solving the inverse Frobenius-Perron problem using stationary densities of dynamical systems with input perturbations. Communications in Nonlinear Science and Numerical Simulation, 90. 105302. ISSN: 1007-5704

https://doi.org/10.1016/j.cnsns.2020.105302

Solving the inverse Frobenius-Perron problem using stationary densities of dynamical systems with input perturbations

Xiaokai Nie [a,b,c,d*], Daniel Coca [e], Jingjing Luo [f,g,h], Mark Birkin [d]

[a] *School of Automation, Southeast University, Nanjing, 210096, P.R. China*

[b] *Key Laboratory of Measurement and Control of Complex Systems of Engineering, Ministry of Education, Southeast University, Nanjing, 210096, P.R. China*

[c] *School of Mathematics, The University of Manchester, Manchester, M13 9PL, United Kingdom*

[d] *Leeds Institute for Data Analytics, University of Leeds, Leeds, LS2 9JT, United Kingdom*

[e] *Department of Automatic Control and Systems Engineering, The University of Sheffield, Sheffield, S1 3JD, United Kingdom*

[f] *Institute of AI and Robotics, Academy for Engineering and Technology, Fudan University, Shanghai,200433, P.R. China*

[g] *Jihua Laboratory, Foshan, Guangdong, 528000, P.R. China*

[h] *Engineering Research Center of AI & Robotics, Ministry of Education, Shanghai, 200433, P.R. China*

[*]Corresponding author: xiaokai.nie@hotmail.com (X. Nie).

Abstract

Stationary density functions statistically characterize the stabilized behavior of dynamical systems. Instead of temporal sequences of data, stationary densities are observed to determine the unknown transformations, which is called the inverse Frobenius-Perron problem. This paper proposes a new approach to determining the unique map from stationary densities generated by a one-dimensional discrete-time dynamical system driven by an external control input, given the input density functions that are linearly independent. A numerical simulation example is used to validate the effectiveness of the developed approach.

Keywords: Nonlinear systems; Chaotic maps; Asymptotic stability; Stationary densities; Inverse Frobenius-Perron Problem

1. Introduction

Chaotic behavior is prevalent in many real-world dynamical systems including physical, biological, chemical and economical systems that can be represented by deterministic equations [1-3]. It is common knowledge that even simple one-dimensional iterated maps can model such complicated dynamical behavior [4]. The limiting statistical behavior is statistically described by the stationary density function, which is obtained by observing the long-term outcomes of the system. There are many practical situations where the underlying system is unknown and only the stationary density of the system is observable [3, 5-10]. To model and analyze the dynamical system raises a challenging problem to determine the unknown transformation of the dynamical system from the given stationary density, which is the well-known inverse Frobenius-Perron problem [11]. Given the nonsingular transformation, the evolution of an initial density function under the action of the deterministic transformation is described by the Frobenius-Perron operator associated with the transformation, whose fixed point is the stationary density of the transformation [12].

A great many attempts have been made to solve the inverse Frobenius-Perron problem, given the knowledge of stationary densities. In general, these solutions to the inverse problem rely on the prescribed stationary densities/statistical properties and incorporate *a priori* knowledge of the unknown transformation. These include an entirely graph-theoretic method introduced in [13] to construct piecewise linear transformations that have piecewise constant stationary densities of relative minima value zero; a more generalized approach given in [14] to deriving an explicit relationship between stationary density functions and a special class of piecewise linear transformations over a partition of interest, and furthermore a matrix-based solution to construct 3-band transformations that preserves a given piecewise constant stationary density. Using this matrix-based algorithm, a recursive Markov state disaggregation scheme was proposed in [15] to construct multiple semi-Markov chaotic maps with stationary densities. The inverse problem was investigated in [4] for a class of symmetric maps that have stationary beta density functions and a unique solution can be obtained under the symmetry constraints of the unknown maps. This method was extended to the class of continuous unimodal maps for which each branch of the map covers the complete unit interval and the stationary density function is a two-parametric asymmetric beta density function in [16]. Given arbitrary stationary densities, but specified forms of maps to be constructed, further approaches were developed for constructing maps including unimodal transformations[17]; two types of one-dimensional symmetric maps [18]; complete chaotic maps with closed functional forms [19]; and multiple segments [20, 21]. In [22] and [23] additional statistical properties of correlation functions were used to construct one-dimensional maps. In [24, 25] the inverse problem was studied as a problem of determining a perturbation applied to the original chaotic map to obtain a desired stationary density function. Based on positive matrix theory a method of synthesizing chaotic maps with prescribed piecewise constant stationary densities and mixing properties was devised in [7], and was applied to constructing synchronized communication networks [5, 26], and generating images through synthesizing higher-dimensional maps with their stationary densities [27]. In [6] to characterize the patterns of activity in olfactory bulbs an optimization algorithm was introduced to synthesize the Markov process using probability distributions of neural signal interspike intervals that represent an odor.

Since different dynamical systems, possessing totally different transient dynamics, may display the same asymptotic time behavior characterized by stationary densities, the uniqueness of the solutions to the inverse problem using stationary densities cannot be guaranteed, that is, there may exist multiple transformations inferred from the same stationary density. As a result, in order to attain a unique result, additional constraints or assumptions, including statistical properties and special forms of the transformations are required. However, in many practical situations it is difficult to have such knowledge pertaining to the unknown transformations, and only the stationary densities of the systems can be observed. In [28] not only stationary density but also a temporal sequence of densities generated by the underlying system are used to estimate the unique transformation, and in [29, 30] chaotic maps are determined from sequences of generated densities, which are both under the assumption that the evolution of an arbitrary density function is observable. This is, however, not always the case in practice, particularly as they require selecting initial conditions to generate densities [15].

In this context, this paper proposes a new method of inferring one-dimensional dynamical systems using stationary densities, which are generated and experimentally observed by applying an external control input of different density functions. To the best of our knowledge, this solution is developed for the first time to determine the unique maps given only stationary densities.

One motivation of developing the method was to infer the chaotic map model that describes the dynamic evolution of heterogeneous stem cell populations using observed equilibrium distributions. These stationary densities were generated by applying different external stimuli to stem cells cultures. The identification facilitates our understanding of the molecular mechanisms underlying the behavior of divergent cell fates. This method also forms a theoretical basis for designing the control strategy to manipulate the differentiation of stem cells into desired types.

The paper is organized as follows: Section 2 introduces the inverse problem for dynamical systems to be addressed in this paper, and the formulation of stationary density functions is provided in Section 3. Section 4 elucidates a methodology for inferring a chaotic map from stationary density functions arising from linearly independent input density functions. Section 5 presents a numerical simulation example to validate the developed algorithm. Conclusions are given in Section 6.

2. Dynamical systems with external input perturbations

Consider a one-dimensional discrete-time dynamical system in the presence of an external input perturbation as follows

$$x_{n+1} = S(x_n) + \omega_n \qquad (\text{mod } b), \tag{1}$$

where $S : R \to R$, $R = [0,b]$ is a measurable and nonsingular transformation, that is $\mu(S^{-1}(A)) \in \mathfrak{B}$ for any $A \in \mathfrak{B}$ and $\mu(S^{-1}(A)) = 0$ for all $A \in \mathfrak{B}$ with $\mu(A) = 0$, where μ denotes a measure on (R, $\mathfrak{B}$) and $\mathfrak{B}$ denotes a Borel σ-algebra of subsets in R; $\omega_n \in R$ denotes the control input, and are independent random variables, each distributed with the same probability density function, specified as $f^{\omega} \in D(R, \mathfrak{B}, \mu)$, $D = \{ f \in L^1(R, \mathfrak{B}, \mu) :$ $f \geq 0, \|f\|_1 = 1\}$.

Let $f_n \in D$ be the probability density function of x_n, and f_* the stationary density function of the artificially perturbed dynamical system with a control input density function f^{ω}. To show the stationary density is observable, that is, the ultimate limit of f_n as $n \to +\infty$ is f_*, the existence of f_* is analyzed in the following section. It is assumed here that the map S which we aim to reconstruct is nonsingular, and that only the stationary densities $\{f_*\}$ can be observed and estimated with step functions over R, after sufficient successive iterations of the process.

The identification problem addressed in this paper is to determine the transformation S from the stationary densities $\{f_*^k\}_{k=1}^K$ generated from the dynamical system in response to the given different input density functions $\{f_k^\omega\}_{k=1}^K$.

3. Formulation of stationary densities

To make use of stationary densities to infer the unknown transformation S, the underlying relationship between S and the stationary density f_* is derived firstly and uniqueness of stationary density is subsequently proven.

The perturbed dynamical system (1) is rewritten as follows

$$x_{n+1} = \begin{cases} S(x_n) + \omega_n, & \text{if } S(x_n) + \omega_n \le b; \\ S(x_n) + \omega_n - b, & \text{if } b < S(x_n) + \omega_n \le 2b, \end{cases} \tag{2}$$

which is illustrated in Fig. 1. Since $S(x_n) + \omega_n \ge 0$, this is equivalent to

$$x_{n+1} = S(x_n) + \omega_n - b\chi_{(b,2b]}(S(x_n) + \omega_n), \tag{3}$$

where $\chi_\Delta(x)$ is the indicator function for a set Δ defined by

$$\chi_\Delta(x) = \begin{cases} 1, & x \in \Delta; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

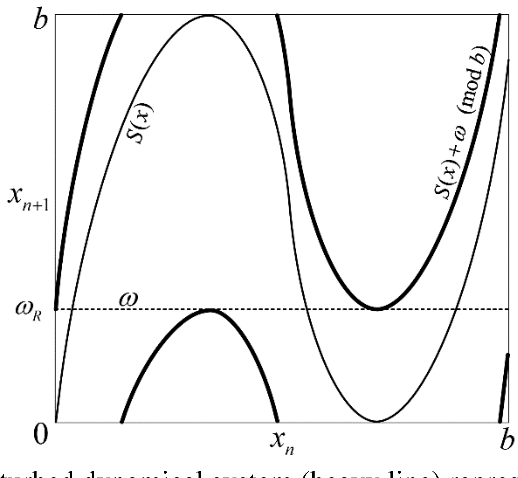

Fig. 1 Illustration of the perturbed dynamical system (heavy line) represented by (2), where a one-dimensional map $S(x)$ (thin line) with an external input perturbation (dotted line) with constant value ω_R over R.

The probability of the generated new state x_{n+1} being in an arbitrary Borel set $B \subset R$ is given by

$$\text{Prob}\{x_{n+1} \in B\} = \int_B f_{n+1}(x)dx\,. \tag{5}$$

Let $x = S(y) + \omega - b\chi_{(b,2b]}(S(y) + \omega_n)$, then $\omega = x - S(y) + b\chi_R(S(y) - x)$. Given $x \in B$, suppose $\omega \in B'$, $B' \subset R$, for $y \in R$. Since $S(x_n)$ and ω_n are independent, the joint density of (x_n, ω_n) is $f_n f^{\omega}$. $\text{Prob}\{x_{n+1} \in B\}$ can also be written as

$$\text{Prob}\{x_{n+1} \in B\} = \int_{B'}\int_R f_n(y) f^{\omega}(\omega)dyd\omega\,. \tag{6}$$

Further, it is obtained that

$$\text{Prob}\{x_{n+1} \in B\} = \int_B\int_R f_n(y) f^{\omega}(x - S(y) + b\chi_R(S(y) - x))dydx\,. \tag{7}$$

Let $\overline{P}f_n = f_{n+1}$, $f_n \in D$, where $\overline{P} : L^1 \to L^1$ is the operator that transforms the probability density function f_n into f_{n+1} under the operation of S and ω. From (5) and (7), it follows that

$$\overline{P}f_n(x) = \int_R f_n(y) f^{\omega}(x - S(y) + b\chi_R(S(y) - x))dy\,, \tag{8}$$

which describes the evolution of densities generated by the dynamical system with an input perturbation. Let $K(x, y) = f^{\omega}(x - S(y) + b\chi_R(S(y) - x))$. Apparently for every $x \in R$ and $y \in R$, $K(x, y) \geq 0$, and by changing variables $z = x - S(y) + b\chi_R(S(y) - x)$, for every $y \in R$, $0 \leq x \leq b$ then $0 \leq z \leq b$, we have

$$\int_R K(x, y)dx = \int_R f^{\omega}(z)dz = 1. \tag{9}$$

K is, therefore, a stochastic kernel. We further have

$$\begin{aligned}\int_R \overline{P}f(x)dx &= \int_R\int_R f(y)K(x, y)dydx \\ &= \int_R f(y)dy\int_R K(x, y)dx \\ &= \int_R f(y)dy.\end{aligned} \tag{10}$$

Hence, $\overline{P}$ is a Markov operator by the definition below [11].

Definition 1. Any linear operator $\overline{P}: L^1 \to L^1$ satisfying that a) $\overline{P}f \geq 0$, for $f \geq 0$, $f \in D$; and b) $\|\overline{P}f\|_1 = \|f\|_1$, for $f \geq 0$, $f \in D$, is called a Markov operator.

For a Markov operator $\overline{P}$, if the sequence $\{\overline{P}^n\}$ is asymptotic stable, there exists a unique stationary density for $\overline{P}$. To prove the existence and also the uniqueness of f_*, the following result is introduced firstly [11].

Theorem 1. *Let* $(R, \mathfrak{B}, \mu)$ *be a measure space,* $k: R \times R \to R$ *a stochastic kernel, and* $\overline{P}$ *the corresponding Markov operator. Denote the kernel corresponding to* $\overline{P}^n$ *by* k^n*. If, for some m,*

$$\int_R \inf_y k^m(x, y) dx > 0, \tag{11}$$

then $\{\overline{P}^n\}$ *is asymptotically stable.*

The asymptotical stability of the sequence of densities generated by the perturbed chaotic systems is proven below.

Theorem 2. *Given any input density function* $f^{\omega} \in D(R, \mathfrak{B}, \mu)$*, for arbitrary initial conditions, the densities generated by the perturbed dynamical system* (1) *are asymptotically stable, that is, the perturbed dynamical system possesses a unique stationary density function corresponding to a given input density function.*

Proof. Let $\overline{P}: L^1 \to L^1$ be the transfer operator of probability density generated by the perturbed dynamical system (1). We have that $k(x, y) = f^{\omega}(x - S(y) + b\chi_R(S(y) - x))$. Let $z(x, y) = x - S(y) + b\chi_R(S(y) - x)$. Since $k(x, y) \geq 0$, and for every $y \in R$, $\int_R k(x, y) dx > 0$, we then have $\int_R \inf_y k(x, y) dx > 0$, this implies that (11) holds. Therefore, the generated densities $\{\overline{P}^n\}$ are asymptotic stable for a given input density function. This completes the proof.

Remark 1. Theorem 2 implies the existence and uniqueness of stationary density functions for the input perturbed discrete-time dynamical systems (1), and that the transformation in the presence of input perturbation are statistically stable. Given the transformation and an input density function, the generated densities converge to a unique stationary density function f_*.

Thus $\lim_{n\to+\infty} \bar{P}^n f = f_*$ for all $f \in D(R,\mathfrak{B},\mu)$. This implies that for arbitrary initial conditions, there exists a unique stationary density function that is observable after sufficient iterations.

To estimate the stationary density function the induction of f_* by the map S and input density function f^{ω} is further explored. A particular Markov operator, called the Frobenius-Perron operator is introduced below, which will be used to infer S based on the formulation of operator $\bar{P}$ corresponding to the perturbed dynamical system.

Definition 2. For a nonsingular transformation $S: R \to R$, the unique Frobenius-Perron operator $P: L^1 \to L^1$ associated with S is given by

$$Pf(x) = \frac{d}{dx}\int_{S^{-1}(A)} f(y)dy\,, \tag{12}$$

where $A = [a, x] \in \mathfrak{B}$.

Let $\Re = \{R_1, R_2, \ldots, R_N\}$ be a partition of R into N intervals, and $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \varnothing$ if $i \neq j$, $1 \le i, j \le N$. Assuming that S is piecewise monotonic and expanding with respect to $\Re$, the associated Frobenius-Perron operator P_S is given by [12]

$$P_S f_n(x) = \sum_{i=1}^{N} \frac{f_n(S_i^{-1}(x))}{\left|S'(S_i^{-1}(x))\right|} \chi_{S(R_i)}(x)\,, \tag{13}$$

where S_i is the monotonic restriction of S on the interval R_i.

To explore the representation of $\bar{P}$ in terms of Frobenius-Perron operator P_S, the Koopman operator is introduced here to rewrite (8).

Definition 3. For a nonsingular transformation $S: R \to R$ and $f: L^1 \to L^1$, the operator $U: L^1 \to L^1$ defined by

$$Uf(x) = f(S(x))\,, \tag{14}$$

is called the Koopman operator corresponding to S.

For every $f: L^1 \to L^1$, $g: L^1 \to L^1$, the following equality holds

$$\langle P_S f, g \rangle = \langle f, Ug \rangle, \tag{15}$$

where $\langle\ ,\ \rangle$ represents the scalar product, i.e. $\langle f, g \rangle = \int_R f(x)g(x)dx$ [11], so that U is adjoint to the Frobenius-Perron operator P_S. Since S is nonsingular, the Frobenius-Perron and Koopman operators corresponding to S both exist. Let $F_x(y) = f^{\omega}(x - y + b\chi_R(y - x))$. Then (8) is rewritten as

$$\overline{P}f_n(x) = \int_R f_n(y)F_x(S(y))dy. \tag{16}$$

From Definition 3, $\overline{P}$ may be given in terms of the corresponding Koopman operator U, $\overline{P}f_n(x) = \langle f_n, UF_x \rangle$. It follows from (15) that $\overline{P}f_n(x) = \langle P_S f_n, F_x \rangle$. Thus we have

$$\overline{P}f_n(x) = \int_R f^{\omega}(x - y + b\chi_R(y - x)) \cdot P_S f_n(y)\, dy. \tag{17}$$

which demonstrates a straightforward relationship between the transfer operator $\overline{P}$ corresponding to the perturbed system (1) and the Frobenius-Perron operator P_S associated with S in terms of input density function f^{ω}.

Remark 2. As $n \to +\infty$, $\overline{P}f_n$ converges to the stationary density function f_*, and the corresponding perturbation free Frobenius-Perron operator $P_S f_n$ associated with S is $P_S f_*$. (17) shows an elegant bridge between $\overline{P}$ and the Frobenius-Perron operator P_S, which forms a basis for exploring the pathway to infer S through $P_S f_*$ using the observed f_*. The problem of inferring S is thus predominantly reduced to that of identifying the Frobenius-Perron operator P_S from the fixed points of $\overline{P}$, i.e. the stationary densities.

Here we use a special class of nonlinear transformations called piecewise linear semi-Markov transformations to estimate unknown maps. Let S be a piecewise linear expanding semi-Markov transformation over the N-interval partition $\Re = \{R_1, R_2, \ldots, R_N\}$.

Definition 4. A transformation $S: R \to R$ is said to be semi-Markov with respect to the partition $\Re$ (or $\Re$-semi-Markov) if there exist disjoint intervals $Q_j^{(i)}$ so that $R_i = \cup_{k=1}^{p(i)} Q_k^{(i)}$, $i = 1, \ldots, N$, the restriction of S to $Q_k^{(i)}$, denoted $S|_{Q_k^{(i)}}$, is monotonic and $S(Q_k^{(i)}) \in \Re$ [14].

In other words the nonlinear transformation satisfies that each restriction $S|_{R_i}$ is a homeomorphism from R_i to a union of intervals of $\Re$, that is $\bigcup_{k=1}^{p(i)} R_{r(i,k)} = \bigcup_{k=1}^{p(i)} S(Q_k^{(i)})$, where $R_{r(i,k)} = S(Q_k^{(i)}) \in \Re$, $Q_k^{(i)} = [q_{k-1}^{(i)}, q_k^{(i)}]$, $i = 1,\ldots,N$, $k = 1,\ldots,p(i)$ and $p(i)$ denotes the number of disjoint subintervals $Q_k^{(i)} \subseteq R_i$ [29]. For a piecewise linear semi-Markov map S with respect to $\Re$, its Frobenius-Perron operator P_S can be represented by a finite rank non-negative square matrix $M = (m_{i,j})_{1\le i,j\le N}$, where

$$m_{i,j} = \begin{cases} |(S|_{Q_j^{(i)}})'|^{-1}, & \text{if } S(Q_k^{(i)}) = R_j; \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Give a piecewise constant density function $f_n = \sum_{i=1}^{N} w_i^n \chi_{R_i}(x)$ defined on $\Re$, the new density function $P_S f_n$ generated by S is also piecewise constant on $\Re$, which is given by $P_S f_n = \sum_{i=1}^{N} v_i^{n+1} \chi_{R_i}(x)$, and the formula $\boldsymbol{v}^{P_S f_n} = \boldsymbol{w}^{f_n} M$ holds for $\boldsymbol{w}^{f_n} = [w_1^n, w_2^n, \ldots, w_N^n]$ and $\boldsymbol{v}^{P_S f_n} = [v_1^{n+1}, v_2^{n+1}, \ldots, v_N^{n+1}]$, which are the coefficient vectors of f_n and $P_S f_n$, respectively. Since an arbitrary stationary density $f_* \in D$ can be approximated by a piecewise constant function on $\Re$ given by

$$f_*^N(x) = \sum_{i=1}^{N} w_i^* \chi_{R_i}(x), \tag{19}$$

and the density function $P_S f_*$ transformed from f_* under the action of S can also be approximated in the piecewise constant form by

$$P_S f_*^N(x) = \sum_{j=1}^{N} v_j^* \chi_{R_j}(x), \tag{20}$$

we then have

$$\boldsymbol{v}^{P_S f_*^N} = \boldsymbol{w}^{f_*^N} M, \tag{21}$$

where $\boldsymbol{w}^{f_*^N} = [w_1^*, w_2^*, \ldots, w_N^*]$ and $\boldsymbol{v}^{P_S f_*^N} = [v_1^*, v_2^*, \ldots, v_N^*]$ that are the coefficient vectors of piecewise constant approximations f_*^N and $P_S f_*^N$ of f^* and $P_S f^*$, respectively. This is equivalent to

$$P_S f_*^N(x) = \sum_{j=1}^{N} \left[\sum_{i=1}^{N} (w_i^* m_{i,j}) \right] \chi_{R_j}(x). \tag{22}$$

Since $\lim_{n\to+\infty} \bar{P} f_n = \bar{P} f_*$, $\bar{P} f_* = f_*$, and f_*^N converges to f_* as $N \to +\infty$, that is, $\lim_{N\to+\infty} \left[f_*(x) - f_*^N(x) \right] = 0$ for $x \in R$, after integration of each side of (17) over an interval $R_i \in \Re$, it is obtained that

$$\lim_{N\to+\infty} \left[\int_{R_i} f_*(x)dx - \int_{R_i} f_*^N(x)dx \right] = 0, \tag{23}$$

and

$$\begin{aligned} \lim_{N\to+\infty} [\int_{R_i} \int_{R} f^{\omega}(x - z + b\chi_R(z - x)) \cdot P_S f_*(z)\, dz dx - \\ \int_{R_i} \int_{R_j} f^{\omega}(x - z + b\chi_R(z - x)) \cdot P_S f_*^N(z)\, dz dx] = 0. \end{aligned} \tag{24}$$

Substituting (20) into (24) gives

$$\begin{aligned} \lim_{N\to+\infty} \{ \int_{R_i} \int_{R} f^{\omega}(x - z + b\chi_R(z - x)) \cdot P_S f_*(z)\, dz dx - \\ \int_{R_i} \sum_{j=1}^{N} [\int_{R_j} f^{\omega}(x - z + b\chi_R(z - x))\, dz \cdot v_j^* dx] \} = 0. \end{aligned} \tag{25}$$

From (19), (23) and (25), it is then obtained

$$\lim_{N\to+\infty} \left\{ \int_{R_i} \sum_{i=1}^{N} w_i^* \chi_{R_i}(x) dx - \int_{R_i} \sum_{j=1}^{N} [\int_{R_j} f^{\omega}(x - z + b\chi_R(z - x))\, dz \cdot v_j^* dx] \right\} = 0. \tag{26}$$

It follows that

$$\lim_{N\to+\infty} \left\{ w_i^* - \frac{1}{\lambda(R_i)} \sum_{j=1}^{N} [\int_{R_i} \int_{R_j} f^{\omega}(x - z + b\chi_R(z - x))\, dz dx \cdot v_j^*] \right\} = 0, \tag{27}$$

for $i = 1, \ldots, N$, where $\lambda(R_i)$ is the Lebesgue measure on R_i.

Let $Q = (q_{i,j})_{1 \le i \le N; 1 \le j \le N}$ be a matrix defined by

$$q_{i,j} = \frac{1}{\lambda(R_i)} \int_{R_i} \int_{R_j} f^{\omega}(x - z + b\chi_R(z - x))\, dzdx. \tag{28}$$

Then, substituting (21) into (27) gives

$$\lim_{N \to +\infty} \left\| \boldsymbol{w}^{f_*^N} - \boldsymbol{w}^{f_*^N} M \cdot Q' \right\|_F = 0, \tag{29}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Let $H = M \cdot Q'$. Thus it can be seen that H is a matrix representation of the operator approximation $\overline{P}_N$ with respect to $\Re$.

Remark 3. It can be seen that the matrix Q is calculated directly from input density function. The stationary density function is related to the transformation S and also the input density function f^{ω}.

Theorem 3. *For a perturbed dynamical system* (1)*, any two linearly independent input density functions denoted by* $\{f_k^{\omega}\}_{k=1}^2$, $f_k^{\omega} \in D$*, give rise to two linearly independent stationary density functions* $\{f_{k*}\}_{k=1}^2$.

Proof. Let $f_1^{\omega}, f_2^{\omega} \in D$ be two input density functions. Suppose that f_1^{ω} and f_2^{ω} are linearly independent and that the corresponding stationary densities $f_{1*} = f_{2*}$ such that $w^{f_{1*}^N} = w^{f_{2*}^N}$. From (29), we have

$$\lim_{N \to +\infty} \left\| w^{f_{1*}^N} M Q_1' - w^{f_{2*}^N} M Q_2' \right\| = 0, \tag{30}$$

where $Q_1 \sim f_1^{\omega}$ and $Q_2 \sim f_2^{\omega}$.

Let $v = [v_1, \ldots, v_N]$ and $v = w^{f_{1*}^N} M = w^{f_{2*}^N} M$. It follows that $\lim_{N \to +\infty} \| vQ_1' - vQ_2' \| = 0$. From (28) it can be found that each column of Q contains the same entry values that satisfies $q_{i,j} = q_{i-1,j-1}$ and

$$\begin{aligned} \sum_{i=1}^{N} q_{i,j} &= \sum_{i=1}^{N} \left[\frac{1}{\lambda(R_i)} \int_{R_i} \int_{R_j} f^{\omega}(x - z + b\chi_R(z - x))\, dzdx \right] \\ &= \frac{1}{\lambda(R_i)} \int_{R} \int_{R_j} f^{\omega}(x - z + b\chi_R(z - x))\, dzdx \\ &= 1. \end{aligned} \tag{31}$$

Hence, the matrix Q_k can be rewritten as

$$Q_k = \begin{bmatrix} d_1^k & d_N^k & d_{N-1}^k & \cdots & d_2^k \\ d_2^k & d_1^k & d_N^k & \cdots & d_3^k \\ d_3^k & d_2^k & d_1^k & \cdots & d_4^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_N^k & d_{N-1}^k & d_{N-2}^k & \cdots & d_1^k \end{bmatrix}, \tag{32}$$

where $d_i = q_{i,1}$ for $i = 1, \dots, N$ that satisfy $\sum_{i=1}^{N} d_i = 1$ by (31). Thus, we have the equality as $n \to +\infty$,

$$\begin{cases} v_1(d_1^1 - d_1^2) + v_2(d_N^1 - d_N^2) + \cdots + v_N(d_2^1 - d_2^2) = 0 \\ v_1(d_2^1 - d_2^2) + v_2(d_1^1 - d_1^2) + \cdots + v_N(d_3^1 - d_3^2) = 0 \\ \cdots \\ v_1(d_N^1 - d_N^2) + v_2(d_{N-1}^1 - d_{N-1}^2) + \cdots + v_N(d_1^1 - d_1^2) = 0. \end{cases} \tag{33}$$

Since $v_i \neq v_j$, $\forall i \neq j$, it follows that $d_i^1 = d_2^1$ for $i = 1, \dots, N$, hence $f_1^\omega = f_2^\omega$, which contradicts the assumption. Thus, for any two different input densities $f_k^\omega \in D$, the corresponding generated stationary densities are different. This completes the proof.

Remark 4. Theorem 3 suggests that, given a nonsingular transformation *S*, the generated stationary density function is uniquely determined by the input density function, that is, a unique stationary density may not be shared by the perturbed dynamical system with multiple different input density functions.

Remark 5. Given the nonsingular transformation and an input density function, there exists a unique stationary density function, which is the fixed point of the corresponding operator $\overline{P}$, so that $f_* = \overline{P} f_*$, which can be expressed with respect to $\Re$ in (29). Hence, $\boldsymbol{w}^{f_*^N}$ can be viewed as the approximate left eigenvector of the eigenvalue 1 of *H*. f_* can be estimated arbitrarily well by f_*^N, therefore we have that f_*^N converge to f_* as $N \to +\infty$, that is

$$\lim_{N \to +\infty} \int_R (f_*^N(x) - f_*(x))^2 dx = 0. \tag{34}$$

The stationary density formulation result can be extended to dynamical systems with multiple external input perturbations.

Theorem 4. Let $S: R \to R$, and $\omega^k \in R$, $k = 1,\dots,K$ the i.i.d. input variables following the corresponding density functions $f^{\omega^k} \in R$, $k = 1,\dots,K$. The dynamical system perturbed by K input variables

$$x_{n+1} = S(x_n) + \omega_n^1 + \ldots + \omega_n^K \pmod{b}, \tag{35}$$

has the stationary density function f_*^N with respect to $\mathfrak{R}$ given by

$$\lim_{N \to +\infty} \left\| w^{f_*^N} - w^{f_*^N} M \prod_{k=1}^{K} Q_k' \right\|_F = 0. \tag{36}$$

Proof. Let $Y_1 = S(x_n) + \omega_n^1$ (mod b). From (29) the matrix representation of the transfer operator approximation $\bar{P}_{1,N}$ corresponding to Y_1 is given by $H_1 = M \cdot Q_1'$. Let $H_1 = (h_{i,j}^1)_{1 \le i,j \le N}$, then we have

$$h_{i,j}^1 = \begin{bmatrix} m_{i,1} \\ \vdots \\ m_{i,j} \\ m_{i,j+1} \\ \vdots \\ m_{i,N} \end{bmatrix}' \begin{bmatrix} d_j^1 \\ \vdots \\ d_1^1 \\ d_N^1 \\ \vdots \\ d_{j+1}^1 \end{bmatrix}. \tag{37}$$

The sum of each row is given by

$$\sum_{j=1}^{N} h_{i,j}^1 = \begin{bmatrix} m_{i,1} \\ \vdots \\ m_{i,j} \\ m_{i,j+1} \\ \vdots \\ m_{i,N} \end{bmatrix}' \begin{bmatrix} d_1^1 + d_2^1 + \ldots + d_N^1 \\ \vdots \\ d_N^1 + d_1^1 + \ldots + d_{N-1}^1 \\ d_{N-1}^1 + d_N^1 + \ldots + d_{N-2}^1 \\ \vdots \\ d_2^1 + d_3^1 + \ldots + d_1^1 \end{bmatrix} = \begin{bmatrix} m_{i,1} \\ m_{i,2} \\ \vdots \\ m_{i,N} \end{bmatrix}' \begin{bmatrix} \sum_{j=1}^{N} d_j^1 \\ \sum_{j=1}^{N} d_j^1 \\ \vdots \\ \sum_{j=1}^{N} d_j^1 \end{bmatrix}. \tag{38}$$

It follows that $\sum_{j=1}^{N} h_{i,j}^1 = \sum_{j=1}^{N} \left(m_{i,j} \sum_{j=1}^{N} d_j^1 \right) = \sum_{j=1}^{N} m_{i,j} = 1$, which suggests H_1 is a stochastic matrix. Since $h_{i,j}^1 \ge 0$, H_1 can be viewed as a Frobenius-Perron matrix corresponding to Y_1.

Then (35) is rewritten as

$$x_{n+1} = Y_1(S(x_n), \omega_n^1) + \omega_n^2 + \ldots + \omega_n^K \pmod{b}. \tag{39}$$

Let $Y_2 = Y_1(S(x_n), \omega_n^1) + \omega_n^2 \pmod{b}$. Likewise, the transfer operator approximation $\bar{P}_{2,N}$ corresponding to Y_2 can be represented by a matrix $H_2 = (h_{i,j}^2)_{1 \le i,j \le N}$ given by $H_2 = H_1 \cdot Q_2'$. It can be further given that

$$\sum_{j=1}^{N} h_{i,j}^2 = \begin{bmatrix} h_{i,1}^1 \\ h_{i,2}^1 \\ \vdots \\ h_{i,N}^1 \end{bmatrix}' \begin{bmatrix} d_1^2 + d_2^2 + \ldots + d_N^2 \\ \vdots \\ d_N^2 + d_1^2 + \ldots + d_{N-1}^2 \\ d_{N-1}^2 + d_N^2 + \ldots + d_{N-2}^2 \\ \vdots \\ d_2^2 + d_3^2 + \ldots + d_1^2 \end{bmatrix} = \begin{bmatrix} h_{i,1}^1 \\ h_{i,2}^1 \\ \vdots \\ h_{i,N}^1 \end{bmatrix}' \begin{bmatrix} \sum_{j=1}^{N} d_j^2 \\ \sum_{j=1}^{N} d_j^2 \\ \vdots \\ \sum_{j=1}^{N} d_j^2 \end{bmatrix}$$
$$= \sum_{j=1}^{N} \left(h_{i,j}^1 \sum_{j=1}^{N} d_j^2 \right) = \sum_{j=1}^{N} h_{i,j}^1 = 1. \tag{40}$$

Hence, H_2 is also a stochastic matrix and can be viewed as the Frobenius-Perron matrix induced by Y_2.

By analogy, (35) may be written as $x_{n+1} = Y_{K-1}(S(x_n), \omega_n^1, \ldots, \omega_n^{K-1}) + \omega_n^K \pmod{b}$ = $Y_K(S(x_n), \omega_n^1, \ldots, \omega_n^K)$, and the induced matrix is then given by

$$\begin{aligned} H_K &= H_{K-1} \cdot Q_K' \\ &= H_{K-2} Q_{K-1}' Q_K' = M \prod_{k=1}^{K} Q_k', \end{aligned} \tag{41}$$

where Q_K denotes the matrix Q_k (32) associated with the input variable ω_n^K. It can be seen that H_K is, therefore, a stochastic matrix and has 1 as the eigenvalue of maximum modulus and the associated eigenvector that is the estimated by the stationary density f_*^N over $\Re$. Thus, (36) is obtained.

4. An approach to estimating the unique transformation of the underlying system from stationary densities

From Remark 4, a discrete-time dynamical system driven by an external control input can have different stationary densities corresponding to different input densities. The objective of the developed approach is to infer a piecewise linear semi-Markov map approximation to the unknown map *S* that is assumed to be general nonlinear continuous using multiple different stationary densities observed experimentally. From (29) the problem is reduced to firstly determining the Frobenius-Perron matrix *M* associated with the piecewise linear semi-Markov map approximation and then constructing the map. A nonsingular transformation *S* having infinitely many pieces of monotonicity can be approximated by a sequence of piecewise linear functions $\{S_N\}_{N\geq 2}$ [31], and it follows from (34) that given the input density function, the stationary densities of the perturbed dynamical system can be approximated arbitrarily well by stationary densities of the perturbed finite approximations $\{S_N\}_{N\geq 2}$. Therefore, the Frobenius-Perron operator associated with *S* can be approximated arbitrarily well using the matrices $\{M_N\}_{N\geq 2}$ which can be estimated from the observed stationary densities.

The algorithm is summarized as follows:

Step 1: Specify *K* linearly independent input density functions $\{f_k^{\omega}\}_{k=1}^{K}$ to yield *K* stationary densities $\{f_{k*}^{N}\}_{k=1}^{K}$;

Step 2: Compute the matrix Q_k for each f_k^{ω}, and estimate the coefficient vectors $v^{P_S f_{k*}^N} = [v_1^k, v_2^k, \ldots, v_N^k]$ of $P_S f_{k*}^N$, for $k = 1, \ldots, K$;

Step 3: Determine the indices of positive entries of the Frobenius-Perron matrix *M* and identify the Frobenius-Perron matrix associated with the piecewise linear semi-Markov map $\hat{S}$ that approximates *S*.

Step 4: Construct the piecewise linear nonlinear map $\hat{S}$ corresponding to the identified Frobenius-Perron matrix *M* and smooth $\hat{S}$ to obtain the nonlinear continuous approximate map $\tilde{S}$.

Details of these steps are introduced below.

Step 1: Sample *K* linearly independent input densities $\{f_k^{\omega}\}_{k=1}^{K}$ to generate *K* sets of input data $\Omega^k = \{\omega_i^k\}_{i=1}^{\theta}$, $k = 1, \ldots, K$ respectively. Let $X_0 = \{x_j^0\}_{j=1}^{\theta}$ be a set of initial conditions and $X^{k*} = \{x_i^{k*}\}_{i=1}^{\theta}$ a set of final states computed by applying sufficient times the iterations (1) using the initial conditions X_0 and the corresponding input data set Ω^k. The stationary densities

$\{f_{k^*}\}_{k=1}^{K}$ are estimated with piecewise constant density functions $\{f_{k^*}^{N}\}_{k=1}^{K}$ over the partition $\Re$ given by

$$f_{k^*}^{N}(x)=\sum_{i=1}^{N} w_i^{k^*}\chi_{R_i}(x),\ w_i^{k^*}=\frac{N}{\theta b}\sum_{j=1}^{\theta}\chi_{R_i}(x_j^{k^*}), \tag{42}$$

which form the following matrix

$$W=\begin{bmatrix} w^{f_N^{1^*}} \\ w^{f_N^{2^*}} \\ \vdots \\ w^{f_N^{K^*}} \end{bmatrix}=\begin{bmatrix} w_1^{1^*} & w_2^{1^*} & \cdots & w_N^{1^*} \\ w_1^{2^*} & w_2^{2^*} & \cdots & w_N^{2^*} \\ & \cdots & & \\ w_1^{K^*} & w_2^{K^*} & \cdots & w_N^{K^*} \end{bmatrix}. \tag{43}$$

Remark 6. In practice it is typically required that $K \geq N$ so that more stationary densities are observed than the order of the Frobenius-Perron matrix to be determined. This ensures that sufficient dynamical behavior in response to different density functions of the control input, exhibiting the asymptotic dynamics of the perturbed system, is observed for referring the unknown map with satisfactory performance.

Step 2. The matrix Q^k induced by each input density f_k^{ω} is given by $(q_{i,j}^k)_{1\leq i,\, j\leq N}$, where

$$q_{i,j}^{k}=\frac{1}{\lambda(R_i)}\int_{R_i}\int_{R_j} f_k^{\omega}(x-z+b\chi_R(z-x))\,dzdx. \tag{44}$$

Given the matrix *W*, to determine the Frobenius-Perron matrix *M*, from (29) the coefficient vectors $v^{P_S f_{k^*}^N}=[v_1^{k^*},v_2^{k^*},\ldots,v_N^{k^*}]$ are obtained by solving the following constrained optimization problem

$$\min\left\| W-\begin{bmatrix} v^{P_S f_N^{1^*}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & v^{P_S f_N^{2^*}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & v^{P_S f_N^{K^*}} \end{bmatrix}\begin{bmatrix} Q_1' \\ Q_2' \\ \vdots \\ Q_K' \end{bmatrix}\right\|_F, \tag{45}$$

subject to $\sum_{i=1}^{N} v_i^{k^*}=N/b$, and $0\leq v_i^{k^*}\leq N/b$ for $i=1,\ldots,N$, and $k=1,\ldots,K$.

Step 3. For a continuous nonlinear map, it is required that the positive entries in each row of *M* are contiguous. However, it is generally difficult to guarantee the resulting matrix satisfies this condition. Hence, a trial Frobenius-Perron matrix $\hat{M} = (\hat{m}_{i,j})_{1\le i,j\le N}$ is obtained to determine the indices of contiguous positive entries in each row, and then this will be used to refine the matrix. Specifically, this is carried out in two steps. Firstly, given $v^{P_S f_N^{k*}}$ from *Step 2*, $\hat{M} = (\hat{m}_{i,j})_{i=1;j=1}^{N;N}$ is obtained by solving the constrained optimization problem below

$$\min \left\| \begin{bmatrix} v^{P_S f_{1*}^N} \\ v^{P_S f_{2*}^N} \\ \vdots \\ v^{P_S f_{K*}^N} \end{bmatrix} - W \begin{bmatrix} \hat{m}_{1,1} & \hat{m}_{1,2} & \cdots & \hat{m}_{1,N} \\ \hat{m}_{2,1} & \hat{m}_{2,2} & \cdots & \hat{m}_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{m}_{N,1} & \hat{m}_{N,2} & \cdots & \hat{m}_{N,N} \end{bmatrix} \right\|_F , \tag{46}$$

subject to $0 \le \{m_{i,j}\}_{i,j=1}^{N} \le 1$, and $\sum_{j=1}^{N} \hat{m}_{i,j} = 1$, for $i = 1,\ldots,N$.

Let $\mathfrak{r}^i = \{r_s^i, r_s^i + 1,\ldots, r_e^i\}$ be the set of column indices of the consecutive positive entries in *i*-th row and include the index of the maximum value in *i*-th row of $\hat{M}$, that is, $r_m^i \in \mathfrak{r}^i$ satisfying $\hat{m}_{i,r_m^i} = \max\{\hat{m}_{i,j}\}_{j=1}^{N}$. The approximated piecewise linear $\Re$ -semi-Markov map associated with a refined Frobenius-Perron matrix *M* should satisfy that $R_{r(i,k)} = S(Q_k^{(i)}) \in \Re$, where $\overset{p(i)}{\underset{k=1}{\cup}} R_{r(i,k)}$ is a connected interval and the image of the interval R_i , $i = 1,\ldots,N$, $p(i) = r_e^i - r_s^i + 1$ and $r(i,k) \in \mathfrak{r}^i$ are the column indices of the positive entries on the *i*-th row of *M* satisfying $r(i,k+1) = r(i,k) + 1$ for $i = 1,\ldots,N$, $k = 1,\ldots, p(i) - 1$.[30]

Subsequently, a refined Frobenius-Perron matrix *M* is obtained by solving the following optimization problem

$$\min \left\| \begin{bmatrix} v^{P_S f_{1*}^N} \\ v^{P_S f_{2*}^N} \\ \vdots \\ v^{P_S f_{K*}^N} \end{bmatrix} - W \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,N} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N,1} & m_{N,2} & \cdots & m_{N,N} \end{bmatrix} \right\|_F , \tag{47}$$

subject to $\sum_{k=1}^{p(i)} m_{i,r(i,1)+k-1} = 1$ and $0 < m_{i,r(i,k)} < 1$, for $i = 1,\ldots,N$, and $m_{i,j} = 0$, if $j \neq r(i,k)$, for $k = 1,\ldots,p(i)$.

In the following, the uniqueness of the solution to the proposed approach is proved given linearly independent input densities.

Theorem 5. *For a perturbed dynamical system* (1)*, given K linearly independent input density functions, the solution to the inverse problem of reconstructing the map S is unique.*

Proof. Let

$$\Omega = \begin{bmatrix} Q_1' \\ Q_2' \\ \vdots \\ Q_K' \end{bmatrix}, \tag{48}$$

$C_1 = \Omega\Omega'$, and $C_2 = W'W$. To make sure the identification result is unique, we need to have that the matrices C_1 and C_2 are invertible. C_1 is written as

$$C_1 = \begin{bmatrix} Q_1' \\ Q_2' \\ \vdots \\ Q_K' \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_K \end{bmatrix} = \begin{bmatrix} Q_1'Q_1 & Q_1'Q_2 & \cdots & Q_1'Q_K \\ Q_2'Q_1 & Q_2'Q_2 & \cdots & Q_2'Q_K \\ \vdots & \vdots & \ddots & \vdots \\ Q_K'Q_1 & Q_K'Q_2 & \cdots & Q_K'Q_K \end{bmatrix}. \tag{49}$$

For *K* linearly independent input density functions, and from (32) $\det(Q_k) > \frac{(d_1^k)^N}{2^{N-1}} > 0$. Hence, the matrix Q_k is invertible. Since $\operatorname{rank}(Q_k) = \operatorname{rank}(Q_k'Q_k)$, $Q_k{}'Q_k$ is also invertible. Let $\Phi_{i,j} = Q_i'Q_j$, thus $\det(\Phi_{k,k}) > 0$. The matrix C_1 can be further decomposed as

$$C_1 = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,K} \\ \Phi_{2,1} & \Phi_{2,2} & \cdots & \Phi_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K,1} & \Phi_{K,2} & \cdots & \Phi_{K,K} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \Phi_{2,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K,1} & \Phi_{K,2} & \cdots & \mathbf{0} \end{bmatrix}}_{C_1^1} + \tag{50}$$

$$\underbrace{\begin{bmatrix} \mathbf{0} & \Phi_{1,2} & \cdots & \Phi_{1,K} \\ \mathbf{0} & \mathbf{0} & \cdots & \Phi_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}}_{C_1^2} + \underbrace{\begin{bmatrix} \Phi_{1,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Phi_{2,2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Phi_{K,K} \end{bmatrix}}_{C_1^3}.$$

We can find that $\det(C_1^1) = \det(C_1^2) = 0$, and that $\det(C_1^3) = \prod_{k=1}^{K} \det(\Phi_{k,k})$. It follows that $\det(C_1^3) > 0$. Then we have that $\det(C_1) \geq \det(C_1^1) + \det(C_1^2) + \det(C_1^3) > 0$. Therefore C_1 is nonsingular.

It can be further obtained that $\det(C_2) > \frac{1}{2^{N-1}} \prod_{i=1}^{N} \sum_{k=1}^{K} (w_i^{k*})^2 > 0$. Hence C_2 is also nonsingular. This completes the proof.

Remark 7. It is easy to see that for $K = N$, C_1 is composed of *N* matrices Q_k', for $k = 1, \ldots, N$. Since columns of each Q_k and $\{f_k^{\omega}\}_{k=1}^{N}$ are linearly independent, the whole rows of C_1 are linearly independent. Also the matrix *W* is a full rank matrix. Thereby, the reconstruction result is guaranteed to be the unique transformation of the underlying system.

Remark 8. In contrast to perturbation free and stochastic noise perturbed dynamical systems whose stationary density function are fixed and dependent on the transformations, the input driven systems can generate different stationary densities given different input density functions. This establishes the feasibility of observing multiple stationary densities to characterise the various asymptotic dynamics resulting from a set of given different input density functions.

Step 4. The piecewise linear semi-Markov map $\hat{S}$ is constructed over $\Re$ based on the identified Frobenius-Perron matrix *M*. For a continuous transformation *S*, the monotonicity of each branch $S|_{Q_k^{(i)}}$ needs to be determined firstly. Let $R_i' = [a_{r(i,1)-1}, a_{r(i,p(i))}]$ be the image of the interval R_i by the semi-Markov transformation $\hat{S}$ associated with the identified Frobenius-Perron matrix *M*, $a_{r(i,1)-1}$ the start point of $R_{r(i,1)}$ mapped from the subinterval $Q_1^{(i)}$, and $a_{r(i,p(i))}$ the end point of $R_{r(i,p(i))}$, the image of the subinterval $Q_{p(i)}^{(i)}$. Let $\bar{c}_i$ be the midpoint of the image R_i', that is, $\bar{c}_i = (a_{r(i,1)-1} + a_{r(i,p(i))})/2$. Let $l(i)$ be a sign representing the monotonicity associated with $\{\hat{S}(x)\big|_{Q_k^{(i)}}\}_{k=1}^{p(i)}$, which is defined by $l(i) = -1$ if $\bar{c}_i - \bar{c}_{i-1} < 0$, $l(i) = 1$ if $\bar{c}_i - \bar{c}_{i-1} \geq 0$, and $l(i) = l(i-1)$, if $\bar{c}_i = \bar{c}_{i-1}$, for $i = 2, \ldots, N$, and $l(1) = l(2)$. The piecewise linear semi-Markov transformation $\hat{S}$ on each subinterval $Q_j^{(i)}$ is then constructed by

$$\hat{S}\big|_{Q_j^{(i)}}(x) = \begin{cases} \dfrac{1}{m_{i,j}}(x - a_{i-1} - \dfrac{b}{N}\sum_{j=1}^{k-1} m_{i,r(i,j)}) + a_{j-1}, & \text{if } l(i) = +1; \\ -\dfrac{1}{m_{i,j}}(x - a_{i-1} - \dfrac{b}{N}\sum_{j=1}^{k-1} m_{i,r(i,p(i)-k+2)}) + a_j, & \text{if } l(i) = -1. \end{cases} \tag{51}$$

$a_0 = 0$, for $m_{i,j} \neq 0$, $i = 1, \ldots, N$, $j = 1, \ldots, N$, $k = 1, \ldots, p(i) - 1$. By fitting a polynomial smoothing spline, a smooth nonlinear map is then obtained. [30]

Remark 9. The method can be readily generalized to unbounded dynamical systems, for which the state space *R* has infinite length. In this case, it can be assumed that the state value never approaches the boundary, which is infinity.

5. Numerical simulation

The proposed algorithm is demonstrated to identify the following perturbed chaotic map

$$x_{n+1} = S(x_n) + \omega_n \pmod 1, \tag{52}$$

where $S(x_n) = 4x_n(1 - x_n)$, $x_n \in [0,1]$, $\omega_n \in [0,1]$.

To infer a piecewise linear semi-Markov transformation $\hat{S}$ over a given uniform partition $\Re$ of $N = 30$ intervals that approximates the unknown transformation *S*, the input density functions are set to be Gaussian distribution functions $f_k^{\omega} : \omega^k \sim \mathcal{N}(u_k, \sigma^2)$ truncated to [0,1], where $u_k = 0.0071k + 0.0729$, $k = 1, \ldots, 120$, examples of which are shown in Fig. 2 . Input data given by $\omega_n^k \in \Omega_k = \{\bar{\omega}_k^i\}_{i=1}^{\theta}$, $\theta = 5 \times 10^3$, is generated by sampling f_k^{ω} for $k = 1, \ldots, 120$. Each of the stationary density functions f_{k*}^N resulting from the corresponding f_k^{ω} is estimated over $\Re$ from the final states of the perturbation process $X^{k*} = \{x_i^{k*}\}_{i=1}^{\theta}$ with initial conditions $X^0 = \{x_i^0\}_{i=1}^{\theta}$ that are sampled from a density function $f_0(x) = \chi_{[0,1]}(x)$ after 2×10^3 iterations, and are shown in Fig. 3.

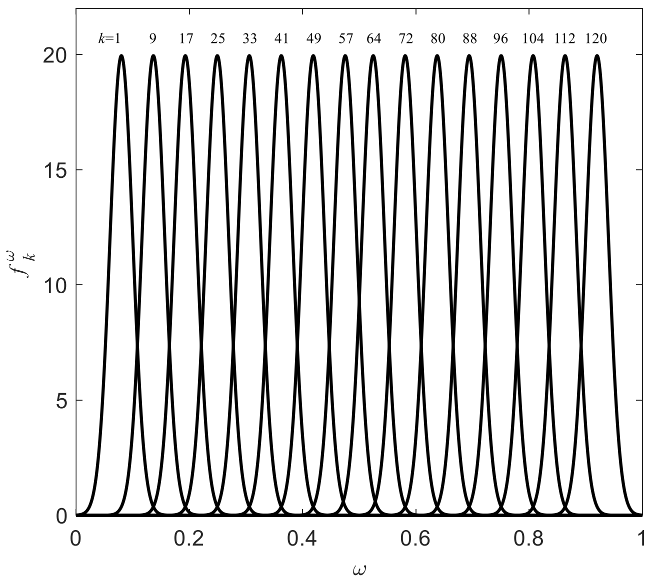

Fig. 2 Examples of input density functions f_k^{ω}.

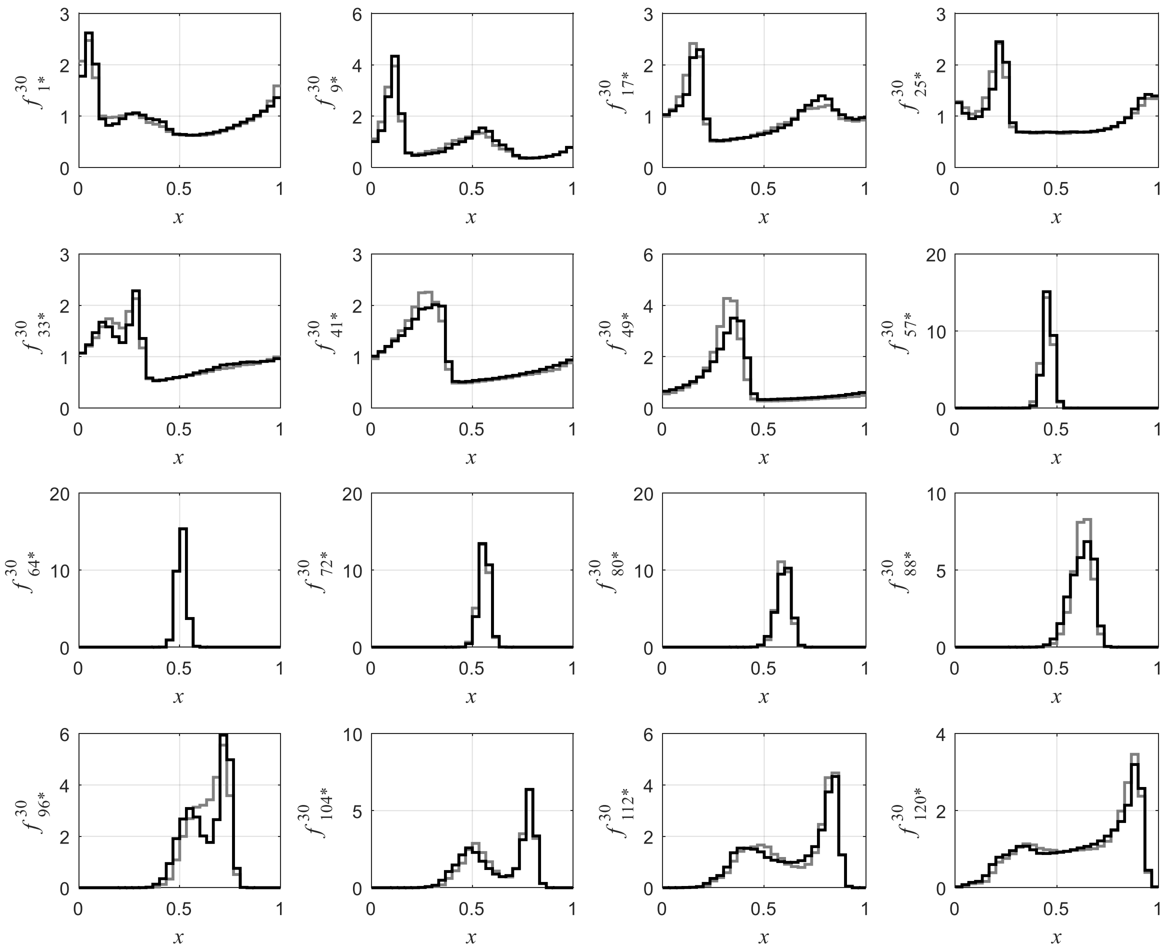

Fig. 3 Examples of stationary densities observed using the original map and used to identify the unknown map (black lines), and generated using the identified map (grey lines).

By solving the constrained linear optimization (45), (46) and (47) using *lsqlin* function in Matlab to estimate the matrix M, the constructed piecewise linear semi-Markov approximation $\hat{S}$ is shown in Fig. 4. The smoothed result with parameter 0.99 is given in Fig. 5.

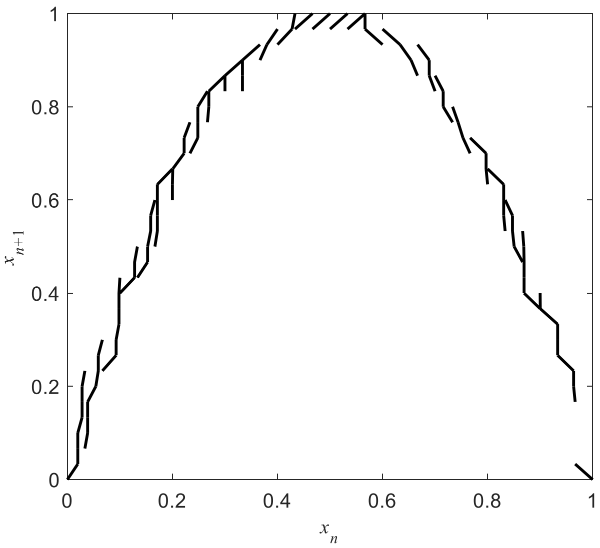

Fig. 4 The identified piecewise linear semi-Markov approximate $\hat{S}$ over the defined uniform partition $\mathfrak{R}$.

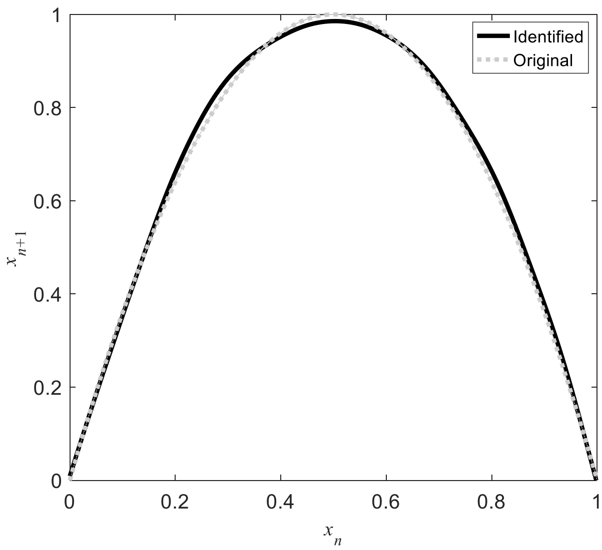

Fig. 5 The identified continuous nonlinear map $\tilde{S}$ by smoothing the inferred piecewise linear semi-Markov map $\hat{S}$, compared with the original map S.

Fig. 6 shows the relative percentage error between the identified and original maps which is calculated by

$$\delta S(x) = 100\left|\frac{S(x) - \tilde{S}(x)}{S(x)}\right| (\%), \tag{53}$$

on the uniformly spaced points. It can be seen the approximation error is low, since the errors on 93% of points are less than 5%. As shown in Fig. 3 the predicted stationary densities using identified map $\tilde{S}$ for each input density function are very close to that generated by the original map. The error between the stationary density functions generated using the identified and original maps, measured by the root mean squared error (RMSE) between the coefficient vectors $w^{f_{k*}^N}$ and $w^{\hat{f}_{k*}^N}$ is given by

$$\mathrm{RMSE}(k) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(w_i^{k*} - \hat{w}_i^{k*})^2}\ , \tag{54}$$

and is shown in Fig. 7.

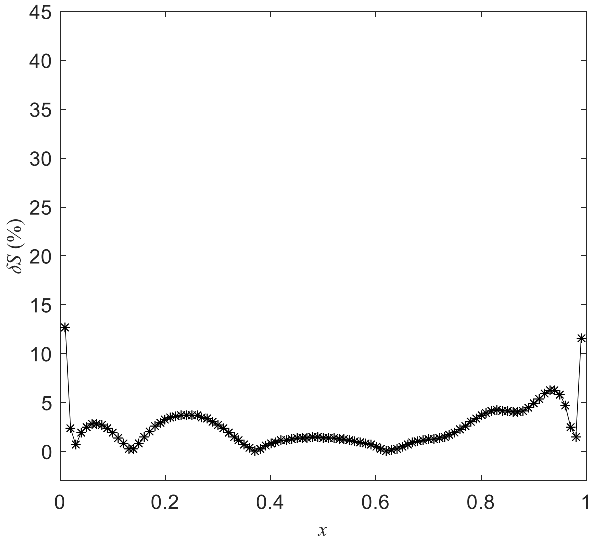

Fig. 6 The relative percentage error δS between the smoothed continuous map $\tilde{S}$ and the original map S.

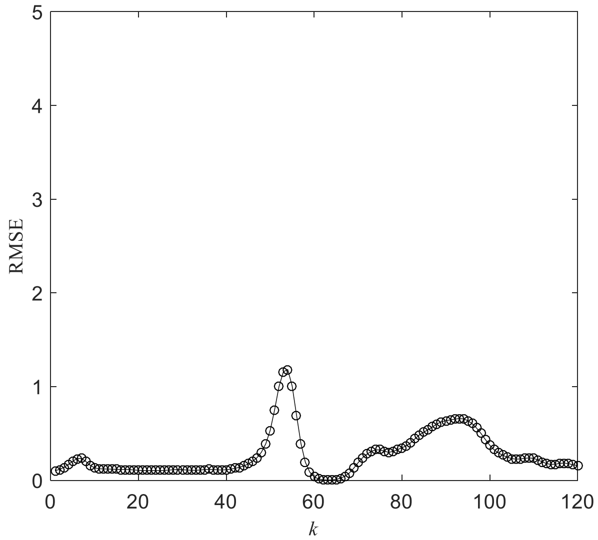

Fig. 7 The root mean squared error between the stationary densities generated using identified and original maps after 2×10^3 iterations for each given input density functions.

Since it is hard to estimate the stochastic noise in practical situations, the proposed algorithm is applied in the presence of additive noise of multiple levels. Assuming the stochastic system to be $x_{n+1} = S(x_n) + \omega_n + \eta\xi_n$ (mod b), where ξ_n is a white Gaussian noise term, the mean relative percentage error $\bar{\delta} S$ for some increasing noise levels is given in Table 1. It can be seen that the error can remain low even as the noise level dramatically increases, which suggests that the algorithm is still well applicable to stochastic chaotic systems with finite noise levels.

Table 1 The mean relative percentage error for four different Gaussian noise levels.

$\eta = \sigma_\xi^2 / \sigma_x^2$	0	0.0978	0.5431	1.1231
$\bar{\delta} S$ (%)	2.40	3.63	8.96	35.41

6. Conclusions

This paper presents a new scheme of reconstructing an unknown one-dimensional chaotic map using the stationary densities generated by applying an additive control input with linearly independent density functions. It is proven that there exists a unique stationary density function for such a perturbed chaotic map, corresponding to a given the input density function. The derived relationship between the stationary density and the input density function suggests the dependence of a stationary density on the input density function. Thus, the stationary density can be estimated with the given transformation and input density function, and for an unknown chaotic system the generated stationary density function varies with different input density functions. Under the assumption that only stationary densities can be observed, compared with the existing solutions to the inverse Frobenius-Perron problem, this method can be used to infer the unique chaotic map of the underlying system.

This paper addressed the challenge of inferring the chaotic map from stationary densities, and also provided a novel heuristic perspective for controlling the dynamics of chaotic systems. From Remark 5, the stationary density function of such perturbed chaotic maps can be estimated by the eigenvector of the matrix representation of the transfer operator that describes the evolution of densities. The control strategy is to determine the input density function so as to drive the chaotic system to attain a desired stationary density function that characterizes the new asymptotic dynamics. Thus, as long as the input density function is specified, the dynamical behavior exhibited by the perturbed system will stabilize at the target one. The main challenge of solving this problem is to estimate the input density function given the target stationary density function and the transformation. From the point of view of applicability to characterizing the dynamic evolution of heterogeneous cell populations and altering cell fates upon differentiation, the introduced identification method forms a theoretical basis for inferring the dynamical model and designing the control strategy to manipulate the differentiation of stem cells into desired types.

Acknowledgements

The author X. N. gratefully acknowledges that the research is supported by the Fundamental Research Funds for the Central Universities. D. C. gratefully acknowledges the support from Human Frontier Science Program (RGP0001/2012), MRC (G0802627), and BBSRC (BB/M025527/1). X. N. and M. B. gratefully acknowledge the support from Innovate UK (RG.GEOG.106459). J. L. gratefully acknowledges the support from Shanghai Municipal Science and Technology Major Project (2017SHZDZX01), Independent Deployment of Scientific Research Projects of Jihua Laboratory (X190051TB190), Shanghai Science and Technology Innovation Action Plan (18441900700 & 19441908200) and National Natural Science Foundation of China (U1913216). X. N. also thanks Dr. Stephen Clark from the University of Leeds for his helpful suggestions on the manuscript. The anonymous reviewers are greatly appreciated for their encouraging and insightful comments that helped improve the paper.

References

[1] S.H. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. 2014: Westview press.

[2] J.E. Skinner, *Low-dimensional Chaos in Biological Systems.* Nature Biotechnology, 1994. **12**(6): p. 596-600.

[3] A. Swishchuk and S. Islam, *Random Dynamical Systems in Finance*. 2013: Taylor & Francis.

[4] F.K. Diakonos and P. Schmelcher, *On the construction of one-dimensional iterative maps from the invariant density: the dynamical route to the beta distribution.* Physics Letters A, 1996. **211**(4): p. 199-203.

[5] A. Berman, R. Shorten, and D. Leith, *Positive matrices associated with synchronised communication networks.* Linear Algebra and Its Applications, 2004. **393**(0): p. 47-54.

[6] A.G. Lozowski, M. Lysetskiy, and J.M. Zurada, *Signal Processing with temporal sequences in olfactory systems.* IEEE Transactions on Neural Networks, 2004. **15**(5): p. 1268-1275.

[7] A. Rogers, R. Shorten, and D.M. Heffernan, *Synthesizing chaotic maps with prescribed invariant densities.* Physics Letters A, 2004. **330**(6): p. 435-441.

[8] D. Leith and R. Shorten, *Analysis and design of synchronised communication networks.* Automatica, 2005. **41**: p. 725-730.

[9] E. Bollt, P. Góra, A. Ostruszka, and K. Zyczkowski, *Basis Markov partitions and transition matrices for stochastic systems.* SIAM Journal on Applied Dynamical Systems, 2008. **7**(2): p. 341-360.

[10] J. Perry, R.H. Smith, I.P. Woiwod, and D.R. Morse, *Chaos in real data: the analysis of non-linear dynamics from short ecological time series*. Vol. 27. 2012: Springer Science & Business Media.

[11] A. Lasota and M.C. Mackey, *Chaos, fractals, and noise : stochastic aspects of dynamics*. 2nd ed. 1994, New York: Springer-Verlag. xiv, 472 p.

[12] A. Boyarsky and P. Góra, *Laws of chaos : invariant measures and dynamical systems in one dimension*. Probability and its applications. 1997, Boston, Mass.: Birkhäuser. xv, 399 p.

[13] N. Friedman and A. Boyarsky, *Construction of Ergodic Transformations.* Advances in Mathematics, 1982. **45**(3): p. 213-254.

[14] P. Góra and A. Boyarsky, *A Matrix Solution to the Inverse Perron-Frobenius Problem.* Proceedings of the American Mathematical Society, 1993. **118**(2): p. 409-414.

[15] A. McDonald and M. van Wyk, *Solution of the Inverse Frobenius–Perron Problem for Semi–Markov Chaotic Maps via Recursive Markov State Disaggregation*, in *25th European Signal Processing Conference*. 2017. p. 1654-1658.

[16] D. Pingel, P. Schmelcher, and F.K. Diakonos, *Theory and examples of the inverse Frobenius-Perron problem for complete chaotic maps.* Chaos, 1999. **9**(2): p. 357-366.

[17] S.V. Ershov and G.G. Malinetskii, *The solution of the inverse problem for the Perron-Frobenius equation.* USSR Computational Mathematics and Mathematical Physics, 1988. **28**(5): p. 136-141.

[18] S. Koga, *The inverse problem of Flobenius-Perron equations in 1D difference systems—1D map idealization.* Progress of Theoretical Physics, 1991. **86**(5): p. 991-1002.

[19] W. Huang, *Constructing chaotic transformations with closed functional forms.* Discrete Dynamics in Nature and Society, 2006. **2006**.
[20] W. Huang, *Constructing multi-branches complete chaotic maps that preserve specified invariant density.* Discrete Dynamics in Nature and Society, 2009. **2009**: p. 14.
[21] W. Huang. *On the complete chaotic maps that preserve prescribed absolutely continuous invariant densities*. in *Topics on Chaotic Systems: Selected Papers from CHAOS 2008 International Conference*. 2009.
[22] A. Baranovsky and D. Daems, *Design of one-dimensional chaotic maps with prescribed statistical properties.* International Journal of Bifurcation and Chaos, 1995. **5**(6): p. 1585-1598.
[23] F.K. Diakonos, D. Pingel, and P. Schmelcher, *A stochastic approach to the construction of one-dimensional chaotic maps with prescribed statistical properties.* Physics Letters A, 1999. **264**(2–3): p. 162-170.
[24] E.M. Bollt, *Controlling chaos and the inverse Frobenius-Perron problem: Global stabilization of arbitrary invariant measures.* International Journal of Bifurcation and Chaos, 2000. **10**(5): p. 1033-1050.
[25] E.M. Bollt and N. Santitissadeekorn, *Applied and Computational Measurable Dynamics*. 2013: SIAM.
[26] A. Rogers, R. Shorten, and D.M. Heffernan, *A novel matrix approach for controlling the invariant densities of chaotic maps.* Chaos Solitons & Fractals, 2008. **35**(1): p. 161-175.
[27] A. Rogers, R. Shorten, D.M. Heffernan, and D. Naughton, *Synthesis of Piecewise-Linear Chaotic Maps: Invariant Densities, Autocorrelations, and Switching.* International Journal of Bifurcation and Chaos, 2008. **18**(8): p. 2169-2189.
[28] X. Nie and D. Coca, *Reconstruction of one-dimensional chaotic maps from sequences of probability density functions.* Nonlinear Dynamics, 2015. **80**(3): p. 1373-1390.
[29] X. Nie and D. Coca, *A matrix-based approach to solving the inverse Frobenius–Perron problem using sequences of density functions of stochastically perturbed dynamical systems.* Communications in Nonlinear Science and Numerical Simulation, 2018. **54**: p. 248-266.
[30] X. Nie, M. Birkin, and J. Luo, *A new approach to identification of input-driven dynamical systems from probability densities.* Inverse Problems, 2018. **34**(8): p. 085004.
[31] P. Góra and A. Boyarsky, *Approximating the invariant densities of transformations with infinitely many pieces on the interval.* Proceedings of the American Mathematical Society, 1989. **105**(4): p. 922-928.