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SPECTRAL ASYMPTOTICS ON STATIONARY SPACE-TIMES

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Abstract. We review our recent relativistic generalization of the Gutzwiller-Duistermaat-Guillemin trace formula and Weyl law on a globally hyperbolic stationary space-times with compact Cauchy hypersurfaces. We also discuss anticipated generalizations to non-compact Cauchy hypersurface cases.

Two of the cornerstone results of spectral asymptotics of the Laplace-Beltrami operator Δ_h on a compact Riemannian manifold (Σ, h) are the Weyl counting law of eigenvalues and the Gutzwiller-Duistermaat-Guillemin singularities trace formula [Gutz71, DG75]. These results are manifestly non-relativistic and have been generalized to non-relativistic Schrödinger operators $-\hbar^2\Delta_h + V$, both on compact and on non-compact Riemannian manifolds. Our recent article [SZ18] gives a different type of generalization, namely to the setting of globally hyperbolic stationary spacetimes with compact Cauchy hypersurface. The purpose of this expository article is to review the main ideas of the relativistic generalization and to discuss two types of generalizations: (i) to joint mass and normal mode asymptotics [SZ19]; and (ii) to globally hyperbolic spacetimes with asymptotically Euclidean Cauchy hypersurfaces [SZ19+].

1. The product or ultra-static case

Before getting to the relativistic setting, let us review the Weyl law and Gutzwiller-Duistermaat-Guillemin trace formula on product (or ultra-static) spacetimes.

Let (Σ, h) be a connected compact Riemannian manifold of dimension d, and let Δ_h be the Laplace operator of (Σ, h). The wave group (resp. half-wave group) of (Σ, h) are the unitary groups on $L^2(\Sigma)$,

$$U(t) := \begin{pmatrix} \cos t\sqrt{-\Delta} & \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \\ -\sqrt{-\Delta}\sin t\sqrt{-\Delta} & \cos t\sqrt{-\Delta} \end{pmatrix},$$

resp.

$$V(t) := \exp(\mathrm{i}t\sqrt{-\Delta}).$$

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$ be the eigenvalues of $\sqrt{-\Delta}$ repeated according to their multiplicities.

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The 'trace' of the wave group is,

$$\mathrm{Tr}\, U(t) = 2\sum_j \cos\lambda_j t = 2\Re\sum_j e^{\mathrm{i}t\lambda_j}, \text{ resp. } \mathrm{Tr}\, V(t) = \sum_j e^{\mathrm{i}t\lambda_j},$$

It is a distribution on $\mathbb{R}$ with singularities at $t = 0$ and at times $t \in \mathrm{Lsp}(\Sigma, h)$, i.e. the set of lengths L_γ of closed geodesics γ.

The singularity of $V(t)$ at $t = 0$ has leading term $\mathrm{i}^d C_d \Gamma(d+1)\mathrm{Vol}(\Sigma, h)(t+\mathrm{i}0)^{-d}$, where $C_d = (2\pi)^{-d}\mathrm{Vol}(B_1)$ and B_1 is the unit ball in $\mathbb{R}^d$. If $f \in \mathcal{S}(\mathbb{R})$ is an even Schwartz function its Fourier transform $\hat{f}$ is also a Schwartz function and by spectral calculus $f(\sqrt{-\Delta}) = \frac{1}{2\pi}\int \cos(t\sqrt{-\Delta})\hat{f}(t)dt$. This is a trace-class operator and $\mathrm{Tr}\,(f(\sqrt{-\Delta})) = \sum_j f(\lambda_j)$. Therefore, in the sense of distributions, we have

$$\sum_j f(\lambda_j) = \int f(\lambda)dN(\lambda) = \frac{1}{2\pi}\int \hat{f}(t)\mathrm{Tr}\, U(t)\mathrm{d}t = \frac{1}{2\pi}\int f(t)\widehat{\mathrm{Tr}\, U(t)}\mathrm{d}t,$$

where $N(\lambda) := \{j : \lambda_j \leq \lambda\}$ is the spectral counting function. To see the qualitative behavior of $N(\lambda)$ as $\lambda \to \infty$ one chooses f suitably to approximate the indicator function of the interval $[-1, 1]$, so that $f_\lambda(x) = f(\lambda^{-1}x)$ approximates the indicator function of the interval $[-\lambda, \lambda]$. As $\lambda \to \infty$ the distribution $\hat{f}_\lambda$ concentrates near zero and it is therefore the main singularity of $\mathrm{Tr}\, U(t)$ at zero that determines the asymptotic behavior of $N(\lambda)$ as $\lambda \to \infty$. A precise statement is provided by Hörmander's Fourier Tauberian theorems. We refer to [S01] for an overview and a quantitative statement with lower and upper bounds. An application yields the Weyl law

$$N(\lambda) \simeq C_d \mathrm{Vol}(\Sigma, h)\lambda^d + O(\lambda^{d-1}).$$

The other singularities of $\mathrm{Tr}\, U(t)$ occur at times $t \in \mathrm{Lsp}(\Sigma, h)$. When the closed geodesics are non-degenerate, $\mathrm{Tr}\, V(t)$ admits a singularity expansion around $0 < t = L$ with leading order given by

$$\mathrm{Tr}\, V(t) = \sum_{\gamma, L_\gamma = L} a_{\gamma,-1}(t - L_\gamma + \mathrm{i}0)^{-1} + \psi, \text{ with } \mathrm{a}_{\gamma,-1} = \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\mathrm{m}_\gamma}\mathrm{L}_\gamma^{\#}}{|\det(\mathrm{id} - \mathrm{P}_\gamma)|^{\frac{1}{2}}} \tag{1}$$

in case $L \in \mathrm{Lsp}(\Sigma, h)$, where ψ is a bounded function near $t = L$. Let us briefly explain the notation. We think of γ as a periodic orbit of the geodesic flow G^t on the unit-cotangent bundle $T_1^*\Sigma$. Picking a point $\xi \in \gamma$ the smallest positive time t for which $G^t(\xi) = \xi$ is the length $L_\gamma^{\#}$ of the primitive closed geodesic through ξ. The length L_γ is then of course an integer multiple of $L_\gamma^{\#}$. Given a small transversal V to ξ in $T_1^*\Sigma$ one can define the Poincaré map as the return map $G^{L_\gamma} : V \to V$. Its derivative at ξ is, by definition, the linear Poincaré map $P_\gamma(\xi)$. Note that $\det(\mathrm{id} - P_\gamma(\xi))$ does not depend on the chosen point ξ since other choices lead to conjugate linear Poincaré maps. Finally, m_γ is the Maslov index of the path γ. In particular, $m_\gamma = 0$ in case there are no conjugate points. The geodesic is called non-degenerate if the linear Poincaré map does not have eigenvalue 1.

Thus, one obtains geometric information about geodesics from the eigenvalues. This wave trace formula is the basis of most inverse spectral results for the Laplacian. One may envision similar applications of the generalized Gutzwiller trace formula of this article. For a boundary inverse result in the setting of relativistic stationary spacetimes we refer to [FIO].

The Weyl law and Gutzwiller-(Duistermaat-Guillemin) trace formula are formulated in the setting of the Riemannian geometry of (Σ, h). The underlying symplectic geometry of the geodesic flow G^t is on the unit cosphere bundle $S^*\Sigma$ of Σ. Hence, these results are non-relativistic in the sense that space and time have been separated in a very specific way. To see this it is instructive to look at these results in the setting of Lorentzian geometry by passing to the product or ultra-static spacetime, i.e. a spacetime that of the form $\Sigma \times \mathbb{R}$ equipped with product-Lorentz metric $-dt^2 + h$.

Eigenfunctions φ_j of Δ_h give null solutions of $\Box = \frac{\partial^2}{\partial t^2} - \Delta_h$, namely,

$$u_j^{\pm}(x,t) = \varphi_j(x) e^{\pm it\lambda_j}.$$

The vector field is $Z = \frac{\partial}{\partial t}$ is a Killing vector field and obviously

$$D_Z u_j^{\pm} := \mathrm{i}^{-1} Z u_j^{\pm} = \pm\lambda_j u_j^{\pm}.$$

Thus,

$$\operatorname{Tr} e^{tZ} = \sum_{j,\pm} e^{\pm \mathrm{i}t\lambda_j},$$

if the trace is taken over the space of solutions of the wave equation.

2. Relativistic setting

We briefly recall some notions from general relativity and refer the reader to [ON] for more details. A Lorentzian manifold (M,g) with metric g of signature $(-1,1,\ldots 1)$ will be called a spacetime if it is connected, oriented, and time-oriented. Recall that a non-zero vector $v \in T_xM$ is called timelike (spacelike, causal, lightlike), if $g(v,v) < 0$ ($g(v,v) > 0, g(v,v) \leq 0, g(v,v) = 0$). A C^1-curve is called timelike of its tangent vector is timelike at every point. In the same way one defines the notion of a spacelike, causal, and lightlike curve. One of the principals of general relativity is that world-lines of particles need to be causal. The time-orientation gives us a notion of future and past direction for causal curves. For a subset $K \subset M$ the causal future/past $J_{\pm}(K)$ is the set of points x that can be reached by future/past directed causal curves emanating from K. We define a Cauchy surface Σ to be a smooth spacelike hypersurface such that each maximal causal curve intersects Σ exactly once. Spacetimes that admit a Cauchy surface are called globally hyperbolic. Globally hyperbolic spacetimes can be smoothly foliated into Cauchy surfaces and are therefore, diffeomorphic to $\mathbb{R} \times \Sigma$. Globally hyperbolic spacetimes are a good category of spacetimes for the treatment of hyperbolic evolution equations. Indeed, the Cauchy problem for the d'Alembert operator is well posed.

Our purpose is to give a relativistic Gutzwiller trace formula for an n-dimensional spacetime (M,g) satisfying:

- (M, g) is globally hyperbolic, i.e. possesses a Cauchy hypersurface Σ;
- there exists a compact Cauchy surface Σ, i.e. M is spatially compact;
- (M, g) is stationary, i.e. has a complete timelike Killing vector field Z. This Killing vector field then generates a one-parameter group of isometries e^{tZ}.

In place of the wave group $U(t)$ above we use the translation operator by the flow e^{tZ} of the Killing vector field Z acting on the nullspace $\ker \Box_g$ of a wave operator (d'Alembertian) $\Box_g$.

In place of $L^2(\Sigma, d\sigma_h)$ we use $\ker \Box_g$. We need to endow it with the topology of a Hilbert space. One has $[Z, \Box_g] = 0$, so e^{tZ} acts by composition or time-translation on solutions of $\Box_g u = 0$. The cotangent bundle $T^*\Sigma$ is replaced by the symplectic manifold $\mathcal{N}$ of null-geodesics of (M, g), i.e. geodesics with lightlike tangent vectors $\dot\gamma$ with $g(\dot\gamma, \dot\gamma) = 0$. Thus, $\ker \Box_g$ is the 'quantization' of $\mathcal{N}$, just as $L^2(\Sigma)$ is the quantization of $T^*\Sigma$.

A spatially compact stationary globally hyperbolic spacetime can be put in the form,

$$(M, g) \simeq (\mathbb{R} \times \Sigma, g), \quad g = -N^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{2}$$

where (Σ, h) is a Riemannian manifold, $N : \Sigma \to \mathbb{R}_+$ is a positive smooth function, and β a vector field on Σ. Then $Z = \partial_t$ is a Killing vector field. If the metric can be put in the above form with $\beta = 0$, the spacetime is called *static*.

To avoid local coordinates, we consider the space $\mathcal{K} = M/\mathbb{R}$ of Killing orbits. Thus, $\pi : M \to \mathcal{K}$ is a principal $\mathbb{R}$ bundle. The orthogonal distributions to the fibers determine a connection 1-form θ for which $\theta(Z) = 1, \mathcal{L}_Z\theta = 0$. The metric g on the horizontal spaces $\ker\theta$ induces a metric $g_{\mathcal{K}}$ on $\mathcal{K}$. Define u by $u^2 = -g(Z, Z)$. It is constant along the fibers (Killing orbits), hence defines a function on $\mathcal{K}$. The spacetime metric is then:

$$g = -u^2\theta \otimes \theta + \pi^* g_{\mathcal{K}}, \;\; u^2 = -g(Z, Z). \tag{3}$$

The next issue is to find analogues for the geodesic flow, periodic orbits and the symplectic geometry of the Poincaré map. The geodesic flow on Σ is replaced by the null-bicharacteristic flow G^t, i.e. the Hamiltonian flow with Hamiltonian,

$$\frac{1}{2}\sigma_{\Box_g}(x, \xi) = \frac{1}{2}|\xi|_g^2,$$

on T^*M, where $|\xi|_g^2$ is the Lorentzian 'norm' squared.

Let

$$\mathrm{Char}(\Box_g) = \{(x, \xi) \in T^*M \setminus 0 : \sigma_{\Box_g}(x, \xi) = 0\}.$$

We denote the restriction of G^t to $\mathrm{Char}(\Box_g)$ by G_0^t. $\mathrm{Char}(\Box_g)$ is a (co-isotropic) hypersurface whose null-foliation consists of orbits of G_0^t, i.e. of scaled null-geodesics. The space $\mathcal{N}$, of future-directed scaled null-geodesics, is naturally a symplectic cone (invariant under multiplication by positive reals in the ξ variables). The quotient by its $\mathbb{R}_+$-action is the space $\mathcal{N}_p$ of unparametrized null-geodesics.

The flow e^{tZ} of the Killing vector field commutes with the null-bicharacteristic flow and defines a quotient (reduced) symplectic flow on $\mathcal{N}$. We denote the quotient flow by $\Psi^t_{\mathcal{N}}$.

LEMMA **1.** *$\Psi^t_{\mathcal{N}}$ is a Hamiltonian flow on $\mathcal{N}$ with Hamiltonian*

$$H(\zeta) = \xi(Z), \quad \text{where } \zeta = \{G^t(x,\xi), t \in \mathbb{R}\}.$$

The value $\xi(Z)$ is independent of the lift of ζ to (x,ξ).

Since Z is timelike, the Hamiltonian is positive and homogeneous. Hence, for any $E > 0$ the contact manifold $\mathcal{N}_p$ can be identified with the energy surface $\mathcal{N}_E := \{\zeta : H(\zeta) = E\}$. As with any Hamiltonian flow, $\Psi^t_{\mathcal{N}}$ preserves level sets of H and therefore also acts on $\mathcal{N}_E := \{\zeta : H(\zeta) = E\}$. The induced flow on the quotient space $\mathcal{N}_p$ will be denoted by $\Psi^t_{\mathcal{N}_p}$.

We then define the periods and periodic points of $\Psi^t_{\mathcal{N}_p}$ by

$$\mathcal{P} := \{T \neq 0 : \exists \zeta \in \mathcal{N}_p : \Psi^T_{\mathcal{N}_p}(\zeta) = \zeta\}, \;\; \mathcal{P}_T = \{\zeta \in \mathcal{N}_p : \Psi^T_{\mathcal{N}_p}(\zeta) = \zeta\}. \tag{4}$$

PROPOSITION **2.** *Let (M,g) be a globally hyperbolic, stationary spacetime with a compact Cauchy hypersurface. Then $\operatorname{Tr} e^{itD_Z}|_{\ker \Box_g}$ is a distribution on $\mathbb{R}$, and its singular support is a subset of $\mathcal{P}$.*

The Poincaré map is defined precisely as in the product case. Orbits are classified as non-degenerate, elliptic, hyperbolic and so on as for any Hamiltonian flow.

We now consider the quantum mechanics. The d'Alembertian $\Box_g$ is the hyperbolic analogue of Δ_h. In local coordinates (x,t),

$$\Box_g = -\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ik}\partial_k\right),$$

where we have used Einstein's sum convention. More generally we consider the massive Klein-Gordon operator $\Box_g + m^2$. [1]

Since $[D_Z, \Box_g + m^2] = 0$,

$$U(t) := e^{\mathrm{i}tD_Z} : \ker(\Box_g + m^2) \to \ker(\Box_g + m^2). \tag{5}$$

The Killing flow e^{tZ} acts on functions u by pull-back $e^{\mathrm{i}tD_Z}u = u \circ e^{tZ}$. The eigenfunctions of D_Z in $\ker(\Box_g + m^2)$ are joint eigenfunctions,

$$\begin{cases} (\Box_g + m^2)u = 0, \\ \\ D_Z u = \lambda u. \end{cases}$$

For each $m \in \mathbb{R}$, the spectrum of D_Z in $\ker(\Box_g + m^2)$ is a discrete set $\{\lambda_j(m)\}_{j\in\mathbb{Z}}$. In [SZ18] we consider the asymptotic properties of the eigenvalues for fixed m. In [SZ19] we consider the 'ladder' asymptotics as $m \to \infty$, $\lambda_j(m) \to \infty$ with $\frac{\lambda_j(m)}{m} \to \nu$ for some $\nu \in \mathbb{R}$.

We will define the 'trace' below. First, let us state the main result:

[1] In [SZ18] we also consider $\Box_g + V$ where $V \in C^\infty(M)$ with $D_Z V = 0$. Important examples include $\Box = \Box_g + m^2 + \kappa R$ where $\kappa, m \in \mathbb{R}$ and R denotes the scalar curvature.

Theorem **3.** *For general spatially compact stationary globally hyperbolic spacetimes we have that*

$$\operatorname{Tr} e^{\mathrm{i}tD_Z} \mid_{\ker \Box_g} = e_0(t) + \psi(t)$$

where ψ is a distribution that is smooth near 0, and $e_0(t)$ is a Lagrangian distribution with singularity at $t = 0$ of the form

$$e_0(t) \sim 2(2\pi)^{-n+1}(n-1)\mathrm{Vol}(\mathcal{N}_{H\leq 1})\mu_{n-1}(t) + c_1\mu_{n-2}(t) + \dots,$$

where the homogeneous distribution $\mu_k(t)$ is defined by the oscillatory integral

$$\mu_k(t) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\tau}|\tau|^{k-1}\mathrm{d}\tau,$$

Theorem **4.** *Let $T \in \mathcal{P}$ and assume that the fixed point sets $\mathcal{P}_T$ of $\Psi^s_{\mathcal{N}_p}$ on $\mathcal{N}_p$ are non-degenerate. Then,*

$$\operatorname{Tr} e^{\mathrm{i}tD_Z} \mid_{Char(\Box)} = 2\sum_{\gamma:T_\gamma = T} \Re(e_\gamma(t)) + \psi_T,$$

where ψ_T is a distribution smooth near $t = T$ and $e_\gamma(t)$ are Lagrangian distributions with singularities at $t = T_\gamma$. If γ is non-degenerate, we have

$$e_\gamma(t) \sim \frac{1}{2\pi\mathrm{i}}\frac{e^{-\mathrm{i}\frac{\pi}{2}m_\gamma}T^{\#}_\gamma}{|\det(I - P_\gamma)|^{\frac{1}{2}}}(t - T_\gamma + \mathrm{i}0)^{-1} + \dots,$$

where m_γ is the Conley-Zehnder index of the periodic orbit γ. The sum is over all periodic orbits of period T. The expansion above is a singularity expansion around $t = T_\gamma$.

The factor $e^{-\mathrm{i}\frac{\pi}{2}m_\gamma}$ is the Maslov factor.

2.1. **Weyl law.** By a standard Fourier Tauberian argument, we may derive a Weyl law for the growth of the spectrum of D_Z in $\ker(\Box_g)$.

Corollary **5.** *For general spatially compact stationary globally hyperbolic spacetimes the spectrum of D_Z in $\ker(\Box_g)$ is discrete. Moreover the Weyl eigenvalue counting function*

$$N_Z(\lambda) := \#\{j : 0 \leq \lambda_j \leq \lambda\},$$

has the asymptotics,

$$N_Z(\lambda) = \frac{1}{(2\pi)^{n-1}}\,\mathrm{Vol}(\mathcal{N}_{H\leq 1})\lambda^{n-1} + O(\lambda^{n-2}),$$

as $\lambda \to \infty$.

As in the product case, we conjecture that the remainder term is $o(\lambda^{n-2})$ when the set of periodic orbits has Liouville measure zero.

2.2. **How to define the trace.** We still need to define the trace of the Killing flow on $\ker \Box_g$ or more generally on $\ker(\Box_g + m^2)$. This requires an inner product on $\ker(\Box_g + m^2)$ (actually, just a Hilbert space topology).

The resulting distributional trace is then defined:

$$\mathrm{Tr}\ U(t) = \mathrm{Tr}\, e^{\mathrm{i}tD_Z}|_{\ker(\Box+m^2)} \tag{6}$$

There are several related energy inner product topologies, each of them induced by the topology of $H^s(\Sigma) \oplus H^{s-1}(\Sigma)$ on the space of Cauchy data on the Cauchy surface Σ for some $s \in \mathbb{R}$. Of course the inner products depend on the choice of Cauchy surface and the choice of inner product for the Sobolev spaces. Naturally defined inner products are

(1) the energy inner product defined below by the stress energy tensor; this inner product induces the topology of $H^1(\Sigma) \oplus L^2(\Sigma)$ on the Cauchy data space.
(2) the Hadamard state inner product; this inner product can be constructed when $m > 0$ and corresponds to the topology of $H^{\frac{1}{2}}(\Sigma) \oplus H^{-\frac{1}{2}}(\Sigma)$ on the Cauchy data space.

All of these give the same resulting distributional trace for $U(t)$.

2.3. **Inner product on** $\ker(\Box_g + m^2)$**.** If (M, g) is a spatially compact stationary globally hyperbolic spacetime and $m > 0$ then one can define an inner product on $\ker(\Box_g + m^2)$ using the stress energy tensor,

$$T(u) := du \otimes du - \frac{1}{2}|du|^2 g - \frac{1}{2} g\, m^2 u^2$$

If Z is Killing, and $(\Box_g + m^2)u = 0$ then $T(u)(Z)$ is a divergence free covector field.

Definition: The energy (quadratic) form of an element in $\ker(\Box_g + m^2) \cap C^\infty(M, \mathbb{R})$ is defined by

$$Q(u) = \int_\Sigma \langle T(u)(Z), \nu\rangle \mathrm{d}S$$

$$= \frac{1}{2}\int_\Sigma \frac{1}{N}\left(|\partial_t u|^2 + (N^2 h^{ij} - \beta^i\beta^j)(\partial_i u)(\partial_j u) + m^2|u|^2\right)\mathrm{dVol}_h.$$

where ν is the unit normal to Σ.

The space of smooth solutions of $(\Box_g + m^2)u = 0$ is naturally a symplectic space, with symplectic form defined by

$$\sigma(u, v) = \int_\Sigma (\nu_x u)(x)v(x) - v(x)(\nu_x u)\mathrm{dVol}_\Sigma, \tag{7}$$

where ν_x denotes the future directed unit-normal derivative to Σ at $x \in \Sigma$. We then have

- $Q(e^{itD_Z}u, e^{itD_Z}u) = Q(u, u)$ for all $u \in \ker(\Box_g + m^2)$.
- Suppose $u, v \in \ker(\Box + m^2)$. Then $Q(u, v) = \frac{i}{2}\sigma(\bar{u}, D_Z v) = \frac{1}{2}\sigma(\bar{u}, \mathcal{L}_Z v)$.

To define "$\operatorname{Tr} U(t)$" we need an explicit formula for the solution of the Cauchy problem of the wave equation in terms of Cauchy data on a Cauchy hypersurface:

$$\begin{cases} (\Box_g + m^2)u = 0, \\ u|_\Sigma = f, \quad \partial_\nu u|_\Sigma = g, \end{cases}$$

It is defined in terms of a homogeneous solution $E \in \mathcal{D}'(M \times M)$ defined by

$$E = E_{\mathrm{ret}} - E_{\mathrm{ad}}.$$

Here $E_{\mathrm{ad}}, E_{\mathrm{ret}}$ are respectively the advanced/retarded Green's functions ($\operatorname{Supp}(E_{\mathrm{ret/adv}} f) \subset J_\pm(\operatorname{Supp} f)$) of the Klein-Gordon operator $\Box_g + m^2$ and we identified the maps $E_{\mathrm{ret}} : C_0^\infty(M) \to C^\infty(M)$ and $E_{\mathrm{ad}} : C_0^\infty(M) \to C^\infty(M)$ with their integral kernels in $\mathcal{D}'(M \times M)$.

The following theorems are well known (see for example [D80] for the first statement and [DH] for the second).

Theorem **6.** *The map E extends by duality to a map $\mathcal{E}'(M) \to \mathcal{D}'(M)$ and*

$$u := E(f \otimes \delta'_\Sigma + g \otimes \delta_\Sigma)$$

is the unique solution of the Cauchy problem

$$\Box u = 0, \quad (f, g) = (u|_\Sigma, \nu_\Sigma u|_\Sigma).$$

Theorem **7.** *The map E is a Fourier integral operator in $I^{-\frac{3}{2}}(M \times M, C')$, where C' is the graph of the null bicharacteristic flow.*

We refer to Appendix A for a review of the definition of Fourier integral operators and Lagrangian distributions.

2.4. Distribution trace of the Killing group on $\ker \Box_g + m^2$.

Proposition **8.** *Let $E_t(x,y) = \mathrm{e}^{-\mathrm{i}(D_Z)_x t} E(x,y)$. Then,*

$$\operatorname{Tr}(U(t)) = \textstyle\int_\Sigma * \left(\mathrm{d}_x E_t(x,y) - \mathrm{d}_y E_t(x,y)\right)|_{y=x},$$

$$= \textstyle\int_\Sigma * \left(\mathrm{d}_x (E_t(x,y) + E_{-t}(x,y))\right)|_{y=x}.$$

where $*$ is the Hodge star operator on M.

The form $* \left(\mathrm{d}_x E_t(x,y) - \mathrm{d}_y E_t(x,y)\right)|_{y=x}$ (with values in $\mathcal{D}'(\mathbb{R})$) is closed, and its integral over Σ is independent of the chosen Cauchy surface. The identity holds because E is skew-symmetric and commutes with the flow, hence $E_t(x,y) = -E_{-t}(y,x)$.

2.5. **Traces as products of FIOs.** The basic idea of [DG75] is to express $\operatorname{Tr} U(t)$ as a composition of FIOs. In generic cases, the composition is an FIO on $\mathbb{R}$ and at each singularity $t \in \operatorname{Lsp}(M, g)$ it is a Lagrangian distribution. The leading order term is the principal symbol. We follow the same procedure.

The trace in the energy inner product is given by,

$$\operatorname{Tr}(U(t)) = \pi_* R_\Sigma \circ \Delta^* \mathrm{d}_x (E_t(x,y) + E_{-t}(y,x)) \tag{8}$$

where R_Σ is restriction to Σ, $\Delta(x) = (x, x)$ is the diagonal empedding and $\pi : \Sigma \times \mathbb{R} \to \mathbb{R}$ is the natural projection.

To prove the trace formula we need to compose the canonical relations and symbols of each factor.

2.6. **Proof of trace formula.** Let us denote $\Box = \Box_g + m^2$. Fix a Cauchy surface Σ to identify $\ker \Box$ with $H^1(\Sigma) \oplus L^2(\Sigma)$. Denote by R the corresponding restriction map

$$R : \ker \Box \to H^1(\Sigma) \oplus L^2(\Sigma).$$

One can describe the induced map $V(t) = R \circ U(t) \circ R^{-1}$ as a Fourier integral operator as follows. The surface $\Sigma_t = \Phi_t \Sigma_t$ is again a Cauchy surface and we therefore obtain a foliation of a compact subset $M_T = \cup_{t\in[-T,T]} \Sigma_t$ of M. We can use this foliation to identify M_T as a smooth manifold with the product $[-T, T] \times \Sigma$. This gives a global time coordinate t on M_T and the vector field Z is given by ∂_t. The unit-normal to Σ defines a vector field ν on M_T. The map $V(t)$ is then identified with the Cauchy evolution map

$$V(t) = R_t \circ R^{-1}. \tag{9}$$

Let $E_t(x, y) = E(e^{tZ} x, y)$.

Theorem **9.** *We have,*

$$E_t(x, y) \in I^{-\frac{7}{4}}(\mathbb{R} \times M \times M, \mathcal{C}),$$

where

$$\mathcal{C} = \{(t, \tau, \zeta_1, \zeta_2) \in T^*(\mathbb{R} \times M \times M) \mid \tau + \zeta_1(Z) = 0, (e^{tZ}(\zeta_1), \zeta_2) \in C\}.$$

The canonical relation is now parametrized by

$$\mathbb{R}_t \times C \to \mathcal{C}, (t, \zeta_1, \zeta_2) \to (t, \zeta_1(Z), e^{tZ}(\zeta_1), \zeta_2).$$

The principal symbol under the parametrization is given by,

Lemma **10.**

$$\sigma_{E_t}|_{C_\pm} = \mp \frac{\mathrm{i}}{2}(2\pi)^{\frac{3}{4}} |\mathrm{d}t|^{\frac{1}{2}} \otimes |\mathrm{d}_C|^{\frac{1}{2}}.$$

One can also derive this by using the properties of the restriction map to codimension one hypersurfaces as explained below.

The operator with kernel $\mathrm{d}_x E_t(x, y)$ has order $-\frac{3}{4}$ and its principal symbol is equal to $\frac{1}{2}(2\pi)^{\frac{3}{4}}(\xi \wedge)|\mathrm{d}t|^{\frac{1}{2}} \otimes |\mathrm{d}_C|^{\frac{1}{2}}$ on each component C_+ and C_-. Restriction to $\mathbb{R} \times \Sigma \times \Sigma$ gives an operator of order $-\frac{1}{4}$ with principal symbol $(2\pi)^{\frac{1}{4}} |\mathrm{d}t|^{\frac{1}{2}} \otimes |\mathrm{d}V_{T^*\Sigma}|^{\frac{1}{2}}$, where we have used the natural parametrisation of the canonical relation on $\mathbb{R} \times \Sigma \times \Sigma$. Restriction to the diagonal and integration over Σ gives an element in $I^{(n-1)-3/4}(\mathbb{R})$ with principal symbol $\mathrm{res}(H^{-n+1})(2\pi)^{3/4-(n-1)} |\tau|^{n-2} |\mathrm{d}t|^{1/2}$ at $T_0^*\mathbb{R}$. Hence, the principal part of the distribution $\mathrm{Tr}\, U(t)$ is given by

$$2(n-1)\mathrm{Vol}(\mathcal{N}_{H\le 1}) \int_{\mathbb{R}} \mathrm{e}^{-i\tau t} |\tau|^{n-2} \mathrm{d}\tau.$$

Residue means 'symplectic residue' of the fixed point set of the Hamilton flow. An application of Fourier Tauberian theory then implies the Weyl law.

2.7. **Joint mass-energy asymptotics.** The results of the previous section are generalized to 'ladder asymptotics' for $\lambda_j(m)$ when $m \to \infty, \lambda_j(m) \to \infty$ with $\frac{\lambda_j(m)}{m} \to \infty$. The main observation is that the mass plays the role of h^{-1} (the inverse Planck constant) in semi-classical asymptotics for Schrödinger operators, i.e. we consider the eigenvalues of D_Z in the kernel of $m^{-2}\Box_g + 1$. This requires a generalization of the microlocal description of E_t to a semi-classical one. It is standard in the case of Minkowski space that the massive property is given by a semi-classical parametrix; to our knowledge, the generalization to stationary spacetimes is a novel observation in [SZ19]. Roughly speaking, instead of the flow of e^{tZ} on the space of null-geodesics, one studies the Killing flow on the space of massive geodesics instead of null-geodesics. Here we call a geodesic massive $g(\dot\gamma, \dot\gamma) + m^2 = 0$, i.e. its tangent vector is on the mass hyperboloid. Since such geodesics are timelike and not lightlike the dynamics corresponds to that of massive particles moving in a curved background.

2.8. **Non-compact Cauchy surfaces.** For non-compact Cauchy surfaces the spectrum of D_Z on $\ker(\Box_g + m^2)$ may fail to be discrete and one expects the trace formula on the spectral side to have contributions from the essential spectrum. A natural class of stationary spacetimes to which scattering theory can be applied is the class of spacetimes that, as stationary spacetimes, coincide with Minkowski space with Killing field ∂_t, outside the Killing orbit of a compact set. This is a natural Lorentzian generalization of non-compact Riemannian manifolds that are Euclidean at infinity. In [SZ19+] we analyse these spacetimes and study their scattering theory. This version of trace-formula has the same geometric side as the one for compact Cauchy surfaces, but the spectral side involves an appropriately defined spectral shift function.

3. Remarks on separation of variables

A natural question is to ask, how the results of this article and of [SZ18] appear if one chooses a Cauchy hypersurface Σ and does both the classical and quantum analysis on it. The classical analysis refers to the projection of null geodesics to Σ. We briefly respond to this question by recording the equations for projections of null geodesics to Σ from [Ger07, page 262]. That article gives a detailed analysis of these equations.

We employ a somewhat different notation from (2) in which the equations simplify. Let $z = (t, x) \in M = \mathbb{R} \times \Sigma$ and let $\zeta = (\tau, \xi), \zeta' = (\tau', \xi') \in T_z M$. Denote the Lorentzian inner product by

$$\langle \zeta, \zeta' \rangle = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau - \alpha(x) \tau \tau'$$

Here, $\delta_i = h_{ij}\beta^j$, and $\alpha = N^2 - h_{ij}\beta^i\beta^j$.

We next endow Σ with a new Riemannian metric,

$$\langle \xi, \xi' \rangle_1 = \langle \xi, \xi' \rangle + \frac{1}{\alpha(x)} \langle \delta(x), \xi \rangle \langle \delta(x), \xi' \rangle,$$

i.e. $g_1 = g + \frac{1}{\alpha}\delta \otimes \delta$. Due to the Killing symmetry, there is a conserved quantity,

$$\alpha(x(s))\dot{t}(s) - \langle \delta(x(s)), \dot{x}(s)\rangle.$$

In the notation of [SZ18] it is the Clairaut integral $\langle Z, \zeta\rangle$. A geodesic for g_1 is a Lorentzian geodesic for which the Clairaut integral vanishes.

We now write down the equations for geodesics in terms of the metric g_1. Let

$$A(x) = \frac{\delta(x)}{\alpha(x) + \langle \delta(x), \delta(x)\rangle}, V = -\frac{1}{\alpha}.$$

Let $\hat{F}^1(X,Y) = \nabla \times A = \langle \nabla^1_X A, Y\rangle - \langle X, \nabla^1_Y A\rangle_1$ be the curl of A. Then, the projection of a geodesic of M to Σ is a curve $x(s)$ satisfying,

$$D^1_s \dot{x} + \frac{1}{2}K^2 \nabla^1 V(x) = K\hat{F}^1(x)[\dot{x}],$$

where the superscript denotes covariant differentiation with respect to g_1 and where K is the Clairaurt constant of the geodesic. As explained below [Ger07, (1.9)], these are the equations of motion of a particle on Σ moving in the potential $\frac{K^2}{2}V$ and in the magnetic field corresponding to KA.

From the viewpoint of analysing the dynamics of the Killing flow on the space of null geodesics that arises in the Gutzwiller trace formula, it seems advantageous to use the relativistically invariant description of the dynamics in [SZ18], rather than to attempt to relate the wave group of an operator pencil on Σ to the separation-of-variable dynamics above.

Appendix A. Lagrangian distributions and Fourier integral operators

For completeness and the convenience of the reader we briefly sketch the theory of Lagrangian distributions and their symbols. We refer to [HoI-IV, Section 25.1] for background. For an open subset $U \subset \mathbb{R}^n$ and $N \in \mathbb{N}$ we consider distribution $I \in \mathcal{D}'(U)$ that are given by oscillatory integrals of the form as

$$I(x) = (2\pi)^{-\frac{n+2N-2e}{4}} \int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}\varphi(x,\theta)} a(x,\theta) \mathrm{d}\theta. \tag{10}$$

Here φ is assumed to be a phase function, i.e. $\varphi \in C^\infty(U \times \mathbb{R}^N \setminus \{0\})$, $\varphi(x,\theta)$ is positively homogeneous of degree one in θ, and φ has no critical points in $U \times \mathbb{R}^N \setminus \{0\}$. Moreover, we assume φ to be clean with excess e. This means that critical manifold

$$C_\varphi = \{(x,\theta) \mid \partial_\theta \varphi(x,\theta) = 0\}$$

is a $(n+e)$-dimensional submanifold of $U \times \mathbb{R}^N \setminus \{0\}$ and the tangent space $T_{(x,\theta)}C_\varphi$ is given by the kernel of $d\partial_\theta\varphi(x,\theta)$. The function $a \in C^\infty(U \times \mathbb{R}^N)$ is a polyhomogeneous symbol of order $q = m + (n - 2N - 2e)/4$. In other words a has, as a symbol, an asymptotic expansion of the form

$$a(x, \theta) \sim \sum_{k=0}^{\infty} a_{q-k}(x, \theta)\chi(\theta),$$

where $a_k(x, \theta)$ is smooth and positively homogeneous of degree k in θ and χ is a cut-off function vanishing near zero and with $\chi(\theta) = 1$ for $|\theta| > 1$. The cutoff function removes possible singularities at $\theta = 0$ of the functions $a_k(x, \theta)$. A clean phase function with excess e defines an immersed homogeneous Lagrangian submanifold of $T^*U \setminus \{0\}$ by

$$\Lambda_\varphi = \{(x, \partial_x \varphi(x, \theta) \mid \partial_\theta \varphi(x, \theta) = 0\}.$$

A distribution is said to be a Lagrangian distribution of order m if (modulo smooth functions) it can be written in the above form with a clean phase function φ of excess e and a polyhomogeneous symbol a of order $q = m + (n - 2N - 2e)/4$. We write $I \in I^m(U, \Lambda_\varphi)$. If M is a manifold and $\Lambda \subset T^*M$ a homogeneous Lagrangian submanifold, then we say $I \in I^m(M, \Lambda)$ if it can be constructed by patching together distributions in $I^m(U, \Lambda_\varphi)$ in local coordinate charts in which Λ_φ locally defines Λ.

If M is a smooth manifold a map $C_0^\infty(M) \to C^\infty(M)$ is called a Fourier integral operator of order $m \in \mathbb{R}$ if its integral kernel is a Lagrangian distribution in $I^m(M \times M, \Lambda')$, where $\Lambda \subset (T^*M \setminus 0) \times (T^*M \setminus 0)$ is a homogeneous canonical relation and as usual $\Lambda' = \{(x_1, \xi_1, x_2, \xi_2) \mid (x_1, \xi_1, x_2, -\xi_2) \in \Lambda\}$.

In order to describe the principal symbol of a Lagrangian distribution we assume that we are given a Lagrangian distribution in the form (10) with a non-degenerate phase function, i.e. a clean phase function with excess $e = 0$. In this case the principal symbol is a half-density taking values in the Maslov bundle. The Maslov bundle is a $\mathbb{Z}_4$-principal bundle and it appears because of additional factors of the form i^σ when changing phase functions. In particular any choice of local phase function defines a local trivialisation of the Maslov bundle and modulo this Maslov factor the principal symbol is the transport to the Lagrangian $\Lambda_\varphi = \iota_\varphi(C_\varphi)$ of the half-density

$$a(\lambda)|\mathrm{d}_{C_\varphi}|^{\frac{1}{2}} := \frac{a(\lambda)|\mathrm{d}\lambda|^{\frac{1}{2}}}{|D(\lambda, \varphi'_\theta)/D(x, y, \theta)|^{\frac{1}{2}}} \tag{11}$$

on C_φ, where $\lambda = (\lambda_1, ..., \lambda_n)$ are local coordinates on the critical manifold

$$C_\varphi = \{(x, y, \theta); \mathrm{d}_\theta \varphi(x, y, \theta) = 0\}, \tag{12}$$

and where $\iota_\varphi : C_\varphi \to T^*(X \times Y) \setminus \{0\}$ is the map $(x, y, \theta) \to (x, d_x\varphi, y, -d_y\varphi)$.

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