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Operator-Valued Continuous Gabor Transforms over Non-unimodular Locally Compact Groups

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Abstract. In this article, we present the abstract harmonic analysis aspects of the operator-valued continuous Gabor transform (CGT) on second countable, non-unimodular, and type I locally compact groups. We show that the operator-valued continuous Gabor transform CGT satisfies a Plancherel formula and an inversion formula. As an example, we study these results on the continuous affine group.

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Keywords. Continuous Gabor transform, Fourier transform, Plancherel formula, Plancherel measure, unitary representation, irreducible representation, primary representation, type I group, non-unimodular group, measurable field of operators.

1. Introduction

The abstract aspects of non-commutative harmonic analysis play classical role in mathematical (theoretical) physics and geometric analysis [5,6,12,19,27]. Over the last decades, abstract non-commutative harmonic analysis has achieved a significant popularity in coherent state transforms such as time-scale (wavelet) transform and time-frequency (Gabor) transform and continuous frame theory, see [3,8–11,21–23,26] and standard references therein.

The theoretical, computational, and applied aspects of time-frequency (Gabor) analysis have been studied at depth by many researchers and authors, see [1,2,4,7,17,18] and references therein. The mathematical theory of Gabor analysis on the real line is based on the modulations and translations of a given window signal (atom). The phase space (time-frequency plane) has a unified group structure, which implies a concrete discretization and quantization. Abstract harmonic analysis aspects of Gabor analysis on Euclidean spaces imply a unified operator-valued generalizations of the Gabor analysis to the set up of locally compact Abelian (LCA) groups, and non-Abelian, unimodular, and type I locally compact groups, see [13–16] and references therein.

The following article introduces the abstract notion of continuous Gabor transforms for classical Hilbert function spaces over non-unimodular and type I groups. We aim to address abstract harmonic analysis aspects of the operator-valued continuous Gabor transform (CGT) on second countable, non-unimodular, and type I locally compact groups using tools from representation theory. Throughout this paper which contains four sections, it is assumed that G is a second countable, type I, and non-unimodular locally compact group. Section 2 is devoted to fix notations and a brief summary on non-Abelian Fourier analysis. Then, we define the continuous Gabor transform of a square integrable function f on G, with respect to the window function ψ, as a measurable field of operators defined on $G\times\widehat{G}$. Finally, in Sect. 4, we study examples of continuous Gabor transform for the continuous affine group.

2. Preliminaries and Notations on Non-Abelian Fourier Analysis

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T\in\mathcal{B}(\mathcal{H})$ is called a Hilbert–Schmidt operator if for one, and hence, for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$, we have $\sum_k\|Te_k\|^2<\infty$. The set of all Hilbert–Schmidt operators on $\mathcal{H}$ denoted by $\mathrm{HS}(\mathcal{H})$, and for $T\in\mathrm{HS}(\mathcal{H})$, we define Hilbert–Schmidt norm of T as $\|T\|_{\mathrm{HS}}^2:=\sum_k\|Te_k\|^2$. It can be checked that $\mathrm{HS}(\mathcal{H})$ is a self-adjoint and two sided ideal in $\mathcal{B}(\mathcal{H})$, and when $\mathcal{H}$ is finite-dimensional, we have $\mathrm{HS}(\mathcal{H}_\pi)=\mathcal{B}(\mathcal{H})$, also we call an operator $T\in\mathcal{B}(\mathcal{H})$ of trace-class, whenever $\|T\|_{\mathrm{tr}}:=\mathrm{tr}[|T|]<\infty$, where $\mathrm{tr}[T]:=\sum_k\langle Te_k,e_k\rangle$, and $|T|=(TT^*)^{1/2}$. For more details about trace-class and Hilbert–Schmidt operators, we refer the readers to [25].

Let $(A,\mathcal{M})$ be a measurable space. A family $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ of non-zero separable Hilbert spaces indexed by A will be called a field of Hilbert spaces over A. A map Φ on A, such that $\Phi(\alpha)\in\mathcal{H}_\alpha$ for each $\alpha\in A$ will be called a vector field on A. We denote the inner product and norm on $\mathcal{H}_\alpha$ by $\langle .,.\rangle_\alpha$ and $\|.\|_\alpha$, respectively. A measurable field of Hilbert spaces over A is a field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ together with a countable set $\{e_j\}$ of vector fields, such that the functions $\alpha\mapsto\langle e_j(\alpha),e_k(\alpha)\rangle$ are measurable for all j,k and also the linear span of $\{e_j(\alpha)\}$ is dense in $\mathcal{H}_\alpha$ for each $\alpha\in A$. Given a measurable field of Hilbert spaces $(\{\mathcal{H}_\alpha\}_{\alpha\in A},\{e_j(\alpha)\})$ on A, a vector field Φ on A will be called measurable if $\langle\Phi(\alpha),e_j(\alpha)\rangle_\alpha$ is measurable function on A for each j. The direct integral of the spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ with respect to a measure $\mathrm{d}\alpha$ on A is denoted by $\int_A^\oplus\mathcal{H}_\alpha\mathrm{d}\alpha$. This is the space of measurable vector fields Φ on A, such that we have $\|\Phi\|^2=\int_A\|\Phi(\alpha)\|_\alpha^2\mathrm{d}\alpha<\infty$. Then, it is easily follows that $\int_A^\oplus\mathcal{H}_\alpha\mathrm{d}\alpha$ is a Hilbert space with the inner product $\langle\Phi,\Psi\rangle=\int_A\langle\Phi(\alpha),\Psi(\alpha)\rangle_\alpha\mathrm{d}\alpha$.

If G is a locally compact group, the notation Δ_G stands for the modular function of G, see [6,19]. The group G is called unimodular, if $\Delta_G=1$. Henceforth, when G is a locally compact group and dx is a left Haar measure on G, $\mathcal{C}_c(G)$ consists of all continuous complex-valued functions on G with compact supports, and for each $1 \leq p < \infty$, the notation $L^p(G)$ stands for $L^p(G, dx)$, that is the Banach space of equivalence classes of measurable complex-valued functions on G whose pth powers are integrable.

Let π be a continuous unitary representation of G on the Hilbert space $\mathcal{H}_\pi$ (for more details and elementary descriptions about the topological group representations, see [6,19,20]). The representation π is called primary, if only scaler multiples of the identity belong to center of $\mathcal{C}(\pi)$. Primary representations are also known as factor representations. According to the Schur's lemma, Theorem 3.5 of [6], every irreducible representation is primary. More generally, if π is a direct sum of irreducible representations, π is primary if and only if all its irreducible subrepresentations are unitarily equivalent. The group G is said to be type I, if every primary representation of G is a direct sum of copies of some irreducible representation. The dual space $\widehat{G}$ is the set of all equivalence classes $[\pi]$ of irreducible unitary representations π of G and we still use π to denote its equivalence class $[\pi]$. The dual space $\widehat{G}$ is usually equipped with the Fell topology, see [6,24] for a discussion of this topology on $\widehat{G}$.

If G is unimodular, there is a measure $\mathrm{d}\pi$ on $\widehat{G}$, called the Plancherel measure, uniquely determined once the Haar measure on G is fixed. The family $\{\mathrm{HS}(\mathcal{H}_\pi)\}_{\pi\in\widehat{G}}$ of Hilbert spaces indexed by $\widehat{G}$ is a field of Hilbert spaces over $\widehat{G}$. Recall that, $\mathrm{HS}(\mathcal{H}_\pi)$ is a Hilbert space with the inner product $\langle T, S\rangle_{\mathrm{HS}(\mathcal{H}_\pi)} = \mathrm{tr}(S^*T)$. The direct integral of the spaces $\{\mathrm{HS}(\mathcal{H}_\pi)\}_{\pi\in\widehat{G}}$ with respect to $\mathrm{d}\pi$ is denoted by $\int_{\widehat{G}}^{\oplus} \mathrm{HS}(\mathcal{H}_\pi)\mathrm{d}\pi$, and for convenience, we use the notation $\mathcal{H}^2(\widehat{G})$ for it. If $f \in L^1(G)$, the unimodular Fourier transform of f is a measurable field of operators over $\widehat{G}$ given by

$$\mathcal{F}f(\pi) = \widehat{f}(\pi) = \int_G f(x)\pi(x)^*\mathrm{d}x. \tag{2.1}$$

Let $\mathcal{J}^1(G) := L^1(G) \cap L^2(G)$ and $\mathcal{J}^2(G)$ be the finite linear combinations of convolutions of elements of $\mathcal{J}^1(G)$. In [28], Segal proved that, when G is a second countable, non-Abelian, unimodular, and type I group, there is a measure $\mathrm{d}\pi$ on $\widehat{G}$, uniquely determine once the Haar measure dx on G is fixed, which is called the Plancherel measure and satisfies the following properties:

(1) (**Unimodular Plancherel theorem**) The Fourier transform $f \mapsto \widehat{f}$ maps $\mathcal{J}^1(G)$ into $\mathcal{H}^2(\widehat{G})$ and it extends to a unitary map from $L^2(G)$ onto $\mathcal{H}^2(\widehat{G})$.
(2) (**Unimodular Fourier inversion formula**) Each $h \in \mathcal{J}^2(G)$ satisfies the Fourier inversion formula $h(x) = \int_{\widehat{G}} \mathrm{tr}[\pi(x)\widehat{h}(\pi)]\mathrm{d}\pi$.

In non-unimodular case, the Fourier transform of $f \in L^1(G)$ at $\pi \in \widehat{G}$ is redefined via

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)D_\pi\mathrm{d}x = \pi(f)D_\pi, \tag{2.2}$$

where the measurable field of densely defined self-adjoint positive operators with densely defined inverses $\{D_\pi\}_{\pi\in\widehat{G}}$ is such that for all $f \in L^1(G)\cap L^2(G)$, we have $\pi(f)D_\pi^{-1} \in \mathrm{HS}(\mathcal{H}_\pi)$. We also have the following Plancherel formula and Fourier inversion formula in the non-unimodular case. For more details on the Fourier analysis of non-unimodular type I groups and also proofs of the following results, we refer readers to [24,29] and references therein.

Theorem 2.1 (Non-unimodular Plancherel theorem). *Let G be a second countable locally compact group, such that $H := \ker(\Delta_G)$ is a type I group in which G acts regularly on $\widehat{H}$. Then, there exists a Plancherel measure $\mathrm{d}\lambda$ (for summary $\mathrm{d}\pi$) on $\widehat{G}$ and also a measurable field $\{D_\pi\}_{\pi\in\widehat{G}}$ of densely defined self-adjoint positive operators with densely defined inverses, such that for all $f \in L^1(G) \cap L^2(G)$, we have $\pi(f)D_\pi^{-1} \in HS(\mathcal{H}_\pi)$ with*

$$\|f\|_{L^2(G)}^2 = \int_{\widehat{G}} \left\|\pi(f)D_\pi^{-1}\right\|_{HS}^2 \mathrm{d}\pi, \tag{2.3}$$

also the linear map $f \mapsto \widehat{f}$ on $L^1(G) \cap L^2(G)$ given by

$$\widehat{f}(\pi) := \pi(f)D_\pi^{-1},$$

extends uniquely to the unitary operator (non-unimodular Fourier transform):

$$\widehat{\ }: L^2(G) \to \mathcal{H}^2(\widehat{G}) = \int_{\widehat{G}}^{\oplus} HS(\mathcal{H}_\pi)\mathrm{d}\pi.$$

Theorem 2.2 (Non-unimodular Fourier inversion formula). *Let G be a second countable locally compact group, such that $H := \ker(\Delta_G)$ is a type I group in which G acts regularly on $\widehat{H}$. Then, the Plancherel measure $\mathrm{d}\lambda$ (for summary $\mathrm{d}\pi$) and the operator field $\{D_\pi\}_{\pi\in\widehat{G}}$ can be chosen to satisfy the following inversion formula:*

$$f(x) = \int_{\widehat{G}} \mathrm{tr}[\widehat{f}(\pi)D_\pi^{-1}\pi(x)^*]\mathrm{d}\pi = \int_{\widehat{G}} \mathrm{tr}[\pi(f)D_\pi^{-2}\pi(x)^*]\mathrm{d}\pi, \tag{2.4}$$

for all f in a dense subset of $L^2(G)$. The inversion formula (2.4) converges absolutely in the sense that λ-almost every $\widehat{f}(\pi)D_\pi^{-1} = \pi(f)D_\pi^{-2}$ extends to a trace-class operator, and the integral over the trace-class norms is finite.

3. Non-Unimodular Continuous Gabor Transform

Throughout this paper, we assume that G is a second countable, non-unimodular, and type I group in which G acts regularly on $\widehat{H}$, with $H = \ker(\Delta_G)$ where Δ_G is the modular function of G. Suppose that for each $\pi \in \widehat{G}$, there is a (probably unbounded) self-adjoint operator D_π on $\mathcal{H}_\pi$, such that for all $x \in G$, we have (see [6] and references therein):

$$D_\pi\pi(x) = \Delta_G(x)^{1/2}\pi(x)D_\pi. \tag{3.1}$$

Let $\mathrm{d}\sigma$ be the product of the left Haar measure dx on G and the Plancherel measure $\mathrm{d}\pi$ on $\widehat{G}$. For each $(x,\pi) \in G \times \widehat{G}$, let

$$\mathcal{H}_{(x,\pi)} := \pi(x)D_\pi\mathrm{HS}(\mathcal{H}_\pi), \tag{3.2}$$

where

$$\pi(x)D_\pi \mathrm{HS}(\mathcal{H}_\pi) = \{\pi(x)D_\pi T : T \in \mathrm{HS}(\mathcal{H}_\pi)\}.$$

It can be checked that $\mathcal{H}_{(x,\pi)}$ is a Hilbert space with respect to the inner product

$$\langle \pi(x)D_\pi T, \pi(x)D_\pi S\rangle_{\mathcal{H}_{(x,\pi)}} := \mathrm{tr}(S^*T), \quad \text{for } S,T \in \mathrm{HS}(\mathcal{H}_\pi). \tag{3.3}$$

The family $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ of Hilbert spaces indexed by $G\times\widehat{G}$ is a field of Hilbert spaces over $G\times\widehat{G}$. The direct integral of the spaces $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ with respect to σ, is denoted by $\mathcal{H}^2(G\times\widehat{G})$, that is the space of all measurable vector fields F on $G\times\widehat{G}$, such that

$$\|F\|^2_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \|F(x,\pi)\|^2_{(x,\pi)} \mathrm{d}a\sigma(x,\pi) < \infty.$$

It can also be checked that $\mathcal{H}^2(G\times\widehat{G})$ becomes a Hilbert space, with the inner product

$$\langle F, K\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \mathrm{tr}[K(x,\pi)^*F(x,\pi))]\mathrm{d}a\sigma(x,\pi).$$

Let ψ be a window function [a fixed non-zero function in $L^2(G)$] and $f\in L^2(G)$. Define the continuous Gabor transform of f with respect to the window function ψ, as a measurable field of operators $\{\mathcal{G}_\psi f(x,\pi)\}_{(x,\pi)\in G\times\widehat{G}}$ on $G\times\widehat{G}$ by

$$\mathcal{G}_\psi f(x,\pi) := \Delta_G(x)^{1/2}\int_G f(y)\overline{\psi(x^{-1}y)}\pi(y)D_\pi^{-1}\mathrm{d}y. \tag{3.4}$$

The operator-valued integral (3.4) is considered in the weak sense. In other words, for each $(x,\pi)\in G\times\widehat{G}$ and $\zeta,\xi\in\mathcal{H}_\pi$, we have

$$\langle \mathcal{G}_\psi f(x,\pi)\zeta,\xi\rangle = \int_G f(y)\overline{\psi(x^{-1}y)}\langle\pi(y)D_\pi^{-1}\zeta,\xi\rangle\mathrm{d}y.$$

Thus, we have

$$\begin{aligned}
|\langle \mathcal{G}_\psi f(x,\pi)\zeta,\xi\rangle| &= \left|\int_G f(y)\overline{\psi(x^{-1}y)}\langle\pi(y)D_\pi^{-1}\zeta,\xi\rangle\mathrm{d}y\right| \\
&= \int_G |f(y)\overline{\psi(x^{-1}y)}||\langle\pi(y)D_\pi^{-1}\zeta,\xi\rangle|\mathrm{d}y \\
&\le \int_G |f(y)\overline{\psi(x^{-1}y)}|\|\pi(y)D_\pi^{-1}\zeta\|\|\xi\|\mathrm{d}y \\
&\le \|D_\pi^{-1}\zeta\|\|\xi\|\int_G |f(y)\overline{\psi(x^{-1}y)}|\mathrm{d}y \\
&= \|D_\pi^{-1}\zeta\|\|\xi\|\|f\|_{L^2(G)}\|\psi\|_{L^2(G)}.
\end{aligned}$$

For $(x,\pi)\in G\times\widehat{G}$, we can write

$$\mathcal{G}_\psi f(x,\pi) = \Delta_G(x)^{1/2}\int_G f(y)\overline{\psi(x^{-1}y)}\pi(y)D_\pi^{-1}\mathrm{d}y$$

$$
\begin{aligned}
&= \Delta_G(x)^{1/2}\left(\int_G f(y)\overline{\psi(x^{-1}y)}\pi(y)\mathrm{d}y\right)D_\pi^{-1}\\
&= \Delta_G(x)^{1/2}\left(\int_G f(xy)\overline{\psi(y)}\pi(xy)\mathrm{d}y\right)D_\pi^{-1}\\
&= \Delta_G(x)^{1/2}\pi(x)\left(\int_G f(xy)\overline{\psi(y)}\pi(y)\mathrm{d}y\right)D_\pi^{-1}.
\end{aligned}
$$

If $f\in\mathcal{C}_c(G)$ and $\psi\in L^2(G)$, we have $f.L_x\psi\in L^1(G)\cap L^2(G)$ for each $x\in G$. Hence, the non-unimodular Plancherel theorem implies that $\widehat{f.L_x\psi}(\pi)=\pi(f.L_x\psi)D_\pi^{-1}$ is a Hilbert–Schmidt operator for almost everywhere $\pi\in\widehat{G}$. Thus, for σ-almost every (x,π) in $G\times\widehat{G}$, we have $\mathcal{G}_\psi f(x,\pi)\in\mathcal{H}_{(x,\pi)}$.

In the next proposition, we state concrete and unified representations of the continuous Gabor transform defined in (3.4).

If G is a locally compact and non-unimodular group with the modular function Δ_G and $1\leq p<\infty$, the involution for $g\in L^p(G)$ is $\widetilde{g}(x)=\Delta_G(x)^{-1/p}\overline{g(x^{-1})}$.

Proposition 3.1. *Let $\psi\in L^2(G)$ be a window function and $f\in\mathcal{C}_c(G)$. Then, for each $(x,\pi)\in G\times\widehat{G}$, we have*

(1) $\mathcal{G}_\psi f(x,\pi)=\widehat{\mathcal{L}_x^\psi(f)}(\pi)$, *where* $\mathcal{L}_x^\psi(f):=f(y)\overline{\psi(x^{-1}y)}$ *for* $y\in G$.

(2) $\mathcal{G}_\psi f(x,\pi)^*=\mathcal{F}\left(\widetilde{\mathcal{L}_x^\psi(f)}\right)(\pi)$.

The representation (1) *sometimes called as the Fourier representation of the continuous Gabor transform* (3.4).

Proof. (1) follows from the definition of redefined non-unimodular Fourier transform. (2) If $f\in\mathcal{C}_c(G)$ and $x\in G$, we have $L_x\psi\in L^2(G)$. Then, the Hölder's inequality guarantees that $\mathcal{L}_x^\psi(f)=f.\overline{L_x\psi}\in L^1(G)$, and also, we have

$$
\widetilde{\mathcal{L}_x^\psi(f)}=\widetilde{f}.\overline{\widetilde{L_x\psi}}. \tag{3.5}
$$

Note that in Eq. (3.5), the left-side involution is as an element of $L^1(G)$ and also the right-side involutions are as elements of $L^2(G)$. Let $y\in G$. Then, we can write

$$
\begin{aligned}
\widetilde{\mathcal{L}_x^\psi(f)}(y)&=\Delta_G(y^{-1})\overline{\mathcal{L}_x^\psi(f)(y^{-1})}\\
&=\Delta_G(y^{-1})\overline{f(y^{-1})}\psi(x^{-1}y^{-1})\\
&=\Delta_G(y)^{-1}\overline{f(y^{-1})}L_x\psi(y^{-1})\\
&=\Delta_G(y)^{-1/2}\overline{f(y^{-1})}\Delta_G(y)^{-1/2}L_x\psi(y^{-1})=\widetilde{f}(y)\overline{\widetilde{L_x\psi}(y)}=\widetilde{f}.\overline{\widetilde{L_x\psi}}(y).
\end{aligned}
$$

(2) Let $(x,\pi)\in G\times\widehat{G}$ and $\zeta,\xi\in\mathcal{H}_\pi$. Using the identity $\widetilde{\mathcal{L}_x^\psi(f)}=\widetilde{f}.\overline{\widetilde{L_x\psi}}$, we get

$$
\begin{aligned}
\langle\mathcal{G}_\psi f(x,\pi)^*\zeta,\xi\rangle&=\langle\zeta,\mathcal{G}_\psi f(x,\pi)\xi\rangle\\
&=\int_G\langle\zeta,f(y)\overline{\psi(x^{-1}y)}\pi(y)D_\pi^{-1}\xi\rangle\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= \int_G \langle \overline{f(y)} L_x\psi(y)\pi(y)^*\zeta, D_\pi^{-1}\xi\rangle \mathrm{d}y \\
&= \int_G \langle \overline{f(y)} L_x\psi(y)\pi(y^{-1})\zeta, D_\pi^{-1}\xi\rangle \mathrm{d}y \\
&= \int_G \langle \overline{f(y^{-1})} L_x\psi(y^{-1})\pi(y)\zeta, D_\pi^{-1}\xi\rangle \Delta_G(y^{-1})\mathrm{d}y \\
&= \int_G \langle \widetilde{f}(y)\overline{\widetilde{L_x\psi}}(y)\pi(y)\zeta, \xi\rangle \mathrm{d}y \\
&= \int_G \langle \widetilde{\mathcal{L}_x^{\psi}(f)}(y)\pi(y)\zeta, \xi\rangle \mathrm{d}y = \left\langle \mathcal{F}\left(\widetilde{\mathcal{L}_x^{\psi}(f)}\right)(\pi)\zeta, \zeta \right\rangle.
\end{aligned}
$$

In the next theorem, we shall show that the continuous Gabor transform satisfies a Plancherel formula. From operator theory aspects, the next theorem guarantees that the continuous Gabor transform (3.4) is a multiple of an isometry, and hence, it has closed range.

Theorem 3.2. *Let $\psi \in L^2(G)$ be a given window function. Then, for each $f \in \mathcal{C}_c(G)$, we have*

$$\|\mathcal{G}_\psi f\|_{\mathcal{H}^2(G\times\widehat{G})} = \|f\|_{L^2(G)}\|\psi\|_{L^2(G)}. \tag{3.6}$$

Proof. Using Proposition 3.1, Theorem 2.1 of [24], and Fubini's theorem, we have

$$
\begin{aligned}
\|\mathcal{G}_\psi f\|^2_{\mathcal{H}^2(G\times\widehat{G})} &= \int_{G\times\widehat{G}} \|\mathcal{G}_\psi f(x,\pi)\|^2_{(x,\pi)} \mathrm{d}a\sigma(x,\pi) \\
&= \int_{G\times\widehat{G}} \mathrm{tr}[\mathcal{G}_\psi f(x,\pi)^*\mathcal{G}_\psi f(x,\pi)] \mathrm{d}a\sigma(x,\pi) \\
&= \int_G \left(\int_{\widehat{G}} \mathrm{tr}[\mathcal{G}_\psi f(x,\pi)^*\mathcal{G}_\psi f(x,\pi)]\mathrm{d}\pi\right)\mathrm{d}x \\
&= \int_G \Delta_G(x)\left(\int_{\widehat{G}} \mathrm{tr}[\widehat{\widetilde{\mathcal{L}_x^{\psi}(f)}}(\pi)\widehat{\mathcal{L}_x^{\psi}(f)}(\pi)]\mathrm{d}\pi\right)\mathrm{d}x \\
&= \int_G \Delta_G(x)\left(\int_{\widehat{G}} \mathrm{tr}[\widehat{\mathcal{L}_x^{\psi}(f)}(\pi)^*\widehat{\mathcal{L}_x^{\psi}(f)}(\pi)]\mathrm{d}\pi\right)\mathrm{d}x.
\end{aligned}
$$

Now, since $\mathcal{L}_x^{\psi}(f)$ belongs to $L^1(G)\cap L^2(G)$, we get

$$
\begin{aligned}
&\int_G \Delta_G(x)\left(\int_{\widehat{G}} \mathrm{tr}[\widehat{\mathcal{L}_x^{\psi}(f)}(\pi)^*\widehat{\mathcal{L}_x^{\psi}(f)}(\pi)]\mathrm{d}\pi\right)\mathrm{d}x \\
&\quad= \int_G \Delta_G(x)\left(\int_G \overline{\mathcal{L}_x^{\psi}(f)(y)}\mathcal{L}_x^{\psi}(f)(y)\mathrm{d}y\right)\mathrm{d}x \\
&\quad= \int_G \Delta_G(x)\left(\int_G f(y)\overline{f(y)}\psi(x^{-1}y)\overline{\psi(x^{-1}y)}\mathrm{d}y\right)\mathrm{d}x \\
&\quad= \int_G f(y)\overline{f(y)}\left(\int_G \Delta_G(x)\,\psi(x^{-1}y)\overline{\psi(x^{-1}y)}\mathrm{d}x\right)\mathrm{d}y = \|f\|^2_{L^2(G)}\|\psi\|^2_{L^2(G)},
\end{aligned}
$$

which implies (3.6). □

According to Theorem 3.2, the continuous Gabor transform $\mathcal{G}_\psi : \mathcal{C}_c(G) \to \mathcal{H}^2(G \times \widehat{G})$ defined by $f \mapsto \mathcal{G}_\psi f$ is a multiple an isometry. Therefore, we can extend $\mathcal{G}_\psi$ uniquely to a bounded linear operator from $L^2(G)$ into a closed subspace of $\mathcal{H}^2(G \times \widehat{G})$ which we still use the notation $\mathcal{G}_\psi$ for this extension, and this extension for each $f \in L^2(G)$ satisfies

$$\|\mathcal{G}_\psi f\|_{\mathcal{H}^2(G\times\widehat{G})} = \|f\|_{L^2(G)}\|\psi\|_{L^2(G)}.$$

The vector field $\mathcal{G}_\psi f$ is called the continuous Gabor transform of $f \in L^2(G)$ with respect to the window function ψ, which can also be considered as the sesquilinear map $(f, \psi) \mapsto \mathcal{G}_\psi f$ from $L^2(G) \times L^2(G)$ into $\mathcal{H}^2(G \times \widehat{G})$.

Proposition 3.3. *Let ψ and φ be two window functions. The continuous Gabor transform satisfies the following orthogonality relation:*

$$\langle \mathcal{G}_\psi f, \mathcal{G}_\varphi g\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \langle \varphi, \psi\rangle_{L^2(G)}\langle f, g\rangle_{L^2(G)},$$

for all $f, g \in L^2(G)$. Moreover, the normalized Gabor transform $\|\psi\|^{-1}_{L^2(G)}\mathcal{G}_\psi$ is an isometry from $L^2(G)$ onto a closed subspace of $\mathcal{H}^2(G \times \widehat{G})$.

Let ψ be a window function and $K \in \mathcal{H}^2(G \times \widehat{G})$. The conjugate linear functional

$$g \mapsto \ell^K_\psi(g) := \int_{G\times\widehat{G}} \mathrm{tr}[K(y, \pi)\mathcal{G}_\psi g(y, \pi)^*]\mathrm{d}a\sigma(y, \pi),$$

is a bounded functional on $L^2(G)$. Because, using the Cauchy–Schwartz inequality and also Theorem 3.2, we can write

$$\begin{aligned} |\ell^K_\psi(g)| &= \left|\int_{G\times\widehat{G}} \mathrm{tr}[K(y, \pi)\mathcal{G}_\psi g(y, \pi)^*]\mathrm{d}a\sigma(y, \pi)\right| \\ &\le \int_{G\times\widehat{G}} |\mathrm{tr}[K(y, \pi)\mathcal{G}_\psi g(y, \pi)^*]|\, \mathrm{d}a\sigma(y, \pi) \\ &\le \|K\|_{\mathcal{H}^2(G\times\widehat{G})}\|\mathcal{G}_\psi g\|_{\mathcal{H}^2(G\times\widehat{G})} = \|K\|_{\mathcal{H}^2(G\times\widehat{G})}\|\psi\|_{L^2(G)}\|g\|_{L^2(G)}. \end{aligned}$$

Thus, ℓ^K_ψ defines a unique element in $L^2(G)$. From now on, we use the notation

$$\int_{G\times\widehat{G}} \mathrm{tr}[K(y, \pi)M_\pi(L_y\psi)]\mathrm{d}a\sigma(y, \pi),$$

for this element of $L^2(G)$. According to this notation, for each $g \in L^2(G)$, we have

$$\begin{aligned} &\left\langle \int_{G\times\widehat{G}} \mathrm{tr}[K(y, \pi)M_\pi(L_y\psi)]\mathrm{d}a\sigma(y, \pi), g\right\rangle_{L^2(G)} \\ &\quad = \int_{G\times\widehat{G}} \mathrm{tr}[K(y, \pi)\mathcal{G}_\psi g(y, \pi)^*]\mathrm{d}a\sigma(y, \pi). \end{aligned}$$

In the next theorem, we prove an inversion formula.

Theorem 3.4. *Let ψ, φ be two window functions, such that $\langle \varphi, \psi\rangle_{L^2(G)} \neq 0$. Then, for each $f \in L^2(G)$, we have*

$$f = \langle \varphi, \psi\rangle^{-1}_{L^2(G)} \int_{G\times\widehat{G}} \mathrm{tr}[\mathcal{G}_\psi f(y,\pi) M_\pi(L_y\varphi)] \mathrm{d}a\sigma(y,\pi).$$

Proof. By Theorem 3.2, we have $\mathcal{G}_\psi f \in \mathcal{H}^2(G\times\widehat{G})$. As it is mentioned, the integral

$$\langle \varphi, \psi\rangle^{-1}_{L^2(G)} \int_{G\times\widehat{G}} \mathrm{tr}[\mathcal{G}_\psi f(y,\pi) M_\pi(L_y\varphi)] \mathrm{d}a\sigma(y,\pi),$$

denotes a well-defined function in $L^2(G)$. Let us use the notation f_ψ^φ for this function. Using Corollary 3.3, for each $g \in L^2(G)$, we have

$$\begin{aligned}\langle f_\psi^\varphi, g\rangle_{L^2(G)} &= \langle \varphi, \psi\rangle^{-1}_{L^2(G)} \int_{G\times\widehat{G}} \mathrm{tr}[\mathcal{G}_\psi f(y,\pi) \mathcal{G}_\varphi g(y,\pi)^*] \mathrm{d}a\sigma(y,\pi) \\ &= \langle \varphi, \psi\rangle^{-1}_{L^2(G)} \langle \mathcal{G}_\psi f, \mathcal{G}_\varphi g\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \langle f, g\rangle_{L^2(G)},\end{aligned}$$

which implies that $f = f_\psi^\varphi$ in $L^2(G)$. □

Corollary 3.5. *Let ψ be a window function, such that $\|\psi\|_{L^2(G)} = 1$. Then, for each $f \in L^2(G)$, we have*

$$f = \int_{G\times\widehat{G}} \mathrm{tr}[\mathcal{G}_\psi f(y,\pi) M_\pi(L_y\psi)] \mathrm{d}a\sigma(y,\pi).$$

The following proposition presents a formula concerning the continuous Gabor transform with respect to two non-orthogonal window functions.

Proposition 3.6. *For window functions ψ and φ with $\langle \varphi, \psi\rangle_{L^2(G)} \neq 0$, we have*

$$\mathcal{G}_\varphi^* \mathcal{G}_\psi = \langle \varphi, \psi\rangle_{L^2(G)} I_{L^2(G)}. \tag{3.7}$$

Proof. Let $S_\varphi : \mathcal{H}^2(G\times\widehat{G}) \to L^2(G)$ be the bounded linear operator given by

$$S_\varphi(K) = \int_{G\times\widehat{G}} \mathrm{tr}[K(y,\pi) M_\pi(L_y\varphi)] \mathrm{d}a\sigma(y,\pi).$$

Then, S_φ is the adjoint operator of $\mathcal{G}_\varphi$. Using Proposition 3.1, for each $f \in L^2(G)$ and $K \in \mathcal{H}^2(G\times\widehat{G})$, we have

$$\langle S_\varphi(K), f\rangle_{L^2(G)} = \int_{G\times\widehat{G}} \mathrm{tr}[K(y,\pi)\mathcal{G}_\varphi f(y,\pi)^*]\mathrm{d}a\sigma(y,\pi)$$
$$= \langle K, \mathcal{G}_\varphi f\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \langle \mathcal{G}_\varphi^*(K), f\rangle_{L^2(G)}.$$

Now, Theorem 3.4 implies (3.7). □

4. Continuous Affine Group

Let $G_\tau = (0,\infty)\ltimes_\tau \mathbb{R}$ be the affine group $a\mathbf{x}+b$, which is the group of all affine transformations $\mathbf{x}\to a\mathbf{x}+b$ of $\mathbb{R}$ with $a\in(0,\infty)$ and $b\in\mathbb{R}$ or with the semi-direct approach the semi-direct group of $H\ltimes_\tau K$, where $H=(0,\infty)$, $K=\mathbb{R}$, and the continuous homomorphism $\tau: H\to Aut(K)$ given by $a\mapsto\tau_a$, where $\tau_a(b):=ab$ for all $b\in\mathbb{R}$. The group law for all $q=(a,b), p=(\alpha,\beta)\in G_\tau=(0,\infty)\ltimes_\tau\mathbb{R}$ is

$$q\ltimes_\tau p = (a,b)\ltimes_\tau(\alpha,\beta) := (a\alpha, b+\tau_a(\beta)) = (a\alpha, b+a\beta),$$

$$q^{-1} = (a,b)^{-1} := (a^{-1}, \tau_{a^{-1}}(-b)) = (1/a, -b/a).$$

Then, $d_lp = \mathrm{d}\mu_l(a,b) = dadb/a^2$ is a left Haar measure and $d_rp=\mathrm{d}\mu_r(a,b) = dadb/a$ is a right Haar measure for G, and also the modular function for $p=(a,b)\in G$ is $\Delta_{G_\tau}(a,b)=1/a$. All one-dimensional irreducible representations of G_τ are of the form π_λ for some $\lambda\in\mathbb{R}$, where $\pi_\lambda(a,b)=a^{i\lambda}$ for all $(a,b)\in G_\tau$ and $\lambda\in\mathbb{R}$ ([6]). Let $\pi: G_\tau\to\mathcal{U}(L^2(\mathbb{R}))$ be the continuous unitary representation of G_τ given by

$$[\pi(a,b)\mathbf{g}](x) = a^{1/2}e^{2\pi ibx}\mathbf{g}(ax),\quad \text{for } x\in\mathbb{R} \quad\text{and}\quad \mathbf{g}\in L^2(\mathbb{R}). \tag{4.1}$$

Let the continuous unitary representations π_+ and π_- be the subrepresentations of the continuous unitary representation π on the subspaces $\mathcal{H}_+ = L^2(\Omega_+)$ and $\mathcal{H}_+ = L^2(\Omega_-)$, respectively, where $\Omega_+=(0,+\infty)$ and $\Omega_-=(-\infty,0)$. Then

$$\widehat{G_\tau} = \{\pi_\lambda : \lambda\in\mathbb{R}\}\cup\{\pi_\pm\}. \tag{4.2}$$

Let $D_\pm:\mathcal{H}_\pm\to\mathcal{H}_\pm$ be given by

$$[D_\pm\mathbf{g}](t) = |t|^{1/2}\mathbf{g}(t),\quad \text{for } \mathbf{g}\in\mathcal{H}_\pm = L^2(\Omega_\pm). \tag{4.3}$$

Then, the operators $D_\pm:\mathcal{H}_\pm\to\mathcal{H}_\pm$ satisfy

$$D_\pm\pi_\pm(q) = D_\pm\pi_\pm(a,b) = a^{-1/2}\pi_\pm(a,b)D_\pm, \tag{4.4}$$

for all $q=(a,b)\in G_\tau=(0,\infty)\ltimes\mathbb{R}$. The modified Fourier transform will be

$$\widehat{f}(\pi_\pm) = \pi_\pm(f)D_\pm,\quad \text{for } f\in L^1(G_\tau)\cap L^2(G_\tau). \tag{4.5}$$

For $f\in L^1(G_\tau)\cap L^2(G_\tau)$ and $q=(a,b)\in G_\tau$, $\pi\in\widehat{G_\tau}$, we have

$$\mathcal{G}_\psi f(q,\pi) = \Delta_{G_\tau}(q)\int_{G_\tau} f(p)\overline{\psi(p^{-1}q)}\pi(p)D_\pi^{-1}\mathrm{d}\mu_l(p)$$
$$= \Delta_{G_\tau}(a,b)^{1/2}\int_{G_\tau} f(\alpha,\beta)\overline{\psi((a,b)^{-1}\ltimes_\tau(\alpha,\beta))}\pi(\alpha,\beta)D_\pi^{-1}\mathrm{d}\mu_l(\alpha,\beta)$$
$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi((a,b)^{-1}\ltimes_\tau(\alpha,\beta))}\pi(\alpha,\beta)D_\pi^{-1}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}$$

$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi((1/a,-b/a)\ltimes_\tau(\alpha,\beta))}\pi(\alpha,\beta)D_\pi^{-1}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}$$
$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\pi(\alpha,\beta)D_\pi^{-1}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}.$$

Example 4.1. Let $f \in L^1(G_\tau)\cap L^2(G_\tau)$ and $\psi \in \mathcal{C}_c(G_\tau)$. If $q=(a,b)\in G_\tau$, we have

$$\mathcal{G}_\psi f(q,\pi_\pm) = a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\pi_\pm(\alpha,\beta)D_{\pi_\pm}^{-1}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}$$
$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\pi_\pm(\alpha,\beta)D_\pm^{-1}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}.$$

Then, for all $\mathbf{f},\mathbf{g}\in\mathcal{H}_\pm$, we get

$$\begin{aligned}
\langle \mathcal{G}_\psi f(q,\pi_\pm)\mathbf{f},\mathbf{g}\rangle_{\mathcal{H}_\pm} &= \langle \mathcal{G}_\psi f(a,b,\pi_\pm)\mathbf{f},\mathbf{g}\rangle_{L^2(\Omega_\pm)}\\
&= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\\
&\quad\times\ \langle \pi_\pm(\alpha,\beta)D_\pm^{-1}\mathbf{f},\mathbf{g}\rangle_{L^2(\Omega_\pm)}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}\\
&= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\\
&\quad\times\left(\int_{\Omega_\pm}[\pi_\pm(\alpha,\beta)D_\pm^{-1}\mathbf{f}](t)\overline{\mathbf{g}(t)}\mathrm{d}t\right)\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}\\
&= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\\
&\quad\times\left(\int_{\Omega_\pm}\alpha^{1/2}e^{2\pi i\beta t}[D_\pm^{-1}\mathbf{f}](\alpha t)\overline{\mathbf{g}(t)}\mathrm{d}t\right)\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}\\
&= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\\
&\quad\times\left(\int_{\Omega_\pm}\alpha^{1/2}e^{2\pi i\beta t}|\alpha t|^{-1/2}\mathbf{f}(\alpha t)\overline{\mathbf{g}(t)}\mathrm{d}t\right)\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}\\
&= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\\
&\quad\times\left(\int_{\Omega_\pm}|t|^{-1/2}e^{2\pi i\beta t}\mathbf{f}(\alpha t)\overline{\mathbf{g}(t)}\mathrm{d}t\right)\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}.
\end{aligned}$$

Example 4.2. Let $f\in L^1(G_\tau)\cap L^2(G_\tau)$, $\psi\in\mathcal{C}_c(G_\tau)$, $\lambda\in\mathbb{R}$, and $q=(a,b)\in G_\tau$. Then

$$\mathcal{G}_\psi f(q,\lambda) = \mathcal{G}_\psi f(a,b,\pi_\lambda)$$
$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\pi_\lambda(\alpha,\beta)D_{\pi_\lambda}^{-1}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}$$

$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\pi_\lambda(\alpha,\beta)\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}$$
$$= a^{-1/2}\int_0^{+\infty}\int_{-\infty}^{+\infty} \alpha^{i\lambda}f(\alpha,\beta)\overline{\psi(\alpha/a,(-b+\beta)/a)}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{\alpha^2}.$$

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