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Modelling category inflation with multiple inflation processes: Estimation, specification, and testing*

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Abstract

Zero-inflated ordered probit (*ZIOP*) and middle-inflated ordered probit (*MIOP*) models are finding increasing favour in the discrete choice literature. We propose generalisations to these models—which collapse to their *ZIOP*/*MIOP* counterparts under a set of simple parameter restrictions—with respect to the inflation process. These generalisations form the basis of a new specification test of the inflation process in *ZIOP* and *MIOP* models. Support for our generalisation framework is principally demonstrated by revisiting a key *ZIOP* application from the economics literature, and reinforced by the reassessment of an important *MIOP* application from political science. Our specification test supports the generalised models over the original *ZIOP*/*MIOP* ones, suggesting an important role for it in modelling zero- and middle-inflation processes.

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I Introduction

Recent advances in discrete choice modelling have witnessed the development of *inflated ordered probit* models. These models draw inspiration from the suite of hurdle and double-hurdle models for continuous and count outcome variables developed to address an excess of zero observations (Cragg 1971, Mullahey 1986, Lambert 1992, Heilbron 1994, Mullahey 1997). Their use, which hinges on an assumption that the data are generated by two distinct data generation processes, is typically motivated by the fact that in some ordered choice situations, a large proportion of empirical observations fall into a single choice category which appears 'inflated' relative to the others. Underpinning the importance of accounting for the presence of suspected category inflation is the fact that failing to do so can lead to model mis-specification, biased estimates, and incorrect inference.

Inflated ordered probit models have been applied in fields such as economics, political science, and medical statistics, and can be divided into two main variants. The first is the *zero-inflated ordered probit* (*ZIOP*) model, in which an excess of observations is observed at one end of the choice spectrum (Harris and Zhao 2007; Meyerhoefer and Zuvekas 2010; Downward et al. 2011; Gurmu and Dagne 2012; Habib et al. 2012; Jiang et al. 2013; Peng et al. 2013; Akcura 2015; Bagozzi et al. 2015; and Falk and Katz-Gerro 2016). The second is the more recently developed *middle-inflated ordered probit* (*MIOP*) model, which is characterised by a middle outcome being inflated (Bagozzi and Mukherjee 2012; Brooks et al. 2012; Bagozzi et al. 2014; Miwa 2015; and Zirogiannis et al. 2015).

This paper adds to this growing literature in several important ways. We propose generalisations of inflated ordered probit models that preserve the ordering of outcomes whilst still explicitly accounting for the maintained inflation process. Instead of having a single 'splitting equation' in a setting with J categorical outcomes (see Harris and Zhao 2007), our generalisations require J–1 of these latent equations to be estimated. These equations capture the propensity to be pushed away from the model's non-inflated outcomes towards the inflated one. We refer to these models as the *generalised zero-inflated ordered probit* (*GZIOP*) and the *generalised middle-inflated ordered probit* (*GMIOP*). These model collapse to their associated *ZIOP* and *MIOP* counterparts when the parameter vectors of the J–1 splitting equations are restricted to be equal. As these generalised models nest their *ZIOP*/*MIOP* counterparts, it is possible to use standard testing paradigms to test if the nested model specifications are too restrictive.

This aspect of our contribution is significant, as insufficient attention is devoted to this issue in the literature.[1] We derive the appropriate Lagrange multiplier (*LM*) tests, which can be used without having to estimate the more general models (*c.f.*, the likelihood ratio

[1] Our testing framework focuses on instances where one inflated model nests another. In relation to the problem of zero-inflation in the Poisson counts literature, Wilson (2015) argues that the widespread practice of using the Vuong test as a test of zero-inflation in a non-nested setting is erroneous.

(*LR*) test, for example). To explore the performance of our proposed generalisation and testing framework, we consider the original data and model specifications from Harris and Zhao (2007), who model tobacco consumption at the individual level. *LM* tests based on the data in their *ZIOP* application appear correctly sized in Monte Carlo experiments and have good power properties, and typically exhibit good *quasi*-power in identifying mis-specified models. This suggests that our *LM* tests are good general specification tests. When the generalised model is estimated using the original data and specification in Harris and Zhao (2007), our specification test favours the generalised model.

To complement the above analysis, we also explored the performance of our generalisation framework in a middle-inflation setting. The same methodological approach was applied to a dataset from Bagozzi and Mukherjee (2012), who use a *MIOP* framework to model 'face-saving' middle-category responses in a commonly studied Eurobarometer survey question (European Commission 2002a,b). The associated *LM* tests were also found to have desirable statistical properties, and favoured the generalised model when it was estimated on the original data and specification in their paper. Taken together, our findings are important, as both Harris and Zhao (2007) and Bagozzi and Mukherjee (2012) claim to have demonstrated the superiority of the respective *ZIOP* and *MIOP* approaches over the ordered probit (*OP*) one. We establish that further improvements in modelling category inflation can be realised by increasing the flexibility of *ZIOP* and *MIOP* models.

Our focus is restricted to inflation in a single categorical outcome deriving from multiple data generation processes (DGPs), and most closely relates to earlier work by Gillman et al. (2013), who propose a three-outcome case of the generalised *MIOP* model, which is developed specifically to account for the prevalence of 'no-change' monetary policy decisions. Related work by Sirchenko (2019) develops an endogenous switching model of monetary policy in a discrete ordinal setting with three latent regimes. In both contributions, the nature of the middle-inflation means that the decision to leave the policy stance unchanged arises due to one of three distinct scenarios occurring.

Significantly, our work is distinct from contributions where category inflation is characterised by more than one categorical outcome being inflated. For instance, Greene, Harris, and Hollingsworth (2015) estimate a discrete ordered model of self-assessed health in which two outcomes are subject to category inflation. Cai, Xia, and Zhou (2019) explore the consequences of 'generalised' category inflation for multinomial, ordinal, Poisson, and zero-truncated Poisson outcomes and allow for inflation in multiple categories from a single source;[2] unlike our contribution, no testing framework is proposed. In this regard, our proposed generalisations and associated specification tests potentially have widespread applicability across the social and related sciences. For example, the empirical *MIOP* appli-

[2] As noted by a referee, the *ZIOP* modelling approach is distinct in a number of ways from zero-inflated approaches to modelling count data.

cation in Bagozzi and Mukherjee (2012) focuses on a type of survey question where the response options range from feeling negative to positive about an issue, such that a middle category captures feelings of neutrality or indifference. Such questions are commonplace in questionnaires, which suggests there is considerable scope for the analysis of such data using our proposed models. We now say more about the motivation underlying our statistical approach.

II Motivation

Accounting for the presence of category inflation raises salient issues regarding how it should be modelled. Even if an ordered categorical outcome is characterised by a considerable amount of observations relative to all others, a *ZIOP* or *MIOP* modelling approach may not be warranted. Instead, a standard ordered probit model may suffice, in that any category can be 'inflated' through adjustment of the relevant threshold parameters. It amounts to assuming that the data are generated by a single DGP.

This highlights a defining feature of the *ZIOP* and *MIOP* modelling approach: an assumption that category inflation is generated by two distinct DGPs. It also leads to a second equally important characteristic of *ZIOP* and *MIOP* modelling that is commonly overlooked in the literature: namely, a categorical outcome need not exhibit a build-up of observations to warrant using a *ZIOP* or *MIOP* approach. All that is required is a belief that one of the observed categories is generated by two distinct DGPs. This need not manifest itself in a noticeable spike in the number observations for a given category, although in most cases, *ZIOP* and *MIOP* modelling strategies are used when a relative build-up of observations is observed. In this regard, its application should be strictly hypothesis driven. In turn this will have significant implications for the choice of the model's exclusion restrictions.

In this contribution, the nature of the inflation process underpinning the *ZIOP* and *MIOP* models motivates us to ask two pertinent questions. First, can category inflation be the product of more than two DGPs, and if so, how can this be modelled? Second, if category inflation is generated by more than two DGPs, is it possible to test if using a *ZIOP* or *MIOP* approach is too restrictive? As the DGPs that comprise the *ZIOP* and *MIOP* models are unobserved, the question of whether the processes driving category inflation are correctly specified is apposite; as noted, insufficient guidance is provided in the extant literature. In developing a statistical framework that maintains the ordering of categorical outcomes, accounts for the presence of category inflation with more than two DGPs, and nests the respective *ZIOP* and *MIOP* models as special cases under certain parameter restrictions, our contribution explicitly addresses the above questions. The model generalisations which form the basis of specification tests of the *ZIOP* and *MIOP* models do indeed permit us to determine if a *ZIOP* or *MIOP* inflation process is overly restrictive.

Our generalised frameworks are also attractive natural extensions to the *ZIOP* and *MIOP* models in their own right. For instance, some ordered choice situations may be characterised by '*status quo*' bias in which individuals select the alternative that implies 'doing nothing or maintaining one's current or previous decision' (Samuelson and Zeckhauser 1988, p.7). If such bias is suspected of generating category inflation, exclusion restrictions would require that the covariates responsible for this phenomenon will feature in the splitting equations, the hypothesised effect of which will be to push individuals to choose the 'status quo' outcome. In other situations, more overt forms of psychological group and peer pressure may push an individual to select a particular option from an ordered set of alternatives over the option which would otherwise be chosen. Here, the splitting equations would contain variables that might proxy for such influences, which would push individuals towards choosing the inflated outcome. Lastly, one might consider a set of ordered outcomes, where the splitting equations contain proxies capturing 'nudge' effects (Thaler and Sunstein 2008), which push individuals towards selecting a socially or commercially desirable outcome over all other alternatives. These examples highlight the potential widespread applicability of our generalisations. Our statistical approach, which focuses on the generalisation of the *ZIOP* model, but which is straightforwardly applicable to the *MIOP* model, is now set out in more detail.

III Generalising Inflated Ordered Probit Models

Prior to developing our generalisation, it is useful to set out the key features of the *ZIOP* model for comparative purposes. Consider a discrete random variable y that assumes the discrete ordered values of $y \in 0, 1, ..., J–1$. A standard *OP* approach would map a single latent variable to the observed outcome y via so-called boundary parameters, with the latent variable being related to a set of covariates. Let r denote a binary variable indicating the split between regimes 0 and 1. r is related to a latent variable r^* via the mapping: $r = 1$ for $r^* > 0$ and $r = 0$ for $r^* \leq 0$. The latent variable r^* represents the propensity to be in regime 1 and is defined as

$$r^* = \mathbf{x}'\boldsymbol{\beta} + \varepsilon, \tag{1}$$

where $\mathbf{x}$ is a k_x vector of covariates that determine the choice between the two regimes, $\boldsymbol{\beta}$ a vector of unknown coefficients, and ε a standard-normally distributed error term. The probability of being in regime 1 is given by

$$\Pr(r = 1\,|\mathbf{x}) = \Pr(r^* > 0|\mathbf{x}) = \Phi(\mathbf{x}'\boldsymbol{\beta}), \tag{2}$$

where $\Phi(.)$ is the cumulative distribution function (CDF) of the univariate standard normal distribution. Outcomes in regime 1 are represented by a discrete variable $\widetilde{y}$ ($\widetilde{y} = 0, 1, ..., J–1$)

that is generated by an *OP* model via a second underlying latent variable $\widetilde{y}^*$, where

$$\widetilde{y}^* = \mathbf{z}'\boldsymbol{\gamma} + u. \tag{3}$$

In expression (3), $\mathbf{z}$ is a k_z vector of explanatory variables with unknown weights $\boldsymbol{\gamma}$, and u is a standard normal error term. Under the assumption that ε and u identically and independently follow standard Gaussian distributions, the full probabilities for y are

$$\Pr(y) = \begin{cases} \Pr\left(y = 0\,|\mathbf{z}, \mathbf{x}\right) = \Pr\left(r = 0|\mathbf{x}\right) + \Pr(r = 1|\mathbf{x})\Pr(\widetilde{y} = 0|\mathbf{z}, r = 1); \\ \Pr\left(y = j\,|\mathbf{z}, \mathbf{x}\right) = \Pr\left(r = 1|\mathbf{x}\right)\Pr\left(\widetilde{y} = j\,|\mathbf{z}, r = 1\right), \quad (j = 1, 2, ..., J\text{–}1) \end{cases} \tag{4}$$

which, by independence of ε and u are given by

$$\Pr(y) = \begin{cases} \Pr\left(y = 0\,|\mathbf{z}, \mathbf{x}\right) = \left[1 - \Phi\left(\mathbf{x}'\boldsymbol{\beta}\right)\right] + \Phi\left(\mathbf{x}'\boldsymbol{\beta}\right)\Phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right); \\ \Pr\left(y = j\,|\mathbf{z}, \mathbf{x}\right) = \Phi\left(\mathbf{x}'\boldsymbol{\beta}\right)\begin{bmatrix} \Phi\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right) - \\ \Phi\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right) \end{bmatrix}, \ (j = 1, 2, ..., J\text{–}2); \\ \Pr\left(y = J\text{–}1\,|\mathbf{z}, \mathbf{x}\right) = \Phi\left(\mathbf{x}'\boldsymbol{\beta}\right)\left[1 - \Phi\left(\mu_{J-2} - \mathbf{z}'\boldsymbol{\gamma}\right)\right]. \end{cases} \tag{5}$$

The framework depicted in expression (5) is the *ZIOP* model of Harris and Zhao (2007). These authors refer to their contribution as a 'double-hurdle combination of a split probit model and an ordered probit model' (p.1073). Here, the probability that a zero observation has been inflated is captured by a combination of the probability of zero from the *OP* process plus the probability of zero from the splitting equation. This central feature of the model also holds when the model is extended to allow for correlated errors,

$$\Pr(y) = \begin{cases} \Pr\left(y = 0\,|\mathbf{z}, \mathbf{x}\right) = \left[1 - \Phi\left(\mathbf{x}'\boldsymbol{\beta}\right)\right] + \Phi_2(\mathbf{x}'\boldsymbol{\beta}, \mu_0 - \mathbf{z}'\boldsymbol{\gamma}; -\rho); \\ \Pr\left(y = j\,|\mathbf{z}, \mathbf{x}\right) = \begin{bmatrix} \Phi_2\left(\mathbf{x}'\boldsymbol{\beta}, \mu_j - \mathbf{z}'\boldsymbol{\gamma}; -\rho\right) - \\ \Phi_2\left(\mathbf{x}'\boldsymbol{\beta}, \mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}; -\rho\right) \end{bmatrix}, \ (j = 1, 2, ..., J\text{–}2); \\ \Pr\left(y = J\text{–}1\,|\mathbf{z}, \mathbf{x}\right) = \Phi_2\left(\mathbf{x}'\boldsymbol{\beta}, \mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}; \rho\right) \end{cases} \tag{6}$$

where ρ is the correlation coefficient $(-1 \leq \rho \leq 1)$, and Φ_2 denotes the CDF of the bivariate normal distribution.[3] We refer to the correlated model in (6) as the *ZIOPC*.

Diagrammatically, the *ZIOP* model is illustrated in Panel A of Figure 1, and comprises the binary probit 'splitting equation', which comprises regimes $r = 0$ and $r = 1$; and an ordered probit (*OP*) model comprising J categorical outcomes ranging from $\widetilde{y} = 0, 1, 2, ..., J$–1. In many empirical applications, the splitting equation is treated as distinguishing between individuals who are willing to participate ($r = 1$) or not ($r = 0$) in the consumption of a

[3]In estimation, to ensure the required ordering of the boundary parameters, we specify them as $\mu_j = \mu_{j-1} + \exp\left(\xi_j\right), \quad j = 1, 2, ..., J\text{–}2$, where μ_0 is freely estimated (Greene and Hensher 2010). In our J outcome setting, we therefore have a full set of boundary parameters that are denoted μ_0, $\mu_1, ..., \mu_{J-2}$. This convention is carried through to our generalisations.

good, typically a social bad. Non-participation decisions may be governed by factors such as health concerns, religious beliefs, ethical considerations, or societal norms. Many real-world examples reflect such behaviour: consider decisions not to consume drugs and recreational substances such as alcohol, tobacco, and cannabis. However, non-consumption may still arise if individuals who are prepared to consume the good in regime $r = 1$ are unable to do so because of income or price constraints. Here, the 'Observed outcome' column in Panel A shows that an observational unit for whom $r = 1$ and $\widetilde{y} = 0$ will still choose outcome $y = 0$; this outcome is also shown as being realised for observational units in regime $r = 0$. Zero consumption is thus driven by a mixture of non-participants, and participants who are unable to consume.

Now consider the latent class model depicted in Panel B of Figure 1, which comprises a single *OP* model comprising J categorical outcomes ranging from $\widetilde{y} = 0, 1, 2, ..., J–1$, and $J–1$ splitting equations ranging from r_1^* through to r_{J-1}^*. Here, for each $j > 0$ category in the *OP* model, the individual has a propensity to be pushed towards choosing the zero outcome by a category-specific splitting equation, an effect that we describe as 'tempering'. For instance, the 'Observed outcome' column in Panel B shows that an observational unit for whom $\widetilde{y} = 2$ and $r_2 = 0$ will still select outcome $y = 0$; this outcome is also shown as being realised for observational units in regime $\widetilde{y} = 0$. We refer to this econometric model as the 'generalised *ZIOP*' (hereafter *GZIOP* model). The observed data is generated from the joint outcome of J DGPs, namely the $J–1$ binary probit equations and the single *OP* one; this contrasts with the *ZIOP* model, which is characterised by two DGPs.

The $J–1$ splitting equations of the *GZIOP* have the form

$$r_j^* = \mathbf{x}'\boldsymbol{\beta}_j + \varepsilon_j, \tag{7}$$

which allow for differentiated tempering effects across the $j = 1, 2, ..., J–1$ outcome equation propensities. The associated observability criteria are given by

$$y = \widetilde{y}r_j = \begin{cases} 0 \text{ if } \left[\begin{array}{c} (\widetilde{y}^* \leq \mu_0) \text{ or} \\ (\mu_{j-1} < \widetilde{y}^* \leq \mu_j \text{ and } r_j^* \leq 0),\ j = 1, 2, ..., J–2 \text{ or} \\ (\mu_{J-2} < \widetilde{y}^* \text{ and } r_{J-1}^* \leq 0) \end{array} \right]; \\ j \text{ if } (\mu_{j-1} < \widetilde{y}^* \leq \mu_j \text{ and } r_j^* > 0), \quad j = 1, 2, ..., J–2; \\ J–1 \text{ if } (\mu_{J-2} < \widetilde{y}^* \text{ and } r_{J-1}^* > 0). \end{cases} \tag{8}$$

Under independence, generalising the *ZIOP* in this manner yields the *GZIOP* model which has probabilities of the form

$$\Pr(y) = \begin{cases} \Pr\left(y = 0 \left| \mathbf{z}, \mathbf{x}\right.\right) = \begin{pmatrix} \Pr\left(\widetilde{y} = 0 | \mathbf{z}\right) \\ + \Pr\left(\widetilde{y} = j | \mathbf{z}\right) \Pr\left(r_j = 0 | \mathbf{x}, \widetilde{y} = j\right), \ j = 1, 2, ..., J\text{–}1 \end{pmatrix}; \\ \Pr\left(y = j \left| \mathbf{z}, \mathbf{x}\right.\right) = \Pr\left(\widetilde{y} = j | \mathbf{z}\right) \Pr\left(r_j = 1 | \mathbf{x}, \widetilde{y} = j\right), \quad j > 0 \end{cases} \tag{9}$$

such that

$$\Pr(y) = \begin{cases} \Pr\left(y = 0 \left| \mathbf{z}, \mathbf{x}\right.\right) = \begin{cases} \Phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) + \sum_{j=1}^{J-2} \begin{pmatrix} \Phi\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right) - \\ \Phi\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right) \end{pmatrix} \Phi\left(-\mathbf{x}'\boldsymbol{\beta}_j\right) \\ + \left[1 - \Phi\left(\mu_{J-2} - \mathbf{z}'\boldsymbol{\gamma}\right)\right] \Phi\left(-\mathbf{x}'\boldsymbol{\beta}_{J-1}\right) \end{cases}; \\ \Pr\left(y = j \left| \mathbf{z}, \mathbf{x}\right.\right) = \left[\Phi\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right) - \Phi\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right)\right] \Phi\left(\mathbf{x}'\boldsymbol{\beta}_j\right), \ j = 1, 2, ..., J\text{–}2; \\ \Pr\left(y = J\text{–}1 \left| \mathbf{z}, \mathbf{x}\right.\right) = \left[1 - \Phi\left(\mu_{J-2} - \mathbf{z}'\boldsymbol{\gamma}\right)\right] \Phi\left(\mathbf{x}'\boldsymbol{\beta}_{J-1}\right) \end{cases} \tag{10}$$

which embodies the required zero-inflation due to the terms $\Pr\left(\widetilde{y} = j | \mathbf{z}\right) \Pr\left(r_j = 0 | \mathbf{x}, \widetilde{y} = j\right)$ for $j = 1, 2, ..., J\text{–}1$. The generalised *ZIOP* with correlated errors has probabilities

$$\Pr(y) = \begin{cases} \Pr\left(y = 0 \left| \mathbf{z}, \mathbf{x}\right.\right) = \begin{cases} \Phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) + \sum_{j=1}^{J-2} \begin{bmatrix} \Phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}_j; \rho_j\right) - \\ \Phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}; -\mathbf{x}'\boldsymbol{\beta}_j; \rho_j\right) \end{bmatrix} \\ + \Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, -\mathbf{x}'\boldsymbol{\beta}_{J-1}; -\rho_{J-1}\right) \end{cases}; \\ \Pr\left(y = j \left| \mathbf{z}, \mathbf{x}\right.\right) = \begin{bmatrix} \Phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}_j; -\rho_j\right) - \\ \Phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}_j; -\rho_j\right) \end{bmatrix}, \ j = 1, 2, ..., J\text{–}2; \\ \Pr\left(y = J\text{–}1 \left| \mathbf{z}, \mathbf{x}\right.\right) = \Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, \mathbf{x}'\boldsymbol{\beta}_{J-1}; \rho_{J-1}\right). \end{cases} \tag{11}$$

and is referred to as *GZIOPC*, and characterised by J–1 correlation coefficients $\rho_j \ \forall \ j > 0$.

Given this assumed form for the probabilities and an independent and identically distributed sample of size $i = 1, \ldots, N$ from the population on $(y_i, \mathbf{z}_i, \mathbf{x}_i)$, this satisfies all of the standard regularity conditions for maximum likelihood estimation (see Greene 2012). The full parameter set $\boldsymbol{\theta} = \left(\boldsymbol{\gamma}', \boldsymbol{\beta}', \boldsymbol{\mu}', \boldsymbol{\rho}'\right)'$ of the model can be consistently and efficiently estimated using standard maximum likelihood techniques, with the log-likelihood function given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{N} \sum_{j=0}^{J-1} h_{ij} \ln\left[\Pr\left(y_i = j \left| \mathbf{z}, \mathbf{x}, \boldsymbol{\theta}\right.\right)\right], \tag{12}$$

where in (12) the indicator function h_{ij} is given by

$$h_{ij} = \begin{cases} 1 \text{ if individual } i \text{ chooses outcome } j \\ 0 \text{ otherwise.} \end{cases} \quad (i = 1, ..., N; \ j = 0, 1, ..., J\text{–}1). \tag{13}$$

Our empirical applications use the common sandwich estimator (White 1982) to compute the standard errors of parameters.[4] Standard errors of secondary estimated quantities, such as partial effects and summary probabilities are estimated using the delta method. All subsequent models differ only with respect to the probabilities entering the likelihood function and the contents of $\boldsymbol{\theta}$. All latent equations are estimated simultaneously and not sequentially, such that only the joint outcomes of the J DGPs captured by expression (9) are observed. Such a model is an example of a partial observability one involving J latent equations.[5]

The generalised models collapse to their non-generalised counterparts under a set of simple linear parameter restrictions. Imposing the restrictions that $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_{J-1}$ and $\rho_1 = \rho_2 = \cdots = \rho_{J-1}$ on (11) collapses the *GZIOPC* to the *ZIOPC* in (6). Imposing the additional restriction that $\rho_1 = \rho_2 = \cdots = \rho_{J-1} = 0$ collapses the *GZIOPC* to the *ZIOP* in (5).[6] Lastly, imposing the restriction that $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_{J-1}$ in (10) collapses the *GZIOP* to the *ZIOP*. The sets of parameter restrictions described above thus provide tests of: (i) the more flexible functional form of the *GZIOPC* model versus the simpler nested forms of the usual *ZIOPC* and *ZIOP* models; and (ii) the *GZIOP* versus the *ZIOP* model.

This implies that the model on the right side of Figure 1 can nest the model depicted on the left. In a generalised model, identification requires the data to identify J–1 splitting equations as opposed to a single one. One implication of this model characteristic is that compared to a non-generalised model, the choice of exclusion restrictions assumes a more prominent role, as several splitting equations require identification instead of one. More generally, what we refer to as 'behavioural identification' requires that there are no empty sets of individuals in expression (3) that are pushed towards an inflated outcome for each of a generalised model's J–1 splitting equations.[7] This contrasts with the *ZIOP* approach, which assumes that the zero observations are comprised of two types of zero. The issue of behavioural identification is revisited in Section VI.

In similar fashion to the *ZIOP*, we can generalise the *MIOP*, noting that the *generalised middle-inflated ordered probit* (*GMIOP*) model is a total analogue to the generalization of the *ZIOP* presented above.[8] Further, as the *MIOP* and its generalisation are related in an analogous way to that of the *ZIOP* and the *GZIOP*, we can also consider model variants with correlated errors which we label *MIOPC* and *GMIOPC*, respectively. Testing the restrictions associated with these model variants entails testing (i) the more flexible functional

[4] Greene and Hensher (2010, p.31) state that '...in almost any case, the sandwich estimator provides an appropriate asymptotic covariance matrix for an estimator that is biased in an unknown direction'.

[5] Poirier (1980) applies this concept in the context of a bivariate probit model.

[6] For a proof of these claims see Online Appendix A.

[7] This is likely to be evidenced by instances of model non-convergence and/or estimated model probabilities close to zero. As Greene, Rose, and Hensher (2015, p.719) note in the context of a latent class ordered choice model: 'Signature features of a model that has been over-fit will be exceedingly small estimates of the class probabilities, wild values of the structural parameters and huge estimated standard errors.' We encountered no such issues in our applications.

[8] If the *MIOP* was to additionally incorporate the first and the last categorical outcomes, the *ZIOP* could be viewed as a 'special case' of the *MIOP*; the same applies to its respective generalisations.

form of the *GMIOPC* model versus the simpler nested forms of the *MIOPC* and *MIOP* models and (*ii*) the *GMIOP* versus the *MIOP* model. A detailed exposition of the *MIOP* and its generalisation is given in Online Appendix B.

A specification test of the ZIOP and MIOP models

To test the hypotheses associated with the various sets of parameter restrictions described above, two approaches are used. First, we use the standard *LR* test. Second, an *LM* test is proposed. This is an appealing specification test for the *ZIOP* and *MIOP* models and their correlated versions *versus* their generalised alternatives as it only requires estimation of the simpler nested models. It involves evaluation of the score vector of the more general model evaluated at parameter values under the null hypothesis. The generic form for the *LM* statistic is given by

$$LM = (\boldsymbol{\nabla\beta}, \boldsymbol{\nabla\gamma}, \nabla\mu_0, \boldsymbol{\nabla\xi}, \boldsymbol{\nabla\rho})' \left[\mathbf{I}\left(\hat{\theta}_R\right)\right]^{-1} (\boldsymbol{\nabla\beta}, \boldsymbol{\nabla\gamma}, \nabla\mu_0, \boldsymbol{\nabla\xi}, \boldsymbol{\nabla\rho}) \sim \chi^2_q, \tag{14}$$

which is evaluated at the relevant parameter restrictions as defined by the appropriate null hypothesis, and where q denotes the appropriate number of parameter restrictions. If the alternative model is the uncorrelated generalised version, one would omit the relevant partition of the score vector ($\boldsymbol{\nabla\rho}$). As is common practice, the outer product of gradients is used to estimate the inverse of the variance of the score vector, $\left[\mathbf{I}\left(\hat{\theta}_R\right)\right]^{-1}$ (Greene 2012). Reassuringly, the results of the *LR* and *LM* tests are very similar in both empirical applications, suggesting that standard asymptotic theory performs well.[9]

IV Data

To explore the performance of our generalisation and testing framework, we consider the original data and model specifications from the original *ZIOP* contribution of Harris and Zhao (2007), who consider tobacco consumption. Zero tobacco consumption is assumed to be determined by two DGPs: non-participation due to, for example, health and legal concerns; and zeros who are the corner solution associated with a standard consumer demand problem, whereby individuals will not smoke if the price rises above a certain threshold, or income falls below a certain threshold. This is reflected in the nature of the exclusion restrictions governing the respective splitting (or 'inflation') and *OP* (or 'consumption') equations, which also include standard controls capturing socio-economic and personal characteristics. Their data is drawn from the 1995, 1998 and 2001 surveys of the Australian *National Drug Strategy Household Survey*, in which information on tobacco consumption is available via a discrete

[9] Derivations of the score vectors for the *LM* test can be found in Online Appendix C. Online Appendix D establishes that our proposed generalisations are coherent and demonstrates that our models neither nest, nor are nested, by the *generalised ordered probit* ('GOP') model (Terza 1985).

variable measuring consumption intensity. Respondents are asked: 'How often do you now smoke cigarettes, pipes or other tobacco products?', where the responses take the form of one of the following: not at all ($y = 0$); smoking less frequently than daily ($y = 1$); smoking daily with less than 20 cigarettes per day ($y = 2$); and smoking daily with 20 or more cigarettes per day ($y = 3$). 76% of observations are non smokers, 4% smoke weekly or less, 13.8% smoke daily but less than 20 per day, and 6.2% smoke daily and consume more than 20 cigarettes a day. The specification shares 13 common variables in the splitting and *OP* equations, and is characterised by: $N = 28,813$; $J = 4$; $k_x = 16$; and $k_z = 18$. For a full description of the variable set, see Harris and Zhao (2007). In this empirical application, which is characterised by four categorical outcomes, our generalisation requires that three splitting equations—which capture the extent to which the individual is pushed towards zero consumption—require estimation. Prior to these estimations, the performance of our proposed models is explored using Monte Carlo (*MC*) experiments.

V Finite sample performance

To ascertain the finite sample performance of our tests, we consider a range of Monte Carlo (*MC*) experiments based on the original data in the *ZIOP* application and the full sample size reported above. The number of repetitions was set to 2,000, where all simulation 'noise' had effectively settled after 1,000 repetitions. Table 1 presents our findings, which include results relating to empirical size and quasi-power. The first column identifies the true DGP and the respective degrees of freedom for each test (*df*). For each DGP, three tests—each between a generalised model and a null, non-generalised variant—are performed.

Panel A considers the zero-inflated application and tests between: *GZIOP vs. ZIOP*; *GZIOPC vs. ZIOPC*; and *GZIOPC vs. ZIOP*, with $J = 3$ outcomes.[10] Row 1 has a *ZIOP* DGP with *df* = 13, 14, 15, respectively. All empirical sizes are very close to a nominal 5% size, even when the null model is the *ZIOPC*. Row 2 repeats the exercise, but for a true DGP of *ZIOPC*. The empirical size is again very close to the nominal one (at 5.8%). The tests also have good 'power' in correctly rejecting the uncorrelated versions of the model (38% and 49%, respectively). Row 3 considers the implications of extending the choice set to a larger number of outcomes, one of which is relatively sparsely populated; here, the empirical sizes remain very close to the nominal ones.

As rejection of the null model(s) may reflect other forms of model mis-specification, we also generate under ordered probit and parallel regression assumption-relaxed (Brant 1990) models.[11] These *quasi*-power experiments reflect likely forms of serious model mis-

[10] Initially, we combine two contiguous sparsely populated categorical outcomes.

[11] Although our interest with the parallel regression model is restricted to its use as a DGP, we note that it is incoherent (see Greene and Hensher 2010, p.144). We thank a referee for bringing this to our attention.

specification encountered with our type of data. The *OP* model is based on an equation of the form of expression (3). For the parallel regression model, the data is generated by multiple $\boldsymbol{\gamma}_j$ vectors generated by independent binary models for all observed values of j. The results are presented in rows 4 and 5, respectively. All tests have good general 'power' (24%–36%) against the *OP* DGP. Against the parallel regression model, all tests similarly exhibit reasonable 'power' (at around 14%).

We also considered a variant of the *GZIOP* model with no tempering corresponding to the $j = 3$ splitting equation. Such a model does not collapse to the null *ZIOP* model under any set of simple linear parameter restrictions. In experiments, this model variant failed to converge in nearly 50% of instances. When convergence was achieved, the *LM* test always rejected the null model, and the estimated probabilities of being pushed towards the true zero amount were very close to zero. Clear evidence of model mis-specification in the splitting equation of the non-tempered outcome presented itself in the form of very large coefficients and extremely high standard errors. These findings add to the evidence that the *LM* test performs well as a general specification test: they suggest that model failure in estimation would also indicate a mis-specified model, as would obtaining splitting equation tempering probabilities that are very close to zero. Significantly, all *LM* tests therefore appear correctly sized, and typically have good 'power' in identifying mis-specified models.

Using the covariate data we also conducted genuine experiments based on the null model of *ZIOP*. In all experiments we take the estimated value of $\boldsymbol{\beta}$ in each null model, setting $\boldsymbol{\beta}_j = \boldsymbol{\beta} \; \forall \; j$ in the corresponding generalised set-up, and perturb a single parameter β_0 in a single splitting equation by successively larger increments. For brevity, we only report power runs for the non-correlated DGPs. The power curves are presented in Figure 2, and cover experiments performed using alternative *df*.

We consider two curves for the *ZIOP* model, both of which utilise the full data sample: one corresponds to J=3 categorical outcomes (*df*=13); and another to J=4 (*df*=26). The curve corresponding to the higher *df* has uniformly higher power, where increasing the number of categorical outcomes from three to four is responsible for the increase in the *df*. Whilst relatively larger parameter perturbations are required to induce rejections under J=3, both tests have the 'usual' shaped power curves and our analysis suggests both tests have good power. Significantly, our results demonstrate that the ability of the tests to identify *ZIOP* model mis-specification in the direction of the *GZIOP* one is an increasing function of both the number and size of perturbations from the null. All of the above findings are reinforced by the results of experiments for the *MIOP* application, which are reported in part E.2 of the Online Appendix E. We now turn to model estimation.

VI Estimation

To explore the validity of using a *GZIOP* approach, we first turn to the *LM* test results, which are presented in Table 2. All *ZIOP* variants are overwhelmingly rejected in favour of the generalised models, and the *GZIOP* is rejected in favour of *GZIOPC*. Results of the corresponding *LR* tests closely mirror the *LM* ones. Using a generalised framework thus appears to be a more appropriate modelling strategy.

To rationalise the *GZIOP* model conceptually, the ordered consumption levels can be thought of as being driven by an *OP* process, where the propensity for zero-consumption corresponds to non-participation. Here, rational addiction theory (Becker and Murphy 1988) assumes that some individuals are rational in going 'cold-turkey'—that is, switching from positive consumption levels, as captured by the latent ordered probit equation, to zero, as captured by the $j = 1, 2, 3$ splitting equations. Corresponding to each positive consumption level is a splitting equation which divides individuals into two types: those remaining at a positive consumption level, and 'quitters' who are pushed towards zero. The *GZIOPC* model explicitly accounts for such behaviour. *LM* and *LR* tests permit us to determine if a single splitting equation representing non-participation as in Harris and Zhao (2007) is sufficient to represent all of the zero types corresponding to non-participation that could arise.

Table 3 presents a selection of overall partial effects for the *ZIOPC* and *GZIOPC* evaluated at sample means.[12] Clearly, the choice of modelling approach has important implications for inference. For example, a one-unit increase in the own price of tobacco increases the probability of zero-consumption by 0.1139. In the *GZIOPC,* the figure is 0.164. The corresponding figures for high consumption levels ($j = 3$) are 0.083 for the *ZIOPC* and 0.101 for *GZIOPC.* Income has a statistically significant effect on outcomes $y = 0, 1, 2$ in the *ZIOPC,* whereas for the generalised variant, only high level consumers ($j = 3$) are significantly affected. Here, we note that Harris and Zhao (2007) decompose the partial effects for income for the *ZIOPC*, reporting that a 10% increase in personal income causes a 0.0027 rise in the probability of non-participation ($r = 0$), but a 0.0017 fall in the probability of participation with zero consumption ($r = 1$, $\widetilde{y} = 0$). This latter effect indicates that tobacco is a normal good for participants. Conducting a similar exercise for the *GZIOPC* yields a larger fall of 0.0024 in the probability of participation with zero consumption, with the probability of non-participation increasing by 0.0037 for the heaviest smokers ($\widetilde{y} = 3, r_3 = 0$). This effect is considerably smaller in magnitude for individuals who smoke less frequently. Policy recommendations based on the *ZIOPC* model would fail to account for the non-participation effects of income changes being differentiated across individuals with different consumption patterns.

To further investigate the consequences of estimating the mis-specified *ZIOP* and *ZIOPC*

[12] We focus on the partial effects as the model coefficients (available from the authors on request) are difficult to interpret in isolation.

models over their generalisations, Table 4 presents a series of estimated model probabilities averaged over all individuals, which quantify the extent to which non-participatory effects contribute to decision outcomes. Such effects are obtained by estimating the probabilities solely associated with the underlying *OP* components of the respective models. These probabilities effectively 'purge' or 'net-out' any inflation effects. For the correlated versions the estimated *OP* parameters were used to estimate these in isolation from the inflation equations(s)—essentially setting the correlation coefficients to zero. We estimate the amount of zero-inflation in the model—denoted *Amount* (Zero-inflation)—as the difference between the overall predicted probability of zero consumption and the corresponding purged amount. This quantity is then used to calculate the proportion of overall zero-consumption that is attributable to the effects of model inflation. Expressed as a percentage, we denote this quantity $Amount(\%)$.

The purged probabilities differ substantially for the *GZIOP* and *ZIOP* models, especially for higher consumption levels. Whilst the *GZIOP* suggests some nearly 50% of the zero observations can be attributed to zero-inflation, this figure is just over 45% for the *ZIOP*. By comparison, the correlated models suggest greater levels of zero-inflation, with the generalised variant indicating a relatively higher contribution to overall zero consumption (72% versus 63%). These findings point to the non-generalised models underestimating the degree of overall model inflation.

To further evaluate the predictive performance of our models we construct hit-and-miss tables, which provide information about the proportion of correct predictions. This involves cross-tabulating the predictions of a given model obtained using the maximum probability rule. Table 5 presents summary measures for both within sample and for a 10% 'hold-out' sample. Two approaches are used to measure model performance. First, the traditional approach, in which unconditional model forecasts are compared to observed outcomes (Greene 2012). For each $j = 0, 1, 2, ..., J–1$ this is obtained by dividing the number of correct predictions within each category by the total number of predictions for that category. We denote this measure *CP*, such that $0 \leq CP \leq 1$, where a larger value implies greater predictive accuracy. The second approach follows Henriksson and Merton (1981), and mitigates the problem of a so-called 'stopped-clock' strategy when evaluating forecasts. In our example, this translates to the traditional hit-and-miss approach placing too much weight on the most heavily chosen outcome. This measure, denoted CP^*, lies between $–1/J–1$ and 1: a value of $–1/J–1 \leq CP^* < 0$ implies a forecasting performance worse than the stopped–clock strategy; $CP^* = 0$ suggests zero predictability, which is consistent with the 'stopped clock' strategy; and $CP^* = 1$ implies a perfect forecasting model. Both approaches indicate that the generalised models perform best.

Finally, it is informative to consider the behavioural assumptions required for model identification. The *ZIOP* model is only identified if the inflated category observed in the data is composed of two types of observation: non-participants associated with the inflation equation (1); and infrequent smokers associated with the consumption equation (3). Behavioural identification for the generalised model is stricter. This requires that there are no empty sets of individuals in expression (3) that are pushed towards zero-consumption via (7), $\forall\ j \geq 1$. In practice, the presence of empty sets may manifest itself in the form of one or more of the r_j^* splitting equations having negligible tempering probabilities. That is, the model will appear to be 'weakly identified'. We find no evidence of this form of weak identification, in that all of the estimated tempering probabilities associated with the J–1 splitting equations diverge from zero.

Significantly, no evidence of this type of weak identification is found when generalised middle–inflated models are estimated using the Bagozzi and Mukherjee (2012) data and specifications. In addition to this finding, the associated specification tests and measures of predictive performance also support a generalised modelling strategy; and as is the case with our zero-inflation application, the non-generalised middle-inflated models underestimate the extent of overall model inflation.[13]

VII Conclusion

This paper has proposed generalisations to the *ZIOP* and *MIOP* models, which form the basis of a specification test relating to the underlying inflation process. As this issue has been insufficiently explored in the literature, this development is important. Our specification test favours the generalised model in our empirical applications, highlighting the potential for model mis-specification in applications which assume that a *ZIOP* or *MIOP* modelling strategy is appropriate. Just as saliently, the nature of our generalisation framework facilitates testing the predictions of economic theory, such as the decision to go 'cold turkey'. In this regard, our generalisations constitute valid and flexible modelling frameworks in their own right.

Our contribution also raises issues that merit further exploration. Consider the tobacco consumption application: tempering may be characterised *not* by a binary 'to quit or not quit' decision—as captured by each of the J–1 splitting equations—but a movement down from higher levels of tobacco consumption to lower levels, which may, or may not, include zero consumption. Amending our generalisations to accommodate this kind of behaviour would represent a move towards a latent class set-up, which would require even stricter conditions for identification. Similarly, if evidence of empty sets is found, a generalised model may be re-specified by omitting the affected r_j^* splitting equations, and re-estimating without them. Whilst the resulting specifications will still be inflated models, they will no longer be 'generalised', in that the standard *ZIOP* model will no longer be nested. Consequently, our

[13] See Online Appendix E.3.

proposed specification tests would be inappropriate. The possibility of refining the *GZIOP* in these ways suggests that the generalised models developed in this contribution form part of a much broader model class for analysing category inflation, which extends beyond the focus of the current contribution. Accordingly, as zero- and middle-inflated models have been used effectively to model behaviour in an array of social, economic, and political settings, the possibility of using these suggested innovations represents an interesting avenue for future research.

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Figures and Tables

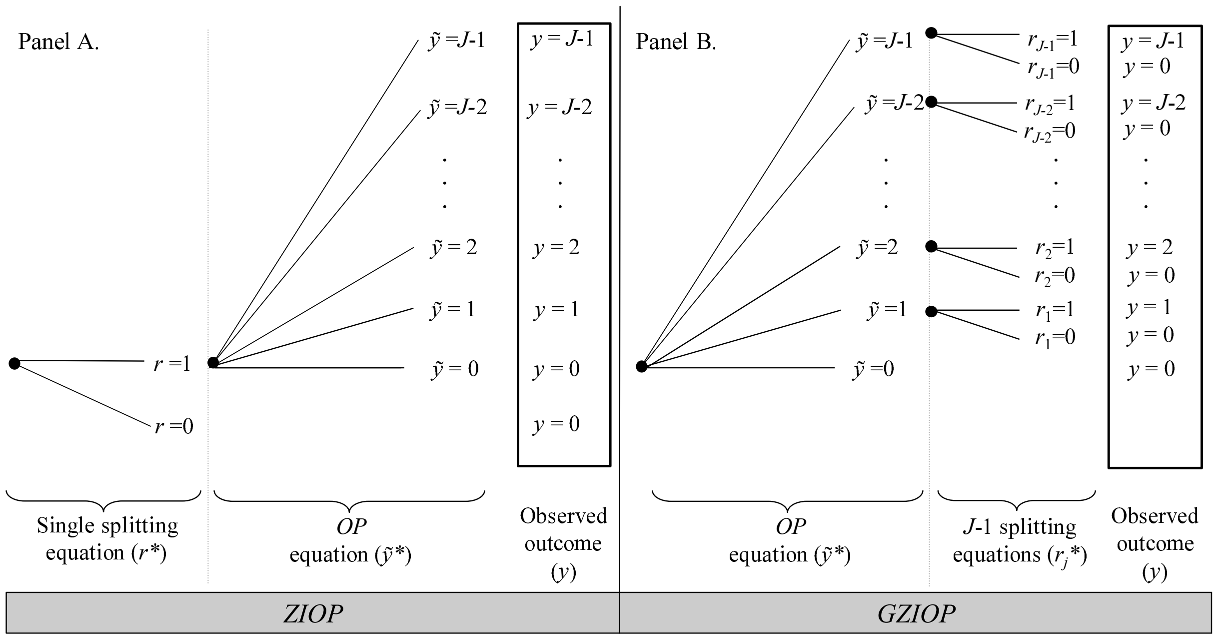

Figure 1: The Zero-Inflated Ordered Probit (*ZIOP*) model and its generalisation (*GZIOP*)

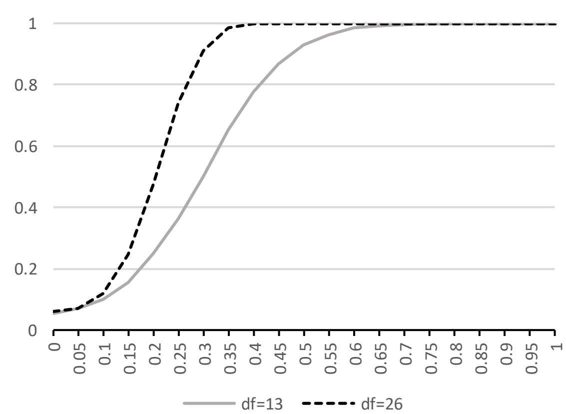

Figure 2: Empirical power curves for the *ZIOP* model

Table 1: Monte Carlo rejection probabilities

True model	Rejection probability		
	GZIOP *vs.* *ZIOP*	*GZIOPC* *vs.* *ZIOPC*	*GZIOPC* *vs.* *ZIOP*
1.$ZIOP(df = 13, 14, 15)$	0.053	0.058	0.056
2.$ZIOPC(df = 13, 14, 15)$	0.381	0.058	0.489
3.$ZIOP(df = 26, 28, 29)$	0.059	0.061	0.063
4.$OP(df = 13, 14, 15)$	0.252	0.358	0.239
5.$Parallel(df = 13, 14, 15)$	0.141	0.140	0.144

Table 2: Specification test results: competing zero-inflated models

Model	*LM* Test statistic	*df*	*p*-value	*LR* Test statistic	*p*-value
ZIOP vs. GZIOP	194	32	$4.27E-25$	178	$3.56E-22$
ZIOPC vs. GZIOPC	207	34	$1.68E-26$	202	$9.09E-26$
ZIOP vs. GZIOPC	221	35	$7.29E-29$	212	$3.33E-27$
GZIOP vs. GZIOPC	27	3	$5.89E-06$	34	$1.98E-07$

Table 3: Selected overall partial effects for the *ZIOPC* and *GZIOPC* models; smoking data[a]

	ZIOPC				*GZIOPC*			
	$j=0$	$j=1$	$j=2$	$j=3$	$j=0$	$j=1$	$j=2$	$j=3$
ln(*Income*)	0.013	−0.003	−0.008	−0.001	0.007	0.003	−0.005	−0.005
	(0.005)***	(0.001)***	(0.003)***	(0.002)	(0.005)	(0.002)	(0.004)	(0.003)**
Male	−0.077	0.009	0.044	0.024	−0.069	0.015	0.013	0.042
	(0.007)***	(0.002)***	(0.004)***	(0.003)***	(0.008)***	(0.003)***	(0.006)**	(0.004)***
Married	0.124	−0.016	−0.071	−0.037	0.128	−0.016	−0.063	−0.049
	(0.007)***	(0.002)***	(0.004)***	(0.003)***	(0.008)***	(0.004)***	(0.006)***	(0.004)***
Pre school	0.038	−0.006	−0.022	−0.01	0.035	−0.005	−0.008	−0.022
	(0.009)***	(0.002)***	(0.006)***	(0.004)**	(0.01)***	(0.004)	(0.008)	(0.006)**
Capital	0.012	0.002	−0.005	−0.009	0.007	0.005	0.001	−0.013
	(0.007)*	(0.001)	(0.004)	(0.003)***	(0.007)	(0.003)*	(0.006)	(0.004)***
Work	0.036	0.004	−0.016	−0.024	0.044	0.001	−0.017	−0.029
	(0.009)***	(0.002)	(0.005)***	(0.004)***	(0.011)***	(0.005)	(0.009)*	(0.006)***
Unemployed	−0.059	0.005	0.032	0.021	−0.072	0.005	0.057	0.01
	(0.019)***	(0.004)	(0.011)***	(0.007)***	(0.025)***	(0.018)	(0.018)***	(0.011)
English-speaking	−0.068	0.004	0.036	0.027	−0.063	0.007	0.024	0.032
	(0.015)***	(0.003)	(0.009)***	(0.006)***	(0.015)***	(0.005)	(0.012)**	(0.009)***
Degree	0.205	0.001	−0.102	−0.104	0.226	0.012	−0.135	−0.102
	(0.01)***	(0.003)	(0.006)***	(0.005)***	(0.012)***	(0.005)**	(0.009)***	(0.007)***
Diploma	0.062	0	−0.031	−0.031	0.076	0	−0.043	−0.033
	(0.008)***	(0.002)	(0.005)***	(0.004)***	(0.01)***	(0.005)**	(0.008)	(0.005)***
Year 12	0.076	0.002	−0.037	−0.041	0.091	0.004	−0.058	−0.037
	(0.01)***	(0.002)	(0.006)***	(0.004)***	(0.011)***	(0.006)	(0.009)***	(0.006)***
Young female	−0.023	0.003	0.013	0.007	−0.012	0	0.013	−0.002
	(0.011)**	(0.002)*	(0.006)**	(0.003)**	(0.01)	(0.003)	(0.006)**	(0.007)
$\ln(P_A)$	0.28	0.017	−0.13	−0.168	0.327	0.003	−0.127	−0.202
	(0.068)***	(0.006)***	(0.032)***	(0.041)***	(0.083)***	(0.013)	(0.036)***	(0.051)***
$\ln(P_M)$	−0.005	0	0.002	0.003	−0.004	0	0.002	0.003
	(0.01)	(0.001)	(0.005)	(0.006)	(0.012)	(0)	(0.005)	(0.007)
$\ln(P_T)$	0.139	0.009	−0.064	−0.083	0.164	0.001	−0.064	−0.101
	(0.018)***	(0.003)***	(0.009)***	(0.011)***	(0.023)***	(0.007)	(0.012)***	(0.014)***

[a]Standard errors in parentheses.***, ** and * denote significance at 1%,5%, and 10%, respectively. $\ln(P_{A/M/T})$ denotes the natural log of the price of alcohol/marijuana/tobacco, respectively.

Table 4: Summary probabilities from the *ZIOP* and *GZIOP* models; and *ZIOPC* and *GZIOPC* models[a]

		Independent errors				Correlated errors			
		Overall		Purged		Overall		Purged	
Outcome	Sample	*ZIOP*	*GZIOP*	*ZIOP*	*GZIOP*	*ZIOPC*	*GZIOPC*	*ZIOPC*	*GZIOPC*
$j = 0$	0.748	0.747	0.748	0.403	0.383	0.747	0.748	0.279	0.206
		(0.002)***	(0.002)***	(0.016)***	(0.027)***	(0.002)***	(0.002)***	(0.032)***	(0.024)***
$j = 1$	0.043	0.043	0.043	0.094	0.109	0.043	0.043	0.078	0.063
		(0.001)***	(0.001)***	(0.003)***	(0.015)***	(0.001)***	(0.001)***	(0.006)***	(0.007)***
$j = 2$	0.145	0.145	0.145	0.340	0.372	0.145	0.145	0.347	0.437
		(0.002)***	(0.002)***	(0.010)***	(0.022)***	(0.002)***	(0.002)***	(0.013)***	(0.033)***
$j = 3$	0.065	0.064	0.064	0.163	0.136	0.064	0.064	0.297	0.295
		(0.001)***	(0.001)***	(0.006)***	(0.019)***	(0.001)***	(0.001)***	(0.046)***	(0.046)***
		ZIOP	*GZIOP*			*ZIOPC*	*GZIOPC*		
Amount (Zero-inflation)		0.344	0.365			0.432	0.460		
		(0.016)***	(0.027)***			(0.030)***	(0.025)***		
Amount(%)		46.09%	48.77%			62.72%	72.48%		

[a]Standard errors in parentheses.***, ** and * denote significance at 1%,5%, and 10%, respectively.

Table 5: In-sample and out-of-sample hit-and-miss tables for *ZIOP* applications

Specification	*OP*				*ZIOP*				*ZIOPC*				*GZIOP*				*GZIOPC*				
Predicted ($\widehat{y}_i$): In-sample																					
Actual (y_i)	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	Total
0	21535	0	0	4	21508	0	25	6	21518	0	15	6	21484	0	46	9	21492	0	43	4	21539
1	1244	0	0	0	1239	0	5	0	1241	0	3	0	1237	0	7	0	1236	0	8	0	1244
2	4167	0	0	5	4136	0	32	4	4148	0	19	5	4105	0	61	6	4107	0	63	2	4172
3	1854	0	0	4	1838	0	17	3	1847	0	8	3	1830	0	22	6	1829	0	28	1	1858
Total	28800	0	0	13	28721	0	79	13	28754	0	45	14	28656	0	136	21	28664	0	142	7	28813
CP	0.7475				0.7477				0.7476				0.7480				0.7481				
*CP**	0.0007				0.0026				0.0017				0.0051				0.0045				
Predicted ($\widehat{y}_i$): Out-of-sample – 10% 'hold-out' sample																					
Actual (y_i)	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	Total
0	2200	0	0	0	2199	0	1	0	2199	0	1	0	2196	0	3	1	2196	0	4	0	2200
1	135	0	0	0	135	0	0	0	135	0	0	0	135	0	0	0	135	0	0	0	135
2	419	0	0	0	415	0	2	2	416	0	1	2	413	0	4	2	414	0	5	0	419
3	180	0	0	0	179	0	1	0	179	0	1	0	177	0	3	0	177	0	3	0	180
Total	2934	0	0	0	2928	0	4	2	2929	0	3	2	2921	0	10	3	2922	0	12	0	2934
CP	0.7498				0.7502				0.7498				0.7498				0.7502				
*CP**	0				0.0014				0.0006				0.0026				0.0034				

Notes: The above table provides information about the proportion of correct predictions for alternative models. The predictions of each model are cross-tabulated using the maximum probability rule $\widehat{y}_i = m$ if $\widehat{P}_{im} = \max(\widehat{P}_{i0}, \widehat{P}_{i1}, \widehat{P}_{i2}, ..., \widehat{P}_{iJ-1})$ against the observed outcomes in a $J \times J$ hit-and miss table, where $\widehat{P}_{ij}$ denotes the predicted probability of outcome j arising for observation i. *CP* denotes the proportion of correct predictions calculated by summing across all J diagonal elements and dividing by the total number of observations N, such that $CP= (1/N)\Sigma_{i=1}^{N} 1\,(\widehat{y}_i = y_i)$. CP^* follows Kim, Mizen, and Chevapatrakul (2008) and Rosa (2009) who adapt the 'stopped clock' methodological approach of Henriksson and Merton (1981) for a discrete choice setting. This criterion is calculated as $CP^* = 1/(1-J)\left[\Sigma_{j=0}^{J-1} CP_j - 1\right]$, where CP_j denotes the proportion of the correct predictions made by $\widehat{y}_i$ when the true state is given by $y_i = j$, such that $CP_j = \big((1/N)\Sigma_{i=1}^{N} 1\,(\widehat{y}_i = j)\,1\,(y_i = j)\big) \,/\, \big((1/N)\Sigma_{i=1}^{N} 1\,(y_i = j)\big)$.

Online Appendix

A Collapsing the *GZIOPC* to the *ZIOPC* and *ZIOP*

Consider imposing the linear set of restrictions that $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_{J-1}$ and $\rho_1 = \rho_2 = \cdots = \rho_{J-1}$ on the *GZIOPC* model in expression (11). This yields

$$\begin{cases} \Pr(y=0\,|\mathbf{z},\mathbf{x}) = \begin{cases} \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}) + \sum_{j=1}^{J-2} \begin{bmatrix} \Phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta};\rho\right) - \\ \Phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}; -\mathbf{x}'\boldsymbol{\beta};\rho\right) \end{bmatrix}; \\ +\Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, -\mathbf{x}'\boldsymbol{\beta}; -\rho\right) \end{cases} \\ \Pr(y=j\,|\mathbf{z},\mathbf{x}) = \Phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) - \Phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right), \; j = 1,2,...,J\text{–}2; \\ \Pr(y=J\text{–}1\,|\mathbf{z},\mathbf{x}) = \Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, \mathbf{x}'\boldsymbol{\beta}; \rho\right) \end{cases} \tag{A.1}$$

where we note that while the expressions for $\Pr(y=j\,|\mathbf{z},\mathbf{x})$ and $\Pr(y=J\text{–}1\,|\mathbf{z},\mathbf{x})$ immediately collapse to those in expression (6), the $\Pr(y=0)$ expression in (A.1) can be constructed using 1 minus the sum of the $\Pr(y=J\text{–}1\,|\mathbf{z},\mathbf{x})$ and all $\Pr(y=j\,|\mathbf{z},\mathbf{x})$, $\forall\; j = 1,2,...,J\text{–}2$ terms to give

$$\Pr(y=0\,|\mathbf{z},\mathbf{x}) = [1 - \Phi(\mathbf{x}'\boldsymbol{\beta})] + \Phi_2(\mathbf{x}'\boldsymbol{\beta}, \mu_0 - \mathbf{z}'\boldsymbol{\gamma}; -\rho). \tag{A.2}$$

This also yields the result in (6), and is straightforward to verify. Using (A.1) and (A.2) yields

$$\Pr(y=0) = 1 - \overbrace{\sum_{j=1}^{J-2}\left[\Phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) - \Phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right)\right]}^{\Pr(y=j\ \forall\ j=1,2,...J-2)} - \overbrace{\Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, \mathbf{x}'\boldsymbol{\beta}; \rho\right)}^{\Pr(y=J-1)}, \tag{A.3}$$

which can be expanded as follows

$$\Pr(y=0) = 1 - \left\{ \begin{array}{c} [\Phi_2(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho) - \Phi_2(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho)] \\ + [\Phi_2(\mu_2 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho) - \Phi_2(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho)] \\ + [\Phi_2(\mu_3 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho) - \Phi_2(\mu_2 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho)] \\ \vdots \\ + \left[\Phi_2\left(\mu_{J-2} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) - \Phi_2\left(\mu_{J-3} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right)\right] \\ + \left[\Phi(\mathbf{x}'\boldsymbol{\beta}) - \Phi_2\left(\mu_{J-2} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right)\right] \end{array} \right\}. \tag{A.4}$$

After cancelling terms and algebraic manipulation, it can be verified that

$$\Pr(y=0) = [1 - \Phi(\mathbf{x}'\boldsymbol{\beta})] + \Phi_2(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho). \tag{A.5}$$

Substituting (A.5) into (A.1) results in *GZIOPC* probabilities that are identical to the *ZIOPC* probabilities in expression (5). That is, the *GZIOPC* collapses to—and therefore nests—the *ZIOPC*. Further, setting $\rho = 0$ in (A.5) yields probabilities that are identical to the *ZIOP* probabilities in expression (5), *viz.*

$$\Pr(y = 0) = [1 - \Phi(\mathbf{x}'\boldsymbol{\beta})] + \Phi(\mathbf{x}'\boldsymbol{\beta})\Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}). \tag{A.6}$$

The *GZIOPC* also collapses to the *ZIOP* if additionally $\rho_j = 0 \; \forall \; j = 1, 2, ..., J\text{–}1$. As noted, the sets of parameter restrictions described above provide tests of: (i) the more flexible functional form of the *GZIOPC* model versus the simpler nested forms of the usual *ZIOPC* and *ZIOP* models; and (ii) the *GZIOP* versus the *ZIOP* model.

B The *GMIOP* model

Building on the *ZIOP* model, two contributions, Bagozzi and Mukherjee (2012) and Brooks et al. (2012), independently suggested the *middle-inflated ordered probit* (*MIOP*) model to allow for inflation in an arbitrary middle category. In this section we develop a generalised framework for middle-inflation in the context of J outcomes. Consider a discrete variable y that assumes the discrete ordered values of $y \in \{0, 1, ..., J\text{–}1\}$. Bagozzi and Mukherjee (2012) and Brooks et al. (2012) assume that inflation in a middle category $y \in \{1, 2, ..., J\text{–}2\}$ arises due to the presence of two DGPs. We label the inflated middle category m, and define r^* as an underlying latent variable that represents an overall propensity to choose the inflated category m over any other, which translates into an 'observed' binary outcome given by

$$r^* = \mathbf{x}'\boldsymbol{\beta} + \varepsilon. \tag{B.1}$$

A propensity to choose m occurs when $r^* \leq 0$, which following the exposition of the *ZIOP* model we label regime $r = 0$; a propensity to choose any outcome other than m arises when $r^* > 0$, and is labelled regime $r = 1$. $\mathbf{x}$ is a k_x vector of covariates that determine the choice between the two regimes, $\boldsymbol{\beta}$ a vector of unknown coefficients, and ε a standard-normally distributed error term. A second latent variable $\widetilde{y}^*$ is given by

$$\widetilde{y}^* = \mathbf{z}'\boldsymbol{\gamma} + u \tag{B.2}$$

such that $\mathbf{z}$ is a k_z vector of explanatory variables with unknown weights $\boldsymbol{\gamma}$, and u is a standard normal error term. $\widetilde{y}$ assumes the form of a discrete ordered variable which can assume the values $\widetilde{y} \in \{0, 1, ..., J\text{–}1\}$. A two-regime scenario arises where for observations in regime $r = 0$, the inflated middle outcome is observed; but for those in $r = 1$, any of the possible outcomes in the choice set $j = \{0, 1, 2, ..., J\text{–}1\}$—*including the inflated category*

m—can be observed. Assuming that the normal error terms are correlated will yield the overall probabilities for the *MIOPC* model, which are given by

$$\Pr(y)=\begin{cases}\Pr\left(y=0\,|\mathbf{x},\mathbf{z}\right)=\Phi_2\left(\mu_0-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta};-\rho\right);\\ \Pr\left(y=j\,|\mathbf{x},\mathbf{z}\right)=\Phi_2\left(\mu_1-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta};-\rho\right)-\Phi_2\left(\mu_0-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta};-\rho\right)+M;\\ \Pr\left(y=J\text{–}1\,|\mathbf{x},\mathbf{z}\right)=\Phi_2\left(\mathbf{x}'\boldsymbol{\beta},\mathbf{z}'\boldsymbol{\gamma}-\mu_{J-2};\rho\right),\end{cases}\tag{B.3}$$

where $M=0$ if $y\neq m$ and $M=\Phi\left(-\mathbf{x}'\boldsymbol{\beta}\right)$ *iff* $y=m$. This implies that

$$\Pr\left(y=m\,|\mathbf{x},\mathbf{z}\right)=\Phi_2\left(\mu_1-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta};-\rho\right)-\Phi_2\left(\mu_0-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta};-\rho\right)+1-\Phi\left(\mathbf{x}'\boldsymbol{\beta}\right).\tag{B.4}$$

The probability of a single, middle category has therefore been inflated. Setting $\rho=0$ in (B.3) and (B.4) collapses the *MIOPC* to the *MIOP* model, which is characterised by independent errors.

We now generalise the inflation process for m. For any given propensity towards a given category $j\neq m$ in the outcome equation (B.2), it is possible to be pushed towards an inflated middle category, m. Let these propensities towards m be determined, respectively, by J–1 splitting equations, each corresponding to a non-inflated category, namely

$$r^*_{j\neq m}=\mathbf{x}'\boldsymbol{\beta}_j+\varepsilon_j\tag{B.5}$$

such that the probability of a movement towards the inflated middle category, m, is given by

$$\Pr(r_{j\neq m}=0)=\Phi\left(-\mathbf{x}'\boldsymbol{\beta}_j\right).\tag{B.6}$$

Generalising the *MIOPC* in this way yields the *generalised middle-inflated ordered probit* (*GMIOPC*) which has probabilities of the form

$$\Pr(y)=\begin{cases}\Pr\left(y=0\,|\mathbf{z},\mathbf{x}\right)=\Pr\left(\widetilde{y}=0|\mathbf{z}\right)\Pr\left(r_{\widetilde{j}}=1|\mathbf{x},\widetilde{y}=0\right);\\ \Pr\left(y=m\,|\mathbf{z},\mathbf{x}\right)=\begin{pmatrix}\Pr\left(\widetilde{y}=m|\mathbf{z}\right)\\ +\Pr\left(\widetilde{y}=\widetilde{j}|\mathbf{z}\right)\Pr\left(r_{\widetilde{j}}=0|\mathbf{x},\widetilde{y}=\widetilde{j}\right),\ \forall\,\widetilde{j}\\ +\Pr\left(\widetilde{y}=J\text{–}1|\mathbf{z}\right)\Pr\left(r_{J-1}=0|\mathbf{x},\widetilde{y}=J\text{–}1\right)\end{pmatrix};\\ \Pr\left(y=\widetilde{j}\,|\mathbf{z},\mathbf{x}\right)=\Pr\left(\widetilde{y}=\widetilde{j}|\mathbf{z}\right)\Pr\left(r_{\widetilde{j}}=1|\mathbf{x},\widetilde{y}=j\right),\ \forall\,\widetilde{j};\\ \Pr\left(y=J\text{–}1\,|\mathbf{z},\mathbf{x}\right)=\Pr\left(\widetilde{y}=J\text{–}1|\mathbf{z}\right)\Pr\left(r_j=1|\mathbf{x},\widetilde{y}=J\text{–}1\right).\end{cases}\tag{B.7}$$

where $\widetilde{j}$ excludes the first ($j=0$) and the last category ($j=J$–1), and includes all middle categories excluding the inflated one (i.e., where $j=m$). The associated observability criteria are given by

$$y=\widetilde{y}r_j=\begin{cases} 0 \text{ if } (\widetilde{y}^*\leq \mu_0 \text{ and } r_0^*>0); \\ m \text{ if } \left[\begin{array}{c} (\widetilde{y}^*\leq \mu_0 \text{ and } r_0^*\leq 0) \text{ or} \\ (\mu_{\widetilde{j}-1}<\widetilde{y}^*\leq \mu_{\widetilde{j}} \text{ and } r_{\widetilde{j}}^*\leq 0),\ \forall\, \widetilde{j} \text{ or} \\ (\mu_{J-2}<\widetilde{y}^* \text{ and } r_{J-1}^*\leq 0) \text{ or} \\ (\mu_{j-1}<\widetilde{y}^*\leq \mu_{j=m}) \end{array}\right] \\ \widetilde{j} \text{ if } (\mu_{\widetilde{j}-1}<\widetilde{y}^*\leq \mu_{\widetilde{j}} \text{ and } r_{\widetilde{j}}^*>0),\ \forall\, \widetilde{j}; \\ J\text{–}1 \text{ if } (\mu_{J-2}<\widetilde{y}^* \text{ and } r_{J-1}^*>0). \end{cases}; \tag{B.8}$$

We note here that by construction, the *GMIOPC* model must include splitting equations for the first and final category; in expression (B.8), these are denoted by the latent equations r_0^* and r_{J-1}^*, respectively. *GMIOPC* probabilities are given by

$$\Pr(y)=\begin{cases} \Pr(y=0\,|\mathbf{x},\mathbf{z})=\Phi_2(\mu_0-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta}_0;-\rho_0) \\ \Pr\left(y=\widetilde{j}\,|\mathbf{x},\mathbf{z}\right)=\Phi_2\left(\mu_{\widetilde{j}}-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta}_{\widetilde{j}};-\rho_{\widetilde{j}}\right)-\Phi_2\left(\mu_{\widetilde{j}-1}-\mathbf{z}'\boldsymbol{\gamma},\mathbf{x}'\boldsymbol{\beta}_{\widetilde{j}};-\rho_{\widetilde{j}}\right) \\ \Pr(y=m\,|\mathbf{x},\mathbf{z})=\begin{cases} \left[\Phi\left(\mu_m-\mathbf{z}'\boldsymbol{\gamma}\right)-\Phi\left(\mu_{m-1}-\mathbf{z}'\boldsymbol{\gamma}\right)\right] \\ +\underbrace{\Phi_2\left(\mu_0-\mathbf{z}'\boldsymbol{\gamma},-\mathbf{x}_i'\boldsymbol{\beta}_0;\rho_0\right)}_{a} \\ +\underbrace{\sum_{\widetilde{j}=1}^{J-2}\left[\begin{array}{c}\Phi_2\left(\mu_{\widetilde{j}}-\mathbf{z}'\boldsymbol{\gamma},-\mathbf{x}'\boldsymbol{\beta}_{\widetilde{j}};\rho_{\widetilde{j}}\right) \\ -\Phi_2\left(\mu_{\widetilde{j}-1}-\mathbf{z}'\boldsymbol{\gamma},-\mathbf{x}'\boldsymbol{\beta}_{\widetilde{j}};\rho_{\widetilde{j}}\right)\end{array}\right]}_{b} \\ +\underbrace{\Phi_2\left(\mathbf{z}'\boldsymbol{\gamma}-\mu_{J-2},-\mathbf{x}'\boldsymbol{\beta}_{J-1};-\rho_{J-1}\right)}_{c} \end{cases} \\ \Pr(y=J\text{–}1\,|\mathbf{x},\mathbf{z})=\Phi_2\left(\mathbf{z}'\boldsymbol{\gamma}-\mu_{J-2},\mathbf{x}'\boldsymbol{\beta}_{J-1};\rho_{J-1}\right) \end{cases} \tag{B.9}$$

where inflation in category m is allowed for by the additional terms of a, b and c in equation (B.9). The *GMIOPC* model is characterised by J–1 correlation coefficients $\rho_j\ \forall\ j\neq m$, which correspond to all categories apart from the middle-inflated one. Specifically, these encompass the categories at each end of the choice spectrum, for which we have ρ_0 and ρ_{J-1}; and all of the $\widetilde{j}$ non-inflated middle categories, namely $\rho_{\widetilde{j}}\ \forall\ \widetilde{j}$. Here, we note that setting $\rho_0=\rho_{\widetilde{j}}=\rho_{J-1}=0,\ \forall\ \widetilde{j}$, in (B.9) will yield the *generalised middle-inflated ordered probit* (*GMIOP*) model, which is characterised by independent error terms.

Consider imposing the linear set of restrictions that $\boldsymbol{\beta}_0=\boldsymbol{\beta}_{\widetilde{j}}=\boldsymbol{\beta}_{J-1}=\boldsymbol{\beta}$ and $\rho_0=\rho_{\widetilde{j}}=\rho_{J-1}=\rho$ on equation (B.9); setting $\boldsymbol{\beta}_0=\boldsymbol{\beta}_{\widetilde{j}}=\boldsymbol{\beta}_{J-1}=\boldsymbol{\beta}$ implies that the tempering propensities for all of the J–1 splitting equations are identical. This yields

$$\Pr(y) = \begin{cases} \Pr\left(y=0\,|\mathbf{x},\mathbf{z}\right) = \Phi_2(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho) \\ \Pr\left(y=\widetilde{j}\,|\mathbf{x},\mathbf{z}\right) = \Phi_2\left(\mu_{\widetilde{j}} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) - \Phi_2\left(\mu_{\widetilde{j}-1} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) \\ \Pr\left(y=m\,|\mathbf{x},\mathbf{z}\right) = \begin{cases} \left[\Phi\left(\mu_m - \mathbf{z}'\boldsymbol{\gamma}\right) - \Phi\left(\mu_{m-1} - \mathbf{z}'\boldsymbol{\gamma}\right)\right] \\ \quad + \underbrace{\Phi_2\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}; \rho\right)}_{a} \\ \underbrace{+\sum_{\widetilde{j}} \left[\begin{array}{l} \Phi_2\left(\mu_{\widetilde{j}} - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}; \rho\right) - \\ \Phi_2\left(\mu_{\widetilde{j}-1} - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}; \rho\right) \end{array} \right]}_{b} \\ \quad + \underbrace{\Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, -\mathbf{x}'\boldsymbol{\beta}; -\rho\right)}_{c} \end{cases} \\ \Pr\left(y=J\text{–}1\,|\mathbf{x},\mathbf{z}\right) = \Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, \mathbf{x}'\boldsymbol{\beta}; \rho\right) \end{cases} \tag{B.10}$$

where the expressions for $\Pr\left(y=0\,|\mathbf{z},\mathbf{x}\right)$, $\Pr\left(y=\widetilde{j}\,|\mathbf{z},\mathbf{x}\right)$ and $\Pr\left(y=J\text{–}1\,|\mathbf{z},\mathbf{x}\right)$ immediately collapse to those in the *MIOPC*, given in expression (B.3). The $\Pr\left(y=\widetilde{j}\,|\mathbf{z},\mathbf{x}\right)$ are equivalent to cases of $\Pr\left(y=j\,|\mathbf{z},\mathbf{x}\right)\ \forall j=1,2,...,J\text{–}2$ where $M=0$. Using (B.3), subtracting these terms from one yields

$$\Pr\left(y=m\,|\mathbf{x},\mathbf{z}\right) = \Phi_2\left(\mu_m - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) - \Phi_2\left(\mu_{m-1} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}; -\rho\right) + 1 - \Phi\left(\mathbf{x}'\boldsymbol{\beta}\right). \tag{B.11}$$

That is, the *GMIOPC* collapses to and therefore nests the *MIOPC*. As setting $\rho = 0$ in (B.11) yields probabilities that are identical to the *MIOP* model, it follows that the *GMIOPC* also collapses to the *MIOP* under the alternative set of parameter restrictions $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3... = \boldsymbol{\beta}_{J-1}$ and $\rho_j = 0\ \forall\ j = 0, \widetilde{j}, J\text{–}1$. Applying only the latter set of restrictions to the *GMIOPC* implicitly reduces it to the *GMIOP*. Equivalently, imposing the parameter restrictions $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_{\widetilde{j}} = \boldsymbol{\beta}_{J-1}$ on the *GMIOP* model leads it to nest the *MIOP*. As with the *GZIOP* model, the *GMIOP* is still an inflated ordered probit model. The ordering of outcomes is still preserved, and middle-inflation arises due to J–1 distinct DGPs, as opposed to just one. Further, as with the *GZIOP*, a straightforward test of hypotheses can be undertaken using *LR* or *LM* tests.

Testing the restrictions associated with these model variants therefore entails testing (i) the more flexible functional form of the *GMIOPC* model versus the simpler nested forms of the *MIOPC* and *MIOP* models and (ii) the *GMIOP* versus the *MIOP* model. As with the *GZIOP* model, the *GMIOP* is still an inflated ordered probit model. The ordering of outcomes is still preserved, middle-inflation arises due to J–1 distinct DGPs as opposed to just one, and all (latent) equations in the model are estimated simultaneously.

Diagrammatically, the *MIOP/MIOPC* is depicted in Panel A of Figure B.1. It comprises

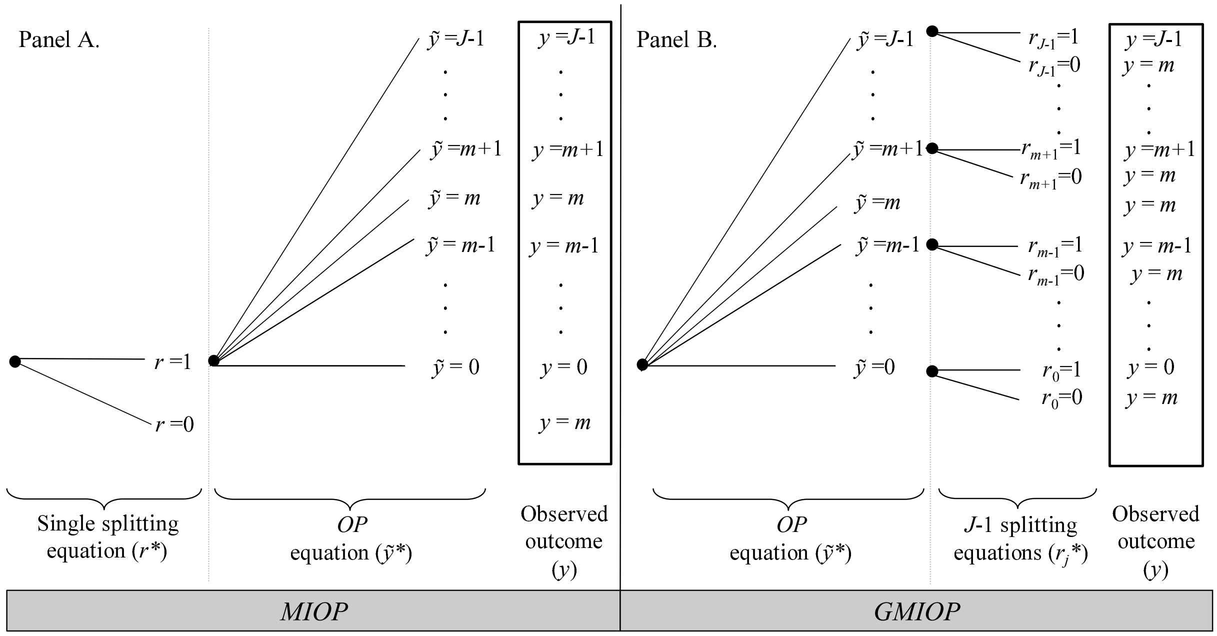

Figure B.1: The Middle-Inflated Ordered Probit (*MIOP*) model and its generalisation (*GMIOP*)

a single splitting equation and an *OP* model, both of which are unobserved. Here, m denotes an inflated middle category, which can assume any of the values in the set $j \in \{1, 2, ..., J–2\}$; the splitting equation now distinguishes between observational units in the inflated middle category ($r = 0$) and those in all other categories ($r = 1$). Here, the 'Observed outcome' column in Panel A shows that an observational unit for whom $r = 1$ and $\widetilde{y} = m$ will select outcome $y = m$; this outcome is also depicted for observational units in regime $r = 0$. The *GMIOP/GMIOPC* is illustrated in Panel B of Figure B.1: it shows that for any given propensity towards a particular category $j \neq m$ in the *OP* equation, it is possible to be pushed towards the inflated middle category m, due to the presence of J–1 splitting equations. For instance, the 'Observed outcome' column in Panel B shows that an observational unit for whom $\widetilde{y} = m$–1 and $r_{m-1} = 0$ will still select outcome $y = m$; this outcome is also shown as being realised for observational units in regime $\widetilde{y} = m$. As with the case of the *GZIOP*, all categories in the *OP* equation other than the one being inflated have a corresponding splitting equation.[14] Intuitively, the nature of our generalisations means that the model depicted on the right hand side of Figure B.1 can nest the non-generalised model depicted on the left hand side through imposing the linear parameter restrictions described above.

[14] In terms of the diagram in Panel B of Figure B.1, if the m–1^{th} category is equal to category $j = 0$ and the m+1^{th} category is equal to the J–1^{th} category, the model collapses to the three outcome case where $j = \{0, 1, 2\}$. This is precisely the set-up we consider in our empirical *GMIOP* application.

C Lagrange multiplier tests: Score vectors

C.1 Zero-inflated models

A highly appealing specification test for the *ZIOPC* models versus their generalised alternatives is the *LM* test, as this only requires estimation of the simpler nested models. This involves evaluation of the score vector of the more general model evaluated at parameter values under the null (*i.e.*, at *ZIOPC* ones). Here we present the score for the case of correlated errors. As noted above, the *GZIOPC* model of equation (11) can form the basis of an *LM* test of the *GZIOPC* versus the *ZIOP* and *ZIOPC* models. The former is tested using $H_0 : \boldsymbol{\beta}_j = \boldsymbol{\beta}$ and $\rho_j = 0, \forall j$ and the latter by $H_0 : \boldsymbol{\beta}_j = \boldsymbol{\beta}$ and $\rho_j = \rho, \forall j$.

Using the matrix version of the general result for bivariate normal distributions that

$$\frac{\partial \Phi_2 (a, b; \rho)}{\partial a} = \phi (a) \Phi \left(\frac{b - \rho a}{\sqrt{1-\rho^2}} \right), \tag{C.1}$$

where $\Phi_2 (a, b; \rho)$ denotes the standardised bivariate normal cumulative density function (CDF), we can define the following quantities of interest. First, define $\Phi^+_{b,j}$ as

$$\Phi^+_{b,j} = \Phi \left(\frac{(\mu_j - \mathbf{z}'\boldsymbol{\gamma}) - \rho_j (-\mathbf{x}'\boldsymbol{\beta}_j)}{\sqrt{1-\rho_j^2}} \right) - \Phi \left(\frac{(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}) - \rho_j (-\mathbf{x}'\boldsymbol{\beta}_j)}{\sqrt{1-\rho_j^2}} \right) \tag{C.2}$$

for $j = 1, \ldots, J$–2 and

$$\Phi^+_{b,J-1} = \Phi \left(\frac{(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}) - \rho_{J-1} (\mathbf{x}'\boldsymbol{\beta}_{J-1})}{\sqrt{1-\rho_{J-1}^2}} \right) \tag{C.3}$$

for $j = J$–1; and then $\Phi^-_{b,j}$ as

$$\Phi^-_{b,j} = \Phi \left(\frac{(\mu_j - \mathbf{z}'\boldsymbol{\gamma}) + \rho_j (\mathbf{x}'\boldsymbol{\beta}_j)}{\sqrt{1-\rho_j^2}} \right) - \Phi \left(\frac{(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}) + \rho_j (\mathbf{x}'\boldsymbol{\beta}_j)}{\sqrt{1-\rho_j^2}} \right) \tag{C.4}$$

for $j = 1, \ldots, J$–2 and

$$\Phi^-_{b,J-1} = \Phi \left(\frac{(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}) + \rho_{J-1} (-\mathbf{x}'\boldsymbol{\beta}_{J-1})}{\sqrt{1-\rho_{J-1}^2}} \right) \tag{C.5}$$

for $j = J$–1. Labelling the probabilities of the *GZIOPC* model P^{GZIOPC}, and using expressions (C.2) to (C.5), the score with respect to the elements of $\boldsymbol{\beta}$ can be written as

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}_j} = \left[\begin{array}{c} \sum\limits_{y_i=0} -\mathbf{x}\phi\left(-\mathbf{x}'\boldsymbol{\beta}_j\right)\Phi^{+}_{b,j} + \sum\limits_{y_i=0} -\mathbf{x}\phi\left(-\mathbf{x}'\boldsymbol{\beta}_{J-1}\right)\Phi^{-}_{b,J-1} + \\ \sum\limits_{y_i>0}^{y_i=J-2} \mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_j\right)\Phi^{-}_{b,j} + \sum\limits_{y_i=J-1} \mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_{J-1}\right)\Phi^{+}_{b,J-1} \end{array} \right] \div P^{GZIOPC}_{j=y_i} \tag{C.6}$$

for $\boldsymbol{\beta}_j$, $j = 1, \ldots, J$–1. Similarly, defining $\phi^{+}_{a,j}$ as

$$\phi^{+}_{a,j} = \phi\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right)\Phi\left(\frac{\left(-\mathbf{x}'\boldsymbol{\beta}_j\right) - \rho_j\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\rho_j^2}}\right) - \phi\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right)\Phi\left(\frac{\left(-\mathbf{x}'\boldsymbol{\beta}_j\right) - \rho_j\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\rho_j^2}}\right) \tag{C.7}$$

for $j = 1, \ldots, J$–2 and

$$\phi^{+}_{a,J-1} = \phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}\right)\Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_{J-1} - \rho_{J-1}\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}\right)}{\sqrt{1-\rho_{J-1}^2}}\right) \tag{C.8}$$

for $j = J$–1; and then $\phi^{-}_{a,j}$ as

$$\phi^{-}_{a,j} = \phi\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right)\Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_j + \rho_j\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\rho_j^2}}\right) - \phi\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right)\Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_j + \rho_j\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\rho_j^2}}\right) \tag{C.9}$$

for $j = 1, \ldots, J$–2 and

$$\phi^{-}_{a,J-1} = \phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}\right)\Phi\left(\frac{\left(-\mathbf{x}'\boldsymbol{\beta}_{J-1}\right) + \rho_{J-1}\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}\right)}{\sqrt{1-\rho_{J-1}^2}}\right) \tag{C.10}$$

for $j = J$–1 permits us to write the score with respect to $\boldsymbol{\gamma}$ as

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}} = \left[\begin{array}{c} \sum\limits_{y_i=0} \left[-\mathbf{z}\phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) + \sum\limits_{j=1}^{J-2} -\mathbf{z}\phi^{+}_{a,j} + \mathbf{z}\phi^{-}_{a,J-1} \right] + \\ \sum\limits_{y_i>0}^{y_i=J-2} \left[-\mathbf{z}\phi^{-}_{a,j}\right] \times 1\left[y_i = j\right] + \\ \sum\limits_{y_i=J-1} \mathbf{z}\phi^{+}_{a,J-1} \end{array} \right] \div P^{GZIOPC}_{j=y_i}. \tag{C.11}$$

The required ordering of the boundary parameters is ensured by specifying them as

$$\mu_j = \mu_{j-1} + \exp\left(\xi_j\right), \quad j = 1, 2, ..., J\text{–}2, \tag{C.12}$$

where μ_0 is freely estimated (Greene and Hensher 2010). The associated scores with respect to $\mu_0, \xi_1, \xi_2, \ldots \xi_{J-2}$ are given by,

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \mu_0} &= \left[\sum_{y_i=0} \phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) + \phi^+_{a,j} - \phi^-_{a,J-1}\right] \div P^{GZIOPC}_{j=0} \\
&\quad + \left[\sum_{y_i>0}^{y_i=J-2} \left[\phi^-_{a,j}\right] \times 1\left[y_i = j\right]\right] \div P^{GZIOPC}_{j=y_i} \\
&\quad - \left[\sum_{y_i=J-1} \phi^+_{a,J-1}\right] \div P^{GZIOPC}_{j=J-1}
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \xi_1} &= \left[\sum_{y_i=0} \left\{ \begin{array}{c} \sum_{j=1} \exp(\xi_1)\, \phi\left(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}\right) \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_1 + \rho_j(\mu_1 - \mathbf{z}'\boldsymbol{\gamma})}{\sqrt{1-\rho_1^2}}\right) + \\ \sum_{j=2}^{J-2} \exp(\xi_1)\, \phi^+_{a,j} - \exp(\xi_1)\, \phi^-_{a,J-1} \end{array} \right\}\right] \div P^{GZIOPC}_{j=0} \\
&\quad + \left[\sum_{y_i=1} \exp(\xi_1)\, \phi\left(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}\right) \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_j + \rho_j\left(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\rho_1^2}}\right)\right] \div P^{GZIOPC}_{j=1} \\
&\quad + \left[\sum_{y_i>1}^{y_i=J-2} \exp(\xi_1)\, \phi^-_{a,j}\right] \div P^{GZIOPC}_{j=y} + \left[\sum_{y_i=J-1} -\exp(\xi_1)\, \phi^+_{a,J-1}\right] \div P^{GZIOPC}_{j=J}
\end{aligned} \tag{C.14}$$

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \xi_2} &= \left[\sum_{y_i=0} \left\{ \begin{array}{c} \sum_{j=2} \exp(\xi_2)\, \phi\left(\mu_2 - \mathbf{z}'\boldsymbol{\gamma}\right) \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_2 + \rho_2(\mu_2 - \mathbf{z}'\boldsymbol{\gamma})}{\sqrt{1-\rho_2^2}}\right) \\ + \sum_{j=2}^{J-2} \exp(\xi_2)\, \phi^+_{a,j} - \exp(\xi_2)\, \phi^-_{a,J-1} \end{array} \right\}\right] \div P^{GZIOPC}_{j=y_i} \\
&\quad + \left[\sum_{y_i=2} \exp(\xi_2)\, \phi\left(\mu_2 - \mathbf{z}'\boldsymbol{\gamma}\right) \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_2 + \rho_2\left(\mu_2 - \mathbf{z}'\boldsymbol{\gamma}\right)}{\sqrt{1-\rho_2^2}}\right)\right] \div P^{GZIOPC}_{j=2} \\
&\quad + \left[\sum_{y_i>2}^{y_i=J-2} \exp(\xi_2)\, \phi^-_{a,j}\right] \div P^{GZIOPC}_{j=y} + \left[\sum_{y_i=J-1} -\exp(\xi_2)\, \phi^+_{a,J-1}\right] \div P^{GZIOPC}_{j=J-1}
\end{aligned} \tag{C.15}$$

$$\vdots$$

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \xi_{J-1}} = \left[\sum_{y_i=J-1} -\exp\left(\xi_{J-1}\right) \phi^+_{a,J-1}\right] \div P^{GZIOPC}_{j=J-1} \tag{C.16}$$

Finally, the derivatives of the elements of $\boldsymbol{\rho}$ $\forall j = 1, 2, ..., J–2$ are given by

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \rho_j} &= \left[\sum_{y_i=0} \left[\phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}_j; \rho_1\right) - \phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}_j; \rho_j\right)\right]\right] \div P^{GZIOPC}_{j=0} \\
&\quad + \left[\sum_{y_i=j} -\left[\phi_2\left(\mu_j - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}_j; -\rho_1\right) - \phi_2\left(\mu_{j-1} - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}_j; -\rho_j\right)\right]\right] \div P^{GZIOPC}_{j=y_i}
\end{aligned} \tag{C.17}$$

whereas for ρ_{J-1} we have

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \rho_{J-1}} = \left[\sum_{y_i=0} -\phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, -\mathbf{x}'\boldsymbol{\beta}_{J-1}; -\rho_{J-1}\right)\right] \div P_{j=0}^{GZIOPC} + \left[\sum_{J-1} \phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_{J-2}, \mathbf{x}'\boldsymbol{\beta}_{J-1}; \rho_{J-1}\right)\right] \div P_{j=J-1}^{GZIOPC} \tag{C.18}$$

In estimation we ensure a well-defined ρ_j, $j = 1, \ldots, J$–1, such that for $-1 < \rho_j < 1$ we use the hyperbolic tangent function transformation, $\rho_j = \tanh \rho_j^*$, where ρ_j^* is freely estimated. If such a transformation is followed, then the above derivatives for $\boldsymbol{\rho}$ need to be multiplied by $\partial \tanh \rho_j^* / \rho_j^* = 1 - \tanh^2 \rho_j^*$. Using all of the above quantities, the *LM* statistic is given by expression (14).

C.2 *MIOP* score vector

Assume that $J = 3$, and label the ordered choices as $j = 0, 1, 2$ (negative, indifferent, positive), where $j = 1$ is the hypothesised inflated category. Here the explicit form of the *GMIOPC* probabilities will be

$$\Pr(y_i) = \begin{cases} 0 = \Phi_2\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}, \mathbf{x}'\boldsymbol{\beta}_0; -\rho_0\right); \\ 1 = \begin{cases} \Phi\left(\mu_0 + \exp(\xi_1) - \mathbf{z}'\boldsymbol{\gamma}\right) - \Phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) \\ +\Phi_2\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}, -\mathbf{x}'\boldsymbol{\beta}_0; \rho_0\right) \\ +\Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1), -\mathbf{x}'\boldsymbol{\beta}_2; -\rho_2\right) \end{cases}; \\ 2 = \Phi_2\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1), \mathbf{x}'\boldsymbol{\beta}_2; \rho_2\right). \end{cases} \tag{C.19}$$

The score with respect to $\boldsymbol{\gamma}$ ($\boldsymbol{\nabla\gamma}$) will be

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}} = \left[\begin{array}{l} \sum_{y_i=0}\left[-\mathbf{z}\phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) \times \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_0 + \rho_0(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})}{\sqrt{1-\rho_0^2}}\right)\right] + \\ \sum_{y_i=1}\left[\begin{array}{c} \left(-\mathbf{z}\phi\left(\mu_0 + \exp(\xi_1) - \mathbf{z}'\boldsymbol{\gamma}\right) + \mathbf{z}\phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right)\right) + \\ -\mathbf{z}\phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) \times \Phi\left(\frac{(-\mathbf{x}'\boldsymbol{\beta}_0) - \rho_0(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})}{\sqrt{1-\rho_0^2}}\right) + \\ \mathbf{z}\phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1)\right) \times \Phi\left(\frac{(-\mathbf{x}'\boldsymbol{\beta}_2) + \rho_2(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1))}{\sqrt{1-\rho_2^2}}\right) \end{array}\right] + \\ \sum_{y_i=J-1}\left[\mathbf{z}\phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1)\right) \times \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_2 - \rho_2(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1))}{\sqrt{1-\rho_2^2}}\right)\right] \end{array}\right] \div P_{j=y_i}^{GMIOPC}. \tag{C.20}$$

And for the boundary parameters, $\nabla\mu_0, \nabla\xi_1$

$$\nabla\mu_0 = \begin{bmatrix} \sum_{y_i=0}\left[\phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) \times \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_0 + \rho_0(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})}{\sqrt{1-\rho_0^2}}\right)\right] + \\ \sum_{y_i=1}\begin{bmatrix} \phi\left(\mu_0 + \exp\left(\xi_1\right) - \mathbf{z}'\boldsymbol{\gamma}\right) - \phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) + \\ \phi\left(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}\right) \times \Phi\left(\frac{(-\mathbf{x}'\boldsymbol{\beta}_0) - \rho_0(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})}{\sqrt{1-\rho_0^2}}\right) + \\ \phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp\left(\xi_1\right)\right) \times \Phi\left(\frac{(-\mathbf{x}'\boldsymbol{\beta}_2) + \rho_2(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1))}{\sqrt{1-\rho_2^2}}\right) \end{bmatrix} + \\ \sum_{y_i=J-1}\left[\phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp\left(\xi_1\right)\right) \times \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_2 - \rho_2(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1))}{\sqrt{1-\rho_2^2}}\right)\right] \end{bmatrix} \div P_{j=y_i}^{GMIOPC}. \tag{C.21}$$

and

$$\nabla\xi_1 = \begin{bmatrix} \sum_{y_i=1}\begin{bmatrix} \exp\left(\xi_1\right)\phi\left(\mu_0 + \exp\left(\xi_1\right) - \mathbf{z}'\boldsymbol{\gamma}\right) + \\ \left(-\exp\left(\xi_1\right)\right)\phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp\left(\xi_1\right)\right) \times \Phi\left(\frac{(-\mathbf{x}'\boldsymbol{\beta}_2) + \rho_2(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1))}{\sqrt{1-\rho_2^2}}\right) \end{bmatrix} + \\ \sum_{y_i=J-1}\left[\left(-\exp\left(\xi_1\right)\right)\phi\left(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp\left(\xi_1\right)\right) \times \Phi\left(\frac{\mathbf{x}'\boldsymbol{\beta}_2 - \rho_2(\mathbf{z}'\boldsymbol{\gamma} - \mu_0 - \exp(\xi_1))}{\sqrt{1-\rho_2^2}}\right)\right] \end{bmatrix} \div P_{j=y_i}^{GMIOPC}. \tag{C.22}$$

The score with respect to $\boldsymbol{\beta}_0$ ($\boldsymbol{\nabla\beta_0}$) and $\boldsymbol{\beta}_2$ ($\boldsymbol{\nabla\beta_2}$) will respectively be

$$\boldsymbol{\nabla\beta_0} = \begin{bmatrix} \sum_{y_i=0}\left[\mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_0\right) \times \Phi\left(\frac{(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}) + \rho_0(\mathbf{x}'\boldsymbol{\beta}_0)}{\sqrt{1-\rho_0^2}}\right)\right] + \\ \sum_{y_i=1}\left[-\mathbf{x}\phi\left(-\mathbf{x}'\boldsymbol{\beta}_0\right) \times \Phi\left(\frac{(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}) - \rho_0(-\mathbf{x}'\boldsymbol{\beta}_0)}{\sqrt{1-\rho_0^2}}\right) + \right] \end{bmatrix} \div P_{j=y_i}^{GMIOPC} \tag{C.23}$$

and

$$\boldsymbol{\nabla\beta_1} = \begin{bmatrix} \sum_{y_i=1}\left[-\mathbf{x}\phi\left(-\mathbf{x}'\boldsymbol{\beta}_2\right) \times \Phi\left(\frac{(\mathbf{z}'\boldsymbol{\gamma} - \mu_1) + \rho_2(-\mathbf{x}'\boldsymbol{\beta}_2)}{\sqrt{1-\rho_2^2}}\right)\right] \\ \sum_{y_i=J-1}\left[\mathbf{x}\phi\left(-\mathbf{x}'\boldsymbol{\beta}_2\right) \times \Phi\left(\frac{(\mathbf{z}'\boldsymbol{\gamma} - \mu_1) - \rho_2(\mathbf{x}'\boldsymbol{\beta}_2)}{\sqrt{1-\rho_2^2}}\right)\right] \end{bmatrix} \div P_{j=y_i}^{GMIOPC} \tag{C.24}$$

Deriving the score vector for the *LM* test is again, straightforward. Define: P_j^{OP} as the standard *OP* probabilities implied by equation (3); P_j^{MIOP} as those for the *MIOP* model; P_j^{GMIOP} as those for the *GMIOP* model; and finally, P^0 as the splitting equation probability of $\Phi\left(\mathbf{x}'\boldsymbol{\beta}_0\right)$, P^{J-1} as the splitting equation probability of $\Phi\left(\mathbf{x}'\boldsymbol{\beta}_{J-1}\right)$, and $P^{\widetilde{j}}$ as the splitting equation probabilities of $\Phi\left(\mathbf{x}'\boldsymbol{\beta}_{\widetilde{j}}\right)$, where $\widetilde{j}$ captures all middle outcomes that are *not* inflated.

As with the case of the *GZIOP*, we maintain the necessary ordering of the boundary parameters by specifying them as $\mu_j = \mu_{j-1} + \exp\left(\xi_j\right)$, where μ_0 is freely estimated. The elements of the score vector are given by

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}} = \left[\begin{array}{c} \sum_{y_i=0} -\mathbf{z}\widetilde{\mu}_{-1}P^0 \\ +\sum_{y_i=1} \left(-\mathbf{z}\widetilde{\mu}_0 - \mathbf{z}\widetilde{\mu}_{-1}\left(1-P^0\right) + \mathbf{z}\widetilde{\mu}_1\left(1-P^2\right)\right) \\ +\sum_{y_i=2} \mathbf{z}\widetilde{\mu}_1 P^2 \end{array} \right] \div P_{j=y_i}^{GMIOP} \tag{C.25}$$

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \mu_0} = & \left[\sum_{y_i=0} \widetilde{\mu}_{-1}P^0\right] \div P_{j=0}^{GMIOP} + \\ & \left[\sum_{y_i=1} \widetilde{\mu}_0 + \widetilde{\mu}_0\left(1-P^0\right) - \widetilde{\mu}_1\left(1-P^2\right)\right] \div P_{j=0}^{GMIOP} + \\ & \left[\sum_{y_i=1} -\widetilde{\mu}_1 P^2\right] \div P_{j=2}^{GMIOP} \end{aligned} \tag{C.26}$$

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \xi} = & \left[\sum_{y_i=1} \exp(\xi)\widetilde{\mu}_1 - \exp(\xi)\widetilde{\mu}_1\left(1-P^2\right)\right] \div P_{j=1}^{GMIOP} + \\ & \left[\sum_{y_i=2} -\exp(\xi)\widetilde{\mu}_1 P^2\right] \div P_{j=2}^{GMIOP} \end{aligned} \tag{C.27}$$

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}_0} = & \left[\sum_{y_i=0} \mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_0\right) P_{j=0}^{OP}\right] \div P_{j=0}^{GMIOP} + \\ & \left[\sum_{y_i=1} -\mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_0\right) \times P_{j=0}^{OP}\right] \div P_{j=1}^{GMIOP} \end{aligned} \tag{C.28}$$

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}_2} = & \left[\sum_{y_i=1} -\mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_0\right) P_{j=2}^{OP}\right] \div P_{j=1}^{GMIOP} + \\ & \left[\sum_{y_i=2} \mathbf{x}\phi\left(\mathbf{x}'\boldsymbol{\beta}_2\right) \times P_{j=2}^{OP}\right] \div P_{j=2}^{GMIOP} \end{aligned} \tag{C.29}$$

Finally, as with the *GZIOP*, in estimation we ensure a well-defined ρ_j, $j = 1, \ldots, J$–1, such that $\rho_j \in (-1, 1)$ where we use the hyperbolic tangent function transformation, $\rho_j = \tanh \rho_j^*$, where ρ_j^* is freely estimated. Following such a transformation the above derivatives for $\boldsymbol{\rho}$ require multiplication by $\partial \tanh \rho_j^* / \rho_j^* = 1 - \tanh^2 \rho_j^*$.

D Model coherency and identification

It is important to ascertain whether the proposed discrete choice model generalisations are 'coherent' or 'logically consistent' (see, for instance, Maddala 1983). This entails ensuring that the model's parameters are uniquely identified and the associated probabilities are well-defined and sum to unity. We demonstrate this using the *GZIOP* model with uncorrelated errors. We also demonstrate that the *generalised ordered probit* (*GOP*) model of Terza (1985), which is arguably characterised by incoherency (Greene, Harris, Hollingsworth, and Weterings 2014), neither nests, nor is nested by our proposed generalizations.

D.1 Unique identification

Ensuring that the model parameters are uniquely identified is akin to ensuring that the model cannot generate more than one value of y simultaneously. In this respect, if one can simulate the dependent variable, then this suggests that the model is, indeed, coherent (implying that the parameters are uniquely identified). Here, consider simulating along the lines of the models described above:

1. Consider the $\widetilde{y}^* = \mathbf{z}'\boldsymbol{\gamma} + u$ equation. With known $\boldsymbol{\gamma}$ and boundary parameters $\boldsymbol{\mu}$, 'first stage' $\widetilde{y}$ values can be straightforwardly simulated by simply simulating u from an assumed $N(0,1)$ distribution by the usual relationship between the simulated $\widetilde{y}^*$ and $\boldsymbol{\mu}$.

2. This uniquely places an individual in one, and only one, of the $j = 0, 1, ..., J–1$ $\widetilde{y}$ outcomes.

3. Individuals in the $\widetilde{y} = 0$ category are allocated to the observed $y = 0$.

4. For individuals falling uniquely into the $\widetilde{y} = 1$ category one can simulate their observed outcome by consideration of $r^*_{j=1} = \mathbf{x}'\boldsymbol{\beta}_{j=1} + \varepsilon_{j=1}$:

   (a) With known $\boldsymbol{\beta}_{j=1}$ it is straightforward to simulate $r^*_{j=1}$ by simulating $\varepsilon_{j=1}$, again from an assumed $N(0,1)$ distribution.

   (b) The position of the simulated index $r^*_{j=1}$ with respect to 0, uniquely simulates $r_{j=1}$; $r_{j=1} = 1\left(r^*_{j=1} > 0\right)$, where 1(.) is an indicator function taking the value 1 if the condition inside the parentheses is true, and 0 otherwise.

   (c) With $\widetilde{y} = 1$ and $r_{j=1}$ in hand, $y_{j=1}$ is uniquely determined by the observability criteria defined above, here explicitly, $y_{j=1} = \widetilde{y}r_{j=1}$.

5. Similarly, for all individuals uniquely falling into the $\widetilde{y} = 2$ category, observed $y_{j=2} = \widetilde{y}r_{j=2}$, with $r_{j=2}$ being determined as above by $1\left(r^*_{j=2} > 0\right)$.

6. And so on, for all other $j \geq 3$.

7. Equivalently, y_0 can be also be simulated as

$$1(\mathbf{z}'\boldsymbol{\gamma} + u < \mu_0) + \sum_{j=1}^{J-1} 1\left([\mu_{j-1} < \mathbf{z}'\boldsymbol{\gamma} + u < \mu_j]\ [\mathbf{x}'\boldsymbol{\beta}_j + \varepsilon_j < 0]\right)$$

with the usual convention of $\mu_{J-1} = \infty$. As all components of this are mutually exclusive, this uniquely maps onto a single value for all observed y (similar expressions apply for the remaining j).

Thus although there is nothing in the model to prevent the 'existence' of several of the r_j variables 'being equal to one', apart from the one corresponding to the uniquely determined $\widetilde{y}_j$ value, all others are redundant. The reason for this follows from the more general latent class models (of which our approach is a special case). Individuals can only be in any one particular class at a given point in time, therefore behaviour in any other class simply does not exist. In this way our approach is consistent with that of the standard latent class approach.

D.2 Well-defined probabilities

We now explore if our proposed models have well-defined probabilities. It is straightforward to show that model probabilities all lie within the unit circle and sum to unity. Consider the *GZIOP* with $J = 3$:

$$\begin{aligned}
P_0 &= \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}) + [\Phi(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}) - \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})]\,\Phi(-\mathbf{x}'\boldsymbol{\beta}_1) + \Phi(\mathbf{z}'\boldsymbol{\gamma} - \mu_1)\,\Phi(-\mathbf{x}'\boldsymbol{\beta}_2) \\
P_1 &= [\Phi(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}) - \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})]\,\Phi(\mathbf{x}'\boldsymbol{\beta}_1) \\
P_2 &= \Phi(\mathbf{z}'\boldsymbol{\gamma} - \mu_1)\,\Phi(\mathbf{x}'\boldsymbol{\beta}_2)
\end{aligned}$$

So $\Sigma_j P_j$ is given by

$$\sum_j P_j = \begin{array}{c} \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma}) + [\Phi(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}) - \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})]\,\Phi(-\mathbf{x}'\boldsymbol{\beta}_1) + \Phi(\mathbf{z}'\boldsymbol{\gamma} - \mu_1)\,\Phi(-\mathbf{x}'\boldsymbol{\beta}_2) \\ + [\Phi(\mu_1 - \mathbf{z}'\boldsymbol{\gamma}) - \Phi(\mu_0 - \mathbf{z}'\boldsymbol{\gamma})]\,\Phi(\mathbf{x}'\boldsymbol{\beta}_1) + \Phi(\mathbf{z}'\boldsymbol{\gamma} - \mu_1)\,\Phi(\mathbf{x}'\boldsymbol{\beta}_2)\,. \end{array} \tag{D.1}$$

Manipulating all of the terms in (D.1), it is straighforward to demonstrate that $\Sigma_j P_j = 1$. Finally, it is evident that all individual outcome probabilities must lie in the unit circle. They are all composed of positive components, or sums of positive components (due to the $\Phi(.)$ transformation) and therefore are all positive. As the sum across j has been shown to sum to unity, the individual outcome probabilities are in the $(0, 1)$ space, and are therefore well-defined.

D.3 Comparison with the *generalised ordered probit* (*GOP*) model

The literature on discrete choice is characterised by a number of contributions which propose generalisations of the ordered probit model. A well-known and popular approach is found in the *generalised ordered probit* (*GOP*) model of Terza (1985), in which the threshold parameters are allowed to vary. Greene, Harris, Hollingsworth, and Weterings (2014) argue that because the ordering of the thresholds is not enforced in this model, the predicted probabilities can lie outside the range of zero and one. As demonstrated above, our proposed generalisations do not suffer from this form of incoherency. Here we show that our proposed extensions to the *ZIOP* and *MIOP* models, and their generalisations are not re-parameterisations of the *GOP* model. We now discount this possibility using the example of the *GZIOP* model.

In its most usual form, the boundary parameters in a *GOP* model would be specified as

$$\begin{aligned} \mu_{i0} &= \mathbf{x}_i' \boldsymbol{\delta}_0 \\ \mu_{i1} &= \mu_{i0} + \exp\left(\mathbf{x}_i' \boldsymbol{\delta}_1\right) \\ &\vdots \end{aligned}$$

so that P_0 in a *GOP* would be

$$P_{i0} = \Phi\left(\mathbf{x}_i' \boldsymbol{\delta}_0 - \mathbf{z}_i' \boldsymbol{\gamma}\right)$$

so compared to the same for the *GZIOP* means that equivalence would imply that

$$\begin{aligned} \Phi\left(\mathbf{x}_i' \boldsymbol{\delta}_0 - \mathbf{z}_i' \boldsymbol{\gamma}\right) &= \Phi\left(\mu_0 - \mathbf{z}' \boldsymbol{\gamma}\right) + \\ &\quad \left[\Phi\left(\mu_1 - \mathbf{z}' \boldsymbol{\gamma}\right) - \Phi\left(\mu_0 - \mathbf{z}' \boldsymbol{\gamma}\right)\right] \Phi\left(-\mathbf{x}' \boldsymbol{\beta}_1\right) + \\ &\quad \Phi\left(\mathbf{z}' \boldsymbol{\gamma} - \mu_1\right) \Phi\left(-\mathbf{x}' \boldsymbol{\beta}_2\right). \end{aligned}$$

There are no obvious restrictions under which this condition would hold. On this basis, the proposed new models are evidently not simple re-parameterisations of the *GOP* model.

References

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Maddala, G. S. (1983). *Limited Dependent and Qualitative Variables in Econometrics.* Cambridge, U.K.: Cambridge University Press.

Terza, J. V. (1985). Ordered probit: A generalization. *Communications in Statistics – Theory and Methods* 14(1), 1–11.

E *MIOP* empirical application

E.1 Data

Bagozzi and Mukherjee (2012) use a *MIOP* framework to analyze individual responses in a data set that explores respondents' attitudes towards European Union (EU) membership in EU accession countries. When asked about their attitudes towards joining the EU, respondents choose from one of three alternatives: *a bad thing*; *neither good nor bad*; or *a good thing*. The associated response frequencies for these are 10.83%, 33.07% and 56.10%, respectively.

The authors hypothesise that the middle category contains responses from two distinct sources: 'informed' respondents with good knowledge of the impact of EU membership; and 'uninformed' respondents, who select *neither good nor bad* as a 'face-saving measure'.[15] The splitting equation covariates capture if a respondent is knowledgeable about the EU and its impact, wheras the outcome (OP) equation contains variables standard in the political science literature used to measure EU membership support. This is in addition to the inclusion of standard controls capturing socio-economic and personal characteristics. The specification shares 8 common variables in the two equations, and is characterised by: $N = 9,113$; $J = 3$; $k_x = 12$; and $k_z = 16$. See Bagozzi and Mukherjee (2012) for a full description of the variable set.

In this three categorical outcome application, our generalisation requires that two splitting equations—which capture the extent to which the respondent is pushed towards the middle-inflated option of *neither good nor bad*—require estimation. Prior to estimation, Monte Carlo experiments are used to investigate the performance of our specification test.

E.2 Finite sample performance

To ascertain the finite sample performance of our specification tests, we consider a range of Monte Carlo (MC) experiments. As with the case of the *ZIOP* model, these experiments are based on the original *MIOP* data application data, and unless stated, are based on the full sample sizes reported above. The number of repetitions was set to $2,000$, and all simulation 'noise' had effectively settled after $1,000$ repetitions. Table E.1 presents our findings, which include results relating to empirical size and quasi-power. The first column identifies the true DGP and the respective degrees of freedom for each test (df). For each DGP, three tests—each between a generalised model and a null, non-generalised variant—are performed.

Results for three *MIOP* experiments and tests are presented: *GMIOP vs. MIOP*;

[15] A fourth '*do not know*' category is treated as being a *neither good nor bad* response by Bagozzi and Mukherjee (2012). We emphasise the hypothesis driven nature of category inflation in this application: in keeping with the discussion in Section II the inflated category is not characterised by an excess of middle category observations relative to other categories.

Table E.1: Monte Carlo rejection probabilities

True model	Rejection probability		
	GMIOP *vs.* *MIOP*	*GMIOPC* *vs.* *MIOPC*	*GMIOPC* *vs.* *MIOP*
1.*MIOP*($df = 12, 13, 14$)	0.057	0.061	0.062
2.*MIOPC*($df = 12, 13, 14$)	0.181	0.061	0.241
3.*MIOP*($df = 7, 8, 9$)	0.056	0.055	0.053
4.*MIOP*($df= 12, 13, 14$)	0.076	0.077	0.0795
5.*OP*($df = 12, 13, 14$)	0.484	0.788	0.657
6.*Parallel*($df = 12, 13, 14$)	0.253	0.677	0.503

GMIOPC vs. MIOPC; and *GMIOPC vs. MIOP*. Row 1 corresponds to a *MIOP* DGP and has $J = 3$. Here, all empirical sizes are very close to nominal ones. Row 2 considers a *MIOPC* DGP. At 6%, empirical size is again very close to the nominal one. These tests have reasonable 'power' at picking-up the mis-specified uncorrelated model, with rejection probabilities of around 18% and 24%. The effect of reducing the *df* is explored here in row 3, where the *MIOP* is re-estimated and all statistically insignificant variables are removed. This respectively yields $df = 7, 8, 9$; again, all tests are correctly sized.

As the tests are asymptotic, the implications for their properties of estimating using a much smaller sample are also explored in row 4. This is achieved by taking the (already) relatively small sample in the *MIOP* example and randomly removing 50% of the observations, yielding $N = 4,556$. The re-sized sample marginally worsens the performance of the tests, with all of them being slightly over-sized at around 7%–8%. Finally, *quasi*-power experiments were once again performed by generating under an *OP* model and a parallel regressions framework (rows 5 and 6, respectively). Again, all tests behave exceptionally well as general specification ones, as indicated by high rejection probabilities of up to nearly 80% in some instances. In summary, for both the zero-inflated and middle inflated experiments, all *LM* tests appear correctly sized, and typically have good 'power' in identifying mis-specified models.

Using the covariate data we also conducted genuine experiments based on the null model of *MIOP*. In all experiments we take the estimated value of $\boldsymbol{\beta}$ in each null model, setting $\boldsymbol{\beta}_j = \boldsymbol{\beta} \; \forall \; j$ in the corresponding generalised set-up, and perturb a single parameter β_0 in a single splitting equation by successively larger increments. For brevity, we only report power runs for the non-correlated DGPs. The *MIOP* power curves are presented in Figure E.1, and cover experiments performed using alternative *df* and sample size as described below.

For the *MIOP* experiments, which all have $J = 3$ categories, we initially focus on two experiments that use the full sample but which are differentiated by one experiment dropping insignificant variables from the splitting equations. This has the impact of reducing the *df* from $df = 12$ to $df = 7$. The difference in the *df* has no discernible effect on power and is

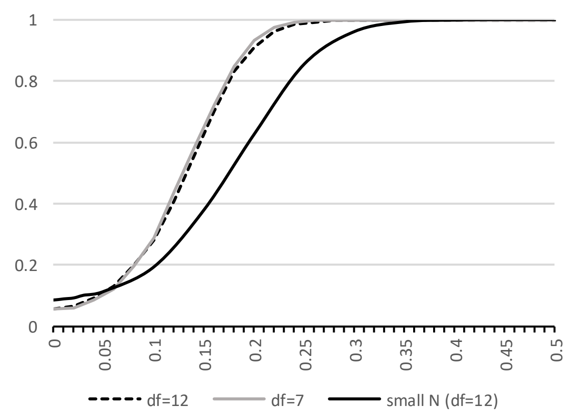

Figure E.1: Empirical power curves for the *MIOP* model

arguably to be expected given the nature of our perturbations. Specifically, in each *MIOP* experiment the single perturbed parameter differs from only a single estimated parameter. This is unlike the *ZIOP* experiments, where each experiment is distinguished by a different number of splitting equations. In the *df*= 26 experiment, there are three such equations, and the single perturbed parameter thus differs from two single estimated parameters. However, in the *df*= 13 experiment, the presence of only two splitting equations means that the single perturbed parameter differs from only a single estimated parameter. One might have anticipated greater differences in power gains here, given the large difference in *df* in the *ZIOP* experiments. When the *df* is smaller, model failure associated with a single parameter could be interpreted as being more severe.[16]

For reasons noted above we also conducted experiments with a relatively small sample size (small N, *df*= 12) under the null of *MIOP*. With the reduced sample, a reduction in power is observed relative to other *MIOP* experiments, in that relatively larger parameter perturbations are required to lead to model rejection. Despite the relative reduction in power, we note that all *MIOP* tests have the 'usual' shaped power curves, and like the *ZIOP* experiments, exhibit good power.

E.3 Estimation

Table E.2 presents the *LM* and *LR* test results for our middle inflated application. For both tests, the *MIOP* model is rejected in favour of the *GMIOP* and *GMIOPC*, and the *GMIOP* is rejected in favour of the *GMIOPC*. The *LR* and *LM* tests are generally similar. However, unlike the zero-inflated application, the non-generalised models are not unanimously rejected by both tests in favour of their corresponding generalised variants at 5% significance levels.

[16] Although not reported here, significant power gains also occurred in cases where (i) a full, single vector was perturbed and (ii) all vectors were perturbed. Both of these alternative scenarios showed comparatively higher power compared to the single-parameter experiment. This is because the single parameter experiment represents the scenario where the test is most likely not to perform well, as it is closest to the null.

Table E.2: Specification test results: competing middle-inflated models

Model	*LM* Test statistic	*df*	*p*-value	*LR* Test statistic	*p*-value
MIOP vs. GMIOP	32.1	12	0.001	39.3	0.000
MIOPC vs. GMIOPC	20.4	13	0.086	26.2	0.016
MIOP vs. GMIOPC	37.0	14	0.001	46.0	0.000
GMIOP vs. GMIOPC	9.5	2	0.009	6.7	0.035

Specifically, the *LM* test of the *MIOPC* versus the *GMIOPC* fails to reject the former at the 5% level, although it does reject at the 10% level. In contrast, all LR tests reject at the 5% significance levels.

The *GMIOP* approach remains consistent with accounting for the presence of 'face saving' behaviour. Whereas 'uninformed' respondents are still likely to have preferences across the full range of alternatives in the latent *OP* equation, the impact of the splitting equations is to push such individuals towards the face-saving '*neither good nor bad*' outcome. In contrast, 'informed' respondents with propensities to select '*a bad thing*' or '*a good thing*' in the *OP* equation will be unlikely to move from these choices.[17] The *GMIOP* allows the potency of these effects to differ across the splitting equations corresponding to these respective choices.

The overall partial effects for the *MIOPC* and *GMIOPC* models are given in Table E.3. The reported effects across all specifications are similar, being comparable in magnitude, direction of effect and statistical significance levels. There are a few exceptions to this. For example, higher education-level effects appear more pronounced in the *GMIOPC* model for outcomes $j = 1, 2$ whereas the effects of EU-bid knowledge ($j = 1, 2$) are comparatively stronger in the *MIOPC* model. These results align with the estimated model parameters which indicate that the face-saving effects in the *GMIOPC* model derive from the $j = 2$ splitting equation.[18] Specifically, there are no significant drivers of face-saving behaviour in the $j = 0$ splitting equation. This suggests the possible presence of an 'asymmetry' with respect to the source of the middle-inflation, in that the *GMIOPC* can be viewed as being characterised by having only a single statistically significant splitting equation. This may account for why the *LM* test for the *MIOPC* model—which by construction has a single splitting equation—was not rejected at the 5% significance level. From a policymaking perspective, this result suggests that if individuals are more informed about the nature and role of the EU and its institutions, the share of responses in favour of EU membership would be even greater; in contrast, the share of respondents selecting '*a bad thing*' would remain unchanged, and increasing the extent to which such individuals are informed will not affect

[17] Here, we note that the latent *OP* equation does not explicitly distinguish between informed and uninformed respondents; it is the splitting equations that contain proxies capturing the extent to which respondents are informed, which in turn determines the extent to which individuals are pushed towards the '*neither good nor bad*' outcome.

[18] Results available from the authors on request.

their views.[19] Despite there being very little to choose between with respect to the *GMIOPC* and the *MIOPC* models, there is a benefit to estimating the former model in that it helps to uncover asymmetries which the single-equation splitting equation of the *MIOPC* may mask.

Model summary probabilities are given in Table E.4. The overall probabilities associated with the underlying *OP* component of each model are again calculated alongside the corresponding probabilities 'purged' of inflation effects. As was the case under zero inflation, for the correlated versions the implied independent *OP* is used in these calculations. Once more, the difference between the overall $j = 1$ probabilities and these purged ones, are denoted *Amount* (Middle-inflation), which can be interpreted as the amount of middle category inflation due to face-saving behaviour.

Turning to the *Amount*(%) statistic, of the total responses to the *neither good nor bad* outcome, some 33% of these can be attributed to face-saving responses for the *MIOP* model, a figure that rises to around 53% for the *GMIOP* model. These percentages rise for the correlated versions, to 43% and 54%, respectively. As with the tobacco consumption application, the extent of overall model inflation in the non-generalised models is underestimated relative to the generalised models. Once again, the results for the hit-and-miss analysis (Table E.5) suggest that the generalised models out-perform the non-generalised variants. Regarding our middle-inflated application, when the response options in survey questions range from feeling negative to feeling positive about an issue, middle categories are often used to capture feelings of neutrality or indifference. As noted earlier, because such questions are commonplace in questionnaires, there is considerable scope for the analysis of such data using a *GMIOP* framework.

[19] Accordingly, whilst policies aimed at better informing and educating people about the EU may improve individuals' reported opinions of it, alternative policies may be required to persuade individuals who select '*a bad thing*' that EU membership is beneficial; our findings suggest that for such individuals, increasing the extent to which they are informed may not change their mind. This type of finding may have implications for public policies designed to mitigate the rising populism and anti-EU sentiment witnessed across the EU in recent years.

Table E.3: Overall partial effects for the *MIOPC* and *GMIOPC* models

	MIOPC			*GMIOPC*		
Common variables	$j=0$	$j=1$	$j=2$	$j=0$	$j=1$	$j=2$
Rural	−0.005	0.012	−0.007	−0.006	0.014	−0.008
	(0.004)	(0.006)**	(0.006)	(0.004)	(0.006)**	(0.007)
Female	−0.015	0.052	−0.036	−0.017	0.054	−0.037
	(0.006)**	(0.01)***	(0.011)***	(0.006)***	(0.01)***	(0.011)***
Age	$8.8e-05$	0.001	−0.001	$1.6e-04$	0.001	−0.001
	$(1.9e-04)$	$(2.8e-04)$***	$(3.6e-04)$***	$(2.1e-04)$	$(3.8e-04)$***	$(4.3e-04)$***
Student	−0.028	0.035	−0.007	−0.025	0.032	−0.007
	(0.014)*	(0.023)	(0.027)	(0.022)	(0.034)	(0.03)
Educ high	−0.017	0.023	−0.006	−0.025	0.042	−0.018
	(0.011)	(0.02)	(0.022)	(0.013)*	(0.024)*	(0.024)
Educ high-mid	−0.01	0.081	−0.071	−0.016	0.099	−0.083
	(0.013)	(0.019)***	(0.023)***	(0.015)	(0.026)***	(0.026)***
Educ low-mid	−0.005	0.083	−0.079	−0.011	0.105	−0.094
	(0.009)	(0.014)***	(0.016)***	(0.011)	(0.018)***	(0.018)***
Discuss politics	0.005	−0.033	0.028	0.008	−0.037	0.03
	(0.004)	(0.007)***	(0.008)***	(0.005)	(0.008)***	(0.008)***
Outcome equation only variables						
Political trust	−0.142	−0.144	0.285	−0.137	−0.162	0.299
	(0.008)***	(0.011)***	(0.016)***	(0.009)***	(0.017)***	(0.018)***
Xenophobia	0.088	0.09	−0.178	0.089	0.105	−0.194
	(0.009)***	(0.01)***	(0.018)***	(0.01)***	(0.012)***	(0.019)***
Professional	0.015	0.015	−0.03	0.012	0.015	−0.027
	(0.013)	(0.012)	(0.025)	(0.012)	(0.014)	(0.026)
Executive	−0.019	−0.02	0.039	−0.017	−0.02	0.037
	(0.016)	(0.016)	(0.032)	(0.015)	(0.018)	(0.033)
Manual	0.021	0.021	−0.042	0.020	0.024	−0.044
	(0.007)***	(0.008)***	(0.015)***	(0.007)***	(0.009)***	(0.016)***
Farmer	0.007	0.007	−0.015	0.009	0.011	−0.02
	(0.015)	(0.016)	(0.031)	(0.016)	(0.018)	(0.033)
Unemployed	−0.018	−0.018	0.036	−0.017	−0.02	0.037
	(0.009)**	(0.009)**	(0.017)**	(0.009)**	(0.01)**	(0.019)**
Income	−0.011	−0.011	0.023	−0.011	−0.013	0.024
	(0.001)***	(0.001)***	(0.002)***	(0.001)***	(0.002)***	(0.002)***
Splitting equation only variables						
EU-bid knowledge	$4.6e-05$	−0.081	0.081	0.006	−0.071	0.065
	$(1.3e-04)$	(0.017)***	(0.017)***	(0.013)	(0.018)***	(0.016)***
True EU knowledge	$1.5e-05$	−0.025	0.025	−0.001	−0.022	0.024
	$(3.9e-05)$	(0.003)***	(0.003)***	(0.002)	(0.003)***	(0.003)***
Media	$5.2e-06$	−0.009	0.009	−0.005	−0.005	0.011
	$(1.5e-05)$	(0.005)*	(0.005)*	(0.003)	(0.005)	(0.005)**

[a]Standard errors in parentheses.***, ** and * denote significance at 1%, 5%, and 10%, respectively.

Table E.4: Summary probabilities from the *MIOP* and *GMIOP* models; EU data

		Independent errors				Correlated errors			
		Overall		Purged		Overall		Purged	
Outcome	Sample	*MIOP*	*GMIOP*	*MIOP*	*GMIOP*	*MIOPC*	*GMIOPC*	*MIOPC*	*GMIOPC*
$j = 0$	0.108	0.108	0.108	0.128	0.189	0.108	0.108	0.109	0.145
		(0.003)***	(0.003)***	(0.004)***	(0.028)***	(0.003)***	(0.003)***	(0.003)***	(0.015)***
$j = 1$	0.331	0.331	0.331	0.222	0.155	0.331	0.331	0.190	0.153
		(0.005)***	(0.005)***	(0.015)***	(0.040)***	(0.005)***	(0.005)***	(0.018)***	(0.021)***
$j = 2$	0.561	0.561	0.561	0.650	0.656	0.561	0.561	0.701	0.702
		(0.005)***	(0.005)***	(0.013)***	(0.022)***	(0.005)***	(0.005)***	(0.018)***	(0.021)***
		MIOP	*GMIOP*			*MIOPC*	*GMIOPC*		
Amount (Middle-inflation)		0.109	0.176			0.141	0.176		
		(0.014)***	(0.040)***			(0.018)***	(0.021)***		
Amount(%)		32.83%	53.14%			42.59%	53.67%		

[a]Standard errors in parentheses.***, ** and * denote significance at 1%, 5%, and 10%, respectively.

Table E.5: In-sample and out-of-sample hit-and-miss tables for *MIOP* applications[a]

	Predicted ($\widehat{y}_i$): In-sample															
Specification	*OP*			*MIOP*			*MIOPC*			*GMIOP*			*GMIOPC*			
Actual (y_i)	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	Total
0	2	318	667	14	290	683	10	307	670	6	328	653	8	311	668	987
1	2	640	2372	18	803	2193	9	832	2173	5	916	2093	6	860	2148	3014
2	2	470	4640	11	544	4557	3	574	4535	3	633	4476	4	589	4519	5112
Total	6	1428	7679	43	1637	7433	22	1713	7378	14	1877	7222	18	1760	7335	9113
CP	0.5796			0.5897			0.5900			0.5923			0.5911			
*CP**	0.0610			0.0860			0.0867			0.0928			0.0887			
	Predicted ($\widehat{y}_i$): Out-of-sample – 10% 'hold-out' sample															
Actual (y_i)	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**	Total
0	0	35	75	1	33	76	1	33	76	0	32	78	1	30	79	110
1	0	59	221	2	71	207	1	78	201	0	86	194	0	75	205	280
2	0	48	487	1	60	474	0	64	471	0	69	466	0	61	474	535
Total	0	142	783	4	164	757	2	175	748	0	187	738	1	166	758	925
CP	0.5903			0.5903			0.5946			0.5968			0.5946			
*CP**	0.0605			0.0743			0.0840			0.0891			0.0815			

[a]See notes to Table 5.