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Low-complexity robust decentralized MPC: a foundational algorithm for constrained coalitional control

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Abstract— **We present a low-complexity robust decentralized MPC formulation for linear time-invariant subsystems that are subject to state and input constraints and coupled via dynamics. The proposed approach is a simple application of tube-based robust MPC to each subsystem, but with some enhancements that make the scheme more applicable to problems with higher-order subsystem dynamics, such as those arising in coalitional control: we remove explicit reliance on invariant sets, and achieve robust stability and feasibility via simple constraint scalings, determined by solving an LP. In the second part of the paper, we apply the approach to coalitional constrained control, and develop theoretical results on recursive feasibility under time-varying coalitions, including the existence of finite dwell times for coalitional switching.**

I. Introduction

Decentralized and distributed forms of model predictive control have emerged as techniques for controlled large-scale constrained systems of subsystems [1]. The main challenge is how to handle the disturbances between subsystems that arise as a consequence of coupled dynamics, in order that guarantees of constraint satisfaction, recursive feasibility, and closed-loop stability are achieved. In addressing this, the literature is broadly split in two [2]: iterative methods, based upon distributed optimization, employ the repeated exchange of information between subsystems as they solve their optimal control problems, while non-iterative methods use the tools of robust control to achieve the same guarantees.

In this paper, following the latter class of approaches, we develop a simple and low-complexity approach to decentralized MPC (DMPC) for state-coupled linear subsystems subject to state and input constraints. Each subsystem is controlled by a model predictive controller, and no on-line communication between controllers takes place. We employ the well-known concept of tube-based MPC [3]–[6], which has found application to distributed MPC before [7]–[10], but with some modifications that enhance the applicability of the technique to higher-order dynamics: we employ simple scalings of constraints in the optimal control problems, rather than exact constraint restrictions, to handle the additive uncertainty in the dynamics, and we show how these scalings may be computed by solving by a single linear programming (LP) problem for each controller. There is *no need* to explicitly characterize or compute a robust invariant set, and the optimal control problem for each subsystem employs no invariant sets, which are a bottleneck for tube-based MPC of higher-order systems. We note that [11, Chapter 3] describes a scheme for obtaining simple constraint restrictions, via solving an LP, that also avoids computing a robust invariant set; it assumes a linear control law for the tube control law, while here we use optimized robust control invariant set synthesis method [12] to obtain a similar result. The price of broader applicability is conservatism: we design robustness of each subsystem to the whole interaction of each coupled neighbour—an aspect improved upon in [7]–[10]—although this is mitigated by the choice of tube controller; for this, we employ the optimized robust control invariance method of [12], which is, for the same disturbance set, known to result in smaller uncertainty tubes than those associated with a linear tube controller [13].

An example instance of higher-order subsystem dynamics is those arising in *coalitional* control [14]–[16]. The coalitional control paradigm considers that, for a large-scale system, a basic decomposition into subsystems is known, but that it may be advantageous to system performance to group subsystems into *coalitions* and control each coalition as a single, coordinated entity. We consider this problem in the second part of the paper, assuming that the overall system is partitioned into time-varying coalitions, each controlled by the proposed low-complexity DMPC. We study the properties of the feasibility regions of the time-varying controlled system, and establish conditions under which recursive feasibility does or does not hold. Moreover, we show that there exist finite dwell times that allow recursively feasible partition switching. These results hold independent to how the coalitions are selected over time; as such, they are fundamental in that apply to all coalition selection schemes. Finally, we discuss some practical options for ensuring recursive feasibility of the overall coalitional control scheme, which may serve to guide the development of any coalition selection scheme.

The organization of this paper is as follows. The next section outlines the setting, including some preliminary notions and definitions relevant to decentralized control and coalitional control, and defines the problem. In Section III, the proposed decentralized tube-based MPC approach is developed, including its design algorithm, and analysed to establish its closed-loop properties of recursive feasibility and stability. Section IV considers the configuration of the system of subsystems into time-varying coalitions, and analyses and discusses the issue of recursive feasibility of the overall control scheme. Conclusions are made in Section V.

Notation and basic definitions: The sets of non-negative and positive reals are denoted, respectively, $\mathbb{R}_{0+}$ and $\mathbb{R}_{+}$. $\mathbb{I}_{\geq 0}$ is the set of non-negative integers, and $\mathbb{I}_{a:b}$ the set of integers between $a < b$. For $a, b \in \mathbb{R}^n$, $a \leq b$ applies element by element. For $X, Y \subset \mathbb{R}^n$, the Minkowski sum is $X \oplus Y \triangleq$

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$\{x + y : x \in X, y \in Y\}$; for $Y \subset X$, the Minkowski difference is $X \ominus Y \triangleq \{x \in \mathbb{R}^n : Y + x \subset X\}$. For $X \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, $X \oplus a$ means $X \oplus \{a\}$. AX denotes the image of a set $X \subset \mathbb{R}^n$ under the linear mapping $A : \mathbb{R}^n \mapsto \mathbb{R}^p$, and is given by $\{Ax : x \in X\}$; similarly, $\kappa(X) = \{\kappa(x) \in \mathbb{R}^p : x \in X\}$ denotes the image of $X \subset \mathbb{R}^n$ under the mapping $\kappa : \mathbb{R}^n \mapsto \mathbb{R}^p$. A set $\mathcal{R}$ is *robust control invariant* (RCI) for a system $x^+ = f(x, u, w)$ and constraint set $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}$ if (i) $\mathcal{R} \subset \mathbb{X}$ and (ii) for all $x \in \mathcal{R}$, there exists a $u = \mu(x) \in \mathbb{U}$ such that $x^+ = f(x, u, w) \in \mathcal{R}$, $\forall w \in \mathbb{W}$; the control law $u = \mu(x)$ is said to be *invariance inducing* over the set $\mathcal{R}$. A polyhedron is an intersection of a finite number of halfspaces, and a polytope is a closed and bounded polyhedron.

II. Setting and problem statement

A. The system of subsystems and control objective

We consider the problem of controlling a large-scale, discrete-time, linear time-invariant system

$$x^+ = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and control input, and x^+ is the state at the next instant of time. We suppose that a basic *decomposition* or *partitioning* of (1) into a number, M, of independently actuated subsystems is known. The dynamics of subsystem $i \in \mathcal{M} \triangleq \{1, \dots, M\}$ are

$$x_i^+ = A_{ii}x_i + B_i u_i + w_i \text{ with } w_i \triangleq \sum_{j \in \mathcal{M}_i} A_{ij}x_j,$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ are the state and input of subsystem $i \in \mathcal{V}$, with $x = (x_1, \dots, x_M)$, $u = (u_1, \dots, u_M)$. That is, the subsystems are non-overlapping in that they share no states or inputs, but interconnected because the off-diagonal block matrices in A give rise to dynamic coupling between subsystems, visible in the exogenous term w_i. The set of coupled *neighbours* of subsystem i is defined as

$$\mathcal{M}_i \triangleq \big\{j \in \mathcal{M} \setminus \{i\} : A_{ij} \neq 0\big\}.$$

Assumption 1 (Controllability). *For each $i \in \mathcal{V}$ the pair (A_{ii}, B_i) is controllable.*

The system is constrained via local, independent constraints on the states and inputs of each subsystem. For subsystem i,

$$x_i \in \mathbb{X}_i \qquad\qquad u_i \in \mathbb{U}_i.$$

Assumption 2 (Constraint sets). *The sets $\mathbb{X}_i \subset \mathbb{R}^{n_i}$ and $\mathbb{U}_i \subset \mathbb{R}^{m_i}$ are polytopes, each containing the origin in its interior.*

The control objective is to regulate the states of the system to the origin, while satisfying all constraints and minimizing the quadratic objective function

$$\sum_{k=0}^{\infty} x^\top(k)Qx(k) + u^\top(k)Ru(k)$$

where $Q \triangleq \operatorname{diag}(Q_1, \dots, Q_M)$ and $R \triangleq \operatorname{diag}(R_1, \dots, R_M)$.

Assumption 3 (Positive definite stage cost). *Q_i and R_i are, for each $i \in \mathcal{M}$, positive definite matrices.*

B. Coalitions of subsystems and partitions of the system

In the context of this problem, it has been shown, via a range of theory, applications and case studies [14]–[16], that closed-loop performance may benefit from grouping subsystems into *coalitions*.

Definition 1 (Coalition of subsystems). *A* coalition *of subsystems is a non-empty subset of $\mathcal{M}$.*

A coalition of subsystems operates as a single entity, controlled by a single coalitional controller that replaces (or coordinates) the local subsystem controllers. The grouping of the subsystems into coalitions in this way induces an alternative partitioning of the system.

Definition 2 (Partition of the system). *A* partition *of the system is an arrangement of the M subsystems into $C \leq M$ coalitions: formally, the partition of $\mathcal{M} = \{1, \dots, M\}$ is the set $\mathcal{C} = \{1, \dots, C\}$, satisfying the following properties:*

1) *Coalition $c \in \mathcal{C}$ contains subsystems $\mathcal{C}_c \subset \mathcal{M}$; the cardinality of $\mathcal{C}_c$ is M_c.*
2) *Coalitions are non-overlapping: $\mathcal{C}_c \cap \mathcal{C}_d = \emptyset$ for all $c \neq d$.*
3) *Coalitions cover the set of subsystems: $\bigcup_{c \in \mathcal{C}} \mathcal{C}_c = \mathcal{M}$.*

These definitions cover the simplest cases of (i) a single, grand coalition of all subsystems ($C = 1$, $\mathcal{C}_1 = \mathcal{V}$) (the *centralized partition*) and (ii) the basic partitioning of the system, in which each subsystem is a coalition ($C = M$, $\mathcal{C} = \mathcal{V}$, $\mathcal{C}_i = \{i\}$ for each $i \in \mathcal{M}$) (the *decentralized partition*). The set of all possible partitions is

$$\Pi_{\mathcal{M}} \triangleq \{\mathcal{C} : \mathcal{C} \text{ is a partition of } \mathcal{M}\}.$$

Given a partition $\mathcal{C} \in \Pi_{\mathcal{M}}$, the state and input of coalition c are, respectively, $x_c = (x_i)_{i \in \mathcal{C}_c}$ and $u_c = (u_i)_{i \in \mathcal{C}_c}$. The states of coalition c evolve as

$$x_c^+ = A_{cc}x_c + B_c u_c + w_c,$$

where the matrices A_{cc} and B_c contain, as sub-blocks, the matrices of the subsystems within the coalition: $A_{cc} = [A_{ij}]_{i,j \in \mathcal{C}_c}$, $B_c = \operatorname{diag}(B_i)_{i \in \mathcal{C}_c}$. Similar to the basic decomposition into subsystems, the coalitions remain coupled via their dynamics: coalition c is coupled with coalition d via the matrices $A_{cd}\big[A_{ij}\big]_{i \in \mathcal{C}_c, j \in \mathcal{C}_d, d \neq c}$ so that

$$w_c \triangleq \sum_{d \in \mathcal{M}_c} A_{cd}x_d \text{ where } \mathcal{M}_c \triangleq \big\{d \in \mathcal{C} : A_{cd} \neq 0\big\}.$$

Assumption 4 (Controllability of coalitions). *For any partition $\mathcal{C} \in \Pi_{\mathcal{M}}$, each pair (A_{cc}, B_c), for $c \in \mathcal{C}$, is controllable.*

III. Simple decentralized MPC for time-invariant coalitions

We first consider the scenario where the set of subsystems $\mathcal{M}$ are arranged into a collection of fixed coalitions $\{\mathcal{C}_1, \dots, \mathcal{C}_C\}$. The aim is for each coalition, acting as a single entity, to regulate its combined state to the origin, while respecting constraints and—to this end—each coalition is equipped with a model predictive controller.[1] Due to the coupling between coalitions, manifested as the disturbance $w_c = \sum_{d\in\mathcal{M}_c} A_{cd}x_d$ for coalition c, consideration needs to be given to handling interactions adequately in order to achieve constraint satisfaction and stability.

Among the numerous distributed and decentralized MPC schemes, algorithms based on robust techniques have the advantage of achieving feasibility and stability guarantees without any inter-agent iterations. The application of a simple tube-based approach [3], [13], as a prototype for the ensuing development, is briefly described next.

If, for the coalition dynamics $x_c^+ = A_{cc} + B_c u_c + w_c$, a constraint-admissible RCI set $\mathcal{R}_c$, with associated control law $\tilde{\kappa}_c(\cdot)$, is available, then it can be guaranteed that the coalitional system, under

$$u_c = \bar{\kappa}_c(\bar{x}_c) + \tilde{\kappa}_c(x_c - \bar{x}_c),$$

is closed-loop stable, recursively feasible and satisfies all constraints. Here, $\bar{u}_c = \bar{\kappa}_c(\bar{x}_c) = u_c^0(0; \bar{x}_c)$ is the first control in the optimized sequence obtained by solving a conventional MPC problem, using the *nominal* (deterministic) model

$$\bar{x}_c^+ = A_{cc}\bar{x}_c + B_c\bar{u}_c \tag{2}$$

and subject to state and input constraint sets *tightened* by, respectively, $\mathcal{R}_c$ and $\tilde{\kappa}_c(\mathcal{R}_c)$.

More sophisticated, less conservative approaches are possible, building on this fundamental technique, *e.g.*, [8]–[10]. There are, however, some features of the coalitional control problem that render such approaches impractical:

1) The coalition dynamics may be of high order, even if the constituent subsystem dynamics are of low order.
2) Though assumed fixed for now, the coalitions will vary in time, both in size and membership.

These features raise significant challenges for controllers built upon invariant sets, for these are difficult to compute, and complex to represent, for higher-order system dynamics, and impractical to re-compute on-line, as would be necessitated in response to a coalition changing in membership.

In the remainder of this section, we address this challenge by developing a simple and low-complexity robust decentralized MPC algorithm. The approach is a simplification of the "nested" DMPC scheme developed in [17]. Constraint restrictions in the primary MPC formulation are achieved via simple *scalings* of $\mathbb{X}_c$ and $\mathbb{U}_c$ rather than $\mathbb{X}_c \ominus \mathcal{R}_c$ and $\mathbb{U}_c \ominus \tilde{\kappa}_c(\mathcal{R}_c)$. We find that the closed-loop properties of the scheme rely only on the *implicit* existence of an RCI set, with the implication for design and implementation that there is no need to either explicitly characterize or compute the RCI set, or impose it anywhere in the MPC constraints. Consequently, the dependency on invariant sets is minimized, while stability and feasibility guarantees are retained, making the approach more suitable for higher-order dynamics.

[1] The MPC problem for a coalition can be solved by a single agent in the coalition (a leader), or distributed among several members, but these details are beyond the scope of this paper.

A. MPC controller for coalition $c \in \mathcal{C}$

The MPC controller, following conventional tube-based MPC, employs the nominal prediction model (2) but simpler constraint restrictions than previously described. For coalition $c \in \mathcal{C}$ with (nominal) state $\bar{x}_c$:

$$\mathbb{P}_c(\bar{x}_c)\colon V_c^0(\bar{x}_c) = \min_{\bar{\mathbf{u}}_c} \big\{ V_c(\bar{x}_c, \bar{\mathbf{u}}_c) : \bar{\mathbf{u}}_c \in \mathcal{U}_c(\bar{x}_c) \big\}$$

where V_c is the finite-horizon regulation cost

$$V_c(\bar{x}_c, \mathbf{u}_c) = \sum_{j=0}^{N-1} \bar{x}_c^\top(j) Q_c \bar{x}_c(j) + \bar{u}_c^\top(j) R_c u_c(j),$$

with $Q_c \triangleq \operatorname{diag}(Q_i)_{i\in\mathcal{C}_c}$, $R_c \triangleq \operatorname{diag}(R_i)_{i\in\mathcal{C}_c}$ and $\mathcal{U}_c(\bar{x}_c)$ is defined by the following constraints for $j \in \mathbb{I}_{0:N-1}$:

$$\begin{aligned} \bar{x}_c(0) &= \bar{x}_c, \\ \bar{x}_c(j+1) &= A_{cc}\bar{x}_c(j) + B_c\bar{u}_c(j), \\ \bar{x}_c(j) &\in \alpha_c^x \mathbb{X}_c, \\ \bar{u}_c(j) &\in \alpha_c^u \mathbb{U}_c, \\ \bar{x}_c(N) &= 0, \end{aligned}$$

where $\mathbb{X}_c \triangleq \prod_{i\in\mathcal{C}_c} \mathbb{X}_i$ and $\mathbb{U}_c \triangleq \prod_{i\in\mathcal{C}_c} \mathbb{U}_i$.

The simple choice of the origin as terminal set is to facilitate the applicability to higher-order dynamics. Conditions on, and selection of, the constraint set scaling parameters $\alpha_c^x, \alpha_c^u \in (0,1)$ is described later.

Solving this problem yields the control sequence

$$\bar{\mathbf{u}}_c^0(\bar{x}_c) \triangleq \{\bar{u}_c^0(0; \bar{x}_c), \ldots, \bar{u}_c^0(N-1; \bar{x}_c)\},$$

and applying the first term of the sequence to the system induces the implicit feedback law

$$\bar{\kappa}_c(\bar{x}_c) = \bar{u}_c^0(0; \bar{x}_c).$$

The domain of problem $\mathbb{P}_c(\bar{x}_c)$, and the control law, is

$$\mathcal{X}_c^N \triangleq \{\bar{x}_c : \mathcal{U}_c(\bar{x}_c) \neq \emptyset\}.$$

B. Overall robust controller

Closing the loop with $u_c = \bar{\kappa}_c(\bar{x}_c)$ alone does not guarantee constraint satisfaction, feasibility and stability for the true coalition dynamics $x_c^+ = A_{cc}x_c + B_c u_c + w_c$, because the disturbance $w_c = \sum_{d\in\mathcal{M}_c} A_{cd}x_d$ is omitted in the prediction model.

The control law is, therefore, and in the spirit of tube-based robust MPC, complemented with a second term $\tilde{\kappa}_c(x_c - \bar{x}_c)$ associated with an admissible RCI set. The resulting two-term policy is a feedback control law on the true state x_c:

$$u_c = \kappa_c(x_c) \triangleq \bar{\kappa}_c(\bar{x}_c) + \tilde{\kappa}_c(x_c - \bar{x}_c). \tag{3}$$

We emphasize that this is a conventional tube-based robust MPC control law; for example, the simple robust controller presented in [13] uses the same construction of a nominal MPC control augmented with an invariance-inducing control law based on RCI sets. A key feature, however, of the proposed scheme over conventional schemes is the removal of the need to have an explicit representation of the RCI set.

Remark 1. *The nominal state $\bar{x}_c$ that parametrizes the optimal control problem and appears in the control law is an internal state of a dynamic feedback controller. For initialization, we assume that $\bar{x}_c(0) = x_c(0)$. Subsequently, the nominal and true states are allowed to evolve independently under their respective dynamics.*

C. Design conditions and closed-loop properties

Recursive feasibility and stability of controlled system are established under some conditions that guide the design of the secondary control law and the selection of suitable scaling factors in the MPC problem.

Assumption 5. *The control law $\tilde{\kappa}_c(\cdot)$ is invariance inducing over a set $\mathcal{R}_c$; the set $\mathcal{R}_c$ is RCI for the system $x_c^+ = A_{cc}x_c + B_c u_c + w_c$ and constraint set $(\xi_c^x \mathbb{X}_c, \xi_c^u \mathbb{U}_c, \mathbb{W}_c)$, for some $\xi_c^x \in [0,1)$ and $\xi_c^u \in [0,1)$, and where $\mathbb{W}_c \triangleq \bigoplus_{d\in\mathcal{M}_c} A_{cd}\mathbb{X}_d$.*

Assumption 6. *The constants (α_c^x, ξ_c^x) and (α_c^u, ξ_c^u) satisfy $\alpha_c^x + \xi_c^x \le 1$ and $\alpha_c^u + \xi_c^u \le 1$.*

Recursive feasibility for time-invariant coalitions is then established in the following proposition.

Proposition 1 (Recursive feasibility)**.** *Suppose that Assumptions 1–6 hold. Then, for each coalition $c \in \mathcal{C}$:*

(i) If $\bar{x}_c \in \mathcal{X}_c^N$ then $\bar{x}_c^+ \in \mathcal{X}_c^N$, where $\bar{x}_c^+ = A_{cc}\bar{x}_c + B_c\bar{\kappa}_c(\bar{x}_c)$.

(ii) Given $\bar{x}_c(0) = x_c(0) \in \mathcal{X}_c^N$, the coalition $x_c^+ = A_{cc}x_c + B_c u_c + w_c$ under the control law $u_c = \bar{\kappa}_c(\bar{x}_c) + \tilde{\kappa}_c(x_c - \bar{x}_c)$ satisfies $x_c(k) \in \mathbb{X}_c$ and $u_c(k) \in \mathbb{U}_c$ for $k \in \mathcal{I}_{\ge 0}$.

Proof. Claim (i) is a straightforward, and well-established, consequence of imposing the terminal equality constraint. For claim (ii), if $\bar{x}_c \in \mathcal{X}_c^N$ and $x_c - \bar{x}_c \in \mathcal{R}_c$, then $x_c^+ - \bar{x}_c^+ \in \mathcal{R}_c$, since $\mathcal{R}_c$ is RCI for the uncertain dynamics. Since $\mathcal{X}_c^N \oplus \mathcal{R}_c \subseteq \alpha_c^x \mathbb{X}_c \oplus \xi_c^x \mathbb{X}_c$ and $\bar{\kappa}_c(\mathcal{X}_c) \oplus \tilde{\kappa}_c(\mathcal{R}_c) \subseteq \alpha_c^u \mathbb{U}_c \oplus \xi_c^u \mathbb{U}_c$, all constraints remain satisfied. Finally, since $\bar{x}_c(0) = x_c(0) \in \mathcal{X}_c^N$ and, because of Assumption 2, $\mathcal{R}_c$ contains the origin, the hypothesis that initially $x_c - \bar{x}_c \in \mathcal{R}_c$ holds. ☐

Having established recursive feasibility and constraint satisfaction, stability follows under a further assumption.

Assumption 7 (Decentralized stabilizability)**.** *The RCI control laws $u_c = \tilde{\kappa}_c(x_c)$ asymptotically stabilize the system $x^+ = Ax + Bu$.*

Theorem 1 (Stability)**.** *Suppose that Assumptions 1–7 hold. Then, for each $c \in \mathcal{C}$, the origin is exponentially stable for the nominal coalition system $\bar{x}_c^+ = A_{cc}x_c + B_c\bar{\kappa}_c(\bar{x}_c)$ and asymptotically stable for the true coalition system $x_c^+ = A_{cc}x_c + B_c\kappa_c(x_c) + w_c$. The region of attraction for $(\bar{x}_c, x_c)$ is $\mathcal{X}_c^N \times \mathcal{X}_c^N$.*

Proof. By Proposition 1, $\bar{x}_c \in \mathcal{X}_c^N$ implies $\bar{x}_c^+ = A_{cc}\bar{x}_c + B_c\bar{\kappa}_N(\bar{x}_c) \in \mathcal{X}_c^N$ and $V_c^0(\bar{x}_c^+) \le V_c^0(\bar{x}_c) - \ell_c(\bar{x}_c, \bar{\kappa}_c(\bar{x}_c))$, where $\ell_c(x_c, u_c) \triangleq x_c^\top Q_c x_c + u_c^\top R_c u_c$. By Assumption 3, there exists a constant $a_c > 0$ such that $V_c^0(\bar{x}_c) \ge \ell_c(x_c, \bar{\kappa}_c(\bar{x}_c) \ge a_c|\bar{x}_c|^2$ for all $\bar{x}_c \in \mathcal{X}_c^N$ and, moreover, Assumption 1 ensures the existence of a constant $b_c > a_c > 0$ such that $V_c^0(\bar{x}_c) \le b_c|\bar{x}_c|^2$ over the same domain. Then $V_c^0(\bar{x}_c^+) \le \gamma_c V_c^0(\bar{x}_c)$ where $\gamma_c \triangleq (1 - a_c/b_c) \in (0,1)$. If $\bar{x}_c(0) \in \mathcal{X}_c^N$ then $V_c^0\big(\bar{x}_c(k)\big) \le \gamma_c^k V_c^0\big(\bar{x}_c(0)\big)$ and, moreover, $\big|\bar{x}_c(k)\big| \le d_c\delta_c^k\big|\bar{x}_c(0)\big|$ where $\delta_c \triangleq \sqrt{\gamma_c}$ and $d_c \triangleq \sqrt{b_c/a_c}$. This establishes exponential stability for the nominal system.

Now consider the true trajectory $\{x_c(k)\}_k$. We have $x_c(0) = \bar{x}_c(0) \in \mathcal{X}_c^N$, so $|x_c(0)| = |\bar{x}_c(0)|$. Consider some $x_c = \bar{x}_c + e_c$ and the dynamics of e_c

$$e_c^+ = A_{cc}e_c + B_c\tilde{\kappa}_c(e_c) + \sum_{d\in\mathcal{M}_c} A_{cd}x_d.$$

Thus, $e^+ = Ae + B\tilde{\kappa}(e)$, where $\tilde{\kappa}(\cdot)$ denotes the diagonal collection of $\tilde{\kappa}_c(\cdot)$, for which $e(k) \to 0$ as $k \to \infty$, in view of Assumption 7. Finally, since $x_c = \bar{x}_c + e_c$, and both terms decay asympotically to zero, then $x_c(k) \to 0$ as $k \to \infty$. ☐

Corollary 1. *For each $c \in \mathcal{C}$, the sets $\mathcal{X}_c^N$ and $\mathcal{X}_c^{N-1}$ are positively invariant for the nominal dynamics $\bar{x}_c^+ = A_{cc}\bar{x}_c + B_c\bar{\kappa}_c(\bar{x}_c)$.*

Corollary 2. *For each $c \in \mathcal{C}$, starting from $e_c(0) = 0$ the true dynamics $x_c^+ = A_{cc}x_c + B_c\kappa_c(x_c) + w_c$ evolve in a robust positively invariant set $\mathcal{X}_c^N \oplus \mathcal{R}_c \subseteq (\alpha_c^x + \xi_c^x)\mathbb{X}_c$.*

D. Controller design

The design of the controller for each coalition, via the selection of the constraints scaling factors and construction of the invariance-inducing secondary control law, uses the theory and algorithm of optimized robust control invariance [12].

1) Optimized Robust Control Invariance: The optimized robust control invariance approach of [12] proposed a novel characterization of an RCI set for a system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ as

$$\mathcal{R}_h(\mathbf{M}_h) = \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\mathbb{W} \text{ with } \mu(\mathcal{R}_h(\mathbf{M}_h)) = \bigoplus_{l=0}^{h-1} M_l\mathbb{W}.$$

The set $\mu(\mathcal{R}_h(\mathbf{M}_h))$ is the set of invariance-inducing control actions, defined as $\mu(\mathcal{R}_h) \triangleq \{\mu(x) : x \in \mathcal{R}_h\} = \{u \in \mathbb{U} : x^+ \in \mathcal{R}_h, \forall w \in \mathbb{W}\}$. The matrices $D_l(\mathbf{M}_h), l = 0 \ldots h$ are

$$D_0(\mathbf{M}_h) = I, \quad D_l(\mathbf{M}_h) \triangleq A^l + \sum_{j=0}^{l-1} A^{l-1-j}BM_j, l \ge 1$$

with $M_j \in \mathbb{R}^{m\times n}$ and $\mathbf{M}_h \triangleq (M_0, M_1, \ldots, M_{h-1})$, such that $D_h(\mathbf{M}_h) = 0$; the latter is ensured by setting h greater than or equal to the controllability index of (A, B). The set of matrices that satisfy these conditions is given by $\mathbb{M}_h \triangleq \{\mathbf{M}_h : D_h(\mathbf{M}_h) = 0\}$. Constraint satisfaction is guaranteed if $\mathcal{R}_h(\mathbf{M}_h) \subseteq \xi^x\mathbb{X}$ and $\mu(\mathcal{R}_h(\mathbf{M}_h)) \subseteq \xi^u\mathbb{U}$, with $(\xi^x, \xi^u) \in [0,1] \times [0,1]$.

As shown in [12], the linear programming (LP) problem to compute these sets is

$$\min\{\delta : \gamma \in \Gamma\}, \tag{4}$$

where $\gamma = (\mathbf{M}_h, \xi^x, \xi^u, \delta)$, and the set $\Gamma = \{\gamma : \mathbf{M}_h \in \mathbb{M}_h, \mathcal{R}_h(\mathbf{M}_h) \subseteq \xi^x\mathbb{X}, \mu(\mathcal{R}_h(\mathbf{M}_h)) \subseteq \xi^u\mathbb{U}, (\xi^x, \xi^u) \in [0,1] \times [0,1],\ q_x\xi^x + q_u\xi^u \leq \delta\}$; q_x and q_u are weights to express a preference for the relative contraction of state and input constraint sets. Feasibility of this problem is linked to the existence of an RCI set: if (4) is feasible, then $\mathcal{R}_h(\mathbf{M}_h)$ exists and satisfies the RCI properties [12].

2) Design algorithm: In our context, the RCI LP problem is useful because solving for each coalition $c \in \mathcal{C}$ provides, given the disturbance set $\mathbb{W}_c$, an invariance-inducing robust control law for the coalition dynamics—a suitable candidate for the second term in the overall control law—plus scaling constants that outer-bound (with respect to the state and input constraint sets) the size of the RCI set and its corresponding set of control actions. These are used subsequently to deduce the scaling factors to be employed in the MPC problem.

1) The problem (4) associated with the dynamics $x_c^+ = A_{cc}x_c + B_cu_c + w_c$ and constraint set $(\mathbb{X}_c, \mathbb{U}_c, \mathbb{W}_c)$ is solved to yield $\gamma_{c,h} = (\mathbf{M}_{c,h}, \xi_c^x, \xi_c^u, \delta_c)$, where $\mu_c(\cdot)$ is the RCI control law, and ξ_c^x and ξ_c^u are scalings of $\mathbb{X}_c$ and $\mathbb{U}_c$ such that $\mathcal{R}_{c,h} \subset \xi_c^x\mathbb{X}_c$ and $\mu_c(\mathcal{R}_{c,h}) \subset \xi_c^u\mathbb{U}_c$ respectively.
2) Given that, under an invariance-inducing control action $u_c = \mu_c(x_c)$, $x_c \in \mathcal{R}_{c,h} \subset \xi_c^x\mathbb{X}_c$ and $u_c \in \mu(\mathcal{R}_{c,h}) \subset \xi_c^u\mathbb{U}_c$, we select
$$\alpha_c^x = 1 - \xi_c^x$$
$$\alpha_c^u = 1 - \xi_c^u,$$
for the scaling factors in the MPC problem. Then $x_c = \bar{x}_c + e_c \in \alpha_c^x\mathbb{X}_c \oplus \xi_c^x\mathbb{X}_c \subseteq \mathbb{X}_c$, with a similar expression for u_c, satisfying Assumption 6.
3) The control law $\tilde{\kappa}_c(e_c) = \mu_c(e_c)$ is computed from the matrices $\mathbf{M}_{c,h}$, using the minimal selection map procedure described in [18].

It is worth noting that, although the theory of RCI sets is used in the design procedure, no RCI sets are explicitly computed or constructed. In contrast, other iteration-free distributed MPC methods not only compute these sets offline, during design, but also employ them online in the constraints.

IV. Feasibility of time-varying coalitions

The final part of the paper considers that the system partition, or set of coalitions, changes in time; that is, the system partition is $\mathcal{C}(k) \in \Pi_{\mathcal{M}}$ at time k, but may change at the next time step to $\mathcal{C}(k+1) \in \Pi_{\mathcal{M}}$[2]. We do not consider in this paper *how* the system partitions or coalitions are designed; as such, the results developed in this section are fundamental to all partition selection schemes. At the same time, these results may guide the development of a suitable selection scheme, and we aim to address this in future work.

A time-varying system partition introduces significant challenges for an MPC-based coalitional control scheme: recursive feasibility of the optimal control problem for each time-invariant coalition is established by Proposition 1, but this does not continue to hold for a time-varying system partition. However, recursive feasibility is the most basic requirement for any MPC controller, since guaranteeing the continued operation, and then stability, of the controlled system relies on the continued feasibility of the underlying optimal control problem. Our aim is, therefore, to determine conditions under which a new system partition is feasible to adopt and subsequently use.

[2]Upon the instigation of a new system partition, the controller for each partition is designed according to the algorithm given in the previous section; that this requires only the solution of a single LP for each coalition opens up the possibility that this re-design can be done on-line, but the fine details of this are beyond the scope of this paper.

A. Definitions

The set $\mathcal{X}_{\mathcal{C}}^N$ is defined as the product of the individual feasibility sets of the coalitions:
$$\mathcal{X}_{\mathcal{C}}^N \triangleq \prod_{c \in \mathcal{C}} \mathcal{X}_c^N,$$
with a similar definition for the RCI sets $\mathcal{R}_{\mathcal{C}}$ (even if these have not been explicitly computed). Note that the condition $x \in \mathcal{X}_{\mathcal{C}}^N$ is equivalent to the condition $x_c \in \mathcal{X}_c^N$ for all $c \in \mathcal{C}$, but utilizes, for convenience, a more compact notation.

We also introduce a key definition of different notions of feasibility, with respect to a partition. In the context of adopting a new system partition, the notion of *strong feasibility* that we introduce has the advantage of permitting the simple initialization of coalitional controller states that is proposed in the previous section, avoiding what would otherwise be an iterative and coupled design process.

Definition 3 (Feasible and strongly feasible partition)**.** *A partition is said to be* feasible *at a state x if $x \in \mathcal{X}_{\mathcal{C}}^N \oplus \mathcal{R}_{\mathcal{C}}$, and* strongly feasible *at a state x if $x \in \mathcal{X}_{\mathcal{C}}^N$.*

Finally, we define the notions of *refinement* and *coarsening* with respect to the system partition.

Definition 4 (Refinement and coarsening)**.** *Given $(\mathcal{D}, \mathcal{C}) \in \Pi_{\mathcal{M}} \times \Pi_{\mathcal{M}}$, the partition $\mathcal{D} \preceq \mathcal{C}$ ($\mathcal{D}$* refines *$\mathcal{C}$, or $\mathcal{C}$* coarsens *$\mathcal{D}$) if every member of $\mathcal{D}$ is contained in some member of $\mathcal{C}$.*

As an example, the grand coalition corresponding to the centralized partition $\mathcal{C} = \{1, 2, 3\}$ for three subsystems coarsens the decentralized partition $\mathcal{D} = \{\{1\}, \{2\}, \{3\}\}$.

B. How partition coarsening and refinement affect feasibility

A key observation is that change in system partition follows some sequence of coarsenings and/or refinements. The following set of results, up to Proposition 2, are proved in a companion paper [19], and are included here because of the fundamental implications they have for changing the system partition online.

The first result is a direct consequence of the fact that, with coarsening (refinement), the disturbance set that each coalition sees diminishes (grows), leading to a smaller (larger) RCI set and hence the scaling factors ξ_c^x and ξ_c^u.

Lemma 1 (Nesting of RCI sets)**.** *If $\mathcal{C} \succeq \mathcal{D}$, then $\mathcal{R}_{\mathcal{C}} \subseteq \mathcal{R}_{\mathcal{D}}$.*

Lemma 1 then directly implies less (more) restriction of the constraints, via the scaling factors α_c^x, α_c^x, with partition coasening (refinement).

Lemma 2 (Nesting of nominal feasibility regions)**.** *If $\mathcal{C} \succeq \mathcal{D}$, then $\mathcal{X}_{\mathcal{C}}^i \supseteq \mathcal{X}_{\mathcal{D}}^i$ for $i = 0 \dots N$.*

This might seem to suggest that feasibility is trivially maintained with partition coarsening. The reality is, however, not so simple, owing to the following result—a consequence of the fact that the overall domain for the true state x is $\mathcal{X}_{\mathcal{C}}^N \oplus \mathcal{R}_{\mathcal{C}}$ and not merely $\mathcal{X}_{\mathcal{C}}^N$.

Proposition 2 (Feasibility is not necessarily maintained with coarsening)**.** *Suppose $\mathcal{D} \in \Pi_{\mathcal{M}}$ is feasible at x. Then $\mathcal{C} \succeq \mathcal{D}$ is not necessarily feasible at x.*

That is, even though $\mathcal{X}_{\mathcal{C}}^N \supseteq \mathcal{X}_{\mathcal{D}}^N$ when $\mathcal{C} \succeq \mathcal{D}$, there is no clear relation between the sets $\mathcal{X}_{\mathcal{D}} \oplus \mathcal{R}_{\mathcal{D}}$ and $\mathcal{X}_{\mathcal{C}} \oplus \mathcal{R}_{\mathcal{C}}$—we demonstrate this via counterexamples, and hence prove the claim, in our companion paper [19]. On the other hand, *strong* feasibility is guaranteed under the same assumptions, as a consequence of Lemma 2.

Proposition 3 (Strong feasibility is maintained with coarsening)**.** *Suppose $\mathcal{D} \in \Pi_{\mathcal{M}}$ is strongly feasible at a state x. Then $\mathcal{C} \succeq \mathcal{D}$ is strongly feasible at x.*

The situation is more challenging, however, in the case of partition refinement, since a counterpart to Proposition 3 for a movement from $\mathcal{C}$ to $\mathcal{D} \preceq \mathcal{C}$ does not hold.

Proposition 4 (Strong feasibility does not imply feasibility after refinement)**.** *Suppose $\mathcal{C} \in \Pi_{\mathcal{M}}$ is strongly feasible at a state x. Then $\mathcal{D} \preceq \mathcal{C}$ is not necessarily feasible at x.*

This raises at least two questions: firstly, *when* is the hypothesis of Proposition 3, for partition coarsening, met? Indeed, although $\bar{x}(0) \in \mathcal{X}_{\mathcal{C}}^N$ implies the nominal state $\bar{x}(k) \in \mathcal{X}_{\mathcal{C}}^N$ for all $k \in \mathbb{I}_{\geq 0}$, the initialization $x(0) \in \mathcal{X}_{\mathcal{C}}^N$ does *not* imply that the true state $x(k) \in \mathcal{X}_{\mathcal{C}}^N$ for all $k \in \mathbb{I}_{\geq 0}$. Secondly, when is strong feasibility achieved under partition refinement? In the next subsection, we present and discuss some answers to these questions.

C. Schemes for feasible partition switching

We outline three schemes for enabling feasible switching between partitions over time. Our intention is not to develop any scheme into a comprehensive proposal, but to explore the range of options and illustrate the comparative ease, or difficulty, of implementing each.

1) A quest for feasibility by design: With the system in a partition $\mathcal{C}$ that is feasible at a state x, the nominal state $\bar{x} \in \mathcal{X}_{\mathcal{C}}^N$. The successor nominal state $\bar{x}_i^+ \in \mathcal{X}_{\mathcal{C}}^{N-1}$ and the true state $x^+ \in \mathcal{X}_{\mathcal{C}}^{N-1} \oplus \mathcal{R}_{\mathcal{C}} \subset \mathcal{X}_{\mathcal{C}}^N \oplus \mathcal{R}_{\mathcal{C}}$. This motivates and leads to the following proposition concerning switching between partitions.

Proposition 5. *Suppose partition $\mathcal{C} \in \Pi_{\mathcal{M}}$ is feasible at a state x. Partition $\mathcal{D} \in \Pi_{\mathcal{M}}$ is strongly feasible at the successor state x^+ if*

$$\mathcal{X}_{\mathcal{C}}^{N-1} \oplus \mathcal{R}_{\mathcal{C}} \subseteq \mathcal{X}_{\mathcal{D}}^N.$$

With respect to the *usefulness* of this result, the inclusion is not straightforward to verify or enforce, in view of the preference to avoid characterizing and computing $\mathcal{R}_{\mathcal{C}}$. Nevertheless, following the design procedure in Section III-D, $\mathcal{R}_{\mathcal{C}}$ is outer-bounded as

$$\mathcal{R}_{\mathcal{C}} \subset \prod_{c \in \mathcal{C}} \xi_c^x \mathbb{X}_c,$$

leading to the following more practical result.

Proposition 6. *Suppose partition $\mathcal{C} \in \Pi_{\mathcal{M}}$ is feasible at a state x. Partition $\mathcal{D} \in \Pi_{\mathcal{M}}$ is strongly feasible at the successor state x^+ if*

$$\mathcal{X}_{\mathcal{C}}^{N-1} \oplus \prod_{c \in \mathcal{C}} \xi_c^x \mathbb{X}_c \subseteq \mathcal{X}_{\mathcal{D}}^N.$$

The condition is still problematic, and perhaps impossible, to meet as a design constraint because of the fundamental relations governing the relations between sets under refinement and coarsening. For example, for the condition to be met under the refinement $\mathcal{D} \preceq \mathcal{C}$, it is necessary that $\mathcal{X}_{\mathcal{C}}^{N-1} \subset \text{interior}(\mathcal{X}_{\mathcal{D}}^N)$ even though $\mathcal{X}_{\mathcal{C}}^N \supseteq \mathcal{X}_{\mathcal{D}}^N$; satisfaction would be highly problem specific and, even if possible, would require careful design. Even under coarsening, for which $\mathcal{X}_{\mathcal{C}}^N \subseteq \mathcal{X}_{\mathcal{D}}^N$ already, the condition is not trivially met, and relies on weak coupling for satisfaction—the size of the coalitional disturbance set $\mathbb{W}_c = \bigoplus_{d \in \mathcal{M}_c} A_{cd} \mathbb{X}_d$ must be sufficiently small. More constructive alternatives are therefore discussed next.

2) Use of a feasibility dwell time: An attractive option, well established in the switching systems literature, and more recently in the context of robust MPC for switching systems [20]–[22], is the use of a *dwell time* in order to ensure the state lies within the feasibility region associated with the new partition at the moment of switching. The following result, which follows directly from the stability of each coalition in the time-invariant system partition setting (Theorem 1), enables this.

Proposition 7 (Feasibility becomes and remains strong feasibility)**.** *Suppose the system is in a partition $\mathcal{C} \in \Pi_{\mathcal{M}}$ that is feasible at x. The same partition $\mathcal{C}$ becomes, and remains, strongly feasible a finite number of time steps thereafter. Moreover, if $\mathcal{R}_{\mathcal{C}} \subset \mathcal{X}_{\mathcal{C}}^N$, then this happens exponentially fast.*

The hypothesis $\mathcal{R}_{\mathcal{C}} \subset \mathcal{X}_{\mathcal{C}}^N$ is satisfied if the constraint scaling factors follow $\xi_c^x < \alpha_c^x$ for all $c \in \mathcal{C}$; note that this condition is, again, a weak coupling requirement in order for the condition to hold. Where an exponential stability result holds, as it does here, a *dwell time* is possible to characterize and compute by exploiting the exponential decay constants of the system [22].

Once strong feasibility is established for all subsequent times, a similar result establishes that a switch from partition $\mathcal{C}$ to partition $\mathcal{D}$ is, if the coupling is sufficiently weak, possible after a finite number of steps.

Proposition 8 (Strong feasibility dwell time)**.** *Suppose the system is in a partition $\mathcal{C} \in \Pi_{\mathcal{M}}$ that is feasible at x. A partition $\mathcal{D} \neq \mathcal{C}$ becomes strongly feasible a finite number of time steps thereafter. Moreover, if $\mathcal{R}_{\mathcal{C}} \subset \mathcal{X}_{\mathcal{D}}^N$, then this happens exponentially fast.*

3) Use of a simple feasibility check: A key observation is that, depending on the partition selection scheme, the choice of system partition at each time may be a controllable degree of freedom[3]. It follows, then, that if the system partition is $\mathcal{C}$ and, subsequently, a new partition $\mathcal{D}$ is selected, it is not *necessary* (for maintaining feasibility and stability) to adopt the new partition. Indeed, if the new partition $\mathcal{D}$ is not (strongly) feasible, then Proposition 1 already ensures that the current partition $\mathcal{C}$ is.

This motivates the use of a simpler approach than the dwell-time-based one, of checking the strong feasibility of a partition immediately after it is selected. To this end, there are two options: explicit checking via a set-membership test, and implicit checking via convex optimization. For the former, note that it is possible to obtain an explicit characterization of the set $\mathcal{X}_c^N$, for each $c \in \mathcal{C}$, by the performing N iterations of the backwards reachability operation; the test of strong feasibility of a partition $\mathcal{C}$ amounts then to the verification that $x_c \in \mathcal{X}_c^N$ for each $c \in \mathcal{C}$.

Alternatively, feasibility may be checked by attempting to solve the QP $\mathbb{P}_c(\bar{x}_c)$ or, more simply, by solving the associated linear programming (LP) problem obtained by replacing the quadratic cost of $\mathbb{P}_c(\bar{x}_c)$ with a linear one; in the latter case, an infeasibility certificate is easy to obtain.

V. Conclusions and future work

We have presented a robust decentralized MPC scheme for linear dynamically coupled subsystems subject to constraints. The scheme uses the well established tube approach to guarantee robust feasibility and stability for each subsystem and/or coalition, but here the design and formulation avoids the need to explicitly characterize a robust invariant set that represents the uncertainty tube cross section. The MPC formulation and design, aided by a simple (albeit conservative) choice of the origin as a terminal constraint, does not rely on the computation of any invariant sets, which enhances its applicability to higher-order dynamics, such as those found in coalitional schemes, and removes the most significant computational barriers for the on-line controller re-design problem that also emerges in coalitional control. Finally, we analysed the recursive feasibility of a coalitional control scheme built upon the proposed control algorithm, and established conditions under which changing the system partition—*i.e.*, the set of coalitions—can safely take place.

Future work will consider how coalitions may be chosen in time via a suitable selection algorithm, and also improvements to the proposed DMPC scheme to further lower conservatism and enhance its applicability, *e.g.* to more strongly coupled systems. To this end, we aim to employ the full "nested" DMPC algorithm proposed in [17]; more strongly coupled systems may be handled by using the coupling structure to guide the formation of coalitions.

[3] *c.f.* the definition of *switched* systems versus *switching* systems: in the former, the switching signal can be arbitrarily chosen [23].

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