Article:

Ballai, I., Erdélyi, R. and Goossens, M. (2000) Nonlinear theory of non-axisymmetric resonant slow waves in straight magnetic flux tubes. Journal of Plasma Physics, 64 (5). pp. 579-599. ISSN: 0022-3778

J. Plasma Physics (2000), *vol.* 64, *part* 5, *pp.* 579–599. Printed in the United Kingdom

Nonlinear theory of non-axisymmetric resonant slow waves in straight magnetic flux tubes

I. BALLAI,[1] R. ERDÉLYI[2] and M. GOOSSENS[1]

[1]Centre for Plasma Astrophysics, KU Leuven, Celestijnenlaan 200B, B-3001 Leuven-Heverlee, Belgium

[2]Space and Atmosphere Research Center, Department of Applied Mathematics, University of Sheffield, Sheffield S3 7RH, UK

(Received 21 December 1999 and in revised form 7 July 2000)

Abstract. Nonlinear resonant slow magnetohydrodynamic (MHD) waves are studied in weakly dissipative isotropic plasmas for a cylindrical equilibrium model. The equilibrium magnetic field lines are unidirectional and parallel with the z axis. The nonlinear governing equations for resonant slow magnetoacoustic (SMA) waves are derived. Using the method of matched asymptotic expansions inside and outside the narrow dissipative layer, we generalize the connection formulae for the Eulerian perturbation of the total pressure and for the normal component of the velocity. These nonlinear connection formulae in dissipative cylindrical MHD are an important extention of the connection formulae obtained in linear ideal MHD [Sakurai et al., *Solar Phys.* **133**, 227 (1991)], linear dissipative MHD [Goossens et al., *Solar Phys.* **175**, 75 (1995); Erdélyi, *Solar Phys.* **171**, 49 (1997)] and in nonlinear dissipative MHD derived in slab geometry [Ruderman et al., *Phys. Plasmas* **4**, 75 (1997)]. These generalized connection formulae enable us to connect the solutions at both sides of the dissipative layer without solving the MHD equations *in* the dissipative layer. We also show that the nonlinear interaction of harmonics in the dissipative layer is responsible for generating a parallel mean flow outside the dissipative layer.

1. Introduction

The heating of space (e.g. solar and stellar coronal) or laboratory (e.g. tokamak) fusion plasmas is a longstanding problem of fundamental plasma physics. It is now widely recognized that plasma heating for the Sun and solar-like stars, for example, is tied to magnetic fields (Rosner et al. 1978; Acton et al. 1992). Laboratory experiments also show that efficient heating of a confined plasma requires the presence of magnetic field. The magnetic field distribution in space plasmas and in laboratory plasma devices is generally non-uniform. The magnetic field is often structured in elongated loop-like forms (e.g. in tokamaks or the solar corona). The greatest contribution to the heating of such confined plasmas comes from these magnetic loops, which are viewed as basic building blocks of plasma heating.

Two broad possibilities for the heating mechanism are studied in particular: *direct current* (DC) dissipation and *alternating current* (AC) dissipation (Browning 1991;

Narain and Ulmschneider 1996). This division is mainly based on the time scale of the response of the inhomogeneous magnetic plasma loops to the driving motions or forces (Heyvaerts and Priest 1983; Zirker 1993). When the response of the magnetic plasma structures consists of slow motions on a time scale long compared with the Alfvén transit time, the mechanism is called DC. The magnetic flux tube undergoes a slow evolution, and magnetic energy is gradually built up by shearing motions until the field relaxes or reconnects to a lower state of energy. Part of the released energy can be converted into heat. Currents that are dissipated in reconnection models are DC currents (for example, in solar physics, they might be associated with microflares, explosive events, and nanoflares). On the other hand, if driving motions of the footpoints of the magnetic field lines induce motions in the flux tubes with time scales of the order of the Alfvén travel time, the mechanism is called AC. The currents in MHD waves, and in particular in Alfvén waves, fall in the AC category.

A highly non-uniform plasma is a natural medium for MHD waves (see reviews by, e.g. Roberts 1991a,b; Roberts and Ulmschneider 1998). MHD waves may play an important role in the supplementary heating in laboratory plasmas (Chen and Hasegawa 1974; Tataronis 1975; Tataronis and Grossmann 1976; Kappraff and Tataronis 1977; Appert et al. 1984) or the observed high temperatures in solar and stellar coronae (as first suggested by Ionson 1978).

In a non-uniform plasma, a continuous spectrum of Alfvén and slow MHD waves may exist in ideal MHD, and this can lead to resonant absorption in a driven problem. Ionson (1978) proposed resonant MHD waves as a means to heat magnetic loops in the solar corona, and the mechanism has become popular (see e.g. Rae and Roberts 1981; Davila 1987; Poedts et al. 1989, 1990; Goossens 1991; Hollweg 1991; Sakurai et al. 1991; Wright and Allan, 1996; Erdélyi 1997, 1998).

When the condition for resonant absorption is fulfilled, global wave motions are locally in resonance on particular magnetic surfaces, causing energy to build up on these magnetic surfaces at the expense of the global motions. To compute the heating of the plasma, one has to include dissipative effects in the studies.

1.1. Why nonlinear dissipative MHD?

Non-ideal effects such as viscosity and electrical resistivity can remove the singularities in the mathematical solutions found in linear ideal MHD. In principle, we also could include thermal conductivity in our studies, but we neglect this effect in the present model. This approach is entirely suitable for certain plasmas, for example in the lower part of the solar atmosphere. Since the dissipative effects are small, they only act in a narrow layer, called the *dissipative layer*. Within this layer, the perturbations have steep gradients and large amplitudes. It can be shown that, owing to the near-resonant behaviour of the waves in the dissipative layers close to the ideal resonant positions (see e.g. Goossens et al. 1995; Erdélyi and Goossens 1995), linear theory can break down in the vicinity of the resonant positions, and nonlinearity has to be taken into account.

Nonlinear effects are responsible for many important phenomena, such as saturation of linear instabilities and the onset of specific nonlinear instabilities, the interaction of different modes, turbulence, formation of shock waves, etc. A full understanding of the natural resonant wave phenomena in an inhomogeneous magnetic plasma cannot be complete without considering nonlinear effects.

Ruderman et al. (1997a) have studied the nonlinear theory of resonant slow waves in dissipative layers in isotropic plasmas. They derived the governing equations and calculated the jump conditions (which connect the analytical solutions at both sides of the resonant layer; these so-called connection formulae play a similar role as the Rankine–Hugoniot relations for shock waves!). This theory was then applied by Ruderman et al. (1997b) to study the nonlinear interaction of sound waves with resonant slow waves using slab geometry. These studies were extended by Ballai et al. (1998a,b) to anisotropic plasmas. So far, the main conclusion of these nonlinear studies has been that nonlinearity decreases the absolute value of the coefficient of energy absorption for any type of dissipation, and the absorption depends on the actual type of dissipation. This result is in contrast to the linear theories, where the coefficient of energy absorption does not depend on the type of dissipative effect.

Since the magnetic field is mainly confined in loops (or tubes), it is natural to study the nonlinear resonant interaction in cylindrical geometry. The aim of this paper is to develop the nonlinear theory of resonant slow waves in isotropic plasmas in a magnetic cylinder when the azimuthal dependence is taken into account. We show that the nonlinear interaction of harmonics in the dissipative layer generates a shear mean flow outside the dissipative layer. This mean flow is a purely nonlinear effect; it was first found by Ofman et al. (1994) and Ofman and Davila (1995) in a numerical investigation.

In the present paper we use some characteristic quantities such as the thickness of the dissipative layer l_{dis}, the characteristic length scale of the inhomogeneity l_{inh}, and the dimensionless amplitude of the oscillations far away from the dissipative layer ϵ. The total Reynolds number, which measures the importance of dissipation, is defined by

$$\frac{1}{R} = \frac{1}{R_e} + \frac{1}{R_m}, \tag{1.1}$$

where R_e and R_m are the classical and magnetic Reynolds numbers. Since estimations reveal that R is very large in astrophysical plasmas (10^6 in the photosphere and up to 10^{12} in the corona), we can use the approximation of a weakly dissipative plasma.

2. Basic equations

In what follows, we adopt a cylindrical coordinate system (r, φ, z) and we restrict our analysis to a static equilibrium state, i.e. $\mathbf{v}_0 = 0$, and we also neglect the effect of gravity. The components of the background magnetic field are $(0, 0, B_0(r))$, and all other equilibrium quantities depend on the radial coordinate only. The magnetostatic equilibrium is described by the radial force balance

$$\frac{d}{dr}\left(p_0 + \frac{B_0^2}{2\mu}\right) = 0. \tag{2.1}$$

The full set of nonlinear dissipative MHD equations has the form

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \qquad \boldsymbol{\nabla}\cdot\bar{\mathbf{B}} = 0, \tag{2.2}$$

$$\bar{\rho}\frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla}\bar{p} + \frac{1}{\mu}(\boldsymbol{\nabla}\times\bar{\mathbf{B}})\times\bar{\mathbf{B}} + \bar{\rho}\,[\bar{\nu}\nabla^2\mathbf{v} + \zeta'\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})], \tag{2.3}$$

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \bar{\mathbf{B}}) + \bar{\eta}\nabla^2 \bar{\mathbf{B}}, \tag{2.4}$$

$$\frac{\bar{p}}{\gamma - 1} \frac{D}{Dt}\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) = \bar{\eta}\,\frac{(\boldsymbol{\nabla} \times \mathbf{B})^2}{\mu} - \bar{\rho}\,\bar{\nu}\left[\tfrac{1}{2}e_{ij}\,e_{ij} - \tfrac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{v})^2\right], \tag{2.5}$$

where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the convective derivative. We use the conventional notation for the physical variables: $\mathbf{v}$ and $\bar{\mathbf{B}}$ are the velocity and magnetic induction vectors, $\bar{p}$ and $\bar{\rho}$ are the pressure and density, $\bar{\nu}$, $\bar{\zeta}' = \bar{\zeta} - \frac{1}{3}\bar{\nu}$, $\bar{\eta}$ and γ are the kinematic shear coefficient of viscosity, the volume viscosity, the magnetic diffusion and the adiabatic index respectively, and the perturbations of the magnetic field and velocity are denoted by $\mathbf{b} = (b_r, b_\varphi, b_z)$ and $\mathbf{v} = (u, v, w)$;

$$e_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$$

is the rate-of-strain tensor. It can be shown that, from an energetic point of view, resistivity is dominant in the energy equation, so that the effect of viscosity is neglected.

Finite perturbations of the equilibrium quantities take the form

$$\bar{f} = f_0(r) + f(r, \varphi, z, t), \tag{2.6}$$

where f is the Eulerian perturbation for the density, pressure, and magnetic field. With these considerations, the definition of the total pressure becomes

$$P = p + \frac{\mathbf{B}^2}{2\mu} = p + \frac{\mathbf{B}_0\cdot\mathbf{b}}{\mu} + \frac{\mathbf{b}^2}{2\mu}. \tag{2.7}$$

In linear theory, all of the physical variables oscillate with the same real frequency, ω, (a driven problem operates with real frequencies), and they can be Fourier-analysed. However, in nonlinear theory, this procedure cannot be applied, since the oscillations are no longer in phase. To be as close as possible to the linear theory, we suppose that the oscillations are plane-periodic propagating waves with permanent shape with period $L = 2\pi/k$, and we take all the variables to depend on a combination $\theta = m\varphi + kz - \omega t$ of the independent variables φ, z, and t, rather than on φ, z, and t separately. Here m and k are the azimuthal and longitudinal wavenumbers. We define the square of the Alfvén speed, the Alfvén frequency, the sound speed, and the cusp frequency as follows:

$$v_A^2 = \frac{B_0^2}{\rho_0\mu}, \qquad \omega_A^2 = k^2 v_A^2, \qquad c_S^2 = \frac{\gamma p_0}{\rho_0}, \qquad \omega_C^2 = \frac{c_S^2\omega_A^2}{c_S^2 + v_A^2}. \tag{2.8}$$

The slow resonance can take place where the frequency ω of a propagating wave matches the local cusp frequency, i.e. $\omega = \omega_C(r = r_c)$. The waves that fulfil this condition are called *resonant waves*. In ideal MHD, these resonant waves are confined to an individual magnetic surface without any interaction with neighbouring magnetic surfaces. Dissipative effects can cause coupling of the resonant magnetic surface to neighbouring magnetic surfaces. The vicinity of the resonant surfaces will be effectively in resonance if they do not become too much out of phase with the driver, and in this case one can talk about a *resonant layer*. The resonant layer is located inside the dissipative layer. If the local wave that is in resonant interaction with the global wave is a slow wave then the dissipative layer is called a slow (wave) dissipative layer. It was shown by Ruderman et al. (1997b) that the characteristic thickness of the dissipative layer is of the order of $l_{\mathrm{inh}}R^{-1/3}$. By inspecting the nonlinear dissipative MHD equations (2.2)–(2.5), we can see that the largest nonlinear and dissipative terms are of the form of $f\partial f/\partial z$ and $\partial^2 f/\partial z^2$ multiplied by a dissipative coefficient, where f is any of the so-called 'large' variables, which are of the order of $\epsilon l_{\rm inh}/l_{\rm dis}$. In order to explain what a 'large' variable means, we recall the results of linear MHD (see e.g. Sakurai et al. 1991; Goossens et al. 1995; Erdélyi 1997). In linear ideal MHD, 'large' variables have an $(r - r_c)^{-1}$ singularity in the vicinity of the resonant position r_c. For slow resonances, the 'large' variables are the pressure p, the density ρ, the parallel component of the velocity w, and the magnetic field perturbation b_z. The normal component of velocity and the magnetic field perturbation have a logarithmic singularity, while the rest of the perturbed variables have a regular behaviour. Singularities can be removed by dissipation, which we consider in the present paper.

As a consequence, the 'large' variables have a dimensionless amplitude of the order of $\epsilon R^{1/3}$ in the dissipative layer. This means that the ratio of a nonlinear term (taking into account only quadratic terms) to the dissipative term can be written as

$$\frac{f\partial f/\partial z}{\bar{\nu}\partial^2 f/\partial z^2} \sim \epsilon R^{2/3}. \tag{2.9}$$

It is natural to consider $\epsilon R^{2/3}$ also as a measure of nonlinearity (Ruderman et al. 1997a). Hence we can conclude that linear theories give an adequate result for resonant waves if $\epsilon R^{2/3} \ll 1$; otherwise nonlinear effects have to be taken into account. (Note that this remark also means that nonlinearity plays an important role even for $\epsilon R^{2/3} = \mathcal{O}(1)$.) Since our aim is to derive the nonlinear governing equations in the dissipative layer (where both effects are present), we suppose the nonlinear and dissipative terms are of the same order, i.e. $\epsilon R^{2/3} = \mathcal{O}(1)$. With the aid of these considerations, we introduce the scaling laws

$$\bar{\nu} = \epsilon^{3/2}\nu, \qquad \bar{\zeta}' = \epsilon^{3/2}\zeta', \qquad \bar{\eta} = \epsilon^{3/2}\eta, \tag{2.10}$$

where $\nu, \zeta', \eta \sim \omega_C(r_C)l_{\rm inh}$.

In linear theory, all perturbed quantities are harmonic functions of θ, and their mean values over a period vanish. In nonlinear theory, however, the mean values of perturbations can be non-zero owing to the interaction of the different harmonics. This interaction generates a mean flow outside the dissipative layer. The mean value of a quantity $f(\theta)$ over a period L is defined by

$$\langle f \rangle = \frac{1}{L}\int_0^L f(\theta)\,d\theta. \tag{2.11}$$

It follows from (2.2) that

$$\langle \bar{\rho} u \rangle = \langle b_r \rangle = 0. \tag{2.12}$$

The equilibrium state can be chosen in such a way that the mean values of density, pressure, and magnetic field vanish, i.e.

$$\langle \rho \rangle = \langle p \rangle = \langle b_\varphi \rangle = \langle b_z \rangle = 0. \tag{2.13}$$

This procedure is not applicable for the velocity perturbations, since we suppose a static equilibrium. Thus we divide the φ and z components of velocity into mean and oscillatory parts, as follows:

$$v = U_\varphi + \tilde{v}, \qquad w = U_z + \tilde{w}, \qquad \langle v \rangle = U_\varphi, \qquad \langle w \rangle = U_z, \tag{2.14}$$

where quantities with tildes denote the oscillatory parts of the velocities. The quantities U_φ and U_z respectively describe the mean flow components. These components of flow are generated by the nonlinear interaction of harmonics through resonant absorption of wave momentum in the dissipative layer (Ruderman et al. 1997a). The amplitude of the generated mean flow is determined by the balance between the force created by resonant absorption and shear viscosity.

With these considerations, the MHD equations become

$$\rho_0 \left(\frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{m}{r}\frac{\partial \tilde{v}}{\partial \theta} + k\frac{\partial \tilde{w}}{\partial \theta} \right) + u\frac{d\rho_0}{dr} - \omega\frac{\partial \rho}{\partial \theta} = N_1, \qquad \Psi = ur, \tag{2.15}$$

$$\frac{\partial P}{\partial r} - \rho_0 \frac{\partial}{\partial \theta}\left(\omega u + \frac{kB_0}{\mu\rho_0} b_r \right) = N_2 + DT_1, \tag{2.16}$$

$$\frac{m}{r}\frac{\partial P}{\partial \theta} - \rho_0 \frac{\partial}{\partial \theta}\left(\omega \tilde{v} + \frac{kB_0}{\mu\rho_0} b_\varphi \right) = N_3 + DT_2, \tag{2.17}$$

$$k\frac{\partial P}{\partial \theta} - \rho_0 \frac{\partial}{\partial \theta}\left(\omega \tilde{w} + \frac{kB_0}{\mu\rho_0} b_z \right) - \frac{b_r}{\mu}\frac{dB_0}{dr} = N_4 + DT_3, \tag{2.18}$$

$$kB_0\frac{\partial u}{\partial \theta} + \omega\frac{\partial b_r}{\partial \theta} = N_5 + DT_4, \tag{2.19}$$

$$k\frac{\partial}{\partial \theta}\left(B_0 \tilde{v} + \frac{\omega}{k} b_\varphi \right) = N_6 + DT_5, \tag{2.20}$$

$$\frac{B_0}{r}\frac{\partial \Psi}{\partial r} + u\frac{dB_0}{dr} + B_0\frac{m}{r}\frac{\partial \tilde{v}}{\partial \theta} - \omega\frac{\partial b_z}{\partial \theta} = N_7 + DT_6, \tag{2.21}$$

$$\omega\left(\frac{\partial p}{\partial \theta} - c_S^2\frac{\partial \rho}{\partial \theta} \right) - u\left(\frac{dp_0}{dr} - c_S^2\frac{d\rho_0}{dr} \right) = N_8 + DT_7, \tag{2.22}$$

$$\frac{\partial}{\partial \theta}\left(P - p - \frac{B_0}{\mu} b_z \right) = N_9, \tag{2.23}$$

where we have collected all nonlinear terms (N_j, $j = 1, \ldots, 9$) on the right-hand sides of the equations, and DT_i ($i = 1, \ldots, 6$) denote the dissipative terms. The actual forms of the nonlinear and dissipative terms are given in Appendix A.

The expressions for the mean velocity can be obtained by dividing (2.17)–(2.18) by $\bar{\rho}$ and taking the average value over a period; for example

$$\begin{aligned}
\epsilon^{3/2}\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dU_\varphi}{dr} \right) - \frac{U_\varphi}{r^2} \right] = {} & \frac{m}{r}\left\langle \frac{1}{\bar{\rho}}\frac{\partial P}{\partial \theta} \right\rangle - \rho_0\omega\left\langle \frac{1}{\bar{\rho}}\frac{\partial \tilde{v}}{\partial \theta} \right\rangle - \frac{kB_0}{\mu}\left\langle \frac{1}{\bar{\rho}}\frac{\partial b_\varphi}{\partial \theta} \right\rangle \\
& -\omega\left\langle \frac{\rho}{\bar{\rho}}\frac{\partial v}{\partial \theta} \right\rangle + \left\langle u\frac{\partial v}{\partial r} \right\rangle + \frac{m}{r}\left\langle v\frac{\partial v}{\partial \theta} \right\rangle \\
& +k\left\langle w\frac{\partial v}{\partial \theta} \right\rangle + \frac{1}{r}\langle uv \rangle - \frac{1}{\mu}\left\langle \frac{b_r}{\bar{\rho}}\frac{\partial b_\varphi}{\partial r} \right\rangle \\
& -\frac{m}{r}\frac{1}{\mu}\left\langle \frac{b_\varphi}{\bar{\rho}}\frac{\partial b_\varphi}{\partial \theta} \right\rangle - \frac{k}{\mu}\left\langle \frac{b_z}{\bar{\rho}}\frac{\partial b_\varphi}{\partial \theta} \right\rangle - \frac{1}{\mu r}\left\langle \frac{b_r b_\varphi}{\bar{\rho}} \right\rangle,
\end{aligned} \tag{2.24}$$

$$\begin{aligned}\epsilon^{3/2}\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dU_z}{dr}\right)\right] &= k\left\langle\frac{1}{\bar{\rho}}\frac{\partial P}{\partial\theta}\right\rangle - \rho_0\omega\left\langle\frac{1}{\bar{\rho}}\frac{\partial\tilde{w}}{\partial\theta}\right\rangle - \frac{kB_0}{\mu}\left\langle\frac{1}{\bar{\rho}}\frac{\partial b_z}{\partial\theta}\right\rangle \\ &\quad - \frac{1}{\mu}\frac{dB_0}{dr}\left\langle\frac{b_r}{\bar{\rho}}\right\rangle - \omega\left\langle\frac{\rho}{\bar{\rho}}\frac{\partial w}{\partial\theta}\right\rangle + \left\langle u\frac{\partial w}{\partial r}\right\rangle \\ &\quad + \frac{m}{r}\left\langle v\frac{\partial w}{\partial\theta}\right\rangle + k\left\langle w\frac{\partial w}{\partial\theta}\right\rangle - \frac{1}{\mu}\left\langle\frac{b_r}{\bar{\rho}}\frac{\partial b_z}{\partial r}\right\rangle \\ &\quad - \frac{m}{r}\frac{1}{\mu}\left\langle\frac{b_\varphi}{\bar{\rho}}\frac{\partial b_z}{\partial\theta}\right\rangle - \frac{k}{\mu}\left\langle\frac{b_z}{\bar{\rho}}\frac{\partial b_z}{\partial\theta}\right\rangle. \end{aligned} \tag{2.25}$$

3. Governing equations for the external region

To find a solution of the nonlinear dissipative MHD equations we use the *method of matched asymptotic expansions* (see e.g. Nayfeh 1981). The basis of this procedure is to find solutions in two different regions in the form of expansions and then to match these expansions at the boundary of the two regions. Projecting this method to our problem, the two regions are the domains inside and outside the dissipative layer.

In what follows, we derive the equations that describe the wave motion outside the dissipative layer. The dimensionless amplitude of variables in the external region is of the order of ϵ. In addition, the MHD equations contain terms proportional to $\epsilon^{3/2}$. Therefore we can look for a solution in the form of an asymptotic expansion, i.e.

$$f = \epsilon f^{(1)} + \epsilon^{3/2} f^{(2)} + \dots . \tag{3.1}$$

Here f denotes any perturbed quantity with the exception of the φ and z components of the velocity. Below, we shall show that resonant waves generate a shear flow with an amplitude of the order of $\epsilon^{1/2}$ outside the dissipative layer. Therefore we expand $\tilde{v}$ and $\tilde{w}$ in the form of (3.1), and U_φ and U_z in the form of

$$U_\varphi = \epsilon^{1/2}U_\varphi^{(0)} + \epsilon U_\varphi^{(1)} + \epsilon^{3/2}U_\varphi^{(2)} + \dots, \tag{3.2a}$$

$$U_z = \epsilon^{1/2}U_z^{(0)} + \epsilon U_z^{(1)} + \epsilon^{3/2}U_z^{(2)} + \dots . \tag{3.2b}$$

In the first-order approximation, we obtain a system of linear equations for the variables, denoted with superscript '(1)'. The first-order approximation recovers the linear ideal case. All but two variables can be eliminated algebraically from this system, leading to a system of two first order partial differential equations for $\Psi^{(1)} = u^{(1)}r$ and $P^{(1)}$ that describe the wave motion in the outer region, i.e.

$$D\frac{\partial\Psi^{(1)}}{\partial r} = \omega C r\frac{\partial P^{(1)}}{\partial\theta}, \tag{3.3}$$

and

$$\omega r\frac{\partial^2 P^{(1)}}{\partial r\,\partial\theta} = -\rho_0 D_A \Psi^{(1)}, \tag{3.4}$$

where

$$D = \rho_0 D_A D_C = \rho_0(\omega^2 - \omega_A^2)(c_S^2 + v_A^2)(\omega^2 - \omega_C^2), \tag{3.5}$$

$$C = \omega^4 - \left(\frac{m^2}{r^2} + k^2\right)D_C, \qquad D_C = (c_S^2 + v_A^2)(\omega^2 - \omega_C^2). \tag{3.6}$$

The remaining variables can be calculated using these two quantities as given by

$$v = \frac{m}{r}\frac{\omega}{\rho_0 D_A}P^{(1)}, \qquad w = \frac{k\omega c_S^2}{\rho_0 D_C}P^{(1)}, \tag{3.7}$$

$$b_r^{(1)} = -\frac{f_B}{\omega r}\Psi^{(1)}, \qquad b_\varphi = -\frac{m}{r}\frac{kB_0}{\rho_0 D_A}P^{(1)}, \tag{3.8}$$

$$\frac{\partial b_z}{\partial \theta} = \frac{u^{(1)}}{\omega}\frac{dB_0}{dr} + \frac{B_0}{\rho_0 D_C}(\omega^2 - k^2 c_S^2)P^{(1)}, \tag{3.9}$$

$$\frac{\partial p}{\partial \theta} = \frac{\omega^2 c_S^2}{D_C}\frac{\partial P^{(1)}}{\partial \theta} - \frac{u^{(1)} B_0}{\omega \mu}\frac{dB_0}{dr}, \tag{3.10}$$

$$\frac{\partial \rho}{\partial \theta} = \frac{\omega^2}{D_C}\frac{\partial P^{(1)}}{\partial \theta} + \frac{u^{(1)}}{\omega}\frac{d\rho_0}{dr}. \tag{3.11}$$

Eliminating the pressure from (3.3) and (3.4), we obtain a generalised Hain–Lüst equation for the normal component of the velocity:

$$\frac{\partial}{\partial r}\left[F(r)\frac{\partial \Psi^{(1)}}{\partial r}\right] + G(r)\Psi^{(1)} = 0, \tag{3.12}$$

where

$$F(r) = \frac{D}{rC}, \qquad G(r) = \rho_0 D_A. \tag{3.13}$$

Alternatively, eliminating the normal component of the velocity, we obtain the governing equation for the total pressure:

$$\frac{\partial}{\partial r}\left[\hat{F}(r)\frac{\partial P^{(1)}}{\partial r}\right] + \hat{G}(r)\frac{\partial^2 P^{(1)}}{\partial \theta^2} = 0, \tag{3.14}$$

where

$$\hat{F}(r) = \frac{r}{\rho_0 D_A}, \qquad \hat{G}(r) = \frac{rC}{D}. \tag{3.15}$$

We focus our attention on the problem of slow resonance, which takes place at $r = r_C$, i.e. where the condition $\omega^2 = \omega_C^2(r_C)$ is satisfied. We introduce a new radial variable s, defined by

$$s = r - r_c. \tag{3.16}$$

The slow resonant position ($s = 0$) is a regular singular point of (3.12); therefore we are looking for solutions in the form of a Fröbenius expansion around the resonant position $s = 0$. Using this procedure for (3.3)–(3.11), the expressions of the expansions are

$$\begin{aligned} f = {} & \epsilon[f_1^{(1)}(\theta) + f_2^{(1)}(\theta)\, s \ln|s| + f_3^{(1)}(\theta)\, s + \ldots] \\ & + \epsilon^{3/2}[f_1^{(2)}(\theta)\ln|s| + f_2^{(2)}(\theta) + \ldots] \\ & + \sum_{n=3}^{\infty}\epsilon^{(n+1)/2}[f_1^{(n)}(\theta)\, s^{-n+2} + \ldots], \end{aligned} \tag{3.17}$$

for $\tilde{v}$, b_φ, and P;

$$g = \epsilon[g_1^{(1)}(\theta)\ln|s| + g_2^{(1)}(\theta) + g_3^{(1)}(\theta)\, s\ln|s| + g_4^{(1)}(\theta)\, s + \ldots]$$
$$+\epsilon^{3/2}[g_1^{(2)}(\theta)\ln^2|s| + g_2^{(2)}(\theta)\ln|s| + g_3^{(2)}(\theta) + \ldots]$$
$$+\sum_{n=3}^{\infty}\epsilon^{(n+1)/2}[g_1^{(n)}(\theta)\, s^{-n+1} + g_2^{(n)}(\theta)\, s^{-n+2}\ln|s| + g_3^{(n)}(\theta)\, s^{-n+2} + \ldots], \tag{3.18}$$

for u and b_r; and

$$h = \epsilon[h_1^{(1)}(\theta)\, s^{-1} + h_2^{(1)}(\theta)\ln|s| + h_3^{(1)}(\theta) + \ldots]$$
$$+\epsilon^{3/2}[h_1^{(2)}(\theta)\, s^{-1}\ln|s| + h_2^{(2)}(\theta)\, s^{-1} + \ldots]$$
$$+\sum_{n=3}^{\infty}\epsilon^{(n+1)/2}[h_1^{(n)}(\theta)\, s^{-n} + h_2^{(n)}(\theta)\, s^{-n+1}\ln|s|$$
$$+h_3^{(n)}(\theta)\, s^{-n+1} + \ldots], \tag{3.19}$$

for ρ, p, $\tilde{w}$, and b_z.

Using (3.3)–(3.11), we can obtain for the mean flow in the first-order approximation

$$\frac{d^2U_\varphi^{(0)}}{dr^2} = \frac{d^2U_z^{(0)}}{dr^2} = 0. \tag{3.20}$$

In what follows, we can shall see that $U_\varphi^{(0)}$ and $U_z^{(0)}$ are continuous at $s = 0$. We choose a moving coordinate system in such a way that

$$U_\varphi^{(0)}(0) = U_z^{(0)}(0) = 0. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$U_\varphi^{(0)} = V_\varphi^{\pm} s, \qquad U_z^{(0)} = V_z^{\pm} s, \tag{3.22}$$

where $V_\varphi^{\pm}$ and $V_z^{\pm}$ are constants and the superscripts '$-$' and '$+$' refer to $s < 0$ and $s > 0$.

Proceeding in the same way, one can find the solutions of the subsequent higher-order approximations. For the second order, we obtain an equation similar to (3.20), while for $n \geqslant 2$, the following estimates hold:

$$U_\varphi^{(n)} = \mathcal{O}(s^{-n+1}), \qquad U_z^{(n)} = \mathcal{O}(s^{-n+1}\ln|s|). \tag{3.23}$$

These imply that the mean velocity has a singular behaviour only starting with the second-order approximation.

Finally, the expansions for the mean velocity components are

$$U_\varphi = \epsilon^{1/2}V_\varphi\, s + \sum_{n=1}^{\infty}\epsilon^{n/2+1}V_\varphi^{(n)}(s)\, s^{-n}, \tag{3.24}$$

$$U_z = \epsilon^{1/2}V_z\, s + \epsilon^{3/2}V_z^{(1)}(s)\, s^{-1} + \sum_{n=2}^{\infty}\epsilon^{n/2+1}[V_{z1}^{(n)}(s)\ln|s| + V_{z2}^{(n)}(s)]s^{-n}, \tag{3.25}$$

where the functions $V_\varphi^{(n)}(s)$, $V_z^{(1)}(s)$, $V_{z1}^{(n)}(s)$, and $V_{z2}^{(n)}(s)$ have finite limits for $|s| \to 0$. The important property of the expansions determined by (3.24) and (3.25) is that terms of the lowest order of approximation, which are proportional to $\epsilon^{1/2}$, are very small in the dissipative layer but become large far away from the dissipative layer.

4. Solutions in the dissipative layer

In order to obtain solutions in the internal region (dissipative layer), we can introduce a new variable, provided that the thickness of the dissipative layer is of the order of $l_{\rm inh} R^{-1/3}$. Since we assume that $R \sim \epsilon^{-3/2}$, an appropriate choice for the new variable is $\tau = \epsilon^{-1/2} s$, i.e. $r = \tau' = r_c + \epsilon^{1/2}\tau$. Note, however, that the equilibrium quantities still depend on s and not τ, and we approximate them with the first non-zero term in their Taylor expansions. The new form of the MHD equations (2.15)–(2.23) can be found in Appendix B.

Introducing a similar change of variable in the equations describing the expressions for the φ and z components of the mean velocity, we arrive at

$$\begin{aligned}\epsilon\nu\left(\tau'\frac{d^2U_\varphi}{d\tau^2}+\epsilon^{1/2}\frac{dU_\varphi}{d\tau}-\epsilon\frac{U_\varphi}{\tau'}\right) &= m\tau'\left\langle\frac{1}{\bar\rho}\frac{\partial P}{\partial\theta}\right\rangle-\rho_0\omega\tau'\epsilon^{1/2}\left\langle\frac{1}{\bar\rho}\frac{\partial\tilde v}{\partial\theta}\right\rangle\\ &\quad -\frac{kB_0}{\mu}\tau'\epsilon^{1/2}\left\langle\frac{1}{\bar\rho}\frac{\partial b_\varphi}{\partial\theta}\right\rangle-\tau'\epsilon^{1/2}\omega\left\langle\frac{\rho}{\bar\rho}\frac{\partial v}{\partial\theta}\right\rangle\\ &\quad +\tau'\left\langle u\frac{\partial v}{\partial\tau}\right\rangle+m\epsilon^{1/2}\left\langle v\frac{\partial v}{\partial\theta}\right\rangle\\ &\quad +k\tau'\epsilon^{1/2}\left\langle w\frac{\partial v}{\partial\theta}\right\rangle+\epsilon^{1/2}\langle uv\rangle\\ &\quad -\frac{\tau'}{\mu}\left\langle\frac{b_r}{\bar\rho}\frac{\partial b_\varphi}{\partial\tau}\right\rangle-\frac{m\epsilon^{1/2}}{\mu}\left\langle\frac{b_\varphi}{\bar\rho}\frac{\partial b_\varphi}{\partial\theta}\right\rangle\\ &\quad -\frac{k\tau'\epsilon^{1/2}}{\mu}\left\langle\frac{b_z}{\bar\rho}\frac{\partial b_\varphi}{\partial\theta}\right\rangle-\frac{\epsilon^{1/2}}{\mu}\left\langle\frac{b_rb_\varphi}{\bar\rho}\right\rangle,\end{aligned}\tag{4.1}$$

$$\begin{aligned}\epsilon\nu\left(\tau'\frac{d^2U_z}{d\tau^2}+\epsilon^{1/2}\frac{dU_z}{d\tau}\right) &= k\tau'\epsilon^{1/2}\left\langle\frac{1}{\bar\rho}\frac{\partial P}{\partial\theta}\right\rangle-\rho_0\omega\tau'\epsilon^{1/2}\left\langle\frac{1}{\bar\rho}\frac{\partial\tilde w}{\partial\theta}\right\rangle\\ &\quad -\frac{kB_0}{\mu}\tau'\epsilon^{1/2}\left\langle\frac{1}{\bar\rho}\frac{\partial b_z}{\partial\theta}\right\rangle-\frac{\tau'\epsilon^{1/2}}{\mu}\frac{dB_0}{dr}\left\langle\frac{b_r}{\bar\rho}\right\rangle\\ &\quad -\tau'\epsilon^{1/2}\omega\left\langle\frac{\rho}{\bar\rho}\frac{\partial w}{\partial\theta}\right\rangle+\tau'\left\langle u\frac{\partial w}{\partial\tau}\right\rangle\\ &\quad +m\epsilon^{1/2}\left\langle v\frac{\partial w}{\partial\theta}\right\rangle+k\tau'\epsilon^{1/2}\left\langle w\frac{\partial w}{\partial\theta}\right\rangle\\ &\quad -\frac{\tau'}{\mu}\left\langle\frac{b_r}{\bar\rho}\frac{\partial b_z}{\partial\tau}\right\rangle-\frac{m\epsilon^{1/2}}{\mu}\left\langle\frac{b_\varphi}{\bar\rho}\frac{\partial b_z}{\partial\theta}\right\rangle\\ &\quad -\frac{k\tau'\epsilon^{1/2}}{\mu}\left\langle\frac{b_z}{\bar\rho}\frac{\partial b_z}{\partial\theta}\right\rangle.\end{aligned}\tag{4.2}$$

The procedure for obtaining the internal expansions followed by the procedure for matching the asymptotic expansions at the boundaries has been explained in detail by Ruderman et al. (1997a) and Ballai et al. (1998a). Therefore we give only the expressions of these expansions, and we look for solutions in the internal region directly in the form of

$$P = \epsilon P^{(1)} + \epsilon^{3/2} P^{(2)} + \dots, \tag{4.3}$$

with the same form being found for the expansions for $\tilde{v}$ and b_φ;

$$\rho = \epsilon^{1/2}\rho^{(1)} + \epsilon\rho^{(2)} + \dots, \tag{4.4}$$

with the same form being found for expansions of p, $\tilde{w}$, and b_z.

For the azimuthal and longitudinal components of the mean velocity, we find

$$U_\varphi = \epsilon\, U_\varphi^{(1)} + \dots, \tag{4.5}$$

$$U_z = \epsilon\, U_z^{(2)} + \dots. \tag{4.6}$$

The quantities u and b_r behave like $\ln|s|$ in the vicinity of the slow resonant position, which means that they are of the order of $\epsilon \ln \epsilon$ in the dissipative layer. In fact, these expansions should start with a term proportional to $\epsilon \ln \epsilon$ and should contain terms proportional to $\epsilon \ln \epsilon$ and $\epsilon^{(3/2)} \ln \epsilon$. However, using the simplified version of the asymptotic expansion (see e.g. Ballai et al. 1998a), we suppose that $|\ln \epsilon| \ll \epsilon^{-\kappa}$, for any positive κ and $\epsilon \to +0$, so we can consider $\ln \epsilon$ as a quantity of the order of unity. Therefore we can write the expansion for u and b_r in the form (4.3). The variables v and w are expanded in series of the same form as $\tilde{v}$ and $\tilde{w}$.

In the first-order approximation, we obtain a system of equations for the variables with superscript '(1)'. The most important result found here is a conservation law that states that the first term in the expansion of the total pressure is constant, i.e. it does not depend on τ:

$$\frac{\partial P^{(1)}}{\partial \tau} = 0, \quad \text{i.e.} \quad P^{(1)} = P^{(1)}(\theta). \tag{4.7}$$

All other variables can be expressed as functions of $u^{(1)}$, $w^{(1)}$, and $P^{(1)}$, namely

$$v^{(1)} = -\frac{m c_{Sc}^2}{\rho_{0c} r_c v_{Ac}^2 \omega} P^{(1)}, \qquad b_\varphi^{(1)} = -\frac{k B_{0c}}{\omega} v^{(1)}, \tag{4.8}$$

$$b_r^{(1)} = -\frac{k B_{0c}}{\omega} u^{(1)}, \qquad b_z^{(1)} = -\frac{\omega B_{0c}}{k v_{Ac}^2} w^{(1)}, \tag{4.9}$$

$$p^{(1)} = \frac{\omega \rho_{0c}}{k} w^{(1)}, \qquad \rho^{(1)} = \frac{\omega \rho_{0c}}{k c_{Sc}^2} w^{(1)}. \tag{4.10}$$

Here the subscript 'c' means that the equilibrium quantities are calculated at the resonant position, $s = 0$. In addition, the equation that relates the normal and parallel components of the velocity perturbation is

$$\frac{\partial u^{(1)}}{\partial \tau} + \frac{\omega^2}{k v_{Ac}^2} \frac{\partial w^{(1)}}{\partial \theta} = 0. \tag{4.11}$$

In the second-order approximation, we use only the relations obtained from (2.15), (2.18), (2.21), (2.22), and (2.23). With the aid of (4.8)–(4.11), the second-order approximation can be written as

$$-\omega\frac{\partial\rho^{(2)}}{\partial\theta}+\left(\frac{d\rho_0}{ds}\right)_c u^{(1)}+\frac{\rho_{0c}\omega}{kc_{Sc}^2}u^{(1)}\frac{\partial w^{(1)}}{\partial\tau}+\rho_{0c}\frac{\partial u^{(2)}}{\partial\theta}$$
$$+\frac{\omega^2}{kc_{Sc}^2}\left(\frac{d\rho_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}-\frac{\rho_{0c}\omega^3}{k^2c_{Sc}^2v_{Ac}^2}w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}+\frac{\rho_{0c}}{r_c}u^{(1)}$$
$$-\frac{m^2c_{Sc}^2}{r_c^2v_{Ac}^2\omega}\frac{dP^{(1)}}{d\theta}+\rho_{0c}k\frac{\partial w^{(2)}}{\partial\theta}+2\frac{\rho_{0c}\omega}{c_{Sc}^2}w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}=0, \tag{4.12}$$

$$\omega\rho_{0c}\frac{\partial w^{(2)}}{\partial\theta}+\frac{kB_{0c}}{\mu}\frac{\partial b_z^{(2)}}{\partial\theta}=\frac{B_{0c}k}{\mu\omega}\left(\frac{dB_0}{ds}\right)_c u^{(1)}-\omega\left(\frac{d\rho_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}$$
$$+\frac{B_{0c}\omega}{\mu v_{Ac}^2}\left(\frac{dB_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}+k\frac{dP^{(1)}}{d\theta}-\nu\rho_{0c}\frac{\partial^2 w^{(1)}}{\partial\tau^2}, \tag{4.13}$$

$$B_{0c}\frac{\partial u^{(2)}}{\partial\tau}-\omega\frac{\partial b_z^{(2)}}{\partial\theta}=-\left(\frac{dB_0}{ds}\right)_c u^{(1)}+\frac{\omega^2}{kv_{Ac}^2}\left(\frac{dB_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}$$
$$-\frac{B_{0c}}{r_c}u^{(1)}-\frac{\omega B_{0c}}{kc_{Sc}^2}u^{(1)}\frac{\partial w^{(1)}}{\partial\tau}$$
$$+\frac{B_{0c}m^2c_{Sc}^2}{r_c^2\rho_{0c}v_{Ac}^2\omega}\frac{dP^{(1)}}{d\theta}+\frac{B_{0c}\omega^3}{k^2v_{Ac}^2c_{Sc}^2}w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}$$
$$-\frac{\omega B_{0c}}{kv_{Ac}^2}\eta\frac{\partial^2 w^{(1)}}{\partial\tau^2}, \tag{4.14}$$

$$\omega\left(\frac{\partial p^{(2)}}{\partial\theta}-c_{Sc}^2\frac{\partial\rho^{(2)}}{\partial\theta}\right)=\frac{\omega^3\rho_{0c}}{k^2c_{Sc}^2}(\gamma-1)w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}-\frac{\omega^2}{k}\left(\frac{d\rho_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}$$
$$-\frac{B_{0c}}{\mu}\left(\frac{dB_0}{ds}\right)_c u^{(1)}-\frac{\gamma\omega^2B_{0c}}{\mu kc_{Sc}^2}\left(\frac{dB_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}$$
$$-c_{Sc}^2\left(\frac{d\rho_0}{ds}\right)_c u^{(1)}, \tag{4.15}$$

$$\frac{\partial p^{(2)}}{\partial\theta}+\frac{B_{0c}}{\mu}\frac{\partial b_z^{(2)}}{\partial\theta}=\frac{\omega B_{0c}}{\mu kv_{Ac}^2}\left(\frac{dB_0}{ds}\right)_c\tau\frac{\partial w^{(1)}}{\partial\theta}-\frac{\omega^2\rho_{0c}}{k^2v_{Ac}^2}w^{(1)}\frac{\partial w^{(1)}}{\partial\theta}+\frac{dP^{(1)}}{d\theta}. \tag{4.16}$$

We should note here that the expression of the energy equation in the second-order approximation is similar to the adiabatic equation. This result is due to the fact that in the energy equation at this order of approximation, the non-dissipative terms are of the order of $\epsilon^{3/2}$ while the dissipative terms are of the order of ϵ^2. This result simply means that in the theory of resonant slow MHD waves in isotropic plasmas the energy-loss function in the energy equation can be neglected from the very beginning and we can consider the evolution of the energy to be adiabatic. This result cannot be used to study the nonlinear resonant Alfvén waves, since for resonant Alfvén waves the amplitudes of the nonlinear and the dissipative terms in the energy equation are of the same order.

In what follows, we derive the compatibility condition for the system (4.12)–(4.16). In order to derive this condition, we express $b_z^{(2)}$ and $\rho^{(2)}$ in terms of $w^{(2)}$, $\Psi^{(2)}$, and variables of the first order of approximation. Subsequently, we substitute these expressions into (4.12) and (4.14). These two equations have the same left-hand sides. Subtracting these two equations from each other, one arrives at the compatibility condition. The compatibility condition is an equation describing the nonlinear behaviour of the parallel component of the velocity perturbation, for example

$$\Delta\tau\frac{\partial w^{(1)}}{\partial\theta} - \Lambda w^{(1)}\frac{\partial w^{(1)}}{\partial\theta} + \omega\left(\nu + \frac{c_{Tc}^2}{v_{Ac}^2}\eta\right)\frac{\partial^2 w^{(1)}}{\partial\tau^2} = \frac{\omega^3}{k\rho_{0c}v_{Ac}^2}\frac{dP^{(1)}}{d\theta}, \tag{4.17}$$

where

$$\Delta = \frac{d}{ds}(\omega^2 - \omega_T^2)_{s=0}, \qquad \Lambda = \frac{\omega^3[(\gamma+1)v_{Ac}^2 + 3c_{Sc}^2]}{kc_{Sc}^2(c_{Sc}^2 + v_{Ac}^2)}. \tag{4.18}$$

We see that (4.17) in cylindrical geometry is similar to its counterpart obtained by Ruderman et al. (1997b) for isotropic plasmas in slab geometry. Therefore we can conclude that this equation has a universal character for describing the nonlinear dynamics of resonant slow waves for isotropic dissipative effects. In addition, we can observe that the governing equation for the nonlinear resonant slow waves in the dissipative layer does not contain information about the non-axisymmetric character of wave propagation.

To have a complete description of the wave dynamics, we still need another jump condition. In order to obtain this second jump condition, let us introduce scaled variables of the form

$$q = \epsilon^{1/2}\frac{\omega\delta_c}{v_{Ac}^2}w^{(1)}, \qquad \sigma = \delta_c^{-1}\epsilon^{1/2}\tau, \tag{4.19}$$

where δ_c is the thickness of the slow-wave dissipative layer and is defined by

$$\delta_c = \left[\frac{\omega}{|\Delta|}\left(\nu + \frac{c_{Tc}^2}{v_{Ac}^2}\bar{\eta}\right)\right]^{1/3}, \tag{4.20}$$

by analogy to the linear theory. Outside the dissipative layer, the approximations $u \approx \epsilon u^{(1)}$ and $P \approx \epsilon P^{(1)}$ are still valid. Using these new variables, (4.17) can be written as

$$\operatorname{sign}(\Delta)\sigma\frac{\partial q}{\partial\theta} - Nq\frac{\partial q}{\partial\theta} + k\frac{\partial^2 q}{\partial\sigma^2} = \frac{\omega^4}{k\rho_{0c}v_{Ac}^4|\Delta|}\frac{dP_c}{d\theta}, \tag{4.21}$$

where N is defined as

$$N = \frac{\Lambda v_{Ac}^2}{\omega\delta_c^2|\Delta|}, \tag{4.22}$$

and $P_c = P(r = r_c)$. It is straightforward to see from (4.19) and (4.20) that $\delta_c = \mathcal{O}(l_{\text{inh}}R^{-1/3})$, $q = \mathcal{O}(\epsilon^{1/2}\,l_{\text{inh}}\,R^{-1/3})$, and $N = \mathcal{O}(R^{2/3}\,l_{\text{inh}}^{-2})$. Now we can define the precise form of the so-called nonlinearity parameter already introduced in Sec. 1 in general terms. Since the nonlinearity parameter is the ratio of the nonlinear term to the dissipative term in (4.21), its value is

$$\lambda \sim \epsilon^{1/2}R^{1/3}(kl_{\text{inh}})^{-1}. \tag{4.23}$$

Once again, we can conclude that nonlinearity is important if $\lambda \geqslant 1$. If $\lambda \ll 1$, the nonlinear term can be dropped and the wave dynamics is simply described by linear theory.

Let us proceed with calculating the actual form of the jump conditions. In general, the jump of a quantity is defined by

$$[f] = \lim_{\sigma \to \infty} \{f(\sigma) - f(-\sigma)\}. \tag{4.24}$$

One connection formula can be obtained straight away. The quantity P does not change across the dissipative layer – meaning that there is no jump in total pressure, for example,

$$[P] = 0. \tag{4.25}$$

This connection formula coincides with its counterpart found in linear theory (see, e.g. Hollweg and Yang (1988) for planar geometry, and Sakurai et al. (1991) and Erdélyi (1997) for cylindrical geometry) and in nonlinear theory in slab geometry (see e.g. Ruderman et al. 1997a; Ballai et al. 1998a).

In order to calculate the second connection formula, we use (4.21). Inspecting the asymptotic behaviour of this equation and using the property that $q = \mathscr{O}(\sigma^{-1})$ when $|\sigma| \to \infty$, we arrive at

$$q \simeq \frac{\omega^4}{k\rho_{0c}\Delta v_{Ac}^4 \sigma} P_c(\theta). \tag{4.26}$$

From (4.11), using now the new variables, we obtain

$$\frac{\partial u}{\partial \sigma} = -\frac{\omega}{k}\frac{\partial q}{\partial \theta}, \tag{4.27}$$

where the approximation $u \approx \epsilon u^{(1)}$ is applied. Combining the latter two equations for the normal component of velocity, we find

$$u = -\frac{\omega^5}{k^2 \rho_{0c}\Delta v_{Ac}^4}\frac{dP(\theta)}{d\theta}\ln|\sigma| + u_{\pm}(\theta) + \mathscr{O}(\sigma^{-1}), \tag{4.28}$$

where the functions $u_{\pm}(\theta)$ are connected by

$$u_+(\theta) - u_-(\theta) = -\frac{\omega}{k}\mathscr{P}\int_{-\infty}^{\infty}\frac{\partial q}{\partial \theta}\, d\sigma. \tag{4.29}$$

We take the Cauchy principal part (denoted by $\mathscr{P}$), because the integral is divergent at infinity. The jump in u is then

$$[u] = -\frac{\omega}{k}\mathscr{P}\int_{-\infty}^{\infty}\frac{\partial q}{\partial \theta}\, d\sigma. \tag{4.30}$$

This equation is the *nonlinear analogue* of the connection formula for the normal component of the velocity derived in linear theory. However, in contrast to linear theory, where the jump in u was given in terms of the total pressure and equilibrium quantities, here the jump in the normal velocity contains an integral of an unknown function q. Since we cannot find an analytical solution to (4.21), we have to solve this equation *together* with the system (3.3), (3.4) that describes the plasma motion outside the dissipative layer. The two connection formulae (4.25) and (4.30) provide the necessary boundary conditions for a unique solution.

5. Generation of mean flow

Let us recall that the nonlinear interaction of harmonics in the dissipative layer generates a mean flow parallel to the resonant surface outside the dissipative layer.

In order to derive an expression for this mean flow, we should first obtain solutions to (4.1) and (4.2) describing the mean flow components in the dissipative layer, and then match these solutions with the outer solutions obtained from (2.24) and (2.25). Using the expansions for the φ component of the mean flow and the relations (4.8)–(4.11), we obtain

$$\nu \frac{d^2 U_\varphi^{(1)}}{d\tau^2} = \frac{m\omega}{k r_c \rho_{0c} v_{Ac}^2} \left\langle \tilde{w}^{(1)} \frac{dP^{(1)}}{d\theta} \right\rangle . \tag{5.1}$$

With the aid of (4.17), we can rewrite the above equation and arrive at

$$\frac{d^2 U_\varphi^{(1)}}{d\tau^2} = \frac{m}{\nu r_c \omega} \left(\nu + \frac{\omega_{Tc}^2}{\omega_{Ac}^2} \eta \right) \left\langle \tilde{w}^{(1)} \frac{\partial^2 \tilde{w}^{(1)}}{\partial \tau^2} \right\rangle . \tag{5.2}$$

Now, with the aid of the matching condition, we obtain

$$\begin{aligned} [V_\varphi] = V_\varphi^+ - V_\varphi^- &= \lim_{\tau\to\infty} \frac{dU_\varphi^{(1)}}{d\tau} - \lim_{\tau\to-\infty} \frac{dU_\varphi^{(1)}}{d\tau} \\ &= -\frac{m}{r_c\omega} \left(1 + \frac{\omega_{Tc}^2}{\omega_{Ac}^2} P_m^{-1} \right) \int_{-\infty}^{\infty} \left\langle \left(\frac{\partial \tilde{w}^{(1)}}{\partial \tau} \right)^2 \right\rangle d\tau, \end{aligned} \tag{5.3}$$

where $P_m = \nu/\eta$ is the magnetic Prandtl number, which is of the order of 10^{-6} in the solar photosphere, 5×10^{-3} in the chromosphere, and 10^8 in the low corona. The φ component of the mean velocity depends on the azimuthal wavenumber and the position where the slow resonance takes place.

Collecting terms of the order ϵ^2 in (4.2) and using (4.7)–(4.11), we obtain for the z component of the mean velocity

$$\begin{aligned} \nu \frac{d^2 U_z^{(2)}}{d\tau^2} = &\left\langle u^{(1)} \frac{\partial \tilde{w}^{(2)}}{\partial \tau} + u^{(2)} \frac{\partial \tilde{w}^{(1)}}{\partial \tau} \right\rangle + \frac{\omega}{k c_{Sc}^2} \left\langle u^{(1)} \tilde{w}^{(1)} \frac{\partial \tilde{w}^{(1)}}{\partial \tau} \right\rangle \\ &+ \frac{1}{\rho_{0c}} \left(\frac{d\rho_0}{ds} \right)_c \left\langle u^{(1)} \frac{\partial (\tau \tilde{w}^{(1)})}{\partial \tau} \right\rangle + \frac{\omega}{k B_{0c}} \left\langle b_r^{(2)} \frac{\partial \tilde{w}^{(1)}}{\partial \tau} \right\rangle + \frac{k B_{0c}}{\mu \rho_{0c} \omega} \left\langle u^{(1)} \frac{\partial b_z^{(2)}}{\partial \tau} \right\rangle . \end{aligned} \tag{5.4}$$

After some cumbersome algebra (using (B 5), (4.11), (4.13), (4.17), and the first- and second-order approximations), the matching conditions yield

$$\begin{aligned} [V_z] = V_z^+ - V_z^- &= \lim_{\tau\to\infty} \frac{dU_z^{(1)}}{d\tau} - \lim_{\tau\to-\infty} \frac{dU_z^{(1)}}{d\tau} \\ &= -\frac{\omega}{k v_{Ac}^2} \left(1 + P_m^{-1} \right) \int_{-\infty}^{\infty} \left\langle \left(\frac{\partial \tilde{w}^{(1)}}{\partial \tau} \right)^2 \right\rangle d\tau. \end{aligned} \tag{5.5}$$

Outside the dissipative layer, we have found the following approximate relations:

$$\bar{U}_\varphi - U_{\varphi c} = \mathcal{O}(\epsilon^2 Rks), \qquad \bar{U}_z - U_{zc} = \mathcal{O}(\epsilon^2 Rks), \tag{5.6}$$

where $U_{\varphi c}$ and U_{zc} are the values of $\bar{U}_\varphi$ and $\bar{U}_z$ at $s = 0$. When $\epsilon R^{2/3} \sim 1$ and $s \sim k^{-1}$, then $\bar{U}_\varphi - U_{\varphi c}$ and $\bar{U}_z - U_{zc}$ are of the order of $\epsilon^{1/2}$, in perfect agreement with the suppositions made earlier. Equations (5.3) and (5.5) give the jumps in the derivatives of the components of the generated mean flow.

6. Conclusions

This paper has concentrated on the nonlinear behaviour of resonant slow MHD waves in magnetic flux tubes when non-axisymmetric propagation is taken into account. The scaling method applied here divides the domain into two regions, where the behaviour and dynamics of the waves are governed by a different set of equations.

In the outer domain, for example outside the dissipative layer, the wave motion is described by the equations of ideal MHD and is governed by two coupled first-order ordinary differential equations derived for example for the normal component of the velocity and the Eulerian perturbation of total pressure. These governing equations obviously coincide with their counterparts found in ideal MHD.

In the inner domain, for example in the dissipative layer, the wave dynamics is governed by an inhomogeneous nonlinear partial differential equation derived for example for the z component of the velocity, where the inhomogeneous part originates from the driving term. Since the dissipative layer is very narrow and embraces the resonant surface, it can be considered as a surface of discontinuity when solving the governing partial differential equations outside the dissipative layer. The connection formulae obtained for the total pressure and the normal component of the velocity give the jumps in these quantities across the dissipative layer, thereby providing boundary conditions at the resonant surface of discontinuity.

The nonlinear interaction between harmonics generates a mean flow outside the dissipative layer. For the mean flow, analytical expressions have been found that are piecewise-linear continuous functions of the radial coordinate r. However, the derivatives of these shear velocities exhibit a jump across the dissipative layer.

In a follow-up paper, we shall apply these results to the problem of resonant interaction of p-modes with slow MHD waves in the magnetic canopy.

Acknowledgements

I. Ballai acknowledges financial support by the 'Onderzoeksfonds KU Leuven' and the warm hospitality received during his visit at the Space and Atmosphere Research Center, Department of Applied Mathematics, University of Sheffield, where part of this work was carried out. R. Erdélyi acknowledges M. Kéray for patient encouragement. R. E. and I. B. also acknowledge financial support obtained from the NSF Hungary (OTKA, ref. TO32462). The authors also thank M. S. Ruderman for valuable discussions and suggestions.

Appendix A. The dissipative and nonlinear terms in the governing equations outside the dissipative layer

The dissipative terms (denoted by DT_i, $i = 1, \ldots, 6$) and the nonlinear terms (denoted by N_j, $j = 1, \ldots, 9$) in the system (2.15)–(2.23) are given by

$$DT_1 = \bar{\rho}\,\epsilon^{3/2}\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \left(\frac{m^2}{r^2} + k^2\right)\frac{\partial^2 u}{\partial \theta^2} - \frac{2m}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right]$$
$$+\bar{\rho}\epsilon^{3/2}\zeta'\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{m}{r}\frac{\partial v}{\partial \theta} + k\frac{\partial w}{\partial \theta}\right], \tag{A 1}$$

$$DT_2 = \bar{\rho}\epsilon^{3/2}\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \left(\frac{m^2}{r^2}+k^2\right)\frac{\partial^2 v}{\partial\theta^2} + \frac{2m}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2}\right]$$

$$+\bar{\rho}\epsilon^{3/2}\zeta'\frac{m}{r}\left[\frac{1}{r}\frac{\partial^2(ur)}{\partial r\partial\theta} + \frac{m}{r}\frac{\partial^2 v}{\partial\theta^2} + k\frac{\partial^2 w}{\partial\theta^2}\right], \tag{A 2}$$

$$DT_3 = \bar{\rho}\epsilon^{3/2}\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \left(\frac{m^2}{r^2}+k^2\right)\frac{\partial^2 w}{\partial\theta^2}\right]$$

$$+\bar{\rho}\,\epsilon^{3/2}\zeta' k\left[\frac{1}{r}\frac{\partial^2(ur)}{\partial r\partial\theta} + \frac{m}{r}\frac{\partial^2 v}{\partial\theta^2} + k\frac{\partial^2 w}{\partial\theta^2}\right], \tag{A 3}$$

$$DT_4 = \epsilon^{3/2}\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_r}{\partial r}\right) + \left(\frac{m^2}{r^2}+k^2\right)\frac{\partial^2 b_r}{\partial\theta^2} - \frac{2m}{r^2}\frac{\partial b_\varphi}{\partial\theta} - \frac{b_r}{r^2}\right], \tag{A 4}$$

$$DT_5 = \epsilon^{3/2}\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_\varphi}{\partial r}\right) + \left(\frac{m^2}{r^2}+k^2\right)\frac{\partial^2 b_\varphi}{\partial\theta^2} + \frac{2m}{r^2}\frac{\partial b_r}{\partial\theta} - \frac{b_\varphi}{r^2}\right], \tag{A 5}$$

$$DT_6 = \epsilon^{3/2}\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_z}{\partial r}\right) + \left(\frac{m^2}{r^2}+k^2\right)\frac{\partial^2 b_z}{\partial\theta^2}\right]. \tag{A 6}$$

$$DT_7 = \frac{\epsilon^{3/2}}{\mu}\eta\bar{\rho}(\gamma-1)\left[\frac{dB_0}{dr}\left(\frac{\partial b_z}{\partial r} - k\frac{\partial b_r}{\partial\theta}\right) + \left(\frac{m}{r}\frac{\partial b_z}{\partial\theta} - k\frac{\partial b_\varphi}{\partial\theta}\right)^2\right.$$

$$\left.+\left(k\frac{\partial b_r}{\partial\theta} - \frac{\partial b_z}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial r}(rb_\varphi) - \frac{m}{r}\frac{\partial b_r}{\partial\theta}\right)^2\right]. \tag{A 7}$$

The nonlinear quantities N_i $(i = 1, \ldots, 9)$ are as follows:

$$N_1 = -u\frac{\partial\rho}{\partial r} - \left(\frac{m}{r}v + kw\right)\frac{\partial\rho}{\partial\theta} - \rho\left(\frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{m}{r}\frac{\partial v}{\partial\theta} + k\frac{\partial w}{\partial\theta}\right), \tag{A 8}$$

$$N_2 = \rho\omega\frac{\partial u}{\partial\theta} - \bar{\rho}\left(u\frac{\partial u}{\partial r} + \frac{m}{r}v\frac{\partial u}{\partial\theta} + kw\frac{\partial u}{\partial\theta} - \frac{v^2}{r}\right)$$

$$+\frac{1}{\mu}\left[b_r\frac{\partial b_r}{\partial r} + \left(\frac{m}{r}b_\varphi + kb_z\right)\frac{\partial b_r}{\partial\theta} - \frac{b_\varphi^2}{r}\right], \tag{A 9}$$

$$N_3 = \rho\omega\frac{\partial v}{\partial\theta} - \bar{\rho}\left(u\frac{\partial v}{\partial r} + \frac{m}{r}v\frac{\partial v}{\partial\theta} + kw\frac{\partial v}{\partial\theta} + \frac{uv}{r}\right)$$

$$+\frac{1}{\mu}\left[b_r\frac{\partial b_\varphi}{\partial r} + \left(\frac{m}{r}b_\varphi + kb_z\right)\frac{\partial b_\varphi}{\partial\theta} + \frac{b_r b_\varphi}{r}\right], \tag{A 10}$$

$$N_4 = \rho\omega\frac{\partial w}{\partial\theta} - \bar{\rho}\left(u\frac{\partial w}{\partial r} + \frac{m}{r}v\frac{\partial w}{\partial\theta} + kw\frac{\partial w}{\partial\theta}\right)$$

$$+\frac{1}{\mu}\left[b_r\frac{\partial b_z}{\partial r} + \left(\frac{m}{r}b_\varphi + kb_z\right)\frac{\partial b_z}{\partial\theta}\right], \tag{A 11}$$

$$N_5 = \frac{m}{r}\frac{\partial}{\partial\theta}(vb_r - ub_\varphi) - k\frac{\partial}{\partial\theta}(ub_z - wb_r), \tag{A 12}$$

$$N_6 = k\frac{\partial}{\partial\theta}(wb_\varphi - vb_z) - \frac{\partial}{\partial r}(vb_r - ub_\varphi), \tag{A 13}$$

$$N_7 = \frac{1}{r}\frac{\partial}{\partial r}\left[r(wb_r - ub_z)\right] - \frac{m}{r}\frac{\partial}{\partial\theta}(vb_z - wb_\varphi), \tag{A 14}$$

$$\begin{aligned} N_8 = \frac{1}{\rho_0}\Bigg[&\omega\left(\gamma p\frac{\partial\rho}{\partial\theta} - \rho\frac{\partial p}{\partial\theta}\right) + u\left(\rho\frac{dp_0}{dr} - \gamma p\frac{d\rho_0}{dr} + \bar{\rho}\frac{\partial p}{\partial r} - \bar{p}\,\gamma\,\frac{\partial\rho}{\partial r}\right)\\ &-\left(\bar{p}\,\gamma\,\frac{\partial\rho}{\partial\theta} - \bar{\rho}\frac{\partial p}{\partial\theta}\right)\left(\frac{m}{r}v + kw\right)\Bigg], \end{aligned} \tag{A 15}$$

$$N_9 = \frac{1}{2\mu}\frac{\partial}{\partial\theta}(b_r^2 + b_\varphi^2 + b_z^2). \tag{A 16}$$

Appendix B. The modified MHD equations in the dissipative layer

$$\begin{aligned} &-\omega\tau'\epsilon^{1/2}\frac{\partial\rho}{\partial\theta} + u\tau'\epsilon^{1/2}\frac{d\rho_0}{ds} + \tau' u\frac{\partial\rho}{\partial\tau} + \epsilon^{1/2}(vm + wk\tau')\frac{\partial\rho}{\partial\theta}\\ &\quad+\bar{\rho}\left[r_c\frac{\partial u}{\partial\tau} + \epsilon^{1/2}\frac{\partial(u\tau)}{\partial\tau} + \epsilon^{1/2}m\frac{\partial v}{\partial\theta} + \tau'\epsilon^{1/2}k\frac{\partial w}{\partial\theta}\right] = 0, \end{aligned} \tag{B 1}$$

$$\begin{aligned} &-\bar{\rho}\omega\tau'\epsilon^{1/2}\frac{\partial u}{\partial\theta} + \bar{\rho}\left(\tau' u\frac{\partial u}{\partial\tau} + m\epsilon^{1/2}v\frac{\partial u}{\partial\theta} + k\tau'\epsilon^{1/2}w\frac{\partial u}{\partial\theta} - \epsilon^{1/2}v^2\right)\\ &= -\tau'\frac{\partial P}{\partial\tau} + \frac{1}{\mu}\left(kB_0\tau'\epsilon^{1/2}\frac{\partial b_r}{\partial\theta} + \tau' b_r\frac{\partial b_r}{\partial\tau}\right.\\ &\quad\left.+\tau'\epsilon^{1/2}kb_z\frac{\partial b_r}{\partial\theta} - \epsilon^{1/2}b_\varphi^2 + \epsilon^{1/2}mb_\varphi\frac{\partial b_r}{\partial\theta}\right)\\ &\quad+\epsilon\nu\bar{\rho}\left[\tau'\frac{\partial^2 u}{\partial\tau^2} + \epsilon^{1/2}\frac{\partial u}{\partial\tau} + \epsilon\tau'\left(\frac{m^2}{\tau'^2} + k^2\right)\frac{\partial^2 u}{\partial\theta^2} - \frac{2m}{\tau'}\,\epsilon\,\frac{\partial v}{\partial\theta} - \frac{\epsilon}{\tau'}u\right]\\ &\quad+\epsilon\zeta'\bar{\rho}\frac{\partial}{\partial\tau}\left[r_c\frac{\partial u}{\partial\tau} + \epsilon^{1/2}\frac{\partial(u\tau)}{\partial\tau} + \epsilon^{1/2}\left(m\frac{\partial v}{\partial\theta} + k\tau'\frac{\partial w}{\partial\theta}\right)\right], \end{aligned} \tag{B 2}$$

$$\begin{aligned} &-\bar{\rho}\,\omega\tau'\epsilon^{1/2}\frac{\partial v}{\partial\theta} + \bar{\rho}\left(\tau' u\frac{\partial v}{\partial\tau} + m\epsilon^{1/2}v\frac{\partial v}{\partial\theta} + k\tau'\epsilon^{1/2}w\frac{\partial v}{\partial\theta} + \epsilon^{1/2}uv\right)\\ &= -\epsilon^{1/2}m\frac{\partial P}{\partial\theta} + \frac{1}{\mu}\left(kB_0\tau'\epsilon^{1/2}\frac{\partial b_\varphi}{\partial\theta} + \tau' b_r\frac{\partial b_\varphi}{\partial\tau}\right.\\ &\quad\left.+\tau'\epsilon^{1/2}kb_z\frac{\partial b_\varphi}{\partial\theta} + \epsilon^{1/2}b_r b_\varphi + \epsilon^{1/2}mb_\varphi\frac{\partial b_\varphi}{\partial\theta}\right)\\ &\quad+\epsilon\nu\bar{\rho}\left[\tau'\frac{\partial^2 v}{\partial\tau^2} + \epsilon^{1/2}\frac{\partial v}{\partial\tau} + \epsilon\tau'\left(\frac{m^2}{\tau'^2} + k^2\right)\frac{\partial^2 v}{\partial\theta^2} - \frac{2m}{\tau'}\,\epsilon\,\frac{\partial u}{\partial\theta} - \frac{\epsilon}{\tau'}v\right]\\ &\quad+\epsilon^{3/2}\zeta'\bar{\rho}m\left[\frac{1}{\tau'}\frac{\partial^2(u\tau')}{\partial\tau\partial\theta} + \epsilon^{1/2}\left(\frac{m}{\tau'}\frac{\partial^2 v}{\partial\theta^2} + k\frac{\partial^2 w}{\partial\theta^2}\right)\right], \end{aligned} \tag{B 3}$$

$$
\begin{aligned}
&-\bar{\rho}\,\omega\tau'\epsilon^{1/2}\frac{\partial w}{\partial\theta}+\bar{\rho}\left(\tau' u\frac{\partial w}{\partial\tau}+m\epsilon^{1/2}v\frac{\partial w}{\partial\theta}+k\tau'\epsilon^{1/2}w\frac{\partial w}{\partial\theta}\right)\\
&\quad=-k\epsilon^{1/2}\tau'\frac{\partial P}{\partial\theta}+\frac{1}{\mu}\left(kB_0\tau'\epsilon^{1/2}\frac{\partial b_z}{\partial\theta}+\tau'\epsilon^{1/2}b_r\frac{dB_0}{ds}\right.\\
&\qquad\left.+\tau' b_r\frac{\partial b_z}{\partial\tau}+\tau'\epsilon^{1/2}kb_z\frac{\partial b_z}{\partial\theta}+\epsilon^{1/2}mb_\varphi\frac{\partial b_z}{\partial\theta}\right)\\
&\qquad+\epsilon\nu\bar{\rho}\left[\tau'\frac{\partial^2 w}{\partial\tau^2}+\epsilon^{1/2}\frac{\partial w}{\partial\tau}+\epsilon\tau'\left(\frac{m^2}{\tau'^2}+k^2\right)\frac{\partial^2 w}{\partial\theta^2}\right]\\
&\qquad+\epsilon^{3/2}\zeta'\bar{\rho}k\left[\frac{\partial^2(u\tau')}{\partial\tau\partial\theta}+\epsilon^{1/2}\left(m\frac{\partial^2 v}{\partial\theta^2}+k\tau'\frac{\partial^2 w}{\partial\theta^2}\right)\right],
\end{aligned}
\tag{B 4}
$$

$$
\begin{aligned}
-\omega\tau'\epsilon^{1/2}\frac{\partial b_r}{\partial\theta}&=kB_0\tau'\epsilon^{1/2}\frac{\partial u}{\partial\theta}+m\epsilon^{1/2}\frac{\partial}{\partial\theta}(ub_\varphi-vb_r)\\
&\quad-k\tau'\epsilon^{1/2}\frac{\partial}{\partial\theta}(wb_r-ub_z)+\epsilon\eta\left[\tau'\frac{\partial^2 b_r}{\partial\tau^2}+\epsilon^{1/2}\frac{\partial b_r}{\partial\tau}\right.\\
&\quad\left.+\epsilon\tau'\left(\frac{m^2}{\tau'^2}+k^2\right)\frac{\partial^2 b_r}{\partial\theta^2}-\frac{2m}{\tau'}\epsilon\frac{\partial b_\varphi}{\partial\theta}-\frac{\epsilon}{\tau'}b_r\right],
\end{aligned}
\tag{B 5}
$$

$$
\begin{aligned}
-\omega\tau'\epsilon^{1/2}\frac{\partial b_\varphi}{\partial\theta}&=kB_0\tau'\epsilon^{1/2}\frac{\partial v}{\partial\theta}+k\tau'\epsilon^{1/2}\frac{\partial}{\partial\theta}(vb_z-wb_\varphi)-\tau'\frac{\partial}{\partial\tau}(ub_\varphi-vb_r)\\
&\quad+\epsilon\eta\left[\tau'\frac{\partial^2 b_\varphi}{\partial\tau^2}+\epsilon^{1/2}\frac{\partial b_\varphi}{\partial\tau}+\epsilon\tau'\left(\frac{m^2}{\tau'^2}+k^2\right)\frac{\partial^2 b_\varphi}{\partial\theta^2}-\frac{2m}{\tau'}\epsilon\frac{\partial b_r}{\partial\theta}-\frac{\epsilon}{\tau'}b_\varphi\right],
\end{aligned}
\tag{B 6}
$$

$$
\begin{aligned}
-\omega\tau'\epsilon^{1/2}\frac{\partial b_z}{\partial\theta}&=-B_0\left[r_c\frac{\partial u}{\partial\tau}+\epsilon^{1/2}\frac{\partial(u\tau)}{\partial\tau}\right]-u\tau'\epsilon^{1/2}\frac{dB_0}{ds}-m\epsilon^{1/2}B_0\frac{\partial v}{\partial\theta}\\
&\quad+\frac{\partial}{\partial\tau}[\tau'(wb_r-ub_z)]-m\epsilon^{1/2}\frac{\partial}{\partial\theta}(vb_z-wb_\varphi)\\
&\quad+\epsilon\eta\left[\tau'\frac{\partial^2 b_z}{\partial\tau^2}+\epsilon^{1/2}\frac{\partial b_z}{\partial\tau}+\epsilon\tau'\left(\frac{m^2}{\tau'^2}+k^2\right)\frac{\partial^2 b_z}{\partial\theta^2}\right],
\end{aligned}
\tag{B 7}
$$

$$
\begin{aligned}
&\omega\tau'\epsilon^{1/2}\left(\frac{\partial p}{\partial\theta}-c_S^2\frac{\partial\rho}{\partial\theta}\right)-u\tau'\epsilon^{1/2}\left(\frac{dp_0}{ds}-c_S^2\frac{d\rho_0}{ds}\right)\\
&\quad=\frac{1}{\rho_0}\left[\omega\tau'\epsilon^{1/2}\left(\gamma p\frac{\partial\rho}{\partial\theta}-\rho\frac{\partial p}{\partial\theta}\right)+u\left(\rho\tau'\epsilon^{1/2}\frac{dp_0}{ds}-\gamma\,p\,\tau'\epsilon^{1/2}\frac{d\rho_0}{ds}\right.\right.\\
&\qquad\left.\left.+\bar{\rho}\,\tau'\frac{\partial p}{\partial\tau}-\bar{\rho}\,\gamma\,\tau'\frac{\partial\rho}{\partial\tau}\right)-\left(\bar{\rho}\,\gamma\frac{\partial\rho}{\partial\theta}-\bar{\rho}\frac{\partial p}{\partial\theta}\right)(vm+wk\tau')\epsilon^{1/2}\right]\\
&\qquad+\frac{\epsilon^{3/2}}{\mu}\eta\,\bar{\rho}\,\tau'(\gamma-1)\left[\frac{dB_0}{ds}\left(\frac{\partial b_z}{\partial\tau}-k\epsilon^{1/2}\frac{\partial b_r}{\partial\theta}\right)\right.\\
&\qquad+\epsilon^{1/2}\left(\frac{m}{\tau'}\frac{\partial b_z}{\partial\theta}-k\frac{\partial b_\varphi}{\partial\theta}\right)^2+\left(k\frac{\partial b_r}{\partial\theta}-\epsilon^{-1/2}\frac{\partial b_z}{\partial\tau}\right)^2\\
&\qquad\left.+\left(\frac{\epsilon^{-1/2}}{\tau'}\frac{\partial}{\partial\tau}(\tau' b_\varphi)-\frac{m}{\tau'}\frac{\partial b_r}{\partial\theta}\right)^2\right],
\end{aligned}
\tag{B 8}
$$

$$P = p + \frac{B_0}{\mu} b_z + \frac{1}{2\mu}(b_r^2 + b_\varphi^2 + b_z^2). \tag{B 9}$$

References

Acton, L., Tsuveta, S., Ogawara, Y., Bentley, R., Bruver, M., Canfield, R. et al. 1992 The YOHKOH mission for high-energy solar physics. *Science* **258**, 618.

Appert, K., Václavík, J. and Villard, L. 1984 Spectrum of low-frequency, nonaxisymmetric oscillations in a cold, current-carrying plasma column. *Phys. Fluids* **27**, 432.

Ballai, I., Ruderman, M. S. and Erdélyi, R. 1998a Nonlinear theory of slow dissipative layers in anisotropic plasmas. *Phys. Plasmas* **5**, 252.

Ballai, I., Erdélyi, R. and Ruderman, M. S. 1998b Interaction of sound waves with slow dissipative layers in anisotropic plasmas in the approximation of weak nonlinearity. *Phys. Plasmas* **5**, 2264.

Browning, P. K. 1991 Mechanisms of solar coronal heating. *Plasma Phys. Contr. Nucl. Fusion* **33**, 539.

Chen, L. and Hasegawa, A. 1974 Plasma heating by spatial resonance of Alfvén waves. *Phys. Fluids* **17**, 1399.

Davila, J. M. 1987 Heating of the solar corona by the resonant absorption of Alfvén waves. *Astrophys. J.* **317**, 514.

Erdélyi, R. 1997 Analytical solutions for cusp resonance in dissipative MHD. *Solar Phys.* **171**, 49.

Erdélyi, R. 1998 Resonant absorption of Alfvén waves in steady coronal loops. *Solar Phys.* **180**, 213.

Erdélyi R. and Goossens, M. 1995 Resonant absorption of Alfvén waves in coronal loops in visco-resistive MHD. *Astron. Astrophys.* **294**, 575.

Goossens, M. 1991 MHD waves and wave heating in non-uniform plasmas. In: *Advances in Solar System Magnetohydrodynamics* (ed. E. R. Priest and A. Hood), p. 135. Cambridge University Press.

Goossens, M., Ruderman, M. S. and Hollweg, J. V. 1995 Dissipative MHD solutions for resonant Alfvén waves in 1-dimensional magnetic flux tubes. *Solar Phys.* **175**, 75.

Heyvaerts, J. and Priest, E. R. 1983 Coronal heating by phase-mixed shear Alfven waves. *Astron. Astrophys.* **117**, 220.

Hollweg, J. V. 1991 Alfvén waves. In: *Mechanism of Chromospheric and Coronal Heating*, (ed. P. Ulmschneider, E. R. Priest and R. Rosner), p. 423. Springer-Verlag, New York.

Hollweg, J. V. and Yang, G. 1988 Resonant absorption of compressible magnetohydrodynamic waves at thin 'surfaces'. *J. Geophys. Res.* **93**, 5423.

Ionson, J. A. 1978 Resonant absorption of Alfvénic surface waves and the heating of solar coronal loops. *Astrophys. J.* **226**, 650.

Kappraff, J. M. and Tataronis, J. A. 1977 Resistive effects on Alfvén wave heating. *J. Plasma Phys.* **18**, 209.

Narain, U. and Ulmschneider, P. 1996 Chromospheric and coronal heating mechanisms II. *Space Sci. Rev.* **75**, 453.

Nayfeh, A. H. 1981 *Introduction to Perturbation Techniques.* Wiley-Interscience, New York.

Ofman, L., Davila, J. M. and Steinolfson 1994 Nonlinear studies of coronal heating by resonant absorption of Alfvén waves. *Geophys. Res. Lett.* **21**, 2259.

Ofman, L. and Davila, J. M. 1995 Nonlinear resonant absorption of Alfvén waves in three dimensions, scaling laws and coronal heating. *J. Geophys. Res.* **100**, 23427.

Poedts, S., Kerner, W. and Goossens, M. 1989 Alfvén-wave heating in resistive MHD. *J. Plasma Phys.* **42**, 27.

Poedts, S., Goossens, M. and Kerner, W. 1990 On the efficiency of the coronal loop heating by resonant absorption. *Astrophys. J.* **360**, 279.

Rae, C.I. and Roberts, I. 1981 Surface waves and the heating of the corona. *Geophys. Astrophys. Fluid Dyn.* **18**, 197.

Roberts, B. 1991a Magnetohydrodynamic waves in the Sun. In: *Advances in Solar System Magnetohydrodynamics* (ed. E. R. Priest and A. W. Hood), p. 105. Cambridge University Press.

Roberts, B. 1991b Waves in the solar atmosphere. *Geophys. Astrophys. Fluid Dyn.* **62**, 83.

Roberts, B. and Ulmschneider, P. 1998 Dynamics of flux tubes in the solar atmosphere: Theory. In: *Solar and Heliospheric Plasma Physics* (ed. C. E. Alissandrakis, G. Simnett and L.Vlahos), p. 37. Lecture Notes in Physics, Vol. 489, Springer-Verlag, Berlin.

Rosner, R., Tucker, W. and Vaiana, G. S. 1978 Dynamics of the quiscent solar corona. *Astrophys. J.* **220**, 643.

Ruderman, M. S., Hollweg, J. V. and Goossens, M. 1997a Nonlinear theory of resonant slow waves in dissipative layers. *Phys. Plasmas* **4**, 75.

Ruderman, M. S., Goossens, M. and Hollweg, J. V. 1997b Nonlinear theory of the interaction of sound waves with an inhomogeneous magnetized plasma in the resonant slow wave layer. *Phys. Plasmas* **4**, 91.

Sakurai, T., Goossens, M. and Hollweg, J. V. 1991 Resonant behaviour of MHD waves on magnetic flux tubes, I. Connection formulae at the resonant surface. *Solar Phys.* **133**, 247.

Tataronis, J. A. 1975 Energy absorption in the continuous spectrum of ideal MHD. *J. Plasma Phys.* **13**, 87.

Tataronis, J. A. and Grossmann, W. 1976 On Alfvén wave heating and transit time magnetic pumping in the guiding-centre model of a plasma. *Nucl. Fusion* **16**, 667.

Wright, A. N. and Allan, W. 1996 Structure, phase motion and heating within Alfvén resonances. *J. Geophys. Res.* **101**, 17399.

Zirker, J. 1993 Coronal heating. *Solar Phys.* **148**, 43.