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Linear and nonlinear resonant interaction of sound waves in dissipative layers

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Abstract. The theory of resonant nonlinear magnetohydrodynamic (MHD) waves in dissipative *steady* plasmas developed by Ballai and Erdélyi is used to study the effect of steady flows on nonlinear resonant heating of MHD waves in (a) linear, (b) weakly and (c) strongly nonlinear approximations. Nonlinear connection formulae for slow MHD waves are derived. This nonlinear theory of driven MHD waves is then used to study the interaction of sound waves with one-dimensional isotropic steady plasmas. We find that a steady equilibrium flow can significantly influence the efficiency of resonant absorption in the considered limits. In the case of strong nonlinearity, the efficiency of the resonant coupling is found to be proportional to the counterpart obtained in linear theory. The factor of proportion is approximately of the order of unity, justifying the commonly applied linear approximations.

1. Introduction

Recent high-resolution satellite observations make it clear that the solar atmosphere is a highly structured dynamical system. Beside the inhomogeneous structure of the solar atmosphere, observations also show that plasmas are moving on almost all time scales. In order to have a better understanding of the different solar phenomena such as MHD wave propagation and MHD wave dissipation (heating), one has to revise previous theoretical attempts by expanding these studies to include the effects caused by steady states where equilibrium flows are present.

Resonant absorption is a common dissipative mechanism in non-uniform plasmas. The resonant transfer of energy is a natural phenomenon in interacting dynamical systems. In the solar atmosphere, where excited and propagating MHD waves interact with an inhomogeneous plasma, these waves can transfer their energy to each other. The plasma–wave transfer is related to instabilities that are not the subject of the present paper. The energy transfer from the waves to the background plasma is related to the fact that in an inhomogeneous plasma, external driving waves can resonantly interact with local oscillation modes. As a result, the wave energy can be converted into heat by means of dissipation. A similar phenomenon can be used to model the observed loss of power of acoustic oscillations in sunspots (see e.g. Hollweg 1988; Lou 1990; Sakurai et al. 1991b; Goossens and Poedts 1992; Erdélyi and Goossens 1994).

Usually resonant absorption is investigated in a *static equilibrium*. However, Hollweg et al. (1990) showed that, for an incompressible linear plasma in planar geometry, the damping or excitation of surface waves can be strongly influenced by an equilibrium shear flow. Later, Goossens et al. (1992), Erdélyi and Goossens (1996) and Erdélyi (1997) studied the effect of shear flow on linear resonant MHD wave heating. They found that an equilibrium flow changes the heating rate and the efficiency of the coupling between the impinging wave and the steady plasma.

The process of resonant absorption has been studied extensively in the linear regime. Wave dynamics, however, becomes nonlinear if the amplitudes reach sufficiently large values. Results of linear studies of velocity scaling laws indicate that, with classical solar dissipation parameters, the velocities at the resonant layer are several orders of magnitude *larger* than the observed non-thermal velocities, indicating that nonlinear theory might be needed. The implication is that linear theory can break down owing to this resonant behaviour of the waves in these regions, as pointed out by Ofman and Davila (1994), Ruderman et al. (1997a) and Ballai et al. (1998), among others.

Nonlinearity gives rise to effects in solar plasmas that have not yet been observed in the linear approximation. A typical nonlinear effect is, for example, the appearance of *solitons*. Solitons are the result of two competing effects in a waveguide: nonlinearity steepens the wave amplitudes, and dispersion tends to broaden the waves by 'smearing' them out. When dispersion *balances* the nonlinear steepening, a pulse-like wave can propagate along the waveguide without changing its form. Solitons are of interest in many areas of physics, including, for example, solar physics (see e.g. Roberts and Mangeney 1982; Roberts 1984; Zhughda and Nakariakov 1999).

Another important effect of nonlinearity appears in problems related to the nonlinear coupling of waves and phase mixing. This coupling can be a source of wave generation, which can provoke the formation of shock waves. At the same time, a perpendicular inhomogeneity in the magnetic field can generate phase mixing, another important nonlinear effect (see e.g. Oraevski 1983; Wentzel 1977; Nakariakov and Oraevsky 1995; Nakariakov et al. 1997). A third effect of nonlinearity in the context of solar plasmas is emerging from the resonant absorption of MHD waves in magnetic structures. Linear theory predicts that the amplitude of perturbations in the vicinity of the *ideal* resonant position can be very large. The implication is that linear theory can break down owing to this resonant behaviour of the waves in these regions (see e.g. Ofman and Davila 1994; Ruderman et al. 1997b; Ballai and Erdélyi 1998).

The aim of the present study is to apply the nonlinear theory of slow dissipative layers in isotropic plasmas with steady equilibrium state, and to assess the effect of an equilibrium flow on the efficiency of resonant energy absorption in the limit of weak and strong nonlinearity. The results are also compared with linear studies.

The paper is organized as follows. In the next section, we introduce the MHD equations and discuss some basic assumptions. In Sec. 3, we derive the system of equations that describes the behaviour of the plasma motion in ideal and dissipative MHD. The connection formulae are recalled. In Sec. 4, we take nonlinearity into account. The applicability of this theory is discussed. The nonlinear governing equation and connection formulae for wave dynamics in the dissipative layer are derived. In Sec. 5, we apply the results found previously to study the nonlinear resonant interaction of sound waves with the slow-wave dissipative layer in the limit of weak and strong nonlinearity.

2. Basic equations

The present study uses the full set of visco-resistive MHD equations

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \qquad \frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \tag{2.1}$$

$$\rho\frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla}p + \frac{1}{\mu}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + \rho\nu\nabla^2\mathbf{v}, \tag{2.2}$$

$$\frac{\partial B}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{2.3}$$

where $D/Dt{=}\partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the convective derivative operator. Here p and ρ are the kinetic pressure and the density, $\mathbf{v}$ is the velocity, $\mathbf{B}$ is the magnetic field, γ is the adiabatic index, and ν and η are the coefficients of kinematic viscosity and magnetic diffusion (assumed to be constants).

We adopt a Cartesian coordinate system where all the equilibrium quantities depend on x only. The equilibrium magnetic field $\mathbf{B}_0$ is unidirectional and its direction is along the z axis. We also suppose that there is an equilibrium flow $\mathbf{C}_0$ parallel to the magnetic field lines.

In the perturbed system, every quantity can be written as $f = f_0 + \hat{f}$, where f_0 denotes an equilibrium quantity and $\hat{f}$ denotes the perturbation. In what follows, we drop the 'hat'.

3. Linear MHD

In linear theory, the perturbations oscillate in time with the same real frequency ω. One can Fourier-analyse the system with respect to z and consider the perturbations to be proportional to $\exp(ikz - i\omega t)$. Therefore it is more convenient to use the Lagrangian displacement ξ_x instead of the normal component of the velocity. Dissipation is relevant only in a narrow layer, called the *dissipative layer*, that embraces the ideal resonant surface. This dissipative layer can be considered as a surface of discontinuity when solving the linear ideal MHD equations on both sides of the dissipative layer. In the first step, let us take the dissipative coefficients to be zero.

With these considerations, from the system (2.1)–(2.3), we obtain a system of two coupled first-order ODEs for total pressure and the normal component of the displacement:

$$\left.\begin{aligned} \frac{dP}{dx} &= \rho_0 D_A \xi_x, \\ \frac{d\xi_x}{dx} &= -\frac{\Omega^2 - k^2C_S^2}{D_T}P, \end{aligned}\right\} \tag{3.1}$$

where $\Omega = \omega - kC_0$ is the Doppler-shifted frequency and

$$\left.\begin{aligned} D_A &= \Omega^2 - k^2C_A^2, \\ D_T &= \rho_0(C_S^2 + C_A^2)(\Omega^2 - k^2C_T^2), \end{aligned}\right\} \tag{3.2}$$

where C_A, C_S and C_T are the Alfvén, sound and cusp velocities defined by

$$C_S^2 = \frac{\gamma p_0}{\rho_0}, \qquad C_A^2 = \frac{B_0^2}{\mu\rho_0}, \qquad C_T^2 = \frac{C_S^2 C_A^2}{C_S^2 + C_A^2}. \tag{3.3}$$

Equations (3.1) define an eigenvalue problem with Ω^2 as eigenvalue parameter when they are supplemented with boundary conditions. The system (3.1) is singular at the slow resonance, i.e. where $\Omega^2 = k^2 C_T^2$. In order to remove this singularity, we consider the effect of dissipation. This effect is small, and is important *only* in the vicinity of the ideal resonant position. Outside this region, the plasma behaviour is well described by the ideal MHD equations. Pioneering work on the analytical description of linear resonant MHD waves for static equilibrium was done by, for example, Sakurai et al. (1991a) and Goossens et al. (1995), while the case of steady equilibrium states was investigated by, for example, Goossens et al. (1992), Erdélyi and Goossens (1996) and Erdélyi et al. (1996). Let us introduce in the dissipative layer a new variable s, defined by $s = x - x_C$, where x_C is the position of the slow resonant surface and is defined by the condition $\Omega^2 = k^2 C_T^2(x_C)$. Expanding the coefficients of the system (3.1) around this singular position in Taylor series, we obtain a simplified version of this system of equations. This system is valid, strictly speaking, in the interval $[-s_C, s_C]$ around the point of resonance where the linear Taylor polynomial is a valid approximation of $\Omega^2 - k^2 C_T^2$; hence s_C has to satisfy

$$s_C \ll \left| \frac{2(C_T^2)'}{(C_T^2)''} \right|,$$

where the prime denotes the derivative with respect to x.

Thus the system obtained by using series expansions of the coefficient functions is

$$\left[s\Delta - i\Omega \left(\nu + \frac{C_{Tc}^2}{C_{Ac}^2}\eta \right) \frac{d^2}{ds^2} \right] \frac{d\xi_x}{ds} = \frac{k^2 C_{Tc}^4}{\rho_{0c} C_{Ac}^4} P, \tag{3.4}$$

$$\left[s\Delta - i\Omega \left(\nu + \frac{C_{Tc}^2}{C_{Ac}^2}\eta \right) \frac{d^2}{ds^2} \right] \frac{dP}{ds} = 0, \tag{3.5}$$

where $\Delta = d(\Omega^2 - k^2 C_T^2)/ds$ and all coefficients in the above equations are evaluated at $s = 0$.

The solutions to the system (3.4) and (3.5) enable us to obtain *the jump conditions*, which connect the solutions across the dissipative layer for the Lagrangian displacement ξ_x and for the total pressure perturbation P:

$$[\xi_x] = -i\pi \frac{k^2 C_{Tc}^4 \operatorname{sgn}(\Omega)}{|\Delta| \rho_{0c} C_{Ac}^4} P \tag{3.6}$$

and

$$[P] = 0, \tag{3.7}$$

where the square brackets denote the jump in the quantities across the dissipative layer. An important property of resonant slow-wave heating is that the jumps are independent of the dissipative coefficients. This implies that the amount of absorbed wave energy and the total amount of dissipative heating in the dissipative layer are also independent of the dissipative coefficients.

Finally, we point out that waves that have a propagation vector with y component and/or an inclined magnetic field relative to the z axis introduce an Alfvén resonance that can be described in an analogous way.

4. Nonlinear MHD

The situation presented in the previous section is somewhat altered by the fact that at the resonance, the amplitudes of variables and spatial gradients reach such large values that in order to have a correct description, the nonlinear terms have to be taken into account.

Let ϵ be the dimensionless amplitude of the large variables far from the dissipative layer. A key quantity in our discussion is the total Reynolds number R, which, under solar conditions, takes very large values and is given by

$$\frac{1}{R} = \frac{1}{R_e} + \frac{1}{R_m}, \tag{4.1}$$

where R_e and R_m are the viscous and magnetic Reynolds numbers. The value of this parameter determines the thickness of the dissipative layer. Nonlinearity in the dissipative layer becomes important if the amplitudes of the nonlinear terms are comparable to the amplitudes of the dissipative terms. Let us denote by f any 'large' variable (e.g. the parallel component of the velocity relative to the field lines); then a typical nonlinear term is of the form $f\partial f/\partial z$, while a typical dissipative term is $\partial^2 f/\partial s^2$ multiplied by one of the dissipative coefficients. Now we can estimate the ratio of the nonlinear terms to the dissipative terms, for example

$$\lambda = \frac{f\partial f/\partial z}{\nu\partial^2 f/\partial s^2} = \mathscr{O}(\epsilon R^{2/3}). \tag{4.2}$$

The parameter λ can be considered as the nonlinear parameter (or scaling factor), and we can conclude that if $\lambda \ll 1$, linear theory gives adequate results, but when $\lambda \geqslant 1$, nonlinearity has to be taken into account.

In contrast to the linear theory, we cannot Fourier-analyse the wave variables in the MHD equations. However, in order to be as close as possible to the linear theory, we suppose that the system supports plane periodic propagating waves with permanent shape. Therefore we introduce a propagating coordinate $\theta = z - Vt$, where V is the phase velocity in the direction of the equilibrium magnetic field and the resonant position x_C is given by the condition $V = C_T(x_C)$.

Away from the dissipative layer, the amplitudes of perturbations are small, so that we can use the linear ideal system of equations. In this case, we obtain a set of two coupled first-order PDEs for the total pressure and the normal component of the velocity (here we use the velocities instead of the Lagrangian displacements):

$$\left.\begin{aligned} \frac{\partial P}{\partial x} &= \frac{\rho_0 D_A}{\mathscr{V}}\frac{\partial u}{\partial \theta}, \\ \frac{\partial u}{\partial x} &= \frac{\mathscr{V}}{D_T}\frac{\partial P}{\partial \theta}, \end{aligned}\right\} \tag{4.3}$$

where D_A and D_T are the same as in (3.2) but we replace Ω by $\mathscr{V} = V - C_0$. All the other variables can be expressed in terms of P and u.

It was shown in Ballai and Erdélyi (1998) that the wave dynamics in the dissipative layer is governed by a nonlinear equation of the form

$$(x - x_c)\,\mathrm{sgn}\,(\Delta)\,\frac{\partial w}{\partial \theta} - \Lambda w \frac{\partial w}{\partial \theta} + \frac{\mathscr{V}}{|\Delta|}\left(\nu + \frac{C_{Tc}^2}{C_{Ac}^2}\eta\right)\frac{\partial^2 w}{\partial x^2} = \frac{\mathscr{V}^3}{\rho_{0c} v_{Ac}^2 |\Delta|}\frac{dP_c}{d\theta}, \tag{4.4}$$

where Δ and Λ are given as functions of the equilibrium quantities, and w is the z component of the perturbed velocity. The term on the right-hand side of (4.4) can be considered as a driving term responsible for the appearance of the resonance, and its form can be prescribed. This equation was obtained assuming that in the dissipative layer the nonlinearity and dissipation are of the same order.

Before we continue the derivation of the problem, let us point out an interesting fact related to the equation governing the wave dynamics in the dissipative layer.

In a homogeneous magnetic slab, the evolution of a *trapped* disturbance along the magnetic field lines is governed by an equation of the form (Benjamin–Ono–Burgers equation – see e.g. Edwin and Roberts 1986)

$$\frac{\partial w}{\partial t} + C_T \frac{\partial w}{\partial z} + \beta w \frac{\partial w}{\partial z} + \alpha \frac{\partial^2}{\partial z^2}\mathscr{H}[v] = \psi L(w), \tag{4.5}$$

where

$$\mathscr{H}[v] = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty} \frac{w(z', t)}{z' - z}\, dz'$$

is the Hilbert transform, α and β are coefficients expressed with the aid of equilibrium quantities, $\psi L(w)$ is a possible energy dissipation, w is the velocity component parallel to the z axis, and $\mathscr{P}$ in the expression of the Hilbert transformation means the Cauchy principal value. The solution of this equation can be a soliton. Equation (4.5) was obtained under the assumption that nonlinearity, dispersion and dissipation are of the same order.

However, the solar atmosphere is an inhomogeneous medium; therefore for a correct description of wave dynamics, this fact has to be taken into account. Ideally, one should try and derive an extension of (4.5) including the effects of inhomogeneity, for example, taking into account resonant absorption as a result of wave excitation by impinging waves (i.e. a driver) at the boundaries of the inhomogeneity. However, this seems to be far more complicated, and to the best of our knowledge has not yet been done. Alternatively, (4.4) describes the slow-wave dynamics in the dissipative layer *without* the effect of a waveguide (e.g. without dispersion) where an equilibrium bulk motion is present.

Equation (4.4) can be considered as a starting point in combining the nonlinear description of a dispersive waveguide with the nonlinear theory of inhomogeneous resonant flux tubes.

Owing to the similarities in forms and coefficients, we speculate that there is a connection between (4.4) and (4.5). We plan to develop a theory that describes wave propagation in an inhomogeneous plasma, when nonlinearity, dispersion and resonant dissipation are of the same order. Our expectation is to obtain an equation similar to (4.5) with coefficients depending on the spatial coordinate of inhomogeneity and having an additional term on the right-hand side that which describes the effect of the resonance. This problem will be the subject of a future paper.

Returning to the main subject of the present study, we are going to derive the behaviour of physical quantities across the dissipative layer. The solution of (4.4) has to vanish at $|x - x_c| \to \infty$, and satisfies the condition that its mean value over a period with respect to θ is zero. The surface $x = x_c$ can be considered as a surface of discontinuity when solving the system of equations that govern the plasma motion outside the dissipative layer. Ballai and Erdélyi (1998) derived the expressions for the jumps in u and P. These are

$$[P] = 0, \tag{4.6}$$

$$[u] = -\frac{\mathscr{V}^2}{C_{Ac}^2}\mathscr{P}\int_{-\infty}^{\infty}\frac{\partial w}{\partial \theta}\,dx. \tag{4.7}$$

In (4.7) we use the Cauchy principal part because $w = \mathscr{O}(x - x_c)^{-1}$ as $x - x_c \to \infty$, so the integral in (4.7) is divergent. The two jump conditions provide boundary conditions when solving simultaneously the system (4.3) and (4.4). The continuity of the total pressure is analogous to the linear theory (see e.g. Hollweg 1988; Sakurai et al. 1991a; Goosens et al. 1992, 1995; Erdélyi 1977). The zero jump in the total pressure means that there has to be a balance of forces at both sides of the dissipative layer. Since the dissipative layer is very thin, it has small inertia and cannot alter drastically the pressure force by crossing the dissipative layer from one side to the other.

5. Interaction of sound waves with slow wave dissipative layers

As an application, we study the nonlinear interaction of incident sound waves with one-dimensional steady plasmas, i.e. the nonlinear coupling of sound waves and slow MHD waves. Monochromatic sound waves are impinging from the unmagnetized half-space $x < 0$ (region I) and penetrate into the inhomogeneous region $0 < x < x_0$ (region II). This inhomogeneous region is bounded on its right by a semi-infinite subspace containing a homogeneously magnetized plasma (region III). The magnetic field is parallel to the z direction, and a field-aligned steady flow C_0 is present in the magnetic regions. In what follows, we use the subscripts e, 0 and i to indicate equilibrium quantities in the three regions I, II and III respectively. All equilibrium quantities are continuous at the boundaries of region II, and they satisfy the equation of total pressure balance

$$p_e = p_0(x) + \frac{B_0^2(x)}{2\mu} = p_i + \frac{B_i^2}{2\mu} \tag{5.1}$$

and

$$\frac{\rho_i}{\rho_e} = \frac{2C_{Se}^2}{2C_{Si}^2 + \gamma C_{Ai}^2} < 1, \tag{5.2}$$

so that the plasma in region III is less dense than in region I. In order to make analytical progress, we assume from the very beginning that the inhomogeneous region is thin or that the wavelength of perturbations is long, i.e. $kx_0 \ll 1$.

In principle, there can also exist an Alfvén-resonant position at $x = x_{\mathrm{A}}$ in region II. However, the objective of the present paper is to study the effect of flow and nonlinearity on the interaction of incoming sound waves with slow dissipative layers only in inhomogeneous magnetized plasmas. The presence of an Alfvén resonance complicates the analysis and would obscure results. To remove the Alfvén resonance, we assume wave propagations and perturbations of all quantities to be independent of y. To achieve this, we rotate the coordinate system around the x axis, transforming the wave vector of an incoming sound wave to lie entirely in the (x, z) plane.

5.1. Weakly nonlinear limit

Weak nonlinearity means that the nonlinearity can be considered as a *perturbation* of the linear problem and the effects of nonlinearity are obtained as corrections to the linear results. Let us introduce some new dimensionless variables defined by

$$\left.\begin{aligned} \sigma &= \frac{x - x_c}{\delta_c}, \qquad q = \frac{k\mathscr{V}\delta_c}{C_A^2} w, \\ \delta_c &= \left[\frac{|\mathscr{V}|}{k|\Delta|}\left(\nu + \frac{C_T^2}{C_A^2}\eta\right)\right]^{1/3}, \end{aligned}\right\} \tag{5.3}$$

where σ is the scaled dimensionless distance, q is the dimensionless velocity parallel to the magnetic field lines and δ_c is the thickness of the dissipative layer. With these new variables, the equation (4.4) governing wave motion in the dissipative layer becomes

$$\operatorname{sgn}(\Delta)\,\sigma\frac{\partial q}{\partial \theta} - \Lambda' q\frac{\partial q}{\partial \theta} + k\operatorname{sgn}(\mathscr{V})\,\frac{\partial^2 q}{\partial \sigma^2} = \frac{k\mathscr{V}^4}{\rho_{0c}C_{Ac}^4|\Delta|}\frac{dP}{d\theta}, \tag{5.4}$$

where Λ' is an expression of the equilibrium quantities. The plasma motion outside the dissipative layer is described by the set of equations (4.3). The solution of these equations can be obtained in the form of a sum of incoming and outgoing waves. It was shown by Ruderman et al. (1997a) and Erdélyi and Ballai (1999) that nonlinearity generates higher harmonics in the dissipative layer in addition to the fundamental mode. The solution in the magnetic field-free region is written in the form

$$P = \epsilon p_e \operatorname{Re}\left[e^{i(k\theta + \chi_e x)} + \sum_{n\geqslant 1} A_n e^{in(k\theta - \chi_e x)}\right], \tag{5.5}$$

$$u = \frac{\epsilon \chi_e p_e}{kV\rho_e} \operatorname{Re}\left[e^{i(k\theta + \chi_e x)} - \sum_{n\geqslant 1} A_n e^{in(k\theta - \chi_e x)}\right], \tag{5.6}$$

where $\chi_e = k_x/k_z$; and since we assume that $V > C_{Se}$, the waves in this region are propagating.

In the homogeneous region (quantities with subscript i), the solution of the system (4.3) is

$$P = \epsilon \operatorname{Re}\left[\sum_{n\geqslant 1} D_n e^{n(ik\theta - \chi_i(x - x_0))}\right], \tag{5.7}$$

$$u = \frac{\epsilon\mathscr{V}\chi_i}{k\rho_i(\mathscr{V}^2 - C_{Ai}^2)} \operatorname{Re}\left[\sum_{n\geqslant 1} D_n e^{n(ik\theta - \chi_i(x - x_0))}\right], \tag{5.8}$$

where

$$\chi_i^2 = -\frac{(\mathscr{V}^2 - C_{Ai}^2)(\mathscr{V}^2 - C_{Si}^2)}{(C_{Si}^2 + C_{Ai}^2)(\mathscr{V}^2 - C_{Ti}^2)}. \tag{5.9}$$

Since $\chi_i^2 > 0$, the waves are evanescent in the homogeneous region.

In the inhomogeneous region (quantities with subscript 0), the solution to the system (4.3) can be written in the form

$$\frac{\partial}{\partial x}\frac{1}{\rho_0(\mathscr{V}^2 - C_A^2)}\frac{\partial P}{\partial x} = \mathscr{O}(k^2 x_0^2), \tag{5.10}$$

and

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= F(x)\frac{\partial P}{\partial \theta}, \\ F(x) &= \frac{\mathscr{V}(\mathscr{V}^2 - C_S^2)}{\rho_0(C_S^2 + C_A^2)(\mathscr{V}^2 - C_T^2)}. \end{aligned}\right\} \tag{5.11}$$

In general, it is impossible to solve analytically the nonlinear equation that governs the wave dynamics in the dissipative layer, but if we assume that the nonlinear term is small in (5.4) in comparison with the dissipative term, we can look for the solution in the form of an expansion in the small parameter $\hat{\lambda} = V\lambda$, i.e.

$$f = \epsilon \sum_{n \geqslant 1} \lambda^{n-1} f_n, \tag{5.12}$$

where f represents any of the quantities P, u and q.

The condition that P and u have to be continuous at $x = 0$ and $x = x_0$, and in addition that P has to be continuous at $x = x_c$, enables us to find that $D_1 = p_e + A_1$ and $D_n = A_n$ ($n \geqslant 2$), and to obtain the following estimate for the amplitudes of the harmonics:

$$A_2 = \mathscr{O}(\hat{\lambda}^3), \qquad A_n = \mathscr{O}(\hat{\lambda}^3) \qquad (n = 3, 4, \ldots). \tag{5.13}$$

To give a quantitative description of the efficiency of the weak resonant interaction, we calculate the coefficient of wave absorption given by

$$\alpha = 1 - \frac{\Phi_1}{\Phi_2}, \tag{5.14}$$

where Φ_1 and Φ_2 are the x components of the wave energy fluxes of the incoming and outgoing waves respectively. Retaining only the monochromatic overtones, we obtain the following approximate relation for the coefficient of energy absorption:

$$\alpha \approx \alpha_L + \hat{\lambda}^2 \alpha_{NL}. \tag{5.15}$$

Taking into account the assumptions (weak nonlinearity and the long-wavelength approximation), we obtain for the coefficients of energy absorption

$$\alpha_L = -\frac{4\xi\zeta}{\zeta^2 + \phi^2}, \qquad \alpha_{NL} = \frac{4p_e^2\xi^3\zeta^3 I}{\pi^2\mathscr{V}^2(\zeta^2 + \phi^2)^2}, \tag{5.16}$$

where I is an integral that can be found in Erdélyi and Ballai (1999) (their equation (101)). The integral I has been estimated, and found to be of the order of unity, and

$$\left.\begin{aligned} \xi &= \frac{\pi k \mathscr{V}^5}{\rho_0 C_A^4 \Delta}, \qquad \zeta = \frac{\chi_e}{\rho_e V}, \\ \phi &= \frac{\chi_i \mathscr{V}}{\rho_i(\mathscr{V}^2 - C_{Ai}^2)} - k\mathscr{P}\int_0^{x_0} F(x)\, dx. \end{aligned}\right\} \tag{5.17}$$

In contrast to the static case, where nonlinearity *decreases* the coefficient of energy absorption, in a steady plasma, nonlinearity can *increase* the absolute value of this coefficient. Moreover, the linear part of this coefficient can be negative for some specific values of the flow strength. These negative values are related to overreflection or resonant instability (see e.g. Tirry et al. 1998).

5.2. Strongly nonlinear limit

The solution obtained in the limit of weak nonlinearity is just a correction to the linear results, since the regular perturbation method used in the previous section provides only small corrections. The results obtained previously are valid if $\epsilon R^{2/3} \ll 1$. For a typical value of 10^{-2} for ϵ, we obtain for the Reynolds number that $R \ll 10^3$, which is unrealistic small under solar conditions (R is 10^6 in the solar photosphere and even 10^{12} in the corona). Therefore for a realistic situation, we should consider that the nonlinearity is strong, i.e. $\epsilon R^{2/3} \gg 1$.

Let us introduce some new dimensionless variables:

$$\left.\begin{aligned} &\vartheta = k\theta, \qquad \sigma = -\frac{(x - x_c)\,\mathrm{sgn}(\Delta)}{\delta_c (2\lambda)^{1/3}}, \\ &q = \frac{\Lambda w}{\delta_c (2\lambda)^{1/3}}, \qquad Q = \frac{2\mathscr{V}^3 \Lambda P_c}{\rho_{0c} C_{Ac}^2 |\Delta| \delta_c^2 (2\lambda)^{2/3}}. \end{aligned}\right\} \tag{5.18}$$

With these variables the jump condition for the normal component of the velocity and the governing equation (4.4) become

$$[u] = \frac{k\mathscr{V}^2 \delta_c^2\,\mathrm{sgn}(\Delta)}{\Lambda C_{Ac}^2} (2\lambda)^{2/3} \frac{d}{d\vartheta} \mathscr{P} \int_{-\infty}^{\infty} q(\sigma, \vartheta)\, d\sigma, \tag{5.19}$$

$$2\sigma \frac{\partial q}{\partial \vartheta} + 2q \frac{\partial q}{\partial \vartheta} - \frac{\mathrm{sgn}(\mathscr{V})}{\lambda} \frac{\partial^2 q}{\partial \sigma^2} = -\frac{dQ}{d\vartheta}. \tag{5.20}$$

In order to obtain the asymptotic behaviour of (5.20), we neglect the last term on the left-hand side, which contains the large parameter λ. A cumbersome derivation of this problem was given by Ruderman (2000) for a *static* equilibrium. We apply his method to study the effect of a mass flow on the resonant absorption of sound waves in the slow-wave dissipative layer.

The solution in the magnetic-field free region (region I) is a sum of the incoming and outgoing waves. We assume that the incoming wave contains only the fundamental harmonic, so the perturbation of the total pressure can be written as

$$P_{\mathrm{in}} = \epsilon\, \mathrm{Re}\big[p_e e^{ik(\theta + \chi_e x)}\big]\,.$$

For the outgoing wave, we have

$$P_{\mathrm{out}} = \epsilon p_e A(\theta - \chi_e x),$$

where A is a periodic function with zero mean over a period. The normal component of the velocity can be found using the system describing the wave motion outside the dissipative layer, namely

$$P = \epsilon p_e \{\cos[k(\theta + \chi_e x)] + A(\theta - \chi_e x)\}, \tag{5.21}$$

$$u = \epsilon \frac{\chi_e p_e}{\rho_e V} \{\cos[k(\theta + \chi_e x)] - A(\theta - \chi_e x)\}. \tag{5.22}$$

The solutions in the internal region (region III) are found in the form of Fourier series

$$P = \sum_{n=-\infty}^{\infty} P_n(x)e^{inkθ}, \qquad u = \sum_{n=-\infty}^{\infty} u_n(x)e^{inkθ}, \tag{5.23}$$

where

$$P_n = P_n(x_0)\exp[-\chi_i k|n|(x-x_0)], \tag{5.24}$$

$$u_n = \frac{i\mathscr{V}\chi_i\,\mathrm{sgn}(n)}{\rho_i(\mathscr{V}^2 - C_{Ai}^2)} P_n(x_0)\exp[-\chi_i k|n|(x-x_0)]. \tag{5.25}$$

The continuity condition for the normal component of the velocity and the total pressure at $x = 0$ and $x = x_0$, and in addition the continuity of the total pressure at the resonant position x_c, yield for the coefficient of wave energy absorption

$$\alpha_{NL} = -\frac{32\chi_e\zeta}{\chi^2 + \xi^2} + \mathscr{O}(k^2x_0^2), \tag{5.26}$$

where now

$$\xi = \frac{\chi_i\rho_e\mathscr{V}^2}{\rho_i(C_{Ai}^2 - \mathscr{V}^2)}, \qquad \zeta = \frac{k\rho_e V^4\mathscr{V}^2}{\pi\rho_{0c}C_{Ac}^4\Delta}. \tag{5.27}$$

For the ratio of the nonlinear to the linear coefficients of energy absorption, we obtain

$$\frac{\alpha_{NL}}{\alpha_L} = \frac{8V}{\pi^2\mathscr{V}}\,\mathrm{sgn}(\mathscr{V}\Delta) + \mathscr{O}(kx_0). \tag{5.28}$$

This is a surprising result, which tells us that the nonlinear coefficient is *proportional* to its counterpart obtained in linear theory. The factor of proportionality is of the order of unity for subsonic mass flows. This result also justifies the efforts made in the studies of *linear* resonant MHD waves, and their applicability to solar physics.

6. Conclusions

This paper has applied the theory of slow resonant waves in steady-state plasmas developed by Ballai and Erdélyi (1998) in the limit of weak and strong nonlinearity. After recalling the main results in linear theory, we pointed out the main differences between linear and nonlinear resonant MHD.

The nonlinear theory was then applied to study the interaction of sound waves with nonlinear resonant slow MHD dissipative layers. We considered a simple equilibrium consisting of an inhomogeneous layer that contains the resonant surface and is sandwiched by a homogeneous magnetic-field-free and a homogeneous magnetic plasma. A planar sound wave is excited far away from the inhomogeneous magnetic slab in the magnetic-free region. The values of the equilibrium quantities were chosen in such a way that the waves are evanescent in the homogeneous magnetic region and an ideal resonant position is present in the slab. The incoming sound wave is partially reflected and partially absorbed in the resonant layer.

The differences between weak and strong nonlinearity are in the dominance of the dissipative or nonlinear terms. Dissipation is important if $\epsilon R^{2/3} \ll 1$ (this is the criterion for the weak nonlinear limit), and can be neglected in comparison with nonlinearity if $\epsilon R^{2/3} \gg 1$. In the first case, nonlinearity can be considered as a perturbation of the linear theory, and the solutions can be written using a regular perturbation expansion in the small parameter. In the second case, nonlinearity dominates dissipation in the dissipative layer.

For the absorption of, for example, sound waves, we found that *nonlinearity in the steady slow dissipative layer generates higher harmonics in the outgoing sound wave in addition to the fundamental harmonic*. The equilibrium flow can *either increase* or *decrease* the coefficient of the wave energy absorption depending on the wave characteristics, and the strength and direction of the field-aligned flow in the slow dissipative layer.

Another important result of this study is that for strong nonlinearity, the resonant energy absorption is of the *same order of magnitude* as its linear counterpart, justifying the linear approximation. This unexpected result is important, since the nonlinear interactions of resonant MHD waves can be simply approximated by the much simpler linear theory, justifying the previous extensive work in linear theory.

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References

Ballai, I. and Erdélyi, R. 1998 Resonant absorption of nonlinear slow MHD waves in isotropic steady plasmas, I. Theory. *Solar Phys.* **180**, 65.

Ballai, I., Ruderman, M. S. and Erdélyi, R. 1998 Nonlinear theory of slow dissipative layers in anisotropic plasmas. *Phys. Plasmas*, **5(1)**, 252.

Edwin, P. M. and Roberts, B. 1986 The Benjamin–Ono–Burgers equation: an application in solar physics. *Wave Motion* **8**, 151.

Erdélyi, R. 1997 Analytical solutions for cusp resonance in dissipative MHD. *Solar Phys.* **171**, 49.

Erdélyi, R. and Ballai, I. 1999 Resonant absorption of nonlinear slow MHD waves in isotropic steady plasmas, II. Application: resonant acoustic waves. *Solar Phys.* **186**, 67.

Erdélyi, R. and Goossens, M. 1994 Viscous computations of resonant absorption of MHD waves in flux tubes. *Astrophys. Space Sci.* **213**, 273.

Erdélyi, R. and Goossens, M. 1996 Effect of flow on resonant absorption of MHD waves in viscous MHD. *Astron. Astrophys.* **313**, 664.

Erdélyi, R., Goossens, M. and Ruderman, M. S. 1996 Effect of flow on dissipative MHD solutions. *Solar Phys.* **161**, 123.

Goossens, M. and Poedts, S. 1992 Linear resistive magnetohydrodynamic computations of resonant absorption of acoustic-oscillations in sunspots. *Astrophys. J.* **384**, 348.

Goossens, M., Hollweg, J. V. and Sakurai, T. 1992 Resonant behaviour of MHD waves on magnetic flux tubes, III. Effect of equilibrium flow. *Solar Phys.* **138**, 233.

Goossens, M., Ruderman, M. S. and Hollweg, J. V. 1995 Dissipative MHD solutions for resonant Alfvén waves in 1-dimensional magnetic flux tubes. *Solar Phys.* **157**, 75.

Hollweg, J. V. 1988 Resonant absorption of solar *p*-modes by sunpots. *Astrophys. J.* **335**, 1005.

Hollweg, J. V., Yang, G., Cadez, V. and Gakovici, B. 1990 Surface waves in an incompressible fluid: resonant istability due to velocity shear. *Astrophys. J.* **349**, 335.

Lou, Y.-Q. 1990 Viscous magnetohydrodynamic modes and *p*-mode absorption by sunspots. *Astrophys. J.* **350**, 452.

Nakariakov, V. M. and Oraevsky, V. N. 1995 Resonant interactions of modes in coronal magnetic flux tubes. *Solar Phys.* **160**, 289.

Nakariakov, V. M., Roberts, B. and Murawski, K. 1997 Alfvén wave phase mixing as a source of fast magnetoacoustic waves. *Solar Phys.* **175**, 93.

Ofman, L. and Davila, J. M. 1994 Nonlinear studies of coronal heating by the resonant absorption of Alfvén waves. *Geophys. Res. Lett.* **21**, 2259.

Oraevski, V. N. 1983 In: *Foundations of Plasma Physics* (ed. A. A. Galeev and R. Sudan), p. 153. Nauka, Moscow.

Roberts, B. 1984 Solitary waves in magnetic flux tube. *Phys. Fluids*, **28**, 3280.

Roberts, B. and Mangeney, A. 1982 Solitons in solar magnetic flux tubes. *Mon. Not. R. Astron. Soc.* **198**, 7.

Ruderman, M. S. 2000 Interaction of sound waves with an inhomogeneous magnetized plasma in a strongly nonlinear resonant slow-wave layer. *J. Plasma Phys.* **63**, 43.

Ruderman, M. S., Hollweg, J. V. and Goossens, M. 1997a Nonlinear theory of resonant slow waves in dissipative layers. *Phys. Plasmas* **4**, 75.

Ruderman, M. S., Goossens, M. and Hollweg, J. V. 1997b Nonlinear theory of interaction of sound waves with an inhomogeneous magnetized plasma in the resonant slow wave layer. *Phys. Plasmas* **4**, 91.

Sakurai, T., Goossens, M. and Hollweg, J. V. 1991a Resonant behaviour of MHD waves on magnetic flux tubes, I. Connection formulae at the resonant surface. *Solar Phys.* **133**, 227.

Sakurai, T., Goossens, M. and Hollweg, J. V. 1991b Resonant behaviour of MHD waves on magnetic flux tubes, II. Absorption of sound waves by sunspots. *Solar Phys.* **133**, 247.

Tirry, W. J., Cadez, V. M., Erdélyi, R. and Goossens, M. 1998 Resonant flow instability of MHD surface waves. *Astron. Astrophys.* **332**, 786.

Wentzel, D. G. 1977 On the role of hydromagnetic waves in the corona and the base of the solar wind. *Solar Phys.* **52**, 163.

Zhugzhda, Y. D. and Nakariakov, V. M. 1999 Linear and nonlinear magnetohydrodynamics waves in twisted magnetic flux tubes. *Phys. Lett.* **252A**, 222.