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THE B-MODEL CONNECTION AND MIRROR SYMMETRY FOR GRASSMANNIANS

R. J. MARSH AND K. RIETSCH

This paper is dedicated to the memory of Andrei Zelevinsky.

Abstract. We consider the Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ and describe a 'mirror dual' Landau-Ginzburg model $(\check{\mathbb{X}}^\circ, W_q : \check{\mathbb{X}}^\circ \to \mathbb{C})$, where $\check{\mathbb{X}}^\circ$ is the complement of a particular anti-canonical divisor in a Langlands dual Grassmannian $\check{\mathbb{X}}$, and we express W succinctly in terms of Plücker coordinates. First of all, we show this Landau-Ginzburg model to be isomorphic to one proposed for homogeneous spaces in a previous work by the second author. Secondly we show it to be a partial compactification of the Landau-Ginzburg model defined in the 1990's by Eguchi, Hori, and Xiong. Finally we construct inside the Gauss-Manin system associated to W_q a free submodule which recovers the trivial vector bundle with small Dubrovin connection defined out of Gromov-Witten invariants of X. We also prove a T-equivariant version of this isomorphism of connections. Our results imply in the case of Grassmannians an integral formula for a solution to the quantum cohomology D-module of a homogeneous space, which was conjectured by the second author. They also imply a series expansion of the top term in Givental's J-function, which was conjectured by Bertram, Ciocan-Fontanine, Kim and van Straten in 1997.

Contents

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1. Introduction

The genus 0 Gromov-Witten invariants of a Grassmannian X answer enumerative questions about rational curves in X and are put together in various ways to define rich mathematical structures such quantum cohomology rings, flat pencils of connections on Frobenius manifolds [16]. These structures are part of the so-called 'A-model' of X. Mirror symmetry in the sense we consider here seeks to describe these structures in terms a mirror dual 'B-model' or Landau-Ginzburg (LG) model associated to X. Explicitly, the data from the A-model should be encoded by singularity theory or oscillating integrals of a regular function W_q (the superpotential) which is defined on a 'mirror dual' affine Calabi-Yau variety $\check{\mathbb{X}}^\circ$ with a holomorphic volume form ω. In the present paper we construct such a mirror datum in canonical and concrete terms for Grassmannians and prove associated mirror conjectures. Our results in particular imply and enhance a conjecture formulated in the 1990s by Batyrev, Ciocan-Fontanine, Kim and van Straten [6, Conjecture 5.2.3] concerning a series expansion for a coefficient of Givental's J-function. This conjecture was restated in a paper of Bertram, Ciocan-Fontanine and Kim [5, Section 3], where the special case of Grassmannians of 2-planes was proved by an entirely different method. The conjecture was again restated as a problem of interest in [73, Problem 14] some 10 years later.

To give a flavour of our results, consider

$$\mathcal{S}(q) = \sum_\lambda \left(\oint e^{\frac{1}{\hbar} W_q} p_\lambda \omega \right) PD(\sigma^\lambda), \tag{1.1}$$

which is an $H^*(X,\mathbb{C})$-valued function depending on a single variable q (and a parameter $\hbar$). The integrand involves the superpotential W_q. This is a particular rational function introduced in this paper via an explicit formula in terms of Plücker coordinates p_λ on an isomorphic (but Langlands dual) Grassmannian $\check{\mathbb{X}}$; see for example (3.1). The integration is over a natural compact torus in an open subvariety $\check{\mathbb{X}}^\circ$ of $\check{\mathbb{X}}$; see Theorem 4.2. Note that the function $\mathcal{S}(q)$ in (1.1) is expressed as a linear combination of Schubert classes σ^λ permuted by Poincaré duality, denoted by PD. By Theorem 4.2 (one of our main results) the function $\mathcal{S}(q)$ satisfies the flat section equation of the Dubrovin connection, i.e. we have $q\frac{d}{dq}\mathcal{S} = \frac{1}{\hbar}\sigma^\square \star_q \mathcal{S}$. Here $\sigma^\square$ denotes the hyperplane class of X, and $\star_q$ denotes the quantum cup product on the small quantum cohomology ring of X.

A conjecture of Batyrev, Ciocan-Fontanine, Kim and van Straten proposes an integral formula $\oint e^{\frac{1}{\hbar} L_q}\omega$ for the coefficient of the top class $PD(\sigma^\emptyset)$ of a flat section of the Dubrovin connection, and employs a Laurent polynomial L_q introduced by Eguchi, Hori and Xiong [21] in place of our W_q. We prove this conjecture by recovering the Laurent polynomial superpotential L_q as a restriction of W_q to a particular open dense torus inside $\check{\mathbb{X}}$ and applying our more general Theorem 4.2.

Note that the Schubert basis and the Plücker coordinates above are indexed by the same set. This has its explanation in the geometric Satake correspondence. The key point is that there is a natural identification $H^0(\check{\mathbb{X}}, \mathcal{O}(1)) = H^*(X, \mathbb{C})$ identifying Schubert classes of X with homogeneous coordinates of $\check{\mathbb{X}}$. This identification comes from the fact that both left-hand and right-hand sides agree with the k-th fundamental representation $\bigwedge^k \mathbb{C}^n$ of $SL_n(\mathbb{C})$, the special linear group which acts on $\check{\mathbb{X}}$. For the left-hand side this is by the Borel-Weil theorem, and for the right-hand side it is by a very special case of the geometric Satake correspondence [31, 45, 55, 62] which constructs representations of $G^\vee = SL_n(\mathbb{C})$ via intersection cohomology of Schubert varieties of the affine Grassmannian $\mathcal{G}r_G$ for the Langlands dual group $G = PSL_n(\mathbb{C})$. (The Schubert variety which arises in the construction of $\bigwedge^k \mathbb{C}^n$ turns out to be homogeneous for G and coincides with the Grassmannian X.)

The methods developed here are useful and adaptable to other co-minuscule G/P, which are precisely the homogeneous spaces which in the geometric Satake correspondence appear as 'minimal' Schubert varieties of the affine Grassmannian $\mathcal{G}r_G$. In particular, since this paper appeared, the methods we use have been employed to obtain results analogous to Theorem 4.2 in the case of even and odd-dimensional quadrics; see [67, 68]. Moreover, in the case of Lagrangian Grassmananians, a partial result in this direction, the 'canonical' description of the superpotential in terms of Plücker coordinates, has been obtained in [66].

The main work in this paper is concerned with constructing the A-model Dubrovin connection in terms of the Gauss-Manin system associated to a mirror LG model, in the most natural way possible, in the case where X is a Grassmannian. Our main theorem, Theorem 4.1, explicitly describes the Dubrovin connection of X in terms of the Gauss-Manin system of the mirror LG model we introduce. This underlies the formula for the global flat section obtained in Theorem 4.2. We note that the superpotential is additionally shown (see Theorem 4.10) to be isomorphic to the Lie-theoretic superpotential of X that was defined for general G/P in [79, Section 4.2]. Our result therefore also implies a version of the mirror conjecture concerning solutions to the quantum differential equations in terms of the superpotential considered there, namely [79, Conjecture 8.1], in the Grassmannian case.

Our results also shed new light on the (small) quantum cohomology ring of X, which is shown to agree with the Jacobi ring of W_q by an isomorphism which identifies the Schubert class σ^λ with the element of the Jacobi ring represented by the Plücker coordinate p_λ. All of our results, including this one, are also stated and proved in the T-equivariant setting; see Proposition 5.2 and Theorem 5.5.

In more recent work of the second author together with L. Williams [81], the superpotential W_q introduced in this paper, together with the cluster structure on $\mathbb{C}[\check{\mathbb{X}}^\circ]$, is shown by tropicalisation to define a set of polytopes which can be identified as Newton-Okounkov convex bodies of X. This can be understood as another form of mirror symmetry relating X and $(\check{\mathbb{X}}^\circ, W_q)$.

A version of the present paper was placed on the arXiv in December 2015 (arXiv:1307.1085v2 [math.AG]). In May 2017 a preprint of Lam and Templier [50] appeared on the minuscule G/P case of the mirror conjecture [79, Conjecture 8.1] for the Lie-theoretic superpotential (see Section 6.2). Their approach, which covers the Grassmannian case, employs completely different methods and uses the minuscule property. From this approach they can in their setting deduce that the Gauss-Manin system has the expected rank. On the other hand, our approach, using that the Grassmannian is *co-minuscule*, gives rise to explicit formulas for all of the components of a flat section of the Dubrovin connection in certain canonical coordinates. As mentioned above, the proof of the mirror conjecture along the same lines as in the present paper has already been carried out for even and odd-dimensional quadrics in [67, 68], the latter of which are not minuscule and hence not covered by the work in [50]. We expect our approach to be applicable, with analogous canonical coordinates, to all co-minuscule G/P.

The outline of the paper is as follows. We begin with a concrete introduction of the A-model structures. Then we give definitions of the B-model structures, as well as a careful statement of the main results. In Section 6.4 we show how our Plücker formula for the superpotential relates to the formulations in [6, 21, 79]. The proof of the main theorem begins in Section 10 and takes up the remainder of the paper. It makes use of deep properties of the cluster algebra structure of the homogeneous coordinate ring of a Grassmannian. In the final sections we prove a version of the main theorem in the torus-equivariant setting.

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2. The A-model introduction

Let us suppose X is the Grassmannian $Gr_{n-k}(\mathbb{C}^n)$. We focus on the example $k = 3$ and $n = 5$ to illustrate our results in the introduction. The cohomology of X has a basis called the Schubert basis, which is indexed by partitions or Young diagrams (see e.g. [27]). In the example of $X = Gr_2(\mathbb{C}^5)$ we denote the Schubert basis of $H^*(X) = H^*(X, \mathbb{C})$ by

$$\sigma^{\emptyset}, \sigma^{\ydiagram{1}}, \sigma^{\ydiagram{2}}, \sigma^{\ydiagram{1,1}}, \sigma^{\ydiagram{3}}, \sigma^{\ydiagram{2,1}}, \sigma^{\ydiagram{3,1}}, \sigma^{\ydiagram{2,2}}, \sigma^{\ydiagram{3,2}}, \sigma^{\ydiagram{3,3}}\ .$$

Here σ^λ is in $H^{2|\lambda|}(X)$ and $|\lambda|$ denotes the number of boxes in the Young diagram λ. Furthermore X^λ will be a Schubert variety associated to λ, representing the Poincaré dual homology class. For example $X^{\ydiagram{1}}$ is a hyperplane section for the Plücker embedding. In general the set of partitions fitting in an $(n-k) \times k$-rectangle indexes the Schubert basis for $Gr_{n-k}(n)$ in an entirely analogous way and is denoted by $\mathcal{P}_{k,n}$.

In classical Schubert calculus, Monk's rule says that the cup product with $\sigma^{\ydiagram{1}}$ takes any Schubert class σ^λ to the sum of all the Schubert classes σ^μ corresponding to the μ's in $\mathcal{P}_{k,n}$ made up of λ and precisely one extra box. In the cohomology of $Gr_2(5)$,

$$\sigma^{\ydiagram{1}} \cup \sigma^{\ydiagram{2,1}} = \sigma^{\ydiagram{2,2}} + \sigma^{\ydiagram{3,1}}, \qquad \sigma^{\ydiagram{1}} \cup \sigma^{\ydiagram{3,2}} = \sigma^{\ydiagram{3,3}}, \qquad \text{and} \qquad \sigma^{\ydiagram{1}} \cup \sigma^{\ydiagram{3,3}} = 0.$$

The combinatorics of adding a box, in this context, just encodes what happens to a Schubert variety X^λ homologically, when it is intersected with a hyperplane section in general position.

In the quantum cohomology ring [85] (at fixed parameter q), quantum Monk's rule says that the quantum cup product with $\sigma^{\ydiagram{1}}$ takes any Schubert class σ^λ to $\sigma^{\ydiagram{1}} \cup \sigma^\lambda$ plus, if it exists, a term $q\sigma^\nu$, where ν is obtained by removing $n-1 (= 4)$ boxes which must form the rim of λ (by the $a = 1$ case of quantum Pieri formula from [12]). For example,

$$\sigma^{\ydiagram{1}} \star_q \sigma^{\ydiagram{2,1}} = \sigma^{\ydiagram{2,2}} + \sigma^{\ydiagram{3,1}}, \qquad \sigma^{\ydiagram{1}} \star_q \sigma^{\ydiagram{3,2}} = \sigma^{\ydiagram{3,3}} + q\sigma^{\ydiagram{1}}. \qquad \text{and} \qquad \sigma^{\ydiagram{1}} \star_q \sigma^{\ydiagram{3,3}} = q\sigma^{\ydiagram{2}}.$$

Very roughly speaking, the extra term $q\sigma^\nu$ in $\sigma^{\ydiagram{1}} \star_q \sigma^\lambda$ says that the space of degree one maps $\phi : \mathbb{C}P^1 \hookrightarrow X$ for which $\phi(0)$ lies in X^λ and $\phi(1)$ lies in a fixed general position hyperplane, is essentially parametrised by a subvariety of X in the class $[X^\nu]$ (via sending ϕ to $\phi(\infty)$).

The quantum products by degree two classes were used by Dubrovin and Givental [32, 20] to define a connection on the trivial bundle

$$H^2(X, \mathbb{C}) \times H^*(X, \mathbb{C}) \to H^2(X, \mathbb{C}).$$

In our setting we have $H^2(X, \mathbb{C}) = \mathbb{C}$ spanned by the class $c_1(\mathcal{O}(1))$, which equals the Schubert class $\sigma^{\ydiagram{1}}$. We denote by τ the coordinate on $H^2(X, \mathbb{C})$ dual to $c_1(\mathcal{O}(1))$. Let us recall the usual definition of the connection in the conventions of Dubrovin [16], depending on an additional parameter z :

$$\nabla_{\partial_\tau} := \frac{d}{d\tau} + \frac{1}{z} \sigma^{\ydiagram{1}} \star_{e^\tau} \_\ . \tag{2.1}$$

We think of this connection as being dual to the one whose flat sections are constructed by Givental [34] in terms of descendent 2-point Gromov-Witten invariants, compare also Section 4.2.

Our main result is a B-model construction of the above connection. However we make some small adjustments first. Instead of τ we prefer to consider $q = e^\tau$, the coordinate on the torus $H^2(X, \mathbb{C})/H^2(X, 2\pi i\mathbb{Z}) \cong \mathbb{C}_q^*$. We write $\mathbb{C}_q$ to mean $\mathbb{C}$ with coordinate q, and similarly $\mathbb{C}_q^*$ for $\mathbb{C} \setminus \{0\}$ with (invertible) coordinate q. Also, following Dubrovin [20] we will extend the connection (2.1) in the z-direction, to give a flat meromorphic connection over a larger base. Namely, let $\mathcal{H}_A$ denote the (sheaf of regular sections of) the trivial vector bundle with fiber $H^*(X, \mathbb{C})$ over the extended base $\mathbf{P} = \mathbb{C}_z \times \mathbb{C}_q$, where the z and q are coordinates.

We identify a Schubert class $\sigma^\lambda \in H^*(X, \mathbb{C})$ with the corresponding constant section of $\mathcal{H}_A$. Using the conventions of Iritani [43, Definition 3.1] we set :

$$
{}^A\nabla_{q\partial_q} \;:=\; q\frac{\partial}{\partial q} + \frac{1}{z}\sigma^{\square} \star_q \_\, , \tag{2.2}
$$

$$
{}^A\nabla_{z\partial_z} \;:=\; z\frac{\partial}{\partial z} + \mathbf{gr} - \frac{1}{z}c_1(TX) \star_q \_\, , \tag{2.3}
$$

where $\mathbf{gr}$ is a diagonal operator on $H^*(X)$ given by $\mathbf{gr}(\sigma) = k\sigma$ whenever $\sigma \in H^{2k}(X)$. These formulas define a flat meromorphic connection on $\mathcal{H}_A$.

The vector bundle $\mathcal{H}_A$ also comes equipped with a flat pairing which is non-degenerate over $\mathbb{C}_z^* \times \mathbb{C}_q$. Namely let $\langle \_\, , \_ \rangle_{H^*(X)}$ denote the Poincaré duality pairing on $H^*(X)$ and define $j : \mathbf{P} \to \mathbf{P}$ by $(z, q) \mapsto (-z, q)$. Then the pairing $S_A : j^*\mathcal{H}_A \otimes \mathcal{H}_A \to \mathcal{O}_{\mathbf{P}_A}$ is given by

$$
S_A(\_\, , \_) \;=\; (2\pi i z)^N \langle \_\, , \_ \rangle_{H^*(X)} ,
$$

where $N = \dim_{\mathbb{C}}(X)$, compare [43]. For the Grassmannian X, the Poincaré duality pairing is concretely described by the following involution. For $\lambda \in \mathcal{P}_{k,n}$ we denote by $PD(\lambda)$ the Young diagram obtained by taking the complement of λ (within its bounding $(n-k) \times k$-rectangle) and rotating by 180°. The resulting Schubert class $\sigma^{PD(\lambda)}$ pairs to 1 with σ^λ and to 0 with all other Schubert classes under $\langle \_\, , \_ \rangle_{H^*(X)}$.

Definition 2.1. We define

$$
H_A = \Gamma(\mathcal{H}_A, \mathbb{C}_z^* \times \mathbb{C}_q^*) = H^*(X, \mathbb{C}[z^{\pm 1}, q^{\pm 1}])
$$

to be the module for $D_{\mathbf{P}} = \mathbb{C}[z^{\pm 1}, q^{\pm 1}] \langle \partial_z, \partial_q \rangle$ where ∂_z, ∂_q act by ${}^A\nabla_{\partial_z}$ and ${}^A\nabla_{\partial_q}$, respectively. Compare equations (2.2) and (2.3). We also define the $\mathbb{C}[z, q]$-submodule $H_{A,0} := H^*(X, \mathbb{C}[z, q])$, which is acted on by the subalgebra $D_{\mathbf{P},0}$ of $D_{\mathbf{P}}$ generated by $z, q, z(z\partial_z)$ and $z(q\partial_q)$. The pairing $H_A \otimes_{\mathbb{C}[z^{\pm 1}, q^{\pm 1}]} H_A \to \mathbb{C}[z^{\pm 1}, q^{\pm 1}]$ defined by S_A is non-degenerate and denoted again by S_A.

The main goal of this paper is to construct the $D_{\mathbf{P}}$-module H_A above, and with it the data, $\mathcal{H}_A, H_{A,0}$ and ${}^A\nabla$, in terms of a Gauss-Manin system defined by a mirror LG model. Descriptions of the LG model and the Gauss-Manin system follow in Section 3. The main results are stated in Section 4.

3. The B-model introduction

3.1. **The mirror LG model.** To give our presentation of the mirror LG model of the Grassmannian $X = Gr_{n-k}(n)$ we need to introduce a new Grassmannian $\check{\mathbb{X}} := Gr_k(n)$. Both X and $\check{\mathbb{X}}$ have dimension $N = k(n-k)$. To be more precise, if $X = Gr_{n-k}(\mathbb{C}^n)$ then we think of $\check{\mathbb{X}}$ as $Gr_k((\mathbb{C}^n)^*)$, which is embedded by a Plücker embedding in $\mathbb{P}(\bigwedge^k(\mathbb{C}^n)^*)$. Here $(\mathbb{C}^n)^*$ denotes the vector space dual to $\mathbb{C}^n$, with an action of the Langlands dual $GL_n(\mathbb{C})$ from the right. The Plücker coordinates p_λ for $\check{\mathbb{X}}$ correspond in a natural way to the Schubert classes σ^λ in $H^*(X)$ and are indexed by $\lambda \in \mathcal{P}_{k,n}$. [1]

We continue with the explicit example of $X = Gr_2(5)$ and $\check{\mathbb{X}} = Gr_3(5)$, where $k = 3$ and $n = 5$. Define the rational function W on $\check{\mathbb{X}} \times \mathbb{C}_q$ by the formula

$$
W = \frac{p_{\ydiagram{1}}}{p_\emptyset} + \frac{p_{\ydiagram{3,1}}}{p_{\ydiagram{3}}} + q\frac{p_{\ydiagram{2}}}{p_{\ydiagram{3,3}}} + \frac{p_{\ydiagram{3,2}}}{p_{\ydiagram{2,2}}} + \frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{1,1}}}, \tag{3.1}
$$

in terms of Plücker coordinates p_λ. To obtain a regular function, remove from $\check{\mathbb{X}}$ the 5 hyperplanes defined by the Plücker coordinates which appear in the denominators. The resulting affine variety is

$$
\check{\mathbb{X}}^\circ := \check{\mathbb{X}} \setminus \{p_\emptyset = 0\} \cup \{p_{\ydiagram{3}} = 0\} \cup \{p_{\ydiagram{3,3}} = 0\} \cup \{p_{\ydiagram{2,2}} = 0\} \cup \{p_{\ydiagram{1,1}} = 0\}.
$$

[1] This coincidence of Plücker coordinates and Schubert classes is due to the identification of $H^*(X)$ with $\bigwedge^k \mathbb{C}^n$, viewed as the k-th fundamental representation of the Langlands dual $GL_n(\mathbb{C})$ afforded by the geometric Satake correspondence [31, 45, 55, 62]. This explains also why $\check{\mathbb{X}}$ should be viewed as a homogeneous space for the Langlands dual of the group used to define the A-model Grassmannian X. Therefore even though X and $\check{\mathbb{X}}$ are isomorphic we do not think of them as being the same. Compare also [66, 67, 68].

Note that the anti-canonical class of $\check{\mathbb{X}} = Gr_3(5)$ is $5\sigma^{\square}$, therefore $\check{\mathbb{X}}^\circ$ is the complement of an anti-canonical divisor in $\check{\mathbb{X}}$. Let $\omega = \omega_{\check{\mathbb{X}}^\circ}$ be a choice of non-vanishing holomorphic volume form on $\check{\mathbb{X}}^\circ$ with simple poles along D. We denote the regular function again by W, and refer to $W : \check{\mathbb{X}}^\circ \times \mathbb{C}_q \to \mathbb{C}$ as the superpotential.

For a general Grassmannian $X = Gr_{n-k}(n)$ we have a completely analogously defined affine subvariety $\check{\mathbb{X}}^\circ$ of $\check{\mathbb{X}} = Gr_k(n)$, along with a non-vanishing holomorphic volume form ω on $\check{\mathbb{X}}^\circ$, and a superpotential $W : \check{\mathbb{X}}^\circ \times \mathbb{C}_q \to \mathbb{C}$, see Sections 6 and 8. Note that ω is a priori only uniquely determined up to a scalar. This scalar is chosen in Theorem 4.2, which is our first result that depends on this choice.

It is interesting to note that the variety $\check{\mathbb{X}}^\circ$ is an open positroid variety in the sense of Knutson, Lam and Speyer [48]. It also plays a special role for a particular Poisson structure on $\check{\mathbb{X}}$, see [28]. We will show in Section 6.4 that this LG model is isomorphic to the one introduced by the second author in [79], and after restriction to an open subtorus becomes isomorphic to one introduced earlier by Eguchi, Hori and Xiong [21] and studied further in [6].

3.2. **The Gauss-Manin system.** Denote by $\Omega^N(\check{\mathbb{X}}^\circ)$ the space of algebraic N-forms on $\check{\mathbb{X}}^\circ$. We write $W_q : \check{\mathbb{X}}^\circ \to \mathbb{C}$ for the superpotential where the coordinate $q \neq 0$ is fixed. There is a Gauss-Manin system associated to W_q, see [18, 19, 82], which should be thought of as describing algebraic N-forms η measured by 'period integrals' of the form $\int_\Gamma e^{\frac{1}{z}W_q}\eta$, see the definition below. Here we let both z and q vary to get a 2-parameter Gauss-Manin connection.

Definition 3.1. Consider the $\mathbb{C}[z^{\pm1}, q^{\pm1}]$-module G^{W_q} defined by

$$G^{W_q} = \Omega^N(\check{\mathbb{X}}^\circ)[z^{\pm1}, q^{\pm1}] / \left((d + z^{-1} dW_q \wedge \_ \,)(\Omega^{N-1}(\check{\mathbb{X}}^\circ)[z^{\pm1}, q^{\pm1}])\right).$$

Note that for fixed $(z_0, q_0) \in \mathbf{P}$ the 'fiber'

$$F_{B,(z_0,q_0)} := \Omega^N(\check{\mathbb{X}}^\circ) / \left((d + z_0^{-1} dW_{q_0} \wedge \_\,)(\Omega^{N-1}(\check{\mathbb{X}}^\circ))\right),$$

is a twisted de Rham cohomology group in the sense going back to Witten [87, (11)]. There is a Gauss-Manin connection on G^{W_q} defined on $\eta \in \Omega^N(\check{\mathbb{X}}^\circ)$ by

$$ {}^B\nabla_{q\partial_q}[\eta] \quad := \quad \frac{1}{z}\left[q\frac{\partial W}{\partial q}\,\eta\right], \tag{3.2}$$

$$ {}^B\nabla_{z\partial_z}[\eta] \quad := \quad -\frac{1}{z}\left[W\,\eta\right], \tag{3.3}$$

and extended using the Leibniz rule. Thanks to the flatness of ${}^B\nabla$ we get a $D_{\mathbf{P}}$-module structure on G^{W_q} by letting ∂_z and ∂_q in $D_{\mathbf{P}} = \mathbb{C}[z^{\pm1}, q^{\pm1}]\,\langle \partial_z, \partial_q\rangle$ act by the operators ${}^B\nabla_{\partial_z}$ and ${}^B\nabla_{\partial_q}$, respectively.

In order to state our main theorem we will define a $\mathbb{C}[z, q]$-submodule of G^{W_q} which is to play the part of regular global sections of a trivial vector bundle $\mathcal{H}_B$ on $\mathbb{C}_q \times \mathbb{C}_z$.

Since the divisor $\{p_\emptyset = 0\}$ is removed in the definition of $\check{\mathbb{X}}^\circ$, we can adopt the convention of setting $p_\emptyset = 1$ on $\check{\mathbb{X}}^\circ$. Therefore, from here on we consider the remaining p_λ as actual functions on $\check{\mathbb{X}}^\circ$ as opposed to homogeneous coordinates.

Definition 3.2. Recall that $\check{\mathbb{X}}^\circ$ has on it a non-vanishing holomorphic volume form ω. Let H_B be the $\mathbb{C}[z^{\pm1}, q^{\pm1}]$-submodule of G^{W_q} spanned by the classes $[p_\lambda\omega]$ where λ runs through $\mathcal{P}_{k,n}$. Furthermore let $H_{B,0}$ be the $\mathbb{C}[z, q]$-submodule inside H_B generated by the $[p_\lambda\omega]$, and let $\overline{F}_{B,(z_0,q_0)}$ be the corresponding $\mathbb{C}$-linear subspace of $F_{B,(z_0,q_0)}$.

We will prove the following lemma in Section 9.2.

Lemma 3.3. *$H_{B,0}$ is a free $\mathbb{C}[z, q]$-module with basis $\{[p_\lambda\omega], \lambda \in \mathcal{P}_{k,n}\}$.*

By this lemma $H_{B,0}$ is indeed the space of regular sections of a trivial vector bundle on $\mathbf{P} = \mathbb{C}_z \times \mathbb{C}_q$. We let $\mathcal{H}_B$ denote the sheaf of regular sections of this trivial bundle on $\mathbf{P}$. The fiber of the vector bundle $\mathcal{H}_B$ at (z_0, q_0) is $\overline{F}_{B,(z_0,q_0)}$ and has a basis given independently of (z_0, q_0) by the classes $[p_\lambda\omega]$.

Conjecture 3.4. [2] We conjecture that $H_B = G^{W_q}$.

The intuitive meaning of this conjecture is that W_q has no additional critical point at ∞ in any compactification. The conjecture would follow from a related technical condition on W_q (cohomological tameness [83, 82]). Note that in the special case of projective space the $[p_\lambda \omega]$ were already known to form a free basis of G^{W_q}, see [38]. Another related example is the the smooth quadric Q_{2n-1}, which is the orthogonal Grassmannian of lines in $\mathbb{C}^{2n+1}$. In this case the mirror LG-model of [79] was expressed in Plücker coordinates in [67], and proved for Q_3 to agree with one given by Gorbounov and Smirnov which they showed with Sabbah and Nemethi to be cohomologically tame [37].

4. Main results

4.1. **Isomorphism of $D_{\mathbf{P}}$-modules.** In the B-model, we consider the Gauss-Manin system on G^{W_q}. The first main theorem shows that the A-model datum, consisting of $H_{A,0} = H^*(X, \mathbb{C}[z,q])$ together with its small Dubrovin connection, can be recovered inside G^{W_q}.

Theorem 4.1. *The $\mathbb{C}[z^{\pm 1}, q^{\pm 1}]$-module H_B is a $D_{\mathbf{P}}$–submodule of G^{W_q}, and the map*

$$\begin{array}{rcl} \Phi : H_A & \to & H_B \\ \sigma^\lambda & \mapsto & [p_\lambda \omega] \end{array}$$

is an isomorphism of $D_{\mathbf{P}}$-modules. Under this isomorphism $H_{A,0} = H^*(X, \mathbb{C}[z,q])$ is identified with $H_{B,0}$ and $\mathcal{H}_A$ is identified with $\mathcal{H}_B$.

The proof of this theorem hinges on verifying the following formulas for the action of ${}^B\nabla$,

$$ {}^B\nabla_{q\partial_q} [p_\lambda \omega] = \frac{1}{z} \left(\sum_\mu [p_\mu \omega] + \sum_\nu q\, [p_\nu \omega] \right), \tag{4.1}$$

$$ {}^B\nabla_{z\partial_z} [p_\lambda \omega] = |\lambda| [p_\lambda \omega] - \frac{1}{z}\, n \left(\sum_\mu [p_\mu \omega] + \sum_\nu q\, [p_\nu \omega] \right), \tag{4.2}$$

where μ and ν are exactly as in the quantum Monk's rule for $\sigma^\square \star_q \sigma^\lambda$; see equations (2.2) and (2.3) in Section 2. The proof of this theorem will be given in Sections 10 to 18.

In the concrete case of $X = Gr_2(5)$, these formulas say for example

$$\begin{array}{rcl} {}^B\nabla_{q\partial_q} \left([p_{\ydiagram{2,1}} \omega] \right) & = & \dfrac{1}{z} \left([p_{\ydiagram{3,1}} \omega] + [p_{\ydiagram{2,2}} \omega] \right), \\ {}^B\nabla_{q\partial_q} \left([p_{\ydiagram{3,3}} \omega] \right) & = & \dfrac{1}{z} q\, [p_{\ydiagram{2}} \omega]. \end{array}$$

Similarly in the $z\partial_z$ direction :

$$\begin{array}{rcl} {}^B\nabla_{z\partial_z} \left([p_{\ydiagram{2,1}} \omega] \right) & = & 3[p_{\ydiagram{2,1}} \omega] - \dfrac{5}{z} \left([p_{\ydiagram{3,1}} \omega] + [p_{\ydiagram{2,2}} \omega] \right), \\ {}^B\nabla_{z\partial_z} \left([p_{\ydiagram{3,3}} \omega] \right) & = & 6[p_{\ydiagram{3,3}} \omega] - \dfrac{5}{z} q\, [p_{\ydiagram{2}} \omega]. \end{array}$$

This reflects precisely the formulas on the A-side arising from (2.2) and (2.3) and quantum Schubert calculus. For example (2.3) implies

$$ {}^A\nabla_{z\partial_z} \left(\sigma^{\ydiagram{3,3}} \right) = 6\, \sigma^{\ydiagram{3,3}} - \frac{5}{z} q\, \sigma^{\ydiagram{2}}. $$

Now we come to some consequences of Theorem 4.1, along with comparison results connecting W_q with previously defined superpotentials.

[2] Update, May 2017. This conjecture now follows from our Theorem 4.10 combined with the recent work [50] of Lam and Templier.

4.2. **Oscillating integrals.** Recall the flat pairing S_A in the A-model. We may think of this pairing as identifying the bundle $H^*(X) \times \mathbb{C}^*_z \times \mathbb{C}_q \to \mathbb{C}^*_z \times \mathbb{C}_q$ with its dual. The flatness condition can then be interpreted as saying that the dual connection to ${}^A\nabla$ is given by formulas analogous to (2.2) and (2.3) but with z replaced by $-z$. In other words the new connection ${}^A\nabla^\vee$ defined by

$$ {}^A\nabla^\vee_{q\partial_q} \quad := \quad q\frac{\partial}{\partial q} - \frac{1}{z}\sigma^\square \star_q \_ \,, \tag{4.3}$$

$$ {}^A\nabla^\vee_{z\partial_z} \quad := \quad z\frac{\partial}{\partial z} + \mathbf{gr} + \frac{1}{z}c_1(TX) \star_q \_ \,, \tag{4.4}$$

satisfies $dS_A(\sigma,\sigma') = S_A({}^A\nabla^\vee\sigma,\sigma') + S_A(\sigma, {}^A\nabla\sigma')$, and is therefore dual to ${}^A\nabla$.

In [34, Corollary 6.3], Givental wrote down a basis of the space of all solutions $s \in H^*(X, \mathbb{C}[z^{-1}, \ln(q)][[q]])$ to the flat sections equation ${}^A\nabla^\vee_{q\partial_q} s = 0$, that is to the equation:

$$ q\frac{\partial}{\partial q} s = \frac{1}{z}\sigma^\square \star_q s. \tag{4.5}$$

Equation (4.5) is referred to as the *(small) quantum differential equation.* Givental's solution is given in terms of two-point descendent Gromov-Witten invariants. Explicitly in our setting, for each $\mu \in \mathcal{P}_{k,n}$ there is a solution

$$ s_\mu = \frac{1}{z^N} \sum_{\lambda\in\mathcal{P}_{k,n}} \sum_{d\geq 0} q^d \left\langle \frac{1}{z-\psi} e^{\frac{\ln(q)}{z}\sigma^\square}\sigma^\mu, \sigma^\lambda \right\rangle_{2,d} \sigma^{PD(\lambda)} \tag{4.6}$$

to (4.5), where to make sense of the $d=0$ term one sets

$$ \left\langle \frac{1}{z-\psi} e^{\frac{\ln(q)}{z}\sigma^\square}\sigma^\mu, \sigma^\lambda \right\rangle_{2,0} := \left\langle e^{\frac{\ln(q)}{z}\sigma^\square}\sigma^\mu, \sigma^\lambda, 1 \right\rangle_{3,0}. \tag{4.7}$$

Here we use the notation $\langle \quad \rangle_{n,d}$ for genus zero n-point degree d Gromov-Witten invariants of X. Moreover ψ is the 'psi-class' on the moduli space of stable maps $\overline{\mathcal{M}_{0,2}}(X,d)$, which is the first Chern class of the line bundle defined by the cotangent line at the first marked point. We refer to [16] or [65, Section 1.3] for this result and relevant definitions. Note that we have added the factor $\frac{1}{z^N}$, with $N = k(n-k)$ for degree reasons (compare equation (4.8) below). Equivalently, the factor is necessary to ensure flatness of s_μ in the z-direction. In the case where μ is the maximal element in $\mathcal{P}_{k,n}$, which we denote μ_{n-k}, the exponential disappears and the formula simplifies to

$$ s_{\mu_{n-k}} = \sum_\lambda \left(\delta_{\lambda,\emptyset} + \sum_{d\geq 1} \left\langle \psi^{dn-|\lambda|-1}\sigma^{\mu_{n-k}}, \sigma^\lambda \right\rangle_{2,d} \left(\frac{q}{z^n}\right)^d \right) z^{-|PD(\lambda)|}\sigma^{PD(\lambda)}. \tag{4.8}$$

The following theorem implies an integral formula for the solution $s_{\mu_{n-k}} \in H^*(X, \mathbb{C}[z^{-1}][[q]])$ to the quantum differential equation (4.5).

Theorem 4.2. *Let $\Gamma_{\mu_{n-k}}$ be a cycle in $H_N(\check{\mathbb{X}}^\circ, \mathbb{Z})$ represented by an oriented, compact torus (homeomorphic to $(S_1)^N$) which is the compact real form of a cluster torus (isomorphic to $(\mathbb{C}^*)^N$) inside $\check{\mathbb{X}}^\circ$. Note that the class $\Gamma_{\mu_{n-k}}$ does not depend on the choice of the cluster torus. We choose ω to be dual to $\Gamma_{\mu_{n-k}}$ in the sense that $\frac{1}{(2\pi i)^N}\int_{\Gamma_{\mu_{n-k}}}\omega = 1$. Then the formula*

$$ \mathcal{S}_{\Gamma_{\mu_{n-k}}}(z,q) := \frac{1}{(2\pi i z)^N} \sum_{\lambda\in\mathcal{P}_{k,n}} \left(\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q} p_\lambda \omega \right) \sigma^{PD(\lambda)} $$

defines a flat section for ${}^A\nabla^\vee$ inside $H^*(X, \mathbb{C}[z^{-1}][[q]])$. In particular $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ satisfies the small quantum differential equation (4.5).

Remark 4.3. Note that $\Gamma_{\mu_{n-k}}$ and ω in the above theorem are uniquely defined up to a common sign. The function $\mathcal{S}_{\Gamma_{\mu_{n-k}}}(z,q)$ is canonical and does not depend on this sign choice.

Proof. This statement follows in a standard way from Theorem 4.1 and the constructions. For any $(z,q)\in\mathbf{P}$ with $z\neq 0$, consider the linear form on the fiber $F_{B,(z,q)}$ of $\mathcal{H}_B$ at the point (z,q) defined by the formula

$$\mathrm{Osc}_{\Gamma_{\mu_{n-k}}}(z,q):[\eta]\mapsto \int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}\eta.$$

This formula defines a section $\mathrm{Osc}_{\Gamma_{\mu_{n-k}}}$ of $\mathcal{H}^\vee_{B,an}$ over $\mathbb{C}^*_z\times\mathbb{C}_q$, where $\mathcal{H}^\vee_{B,an}$ denotes the sheaf of analytic sections of the bundle dual to $\mathcal{H}_B$. The definition of the Gauss-Manin connection (4.1),(4.2) on $\mathcal{H}_B$ is engineered so that $\mathrm{Osc}_{\Gamma_{\mu_{n-k}}}$ is a flat section of $\mathcal{H}^\vee_{B,an}$.

Using Theorem 4.1 together with the pairing S_A, the bundle $\mathcal{H}^\vee_{B,an}$ with its Gauss-Manin connection can be identified with the pair $(\mathcal{H}_{A,an},{}^A\nabla^\vee)$ over $\mathbb{C}^*_z\times\mathbb{C}_q$, where $\mathcal{H}_{A,an}$ denotes the sheaf of analytic sections of the vector bundle $\mathcal{H}_A$. We now denote by $\mathrm{Osc}^A_{\Gamma_{\mu_{n-k}}}$ the flat section of $(\mathcal{H}_{A,an},{}^A\nabla^\vee)$ over $\mathbb{C}^*_z\times\mathbb{C}_q$ corresponding to $\mathrm{Osc}_{\Gamma_{\mu_{n-k}}}$ under this identification. Then $\mathrm{Osc}^A_{\Gamma_{\mu_{n-k}}}$ is the section of $\mathcal{H}_{A,an}$ determined by the property that

$$S_A(\mathrm{Osc}^A_{\Gamma_{\mu_{n-k}}},\sigma^\lambda)=\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}p_\lambda\omega. \tag{4.9}$$

Since the basis dual to $\{\sigma^\lambda\}$ with respect to the pairing S_A is $\{\frac{1}{(2\pi iz)^N}\sigma^{PD(\lambda)}\}$, equation (4.9) implies that

$$\mathrm{Osc}^A_{\Gamma_{\mu_{n-k}}}=\frac{1}{(2\pi iz)^N}\sum_{\lambda\in\mathcal{P}_{k,n}}\left(\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}p_\lambda\omega\right)\sigma^{PD(\lambda)}.$$

So $\mathrm{Osc}^A_{\Gamma_{\mu_{n-k}}}=\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ and we see that $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ is flat for ${}^A\nabla^\vee$. It remains to check that $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ lies in $H^*(X,\mathbb{C}[z^{-1}][[q]])$, in other words that the coefficients m_λ lie in $\mathbb{C}[z^{-1}][[q]]$ as opposed to $\mathbb{C}[[z^{-1},q]]$. This follows by degree considerations. Namely the flatness of $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ implies in particular for the coefficients that

$$\left(z\frac{\partial}{\partial z}+nq\frac{\partial}{\partial q}\right)\left[\frac{1}{(2\pi iz)^N}\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}p_\lambda\omega\right]=\frac{-|PD(\lambda)|}{(2\pi iz)^N}\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}p_\lambda\omega.$$

Therefore $\left(z\frac{\partial}{\partial z}+nq\frac{\partial}{\partial q}\right)$ annihilates

$$z^{|PD(\lambda)|}\frac{1}{(2\pi iz)^N}\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}p_\lambda\omega, \tag{4.10}$$

which implies that in the q-expansion of (4.10) the coefficient of q^d is a scalar multiple of $\frac{1}{z^{dn}}$. As a consequence the coefficient of $\sigma^{PD(\lambda)}$ in $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ has the form

$$\frac{1}{(2\pi iz)^N}\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}p_\lambda\omega=\sum_{d\geq 0}a_d\left(\frac{q}{z^n}\right)^d z^{-|PD(\lambda)|}. \tag{4.11}$$

In particular it lies in $\mathbb{C}[z^{-1}][[q]]$. □

Remark 4.4. Note that by setting $q=1$ in Equation (4.11) we have the series expansion

$$\frac{1}{(2\pi iz)^N}\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_1}p_\lambda\omega=\sum_{d\geq 0}a_d z^{-dn-|PD(\lambda)|}. \tag{4.12}$$

Expanding the exponential $e^{\frac{1}{z}W_1}$ on the left-hand side of the above equation, the coefficient a_d can now be computed by the residue formula

$$a_d=\frac{1}{(dn-|\lambda|)!}\left(\frac{1}{(2\pi i)^N}\int_{\Gamma_{\mu_{n-k}}}(W_1)^{dn-|\lambda|}p_\lambda\omega\right),$$

where we assume $dn \geq |\lambda|$. If $dn < |\lambda|$ then necessarily $a_d = 0$, since in this case $-dn - |PD(\lambda)| > -N$ while the left-hand side of (4.12) is contained in $z^{-N}\mathbb{C}[[z^{-1}]]$. We will make use of this formula for a_d in Proposition 4.9.

Remark 4.5. In this remark we check that the flat section $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ of Theorem 4.2 agrees with Givental's flat section $s_{\mu_{n-k}}$ exactly. First we note that, up to scalar, $s_{\mu_{n-k}}$ is the unique flat section with coefficients in $\mathbb{C}[z^{-1}][[q]]$, as follows recursively from the differential equation (4.5) and grading considerations. From Theorem 4.2 it follows therefore that $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ is a scalar multiple of $s_{\mu_{n-k}}$. The relevant scalar can be determined by computing the coefficient a_0 of

$$\frac{1}{(2\pi i)^N}\int_{\Gamma_{\mu_{n-k}}} e^{\frac{1}{z}W_q}\omega = \sum_{d\geq 0} a_d q^d z^{-dn},$$

where the above equation is a consequence of (4.11) in the case $\lambda = \emptyset$, and we have multiplied out by z^N. This coefficient a_0 is computed on the left-hand side by

$$a_0 = \frac{1}{(2\pi i)^N}\int_{\Gamma_{\mu_{n-k}}} \omega = 1,$$

using the assumptions of Theorem 4.2. Therefore, comparing with the start of the $\lambda = \emptyset$ term of (4.8), we see that $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ and $s_{\mu_{n-k}}$ agree.

Remark 4.6 (Other solutions and the J-function)**.** Further local solutions $\mathcal{S} = \mathcal{S}_\Gamma$ to the equation ${}^A\nabla^\vee_{q\partial_q}\mathcal{S} = 0$ can be obtained by replacing $\Gamma_{\mu_{n-k}}$ by some other, possibly non-compact integration cycle Γ. In this case it would be necessary to have conditions on the decay of $\Re(\frac{1}{z}W_q)$ in unbounded directions of Γ and let Γ vary with z and q, to ensure convergence. According to Givental [35, Section 2] such cycles Γ may be obtained from Morse theory for $\Re(\frac{1}{z}W_q)$. Other than the compact cycle $\Gamma_{\mu_{n-k}}$ associated to $s_{\mu_{n-k}}$, we don't know how to determine specific cycles Γ_μ such that the flat section S_{Γ_μ} recovers Givental's flat section s_μ defined via the A-model. Identifying such cycles would give integral formulas for all the entries of Givental's 'fundamental solution matrix', and in particular for the coefficients of Givental's 'J-function'; see Section 4 in [35]. When $X = \mathbb{C}P^2$, however, explicit integration cycles can be described; see [38].

Although from our results we only obtain a formula for the constant term of Givental's J-function (the 'A-series', see Section 4.3) and not the full J-function, we can still consider the $\mathbb{C}[q, z^{-1}]\langle\partial_q\rangle$-module generated by the coefficients of the J-function, or equivalently generated by the $\sigma^{\mu_{n-k}}$-coefficients of the flat sectionss of ${}^A\nabla^\vee$; see [33], or for example [6, Section 5.1]. This is sometimes called the 'quantum cohomology D-module', and it quantises the part of the quantum cohomology ring generated by degree 2 elements. (It is not to be confused with the $D_{\mathbf{P}}$-module H_A.) With this definition, we note that the property ${}^A\nabla^\vee_{q\partial_q}\mathcal{S}_\Gamma = 0$ implies that the integrals

$$\int_\Gamma e^{\frac{1}{z}W_q}\omega \tag{4.13}$$

are solutions to the quantum cohomology D-module. We remark that other integral expressions for solutions of the quantum cohomology D-module of a Grassmannian which are very different from (4.13) were obtained by Bertram, Ciocan-Fontanine and Kim [5]. Moreover, in the same paper they prove a formula for the J-function generalising the one for projective space due to Givental.

4.3. **A-series conjecture and the superpotential of Eguchi, Hori and Xiong.** In Section 6.3 we will recall the definition of the conjectural Laurent polynomial superpotential of Eguchi, Hori and Xiong [21]. In that section we will also state in more detail the following comparison result.

Theorem 4.7. *The Laurent polynomial L_q associated to the Grassmannian X by Eguchi, Hori and Xiong in* [21] *is isomorphic to the restriction of W_q to a certain open torus $\mathcal{T}$ inside $\check{\mathbb{X}}^\circ$. The holomorphic volume form ω restricted to this torus agrees with the standard torus-invariant volume form $\omega_{\mathcal{T}}$.*

The proof of Theorem 4.7 is contained in Sections 6.3 and 8.

Consider now Givental's special solution $s_{\mu_{n-k}}$ (4.8) to the quantum differential equation (4.5). The coefficient of $\sigma^{\mu_{n-k}}$ in $s_{\mu_{n-k}}$ with z specialised to 1 is an element of $\mathbb{C}[[q]]$ and referred to as the A-series in [6]. Namely for $X = Gr_{n-k}(\mathbb{C}^n)$,

$$A_X(q) = \sum_d \left\langle \frac{1}{1-\psi}\sigma^{\mu_{n-k}}, \sigma^{\emptyset} \right\rangle_{2,d} q^d \;=\; 1 + \sum_{d\geq 1} \left\langle \psi^{dn-2}\sigma^{\mu_{n-k}} \right\rangle_{1,d} q^d, \tag{4.14}$$

where we have used that $\sigma^{\emptyset}$ is the fundamental class of X to simplify the formula.

In [6] Batyrev, Ciocan-Fontanine, Kim and van Straten studied the Laurent polynomial superpotential of [21] and conjectured an explicit combinatorial formula for the A-series (4.14) in the case of a Grassmannian. This conjecture can now be deduced.

We note that in the special case of $Gr_2(n)$ this conjecture was proved earlier in [5] using a formula (also proved in [5]) for the J-function; compare Remark 4.6.

Corollary 4.8. [6, Conjecture 5.2.3] *The A-series of the Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ is given by*

$$A_X(q) = \sum_{d\geq 0} \frac{1}{(d!)^n} \sum_{(s_{i,j})\in \mathcal{S}_d} \left(\prod_{(i,j)\in \mathcal{I}_{k,n}} \binom{s_{i+1,j}}{s_{i,j}} \binom{s_{i,j+1}}{s_{i,j}} \right) q^d. \tag{4.15}$$

Here the indexing sets are

$$\begin{aligned} \mathcal{I}_{k,n} &= \{(i,j) \in \mathbb{Z}\times\mathbb{Z} \mid 0 < i < n-k, 0 < j < k\}, \\ \mathcal{S}_d &= \{(s_{i,j}) \in (\mathbb{Z}_{\geq 0})^{\mathcal{I}_{k,n}} \mid s_{i+1,j} \geq s_{i,j},\ s_{i,j+1} \geq s_{i,j},\ s_{n-k,j} = s_{i,k} = d\}. \end{aligned}$$

Proof. The combinatorial formula (4.15) is obtained in [8] as the residue of the form $e^{L_q}\omega_{\mathcal{T}}$ determined by the EHX superpotential L_q. This corollary therefore follows from Theorem 4.2 together with Theorem 4.7 and the residue calculation in [8, Section 5.1]. □

Note that the corollary uses a residue calculation to give combinatorial formulas for the descendent Gromov-Witten invariants $\langle \psi^{dn-2}\sigma^{\mu_{n-k}} \rangle_{1,d}$ which are certain coefficients of Givental's flat section $s_{\mu_{n-k}}$. As a corollary to Theorem 4.2 we also have the following residue formulas for all of the remaining descendent Gromov-Witten invariants appearing in $s_{\mu_{n-k}}$.

Proposition 4.9. *With notations as in Section 4.2 we have*

$$\langle \psi^{dn-1}\sigma^{\mu_{n-k}}, \sigma^{\lambda} \rangle_{2,d} = \frac{1}{(dn-|\lambda|)!} \left(\frac{1}{(2\pi i)^N} \int_{\Gamma_{\mu_{n-k}}} (W_1)^{dn-|\lambda|} p_\lambda \omega \right), \tag{4.16}$$

assuming $dn \geq |\lambda|$.

Proof. Recall that the flat section $\mathcal{S}_{\Gamma_{\mu_{n-k}}}$ of Theorem 4.2 agrees with $s_{\mu_{n-k}}$, by Theorem 4.2 and Remark 4.5. The residue formula (4.16) now follows from the expansion (4.8) and Remark 4.4. □

4.4. **The LG model on a Richardson variety.** The first LG model for the Grassmannian X which accurately recovers the quantum cohomology ring is one defined in [79]. In this LG model the superpotential can be interpreted as a regular function $\mathcal{F}_q$ defined on an intersection of opposite Bruhat cells $\mathcal{R} = \mathcal{R}_{w_P, w_0}$. Here $\mathcal{R}$ is an affine subvariety of the full flag variety on the B-model side, which is associated to the Grassmannian X interpreted as a homogeneous space GL_n/P. In Section 6.2 we recall the precise definition of the superpotential $\mathcal{F}_q$. Moreover we explain how our superpotential W_q from Section 3 relates to it, by defining a carefully chosen embedding of $\mathcal{R}$ into the Grassmannian $\check{\mathbb{X}}$. We obtain the following comparison result.

Theorem 4.10. *The LG models $(\check{\mathbb{X}}^{\circ}, W_q)$ and $(\mathcal{R}, \mathcal{F}_q)$ are isomorphic via the maps given in Proposition 6.7. This isomorphism also identifies the holomorphic volume form ω on $\check{\mathbb{X}}^{\circ}$ with the holomorphic volume form on $\mathcal{R}$ introduced in* [79, Section 7].

The proof of this theorem is contained in Section 6.4 and Section 8. This proof also makes use of some special coordinates on $\check{\mathbb{X}}^\circ$ and the EHX Laurent polynomial expression recalled in Definition 6.8. This theorem implies that the Jacobi ring of W_q recovers the quantum cohomology ring $qH^*(X)[q^{-1}]$, since this is what was proved for $\mathcal{F}_q$ in [79, Corollary 4.2]. We will show in Proposition 9.2 that the classes of the Plücker coordinates in the Jacobi ring of W_q recover the Schubert basis, see also Proposition 5.2 and the paragraph preceding it.

Finally, as a consequence of Theorem 4.10 and the results on oscillating integrals described in Section 4.1 and Section 4.2, we also obtain the following corollary, compare Remark 4.6.

Corollary 4.11 (Special case of Conjecture 8.1 from [79])**.** *The quantum cohomology D-module of the Grassmannian has a global holomorphic solution on $\mathbb{C}_z^* \times \mathbb{C}_q$ given by the residue integral $\frac{1}{(2\pi i)^N}\oint e^{\frac{1}{z}\mathcal{F}_q}\omega$.*

This concludes the summary of main results which are non-equivariant.

5. Equivariant results

The Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ is a homogeneous space for $G^\vee = GL_n(\mathbb{C})$. In particular, the maximal torus $T^\vee$ of $n\times n$-diagonal matrices and the Borel subgroups of upper-triangular and lower-triangular matrices, denoted by $B_+^\vee$ and $B_-^\vee$ respectively, all act on X. In the A-model this means that we can consider the $T^\vee$-equivariant cohomology of X and the $T^\vee$-equivariant quantum cohomology of X, and we also have a $T^\vee$-equivariant version of the Dubrovin connection; see in particular [33, 54, 61]. Our final results are about extending Theorem 4.1 to describe the equivariant Dubrovin connection via the B-model.

In Section 20 we define a deformation W^{eq} of the superpotential W (Definition 20.1) involving the equivariant parameters $x_1,\dots,x_n$. These parameters are the standard generators of the equivariant cohomology ring of a point, $H^*_{T^\vee}(\{pt\}) = \mathbb{C}[x_1,\dots,x_n]$. In our running example of $X = Gr_2(\mathbb{C}^5)$ the deformation W^{eq} is given by the formula

$$W^{\mathrm{eq}} = W + (x_1+x_2)\ln(q) + (x_2-x_1)\ln(p_{(3)}) + (x_3-x_2)\ln(p_{(3,3)}) + (x_4-x_3)\ln(p_{(2,2)}) + (x_5-x_4)\ln(p_{(1,1)}).$$

Note that we may think of W^{eq} either as a multivalued map

$$W^{\mathrm{eq}} : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \to \mathbb{C} \oplus H^2_{T^\vee}(\{pt\}) \;=\; H^{\leq 2}_{T^\vee}(\{pt\})$$

or as a multivalued function $W^{\mathrm{eq}} : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \times \mathfrak{h}^\vee \to \mathbb{C}$, interpreting the x_i as functions on $\mathfrak{h}^\vee$ via the identification $H^2_{T^\vee}(\{pt\}) \cong (\mathfrak{h}^\vee)^*$. Our first equivariant result is the comparison result, which is proved in Section 20.

Proposition 5.1. *The LG model $(\check{\mathbb{X}}^\circ, W_q^{\mathrm{eq}})$ is isomorphic to the equivariant LG model $(\mathcal{R}, \mathcal{F}_q^{\mathrm{eq}})$ defined in* [79, Section 4.1 and 4.2] *via the maps given in Proposition 6.7.*

Note that although W^{eq} is multivalued, the derivatives of W^{eq} along $\check{\mathbb{X}}^\circ$ are regular and define an ideal $(\partial_{\check{\mathbb{X}}^\circ} W^{\mathrm{eq}})$ in $\mathbb{C}[\check{\mathbb{X}}^\circ][q^{\pm1}, x_1, \dots x_n]$. We call the quotient by the ideal the *Jacobi ring* of $(\check{\mathbb{X}}^\circ, W_q^{\mathrm{eq}})$ and let $[p_\lambda]$ denote the image in the Jacobi ring of the Plücker coordinate p_λ. As in the non-equivariant case, combining our comparison result (Proposition 5.1) with [79, Corollary 4.2] implies an isomorphism between the Jacobi ring of $(\check{\mathbb{X}}^\circ, W_q^{\mathrm{eq}})$ and the (small) equivariant quantum cohomology ring of X with q^{-1} adjoined. This isomorphism has the following very natural description which will be restated and proved in Proposition 21.5.

Proposition 5.2. *The Jacobi ring of $(\check{\mathbb{X}}^\circ, W_q^{\mathrm{eq}})$ is isomorphic to the equivariant quantum cohomology ring $qH^*_{T^\vee}(X,\mathbb{C})[q^{-1}]$ via an isomorphism which takes the form*

$$[p_\lambda] \mapsto [X^\lambda]_{T^\vee}.$$

Here X^λ is the $B_+^\vee$-invariant Schubert variety of codimension $|\lambda|$ associated to λ, and $[X^\lambda]_{T^\vee}$ is its $T^\vee$-equivariant fundamental class, viewed as an element of the equivariant quantum cohomology of X.

Our main equivariant result is an equivariant version of Theorem 4.1 which we now prepare to state. Equivariant quantum cohomology can in this setting be thought of as providing a q-deformed version $\star_{q,\mathbf{x}}$ of the equivariant cup product on $H^*_{T^\vee}(X)$, where $\mathbf{x} = (x_1, \dots, x_n)$. Let us therefore denote by $\sigma^\lambda_{T^\vee}$ the equivariant *or* quantum equivariant Schubert class

$$\sigma^\lambda_{T^\vee} := [X^\lambda]_{T^\vee}.$$

We recall the equivariant quantum Monk's rule [61, Section 1.1] which reads

$$c_1^{T^\vee}(\mathcal{O}(1)) \star_{q,\mathbf{x}} \sigma^\lambda_{T^\vee} = \sum_\mu \sigma^\mu_{T^\vee} + q\sum_\nu \sigma^\nu_{T^\vee} + x_\lambda \sigma^\lambda_{T^\vee},$$

where the μ and ν are as in the non-equivariant quantum Monk's rule described in Section 2, and x_λ is a particular linear combination of the x_i; see (19.1).

This formula determines the equivariant Dubrovin connection on the A-model side. Namely, we have the following definition, which is the $T^\vee$-equivariant analogue of Definition 2.1.

Definition 5.3 (The equivariant version of H_A)**.** Consider the ring of differential operators,

$$D_{\mathrm{eq}} = \mathbb{C}[x_1, \dots, x_n, z^{\pm 1}, q^{\pm 1}]\langle q\partial_q, z\partial_z + \sum_{i=1}^n x_i \partial_{x_i}\rangle.$$

Let H_A^{eq} be the $H^*_{T^\vee}(\{pt\})[z^{\pm1}, q^{\pm1}]$-module defined by

$$H_A^{\mathrm{eq}} := H^*_{T^\vee}(X)[z^{\pm1}, q^{\pm1}].$$

Then H_A is a D_{eq}-module by setting

$$q\partial_q\ \sigma = \frac{1}{z} c_1^{T^\vee}(\mathcal{O}(1)) \star_{q,\mathbf{x}} \sigma \tag{5.1}$$

and

$$\left(z\partial_z + \sum_i x_i\partial_{x_i}\right)\ \sigma = -\frac{1}{z}[X_{ac}]_{T^\vee} \star_{q,\mathbf{x}} \sigma + \mathbf{gr}(\sigma), \tag{5.2}$$

for $\sigma \in H^*(X, \mathbb{C})$. Here, X_{ac} denotes the anticanonical divisor given by the union of n different $T^\vee$-invariant hyperplanes which are permuted by the cyclic $\mathbb{Z}/n\mathbb{Z}$-action on X; see Section 19.2.

Definition 5.4 (The equivariant version of H_B)**.** Let $\Omega^\bullet_{\mathrm{eq}}(\check{\mathbb{X}}^\circ)$ denote the graded algebra of algebraic differential forms on $\check{\mathbb{X}}^\circ$ with coefficients in

$$H^*_{T^\vee}(pt)[z^{\pm1}, q^{\pm1}] = \mathbb{C}[x_1, \dots, x_n, z^{\pm1}, q^{\pm1}].$$

The k-th graded component

$$\Omega^k_{\mathrm{eq}}(\check{\mathbb{X}}^\circ) = \Omega^k(\check{\mathbb{X}}^\circ) \otimes H^*_{T^\vee}(\{pt\})[z^{\pm1}, q^{\pm1}]$$

consists of algebraic k-forms on $\check{\mathbb{X}}^\circ$ with coefficients in $H^*_{T^\vee}(\{pt\})[z^{\pm1}, q^{\pm1}]$. In particular, the $\Omega^k_{\mathrm{eq}}(\check{\mathbb{X}}^\circ)$ are $H^*_{T^\vee}(\{pt\})[z^{\pm1}, q^{\pm1}]$-modules. Then $\frac{1}{z}dW^{\mathrm{eq}}$, where $d = d_{\check{\mathbb{X}}^\circ}$ is the exterior derivative along $\check{\mathbb{X}}^\circ$, can be thought of as an element of $\Omega^1_{\mathrm{eq}}(\check{\mathbb{X}}^\circ)$. Note that dW^{eq} is algebraic despite the fact that W^{eq} is not. An equivariant analogue of the Gauss-Manin system from Definition 3.1 is defined by

$$G^{W_q^{\mathrm{eq}}} := \Omega^N_{\mathrm{eq}}(\check{\mathbb{X}}^\circ)/(d + \frac{1}{z}dW^{\mathrm{eq}} \wedge \_)\Omega^{N-1}_{\mathrm{eq}}(\check{\mathbb{X}}^\circ).$$

The elements of $G^{W_q^{\mathrm{eq}}}$ may be thought of as algebraic N-forms η on $\check{\mathbb{X}}^\circ$ depending (algebraically) on parameters $q^{\pm1}, z^{\pm1}, x_1, \dots, x_n$, which can be measured by integrals $\int_\Gamma e^{\frac{1}{z}W^{\mathrm{eq}}}\eta$.

The ring of differential operators,

$$D_{\mathrm{eq}} = \mathbb{C}[x_1, \dots, x_n, z^{\pm 1}, q^{\pm 1}]\langle q\partial_q, z\partial_z + \sum_{i=1}^n x_i \partial_{x_i}\rangle,$$

acts in a natural way on $G^{W_q^{\mathrm{eq}}}$. This action is given explicitly by

$$q\partial_q[\eta] = \frac{1}{z}\left[q\frac{\partial W^{\mathrm{eq}}}{\partial q}\eta\right]$$

and

$$(z\partial_z + \sum_i x_i\partial_{x_i})[\eta] = -\frac{1}{z}\left[W\eta\right],$$

where the second formula arises from the identity $\left(z\frac{\partial}{\partial z} + \sum_i x_i \frac{\partial}{\partial x_i}\right)\left(\frac{1}{z}W^{\mathrm{eq}}\right) = -\frac{1}{z}W$.

Note that the differential operator $z\partial_z$ no longer acts on $G^{W_q^{\mathrm{eq}}}$. Indeed, $z\frac{\partial}{\partial z}\left(\frac{1}{z}W_q^{\mathrm{eq}}\right) = -\frac{1}{z}W_q^{\mathrm{eq}}$ now involves logarithms of Plücker coordinates, so is no longer algebraic. This is why it was necessary to replace $z\partial_z$ by the differental operator $z\partial_z + \sum_i x_i\partial_{x_i}$.

Finally, we define H_B^{eq} to be the $H^*_{T^\vee}(pt)[z^{\pm 1}][q, q^{-1}]$-submodule of $G^{W_q^{\mathrm{eq}}}$ spanned by the classes $[p_\lambda\omega]$ for $\lambda \in \mathcal{P}_{k,n}$.

The following Theorem is the equivariant version of Theorem 4.1. It will be proved in Section 21.

Theorem 5.5. *H_B^{eq} is a D_{eq}-submodule of $G^{W_q^{\mathrm{eq}}}$. Moreover the $H^*_{T^\vee}(pt)[z^{\pm1}, q^{\pm1}]$-module homomorphism defined by*

$$\begin{array}{ccc} H_A^{\mathrm{eq}} & \longrightarrow & H_B^{\mathrm{eq}} \\ \sigma^\lambda_{T^\vee} & \mapsto & [p_\lambda\omega] \end{array}$$

is an isomorphism of D_{eq}-submodules. In particular

$$q\partial_q[p_\lambda\omega] = \frac{1}{z}\left(\sum_\mu [p_\mu\omega] + q\sum_\nu [p_\nu\omega] + x_\lambda[p_\lambda\omega]\right),$$

where μ, ν and x_λ are as in the equivariant quantum Monk's rule (19.2) *for multiplication by $c_1^{T^\vee}(\mathcal{O}(1))$.*

Remark 5.6 (Alternative superpotential)**.** We also introduce an alternative version of the equivariant superpotential. It is denoted by $\widetilde{W}^{\mathrm{eq}}$ and is the same as W^{eq} but with the $\ln(q)$-term removed. For example in the case of $X = Gr_2(\mathbb{C}^5)$,

$$\widetilde{W}^{\mathrm{eq}} = W + (x_2 - x_1)\ln(p_{\ydiagram{3}}) + (x_3 - x_2)\ln(p_{\ydiagram{3,3}}) + (x_4 - x_3)\ln(p_{\ydiagram{2,2}}) + (x_5 - x_4)\ln(p_{\ydiagram{1,1}}).$$

Unlike W^{eq}, this alternative superpotential is regular in q. Note that the Jacobi ring of $(\check{\mathbb{X}}^\circ, \widetilde{W}^{\mathrm{eq}})$ again agrees with the equivariant quantum cohomology ring $qH^*(X)[q^{-1}]$, since $\widetilde{W}^{\mathrm{eq}}$ has the same Jacobi ring as W^{eq}. However the change from W^{eq} to $\widetilde{W}^{\mathrm{eq}}$ affects the Gauss-Manin system. With this change, an analogue to Theorem 5.5 holds in which the equivariant first Chern class $c_1^{T^\vee}(\mathcal{O}(1))$ appearing in (5.1) is replaced by the equivariant fundamental class of the $B_-^\vee$-invariant Schubert divisor $\widetilde{X}^\square$. Note that the Chern class $c_1^{T^\vee}(\mathcal{O}(1))$ in the original version is in a sense not geometric, because $\mathcal{O}(1)$ has no $T^\vee$-invariant global sections; therefore $c_1^{T^\vee}(\mathcal{O}(1))$ is not the fundamental class of a $T^\vee$-invariant divisor. We also remark that the Schubert divisor $\widetilde{X}^\square$ which appears here is 'opposite' to the one defining $\sigma^\square_{T^\vee}$.

Remark 5.7 (Oscillating integrals)**.** In analogy with Section 4.2, this theorem provides solutions to the equivariant small quantum differential equations

$$q\frac{\partial}{\partial q}\mathcal{S} = \frac{1}{z}\,c_1^T(\mathcal{O}(1)) \star_{q,\mathbf{x}} \mathcal{S}, \tag{5.3}$$

which are of the form

$$\mathcal{S}^{\mathrm{eq}}_\Gamma(z, q, \mathbf{x}) := \frac{1}{(2\pi i z)^N}\sum_{\lambda\in\mathcal{P}_{k,n}}\left(\int_\Gamma e^{\frac{1}{z}W^{\mathrm{eq}}} p_\lambda\omega\right)\sigma^{PD(\lambda)}, \tag{5.4}$$

given a suitable simply-connected choice of Γ in $\check{\mathbb{X}}^\circ$ (allowed to vary continuously with z, q and $\mathbf{x}$ in some domain of $\mathbb{C}^{2+n}$) for which the integral converges, as in Remark 4.6. Moreover, $\mathcal{S} = \mathcal{S}_\Gamma^{\mathrm{eq}}$ also satisfies the differential equation

$$\left(z\frac{\partial}{\partial z} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right)\mathcal{S} = -\frac{1}{z}[X_{ac}]_{T^\vee} \star_{q,\mathbf{x}} \mathcal{S} - \mathbf{gr}(\mathcal{S}). \tag{5.5}$$

Note that for the definition of the solution (5.4) we need to make a choice of a branch of $W^{\mathrm{eq}}|_\Gamma$. This affects $\mathcal{S}_\Gamma^{\mathrm{eq}}$ by a factor of the form

$$\exp\left(2\pi i m \frac{x_1 + \ldots + x_{n-k}}{z} + 2\pi i \sum_{j=2}^{n} m_j \frac{x_j - x_{j-1}}{z}\right), \tag{5.6}$$

where $m, m_j \in \mathbb{Z}$. We remark that the factor (5.6) is annihilated by $q\partial_q$ and $z\partial_z + \sum x_j \partial_{x_j}$, hence the choice of branch doesn't affect the validity of (5.3) or (5.5).

Finally, a solution $\mathcal{S}_\Gamma^{\mathrm{eq}}$ of the quantum differential equations gives rise to a solution,

$$\int_\Gamma e^{\frac{1}{z}W^{\mathrm{eq}}}\omega,$$

of the equivariant quantum cohomology D-module; compare Remark 4.6. Together with the comparison result (Proposition 20.3), Theorem 5.5 implies a version of [79, Conjecture 8.2] about integral solutions to the quantum cohomology D-module of a homogeneous space, in the special case of a Grassmannian.

This concludes the summary of results. We now begin by defining in more detail the versions of the superpotential and showing how they are related to one another.

6. The three versions of the superpotential

We have already mentioned the three different versions of a Landau-Ginzburg model dual to the A-model Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$. In this section their superpotentials are defined in detail and we show how they are related to one another. We proceed in reverse chronological order, beginning with the superpotential introduced in Section 3.

6.1. **The Plücker coordinate superpotential.** The Plücker coordinate formulation gives a very simple-looking expression for the superpotential, therefore it is a natural starting point. We begin by describing its domain. Recall that the A-model Grassmannian, $X = Gr_{n-k}(\mathbb{C}^n)$, is a homogeneous space for $GL_n(\mathbb{C})$ acting from the left. We think of it in the usual way as a Grassmannian of codimension k subspaces in the n-dimensional vector space of column vectors $\mathbb{C}^n$. In this section we define a Landau-Ginzburg model taking place on a B-model Grassmannian. The B-model Grassmannian is a Grassmannian of row vectors, $\check{\mathbb{X}} := Gr_k((\mathbb{C}^n)^*)$, and we view it as homogeneous space for the Langlands dual GL_n under the (right) action of multiplication from the right. Note that X and $\check{\mathbb{X}}$ are isomorphic, but this is a type A coincidence; compare for example [68, 66]. In order to distinguish between the two general linear groups acting on X and $\check{\mathbb{X}}$ we will refer to the group on the A-model side as $GL_n^\vee(\mathbb{C})$ and add a check $^\vee$ to notations pertaining to this group.

Elements of $\check{\mathbb{X}}$ may be represented by maximal rank $(k \times n)$-matrices M in the usual way, with M representing its row-span. We think of $\check{\mathbb{X}}$ as embedded in $\mathrm{Proj}(\bigwedge^k(\mathbb{C}^n)^*)$ by its Plücker embedding and denote its homogeneous coordinate ring by $\mathbb{C}[\check{\mathbb{X}}]$. The Plücker coordinates are all the maximal minors of M, and are determined by a choice of k columns. We index the Plücker coordinates by partitions $\lambda \in \mathcal{P}_{k,n}$ as follows. Associate to any partition $\lambda \in \mathcal{P}_{k,n}$ a k-subset in $[1, n] := \{1, \ldots, n\}$ by interpreting λ as a path from the top right hand corner of its bounding $(n-k) \times k$ rectangle down to the bottom left hand corner. Such a path necessarily consists of k horizontal and $n-k$ vertical steps. The positions of the horizontal steps (numbered from the start of the path to the end) define a subset of k elements in $[1, n]$. We denote this subset, associated to λ, by J_λ. See Figure 1 for an example.

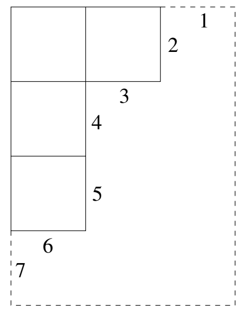


FIGURE 1. The k-subset corresponding to a partition. For $\ydiagram{2,1,1} \in \mathcal{P}_{3,7}$, we have $J_{\ydiagram{2,1,1}} = \{1,3,6\}$.

Suppose that $J_\lambda = \{j_1, \dots, j_k\}$ with $1 \le j_1 < \dots < j_k \le n$. Then the Plücker coordinate associated to λ is defined to be the determinant of a $k \times k$ submatrix of M,

$$p_\lambda(M) = \det\left((M_{i,j_l})_{i,l}\right).$$

We will also sometimes denote the Plücker coordinate corresponding to the k-subset J by p_J.

A special role will be played by the n Plücker coordinates corresponding to the k-subsets which are (cyclic) intervals. These are $J_i := [i+1, i+k]$, where $i \in [1,n]$. Sometimes it will be useful to index such an interval by its last element, in which case we use the notation L_{i+k} for J_i. So

$$J_i = L_{i+k} = [i+1, i+k]. \tag{6.1}$$

Our convention regarding indices is that elements in such k-subsets are interpreted modulo n, and also the subscripts of J_i and L_i. The partition corresponding to J_i is denoted by μ_i. For example since $J_1 = [2, k+1]$ we have $\mu_1 = (k)$, the maximal partition with one part. If $n - k \ge 2$, then μ_2 is the maximal two row partition, (k,k). For $k = 3$ and $n = 7$ we have for example,

$$\mu_1 = \ydiagram{3}\,,\quad \mu_2 = \ydiagram{3,3}\,,\quad \mu_3 = \ydiagram{3,3,3}\,,\quad \mu_4 = \ydiagram{3,3,3,3}\,,\quad \mu_5 = \ydiagram{2,2,2,2}\,,\quad \mu_6 = \ydiagram{1,1,1,1}\,,\quad \mu_7 = \emptyset\,. \tag{6.2}$$

Always μ_{n-k} is the maximal rectangle, and μ_n is the empty partition.

The set $\{p_{\mu_1}, \dots, p_{\mu_n}\}$ of special Plücker coordinates is invariant under the $\mathbb{Z}/n\mathbb{Z}$ action on $\check{\mathbb{X}}$ defined by cyclic shift,

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1,n} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2,n} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{k1} & m_{k2} & m_{k3} & \dots & m_{k,n} \end{pmatrix} \mapsto M[1] = \begin{pmatrix} m_{12} & m_{13} & \dots & m_{1,n} & (-1)^{k-1} m_{11} \\ m_{22} & m_{23} & \dots & m_{2,n} & (-1)^{k-1} m_{21} \\ \vdots & \vdots & & \vdots & \vdots \\ m_{k2} & m_{k3} & \dots & m_{k,n} & (-1)^{k-1} m_{k1} \end{pmatrix}.$$

Indeed, we have $p_J(M[1]) = p_{J+1}(M)$ for any k-subset J of $[1,n]$ (where $J+1$ is obtained by adding 1 to each element of J, modulo n). Therefore also $p_{\mu_i}(M[1]) = p_{\mu_{i+1}}([M])$ as $J_{\mu_i} = J_i$ and $J_{i+1} = J_i + 1$.

Each p_{μ_i} is a section of $\mathcal{O}(1)$ for the Plücker embedding, and the union of the hyperplane sections defined by the p_{μ_i} is an anticanonical divisor

$$D = \{p_{\mu_1} = 0\} \cup \{p_{\mu_2} = 0\} \cup \dots \cup \{p_{\mu_n} = 0\}. \tag{6.3}$$

Indeed, the Plücker embedding of the Grassmannian $\check{\mathbb{X}}$ is minimal, and $\check{\mathbb{X}}$ is Fano of index n, i.e. n is the maximal integer for which the anti-canonical class is divisible by n.

Let $\check{\mathbb{X}}^\circ$ be the Zariski-open subset of $\check{\mathbb{X}}$ obtained by removing the anti-canonical divisor (6.3). So

$$\check{\mathbb{X}}^\circ := \check{\mathbb{X}} \setminus D = \left\{[M] \in \check{\mathbb{X}} \mid p_{\mu_i}(M) \neq 0, \text{ all } i = 1, \dots, n \right\}.$$

Note that the anticanonical divisor D and its complement are invariant under the $\mathbb{Z}/n\mathbb{Z}$-action on $\check{\mathbb{X}}$ defined by $M \mapsto M[1]$, by the discussion above.

The coordinate ring of $\check{\mathbb{X}}^\circ$ is denoted by $\mathbb{C}[\check{\mathbb{X}}^\circ]$. Recall that, to pass from the homogeneous coordinates p_λ on $\check{\mathbb{X}}$ to regular functions on $\mathbb{C}[\check{\mathbb{X}}^\circ]$ we make the convention of setting $p_\emptyset = 1$. Both $\mathbb{C}[\check{\mathbb{X}}^\circ]$ and $\mathbb{C}[\check{\mathbb{X}}]$ are cluster algebras; see Section 7.

Our version of the Landau-Ginzburg model mirror dual to X is a regular function $W : \check{\mathbb{X}}^\circ \times \mathbb{C}_q \to \mathbb{C}$, which we may call the *canonical superpotential*, in analogy with [68, Section 1.1]. It is defined as follows.

Definition 6.1 (The superpotential on $\check{\mathbb{X}}^\circ$)**.** Denote by $\widehat{\mu}_i$ the partition corresponding to

$$\widehat{J}_i = \widehat{L}_{i+k} := [i+1, i+k-1] \cup \{i+k+1\},$$

where $i+k$ has been removed from J_i and replaced by $i+k+1$. Unless $i = n-k$, the Young diagram of $\widehat{\mu}_i$, is obtained from the Young diagram of μ_i by adding a box. The particular shape of μ_i guarantees that there is only one way to do this. The partition $\widehat{\mu}_{n-k}$ is obtained by removing the entire rim from μ_{n-k} to give an $(n-k-1) \times (k-1)$ rectangle. We define

$$W := \sum_{i=1}^{n} \frac{p_{\widehat{\mu}_i}}{p_{\mu_i}} q^{\delta_{i,n-k}} = \left(\sum_{i \neq n-k} \frac{p_{\widehat{\mu}_i}}{p_{\mu_i}} \right) + q \frac{p_{\widehat{\mu}_{n-k}}}{p_{\mu_{n-k}}}. \tag{6.4}$$

Remark 6.2. Notice that in the quantum Schubert calculus of X and for $i \neq n-k$,

$$\sigma^{\square} \star_q \sigma^{\mu_i} = \sigma^{\widehat{\mu_i}}, \qquad \sigma^{\square} \star_q \sigma^{\mu_{n-k}} = q\, \sigma^{\widehat{\mu_{n-k}}},$$

and thus each of the individual summands of W in the above formula formally resembles the hyperplane class $\sigma^{\square}$.

We will next recall the definitions of the two previously conjectured Landau-Ginzburg models for Grassmannians, starting with the LG model of [79] followed by that of Eguchi, Hori and Xiong [21, 6], and explain how these previous definitions relate to this new one.

6.2. **The Lie-theoretic superpotential.** Let the (B-model) group $G = GL_n(\mathbb{C})$ act (now from the left) on a full flag variety. We fix some notation regarding this group. We let B_+, B_- denote its upper-triangular and lower-triangular Borel subgroups, respectively, and T denote the maximal torus of diagonal matrices. The unipotent radicals of B_+ and B_- are denoted by U_+ and U_-. Let $\mathfrak{h}$ denote the Lie algebra of T. We let $E_{i,j}$ denote the matrix with entry 1 in row i and column j and zeros elsewhere. Let $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ be the usual Chevalley elements of $\mathfrak{gl}_n$, and let $\alpha_i \in \mathfrak{h}^*$ be the simple root corresponding to e_i. We define the associated 1-parameter subgroups $x_i : \mathbb{C} \to U_+$ and $y_i : \mathbb{C} \to U_-$,

$$x_i(t) = \exp(te_i), \quad y_i(t) = \exp(tf_i)$$

for $i = 1, \ldots, n-1$. Let $W = N_{GL_n}(T)/T \cong S_n$ be the Weyl group. Namely for every $w \in W$ we have a natural choice of representative $\bar{w}$ which is the permutation matrix in GL_n corresponding to the permutation w. Alternatively we can choose the following representatives. Let

$$\dot{s}_i = x_i(1) y_i(-1) x_i(1) \in N_{GL_n}(T)$$

represent the generator $s_i = \dot{s}_i T$ of W which is the simple transposition $(i, i+1)$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function. If $\ell(w) = m$ and $s_{i_1} \ldots s_{i_m}$ is a reduced expression for w, then the product $\dot{w} = \dot{s}_{i_1} \ldots \dot{s}_{i_m}$ is a well defined element of GL_n and independent of the reduced expression chosen. One advantage of the latter choice of representatives is that its definition is not specific to GL_n. We will also require the root subgroups

$$x_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_k}(a) := \bar{s}_{j+1}\bar{s}_{j+2} \cdots \bar{s}_{k-1} x_k(a) \bar{s}_{k-1}^{-1} \ldots \bar{s}_{j+2}^{-1} \bar{s}_{j+1}^{-1} = \mathbf{1}_n + aE_{j,k+1},$$

where $j < k$ and $\mathbf{1}_n$ is the $n \times n$ identity matrix.

Let $P \supset B_+$ be the maximal parabolic subgroup in G generated by B_+ and the elements $\dot{s}_i$ for $i \neq n-k$. We also have $W_P = \langle s_i \mid i \neq n-k \rangle$, the corresponding parabolic subgroup of the Weyl group W. The longest element in W_P is denoted by w_P, while the longest element of W is denoted by w_0. . Let W^P denote the set of minimal length coset representatives in W/W_P. The longest element in W^P is denoted by w^P. Clearly $w^P w_P = w_0$, the longest element of W.

Consider the Schubert variety inside GL_n/B_- defined by,

$$\check{\mathbb{X}}_{w_P} = \overline{B_+\dot{w}_P B_-/B_-}.$$

We think of $\check{\mathbb{X}}_{w_P} \subset GL_n/B_-$ as a Richardson variety, namely as closure of the intersection,

$$\mathcal{R} = \mathcal{R}_{w_P,w_0} = B_+\dot{w}_P B_- \cap B_-\dot{w}_0 B_-/B_-,$$

of opposite Bruhat cells. This intersection, $\mathcal{R}$, is a smooth irreducible variety of dimension $\ell(w^P)$, which equals to $k(n-k)$ for our choice of P.

The construction of a mirror LG model in [79], applied in the Grassmannian case, yields a superpotential $\mathcal{F}$ on $\mathcal{R}\times\mathbb{C}_q^*$. Here the definition of $\mathcal{F}$ involves first identifying $\mathcal{R}\times\mathbb{C}_q^*$ with a subset of the group $PSL_n(\mathbb{C})$, Langlands dual to the group $SL_n(\mathbb{C})$ acting on the A-model X, see [79, Section 4.1]. Since we are considering our Grassmannians X and $\check{\mathbb{X}}$ as a homogeneous spaces for (Langlands dual) general linear groups, we will replace this subset of $PSL_n(\mathbb{C})$ with a natural choice of lift to the B-model $GL_n(\mathbb{C})$. This will not change the function on $\mathcal{R}\times\mathbb{C}_q^*$, but will make a difference (as it should) when we extend to the $T^\vee$-equivariant case in Section 20.

One further practical difference between PSL_n and GL_n is that in [79] the torus in PSL_n analogous to T^{W_P} is identified with $\mathbb{C}_q^*$, while for GL_n the torus T^{W_P} isn't one-dimensional, but rather is two-dimensional. In order to cut down the dimension let $\widetilde{T}^{W_P}$ be the one-dimensional subtorus of T^{W_P} in $GL_n(\mathbb{C})$ defined by

$$\widetilde{T}^{W_P} := \left\{ \left. \begin{pmatrix} d &&&&& \\ & \ddots &&&& \\ && d &&& \\ &&& 1 && \\ &&&& \ddots & \\ &&&&& 1 \end{pmatrix} \right| d\in\mathbb{C}^* \right\}.$$

Then $\widetilde{T}^{W_P}\cong\mathbb{C}_q^*$ via the identification of α_{n-k} and q. We have an isomorphism

$$\begin{array}{rccc} (6.5) \qquad & \psi_R : B_-\cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+ & \xrightarrow{\sim} & \mathcal{R}\times\mathbb{C}_q^*, \\ & b = u_1 t\dot{w}_P\dot{w}_0^{-1}u_2 & \mapsto & (b\dot{w}_0 B_-, \alpha_{n-k}(t)), \end{array}$$

as in [79, Section 4.1]. To define the superpotential we need the map $e_i^* : U_+\to\mathbb{C}$ which sends $u\in U_+$ to its $(i,i+1)$-entry, so $e_i^*(u) := u_{i,i+1}$. This notation $e_i^*(u)$ stems from the fact that the matrix entry $u_{i,i+1}$ of u can also be thought of as the coefficient of e_i in u after embedding U_+ into the completed universal enveloping algebra of its Lie algebra.

The following definition is an equivalent formulation of the definition from [79, Section 4.2] as follows from [79, Lemma 5.2]. Note that the setting in [79] is that of an arbitrary complex reductive algebraic group G and parabolic subgroup P, and we apply it here to $G = GL_n$ and P a maximal parabolic.

Definition 6.3 (The Lie-theoretic superpotential [79, Lemma 5.2])**.** Let $\widetilde{\mathcal{F}} : B_-\cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+\to\mathbb{C}$ be the map defined by

$$\widetilde{\mathcal{F}} : b = u_1 t\dot{w}_P\dot{w}_0^{-1}u_2 \quad\mapsto\quad \sum_{i=1}^n e_i^*(u_1) + \sum_{i=1}^n e_i^*(u_2).$$

Note that this map $\widetilde{\mathcal{F}}$ is well-defined even though u_1 and u_2 are not uniquely determined by b, see [79, Equation (4.4) and Lemma 5.2]. For the A-model Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ viewed as a homogeneous space $GL_n^\vee/P^\vee$, the Lie-theoretic version of the superpotential is the composition

$$\mathcal{F} = \widetilde{\mathcal{F}}\circ\psi_R^{-1} : \mathcal{R}\times\mathbb{C}_q^*\to\mathbb{C}.$$

Explicitly,

$$\mathcal{F}(b\dot{w}_0 B_-, q) = \sum e_i^*(u_1) + \sum e_i^*(u_2),$$

if b is in B_- and factorizes as $b = u_1 t \dot{w}_P \dot{w}_0^{-1} u_2$ with $u_1, u_2 \in U_+$ where $t \in \widetilde{T}^{W_P}$ is determined by $\alpha_{n-k}(t) = q$. Additionally, we use the notation $\mathcal{F}_q$ for the map $\mathcal{F}_q : \mathcal{R} \to \mathbb{C}$ defined by $\mathcal{F}_q(b\dot{w}_0 B_-) = \mathcal{F}(b\dot{w}_0 B_-, q)$.

We now recall the Dale Peterson presentation of the quantum cohomology ring of a homogeneous space which applies as a special case to $X = Gr_{n-k}(n)$, compare [76].

Theorem 6.4 (Dale Peterson [69])**.** *Associated to the homogeneous space X define a subvariety Y_P^* in $\mathcal{R}$, called the Peterson variety of X, as follows. Let $F \in \mathfrak{g}^*$ be the sum of the dualised positive Chevalley generators,*

$$F = e_1^* + e_2^* + \ldots + e_{n-1}^*,$$

and set

$$Y_P^* := \{gB_- \in \mathcal{R} \mid g^{-1} \cdot F \in [\mathfrak{n}_-, \mathfrak{n}_-]^\perp\},$$

using the coadjoint action of G. Let $\mathbb{C}[Y_P^*]$ denote the coordinate ring of Y_P^*, in the possibly non-reduced sense. Then $\mathbb{C}[Y_P^*]$ is isomorphic to the quantum cohomology ring $qH^*(X, \mathbb{C})[q^{-1}]$ of X by an explicit isomorphism.

We call the isomorphism from Theorem 6.4 the *Peterson isomorphism*. In Proposition 9.1 we will recall where Peterson's isomorphism takes a Schubert class in the Grassmannian case. For a description of the Peterson isomorphism for type A partial flag varieties we refer to [76, 78], in general type see also [79]. We remark that in type A the quantum cohomology rings, and with them the coordinate rings $\mathbb{C}[Y_P^*]$, are always reduced.

The superpotential $\mathcal{F}$ defined in [79] is related to the Peterson variety as follows. Denote again by q the element of the coordinate ring $\mathbb{C}[Y_P^*]$ which corresponds under the Peterson isomorphism to the quantum parameter. Then q is a finite morphism from Y_P^* to $\mathbb{C}^*$.

Theorem 6.5 (Mirror construction of the Peterson variety [79, Theorem 4.1])**.** *The critical points of $\mathcal{F}_q$ inside $\mathcal{R}$ lie in the Peterson variety and precisely recover the fibers of $q : Y_P^* \to \mathbb{C}^*$. Moreover the subvariety of $\mathcal{R} \times \mathbb{C}_q^*$ corresponding to the ideal $(\partial_{\mathcal{R}}\mathcal{F})$ of partial derivatives of $\mathcal{F}$ along $\mathcal{R}$ is isomorphic to Y_P^* by the restriction of the first projection $\mathcal{R} \times \mathbb{C}_q^* \to \mathcal{R}$, and we obtain*

$$\mathbb{C}[\mathcal{R} \times \mathbb{C}_q^*]/(\partial_{\mathcal{R}}\mathcal{F}) \cong \mathbb{C}[Y_P^*].$$

The Lie-theoretic superpotential $(\mathcal{R}, \mathcal{F}_q)$ is therefore related to the quantum cohomology of X by the combination of Theorems 6.4 and 6.5.

In order to set up the comparison of the Lie-theoretic superpotential with our new formulation in terms of Plücker coordinates we define the following maps

$$\check{\mathbb{X}}^\circ \xleftarrow{\pi_L} B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+ \xrightarrow{\pi_R} \mathcal{R}.$$

Here the the map on the left hand side is defined by setting $\pi_L(b) := Pb$, and the map on the right hand side is $\pi_R(b) = b\dot{w}_0 B_-$, where $b \in B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+$. It is straightforward that π_R is well-defined and an isomorphism. The map π_L is a priori a map to $\check{\mathbb{X}}$, but it is known to land in $\check{\mathbb{X}}^\circ$ and moreover π_L is an isomorphism so that we have $\check{\mathbb{X}}^\circ \cong \mathcal{R}$. Namely the following proposition follows from [48, Section 5.4], via a construction from [57, Section 2.1].

Proposition 6.6. *The projection map π_L is a well-defined isomorphism from $B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+$ to $\check{\mathbb{X}}^\circ$.*

In Section 6.4 we will prove the following proposition.

Proposition 6.7. *With the definitions from Section 6.1 and 6.2, the following diagram commutes:*

$$\begin{tikzcd}
\check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \arrow[d, "W"'] & B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+ \times \mathbb{C}_q^* \arrow[l, "\pi_L \times id"'] \arrow[r, "\pi_R \times id"] & \mathcal{R} \times \mathbb{C}_q^* \arrow[d, "\mathcal{F}"'] \\
\mathbb{C} & = & \mathbb{C}.
\end{tikzcd}$$

Propositions 6.6 and 6.7 imply the part of Theorem 4.10 which states that the Landau-Ginzburg model from Section 6.1 is isomorphic to the one from [79] recalled in Definition 6.3. In the case of Lagrangian Grassmannians and for odd-dimensional quadrics, formulas analogous to Definition 6.3 and comparison results analogous to the above Propositions were found by C. Pech and K. Rietsch in [66, 67] and by C. Pech, K. Rietsch and L. Williams in the case of even quadrics [68].

6.3. **The Laurent polynomial superpotential.** The earliest construction of Landau-Ginzburg models for Grassmannians is due to Eguchi, Hori and Xiong [21] and associates to $X = Gr_{n-k}(\mathbb{C}^n)$ a Laurent polynomial L_q in $k(n-k)$ variables (with parameter q). This Laurent polynomial also appeared in [63] in relation to a parabolic analog of the quantum Toda lattice. Let $\mathcal{T} = (\mathbb{C}^*)^{k(n-k)}$ and define $L_q : \mathcal{T} \to \mathbb{C}$ as follows (compare [6, 8]).

Let $\mathcal{Q}_R = (\mathcal{V}, \mathcal{A})$ be a quiver with vertices given by

$$\mathcal{V} = \{(i,j) \in [1, n-k] \times [1,k]\} \sqcup \{(0,1), (n-k, k+1)\}$$

and with two types of arrows $a \in \mathcal{A}$, namely

$$(i,j) \longrightarrow (i, j+1) \quad \text{and} \quad (i,j) \longrightarrow (i+1, j),$$

defined whenever $(i,j), (i,j+1)$, and $(i,j), (i+1,j)$, respectively, are in $\mathcal{V}$. We write $h(a)$ for the head of an arrow a, and $t(a)$ for the tail. To every vertex in the quiver associate a coordinate z_{ij}. We set $z_{0,1} = 1$ and $z_{n-k,k+1} = q$, and let the remaining $(z_{ij})_{i=1,\dots,n-k}^{j=1,\dots,k}$ be the coordinates on the big torus $\mathcal{T} = (\mathbb{C}^*)^{k(n-k)}$.

Definition 6.8 (The EHX Laurent polynomial supotential [6, 21])**.** To every arrow a in the quiver $(\mathcal{V}, \mathcal{A})$ one can associate a Laurent monomial by dividing the coordinate at the head by the coordinate at the tail. The regular function $L_q : \mathcal{T} \to \mathbb{C}$ is defined to be the sum of all of the Laurent monomials obtained in this way,

$$L_q = \sum_{a \in \mathcal{A}} \frac{z_{h(a)}}{z_{t(a)}},$$

keeping in mind that $z_{0,1} = 1$ and $z_{n-k,k+1} = q$ also occur.

Example 6.9. Consider $k = 3$ and $n = 5$. So $X = Gr_2(\mathbb{C}^5)$ and $\check{\mathbb{X}} = Gr_3((\mathbb{C}^5)^*)$, the Grassmannian of 3-planes in the vector space of row vectors. The big torus is $\mathcal{T} \cong (\mathbb{C}^*)^6$ with coordinates $(z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23})$. The superpotential is

$$L_q = z_{11} + \frac{z_{21}}{z_{11}} + \frac{z_{22}}{z_{12}} + \frac{z_{23}}{z_{13}} + \frac{z_{12}}{z_{11}} + \frac{z_{13}}{z_{12}} + \frac{z_{22}}{z_{21}} + \frac{z_{23}}{z_{22}} + \frac{q}{z_{23}}, \tag{6.6}$$

and is encoded in the quiver $\mathcal{Q}_R$ shown below.

$$\begin{tikzcd}
1 \arrow[d] & & & \\
z_{11} \arrow[r] \arrow[d] & z_{12} \arrow[r] \arrow[d] & z_{13} \arrow[d] & \\
z_{21} \arrow[r] & z_{22} \arrow[r] & z_{23} \arrow[r] & q
\end{tikzcd}.$$

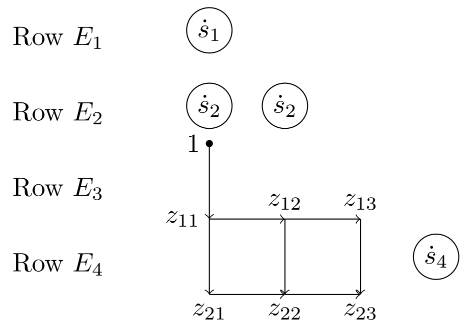
The relationship between this superpotential L_q and the superpotential W_q from Definition 6.1 is given in the proposition below. The Plücker coordinates indexed by rectangular Young diagrams play a special role here, and we denote a Young diagram which is an $i \times j$ rectangle by $i \times j$. For example if $(k,n) = (3,7)$ the Plücker coordinates corresponding to the rectangles μ_i are $p_{\mu_1} = p_{1\times 3}$, $p_{\mu_2} = p_{2\times 3}$ and so forth; compare (6.2). We have $k(n-k)$ rectangular Plücker coordinates, not counting $p_\emptyset$.

Proposition 6.10. *There is a (unique) embedding $\iota : \mathcal{T} \to \check{\mathbb{X}}^\circ$ for which the Plücker coordinates corresponding to rectangular Young diagrams are related to the z_{ij} coordinates as follows,*

$$z_{ij} = \iota^* \left(\frac{p_{i\times j}}{p_{(i-1)\times(j-1)}} \right). \tag{6.7}$$

Moreover, the Laurent polynomial superpotential L_q agrees with the pullback of W_q to $\mathcal{T}$ under ι.

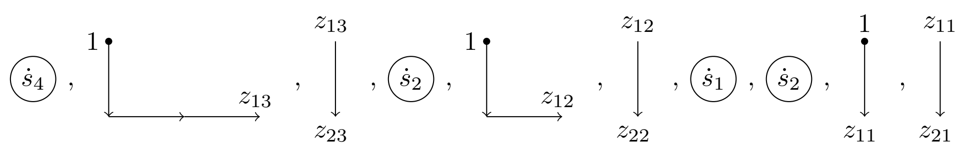
Sketch of proof. We describe the embedding $\iota : \mathcal{T} \to \check{\mathbb{X}}^\circ$ defined in Proposition 6.10 concretely. To explain the construction in our setting we continue with Example 6.9. Let us decorate the above quiver by elements $\dot{s}_i$ as follows and remove the arrow with head labeled q .



The new figure has $n-1$ rows, $E_1, E_2, \ldots, E_{n-1}$ (where, in the example, $n = 5$). For $1 \le i \le k-1$, row E_i contains i copies of $\dot{s}_i$ (written a circle). For $k \le i \le n-1$, row E_i contains all of the downward-pointing arrows with target $z_{i-k+1,j}$ for some j (one arrow if $i = k$; k arrows if $i > k$), followed by $i - k$ copies of $\dot{s}_i$.

We call a path in the quiver which has precisely one vertical step a 1-*path*. Notice that each 1-path contains a downward-pointing arrow from exactly one row. For any vertex v decorated with a z_{ij} there is clearly a unique minimal length 1-path which has this vertex at its lower end. We call this 1-path the *minimal* 1-path with the given vertex v at its base.

Then we read off a sequence of $\dot{s}_i$'s and 1-paths, going column by column from right to left. Namely in each column we list, starting from the top and going down, any $\dot{s}_i$'s, followed by any minimal 1-paths associated to vertices from that column. In the example above the sequence is shown below.



To a 1-path γ in row E_i we associate a factor $x_i\left(\frac{z_{h(\gamma)}}{z_{t(\gamma)}}\right)$, where $t(\gamma)$ is the initial vertex and $h(\gamma)$ is the final vertex of γ. The other elements of the sequence correspond in the obvious way to factors $\dot{s}_i$. These factors are all multiplied together to give an element $g_{(z_{ij})}$ in the big Bruhat double coset $B_-\dot{w}_0B_-$ of GL_n. In the above example we have the element of GL_5 given by

$$g_{(z_{ij})} := \dot{s}_4 x_3\left(z_{13}\right) x_4\left(\frac{z_{23}}{z_{13}}\right) \dot{s}_2 x_3\left(z_{12}\right) x_4\left(\frac{z_{22}}{z_{12}}\right) \dot{s}_1 \dot{s}_2 x_3\left(z_{11}\right) x_4\left(\frac{z_{21}}{z_{11}}\right).$$

Note that one can check that $\dot{w}_0^{-1} g_{(z_{ij})} \in \dot{w}_P \dot{w}_0^{-1} U_+$ by permuting all of the $\dot{s}_i$ factors in $g_{(z_{ij})}$ to the left. Since $g_{(z_{ij})}$ is also in the big Bruhat cell we have $\dot{w}_0^{-1} g_{(z_{ij})} \in \dot{w}_P \dot{w}_0^{-1} U_+ \cap B_+B_-$.

We now define the map $\iota : \mathcal{T} \to \check{\mathbb{X}}^\circ$ by

$$\iota : (z_{ij}) \mapsto P\,\dot{w}_0^{-1}\, g_{(z_{ij})}.$$

To see that this map is the embedding alluded to in the proposition it suffices to consider the $p_{i\times j}$ Plücker coordinates of $P\,\dot{w}_0^{-1}\, g_{(z_{ij})}$ and check that these are related to the z_{ij} as follows,

$$z_{11} = \frac{p_{\square}}{p_{\emptyset}}\ ,\ z_{12} = \frac{p_{\square\square}}{p_{\emptyset}}\ ,\ z_{13} = \frac{p_{\square\square\square}}{p_{\emptyset}}\ , z_{21} = \frac{p_{\substack{\square\\ \square}}}{p_{\emptyset}}\ ,\ z_{22} = \frac{p_{\substack{\square\square\\ \square\square}}}{p_{\square}}\ ,\ z_{23} = \frac{p_{\substack{\square\square\square\\ \square\square\square}}}{p_{\square\square}}. \tag{6.8}$$

It is easy to check that this holds in general, so that we have (6.7). Finally, it is straightforward to compute the $p_{\widehat{\mu}_i}$ Plücker coordinates of $P\,\dot{w}_0^{-1}\, g_{(z_{ij})}$ and see that substituting $P\,\dot{w}_0^{-1}\, g_{(z_{ij})}$ into the formula (6.4) for W recovers the Laurent polynomial L_q. □

Remark 6.11. The construction of the map ι is inspired by the construction of factorisations of elements in the Peterson variety introduced in [77]. In fact the two constructions are essentially related by a reflection of the quiver, see the proof of Proposition 8.6 where both factorisations are needed.

6.4. **Proof of Proposition 6.7.** In this section we prove Proposition 6.7, which says that the superpotential W defined in (6.4) is isomorphic to the Lie-theoretic superpotential $\mathcal{F}$ from [79], see Definition 6.3.

Let $\widetilde{W}$ denote the pullback of $\mathcal{F}$ to $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$ via $(\pi_L \times id)^{-1} \circ (\pi_R \times id)$;

$$\begin{tikzcd}
\check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \arrow[drr, "\widetilde{W}"'] & (B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+) \times \mathbb{C}_q^* \arrow[l, "\pi_L \times id"'] \arrow[r, "\pi_R \times id"] & \mathcal{R} \times \mathbb{C}_q^* \arrow[d, "\mathcal{F}"] \\
 & & \mathbb{C}
\end{tikzcd} \tag{6.9}$$

compare Section 6.2. Here we are keeping in mind Proposition 6.6 which says that π_L is an isomorphism. Then to show Proposition 6.7 we need to prove that $\widetilde{W}$ agrees with W.

Let us assume that $Pg \in \check{\mathbb{X}}^\circ$ is of the form

$$Pg = P\dot{w}_0^{-1} g_{(z_{ij})} \tag{6.10}$$

for an element $(z_{ij}) \in \mathcal{T}$; compare Section 6.3. The subset $\check{\mathbb{X}}^{\circ,fact}$ of $\check{\mathbb{X}}^\circ$ consisting of such factorisable elements Pg is an open dense subset of $\check{\mathbb{X}}^\circ$. Therefore it suffices to show that $\widetilde{W}$ and W agree on $\check{\mathbb{X}}^{\circ,fact}$.

Definition 6.12. Note that any element $\dot{w}_0^{-1} g_{(z_{ij})}$ can also be written in the form $\dot{w}_P \dot{w}_0^{-1} u_2$ for an element $u_2 \in U_+$ which can be factorized as $u(1) \cdots u(k)$, where each $u(j)$ is a product of root subgroups,

$$u(j) = x_{\alpha_j + \cdots + \alpha_k}(m_{jk}) x_{k+1}(m_{j,k+1}) \cdots x_{n-1}(m_{j,n-1}).$$

Let $U_+^{\mathrm{fact}} \subset U_+$ denote the subset of U_+ consisting of such factorized elements $u_2 = u(1) \cdots u(k)$, with nonzero entries $m_{j,l}$ in all of the root subgroup factors.

We have $\check{\mathbb{X}}^{\circ,fact} = P\dot{w}_P \dot{w}_0^{-1} U_+^{\mathrm{fact}}$. Therefore we may rewrite Pg from (6.10) as

$$Pg = P\dot{w}_P \dot{w}_0^{-1} u_2,$$

where $u_2 \in U_+^{\mathrm{fact}}$. We can now define a map $\tilde{\mu} : U_+^{\mathrm{fact}} \to U_+$ by requiring

$$\tilde{\mu}(u_2)^{-1} B_- = \dot{w}_P \dot{w}_0^{-1} u_2 B_-$$

for all $u_2 \in U_+^{fact}$. Notice that $\dot{w}_P \dot{w}_0^{-1} u_2 B_-$ lies in $U_+ B_-/B_-$ since it equals $\dot{w}_0^{-1} g_{(z_{ij})} B_- \in \dot{w}_0^{-1} B_- \dot{w}_0 B_- = U_+ B_-$, and therefore $\tilde{\mu}$ is well-defined. If the context is clear we will write $\tilde{\mu}(u_2) := u_{1,0}$, noting that then $u_{1,0}$ is the unique element in U^+ for which

$$u_{1,0} \dot{w}_P \dot{w}_0^{-1} u_2 \in B_-.$$

Lemma 6.13. *Let $u_2 \in U_+^{\mathrm{fact}}$. We have the following formula for $\widetilde{W}$ on $\check{\mathbb{X}}^{\circ,fact} \times \mathbb{C}_q^*$,*

$$\widetilde{W}(P\dot{w}_P \dot{w}_0^{-1} u_2, q) = \sum_i e_i^*(u_2) + \sum_{i \neq n-k} e_i^*(u_{1,0}) + q e_{n-k}^*(u_{1,0}),$$

where $u_{1,0} = \tilde{\mu}(u_2)$.

Proof. This lemma is straightforward. Let $t_q \in \widetilde{T}^{W_P}$ be the unique element with $\alpha_{n-k}(t_q) = q$. Recall the isomorphism ψ_R from (6.5). We define an analogous isomorphism ψ_L by the following commutative diagram

$$\begin{tikzcd}
\check{\mathbb{X}}^\circ \times \mathbb{C}_q^* & B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+ \times \mathbb{C}_q^* \arrow[l, "\pi_L \times id"'] \arrow[r, "\pi_R \times id"] \arrow[d, "\psi"] & \mathcal{R} \times \mathbb{C}_q^* \\
 & B_- \cap U_+ \widetilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+ \arrow[ul, "\psi_L"] \arrow[ur, "\psi_R"'] &
\end{tikzcd}$$

in which every arrow is an isomorphism; compare Proposition 6.6 along with the paragraph preceding it. The connecting isomorphism ψ in the middle is just the map

$$\begin{array}{rcl}\psi : B_- \cap U_+\dot{w}_P\dot{w}_0^{-1}U_+ \times \mathbb{C}_q^* & \to & B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+ \\ (b_0, q) & \mapsto & b := t_q b_0.\end{array}$$

We can express $\widetilde{W}$ as a composition

$$\begin{array}{rcccccc}\widetilde{W}: & \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* & \xrightarrow{\psi_L^{-1}} & B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_0U_+ & \xrightarrow{\widetilde{\mathcal{F}}} & \mathbb{C} \\ & (P\dot{w}_P\dot{w}_0^{-1}u_2, q) & \mapsto & b = t_q u_{1,0}\dot{w}_P\dot{w}_0^{-1}u_2 & \mapsto & \widetilde{\mathcal{F}}(b) = \mathcal{F}(b\dot{w}_0B_-, q),\end{array}$$

where $\widetilde{\mathcal{F}}$ is as in Definition 6.3. Clearly, since $t_q u_{1,0}\dot{w}_P\dot{w}_0^{-1}u_2 = u_1 t_q\dot{w}_P\dot{w}_0^{-1}u_2$ for $u_1 = t_q u_{1,0}t_q^{-1}$, we have

$$\widetilde{\mathcal{F}}(t_q u_{1,0}\dot{w}_P\dot{w}_0^{-1}u_2) = \sum_i e_i^*(u_2) + \sum_i e_i^*(u_1) = \sum_i e_i^*(u_2) + \sum_{i\neq n-k} e_i^*(u_{1,0}) + qe_{n-k}^*(u_{1,0}),$$

as required. □

Let $\Delta_J^I(g)$ denote the minor with row set I and column set J, and recall that $J_i = [i+1, i+k]$, in interval notation, and $\widehat{J}_i = [i+1, i+k-1] \cup \{i+k+1\}$. We then have the following lemma about minors.

Lemma 6.14. *Let $u_2 \in U_+^{\mathrm{fact}}$ and $u_{1,0} = \tilde{\mu}(u_2)$ and $b = u_{1,0}\dot{w}_P\dot{w}_0^{-1}u_2 \in B_-$. Then we have*

$$e_i^*(u_{1,0}) = \begin{cases} 0 & 1 \leq i \leq n-k-1; \\ \dfrac{\Delta_{\widehat{J}_i}^{[n-k+1,n]}(b)}{\Delta_{J_i}^{[n-k+1,n]}(b)} & n-k \leq i \leq n-1. \end{cases} \tag{6.11}$$

$$e_i^*(u_2) = \begin{cases} 0 & 1 \leq i \leq k-1; \\ \dfrac{\Delta_{\widehat{J}_{i-k}}^{[n-k+1,n]}(b)}{\Delta_{J_{i-k}}^{[n-k+1,n]}(b)} & k \leq i \leq n-1. \end{cases} \tag{6.12}$$

Proof. Since $u_2 \in U_+^{\mathrm{fact}}$, we have a factorization $u_2 = u(1)\cdots u(k)$ as in Definition 6.12. We denote by $v_1, \ldots, v_n$ the standard basis of the defining representation $V = \mathbb{C}^n$ for $GL_n(\mathbb{C})$.

We first consider the proof of (6.12). For $1 \leq i \leq k-1$, it follows from the above factorization of u_2 that $u_2 \cdot v_{i+1} = v_{i+1}$, so $e_i^*(u_2) = 0$ as required. For $k \leq i \leq n-1$, an inductive argument using the factorization of u_2 shows that

$$u_2 \cdot v_{i-k+1} \wedge \cdots \wedge v_{i+1} = 0.$$

It follows that

$$\Delta_{[i-k+1,i+1]}^{[1,k]\cup\{i\}}(u_2) = 0.$$

Expanding this minor along the last row and noting that u_2 is upper unitriangular, this implies that

$$\Delta_{[i-k+1,i-1]\cup\{i+1\}}^{[1,k]}(u_2) - \Delta_{[i-k+1,i]}^{[1,k]}(u_2)e_i^*(u_2) = 0.$$

Hence, using the fact that $b = u_{1,0}\dot{w}_P\dot{w}_0^{-1}u_2$,

$$e_i^*(u_2) = \frac{\Delta_{[i-k+1,i-1]\cup\{i+1\}}^{[1,k]}(u_2)}{\Delta_{[i-k+1,i]}^{[1,k]}(u_2)} = \frac{\Delta_{\widehat{J}_i}^{[n-k+1,n]}(b)}{\Delta_{J_i}^{[n-k+1,n]}(b)}.$$

We now consider the proof of (6.11). The explicit factorization of u_2 implies that the matrix $u_2^{-1}\dot{w}_0\dot{w}_P^{-1}$ has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is a $k \times (n-k)$ matrix with zeros above the leading diagonal (i.e. the entries A_{ij} with $j > i$ are all zero) B is a $k \times k$ identity matrix, C is an upper triangular $(n-k) \times (n-k)$ matrix with $(-1)^k$ on the diagonal, and D is a zero $(n-k) \times k$ matrix.

Since $b^{-1}u_{1,0} = u_2^{-1}\dot{w}_0\dot{w}_P^{-1}$, we have, for $1 \le i \le n-1$, that

$$e_i^*(u_{1,0}) = \frac{\Delta^{[1,i]}_{[1,i-1]\cup\{i+1\}}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1})}{\Delta^{[1,i]}_{[1,i]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1})}.$$

If $1 \le i \le n-k-1$, this is zero since the entries in the first i rows of column $i+1$ of $u_2^{-1}\dot{w}_0\dot{w}_P^{-1}$ are all zero. If $n-k+1 \le i \le n-1$, then, using the above description of $u_2^{-1}\dot{w}_0\dot{w}_P^{-1}$, we have

$$e_i^*(u_{1,0}) = -\frac{\Delta^{\{i-n+k\}\cup[i-n+k+2,i]}_{[1,n-k]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1})}{\Delta^{[i-n+k+1,i]}_{[1,n-k]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1})}.$$

Since $b^{-1}u_{1,0} = u_2^{-1}\dot{w}_0\dot{w}_P^{-1}$ and $u_{1,0} \in U^+$, we obtain

$$e_i^*(u_{1,0}) = -\frac{\Delta^{\{i-n+k\}\cup[i-n+k+2,i]}_{[1,n-k]}(b^{-1})}{\Delta^{[i-n+k+1,i]}_{[1,n-k]}(b^{-1})} = \frac{\Delta^{[n-k+1,n]}_{\widehat{J}_i}(b)}{\Delta^{[n-k+1,n]}_{J_i}(b)},$$

as required (using Jacobi's Theorem for the minors of an inverse matrix). A similar argument can be made in the case $i = n-k$. $\square$

Proposition 6.7 follows from diagram (6.9), Lemma 6.13 and Lemma 6.14, since the $\frac{\Delta^{[n-k+1,n]}_{\widehat{J}_i}(b)}{\Delta^{[n-k+1,n]}_{J_i}(b)}$ are nothing other than the summands of $W(Pb)$ as defined in (6.4). Therefore this concludes the proof of Proposition 6.7.

In these last three sections we have proved that we have an isomorphism $\pi_R \circ \pi_L^{-1} : \check{\mathbb{X}}^\circ \to \mathcal{R}$ (see Proposition 6.6), and that under this isomorphism the superpotentials $\mathcal{F}$ and W are identified (see Proposition 6.7). Also we have demonstrated an embedding of a $k(n-k)$-dimensional torus $\mathcal{T}$ into $\check{\mathbb{X}}^\circ$ for which W restricts to the Laurent polynomial superpotential L_q (see Proposition 6.10). To finish up the proof of Theorems 4.7 and 4.10 from the introduction it remains to compare the holomorphic volume forms on the domains of these three superpotentials. This will be done in Section 8, after we have introduced the cluster structure of the Grassmannian.

7. The coordinate ring $\mathbb{C}[\check{\mathbb{X}}^\circ]$ as a cluster algebra

By [84, Thm. 3], the homogeneous coordinate ring of the Grassmannian $\check{\mathbb{X}}$ has a cluster algebra structure (see also [28, §3],[29, Thm. 4.17]). In the latter this cluster algebra structure is shown to induce a cluster algebra structure on $\mathbb{C}[\check{\mathbb{X}}^\circ]$. We now recall these constructions.

A skew-symmetric *cluster algebra* with frozen variables is defined as follows [23, §5]. Let $r, m \in \mathbb{N}$ and consider the field $\mathbb{F} = \mathbb{C}(u_1, \ldots, u_{r+m})$ of rational functions in $r+m$ indeterminates. A *seed* in $\mathbb{F}$ is a pair $(\widetilde{\mathbf{x}}, \widetilde{Q})$ where $\widetilde{\mathbf{x}} = \{x_1, x_2, \ldots, x_{r+m}\}$ is a set freely generating $\mathbb{F}$ as a field over $\mathbb{C}$ and $\widetilde{Q}$ is a quiver with vertices $1, 2, \ldots, r+m$ which has no loops (1-cycles) or 2-cycles. The vertices $r+1, \ldots r+m$ are said to be *frozen*, and there are no arrows between them. The corresponding variables are called *frozen variables*. The subset $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of $\widetilde{\mathbf{x}}$ is known as a *cluster* while $\widetilde{\mathbf{x}}$ is known as an *extended cluster*.

Given $1 \le k \le r$, the seed $(\widetilde{\mathbf{x}}, \widetilde{Q})$ can be mutated at k to produce a new seed $\mu_k(\widetilde{\mathbf{x}}, \widetilde{Q}) = (\widetilde{\mathbf{x}}', \mu_k\widetilde{Q})$ where $\widetilde{\mathbf{x}}' = (\widetilde{\mathbf{x}} \setminus \{x_k\}) \cup \{x_k'\}$, and

$$x_k x_k' = \prod_{i \to k} x_i + \prod_{k \to i} x_i.$$

The new quiver, $\mu_k\widetilde{Q}$, is obtained from Q as follows:

(1) For every path $i \to k \to j$ in Q, add an arrow $i \to j$ (with multiplicity).
(2) Reverse all arrows incident with k.
(3) Remove a maximal collection of 2-cycles in the resulting quiver.

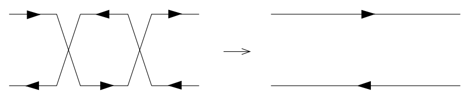


FIGURE 2. The twisting/untwisting move.

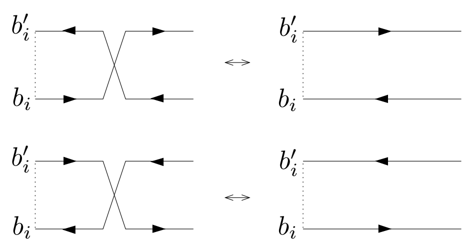


FIGURE 3. The boundary twist.

The *cluster algebra* associated to $(\widetilde{\mathbf{x}}, \widetilde{B})$ is the $\mathbb{C}$-subalgebra of $\mathbb{F}$ generated by the elements of the extended clusters which can be obtained from $(\widetilde{\mathbf{x}}, \widetilde{Q})$ by arbitrary finite sequences of mutations; these elements are called *cluster variables*. Note that the cluster algebra can be defined over $\mathbb{Z}$ or $\mathbb{Q}$.

Recall that in the B-model we are working with $\check{\mathbb{X}} = P\backslash GL_n^\vee$, a Grassmannian of k-planes in $(\mathbb{C}^n)^*$, in its Plücker embedding. We have the following:

Theorem 7.1. [84, Thm. 3] *(see also* [28, §3],[29, Thm. 4.17]*). The homogeneous coordinate ring* $\mathbb{C}[\check{\mathbb{X}}]$ *is a cluster algebra.*

We follow [84], which describes a cluster structure on $\mathbb{C}[\check{\mathbb{X}}]$ in terms of *Postnikov diagrams*, i.e. *alternating strand diagrams* from [70, Defn. 14.1]. We restrict here to the Postnikov diagrams arising in the cluster structure of the Grassmannian.

Definition 7.2. A *Postnikov diagram* of type (k, n) consists of a disk $\mathbb{D}$ with $2n$ marked points $b_1, b'_1, b_2, b'_2, \dots, b_n, b'_n$ marked clockwise on its boundary, together with n smooth oriented curves in the disk, known as *strands*. Strand i starts at b_i or b'_i and ends at b_{i+k} or b'_{i+k}. Here we regard strands (and thus the subscripts of the b_i) as elements of $[1, n]$ interpreted modulo n. The arrangement must satisfy the following additional conditions:

(a) Only two strands can intersect at any given point and all such crossings must be transversal.
(b) There are finitely many crossing points.
(c) If strand i starts at b_i (respectively, b'_i), the first strand crossing it (if such a strand exists) comes from the right (respectively, left). Similarly, if strand i ends at b_{i+k} (respectively, b'_{i+k}), the last string crossing it (if such a strand exists) comes from the right (respectively, left). Following a strand from its starting point to its ending point, the crossings alternate between left and right.
(d) A strand has no self-crossings.
(e) Suppose two strands meet at more than one point. For any two distinct intersection points p and q one strand must be oriented from p to q and the other from q to p.

Postnikov diagrams are considered up to *isotopy* (noting that such an isotopy can neither create nor delete crossings). One may also consider *twisting/untwisting moves* and *boundary twists*; see Figures 2 and 3 (note that these are not isotopies). Two Postnikov diagrams are said to be equivalent if one can be obtained from the other using a sequence of such moves. These moves are local in the sense that no other strands must cross the strands involved in the area where the rule is applied.

When actually drawing Postnikov diagrams, we usually drop the labels of the vertices b_i and b'_i and instead indicate the start of strand i by writing an i in a circle (i.e. at b_i or b'_i) and the end of strand i (i.e. at b_{i+k} or b'_{i+k}) by writing i in a rectangle. We draw a dotted line between b_i and b'_i to make it clearer

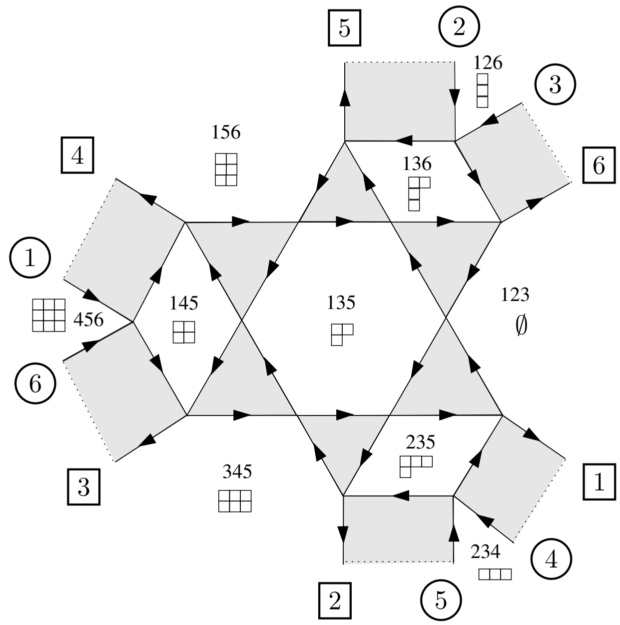


FIGURE 4. A Postnikov diagram for $Gr_3(6)$.

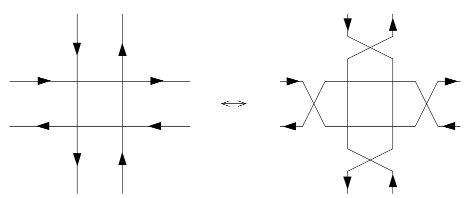


FIGURE 5. Geometric Exchange.

where they are. Thus each i in a rectangle should be linked by a dotted line to $i+k$ in a circle. For an example of a Postnikov diagram, of type $(3,6)$, see Figure 4.

The complement of a Postnikov diagram in the disk is a disjoint union of disks, called *faces* (note that faces often appear as polygons in the Figures, e.g. Figure 4). A face whose boundary includes part of the boundary of the disk $\mathbb{D}$ is called a *boundary face*. A face whose boundary (excluding the boundary of $\mathbb{D}$) is oriented (respectively, alternating) is said to be an *oriented* (respectively, *alternating*) *face*; it is easy to check that all faces are of one of these types.

We label each alternating face F with the subset $L(F)$ of $[1,n]$ which contains i if and only if F lies to the left of strand i. The corresponding Plücker coordinate is denoted by $p_F = p_{L(F)}$.

The *geometric exchange* on a Postnikov diagram is the local move shown in Figure 5.

We recall the following (see [84, Props. 5 and 6]; see also [64, §1] for more recent developments).

Theorem 7.3 (Postnikov)**.** *(a) Each Postnikov diagram of type (k,n) has exactly $k(n-k)+1$ alternating faces.*

(b) Each alternating face is labelled by a k-subset of $[1,n]$.

(c) Every k-subset of $[1,n]$ appears as the label of an alternating face in some Postnikov diagram of type (k,n).

(d) Any two Postnikov diagrams of type (k,n) (up to equivalence) are connected by a sequence of geometric exchanges.

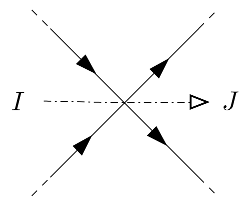


FIGURE 6. Neighbouring faces in a Postnikov diagram.

(e) *The labels of the faces on the boundary of any Postnikov diagram are the* $L_i = [i-k+1, i]$ *for* $i = 1, 2, \ldots, n$. *Indeed,* L_i *labels the boundary face between* b_i' *and* b_{i+1}.

A Postnikov diagram of type (k, n) encodes a seed for the cluster algebra structure of the homogeneous coordinate ring of the Grassmannian $\check{\mathbb{X}}$ as follows. Scott [84, Sect. 5] defines a quiver $Q = Q(D)$ for any Postnikov diagram D. The vertices of Q are the alternating faces of D. The arrows between vertices correspond to points of incidence of the corresponding faces, such that whenever two faces X, Y of D are related as in Figure 6 there is an arrow in Q from X to Y.

We regard Q as being embedded in $\mathbb{D}$, with each vertex mapping to a point in the middle of the corresponding alternating face and each arrow drawn as a line between its endpoints passing through the corresponding point of incidence of the corresponding faces. We will consider the label of an alternating region to also label the corresponding vertex in Q. We refer to the vertices L_i of Q as its *boundary vertices*.

We consider the field $\mathbb{F}$ obtained by adjoining to $\mathbb{C}$ indeterminates u_I for I the label of an alternating face in D. Let $\widetilde{\mathbf{x}}(D)$ be the free generating set for $\mathbb{F}$ containing these indeterminates u_I for I coming from D. We regard the indeterminates corresponding to boundary faces (the L_i) as frozen variables. Note that there are $k(n-k) - n + 1$ non-frozen variables and n frozen variables, making a total of $k(n-k)+1$ variables. Each variable is naturally associated to an alternating face of D and thus to a vertex of $Q(D)$.

Definition 7.4. Fix a Postnikov diagram D_0 of type (k, n). We set $\mathcal{A}$ to be the cluster algebra corresponding to the seed $(\widetilde{\mathbf{x}}(D_0), Q(D_0))$.

Recall for a k-subset I of $[1, n]$ there is an associated Plücker coordinate denoted by p_I; see Section 6.1. For a general Postnikov diagram D we denote by $\mathcal{C}(D)$ the set of Plücker coordinates labelling non-boundary alternating faces of D, and by $\widetilde{\mathcal{C}}(D)$ the set of Plücker coordinates labelling arbitrary alternating faces of D.

Theorem 7.5. [84, Thm. 2]

(a) *There is an isomorphism* φ *from* $\mathbb{C}[\check{\mathbb{X}}]$ *to* $\mathcal{A}$ *taking* p_I *to* u_I *for each* p_I *in* $\widetilde{\mathcal{C}}(D_0)$.

(b) *Let* D *be an arbitrary Postnikov diagram of type* (k, n). *Then* $S(D) := (\varphi(\widetilde{\mathcal{C}}(D)), Q(D))$ *is a seed of* $\mathcal{A}$.

(c) *If* D, D' *are two Postnikov diagrams of type* (k, n) *related by a geometric exchange corresponding to a quadrilateral face* X *of* D' *then* $S(D')$ *is the mutation at* p_X *of* $S(D)$.

We see that the elements $u_I = \varphi(p_I)$ of $\mathcal{A}$, for I a k-subset of $[1, n]$ are all cluster variables.

Let $\mathcal{A}'$ be the cluster algebra defined the same way as for $\mathcal{A}$ except that the elements $u_{J_i}^{-1}$, for $i = 1, 2, \ldots, n$, of $\mathbb{F}$ are added to the generating set. Thus $\mathcal{A}'$ is the localisation of $\mathcal{A}$ obtained by adjoining inverses to the elements u_{J_i} (see [29, Sect. 3.4]). Recall that $\check{\mathbb{X}}^\circ$ is defined to be the subset of $\check{\mathbb{X}}$ where the p_{J_i} do not vanish. We have the following:

Proposition 7.6. (a) *There is an isomorphism* φ' *from* $\mathbb{C}[\check{\mathbb{X}}^\circ]$ *to* $\mathcal{A}'$, *taking* p_I *to* u_I *for each* p_I *in* $\widetilde{\mathcal{C}}(D_0)$.

(b) *Let* D *be an arbitrary Postnikov diagram of type* (k, n). *Then* $S(D) := (\varphi(\widetilde{\mathcal{C}}(D)), Q(D))$ *is a seed of* $\mathcal{A}'$.

(c) *If* D, D' *are two Postnikov diagrams of type* (k, n) *related by a geometric exchange corresponding to a quadrilateral face* X *of* D *then* $S(D')$ *is the mutation at* p_X *of* $S(D)$.

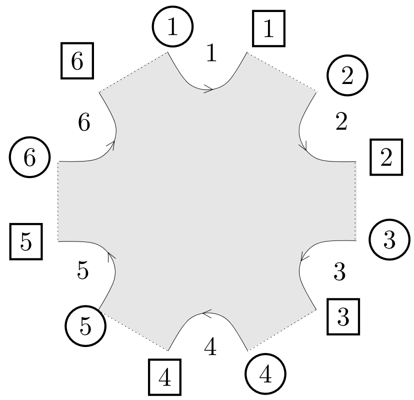


FIGURE 7. A Postnikov diagram for $Gr(1,6)$.

The proof of this result involves applying [29, Prop. 3.37] to get (a) (see also [24, Prop. 11.1]). Parts (b) and (c) can be shown by following the proof in [84] of Theorem 7.5.

Definition 7.7. We identify $\mathbb{C}[\check{\mathbb{X}}^\circ]$ with $\mathcal{A}'$ via the isomorphism φ'. We refer to any seed $(\widetilde{\mathcal{C}}(D), Q(D))$ associated to a Postnikov diagram D as a Postnikov seed and call $\widetilde{\mathcal{C}}(D)$ a Postnikov extended cluster. The set of cluster variables contains the Plücker coordinates p_I. The frozen variables are the p_I, where I is a cyclic interval.

Remark 7.8. If $k = 1$ or $n-1$, there is a unique Postnikov diagram of type (k,n) up to equivalence. It can be chosen to have no crossings at all. The case $k=1$, $n=6$ is shown in Figure 7.

8. THE THREE VERSIONS OF THE HOLOMORPHIC VOLUME FORM

In this section we conclude the proof of the comparison theorems 4.7 and 4.10 which was begun in Section 6. To do this it remains to compare the holomorphic volume forms on the domains, $\mathcal{T}, \check{\mathbb{X}}^\circ$ and $\mathcal{R}$, of the three superpotentials L_q, W_q and $\mathcal{F}_q$. Recall that these domains are related by maps

$$\mathcal{T} \overset{\iota}{\hookrightarrow} \check{\mathbb{X}}^\circ \xrightarrow{\pi_L^{-1}} B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+ \xrightarrow{\pi_R} \mathcal{R}, \tag{8.1}$$

see Section 6.2. We begin by recalling how each of the holomorphic volume forms is constructed.

Definition 8.1 (the holomorphic volume form on $\mathcal{T}$ [35])**.** The domain $\mathcal{T}$ of the Laurent polynomial superpotential L_q is naturally a torus. The holomorphic volume form $\omega_{\mathcal{T}}$ is the standard invariant volume form on the torus,

$$\omega_{\mathcal{T}} = \bigwedge_{i,j} \frac{dz_{ij}}{z_{ij}}, \tag{8.2}$$

where the wedge product is over pairs i,j with $1,\dots,n-k$ and $j = 1,\dots,k$. The sign is determined by the ordering of the indexing set, which we may choose to be lexicographic.

Definition 8.2 (the holomorphic volume form on $\check{\mathbb{X}}^\circ$)**.** We define $\omega_{\check{\mathbb{X}}^\circ}$ to be a choice of holomorphic volume form on $\check{\mathbb{X}}^\circ$ which extends to a meromorphic form on $\check{\mathbb{X}}$ with degree one poles along the $\mathbb{Z}/n\mathbb{Z}$-invariant anticanonical divisor D from (6.3). The normalisation will be chosen after Lemma 8.5.

Definition 8.3 (the holomorphic volume form on $\mathcal{R}$ [79])**.** The holomorphic volume form on $\mathcal{R}$ is defined in outline as follows; for details we refer to [79, Section 7]. Choose a reduced expression $s_{i_1}\dots s_{i_N}$ of w_0, recorded by the sequence $\mathbf{i} = (i_1,\dots,i_N)$. Associated to $\mathbf{i}$ there is an open torus $\mathcal{R}_{\mathbf{i}} \cong (\mathbb{C}^*)^{k(n-k)}$ consisting of 'factorised' elements $u_1\dot{w}_P B_-$ in $\mathcal{R}$; see [59]. We let $z_j^{(\mathbf{i})}$ denote the coordinates on $\mathcal{R}_{\mathbf{i}}$, and consider the standard invariant holomorphic volume form

$$\omega_{\mathbf{i}} = \bigwedge_j \frac{dz_j^{(\mathbf{i})}}{z_j}.$$

Note that the reduced expression $\mathbf{i}$ implies a natural ordering on these coordinates. It is shown in [79, Proposition 7.2] that this form $\omega_{\mathbf{i}}$ extends to $\mathcal{R}$ and (up to sign) is independent of the initial choice of reduced expression $\mathbf{i}$.

For example in the case where $\check{\mathbb{X}}^\circ = Gr_3(\mathbb{C}^5)$ and $\mathbf{i} = (1,2,3,4,1,2,3,1,2,1)$ we have

$$\mathcal{R}_{\mathbf{i}} = \{x_1(m_1)x_2(m_2)\dot{s}_3\dot{s}_4x_1(m_5)x_2(m_6)\dot{s}_3x_1(m_8)x_2(m_9)\dot{s}_1B_- \mid m_j \in \mathbb{C}^*\}, \tag{8.3}$$

and the form $\omega_{\mathcal{R}}$ is the extension of the form $\bigwedge_j \frac{dm_j}{m_j}$ from $\mathcal{R}_{\mathbf{i}}$ to $\mathcal{R}$.

Using the maps in (8.1) we can pull back each one of the three forms to $\mathcal{T}$. For any two of the forms we say they are *compatible* if their pullbacks to $\mathcal{T}$ agree up to a nonzero scalar.

Proposition 8.4. *The forms $\omega_{\mathcal{T}}, \omega_{\check{\mathbb{X}}^\circ}$ and $\omega_{\mathcal{R}}$ are compatible.*

Once we have proved this proposition the normalisations of $\omega_{\check{\mathbb{X}}^\circ}$ and $\omega_{\mathcal{R}}$ can be chosen so that the pullbacks actually agree with $\omega_{\mathcal{T}}$, and the proof of Theorems 4.7 and 4.10 will be complete. Our strategy for proving that the holomorphic volume forms $\omega_{\mathcal{T}}, \omega_{\check{\mathbb{X}}^\circ}$ and $\omega_{\mathcal{R}}$ are compatible is to make use as much as possible of the symmetries of $\check{\mathbb{X}}^\circ$, and of the rectangles cluster.

We begin with the form $\omega_{\check{\mathbb{X}}^\circ}$. Recall the open subset $\check{\mathbb{X}}^{\circ,fact}$ of $\check{\mathbb{X}}^\circ$, and the maps introduced in Sections 6.4 and 6.3,

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{\sim} & \dot{w}_P\dot{w}_0^{-1}U_+^{fact} & \xrightarrow{\sim} & \check{\mathbb{X}}^{\circ,fact} \\ (z_{ij}) & \mapsto & g = \dot{w}_0^{-1}g_{(z_{ij})} & \mapsto & Pg. \end{array}$$

Let $\omega_{\check{\mathbb{X}}^{\circ,fact}}$ denote the holomorphic volume form on $\check{\mathbb{X}}^{\circ,fact}$ which agrees with $\omega_{\mathcal{T}}$ under the above isomorphism $\mathcal{T} \xrightarrow{\sim} \check{\mathbb{X}}^{\circ,fact}$. Proposition 6.10 implies that $\omega_{\check{\mathbb{X}}^{\circ,fact}}$ is given by

$$\omega_{\check{\mathbb{X}}^{\circ,fact}} = \bigwedge_{i,j} \frac{d\,p_{i\times j}}{p_{i\times j}}, \tag{8.4}$$

where the indexing set is as for $\omega_{\mathcal{T}}$ in (8.2).

Lemma 8.5. *The holomorphic volume form $\omega_{\check{\mathbb{X}}^{\circ,fact}}$ extends to a meromorphic form $\omega_{\check{\mathbb{X}}}$ on $\check{\mathbb{X}}$ which is regular on $\check{\mathbb{X}}^\circ$. Moreover $\omega_{\check{\mathbb{X}}}$ has poles of order one along the divisor D.*

Proof. The form $\omega_{\check{\mathbb{X}}^{\circ,fact}}$ extends to a meromorphic form on $\check{\mathbb{X}}$ as a consequence of (8.4), and we need to analyse its poles. We denote this meromorphic form by $\omega_{\check{\mathbb{X}}}$. Note that the rectangular Plücker coordinates form an extended cluster in $\mathbb{C}[\check{\mathbb{X}}^\circ]$ by [60, Lemma 8.1], since, for $1 \le i \le n-k$, $1 \le j \le k$, the k-subset $J_{i\times j}$ associated to $p_{i\times j}$ is given by

$$J_{i\times j} = [1, k-j] \cup [i+k-j+1, i+k], \tag{8.5}$$

and hence coincides with the k-subset $M_{k,n}(k-j,i)$ defined in [60, Lemma 8.3].

It follows that $\omega_{\check{\mathbb{X}}}$ is regular and nonvanishing on $\check{\mathbb{X}}^\circ$ by an argument entirely analogous to the one used in the construction of $\omega_{\mathcal{R}}$ in [79, Section 7], which is now standard in the context of cluster algebras; see for example Section 13 of [49] and references therein. Namely, by applying mutations to $\omega_{\check{\mathbb{X}}}$ one can show that restriction of $\omega_{\check{\mathbb{X}}}$ to any cluster torus is given by the same formula, up to sign, in terms of the corresponding cluster variables. So $\omega_{\check{\mathbb{X}}}$ is regular and nonvanishing on the union of the cluster tori. To extend one uses that the union of cluster tori has complement of codimension ≥ 2 inside $\check{\mathbb{X}}^\circ$. This shows that $\omega_{\check{\mathbb{X}}}$ is regular and nonvanishing on $\check{\mathbb{X}}^\circ$.

We now use this compatibility of $\omega_{\check{\mathbb{X}}}$ with the cluster structure to show that $\pm\omega_{\check{\mathbb{X}}}$ is invariant under the action of $\mathbb{Z}/n\mathbb{Z}$ on $\check{\mathbb{X}}$ defined by the cyclic shift (see Section 6.1). Note that if 1 is added to the label of each strand in a Postnikov diagram D, we obtain another Postnikov diagram D' with the property that the corresponding extended cluster $\widetilde{\mathcal{C}}(D')$ is the pull-back of $\widetilde{\mathcal{C}}(D)$ under the cyclic shift. Since $\omega_{\check{\mathbb{X}}}$ has the same form (up to sign) in terms of the shifted cluster $\widetilde{\mathcal{C}}(D')$ as it does in terms of $\widetilde{\mathcal{C}}(D)$, it follows that the $\mathbb{Z}/n\mathbb{Z}$ action on $\check{\mathbb{X}}^\circ$ preserves $\omega_{\check{\mathbb{X}}}$ up to sign.

We can now determine the poles of $\omega_{\check{\mathbb{X}}}$. Since $\omega_{\check{\mathbb{X}}}$ is regular and nonvanishing on $\check{\mathbb{X}}^\circ$, it must have poles contained in D. We pick an irreducible component D_0 of D and let m_0 be the order of the pole of $\omega_{\check{\mathbb{X}}}$ along D_0. The other irreducible components of D are obtained from D_0 by the $\mathbb{Z}/n\mathbb{Z}$-action. Since $\omega_{\check{\mathbb{X}}}$ is $\mathbb{Z}/n\mathbb{Z}$-invariant up to sign, it must have the same order m_0 of pole along each component of D. Finally, since the index of $\check{\mathbb{X}}$ is n it follows that $m_0 = 1$. Therefore $\omega_{\check{\mathbb{X}}}$ has poles of order one along each component of D. □

We may set $\omega_{\check{\mathbb{X}}^\circ}$ to be the restriction of $\omega_{\check{\mathbb{X}}}$ to $\check{\mathbb{X}}^\circ$, thus fixing the normalisation which was missing from Definition 8.2. The above lemma implies that $\omega_{\check{\mathbb{X}}^\circ}$ is compatible with $\omega_{\mathcal{T}}$, thus proving the first part of Proposition 8.4.

Next we turn out attention to $\omega_{\mathcal{R}}$. In order to be able to make the comparison of forms we introduce another symmetry of $\check{\mathbb{X}}^\circ$, which is in a sense an obstruction to our isomorphism from $\mathcal{R}$ to $\check{\mathbb{X}}^\circ$ extending to a map from the closure of $\mathcal{R}$ to $\check{\mathbb{X}}$. Namely, the main ingredient to proving that $\omega_{\check{\mathbb{X}}^\circ}$ and $\omega_{\mathcal{R}}$ are compatible will be the involution described in the following proposition.

For $b \in B_-\dot{w}_0 \cap U_+\widetilde{T}^{W_P}\dot{w}_P U_-$ and $\lambda \in \mathcal{P}_{k,n}$, let us denote by $P_\lambda(b)$ the minor of B with row set $[n-k+1, n]$ and column set J_λ. We also use the shorthand $P_m(b) = P_{\mu_m}(b)$.

Proposition 8.6. *Transposition of matrices restricts to define an involution* $(\quad)^t$ *on the subvariety* $B_-\dot{w}_0 \cap U_+\widetilde{T}^{W_P}\dot{w}_P U_-$ *of* $GL_n(\mathbb{C})$. *We conjugate this involution by right multiplication by* $\dot{w}_0$ *to obtain an involution*

$$\begin{array}{rccc} \mathrm{Invol}: & B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+ & \longrightarrow & B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+, \\ & b & \mapsto & (b\dot{w}_0)^t\dot{w}_0^{-1}. \end{array} \tag{8.6}$$

The map Invol *satisfies (and is uniquely determined by) the following equality of minors,*

$$P_{i\times j}(\mathrm{Invol}(b)) = (-1)^{ij}\frac{P_{(n-k-i)\times(k-j)}(b)}{P_{n-k+i-j}(b)}(\mathbf{q}(b))^{\min(i,j)}, \tag{8.7}$$

where $1 \le i \le n-k$ *and* $1 \le j \le k$. *Here*

$$\mathbf{q}(b) := \alpha_{n-k}(t),$$

where

$$b = u_1 t\dot{w}_P\dot{w}_0^{-1}u_2 \quad \textit{for} \quad u_1, u_2 \in U_+ \quad \textit{and} \quad t \in \widetilde{T}^{W_P}. \tag{8.8}$$

Remark 8.7. Note that in the setting of Proposition 8.6, transposition of $b = z\dot{w}_0^{-1}$ affects the torus factor t in (8.8) by $t \mapsto \dot{w}_P^{-2}t$. This changes the sign of α_{n-k} precisely if k and $n-k$ have different parity. Hence $\mathbf{q}(\mathrm{Invol}(b)) = (-1)^n\mathbf{q}(b)$

Remark 8.8. The proposition can be interpreted as saying that there exists an involution

$$\widetilde{\tau} : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \to \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \tag{8.9}$$

which (using rectangles cluster coordinates) satisfies

$$(p_{i\times j}, q) \mapsto \left(\frac{p_{(n-k-i)\times(k-j)}}{p_{J_{n-k+j-i}}}q^{\min(i,j)}, q\right). \tag{8.10}$$

Indeed via the identification of $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$ with $B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+$, the involution Invol becomes defined on $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$. We now note that the involution Invol commutes with the $\mathbb{C}_u^*$-action on $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$ given in coordinates by:

$$u \cdot ((p_\lambda)_{\lambda\in\mathcal{P}_{k,n}}, q) = ((u^{|\lambda|}p_\lambda)_{\lambda\in\mathcal{P}_{k,n}}, u^n q). \tag{8.11}$$

This is straightforward to check using the formula (8.10). Let $\mathrm{SignAct} : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \to \check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$ denote the map given by the action of $(-1) \in \mathbb{C}_u^*$. Then

$$\widetilde{\tau} := \mathrm{SignAct} \circ \mathrm{Invol} = \mathrm{Invol} \circ \mathrm{SignAct}\,.$$

The involution $\widetilde{\tau}$ preserves the superpotential $W : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \to \mathbb{C}$. For example this follows from a direct calculation using the formula for the superpotential in the rectangles cluster. Alternatively the invariance of W relates to the symmetry in the formula for $\mathcal{F}$ with regard to u_1 and u_2, see Definition 6.3, via the comparison of superpotentials result (Proposition 6.7).

Finally, we observe that $\widetilde{\tau}$ fixes the critical points of W_q. This is because the critical points are represented by Toeplitz matrices b in $B_- \cap U_+ \widetilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+$; see [79, Equation (5.14)], with $h = 0$ for the non-equivariant case. (Recall that a Toeplitz matrix is a matrix for which the entries along each diagonal are the same, and for $b \in B_-$ this is equivalent to the condition of stabilizing the standard principal nilpotent F from Theorem 6.4). It follows that the matrices $z = b\dot{w}_0$ are almost Hankel matrices (i.e. the entries along each anti-diagonal are the same up to sign, with the signs alternating along the anti-diagonal). Transposition of z swaps all of the signs on even length anti-diagonals. From this we can observe that

$$\mathrm{SignAct}(Pz^t\dot{w}_0^{-1}, q) = (Pz\dot{w}_0^{-1}, q), \qquad \text{if } Pz\dot{w}_0^{-1} \text{ is a critical point of } W_q.$$

In other words, $\widetilde{\tau}$ fixes the critical points $(Pz\dot{w}_0^{-1}, q)$ of W_q.

Remark 8.9. If we set $q = 1$ the restricted domain of (8.9) can be identified with $\check{\mathbb{X}}^\circ$. In this case the proposition can be interpreted as saying that there is an involution τ on $\check{\mathbb{X}}^\circ$ which extends the involution

$$p_{J_j} \mapsto \frac{1}{p_{J_{n-j}}}$$

on frozen variables. Moreover this involution preserves the rectangles cluster torus and is given there by

$$p_{i\times j} \mapsto \frac{p_{(n-k-i)\times(k-j)}}{p_{J_{n-k+j-i}}}. \tag{8.12}$$

Remark 8.10. The involution $\tilde{\tau}$ from Remark 8.8 has an interpretation which involves the quantum cohomology ring $qH^*(X, \mathbb{C})[q^{-1}]$. Namely by the combination of Theorem 6.5 with Theorem 6.4, the quantum cohomology ring is the ring of functions on the critical point locus $\{\partial_{\check{\mathbb{X}}^\circ} W = 0\}$ inside $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$. Since the involution acts trivially on the critical point locus, it also acts trivially on the quantum cohomology ring $qH^*(X, \mathbb{C})[q^{-1}]$. In Section 9.1 we will show that the Plücker coordinates p_λ restricted to the critical locus of W_q are identified with the quantum Schubert classes σ^λ via Peterson's isomorphism (see Proposition 5.2). Therefore in the Grassmannian case, since we have the formula (8.10), this means that we obtain the relations

$$\sigma^{i\times j} \star \sigma^{\mu_{n-k+j-i}} = q^{\min(i,j)} \sigma^{(n-k-i)\times(k-j)} \tag{8.13}$$

in the quantum cohomology ring $qH^*(X, \mathbb{C})$, as a result of the symmetry $\widetilde{\tau}$ of the mirror. These relations also follow from repeated application of the quantum Pieri rule of [12, p. 293].

We note that the involution (8.6) makes sense for arbitrary reductive algebraic groups. By analogous arguments to above this construction gives rise to an involution on the Lie-theoretic mirror $\mathcal{R}_P \times T^{W_P}$ of a general $G^\vee/P^\vee$ such that the superpotential $\mathcal{F}$ is invariant and the critical points are fixed. In particular this involution should again induce relations in the quantum cohomology ring.

We will outline the proof of Proposition 8.6 after proving the following corollary, which is our first application of the proposition.

Corollary 8.11. *The forms $\omega_{\check{\mathbb{X}}^\circ}$ and $\omega_{\mathcal{R}}$ are compatible.*

Proof. To prove the statement of the corollary we need to compare $(\check{\mathbb{X}}^\circ, \omega_{\check{\mathbb{X}}^\circ})$ and $(\mathcal{R}, \omega_{\mathcal{R}})$ under the isomorphisms

$$\check{\mathbb{X}}^\circ \xleftarrow{\pi_L} B_- \cap U_+ \dot{w}_P \dot{w}_0^{-1} U_+ \xrightarrow{\pi_R} \mathcal{R}. \tag{8.14}$$

Let Invol_0 be the restriction of the involution Invol to $B_-\cap U_+\dot w_P\dot w_0^{-1}U_+$,

$$\mathrm{Invol}_0 : B_-\cap U_+\dot w_P\dot w_0^{-1}U_+ \to B_-\cap U_+\dot w_P^{-1}\dot w_0^{-1}U_+$$

given by $b\mapsto \dot w_0^{-1}b^t\dot w_0^{-1}$. Also let

$$\begin{array}{rcl}\tau_0:\check{\mathbb{X}}^\circ & \to & \check{\mathbb{X}}^\circ\\ Pb & \mapsto & P\,\mathrm{Invol}_0(b),\end{array}$$

for $b\in B_-\cap U_+\dot w_P\dot w_0^{-1}U_+$. We consider the commutative diagram

(8.15)
$$\begin{tikzcd}
\check{\mathbb{X}}^\circ \arrow[d, "\tau_0"'] & B_-\cap U_+\dot w_P\dot w_0^{-1}U_+ \arrow[l, "\pi_L"'] \arrow[r, "\pi_R"] \arrow[dl, "\widetilde\pi_L"] & \mathcal{R}. \\
\check{\mathbb{X}}^\circ & &
\end{tikzcd}$$

Recall that we have a torus $\check{\mathbb{X}}^{\circ,fact}$ inside $\check{\mathbb{X}}^\circ$, see Section 6.4, and on the right hand a torus $\mathcal{R}^{fact}:=\mathcal{R}_{\mathbf{i}}$ inside $\mathcal{R}$, where $\mathbf{i}=(1,2,\dots,n;1,2,\dots,n-1;\dots;1,2;1)$ (compare Definition 8.3).

The holomorphic volume form $\omega_{\check{\mathbb{X}}^\circ}$ is characterised up to a scalar by the fact that its restriction to $\check{\mathbb{X}}^{\circ,fact}$ is a torus invariant volume form, and similarly for $\omega_{\mathcal{R}}$ and $\mathcal{R}^{fact}$. Therefore it suffices to prove that the pull-backs of the tori agree, namely that

$$\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right)=\pi_R^{-1}\left(\mathcal{R}^{fact}\right).$$

We have that the preimage $\pi_R^{-1}\left(\mathcal{R}^{fact}\right)$ is described, carrying on with the example from Definition 8.3, by

$$\pi_R^{-1}\left(\mathcal{R}^{fact}\right)=\{b\in B_-\mid b=x_1(m_1)x_2(m_2)\dot s_3\dot s_4x_1(m_5)x_2(m_6)\dot s_3x_1(m_8)x_2(m_9)\dot s_1\dot w_0^{-1}u_2,\ u_2\in U_+,m_i\in\mathbb{C}^*\}.$$

We compare this with the preimage $\widetilde\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right)$. Namely, in the same example, $\widetilde\pi_L(b)=P\dot w_0^{-1}b^t\dot w_0^{-1}$ lies in

$$\check{\mathbb{X}}^{\circ,fact}=\{P\dot w_0^{-1}\dot s_4x_3(n_1)x_4(n_2)\dot s_2x_3(n_4)x_4(n_5)\dot s_1\dot s_2x_3(n_8)x_4(n_9)\mid n_i\in\mathbb{C}^*\}$$

if and only if $\dot w_0^{-1}b^t\in B_-\dot w_0\cap U_+\dot w_P^{-1}U_-$ is of the form

$$\dot w_0^{-1}b^t=b_+\dot w_0^{-1}\dot s_4^{-1}x_3(n_1)x_4(n_2)\dot s_2^{-1}x_3(n_4)x_4(n_5)\dot s_1^{-1}\dot s_2^{-1}x_3(n_8)x_4(n_9)\dot w_0$$

for some $b_+\in B_+$, or equivalently if b is of the form

$$b=x_1(n_9)x_2(n_8)\dot s_3\dot s_4x_1(n_5)x_2(n_4)\dot s_3x_1(n_2)x_2(n_1)\dot s_1\dot w_0^{-1}b_+.$$

Therefore $\widetilde\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right)=\pi_R^{-1}\left(\mathcal{R}^{fact}\right)$. This argument clearly works in general, although we only wrote it out in an example.

Now note that the torus $\check{\mathbb{X}}^{\circ,fact}$ is also characterised by the condition that $p_{i\times j}\neq 0$ for all of the Plücker coordinates in the rectangles cluster, thanks to Proposition 6.10. From Proposition 8.6 it follows that τ_0 is an isomorphism which takes $\check{\mathbb{X}}^{\circ,fact}$ to $\check{\mathbb{X}}^{\circ,fact}$. Therefore

$$\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right)=\widetilde\pi_L^{-1}\left(\tau_0(\check{\mathbb{X}}^{\circ,fact})\right)=\widetilde\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right).$$

and $\widetilde\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right)=\pi_R^{-1}\left(\mathcal{R}^{fact}\right)$ as we saw above. This concludes the proof that $\pi_L^{-1}\left(\check{\mathbb{X}}^{\circ,fact}\right)=\pi_R^{-1}\left(\mathcal{R}^{fact}\right)$, and hence the proof of the Corollary. ☐

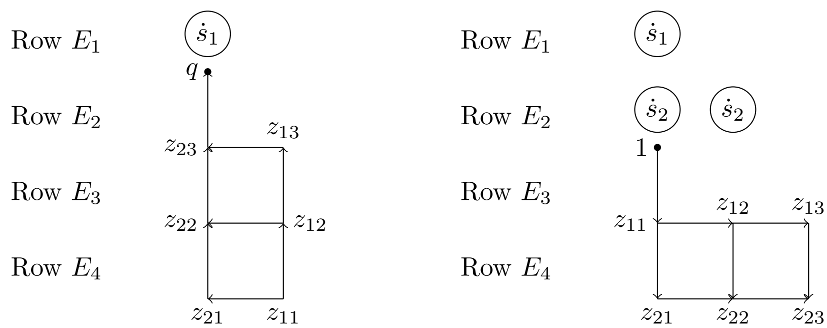
Outline of the proof of Proposition 8.6. Our initial proof of Proposition 8.6 involved a sequence of equalities of minors. However we describe our second proof which involves an extension of the construction from Proposition 6.10.

The idea for this proof is to factorise a generic element of $B_-\cap U_+\widetilde T^{W_P}\dot w_P\dot w_0^{-1}U_-$ in a way that makes it easy to take the transpose. Recall that we have an embedding $\iota:\mathcal{T}\to\check{\mathbb{X}}^\circ$ defined in Section 6.3 which was constructed via multiplying together factors from simple root subgroups. One can 'extend' ι to an embedding

$$\zeta:\mathcal{T}\times\mathbb{C}_q^*\ \hookrightarrow\ B_-\cap U_+\widetilde T^{W_P}\dot w_P\dot w_0^{-1}U_+,$$

for which $b = \zeta(z_{ij}, q)$ satisfies $\mathbf{q}(b) = q$ and $Pb \in \check{\mathbb{X}}^\circ$ is $\iota(z_{ij})$. The map ζ has the following explicit description, which relates to the construction of ι from Section 6.3. As in the proof of Proposition 6.10, we demonstrate our construction of the map ζ in the special case of $X = Gr_2(5)$ and $\check{\mathbb{X}} = Gr_3(5)$, that is, $k = 3$ and $n = 5$.

We now consider both the quiver $\mathcal{Q}_R$ associated to $X = Gr_2(5)$ in Section 6.3 and its reflection through the xy-axis which we denote by $\mathcal{Q}_L$. We decorate the two quivers as shown below. Note that we have purposefully left out some of the s_i's. In both cases we have also dropped the bottom right hand corner arrow.



We remark that the analogue of the quiver from the proof of Proposition 6.10 is the one on the right hand side, above. The left hand side quiver is related to the construction of an open part of the Peterson variety from [77, Theorem 7.2].

To the quiver on the right hand side we simply associate an element $g_R(z_{ij}) \in \dot{s}_2\dot{s}_1\dot{s}_2U_+$ using the construction from the proof of Proposition 6.10. Namely in this example the element is

$$g_R(z_{ij}) = x_3\left(z_{13}\right) x_4\left(\frac{z_{23}}{z_{13}}\right)\dot{s}_2 x_3\left(z_{12}\right) x_4\left(\frac{z_{22}}{z_{12}}\right)\dot{s}_1\dot{s}_2 x_3\left(z_{11}\right) x_4\left(\frac{z_{21}}{z_{11}}\right). \tag{8.16}$$

To the reflected quiver on the left hand side we associate an element $g_L(z_{ij}, q) \in U_+\dot{s}_1$ using the construction from [77] which goes as follows. This time we start from the left-most column at the bottom working upwards and then carry on column by column to the right. Namely to every vertex labelled by a z_{ij} we associate the minimal 1-path starting at that vertex. Then we list these 1-paths one for each z_{ij} vertex in our column, starting from the bottom and working upwards. These are followed by any $\dot{s}_i$'s at the top of the column. Then we repeat for the next column to the right, until the list is complete. As before to any 1-path which crosses row E_i we associate the element $x_i(z_t/z_s) \in U_+$, where z_s is the coordinate at the beginning vertex ('source') of the 1-path, and z_t the vertex at the end ('target'). We again obtain a matrix given as the product of the listed factors. In the running example we have,

$$g_L(z_{ij}, q) = x_4\left(\frac{z_{22}}{z_{21}}\right) x_3\left(\frac{z_{23}}{z_{22}}\right) x_2\left(\frac{q}{z_{23}}\right)\dot{s}_1 x_4\left(\frac{z_{12}}{z_{11}}\right) x_3\left(\frac{z_{13}}{z_{12}}\right) x_2\left(\frac{q}{z_{13}}\right). \tag{8.17}$$

Moreover we set $t_q = \mathrm{diag}(q, q, 1, 1, 1)$ to be the diagonal matrix in $\widetilde{T}^{W_P}$ with $\alpha_2(t_q) = q$. The map ζ is defined by multiplying the matrices as follows,

$$\zeta(z_{ij}, q) = g_L(z_{ij}, q)\, t_q \dot{w}_0^{-1}\; g_R(z_{ij}).$$

From the factorisation of g_R and g_L we see that $\zeta(z_{ij}, q)$ is of the form $u_1 t_q \dot{s}_1\dot{s}_3\dot{s}_4\dot{s}_3\dot{w}_0^{-1}u_2 \in U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+$. One then has to check that the ab-entries of the $n \times n$-matrix $\zeta(z_{ji}, q)$ vanish whenever $a < b$. This can be shown by careful study of the representation of the product $\zeta(z_{ij}, q)$ in terms of a concatenation of 'chips' (corresponding to terms x_i; see [25, Figure 4]) and wiring diagrams (corresponding to products of terms $\dot{s}_i$). We leave out the details. It follows that $\zeta(z_{ij}, q) \in B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+$, as was intended.

We can now work out the minors we are interested in, in the case where $b = \zeta(z_{ij}, q)$. As in Proposition 6.10 we have

$$\frac{P_{i\times j}(\zeta(z_{ij},q))}{P_{(i-1)\times(j-1)}(\zeta(z_{ij},q))} = z_{ij} \tag{8.18}$$

and similarly we obtain

$$\frac{P_{i\times j}(\mathrm{Invol}(\zeta(z_{ij},q)))}{P_{(i-1)\times(j-1)}(\mathrm{Invol}(\zeta(z_{ij},q)))} = \frac{(-1)^{i+j-1}q}{z_{n-k-i+1,k-j+1}}. \tag{8.19}$$

Note that the $P_{i\times j}$ are basically the rectangle Plücker coordinates $p_{i\times j}$ appearing in Proposition 6.10. In particular, note that $P_{i\times 0} = P_{0\times j} = P_\emptyset$, and it is easy to check that $P_\emptyset(\zeta(z_{ij}, q)) = P_\emptyset(\mathrm{Invol}(\zeta(z_{ij}, q)) = 1$.

We now finish the proof of the proposition. Clearly, ζ has open dense image in $B_-\dot{w}_0 \cap U_+\widetilde{T}^{W_P}\dot{w}_P U_-$ (as follows for example from the above formulas). It therefore suffices to prove the identity (8.7) in the case $b = \zeta(z_{ij}, q)$.

From (8.18) and (8.19) it follows that

$$\frac{P_{i\times j}(\mathrm{Invol}(b))}{P_{(i-1)\times(j-1)}(\mathrm{Invol}(b))} = (-1)^{i+j-1}\frac{qP_{(n-k-i)\times(k-j)}(b)}{P_{(n-k-i+1)\times(k-j+1)}(b)}.$$

A telescopic product identity then implies the desired formula,

$$P_{i\times j}(\mathrm{Invol}(b)) = (-1)^{ij}q^{\min(i,j)}\frac{P_{(n-k-i)\times(k-j)}(b)}{P_{n-k+j-i}(b)}.$$

We note that $P_{n-k+j-i}(b)$ is the minor of b associated to the rectangle $(n-k-i+\min(i,j)) \times (k-j+\min(i,j))$, which is indeed a maximal rectangle. □

9. Schubert classes and proof of the free basis lemma

The first aim of this section is to show that the isomorphism between $\mathbb{C}[\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*]/(\partial_{\check{\mathbb{X}}^\circ} W_q)$ and $qH^*(X)[q^{-1}]$ identifies the classes of the Plücker coordinate p_λ with the Schubert class σ^λ. After that we will prove Lemma 3.3.

9.1. **The Schubert basis.**

In this Section we prove the non-equivariant version of Proposition 5.2. As in Section 6.4, let $\Delta^I_J(g)$ denote the minor with row set I and column set J, and recall that to each $\lambda \in \mathcal{P}_{k,n}$ we have associated a k-subset J_λ of $[1, n]$, see Section 6.1. Recall that by Peterson's theorem, Theorem 6.4, there is an explicit isomorphism between $qH^*(X, \mathbb{C})[q^{-1}]$ and the Peterson variety Y_P^* inside G/B_-. We now state in the context of Grassmannians a result from Dale Peterson's theory which makes this isomorphism precise. It can be found with proof as Proposition 11.3 in [76].

Proposition 9.1 (Dale Peterson [69]). *Let $\lambda \in \mathcal{P}_{k,n}$. If $J_\lambda = \{\lambda_1, \dots, \lambda_k\}$ set $J^{\mathrm{opp}}_\lambda = \{n+1-\lambda_1, n+1-\lambda_2, \dots, n+1-\lambda_k\}$. Note that $J_\emptyset = [1, k]$ and $J^{\mathrm{opp}}_\emptyset = [n+1-k, n]$. We write Δ^J for $\Delta^J_{J^{\mathrm{opp}}_\emptyset}$. Define a rational function on G/B_- by*

$$f_\lambda(gB_-) = \frac{\Delta^{J^{\mathrm{opp}}_\lambda}(g)}{\Delta^{J^{\mathrm{opp}}_\emptyset}(g)}.$$

The restriction of f_λ to the Peterson variety Y_P^* is identified with the Schubert class σ^λ under the isomorphism of Theorem 6.4.

Next we recall that by Theorem 6.5, the Peterson variety Y_P^* is isomorphic to the subvariety of $\mathcal{R} \times \mathbb{C}_q^*$ defined by $\partial_{\mathcal{R}}\mathcal{F} = 0$, for the Lie theoretic superpotential $\mathcal{F} : \mathcal{R} \times \mathbb{C}_q^* \to \mathbb{C}$ from Definition 6.3. Moreover by the comparison of superpotentials result, Theorem 4.10, this variety must be isomorphic to the subvariety of $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$ defined by $\partial_{\check{\mathbb{X}}^\circ} W = 0$.

We now apply Proposition 9.1 and isomorphisms above to prove the following proposition.

Proposition 9.2. *The Jacobi ring of $(\check{\mathbb{X}}^\circ, W_q)$ is isomorphic to the quantum cohomology ring $qH^*(X,\mathbb{C})[q^{-1}]$ via an isomorphism which sends the coordinate q to the quantum parameter q and takes the form*

$$[p_\lambda] \mapsto \sigma^\lambda$$

on Plücker coordinates. This isomorphism agrees with the one obtained by combining the isomorphism between $(\check{\mathbb{X}}^\circ, W_q)$ and $(\mathcal{R}, \mathcal{F}_q)$, the isomorphism between the critical locus $\{\partial_{\mathcal{R}}\mathcal{F} = 0\}$ and the Peterson variety Y_P^*, and Peterson's isomorphism between the coordinate ring of the Peterson variety and the quantum cohomology ring $qH^*(X,\mathbb{C})[q^{-1}]$.

Proof. For $\lambda \in \mathcal{P}_{k,n}$ and $b = z\dot{w}_0^{-1} \in B_- \cap U_+\widetilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+$ recall that $P_\lambda(b)$ denotes the minor of b involving the last k rows and with column set given by J_λ. These minors are related to the Plücker coordinates on $\check{\mathbb{X}}^\circ$ as follows:

$$\frac{P_\lambda(b)}{P_\emptyset(b)} = p_\lambda(Pb),$$

keeping in mind our normalisation $p_\emptyset(Pb) = 1$.

The isomorphism between $\check{\mathbb{X}}^\circ$ and $\mathcal{R}$ sends Pb to $b\dot{w}_0B_-$. We want to relate $p_\lambda(Pb)$ to $f_\lambda(zB_-)$, whenever $b\dot{w}_0B_- = zB_-$ is in the Peterson variety Y_P^*.

Recall the involution $b \mapsto \mathrm{Invol}(b)$ from Proposition 8.6 and the related involution

$$\widetilde{\tau} : (Pz\dot{w}_0^{-1}, q) \to \mathrm{SignAct}(Pz^t\dot{w}_0^{-1}, (-1)^n q)$$

from Remark 8.9, where $z = b\dot{w}_0$ and $q = \mathbf{q}(b)$. We have that

$$p_\lambda \circ \widetilde{\tau}\ (Pb, q) = (-1)^{|\lambda|}\frac{P_\lambda(\mathrm{Invol}(b))}{P_\emptyset(\mathrm{Invol}(b))} = (-1)^{|\lambda|}\frac{P_\lambda(z^t\dot{w}_0^{-1})}{P_\emptyset(z^t\dot{w}_0^{-1})} = \frac{\Delta^{J_\emptyset^{\mathrm{opp}}}_{J_\lambda^{\mathrm{opp}}}(z^t)}{\Delta^{J_\emptyset^{\mathrm{opp}}}_{J_\emptyset^{\mathrm{opp}}}(z^t)} = \frac{\Delta^{J_\lambda^{\mathrm{opp}}}_{J_\emptyset^{\mathrm{opp}}}(z)}{\Delta^{J_\emptyset^{\mathrm{opp}}}_{J_\emptyset^{\mathrm{opp}}}(z)} = f_\lambda(zB_-).$$

Assume now that Pb is a critical point of W_q. In this case $\widetilde{\tau}(Pb, q) = (Pb, q)$, as follows from Remark 8.8. Therefore

$$p_\lambda(Pb) = f_\lambda(zB_-) \qquad \text{if } Pb \text{ is a critical point of } W_q \text{ for } q = \mathbf{q}(b).$$

Recall that if Pb is a critical point of W_q then $b\dot{w}_0B_-$ is in the Peterson variety Y_P^*. Therefore by Proposition 9.1, Peterson's isomorphism maps the function f_λ on the right hand side to the quantum Schubert class σ^λ. All in all we have shown that p_λ modulo $(\partial_{\check{\mathbb{X}}^\circ}W)$ maps to f_λ which maps to σ^λ under the relevant isomorphisms. Note that all the isomorphisms also preserve q, therefore the proof is complete. □

9.2. **The free basis lemma.** Recall Definitions 3.1 and 3.2 of the Gauss-Manin system G^{W_q} and its submodule H_B. Now that we are finished comparing the holomorphic volume forms we simply write ω for $\omega_{\check{\mathbb{X}}^\circ}$. In this section we prove the following result and, as a consequence, we show Lemma 3.3.

Lemma 9.3. *$H_{B,0}$ is a free $\mathbb{C}[z,q]$-module with basis $\{[p_\lambda\omega], \lambda \in \mathcal{P}_{k,n}\}$ and*

$$H_B = H_{B,0} \otimes_{\mathbb{C}[z,q]} \mathbb{C}[z^{\pm1}, q^{\pm1}].$$

In particular H_B is a free $\mathbb{C}[z^{\pm1}, q^{\pm1}]$-module with basis $\{[p_\lambda\omega], \lambda \in \mathcal{P}_{k,n}\}$.

Proof. It suffices to show that the elements $[p_\lambda\omega]$ in G^{W_q} are linearly independent over $\mathbb{C}[z, q^{\pm1}]$. To prove this we show first the following claim.

Claim: Suppose we have a relation $\sum_\lambda c_\lambda(z,q,q^{-1})\,[p_\lambda\omega] = 0$ in G^{W_q}. Then for any point $x \in \check{\mathbb{X}}^\circ$ there is a form $\nu \in \Omega^{N-1}(\mathcal{U})[q,q^{-1}]$ defined locally around x such that

$$\sum_\lambda c_\lambda(0,q,q^{-1})\,p_\lambda\omega = dW_q \wedge \nu.$$

Proof of the Claim: The assumption says that in $\Omega^N(\check{\mathbb{X}}^\circ)[z,q^{\pm1}]$ we have

$$\sum_\lambda c_\lambda(z,q,q^{-1})\,p_\lambda\omega = d\eta + \frac{1}{z}dW_q \wedge \eta \tag{9.1}$$

where $\eta \in \Omega^{N-1}(\check{\mathbb{X}}^\circ)[z^{\pm 1}, q^{\pm 1}]$. Let us write

$$\eta = \eta^{(<-1)} + \frac{1}{z}\eta^{[-1]} + \eta^{[0]} + z\eta^{[1]} + \eta^{(>1)}$$

where $\eta^{(<-1)} \in z^{-2}\Omega^{N-1}(\check{\mathbb{X}}^\circ)[z^{-1}, q^{\pm 1}]$ and $\eta^{(>1)} \in z^2\Omega^{N-1}(\check{\mathbb{X}}^\circ)[z, q^{\pm 1}]$, while $\eta^{[-1]}, \eta^{[0]}, \eta^{[1]} \in \Omega^{N-1}(\check{\mathbb{X}}^\circ)[q^{\pm 1}]$. By Equation (9.1) we have that $d\eta + \frac{1}{z}dW_q \wedge \eta \in \Omega^N(\check{\mathbb{X}}^\circ)[z, q^{\pm 1}]$. This implies firstly that

$$d\eta^{(<-1)} + \frac{1}{z}dW_q \wedge \left(\frac{1}{z}\eta^{[-1]} + \eta^{(<-1)}\right) = 0,$$

and secondly that

$$d\eta^{[-1]} + dW_q \wedge \eta^{[0]} = 0,$$

from the z^{-1}-component of (9.1). We record that taking the exterior derivative of the second equality we obtain

$$dW_q \wedge d\eta^{[0]} = 0. \tag{9.2}$$

Now setting z to zero in the non-zero terms of $d\eta + \frac{1}{z}dW_q \wedge \eta$ and rewriting equation (9.1) we obtain,

$$\sum_\lambda c_\lambda(0, q, q^{-1})\, p_\lambda \omega = d\eta^{[0]} + dW_q \wedge \eta^{[1]}.$$

Following [38, Example 2.32], a local argument using the fact that W_q has isolated critical points shows that the sequence of sheaves

$$\Omega^{N-2}_{\check{\mathbb{X}}^\circ} \xrightarrow{dW_q\wedge} \ker\left(\Omega^{N-1}_{\check{\mathbb{X}}^\circ} \xrightarrow{dW_q\wedge} \Omega^N_{\check{\mathbb{X}}^\circ}\right) \xrightarrow{dW_q\wedge} 0$$

is exact. Since $d\eta^{[0]}$ is in the kernel above, by (9.2), we see that there exists locally an $(N-2)$-form ε such that $d\eta^{[0]} = dW_q \wedge \varepsilon$. As a result we obtain the local equation

$$\sum_\lambda c_\lambda(0, q, q^{-1})\, p_\lambda \omega = dW_q \wedge \varepsilon + dW_q \wedge \eta^{[1]} = dW_q \wedge (\eta^{[1]} + \varepsilon).$$

Setting $\nu = \eta^{[1]} + \varepsilon$ this proves the Claim.

We can now finish the proof of the lemma. Suppose that $\sum_\lambda c_\lambda(z, q, q^{-1})[p_\lambda \omega] = 0$. Then by the Claim we have $\sum_\lambda c_\lambda(0, q, q^{-1}) p_\lambda \omega = dW_q \wedge \nu$ locally. Since ω is non-vanishing on $\check{\mathbb{X}}^\circ$ this implies that $\sum_\lambda c_\lambda(0, q, q^{-1}) p_\lambda$ vanishes on the critical points of W_q. Therefore it lies in the ideal $(\partial_{\check{\mathbb{X}}^\circ} W_q)$ in $\mathbb{C}[\check{\mathbb{X}}^\circ][q^{\pm 1}]$ and we have proved that $\sum_\lambda c_\lambda(0, q, q^{-1}) p_\lambda = 0$ in the Jacobi ring of W_q. By Proposition 9.2 this implies the relation $\sum_\lambda c_\lambda(q)\, \sigma^\lambda = 0$ in the quantum cohomology ring $qH^*(X, \mathbb{C})[q^{-1}]$ of X. Since the Schubert classes are linearly independent over $\mathbb{C}[q^{\pm 1}]$ it follows that $c_\lambda(0, q, q^{-1}) = 0$ for all λ.

We have therefore proved that the first term of $c_\lambda(z, q, q^{-1}) = c_\lambda(0, q, q^{-1}) + \sum_{i>0} z^i c_\lambda^{(i)}(q, q^{-1})$ vanishes. Now assume we know that $c_\lambda^{(i)}(q, q^{-1}) = 0$ for all $i < i_0$ and all λ. Then since z is a non-zero-divisor in H_B we can apply the same arguments to

$$z^{-i_0} \sum_\lambda c_\lambda(z, q, q^{-1})[p_\lambda \omega] = 0,$$

which is another a relation in $\Omega^N(\check{\mathbb{X}}^\circ)[z, q^{\pm 1}]$. Again it follows that the left hand side vanishes after setting $z = 0$, giving $c_\lambda^{(i_0)}(q, q^{-1}) = 0$. Therefore by induction all terms vanish and $c_\lambda(z, q, q^{-1}) = 0$ for every λ. It follows that the $[p_\lambda \omega]$ are linearly independent over $\mathbb{C}[z, q^{-1}, q]$.

□

10. Outline of the proof of Theorem 4.1

The remainder of the paper will be devoted to proving Theorem 4.1 and its equivariant counterpart. Here we summarize the proof. We keep all of the notation of the previous sections. We need to show that the following hold for all $\lambda \in \mathcal{P}_{k,n}$:

$$\left[q\frac{\partial W}{\partial q}p_\lambda\omega\right] = \sum_\mu [p_\mu\omega] + q\sum_\nu [p_\nu\omega]; \tag{10.1}$$

$$\frac{1}{z}\left[Wp_\lambda\omega\right] = \frac{n}{z}\left(\sum_\mu [p_\mu\omega] + q\sum_\nu [p_\nu\omega]\right) - |\lambda|[p_\lambda\omega], \tag{10.2}$$

where μ, ν are exactly as in the quantum Monk's rule for $\sigma^\square *_q \sigma^\lambda$. This will be shown in Theorem 18.5.

In Section 7, we have recalled the cluster structure on the Grassmannian, following Scott [84], in terms of Postnikov diagrams. Next, in Section 11, for any partition λ, we construct an extended cluster containing p_λ and also p_μ and p_ν for all μ and ν appearing in the quantum Monk's rule for $\sigma^\square *_q \sigma^\lambda$. This extended cluster corresponds to a Postnikov diagram D_λ with good properties; in particular, strands i and $i+1$ cross at a single point in the diagram.

In order to work with W in this extended cluster we use Theorem 12.3, a special case of [60, Thm. 1.1]. In Section 12, we recall this result in the special case we will need. This gives us an expansion of the numerators $p_{\widehat{\mu}_i}$ occurring in the definition of W_q, in terms of an extended cluster arising from an arbitrary Postnikov diagram, D. Namely, the expansion of $p_{\widehat{\mu}_i}$ is given as a sum of Laurent monomials, with the set of terms in bijection with the perfect matchings on a bipartite graph G_i related to the Postnikov diagram D.

In Sections 13 and 14 we analyse the perfect matchings on the graphs G_i, by constructing a natural initial perfect matching, M_i, and showing that all other perfect matchings can be obtained from M_i by face flips, starting with a face adjacent to the crossing point of i and $i+1$. This involves calculating the elementary components of G_i. In Section 15 we compute the matching monomial corresponding to M_i.

If ξ is a regular vector field on $\check{\mathbb{X}}^\circ$ then it can be inserted into ω to give an $(n-1)$-form $i_\xi\omega$. By the definition of G^W, we have the relation:

$$[di_\xi\omega] + \frac{1}{z}[(\xi W_q)\omega] = 0 \tag{10.3}$$

(see equation 18.3). The proof then proceeds by constructing explicit vector fields for which the relation (10.3) implies equations (10.1) and (10.2).

So, in Section 16 we use the cluster structure to define a regular vector field X_λ on $\check{\mathbb{X}}^\circ$, together with twisted versions $X_\lambda^{(m)}$, $m \in [1,n]$ (satisfying $X_\lambda^{(n)} = X_\lambda$). To obtain the desired relations, we need to compute $X_\lambda^{(m)}W_q$. We compute the action of $X_\lambda^{(m)}$ on each of the monomials in the expansion of W_q in terms of the extended cluster associated to the Postnikov diagram D_λ. This is done by first computing the action on the monomial associated to M_i. Then in Section 17 we use face flips starting from M_i to compute the action of $X_\lambda^{(m)}$ on an arbitrary monomial in the expansion of W_q. We obtain the expression:

$$X_\lambda^{(m)}W = \left(\sum_\mu p_\mu + q\sum_\nu p_\nu\right) - q^{\delta_{mn}}\frac{p_{\widehat{L}_m}}{p_{L_m}}p_\lambda, \tag{10.4}$$

where μ, ν are exactly as in the quantum Monk's rule for $\sigma^\square *_q \sigma^\lambda$ (Theorem 17.3). Recall that $L_i = [i-k+1, i]$ (see (6.1) in Section 6.1).

The case $m = n$ gives:

$$X_\lambda W = \left(\sum_\mu p_\mu + q\sum_\nu p_\nu\right) - q\frac{\partial W}{\partial q}p_\lambda, \tag{10.5}$$

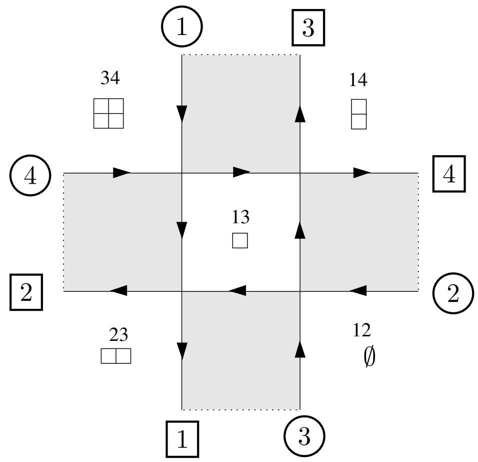


FIGURE 8. The Postnikov diagram $D(\{1,3\})$ for $Gr_2(4)$.

and summing (10.4) over $m = 1, 2, \ldots, n$, we obtain:

$$\sum_{m=1}^{n} X_\lambda^{(m)} W = n\left(\sum_{\mu} p_\mu + q \sum_{\nu} p_\nu\right) - W p_\lambda, \tag{10.6}$$

where, in each case, μ, ν are exactly as in the quantum Monk's rule for $\sigma^{\square} *_q \sigma^{\lambda}$; see Corollary 18.1. In Section 18, we use these identities and equation (10.3) to complete the proof of Theorem 4.1. See Theorem 18.5.

11. A special Postnikov diagram associated to a partition λ

Given $\lambda \in \mathcal{P}_{k,n}$, we denote by $\lambda^{\square}$ any partition obtained from λ by adding a single box. Our aim in this section is, given $\lambda \in \mathcal{P}_{k,n}$, to define a Postnikov diagram containing J_λ and all $J_{\lambda^\square}$ as labels. Moreover, the faces labelled $J_{\lambda^\square}$ should be adjacent to the face labelled J_λ. This diagram will be used later in explicitly computing the action of X_λ on W_q. We start by constructing special, symmetric Postnikov diagrams in the case $n = 2k$ and $J_\lambda = \{1, 3, \ldots, 2k-1\}$; the diagrams for arbitrary (k, n) can then be obtained by adding strands to these in an appropriate way. We assume that $k \neq 1, n-1$.

If J is any k-subset of $[1, n]$ and $1 \leq i \leq n$ with $i \in J$, $i+1 \notin J$, then we can form a k-subset J^i in which i is replaced by $i+1$. In this case we say that i is (clockwise) *moveable* in J. Note that $J = L_i$ for some i if and only if exactly one element of J is moveable. Therefore any J for which p_J is a cluster variable (and not a frozen variable) has at least two moveable elements.

Our aim in this section is to prove the following theorem.

Theorem 11.1. *Let J be a k-subset of $[1, n]$. Then there is a Postnikov diagram $D(J)$ containing an alternating face labelled J such that if i is moveable in J, there is an adjacent alternating face labelled J^i.*

For examples in the cases $J = \{1, 3\}$ in $Gr_2(4)$, $J = \{1, 3, 5\}$ in $Gr_3(6)$ and $J = \{1, 3, 5, 7\}$ in $Gr_4(8)$, see Figures 8, 4 and 9 respectively. We assume for now that $J \neq L_j$ for any j; we will deal with the case $J = L_j$ at the end of the Section.

Our strategy for proving Theorem 11.1 is as follows. Let $K = \{i \in J : i+1 \notin J\}$ be the set of moveable elements in J; then $|K| \geq 2$ by the assumption above on J. For arbitrary $k \geq 2$ and $n = 2k$, we construct an explicit diagram (Proposition 11.2) demonstrating Theorem 11.1 for the case $J = \{1, 3, \ldots, 2k-1\}$. We then show that, given a diagram satisfying Theorem 11.1, it is possible to add a new strand to it around the boundary (clockwise or anticlockwise) to obtain another such diagram (Propositions 11.3 and 11.4). Adding a clockwise strand corresponds to increasing n by 1, while adding an anticlockwise strand corresponds to increasing k by 1.

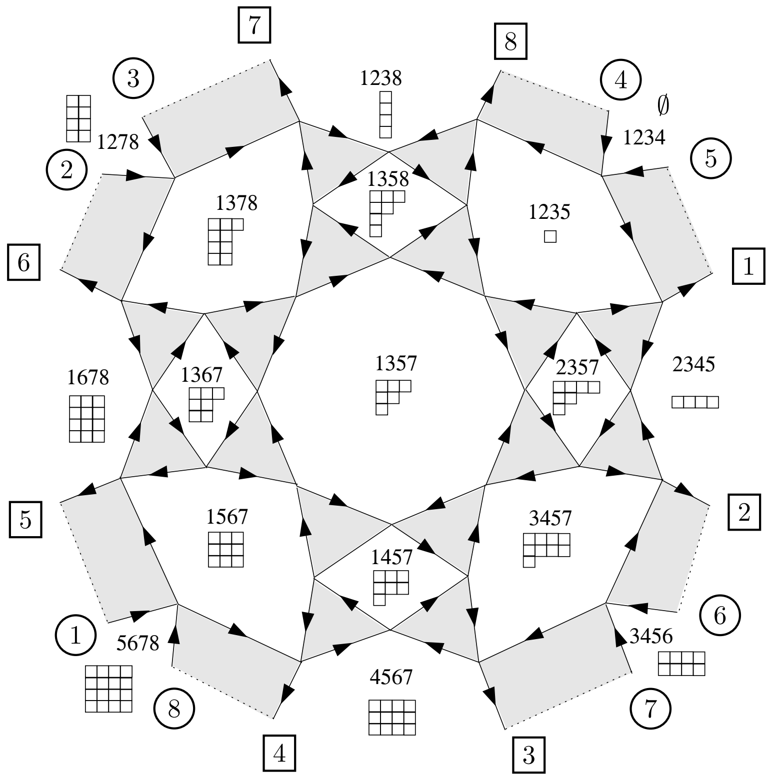


FIGURE 9. The Postnikov diagram $D(\{1,3,5,7\})$ for $Gr_4(8)$.

Proposition 11.2. *Suppose $k \geq 2$ and $n = 2k$. Set $J = \{1, 3, \ldots, 2k-1\}$. Then Theorem 11.1 holds for J.*

Proof. For $k = 2, 3, 4$ it is easy to verify that the diagrams in Figures 8, 4 and 9 are Postnikov diagrams and that they show that the theorem holds in this case. We now show how to construct such diagrams for arbitrary $k \geq 4$.

Consider a tiling $\mathcal{T}$ of the plane by regular hexagons and equilateral triangles in which there are two triangles and two hexagons incident with each vertex (in the order hexagon, triangle, hexagon, triangle) and for which each of the hexagons has a vertical pair of parallel sides. We convert this into a tiling of the upper half-plane by cutting along a horizontal line through the lowest points of a row of hexagons. The faces in the new tiling incident with this line are hexagons, triangles and quadrilaterals, repeating in this order left to right.

We number the edges on the boundary in the order $\ldots -2, -3, 0, -1, 2, 1, 4, 3, 6, 5, \ldots$ from left to right (i.e. switching each pair of integers in which the first is odd in the natural ordering on $\mathbb{Z}$). We orient an edge left to right if it has an even label, and right to left if it has an odd label. This determines orientations for the adjacent triangles. We label all the vertical edges in the tiling upwards and all the remaining edges downwards. This gives an orientation of every edge in the tiling. See Figure 10 for an example. The tiling can be seen as a collection of infinite strands, each of which has precisely one edge incident with the boundary and thus inherits a label from this edge.

Consider the horizontal line in the plane which passes through the vertices at the tops of the triangles above the $(k-3)$rd row of hexagons. Let S be the subset of the plane bounded by the boundary of the half-plane and this horizontal line. Consider also the vertical lines in the strip S bisecting edges -1 and $2k-1$. We identify these to obtain a cylinder. The tiling of the half-plane induces a tiling of this cylinder, with the strands in the half-plane tiling corresponding to $2k$ strands in the cylinder labelled by $1, 2, \ldots, 2k$.

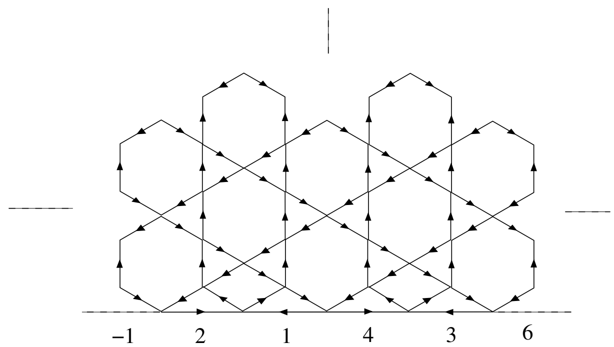


FIGURE 10. A tiling of the upper half-plane.

Since the cylinder is homeomorphic to an annulus, we obtain a corresponding tiling of an annulus, in which the lower boundary of the cylinder corresponds to the inner boundary of the annulus. We glue a disk onto the inner boundary, obtaining a diagram Γ_k on a disk with $2k$ vertices on its boundary. We label the vertex where strand j starts by j. It is clear that parts (a) and (b) in the definition of a Postnikov diagram hold for Γ_k.

Furthermore, it is easy to see inductively that the vertices $1, 2, \ldots, 2k$ are arranged clockwise around the boundary and that if i is even (respectively, odd), strand i goes clockwise (respectively, anticlockwise) around the disk from vertex i to vertex $i+k$. Hence property (d) in the definition of a Postnikov diagram holds.

For each i, we replace vertex i with a pair of vertices b_i and b'_i, with b'_i clockwise of b_i. We move the start of strand i to one of b_i and b'_i and the end of strand $i-k$ to the other in such a way that we don't introduce any additional crossings. Then the odd strands start at b_i and the even strands start at b'_i. We call the resulting diagram $D_{k,2k}$. It is easy to see that it satisfies part (c) in the definition of a Postnikov diagram (and also parts (a), (b) and (d) by the above).

It is clear from the construction that strands $j, j+k$ do not meet, while all other pairs of strands which are both even or both odd meet exactly once. A pair of strands of mixed parity may meet several times, but since one goes clockwise around the disk and the other goes anticlockwise, they satisfy the requirements of part (e) in the definition of a Postnikov diagram. Hence we see that this condition holds for any pair of strands. We have shown that D_k is a Postnikov diagram as required.

The label of the inserted disk is easily seen to be $J = \{1, 3, \ldots, 2k-1\}$ and the labels of the k quadrilaterals sharing a vertex with it are exactly the J^i for i moveable in J. The proof of the proposition is complete. $\square$

Note that, for $k = 4$, the diagram in Figure 9 is equivalent to the diagram we have constructed here using the tiling. Furthermore, the diagrams in 8 and 4 can also be constructed from the tiling by taking appropriate subsets of the tiling near the half-plane boundary. We also remark that none of the faces constructed in Proposition 11.2 (i.e. giving the required labels) is a boundary face.

We next consider the addition of a clockwise strand.

Proposition 11.3. *Let Γ be a Postnikov diagram for $Gr_k(n)$. Then a strand s' can be added to Γ to produce a Postnikov diagram on strands $[1, n] \cup \{s'\}$ (with ordering $1, 2, \ldots, s, s', \ldots, n$) for $Gr_k(n+1)$ in such a way that all labels of non-boundary faces remain after adding the strand.*

Proof. By applying the boundary twist if necessary, we ensure that the oriented regions adjacent to the vertices $b_{s+1}, b'_{s+1}, b_{s+2}, b'_{s+2}, \ldots, b'_{s+k}$ are all oriented clockwise, as shown in Figure 11. We add an extra strand labelled s' with starting point between b_{s+1} and b'_{s+1}. The ending point of s' is placed between b_{s+k} and b'_{s+k}. The strand s' crosses all the strands between these points close to the boundary (i.e. with no additional crossings between it and the boundary). See Figure 12.

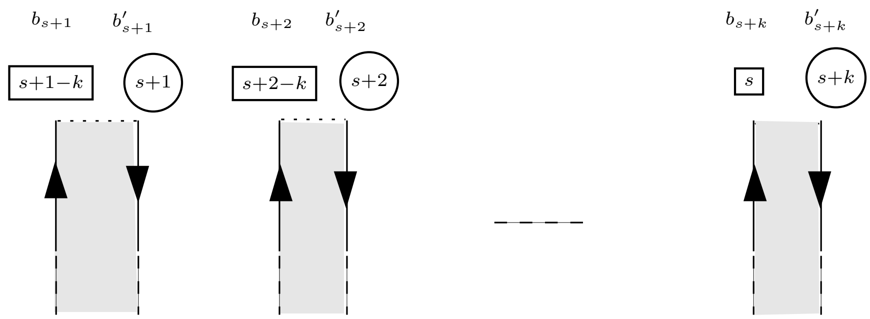


FIGURE 11. Adding a clockwise strand: before.

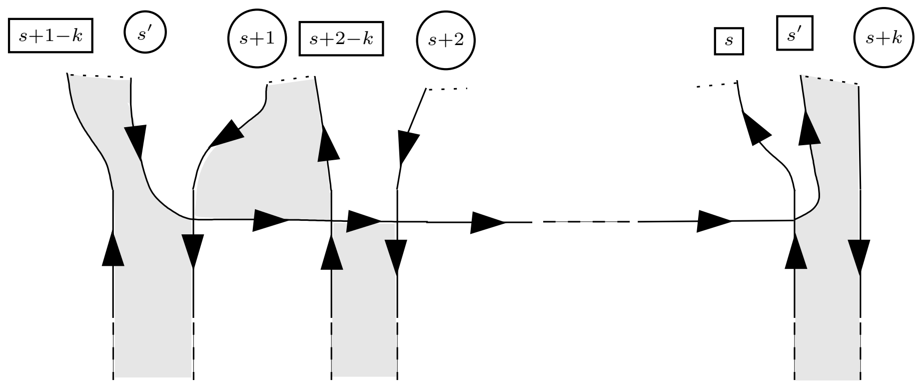


FIGURE 12. Adding a clockwise strand: after.

The new diagram is again a Postnikov diagram. In particular, for (c), note that the crossing nearest to the starting point of strand $s+1$ must cross it from left (before adding the new strand), since $s+1$ starts at b'_{s+1}. After adding the new strand, the starting point of $s+1$ is b_{s+1} and the crossing nearest the starting point comes from the right, as required. The next crossing point on strand $s+1$ is the original first crossing, which is from the left, so the alternation rule, property (c) in Definition 7.2, does not fail. Similar arguments apply to the other strands.

We also note that, relabelling $1, 2, \dots, s, s', \dots, n$ with $1, 2, \dots, n+1$, strand i ends at b_{i+k} or b'_{i+k} for all i, in the new diagram.

Since only boundary alternating faces are on the left of the added strand, s', all of the labels of the non-boundary alternating faces of the original diagram will be labels of non-boundary alternating faces of the new diagram, as required. □

For an example, see Figure 13. Here we add a strand $3'$ to the diagram in Figure 4. Note that we have to first apply the boundary twist to the two strings incident with b_5 and b'_5. We obtain a Postnikov diagram for $Gr_{3,7}$ (with the numbering $1, 2, 3, 3', 4, 5, 6$).

Secondly, we consider adding an anticlockwise strand.

Proposition 11.4. *Let Γ be a Postnikov diagram on strands $[1, n]$ for $Gr_k(n)$. Then a strand a' can be added to Γ to produce a Postnikov diagram on strands $[1, n] \cup \{s'\}$ (with ordering $1, \dots, s, s', \dots, n$) for $Gr_{k+1}(n+1)$ in such a way that, for all labels I of non-boundary alternating faces in the original diagram, $I \cup \{s'\}$ labels a non-boundary alternating face in the augmented diagram.*

Proof. By applying the boundary twist if necessary, we ensure that the oriented regions adjacent to the $b_s, b'_s, b_{s-1}, b'_{s-1} \dots, b'_{s-(n-k-1)} = b'_{s+k+1}$ are all oriented anticlockwise, as shown in Figure 14. We add an

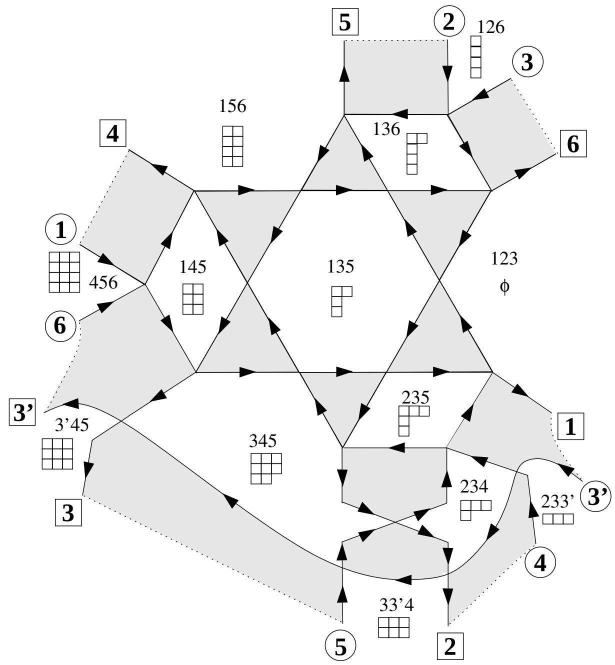


FIGURE 13. Example: adding a clockwise strand 3′.

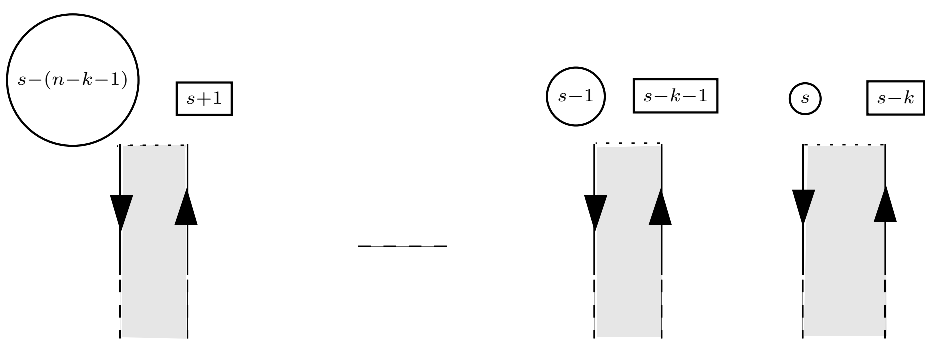


FIGURE 14. Adding an anticlockwise strand: before.

extra strand labelled s' with starting point between b_s and b'_s. The ending point of s' is placed between $b_{s-(n-k-1)}$ and $b'_{s-(n-k-1)}$. It crosses all the strands between these points close to the boundary (i.e. with no additional crossings between it and the boundary). See Figure 15.

It is easy to check that the new diagram is again a Postnikov diagram. The proof is as for Proposition 11.3. Since all non-boundary alternating faces in the original diagram are on the left of the added strand, s', the non-boundary labels of the new diagram are indeed precisely the sets $I \cup \{s'\}$ where I is a non-boundary label of the old diagram.

Finally, it is also easy to check that, relabelling $1, 2, \ldots, s, s', \ldots, n$ with $1, 2, \ldots, n+1$, strand i ends at b_{i+k} or b'_{i+k} for all i, in the new diagram. □

Proof of Theorem 11.1. Suppose we are given a k-subset $J \subseteq [1, n]$ with $J \neq L_j$ for any j. Let $K = \{i \in J : i+1 \notin J\}$ be the set of moveable elements in J; then $|K| \geq 2$. We start with the diagram $D_{|K|}$ as in Proposition 11.2. Let $K+1 = \{i+1 : i \in K\}$; note that K and $K+1$ are disjoint. Renumber the

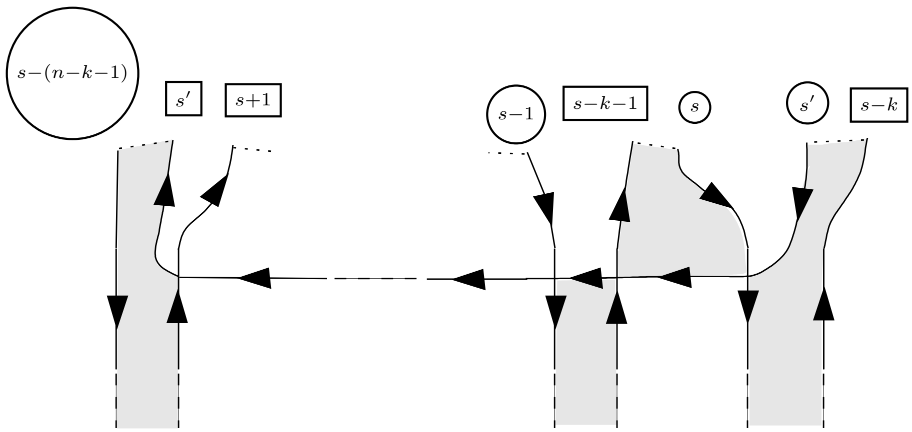


Figure 15. Adding an anticlockwise strand: after.

strands $\{1, 2, 3, \ldots, 2k\}$ with the elements of $K \cup (K+1)$ taken in cyclic order, starting with the smallest element of K. Then, for each $i \in [1, n] \setminus (J \cup (K+1))$ (in numerical order), use Proposition 11.3 to add a clockwise strand i in such a way that all labels of non-boundary faces remain the same after adding the strand. Finally, for each $i \in J \setminus K$ (again in numerical order), use Proposition 11.4 to add an anticlockwise strand i in such a way that for all labels I of non-boundary faces, the resulting diagram has a label of form $I \cup \{i\}$. The diagram constructed in this way satisfies the requirements of Theorem 11.1.

If $J = L_j$ for some j, then $K = \{j\}$. Suppose that j is even. Recall from the proof of Proposition 11.2 that in D_k, strand j goes clockwise around the disk while strand $j+1$ goes anticlockwise around the disk. Hence these two strands must cross on the boundary of L_j and, in fact, this must be the only crossing on the boundary. If j is odd, we may rotate D_k to obtain a diagram with the same property; this shows the result for $Gr(k, 2k)$. A diagram for arbitrary n can then be obtained by adding strands using Propositions 11.3 and 11.4. □

Definition 11.5. If $\lambda \in \mathcal{P}_{k,n}$ corresponds to a k-subset $J = J_\lambda$ of $[1, n]$, we write D_λ for $D(J)$. We denote the face of D_λ labelled J_λ by $F(\lambda)$.

Definition 11.6. For $m \in [1, n]$, let $\mu^{(m)}$ denote the partition defined by

$$J_{\mu^{(m)}} := J_\mu - m \mod n,$$

where $J_\mu - m$ denotes the set obtained from J_μ by subtracting m from every element. For example for the empty partition $\emptyset$, we have $J_\emptyset = \{1, 2, \ldots, k\}$ and $\emptyset^{(m)} = \mu_{n-m}$ with associated subset J_{n-m}, as defined in Section 6.1. Note that $\mu \mapsto \mu^{(m)}$ gives an action of the cyclic group of degree n on the set $\mathcal{P}_{k,n}$.

Remark 11.7. We can arrange that, for each partition $\lambda \in \mathcal{P}_{k,n}$, the Postnikov diagram $D_{\lambda^{(m)}}$ can be obtained from D_λ by relabelling strand i as strand $i - m$ for all i. We follow the proof of Theorem 11.1 as given for a single representative λ in each orbit. Then we define $D_{\lambda^{(m)}}$ to be D_λ with strand i relabelled as $i - m$ for all i. It is easy to check that $D_{\lambda^{(m)}}$ can then still be constructed as in the proof of Theorem 11.1: this holds by construction for $Gr(k, 2k)$. In general, suppose that in the construction, D_λ is obtained from a diagram D_0 for $Gr(k, 2k)$ by adding strands $i_1, i_2, \ldots, i_r$ in order. Then we construct $D_{\lambda^{(m)}}$ from D_0 (with strand i relabelled as $i - m$) by adding strands $i_1 - m, i_2 - m, \ldots, i_r - m$ in order. We shall assume that the construction has been done in this way throughout the rest of the paper.

12. The superpotential written in terms of an arbitrary Plücker extended cluster

Restricting the regular function W_q on $\check{\mathbb{X}}^\circ$ to a cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$ gives a Laurent polynomial in the associated cluster variables. We will need an explicit formula for this in the case where $\widetilde{\mathcal{C}}$ is a Postnikov extended cluster associated to a Postnikov diagram D. To obtain such a formula, we shall use [60, Thm. 1.1] which expresses a twisted version of an arbitrary Plücker coordinate p_μ in terms of an arbitrary Postnikov extended cluster. The Laurent polynomial expansion of $p_\mu|_{\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}}$ is given in terms of perfect matchings on a bipartite graph associated to the Plücker coordinate p_μ and the Postnikov diagram D. The Plücker coordinates appearing in the numerators in the definition of W_q (see (6.4)) are themselves twists of Plücker coordinates, by [60, Prop. 3.5] (up to frozen variables), so it follows that [60, Thm. 1.1] gives an expression for these Plücker coordinates themselves. We now go into more detail.

Let us label the frozen variables by $L_i = \{i-k+1,\dots,i\} = J_{i-k}$ and recall from (6.4) the formula for W_q, which with this notation reads

$$W_q = \sum_{i=1}^{n} \frac{p_{\widehat{L}_i}}{p_{L_i}}\, q^{\delta_{i,n}}. \tag{12.1}$$

We will now explain how to express the $p_{\widehat{L}_i}$ in terms of an arbitrary Plücker extended cluster, by an application of [60, Thm. 1.1].

Definition 12.1. Let $\widetilde{\mathcal{C}}$ be an arbitrary Postnikov extended cluster and D its Postnikov diagram. Fix $1 \le i \le n$ and set

$$N_i = \{i\} \cup [i+2, i+k].$$

Let D_i be the Postnikov diagram obtained from D by applying boundary twists (see Figure 3) where necessary to ensure that the oriented boundary region adjacent to b_j, b'_j is anticlockwise for all $j \in N_i$ and clockwise otherwise. If $k=1$ or $k=n-1$, we take D to be the Postnikov diagram described in Remark 7.8 (see Figure 7). We have $N_i = \{i\}$, so, as in the other cases, to get D_i we add a boundary twist between the end of strand $i-1$ and the start of strand i on the boundary. We also add a double boundary twist between strands $j-1$ and j for all $j \neq i$ to avoid degeneracies.

Figure 16 shows the Postnikov diagram D_3 constructed from the diagram D in Figure 4. Note that $N_3 = \{3,5,6\}$, so in D_3, the oriented regions adjacent to the starting points of strands $1, 2$ and 4 are black and $3, 5$ and 6 are white.

Definition 12.2. We denote the quiver of D_i by Q_i. We also associate a bipartite graph G_i to D_i as follows [70, §14]. Note that any intersection point of two strands is surrounded by four distinct faces, two of which are oriented faces. Let G_i be the graph with vertices corresponding to the oriented faces of D_i and edges determined by the intersection points of strands. The vertex in G_i corresponding to the oriented boundary face adjacent to b_j, b'_j is labelled v_j, for $j \in [1,n]$. Marking a vertex black if corresponds to a clockwise face and white if corresponds to an anticlockwise face makes G_i a bipartite graph. Thus the vertices v_j for $j \in N_i$ are white and the other vertices are black.

The graph G_i can be regarded as being naturally embedded in the disk, with the vertex corresponding to an oriented face of D_i plotted at an interior point in the middle of the face. If there is an edge between two vertices of G_i, it is drawn to pass through the corresponding strand intersection point mentioned above. The faces of G_i are in bijection with the alternating faces of D_i, and hence with the Plücker coordinates in the Postnikov extended cluster $\widetilde{\mathcal{C}}$. We refer to G_i as the *dual bipartite graph* of D_i. See Figure 16, where the dual bipartite graph G_3 of D_3 and the quiver Q_3 have been superimposed on the same diagram.

We assign monomial weights $w_e \in \mathbb{C}[\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}]$ to the edges of G_i as follows. Let v be the unique black vertex incident with an edge e. The weight w_e of e is defined to be the product of the Plücker coordinates labelling the faces of G_i which are incident with v but not with the rest of e (i.e. excluding the two faces on each side of e). See Figure 17 for an illustration of the rule. Figure 18 shows the weighting on the dual bipartite graph of Figure 16.

Recall that a *perfect matching* of a graph Γ with edge-set E is a subset M of E such that each vertex of Γ is incident with precisely one edge in M. If Γ is weighted with weighting w_e for each edge e, its *matching polynomial* is given by:

$$w_\Gamma = \sum_M \prod_{e \in M} w_e,$$

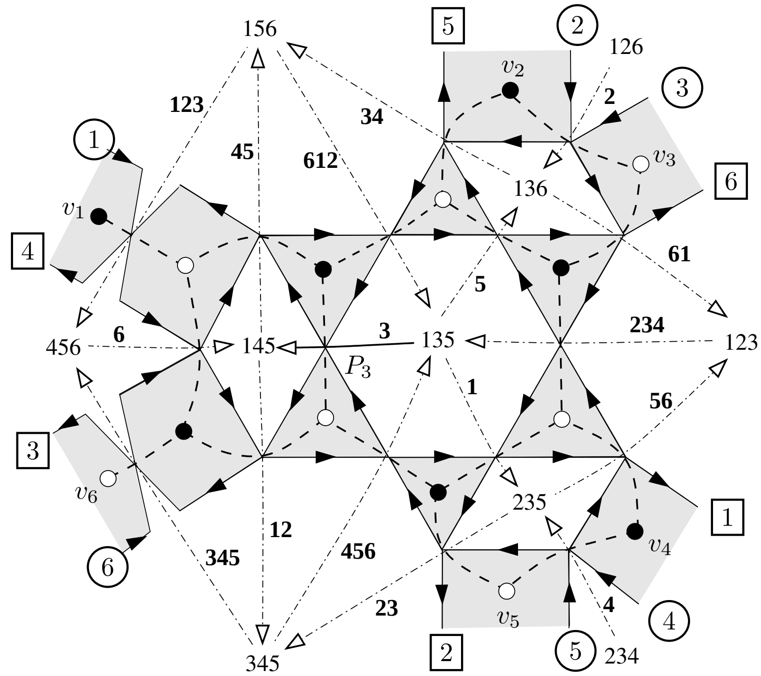


FIGURE 16. The Postnikov diagram D_3 constructed from the diagram in Figure 4, together with its bipartite dual, G_3 (equal dashed lines between coloured vertices) and the quiver Q_3 (uneven dashed arrows). Note that $N_3 = \{3, 5, 6\}$. For an explanation of the notation P_3, see the comment after Lemma 13.1.

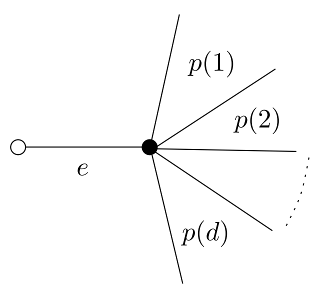


FIGURE 17. Weighting of an edge: $w_e = p(1)p(2)\cdots p(d)$.

where the sum is over all perfect matchings of Γ. We will sometimes write $w_M = \prod_{e\in M} w_e$.

We then have the following theorem, which is the special case of a more general result, [60, Thm. 1.1].

Theorem 12.3. [60]
Let D be a Postnikov diagram, with corresponding Postnikov cluster $\mathcal{C} = \mathcal{C}(D)$ and extended cluster $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}(D)$. Fix $1 \le i \le n$. Let G_i be the bipartite graph defined above. Then the following holds in $\mathbb{C}[\check{\mathbb{X}}^{\circ}_{\widetilde{\mathcal{C}}}]$:

$$p_{\widehat{L}_i} = \frac{w_{G_i}}{\prod_{p\in\mathcal{C}} p} p_{L_{i-1}} p_{L_i} \cdots p_{L_{i+k}} = \sum_M \frac{w_M}{\prod_{p\in\mathcal{C}} p} p_{L_{i-1}} p_{L_i} \cdots p_{L_{i+k}},$$

where the sum is over all perfect matchings M of G_i.

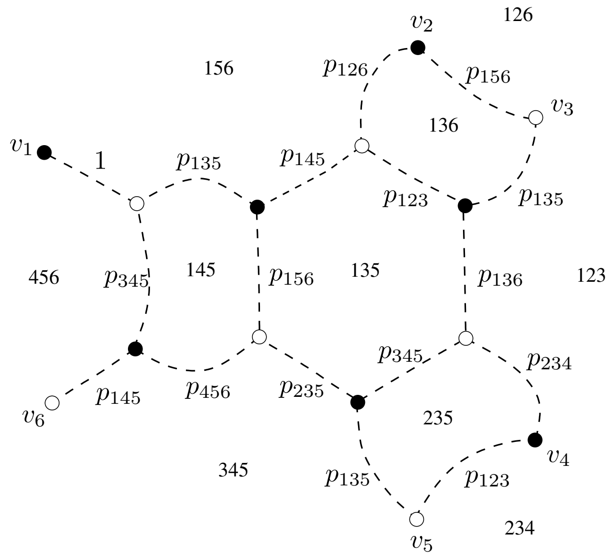


FIGURE 18. The weighting on the dual bipartite graph G_3 in Figure 16.

Recall that the superpotential W_q is given by the formula:

$$W_q = \sum_{i=1}^{n} \frac{p_{\widehat{L}_i}}{p_{L_i}} q^{\delta_{i,n}}.$$

Theorem 12.3 states that

$$\frac{p_{\widehat{L}_i}}{p_{L_i}} = \sum_M \frac{w_M}{\prod_{p\in\mathcal{C}} p} p_{L_{i-1}} p_{L_{i+1}} p_{L_{i+2}} \cdots p_{L_{i+k}}. \tag{12.2}$$

We thus have the following:

Corollary 12.4. *Let D be a Postnikov diagram, with corresponding Postnikov cluster $\mathcal{C} = \mathcal{C}(D)$ and extended cluster $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}(D)$. Let $G_1, G_2, \ldots, G_n$ be the bipartite graphs associated above to D. Then we have the following expression for the superpotential W_q (see equation* (12.1)*) in $\mathbb{C}[\check{\mathbb{X}}^{\circ}_{\widetilde{\mathcal{C}}}]$:*

$$W_q = \frac{1}{\prod_{p\in\mathcal{C}} p} \sum_{i=1}^{n} w_{G_i} p_{L_{i-1}} p_{L_{i+1}} \cdots p_{L_{i+k}} q^{\delta_{in}}. \tag{12.3}$$

13. CONSTRUCTION OF A PERFECT MATCHING

Suppose $\lambda \in \mathcal{P}_{k,n}$ and let $D = D_\lambda$ be the Postnikov diagram constructed by Theorem 11.1. For example, Figure 4 shows the Postnikov diagram $D_{\boxplus}$ for $G(3,6)$, noting that $J_{\boxplus} = \{1,3,5\}$. Let $D_1, D_2, \ldots, D_n$ be the associated boundary-adjusted Postnikov diagrams and $G_1, G_2, \ldots, G_n$ be the corresponding dual bipartite graphs, associated to D in Section 12. We also have corresponding quivers $Q_1, Q_2, \ldots, Q_n$ (see Definitions 12.1 and 12.2). For example, Figure 16 shows D_3, G_3 and Q_3 for the case $D_{\boxplus}$.

Our aim in this section is to construct an explicit perfect matching M_i on each of the G_i. This perfect matching will correspond to a distinguished monomial summand in $p_{\widehat{L}_i}/p_{L_i}$. In Section 14 we will show that every other perfect matching can be obtained from M_i by face flips and in Section 15 we will compute the distinguished monomial summand explicitly. We assume that $k \neq 1, n-1$. We first make the following observation.

Lemma 13.1. *Let $\lambda \in \mathcal{P}_{k,n}$. Then strands $i, i+1$ cross at exactly one point in D_λ.*

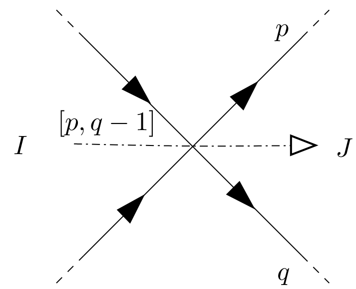


FIGURE 19. The labelling of arrows in $Q(D)$. Strands p and q cross the arrow from I to J, which is given the label $[p, q-1]$.

Proof. Suppose first that $J_\lambda \neq L_j$ for any j. For the case $J_\lambda = \{1, 3, 5, \ldots, 2k-1\}$ in $Gr(k, 2k)$, the result can be observed from the construction of D_λ in Proposition 11.2. Note that, for i even, the intersection point of strands i and $i+1$ is the first intersection point for each of these strands, while for i odd, the intersection point is on the boundary of the central n-sided alternating face in D_λ.

Adding strands does not change this property, since every new strand crosses the existing strands at most once. Because strand i starts at b_i or b'_i and ends at b_{i+k} or b'_{i+k}, every pair of strands $i, i+1$ must cross at least once. The argument for $J_\lambda = L_j$ is similar. $\square$

We define P_i to be the unique crossing point of strands i and $i+1$ in D_λ (and also the corresponding point in D_i). The point P_3 is shown in Figure 16. We note that in the case $k = 1$ or $n-1$, with our convention (see Remark 7.8), Lemma 13.1 does not hold.

Remark 13.2. Perfect matchings for G_i also have an interpretation in terms of the quiver Q_i. This interpretation will be useful for our construction. The quiver Q_i is embedded into a disk in such a way that its complement forms a disjoint union of disks. The boundary of such a disk is either a cycle in Q_i, which we call a *minimal cycle*, or an oriented path in Q_i, together with part of the boundary of the whole disk $\mathbb{D}$. In the latter case, the oriented path goes from a boundary vertex to an adjacent one and is called a *boundary path*.

If E is a set of edges in G_i, we denote by $\Sigma(E)$ the set of arrows $\Sigma(E)$ in Q_i crossing the edges in E. Then E is a perfect matching in G_i if and only if $\Sigma(E)$ contains exactly one arrow in each minimal cycle and one arrow in each boundary path in Q_i. We call such a collection of arrows a *perfect cut* (note that a set of arrows satisfying the first property is referred to as an *admissible cut* in [40] (see also [4, §1]).

In order to construct a perfect cut on Q_i, we use a weighting on the arrows of Q_i from [7, Defn. 4.1]. Recall that for $p, q \in [1, n]$, we denote by $[p, q]$ the cyclic interval $\{p, p+1, \ldots, q\}$.

Definition 13.3. Fix $i \in [1, n]$. Then we give each arrow α in the quiver Q_i a *weight* given by the set

$$d(\alpha) = [p, q-1] = \{p, p+1, \ldots, q-1\},$$

in the case where $\alpha : I \to J$ and $J = I - \{p\} + \{q\}$. Note that the strands p and q cross on the arrow α: a typical arrow in Q_i is shown in Figure 19. This weighting is extended to paths by defining the weight of a path to be the multiset union of the labels of its arrows, regarded as a multisubset of $[1, n]$.

In order to study this weighting, we need to consider a slightly modified version of a Postnikov diagram, D. We define the *closure* $\overline{D}$ of D as follows. Let $\overline{\mathbb{D}}$ be a disk slightly larger than $\mathbb{D}$, and extend the strands in D incident with b_i and b'_i on the boundary of $\mathbb{D}$ to a common point $\overline{b}_i$ on the boundary of $\overline{\mathbb{D}}$. This produces a Postnikov diagram $\overline{D}$ according to the original definition [70, Defn. 14.1].

Let $Q(\overline{D})$ be the quiver associated to $\overline{D}$ in [7, Defn. 2.4]: the vertices are the same as the vertices of $Q(D)$, and we take all of the arrows in $Q(D)$, together with additional *boundary arrows* between the boundary vertices L_j of the quiver. There is a boundary arrow for each point $\overline{b_i}$, oriented and weighted

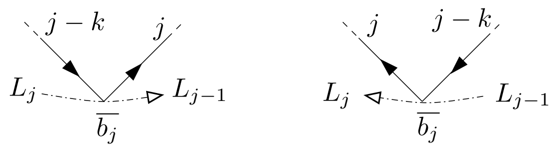


FIGURE 20. Neighbouring faces on the boundary of a closed Postnikov diagram. The weight of the arrow in the left hand figure is $[j, j-k-1]$ and the weight of the arrow in the right hand figure is $[j-k, j-1]$.

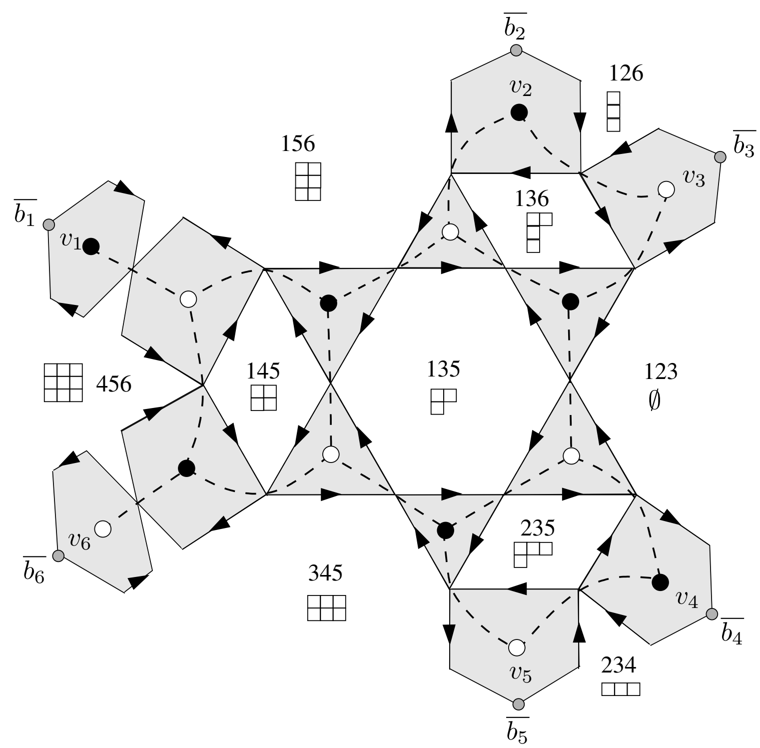


FIGURE 21. The closure $\overline{D_3}$ of the Postnikov diagram D_3 in Figure 16 and the corresponding dual bipartite graph.

according to a one-sided version of the rule for $Q(D)$; see Figure 20. Thus this boundary arrow through $\overline{b_i}$ is oriented clockwise if v_i is black and anticlockwise if v_i is white.

Note that in $Q(\overline{D})$, there is always a boundary arrow between L_i and L_{i+1}, in one direction or the other, corresponding to the intersection point $\overline{b}_i$. We define $\overline{Q_i}$ to be the quiver of $\overline{D_i}$. The closure $\overline{D_3}$ of the Postnikov diagram D_3 in Figure 16 is shown in Figure 21, and the corresponding weighted quiver $\overline{Q_3}$ is shown in Figure 22.

By [64, §9] or [7, Cor. 4.4], we have:

Lemma 13.4. *Let D be a Postnikov diagram. Then the weights of the arrows in a minimal oriented cycle in $Q(\overline{D})$ are of the form $[p_1, p_2-1]$, $[p_2, p_3-1], \ldots, [p_r, p_1-1]$, where $p_1, p_2, \ldots, p_r$ are in cyclic order around $[1, n]$. In particular, the weight of such a cycle is $[1, n]$.*

We note, for future reference, the following corollary:

Corollary 13.5. *Let D be a Postnikov diagram and let c be a cycle in $\overline{D}$. Let r_0 be the number of minimal cycles made up of arrows from c and its interior, whose orientation is the same as that of c. Let r_1 be the*

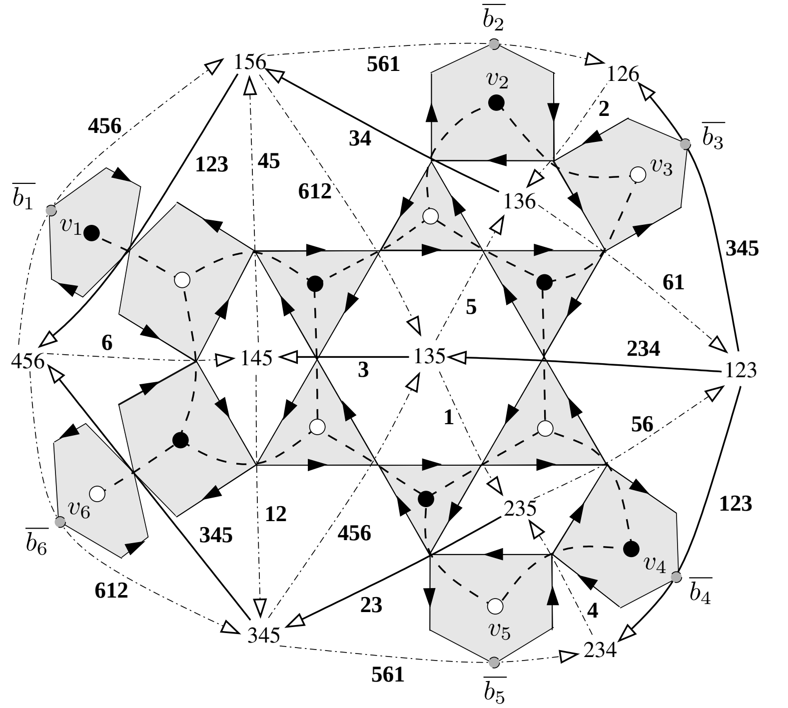


FIGURE 22. The weighted quiver $\overline{Q_3}$ of the Postnikov diagram $\overline{D_3}$ in Figure 21. The arrows in $\overline{S_3}$ (see Definition 13.6) are shown as unbroken arrows.

number of such minimal cycles whose orientation is opposite to that of c. Then $r_0 > r_1$ and the weight of c is equal to the multiset union of $r_0 - r_1$ copies of $[1, n]$.

Proof. The interiors of the minimal cycles in the statement of the corollary tile the interior of c completely. Let $\mathcal{M}_0$ (respectively $\mathcal{M}_1$) denote the set of such minimal cycles which are oriented in the same way as c (respectively, opposite to c). Each arrow in c lies in exactly one minimal cycle in $\mathcal{M}_0$, and each arrow in the interior of c lies on exactly one minimal cycle in $\mathcal{M}_0$ and exactly one minimal cycle in $\mathcal{M}_1$. Hence, considering the weights, we have:

$$\bigcup_{c_0 \in \mathcal{M}_0} d(c_0) = d(c) \cup \bigcup_{c_1 \in \mathcal{M}_1} d(c_1),$$

where the unions are multiset unions. By Lemma 13.4, $d(c_0) = d(c_1) = [1, n]$ for all minimal cycles $c_0 \in \mathcal{M}_0$ and $c_1 \in \mathcal{M}_1$. Since $r_i = |\mathcal{M}_i|$ for $i = 0, 1$, the result follows. □

Definition 13.6. We set S_i to be the set of arrows in Q_i whose weight contains i (see Figure 23 for an example), and $\overline{S_i}$ be the set of arrows in $\overline{Q_i}$ whose weight contains i (see Figure 22 for an example).

Applying Lemma 13.4 to $\overline{D_i}$, it follows that each minimal cycle in $\overline{Q_i}$ contains exactly one arrow in $\overline{S_i}$. Since $\overline{Q_i}$ contains no boundary paths, it follows that $\overline{S_i}$ can be considered as a perfect cut on $\overline{Q_i}$ in the sense of Remark 13.2.

Each minimal cycle in Q_i is a minimal cycle in $\overline{Q_i}$. Therefore, it contains exactly one arrow in S_i. However, S_i is not a perfect cut in Q_i, because there are boundary paths which do not contain any arrows in S_i, as we shall now see. For an example of this, see Figure 23, which shows the quiver Q_3 in our running example and the set of arrows S_3.

Lemma 13.7. *Let D be a Postnikov diagram. Then the boundary path between L_{j-1} and L_j has weight $[j, j-k-1]$ (respectively, $[j-k, j-1]$) if it is oriented towards L_{j-1} (respectively, L_j).*

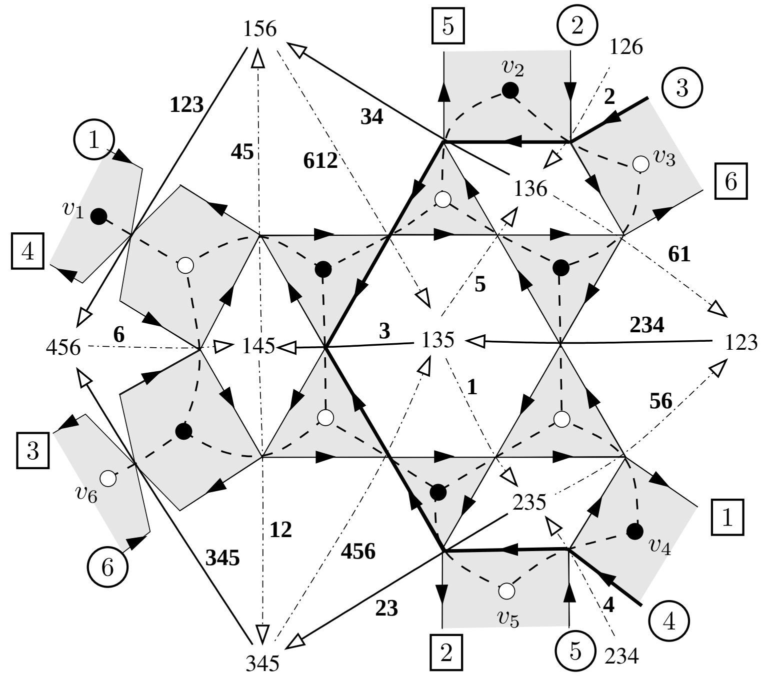


FIGURE 23. The quiver Q_3 from Figure 16. The arrows in S_3 (see Definition 13.6) are shown as unbroken arrows. The path γ_3 is shown as a thickened line, with end points given by the encircled 3 and encircled 4; see the paragraph after Lemma 13.11.

Proof. In $Q(\overline{D})$, there is an arrow between L_j and L_{j-1} crossed by strands $j-k$ and j. This arrow has weight $[j, j-k-1]$ (respectively, $[j-k, j-1]$) if it points towards L_{j-1} (respectively, L_j). Hence the boundary path in $Q(D)$, which completes this arrow to a minimal cycle in $Q(\overline{D})$, has weight as claimed by Lemma 13.4. $\square$

Recall that the vertex $\overline{b_j}$ in $\overline{D_i}$ lies on the boundary of the oriented region of $\overline{D_i}$ corresponding to the vertex v_j in G_i.

Lemma 13.8. *The boundary path around $\overline{b_j}$ in Q_i contains exactly one arrow from S_i if $j \notin \{i, i+1\}$ and contains no arrow from S_i otherwise.*

Proof. We use Lemma 13.7. If $j \in [i+2, i+k]$ then, since v_j is white, the weight of the boundary path around v_j is $[j-k, j-1]$ and contains i. Hence exactly one of the arrows on this path lies in S_i. Since v_{i+1} is black, the boundary path around this vertex has weight $[i+1, i-k]$ and hence does not contain i.

If $j \in [i+k+1, i-1]$, then v_j is black, and the weight of the boundary path around this vertex is $[j, j-k-1]$, so includes i, and exactly one of the arrows on this path lies in S_i. The vertex v_i is white, so the boundary path around around this vertex has weight $[i-k, i-1]$ and has no arrow in S_i. $\square$

Remark 13.9. Since $\overline{S_i}$ is a perfect cut (and S_i is the restriction of $\overline{S_i}$ from $\overline{Q_i}$ to Q_i), it follows from Lemma 13.8 that the boundary arrow in $\overline{Q_i}$ through $\overline{b_j}$ lies in $\overline{S_i}$ if and only if $j \in \{i, i+1\}$.

The proof of Lemma 13.10 arose from discussions of R. Marsh with K. Baur and A. King in an alternative approach towards the results in [7].

We need to alter S_i in order to obtain a perfect cut on Q_i. This will involve reversing membership of S_i for certain arrows in Q_i. In order to make this construction, we need more information about S_i.

Lemma 13.10. *Let D be a Postnikov diagram and let α be an arrow in $Q(D)$ or $Q(\overline{D})$. Then α crosses strand i if and only if the weight of α contains i and not $i-1$, or $i-1$ and not i.*

Proof. If an arrow α crosses strand i from left to right, its weight is $[p, i-1]$ where $p \neq i$ since strands cannot self-intersect. Hence it contains $i-1$ but not i. If α crosses strand i from right to left, its weight is $[i, q-1]$ for some $q \neq i$, and hence contains i but not $i-1$. If α does not cross strand i at all, its weight is $[p, q-1]$ where neither p nor q is equal to i. If $i \in [p, q-1]$, then, since $p \neq i$, we have $i-1 \in [p, q-1]$ also. If $i \notin [p, q-1]$, then, since $q \neq i$, we have $i-1 \notin [p, q-1]$ also. Thus in this case, either $i-1$ and i both lie in the weight of α or neither i nor $i-1$ lies in the weight of α. The result follows. □

Lemma 13.11. *The arrows in $\overline{Q_i}$ crossing strand i (respectively, strand $i+1$), in order from the start of the strand to its end, alternate between lying in $\overline{S_i}$ and not lying in $\overline{S_i}$.*

The first arrow in $\overline{Q_i}$ crossing strand i lies in $\overline{S_i}$. Similarly the first arrow in $\overline{Q_i}$ crossing strand $i+1$ lies in $\overline{S_i}$. The first arrows in Q_i crossing strand i and strand $i+1$, respectively, do not lie in S_i.

The arrow crossing P_i lies in S_i.

Proof. Since any two consecutive arrows crossing strand i lie in the same cycle, it follows from Lemma 13.10 and Lemma 13.4 that the arrows of Q_i (or $\overline{Q_i}$) crossing strand i alternate between lying in S_i and not in S_i. By Lemma 13.8 the boundary path around the white vertex v_i does not contain an arrow in S_i. It follows that the first arrow crossing strand i in Q_i does not lie in S_i. The alternating property implies that the first arrow crossing strand i in $\overline{Q_i}$ does lie in $\overline{S_i}$. A similar argument applies to the case of strand $i+1$.

Since P_i is the unique crossing point of strands i and $i+1$ (Lemma 13.1), it must be the case that strand i crosses the arrow crossing P_i from right to left, while strand $i+1$ crosses it from left to right (looking along the arrow). Hence the weight of this arrow is $\{i\}$ and the second part follows. □

Let γ_i be the path in D_i which proceeds along strand i from the beginning up until the crossing point P_i with strand $i+1$, and then carries on along the reverse of strand $i+1$, to the start of that strand. See Figure 23 for an example. Let $\overline{\gamma_i}$ be the analogously defined path in $\overline{D_i}$.

Corollary 13.12. *The arrows in $\overline{Q_i}$ crossing $\overline{\gamma_i}$ alternate between lying in $\overline{S_i}$ and not in $\overline{S_i}$, starting and ending with the former case. The arrows in Q_i crossing γ_i alternate between lying in S_i and not in S_i, starting and ending with the latter case.*

We define a new set of arrows Σ_i in Q_i as follows. We will see that Σ_i is a perfect cut on $\overline{Q_i}$ and on Q_i.

Definition 13.13. Let Σ_i be the set of arrows α in Q_i which satisfy one of the following:

(a) α does not cross γ_i and α lies in S_i, or

(b) α crosses γ_i and α does not lie in S_i.

Thus Σ_i is obtained from S_i by toggling membership for arrows crossing γ_i. We refer to this operation as the swap ρ.

See Figure 24 for an example of the set Σ_i.

Proposition 13.14. *The set Σ_i of arrows is a perfect cut of Q_i.*

Proof. Note first that γ_i does not self-intersect, since the strands themselves do not self-intersect, and strands i and $i+1$ have a unique crossing point by Lemma 13.1. Consider a minimal oriented cycle in Q_i and the part of the Postnikov diagram lying in the interior of the cycle. If γ_i crosses this cycle, then the part of γ_i in the interior of the cycle is a union of arcs joining mid-points of arrows on the cycle. One end-point of each such arc lies on an arrow in S_i by Corollary 13.12. However, since each minimal oriented cycle contains exactly one arrow in S_i, there must be only one such arc. Then the swap ρ has the effect of swapping which of the two arrows at the ends of this arc lies in the matching. This does not change the property that the oriented cycle contains exactly one arrow chosen by the matching. Therefore Σ_i also contains precisely one arrow from each minimal oriented cycle in Q_i.

A similar argument applies to a path around a boundary vertex v_j for $j \neq i, i+1$, since in this case γ_i does not start or end on the boundary side of such a path. We see that such boundary paths contain exactly one arrow in Σ_i, possibly swapped from the one that was contained in S_i.

However, the two boundary paths around the vertices where strands i (respectively, $i+1$) start have the property that γ_i starts (respectively, finishes) inside the boundary path around the corresponding

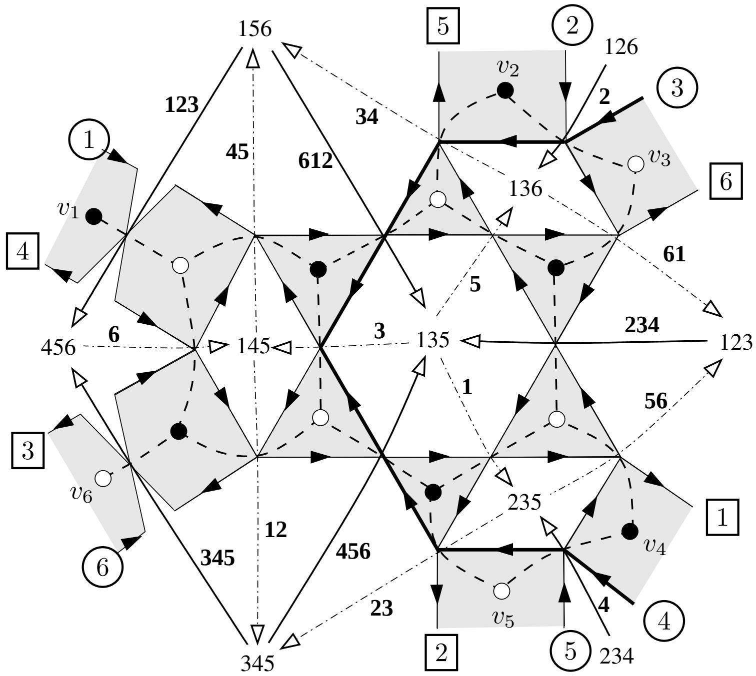


FIGURE 24. The arrows in Σ_3 (see Definition 13.13), in the quiver Q_3 from Figure 16, as unbroken arrows.

white (respectively, black) vertex. Furthermore, the first arrow that γ_i crosses does not lie in S_i, by Corollary 13.12, so it does lie in Σ_i. Similarly, the last arrow that γ_i crosses does not lie in S_i, by Corollary 13.12, so it does lie in Σ_i.

We need to check that γ_i crosses the boundary path around v_{i+1} only once, at its last arrow and that it crosses the boundary path around v_i only once, at its first arrow.

If γ_i were to cross the boundary path around the vertex v_{i+1} a second time, it would be entering the boundary region at the crossing point, describing an arc, and crossing back out at the next arrow (since it does not end at b_{i+1}). By Lemma 13.11, one of these arrows would be contained in S_i, giving a contradiction to Lemma 13.7. An analogous argument shows that γ_i crosses the boundary path around the vertex v_i only once.

It follows that each of these boundary paths contains exactly one element of Σ_i. We have shown that Σ_i is a perfect cut of Q_i as required. □

Corollary 13.15. *The set Σ_i of arrows is a perfect cut of $\overline{Q_i}$.*

Proof. The quiver $\overline{Q_i}$ can be obtained from Q_i by adding the boundary arrows between L_j and L_{j+1} for all j (as in Figure 20). This completes each boundary path in Q_i to a minimal cycle in $\overline{Q_i}$. Each such minimal cycle must contain a single arrow of Σ_i by Proposition 13.14, and the result follows. □

Remark 13.16. If $k = 1$, there is a unique perfect matching (which we also denote by M_i) on G_i (where G_i is as defined at the start of Section 12). It contains the unique edge incident with v_j for each j. A similar description holds for $k = n - 1$.

Definition 13.17. Let M_i be the set of edges in G_i such that $\Sigma(M_i) = \Sigma_i$. By Proposition 13.14 and Remark 13.2, M_i is a perfect matching on G_i.

For an example of the perfect matching M_i, in the case of the diagram D_3 in Figure 16, see Figure 25.

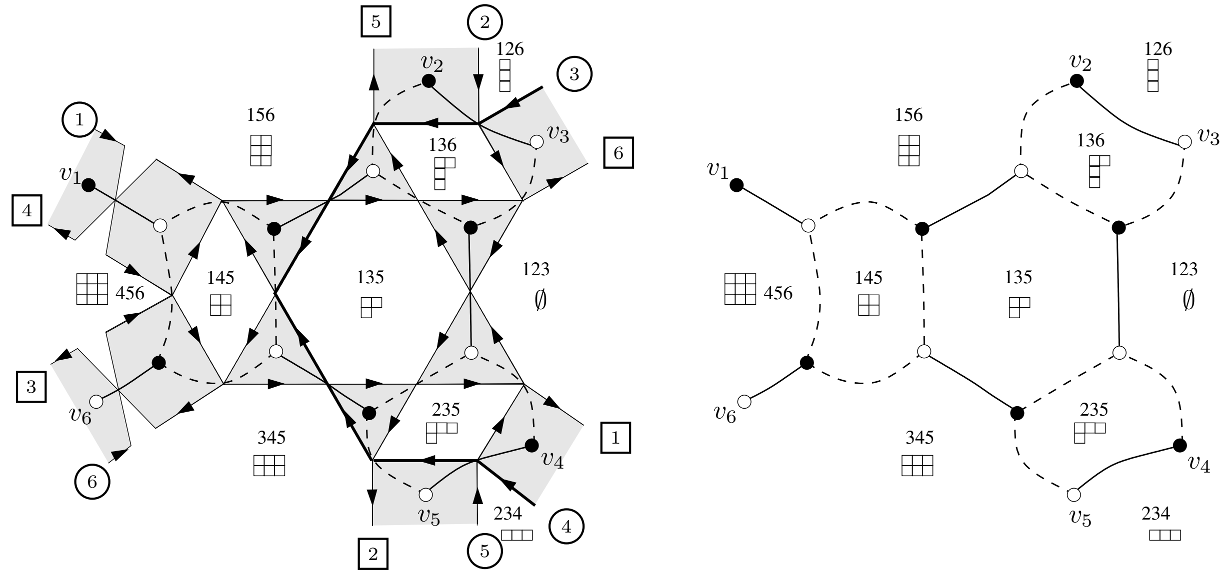


FIGURE 25. The perfect matching M_3 on the graph G_3 in the case of the example in Figure 16. Edges in G_3 are drawn as full edges if they lie in M_3 and as dashed edges otherwise. The path γ_3 is drawn with thick lines.

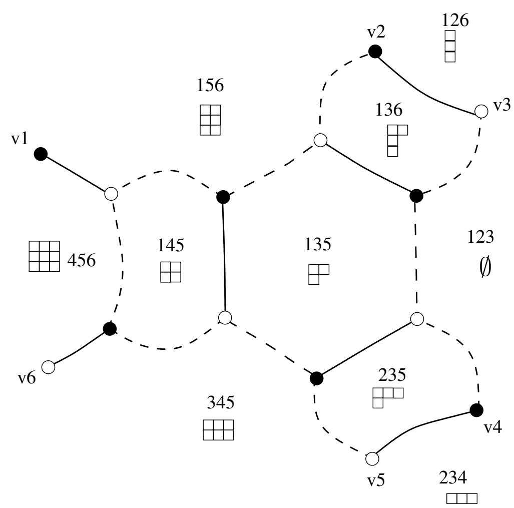


FIGURE 26. The perfect matching on G_3 obtained from the perfect matching shown in Figure 25 by performing a face flip on the hexagonal face. Edges in G_3 are drawn as full edges if they lie in the perfect matching and as dashed edges otherwise.

14. How to obtain all perfect matchings from M_i

If $k = 1$ or $n-1$, M_i is the unique perfect matching on G_i, so we assume in this section that $k \neq 1, n-1$. Our main aim is to show that every other perfect matching on G_i can be obtained from M_i by performing a sequence of face flips, in the sense of Definition 14.1 below, on interior faces. We will also show that the sequences of face flips can be constructed in such a way that a distinguished face F_i of G_i always appears at the beginning of the sequence and then never appears again.

Definition 14.1. Let M be a perfect matching on a plane graph G. An *M-flippable face* of G is a (possibly unbounded) face for which the edges on the boundary alternate between lying in M and not lying in M. If F is an M-flippable face, then we can construct a new perfect matching out of M by reversing membership of M for those edges along the boundary of F. We call this operation a *face flip*.

Recall that the interior faces of G_i correspond bijectively to the internal alternating regions of the Postnikov diagram D_i, so are labelled by the non-boundary vertices of Q_i.

A plane bipartite graph is said to be *factorizable* if it has at least one perfect matching. An edge is said to be *allowed* if it appears in some perfect matching.

For example, the graph G_3 in Figure 25 has a perfect matching, M_3, as shown. The face labelled $\ydiagram{2,1}$ is M_3-flippable in G_3. Performing a face flip on this face produces a new perfect matching M, shown in Figure 26. In M, the faces labelled $\ydiagram{3,1}$ and $\ydiagram{2,1,1}$ are both M-flippable (as well as $\ydiagram{2,1}$). Flipping at either of these two faces in M or both gives three new perfect matchings on G_3. In fact, these five perfect matchings are all of the perfect matchings on G_3. We shall see later that, in general, all perfect matchings on G_i can be obtained by performing sequences of face flips on M_i (see Theorem 14.20). We also see that the allowed edges in the graph G_3 shown in Figure 25 are as shown in Figure 27: these are the edges appearing in the five perfect matchings listed above.

By definition, a connected factorizable plane bipartite graph is *elementary* if and only if every edge is allowed [53, §4]. We recall the following.

Theorem 14.2. [72, Thm. 2], [89, Thm. 3.3] *Let G be a connected elementary plane bipartite graph. Then, given any two perfect matchings M, M' of G, there is a sequence of face flips taking M to M'.*

Note that to apply the result in [72, Thm. 2] to obtain Theorem 14.2, we need the fact that every edge in G appears in some perfect matchings but not others. But this holds for any plane bipartite elementary graph, e.g. by [89, Thm. 2.4], which states that for any face of a plane bipartite elementary graph G, there is a perfect matching in which that face is flippable.

In general, G_i is not elementary, so in order to apply Theorem 14.2 to G_i we need to study the elementary components of G_i. The *elementary components* of a plane bipartite graph G are the connected components of the graph obtained by removing all disallowed edges from G (see the definition before Lemma 2 in [26]). The elementary components of the graph G_3 shown in Figure 25 are shown in Figure 27. We show that the elementary components of G_i are all *elementary blocks* (see [51, §1]), i.e. each interior face of the component is also a face of G_i. In fact, either every elementary component of G_i is a single edge, or G_i has a unique elementary component which is not a single edge (but is an elementary block).

To compute the elementary components of G_i we need to find the allowed edges of G_i. This can be done using a result which we now recall. Suppose that G is a connected factorizable plane bipartite graph with vertex bipartition $W \sqcup B$. Let G^* denote the dual graph of G, oriented in such a way that the boundary of a face in G^* corresponding to a black (respectively, white) vertex of G is oriented clockwise (respectively, anticlockwise). Let M denote a perfect matching of G. We denote by G^*_M the quiver obtained by removing the edges in G^* dual to the edges of M. Then we have the following.

Proposition 14.3. [26, Lemma 5] *Suppose that G is a connected plane factorizable bipartite graph. Then an edge of G not in M is allowed if and only if the dual edge in G^*_M does not belong to a cycle in G^*_M.*

Note that it is clear that all edges in M are allowed. Thus Proposition 14.3 states that the disallowed edges in G are those dual to cycles in G^*_M. Thus in order to apply Proposition 14.3 we need to find out which edges of G^*_M lie in cycles. We do this by first studying Q_i, noting that G^*_i can be obtained from Q_i by identifying all of the boundary vertices L_j (recall that $L_j = \{j-k+1, \ldots, j\}$).

Definition 14.4. Let Q^{in}_i denote the full subquiver of Q_i whose vertices are those which are to the left of strand i and to the right of strand $i+1$, excluding L_i. Let Q^{out}_i denote the full subquiver of Q_i on the remaining vertices of Q_i. In particular, Q^{out}_i contains L_i.

Let G^{in}_i be the subgraph of G_i whose edges are those which either cross an arrow between two vertices in Q^{in}_i or cross an arrow between a vertex in Q^{in}_i and a vertex in Q^{out}_i.

Figure 27 shows Q^{in}_i and G^{in}_i (for $i = 3$) in our running example.

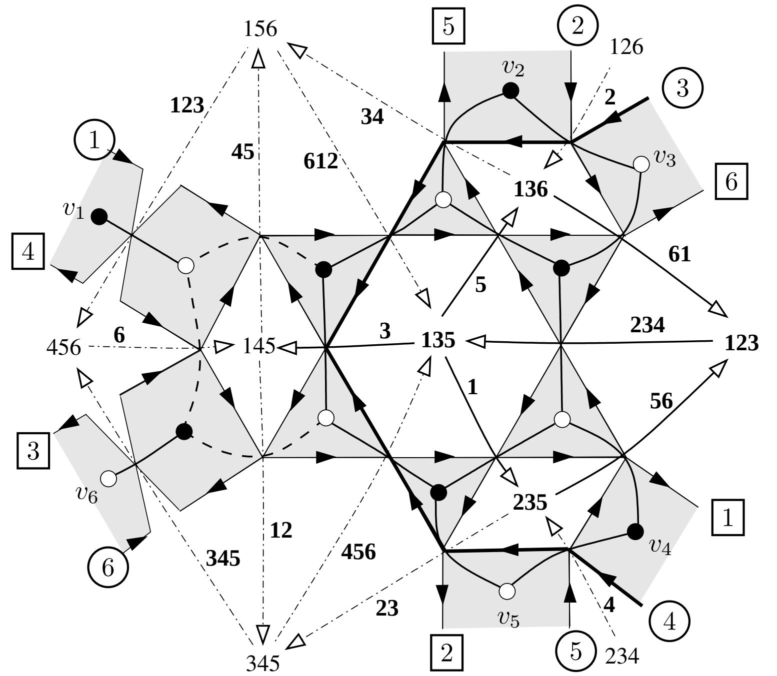


FIGURE 27. The quiver Q_3 from Figure 16. The arrows in Q_3^{in} (see Definition 14.4) are shown as unbroken arrows (its vertices are 123,135,136, 235). As usual, the path γ_3 is shown as a thickened line. The allowed edges in G_i are shown as unbroken edges. We see three elementary components: G_i^{in} and two singleton edges on the left hand side incident with v_1 and v_6.

We shall show that G_i^{in} forms an elementary component of G_i, and that all other elementary components of G_i consist of single edges crossing arrows between vertices of Q_i^{out} which lie in Σ_i. We start with the following.

Lemma 14.5. *Consider the arrows in Q_i incident with the vertex L_i, in order anticlockwise from the boundary. Note that these arrows alternate between starting at L_i and ending at L_i, with the first and last arrows ending at L_i. Arrows starting at L_i lie in S_i, while arrows ending at L_i do not lie in S_i.*

Proof. Since v_i is white and v_{i+1} is black, the first and last arrows in the above ordering must be oriented towards L_i. The fact that these arrows do not lie in S_i follows from Lemma 13.8. An arrow oriented away from L_i must have target labelled $(L_i \setminus \{j\}) \cup \{l\}$ for some $j \in L_i = [i-k+1, i]$ and $l \notin L_i$, and thus has label $[j, l-1]$ containing i. Thus all arrows starting at L_i lie in S_i. Similarly all arrows ending at L_i do not lie in S_i, as required. $\square$

Lemma 14.6. *An arrow in Q_i which has one end-point in Q_i^{in} and one end-point in Q_i^{out} is either*

(a) in the perfect cut Σ_i, and oriented towards the end-point in Q_i^{in}, or
(b) not in the perfect cut Σ_i, and oriented towards the end-point in Q_i^{out}.

Proof. Let α be an arrow as in the statement of the lemma. Suppose first that α crosses γ_i. By the assumption on α, neither end-point of α is L_i (otherwise both of its end-points would lie in Q_i^{out}). In this case, the result follows from the definition of Σ_i and Corollary 13.12, noting that the first and last arrows crossing γ_i are oriented towards Q_i^{in}.

Next, suppose that one of the end-points of α is L_i. Then the other end-point lies in Q_i^{in}, so α does not cross γ_i. The result in this case follows from Lemma 14.5 and the definition of Σ_i, noting that α lies in S_i if and only if it lies in Σ_i, as it does not cross γ_i. $\square$

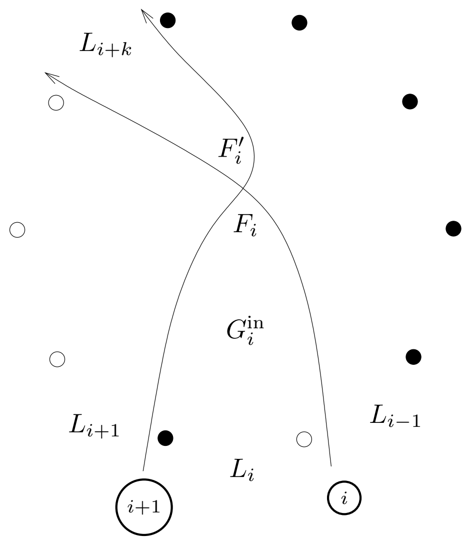


Figure 28. The faces F_i and F_i'.

Note that in the case where $i, i+1$ cross on the boundary of the face labelled by L_i, the subquiver Q_i^{in} is empty; see Figure 32. In this case, Lemma 14.6 does not say anything.

We recall the following result from [7, Prop. 4.9], which gives information concerning the weights of arrows incident with an internal vertex of Q_i which we shall use several times.

Lemma 14.7. [7, Prop. 4.9] *Let I be an internal vertex of Q_i of valency $2r$. Then the weights of the arrows incident with it, taken in order anticlockwise around I, follow on from each other and wrap around $[1, n]$ exactly $r-1$ times.*

For example, the weights of the arrows incident with the vertex 135 in Figure 23, which has valency 3, are, taken in order anticlockwise around the vertex, 1, 234, 5, 612, 3, 456, which wrap around $[1, 6]$ twice.

Definition 14.8. Let F_i be the alternating face adjacent to the crossing point P_i of strands i and $i+1$ which is to the left of strand i and to the right of strand $i+1$. Let I_i be the k-subset labelling this face. Let F_i' be the alternating face adjacent to P_i on the other side of γ_i, i.e. to the right of strand i and to the left of strand $i+1$. Let I_i' be the k-subset labelling this face. Note that if $i, i+1$ cross on the boundary of D_i then $I_i = L_i$ and $I_i' = \widehat{L}_i$. See Figure 28 for a schematic illustration. In Figure 16, we have $I_3 = \{1, 3, 5\}$ and $I_3' = \{1, 4, 5\}$. The faces labelled by these subsets are F_3 and F_3' respectively.

Definition 14.9. We denote by $Q_i^{\text{in}}(\Sigma_i)$ (respectively, $Q_i^{\text{out}}(\Sigma_i)$), the quiver Q_i^{in} (respectively, Q_i^{out}) with all arrows in Σ_i removed. For the example in Figure 27, we show the subquivers $Q_3^{\text{in}}(\Sigma_3)$ and $Q_3^{\text{out}}(\Sigma_3)$ in Figure 29 (recall that the arrows in Σ_i are shown in Figure 24).

Our next step is to obtain more information about paths and cycles in $Q_i^{\text{in}}(\Sigma_i)$ and $Q_i^{\text{out}}(\Sigma_i)$. The proofs of Lemma 14.10(b) and Lemma 14.11 build on discussions of R. Marsh with K. Baur and A. King in an alternative approach to the results in [7].

Lemma 14.10. *(a) The quivers $Q_i^{\text{in}}(\Sigma_i)$ and $Q_i^{\text{out}}(\Sigma_i)$ are acyclic.*
(b) Given any vertex I of $Q_i^{\text{in}}(\Sigma_i)$, there is a path from I_i to I in $Q_i^{\text{in}}(\Sigma_i)$.

Proof. (a) By Corollary 13.5, the weight of any cycle in $\overline{Q_i}$ is a multiset union of $[1, n]$. It follows that the weight of any cycle in Q_i also has this property and, in particular, contains i. Therefore, any cycle in Q_i contains an arrow from S_i. Since the arrows in Q_i^{in} do not cross γ_i, the arrows in Q_i^{in} which lie in Σ_i are exactly those which lie in S_i. Removing these arrows from Q_i^{in} breaks up every cycle in Q_i^{in}. It follows that $Q_i^{\text{in}}(\Sigma_i)$ is acylic. A similar argument applies to $Q_i^{\text{out}}(\Sigma_i)$.

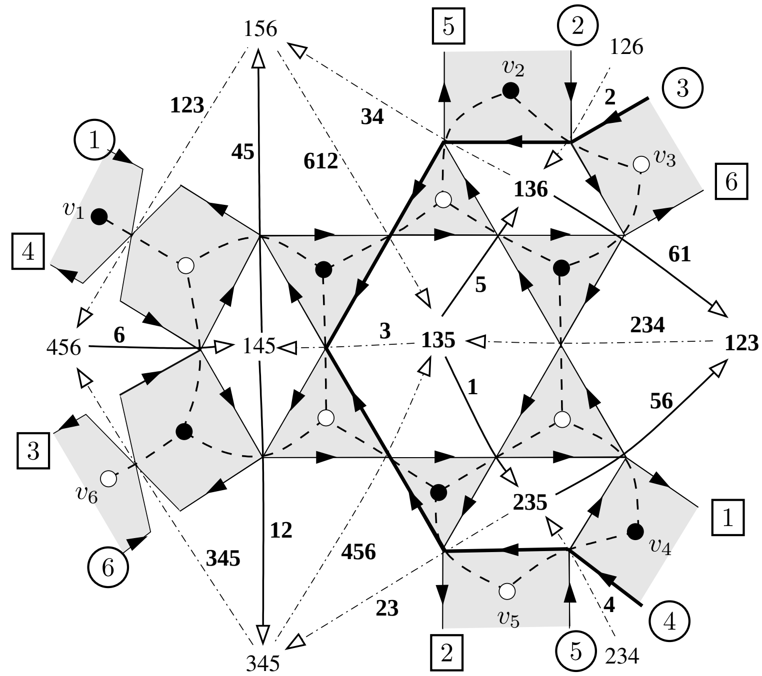


FIGURE 29. The quiver Q_3 from Figure 16. The arrows in $Q_3^{\text{in}}(\Sigma_3)$ and $Q_3^{\text{out}}(\Sigma_3)$ (see Definition 14.9) are shown as unbroken arrows.

(b) We will show that if I is any vertex of $Q_i^{\text{in}}(\Sigma_i)$ not equal to I_i, then there is an arrow in $Q_i^{\text{in}}(\Sigma_i)$ with target I. Since $Q_i^{\text{in}}(\Sigma_i)$ is finite and acyclic, repeated application of this argument must produce a path from I to I_i in $Q_i^{\text{in}}(\Sigma_i)$ as required. We find the required arrow by showing that there is an arrow in Q_i with target I whose weight does not include $i-1, i$ or $i+1$. Such an arrow does not lie in S_i, but also cannot cross γ_i by Lemma 13.10. Hence it also does not lie in Σ_i. Again using the fact that the arrow does not cross γ_i, the source of the arrow must lie in Q_i^{in}. Hence the arrow lies in $Q_i^{\text{in}}(\Sigma_i)$ as required.

We consider an arbitrary vertex I of Q_i^{in}, of valency $2r$ in Q_i. We assume that $I \neq I_i$. This implies that I is not adjacent to the crossing point P_i of strands i and $i+1$, and has no arrow incident with it whose weight is $\{i\}$ (since the arrow through P_i is the unique arrow with weight $\{i\}$).

By Lemma 14.7, there are $r-1$ arrows, $\alpha_1, \ldots, \alpha_{r-1}$ incident with I whose weight contains $i-1$. For every arrow α_j we let β_j refer to the arrow incident with I which is adjacent to α_j in an anti-clockwise direction around I. We obtain a set $\mathcal{A} = \{\alpha_j\} \cup \{\beta_j\}$ consisting of at most $2r-2$ arrows. Clearly (again using Lemma 14.7), any arrow incident with I whose weight contains i must lie in $\mathcal{A}$. Moreover any arrow whose weight contains $i+1$ must also lie in $\mathcal{A}$, since otherwise there would be a β_j with weight $\{i\}$, which contradicts our assumption that $I \neq I_i$.

By its definition, $\mathcal{A}$ contains at most $r-1$ arrows whose target is I. Hence there is an arrow with target I in Q_i whose weight does not contain $i-1, i$ or $i+1$ and we are done. □

Lemma 14.11. *Let I be a non-boundary vertex in $Q_i^{\text{out}}(\Sigma_i)$. Then there is a path in $Q_i^{\text{out}}(\Sigma_i)$ from a boundary vertex to I and a path from I to a boundary vertex.*

Proof. Assume first that I lies to the right of strand i. Let $2r$ be the valency of I in Q_i. Consider the arrows incident with I whose weight contains $i-1$, i or both. By Lemma 14.7, cyclically ordering the arrows incident with I anticlockwise around I, we see that such arrows occur either as singletons or in adjacent pairs, a total of $r-1$ singletons and adjacent pairs. Since I has valency $2r$ in Q_i, we see that there is always an arrow with source I whose weight does not contain i or $i-1$. Similarly, there is always an arrow with target I having this property. These arrows do not intersect strand i by Lemma 13.10.

Since I lies to the right of strand i and γ_i is on or the left of strand i, these arrows do not intersect γ_i. Hence the other end-points of these arrows lie in Q_i^{out}.

A similar argument applies in the case where I lies to the left of strand $i+1$. Since any vertex in $Q_i^{\text{out}}(\Sigma_i)$ lies to the right of strand i or to the left of strand $i+1$, we see that repeating this argument gives the statement in the Lemma, noting that $Q_i^{\text{out}}(\Sigma_i)$ is finite and acyclic (by Lemma 14.10). □

We can now describe the allowed edges in G_i and thus the elementary components.

Lemma 14.12. *An edge in G_i which crosses an arrow α in Q_i is allowed if and only if either at least one endpoint of α lies in $Q_i^{\text{in}}(\Sigma_i)$ or α lies in Σ_i.*

Proof. Since edges crossing arrows in Σ_i are allowed, we are reduced to the case of edges crossing arrows which do not lie in Σ_i. Recall that $(G_i^*)_{M_i}$ can be obtained from $Q_i(\Sigma_i)$ by identifying the boundary vertices. We denote the image of an arrow α in $Q_i(\Sigma_i)$ under this procedure by $\overline{\alpha}$.

Note that G_i is a connected plane bipartite graph, which is factorizable by Proposition 13.14. By Proposition 14.3 we need to show that, for any arrow α in $Q_i(\Sigma_i)$, $\overline{\alpha}$ lies in a cycle in $(G_i^*)_{M_i}$ if and only if both of the endpoints of α lie in Q_i^{out}.

If both endpoints of α lie in Q_i^{out}, then $\overline{\alpha}$ lies in a cycle in $(G_i^*)_{M_i}$ by Lemma 14.11. For the converse, note that by Lemma 14.10(a), $Q_i^{\text{in}}(\Sigma_i)$ is acyclic. Also, by Lemma 14.6, all arrows in $Q_i(\Sigma_i)$ which have one endpoint in $Q_i^{\text{in}}(\Sigma_i)$ and one in $Q_i^{\text{out}}(\Sigma_i)$ are oriented towards $Q_i^{\text{out}}(\Sigma_i)$. It follows that if α has at least one endpoint in Q_i^{in} then $\overline{\alpha}$ does not lie in a cycle in $(G_i^*)_{M_i}$. □

Corollary 14.13. *The elementary components of G_i are as follows:*

(a) *The full subgraph G_i^{in} of G_i (see Definition 14.4) is an elementary component of G_i.*
(b) *Any single edge in M_i which has no end point in G_i^{in} is an elementary component of G_i.*

Proof. This follows from the description above of the allowed edges in G_i and the definition of elementary components. □

Lemma 14.12 and Corollary 14.13 can both be verified in our running example in Figure 27.

In the case where I_i is on the boundary, we see that the elementary components of G_i are exactly the allowed edges (i.e. the edges in M_i), considered as subgraphs, and that M_i is the unique perfect matching on G_i.

An elementary component of a plane graph which has the property that each interior face of the component is also a face of the whole graph is called an *elementary block* (see [51, §1]). We need the following important property of the elementary components of G_i.

Corollary 14.14. *Each elementary component of G_i is an elementary block of G_i.*

Proof. This is trivial for the elementary components of G_i which consist of a single edge, since they have no interior faces. So we consider the elementary component G_i^{in}. By the construction of the Postnikov diagram D, the region to the left of strand i and to the right of strand $i+1$, together with the adjacent vertices in G_i, is a union of interior faces of G_i; these are exactly the interior faces of G_i^{in}. It follows that G_i^{in} is also an elementary block of G_i. □

This gives us the first key result.

Proposition 14.15. *The faces of G_i^{in} are faces of G_i, and the set of perfect matchings of G_i is connected under flips of faces of G_i^{in}.*

Proof. The first statement follows from Corollary 14.14. Let M, M' be arbitrary perfect matchings on G_i. By Lemma 14.12, M and M' must coincide with M_i on all edges of G_i crossing arrows between vertices of Q_i^{out}. By Corollary 14.13 and Theorem 14.2, there is a sequence of face flips taking M to M', noting that every face of the elementary graph G_i^{in} is also a face of G_i, by Corollary 14.14. □

Note that, by [90, Thm. 2.4], this implies that G_i is *weakly elementary* (see e.g. [90, §2] for the definition).

In the remainder of this section, we will show that, as in the example, every perfect matching on G_i can be obtained from M_i by a sequence of face-flips in which the first (and only the first) face is F_i. We first show that F_i is the unique M_i-flippable face of G_i.

Given a vertex I of Q_i, we say that two arrows α, β incident with I are *adjacent* provided one follows the other in the cyclic ordering around I.

Remark 14.16. Let I be the label of an internal alternating face F of D_i and suppose that I has valency $2r$ in Q_i. Since it is not possible for two adjacent arrows incident with I to lie in Σ_i (as it is a perfect cut), we have that F is an M_i-flippable face in G_i if and only if the number of arrows incident with I lying in Σ_i is r.

Lemma 14.17. *Let F be an internal face of D_i, labelled with the k-subset I. Suppose that I has valency $2r$ in Q_i. Then the number of arrows in Σ_i incident with I is r if $F = F_i$, $r-2$ if $F = F_i'$, and is $r-1$ otherwise.*

Proof. By Lemma 14.7, exactly $r-1$ of the arrows incident with the vertex I lie in S_i. Recall that Σ_i is obtained from S_i by applying the swap ρ (see Definition 13.13). Since I is an internal vertex of Q_i, the set of arrows incident with I that are crossed by a fixed strand consists of a number of pairs of adjacent arrows (see Figure 30). This applies, in particular, to the strands i and $i+1$ appearing in the definition of γ_i. The unique crossing point P_i of these strands lies on the boundary of the faces F_i of F_i', and not on the boundary of any other face.

So, if $I \neq I_i, I_i'$, then P_i is not on the boundary of F. It follows that the set of arrows incident with I which cross γ_i consists of a collection (possibly empty) of pairs of adjacent arrows, some pairs arising from the part of γ_i along strand i and some pairs arising from the part of γ_i along (the reverse of) strand $i+1$. The effect of the swap ρ on the set of arrows incident with I is to reverse membership in each such pair. Hence the number of arrows incident with I which lie in Σ_i is the same as the number of such arrows lying in S_i, i.e. $r-1$.

The set of arrows incident with I_i which cross γ_i consists of the arrow $I_i \to I_i'$ and the two adjacent arrows incident with I_i together with a collection (possibly empty) of pairs of adjacent arrows incident with I_i. The flip ρ reverses membership in each of the pairs. It also replaces the arrow $I_i \to I_i'$ with the two adjacent arrows incident with I_i. Hence exactly r of the arrows incident with I_i lie in Σ_i.

Similarly, the set of arrows incident with I_i' which cross γ_i consists of the single arrow $I_i \to I_i'$, together with a collection (possibly empty) of pairs of adjacent arrows incident with I_i'. The flip ρ reverses membership in each of these pairs and deletes the arrow $I_i \to I_i'$. Hence exactly $r-2$ of the arrows incident with I_i' lie in Σ_i. □

Corollary 14.18. *Let I be a vertex in Q_i^{in} labelling an alternating face F. Then F is M_i-flippable if and only if $I = I_i$.*

Proof. Since F must be an internal alternating face of D_i, this follows from Remark 14.16 and Lemma 14.17. □

Let M be a perfect matching on a plane bipartite graph G. Then a *positive M-flippable face* is an M-flippable face with the property that the matched edges, when oriented from black vertices towards white vertices, are oriented in an anticlockwise direction around the face. Otherwise, an M-flippable face is said to be *negative*. For example, in the perfect matching M_3 in Figure 25, the face F_3, labelled $\ydiagram{2,1}$, is an M_3-flippable face and is negative.

Twisting down a face is the operation of flipping a positive M-flippable face. See [72, §1]. We recall the following, which follows from [72, Prop. 1.11, Thm. 2].

Theorem 14.19. [72] *Let G be a connected plane elementary factorizable bipartite graph and fix a face F of G. Let $\mathcal{M}$ be the set of perfect matchings of G. Then the covering relation given by twisting down at a face other than F makes $\mathcal{M}$ into a distributive lattice $\mathcal{M}_F$. The unique minimum element of $\mathcal{M}_F$ is the unique perfect matching M on G which has no positive M-flippable face except for F.*

Note that, since G_i^{in} is an elementary component of G_i, the restriction M_i^{in} of M_i to G_i^{in} is a perfect matching on G_i^{in} by [26, Lemma 2]. We denote by $\mathcal{M}_i^{\text{in}}$ the set of perfect matchings of G_i^{in}.

Theorem 14.20. *Each perfect matching on G_i can be obtained from M_i by a sequence of flips of faces of G_i which are faces of G_i^{in}, starting with the face F_i and never involving that face again.*

Proof. By Proposition 14.15, it is enough to prove the result for the perfect matching M_i^{in} on G_i^{in}. By Corollary 14.18, F_i is the unique internal M_i^{in}-flippable face in G_i^{in}. Since F_i is negative, G_i^{in} has no positive internal M_i^{in}-flippable faces. But, by [72, Prop. 1.11], there must be at least one positive M_i^{in}-flippable face in G_i^{in}. Hence the boundary face of G_i^{in} is a positive M_i^{in}-flippable face.

Taking F to be the boundary face of G_i^{in} in Theorem 14.19, we see that M_i^{in} is the minimum element of the lattice $(\mathcal{M}_i^{\text{in}})_F$. Hence, if M is any perfect matching on G_i^{in} not equal to M_i^{in}, there is a twisting down sequence from M to M_i^{in} involving only internal faces. Since F_i is the unique negative M_i^{in}-flippable face in G_i^{in}, this twisting down sequence must involve F_i as its final flip. By [71, Cor. 4], the number of times that any given face can be flipped in such a twisting down sequence is at most 1, so F_i cannot occur in the twisting down sequence at any other place than the end. □

Note that we can see that Theorem 14.20 holds in the case of example G_3 in Figure 25 from the description of the perfect matchings given in the paragraph after Definition 14.1.

Remark 14.21. Theorem 14.20 holds trivially for $k = 1, n-1$, since there is a unique perfect matching in this case (see Remark 13.16).

15. The matching monomial associated to M_i

Our main aim in this section is to compute the matching monomial associated to M_i, i.e. its contribution towards the matching polynomial of G_i. This, combined with Theorem 14.20, will be used in Section 17 in order to compute the action of X_λ on W_q. We assume in this section that $k \neq 1, n-1$. Recall the definition (Definition 13.3) of the weighting on G_i. We make the following useful definition.

Definition 15.1. Let D be a Postnikov diagram with closure $\overline{D}$. Let I be a vertex of $Q(\overline{D})$. For any minimal cycle

$$I = I_0 \to I_1 \to I_2 \to \cdots \to I_{s-1} \cdots \to I_s = I$$

in $Q(\overline{D})$ containing I, we call the path

$$I_1 \to I_2 \to \cdots \to I_{s-1}$$

a *peripheral path* of I. Note that the underlying unoriented graph of the union of the peripheral paths of I forms a circle around I if I is internal, or an arc of a circle if I is external. Each peripheral path is either oriented clockwise or anticlockwise in this circle (or arc). We call any arrow in a peripheral path of I a *peripheral arrow* of I. The *neighbourhood* of I is the union of the cycles (and their interiors) containing I (i.e. the region bounded by the peripheral paths of I and the boundary arrows incident with I if any). See Figure 30 for an example of the neighbourhood of an internal vertex.

We also say that an arrow in $Q(D)$ is peripheral, etc. as above, if it is peripheral when regarded as an arrow in $Q(\overline{D})$.

The following lemma is a straightforward reformulation of the definitions.

Lemma 15.2. *The exponent of p_I in the matching monomial w_{M_i} is equal to the number of clockwise peripheral arrows of I in Q_i which lie in Σ_i.*

Note that, by Proposition 13.14, each clockwise peripheral path of I contains at most one arrow in Σ_i.

Lemma 15.3. *Let Σ_i' be the set of arrows in $\overline{Q_i}$ obtained from $\overline{S_i}$ by reversing membership for all arrows crossing $\overline{\gamma_i}$. Then $\Sigma_i' = \Sigma_i$.*

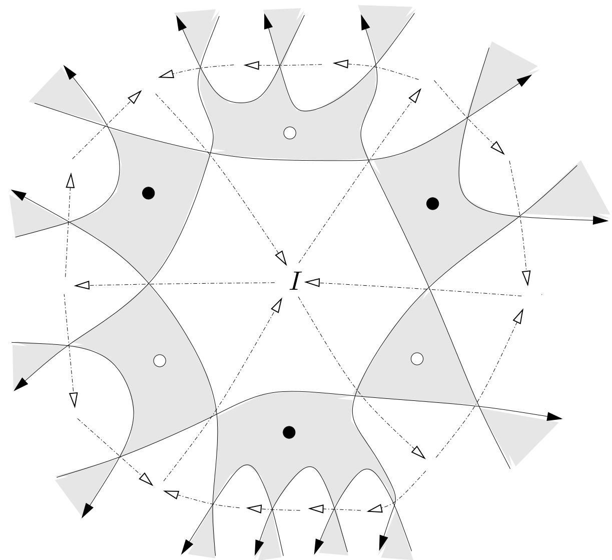


FIGURE 30. The neighbourhood of a internal vertex I.

Proof. The intersection of Σ'_i with Q_i agrees with Σ_i, because both subsets of Q_i are constructed in the same way. It only remains to check that Σ'_i contains no arrows outside of Q_i. The arrows in $\overline{Q_i} \setminus Q_i$ crossing $\overline{\gamma_i}$ are in $\overline{S_i}$ by Corollary 13.12 and therefore are not in Σ'_i by construction. The remaining arrows in $\overline{Q_i} \setminus Q_i$ are also not in Σ'_i. Indeed they are not in $\overline{S_i}$ by Remark 13.9. And Σ'_i agrees with $\overline{S_i}$ away from $\overline{\gamma_i}$. □

We denote the flip (defined on subsets of the set of arrows of $\overline{Q_i}$) which reverses membership for arrows crossing $\overline{\gamma_i}$ by $\overline{\rho}$.

Remark 15.4. Fix a vertex I of Q_i. Then, since every arrow in Σ_i is contained in Q_i, the number of peripheral paths of I in Q_i containing an arrow in Σ_i is the same as the number of peripheral paths of I in $\overline{Q_i}$ containing an arrow in Σ_i.

Lemma 15.5. *Let I be the label of an internal alternating face F of D_i. The exponent e_I of p_I in the matching monomial w_{M_i} is given by:*

$$e_I = \begin{cases} 0 & \textit{if } I = I_i \ ; \\ 2 & \textit{if } I = I'_i \ ; \\ 1 & \textit{otherwise.} \end{cases} \tag{15.1}$$

Proof. By Lemma 14.17, the number of arrows incident with I in Q_i (and hence also in $\overline{Q_i}$, by Remark 15.4) which lie in Σ_i is equal to r if $F = F_i$, $r-2$ if $F = F'_i$, and $r-1$ otherwise. By Corollary 13.15, Σ_i is a perfect cut of $\overline{Q_i}$. In the case where $F = F_i$ the arrows incident with I_i alternate between lying in Σ_i and not. Every minimal cycle passing through I_i therefore has an arrow incident with I_i as its unique arrow in Σ_i. It follows that none of the clockwise peripheral paths of I_i contain an arrow in Σ_i. This shows that $e_I = 0$ if $I = I_i$. The other cases follow similarly. □

We must next consider the boundary vertices L_j of Q_i. The same approach essentially works, but with some additional complications arising from the behaviour at the boundary. For a subset S of $[1, n]$ and $r \in \mathbb{N}$ we denote by rS the multiset union of r copies of S. We recall:

Proposition 15.6. [7, Prop. 4.11] *Let $I = L_j$ be a boundary vertex in the quiver of a closed Postnikov diagram $\overline{D}$. Let α_+ be the arrow with source I which is most anticlockwise (inside the boundary of the disk)* *and let α^- be the arrow with source I which is most clockwise. Let $W^{\mathrm{out}}(L_j)$ be the set of arrows incident with I between α_- and α_+ (including these two arrows). Let r_{out} be the number of arrows in $W^{\mathrm{out}}(L_j)$ with source I. Then the multiset union of weights of the arrows in $W^{\mathrm{out}}(L_j)$ is computed by the formula:*

$$\bigcup_{\alpha \in W^{\mathrm{out}}(L_j)} d(\alpha) = (r_{\mathrm{out}} - 1)[1, n] \cup \{j\}. \tag{15.2}$$

Now we can prove an analogue of Lemma 15.5 for the boundary case:

Lemma 15.7. *The exponent e_{L_j} of p_{L_j} in the matching monomial w_{M_i} is determined as follows. Suppose first that $L_j \notin \{I_i, I_i'\}$. Then*

$$e_{L_j} = \begin{cases} 1, & j \in [i+k+1, i-2] \cup \{i\}; \\ 0, & \textit{otherwise.} \end{cases} \tag{15.3}$$

If $L_j = I_i$ then $j = i$ and $e_{L_j} = 0$. If $L_j = I_i'$ then $j = i+k$ and $e_{L_j} = 1$. Written more compactly, we have

$$e_{L_j} = \begin{cases} 1 - \delta_{L_j I_i} + \delta_{L_j I_i'} & j \in [i+k+1, i-2] \cup \{i\}; \\ -\delta_{L_j I_i} + \delta_{L_j I_i'} & \textit{otherwise.} \end{cases} \tag{15.4}$$

Proof. By Lemma 15.2 and Remark 15.4, it is sufficient to show that the number of clockwise peripheral paths of L_j in $\overline{Q_i}$ containing an arrow in Σ_i is given by the formula (15.3).

Let F be the alternating boundary face of $\overline{D_i}$ labelled L_j. If P_i, the unique crossing point of strands $i, i+1$, lies on the boundary of F, then either $i \in L_j$, $i+1 \notin L_j$, or $i \notin L_j$, $i+1 \in L_j$. It follows that $L_j = L_i = I_i$ (in the first case) or $L_j = L_{i+k} = I_i'$ (in the second case). We assume first that neither of these cases occur. This implies that, since the strand crossing points on the arrows incident with L_j are precisely those on the boundary of F, strands $i, i+1$ cannot cross on an arrow incident with L_j. The separate cases with this assumption are illustrated in Figure 31.

By Corollary 13.15, the number of clockwise peripheral paths of L_j containing an arrow in Σ_i is equal to the number of clockwise minimal cycles incident with L_j minus the number of arrows in Σ_i incident with L_j which lie in such cycles. We denote the number of clockwise minimal cycles incident with L_j by n_c.

Recall that the boundary arrows do not lie in Σ_i, and note that any arrow incident with L_j which is not a boundary arrow lies in some clockwise minimal cycle. Therefore the set of arrows in Σ_i incident with L_j coincides with the set of arrows in Σ_i incident with L_j which lie in a clockwise minimal cycle. Similarly, this set is also equal to the set of arrows in $W^{\mathrm{out}}(L_j)$ which lie in Σ_i.

Suppose first that $j \neq i-1, i, i+1$. Then $\overline{\gamma_i}$ does not start or end on a boundary arrow of $\overline{Q_i}$ incident with L_j. Also P_i does not lie on an arrow incident with L_j. Hence the set of arrows incident with L_j that cross $\overline{\gamma_i}$ consists of a (possibly empty) collection of pairs of adjacent arrows. This collection of arrows coincides with the set of arrows in $W^{\mathrm{out}}(L_j)$ crossing $\overline{\gamma_i}$. By an application of Corollary 13.12, we then see that the number of arrows in $W^{\mathrm{out}}(L_j)$ which lie in Σ_i is the same as the number of arrows in $W^{\mathrm{out}}(L_j)$ which lie in $\overline{S_i}$. This number is $r_{\mathrm{out}} - 1$ by Proposition 15.6. Hence, by the above, $e_{L_j} = n_c - (r_{\mathrm{out}} - 1)$.

We now compute n_c for $j \neq \{i-1, i, i+1\}$ to complete the proof in this case. There are two cases. If the boundary arrow between L_j and L_{j-1} is oriented towards L_{j-1}, then it is an arrow with source L_j which is not part of a clockwise minimal cycle. There is a bijection between the remaining arrows with source L_j and the clockwise minimal cycles incident with L_j. Therefore $n_c = r_{out} - 1$ and $e_{L_j} = 0$ in this case. If on the other hand the boundary arrow between L_j and L_{j-1} is oriented towards L_j, then every arrow with source L_j is part of a clockwise minimal cycle and we have $n_c = r_{out}$. In this case $e_{L_j} = 1$. By definition of the graph G_i we are in the first case if $j \in [i+2, i+k]$ and in the second case if $j \in [i+k+1, i-2]$, as illustrated in Figure 31.

We now consider the cases where $\overline{\gamma_i}$ starts or ends on a boundary arrow of $\overline{Q_i}$ incident with L_j, assuming still that $L_j \notin \{I_i, I_i'\}$. Note that P_i does not lie on an arrow incident with L_j by this assumption. Then we are in the last three cases of Figure 31.

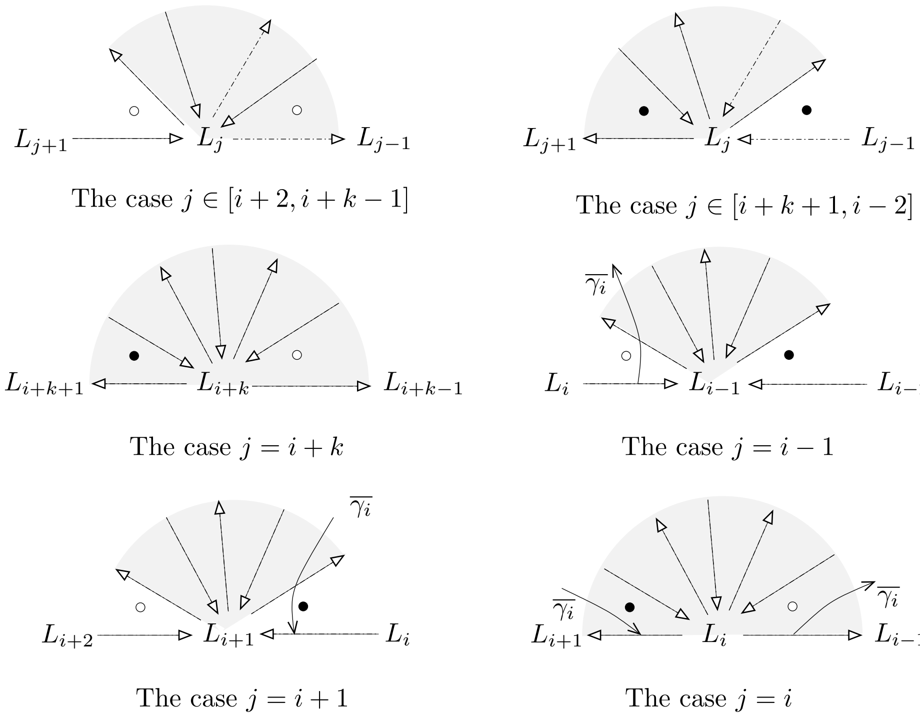


FIGURE 31. Cases in the proof of Lemma 15.7. The shaded region indicates the arrows in $W^{\text{out}}(L_j)$.

Recall that the exponent e_{L_j} is given by the number of clockwise minimal cycles incident with L_j minus the number of arrows in $W^{\text{out}}(L_j)$ which lie in Σ_i. By Proposition 15.6 if $j = i$ there are r_{out} arrows in $W^{\text{out}}(L_j)$ which lie in $\overline{S_i}$; otherwise there are $r_{\text{out}} - 1$. We again use this and the fact that Σ_i is constructed out of $\overline{S_i}$ by the swap $\overline{\rho}$ to compute e_{L_j} in each of the remaining cases.

For the cases $j = i-1$ and $j = i+1$ the path $\overline{\gamma_i}$ crosses one boundary arrow and the adjacent arrow incident with L_j. The perfect cut Σ_i contains the adjacent arrow but not the boundary arrow, while $\overline{S_i}$ contains the boundary arrow but not the adjacent arrow. Therefore the number of arrows in $W^{\text{out}}(L_j)$ which lie in Σ_i is equal to r_{out}. The exponent is then computed by $e_{L_j} = n_c - r_{\text{out}}$, which in both cases equals 0.

For the case $j = i$ the set of arrows incident with L_i that cross $\overline{\gamma_i}$ consists of the two boundary arrows incident with L_i, together with a (possibly empty) collection of pairs of adjacent arrows. The two unpaired boundary arrows incident with L_i lie in $\overline{S_i}$ and do not lie in Σ_i. Both of these boundary arrows lie in $W^{\text{out}}(L_j)$. It follows that the number of arrows in $W^{\text{out}}(L_j)$ which lie in Σ_i is equal to $r_{\text{out}} - 2$. The exponent is then computed by $e_{L_i} = n_c - r_{\text{out}} + 2$, which equals 1.

We are left with the cases $L_j = L_i = I_i$ and $L_j = L_{i+k} = I'_i$. We first consider the case $L_j = L_i = I_i$. In this case, strands $i, i+1$ cross on the boundary of F. For this to happen, the boundary arrows incident with L_i must lie in 2-cycles and the crossing point of strands $i, i+1$ must lie on a unique internal arrow with source L_i, lying in $\overline{S_i}$. See Figure 32. The boundary arrows incident with L_i lie in $\overline{S_i}$, and we see that the arrows completing these arrows to 2-cycles both lie in Σ_i. Therefore, there are two clockwise minimal cycles incident with L_i, and each contains an arrow in Σ_i incident with L_i. It follows that $e_{L_i} = 0$.

Finally, we consider the case $L_j = L_{i+k} = I'_i$. Note that the boundary arrows incident with L_{i+k} point away from L_{i+k}. For the crossing point P_i of strands $i, i+1$ to lie on an arrow incident with L_{i+k}, there must be a unique internal arrow incident with L_{i+k}, necessarily oriented towards L_{i+k}, and P_i must lie on this arrow. See Figure 33. The boundary arrows incident with L_{i+k} do not lie in Σ_i. Also, the arrow

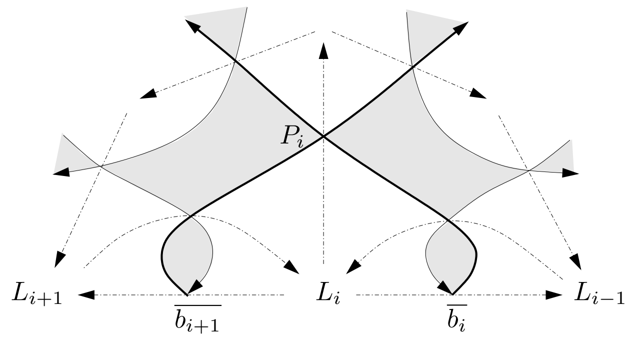


FIGURE 32. Case where strands $i, i+1$ cross on the boundary of the face labelled L_i.

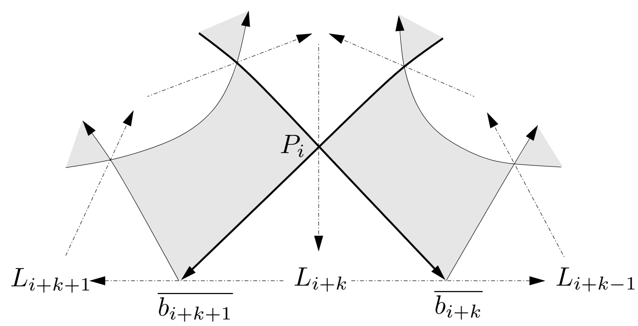


FIGURE 33. Case where strands $i, i+1$ cross on the boundary of the face labelled L_{i+k}.

through P_i does not lie in Σ_i (by Lemma 13.11). Therefore, there is one clockwise minimal cycle incident with L_{i+k}, but no arrows incident with L_{i+k} lie in Σ_i. It follows that $e_{L_{i+k}} = 1$.

We have now considered all possible cases, so the proof is complete. □

We have proved the following theorem, which also holds in the cases $k = 1, n-1$.

Theorem 15.8. *Fix $\lambda \in \mathcal{P}_{k,n}$ and let D_λ be the Postnikov diagram constructed by Theorem 11.1. Let $\mathcal{C}$ be the corresponding Postnikov cluster. Fix $i \in [1, n]$ and let D_i be the Postnikov diagram associated to D_λ in Definition 12.1. Let G_i be the corresponding dual bipartite graph. Let M_i be the perfect matching on G_i defined in Definition 13.17. Let F_i be the alternating face in D_i defined in Definition 14.8. Then the following holds.*

(a) If F_i is a boundary face, then M_i is the unique perfect matching on G_i. If F_i is an internal face, then it is the face of the elementary component G_i^{in} of G_i (see Definition 14.4). In this case, every other perfect matching on G_i can be obtained from M_i by flipping F_i and then applying a further sequence of flips involving faces of G_i^{in} distinct from F_i.

(b) The following identity holds:

$$\frac{w_{M_i}}{\prod_{p \in \mathcal{C}} p} p_{L_{i-1}} p_{L_{i+1}} \cdots p_{L_{i+k}} = \frac{p_{I_i'}}{p_{I_i}},$$

and computes the summand corresponding to M_i in the formula (12.2) *for $\frac{p_{\widehat{L}_i}}{p_{L_i}}$. associated to M_i.*

Proof. Suppose first that $k \neq 1, n-1$. Then part (a) is Theorem 14.20. Part (b) follows from Lemmas 15.5 and 15.7. For the case $k = 1, n-1$, part (a) holds by Remark 13.16. For part (b), note first that for our choice of Postnikov diagram (see Section 12) there is not a unique choice of crossing point for strands $i, i+1$. However, the adjacent faces F_i and F_i' do not depend on this choice, and we have $p_{I_i} = p_i = p_{L_i}$

and $p_{I'_i} = p_{i+1} = p_{\widehat{L}_i}$. Since M_i is the unique perfect matching on G_i in this case, part (b) follows from Theorem 12.3. $\square$

16. The vector fields $X_\lambda^{(m)}$

In this section we define a family of vector fields $X_\lambda^{(m)}$, $\lambda \in \mathcal{P}_{k,n}$, $m \in [1,n]$, on $\check{\mathbb{X}}^\circ$, denoting $X_\lambda^{(n)}$ also by X_λ. We allow the cases $k = 1, n-1$. We start by defining a vector field $X_{\lambda,\widetilde{\mathcal{C}}}^{(m)}$ in a fixed Postnikov extended cluster $\widetilde{\mathcal{C}}$ containing p_λ via an explicit formula involving combinatorially defined coefficients $c_\lambda^{(m)}(\mu)$. We then prove that the coefficients satisfy an additivity property which in particular will imply that the vector field $X_{\lambda,\widetilde{\mathcal{C}}}^{(m)}$ extends to a regular vector field $X_\lambda^{(m)}$ on the whole of $\check{\mathbb{X}}^\circ$.

Fix a Young diagram $\lambda \in \mathcal{P}_{k,n}$. For each $\mu \in \mathcal{P}_{k,n}$ we define a nonnegative integer $c_\lambda(\mu)$ as follows. Write $J_\mu \setminus J_\lambda = \{m_1 < m_2 < \cdots < m_r\}$ and $J_\lambda \setminus J_\mu = \{l_1 < l_2 < \cdots < l_r\}$ in increasing numerical order. Then set

$$c_\lambda(\mu) = |\{1 \le j \le r \,:\, m_j > l_j\}|.$$

Note that $c_\lambda(\lambda) = c_\lambda(\emptyset) = 0$.

Given an extended cluster $\widetilde{\mathcal{C}}$ (for the cluster structure on $\mathbb{C}[\check{\mathbb{X}}^\circ]$ discussed in Section 7), we define:

$$\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}} = \{x \in \check{\mathbb{X}}^\circ \,:\, f(x) \neq 0 \text{ for all } f \in \widetilde{\mathcal{C}}\}.$$

Note that $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$ is isomorphic to $(\mathbb{C}^*)^{k(n-k)}$ by [84, Theorem 4]. We call this a *cluster torus* in $\check{\mathbb{X}}^\circ$. If $\widetilde{\mathcal{C}}$ is an extended cluster of $\mathbb{C}[\check{\mathbb{X}}^\circ]$ coming from a Postnikov diagram containing a region labelled J_λ (so that $p_\lambda \in \widetilde{\mathcal{C}}$), we consider the regular vector field

$$X_{\lambda,\widetilde{\mathcal{C}}} := p_\lambda \sum_{p_\mu \in \widetilde{\mathcal{C}}} c_\lambda(\mu) p_\mu \frac{\partial}{\partial p_\mu} \tag{16.1}$$

on $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$. We shall see later that $X_{\lambda,\widetilde{\mathcal{C}}}$ can be extended to a regular vector field on the whole of $\check{\mathbb{X}}^\circ$.

Example 16.1. Let $\widetilde{\mathcal{C}}(D)$ be the extended cluster associated to the Postnikov diagram D in Figure 4, and take $\lambda = \yng(1,1)$. Note that $J_\lambda = \{1,2,5\}$ We calculate the coefficients $c_\lambda(\mu)$ for $p_\mu \in \widetilde{\mathcal{C}}$ in the following table:

μ	J_μ	$J_\mu \setminus J_\lambda$	$J_\lambda \setminus J_\mu$	$c_\lambda(\mu)$
$\emptyset$	$\{1,2,3\}$	$\{3\}$	$\{5\}$	0
$\yng(3)$	$\{2,3,4\}$	$\{3,4\}$	$\{1,5\}$	1
$\yng(3,3)$	$\{3,4,5\}$	$\{3,4\}$	$\{1,2\}$	2
$\yng(3,3,3)$	$\{4,5,6\}$	$\{4,6\}$	$\{1,2\}$	2
$\yng(2,2,2)$	$\{1,5,6\}$	$\{6\}$	$\{2\}$	1
$\yng(1,1,1)$	$\{1,2,6\}$	$\{6\}$	$\{5\}$	1
$\yng(2,1)$	$\{1,3,5\}$	$\{3\}$	$\{2\}$	1
$\yng(2,1,1)$	$\{1,3,6\}$	$\{3,6\}$	$\{2,5\}$	2
$\yng(2,2)$	$\{1,4,5\}$	$\{4\}$	$\{2\}$	1
$\yng(3,1)$	$\{2,3,5\}$	$\{3\}$	$\{1\}$	1

We see that

$$X_{\yng(1,1),\widetilde{\mathcal{C}}(D)} = p_{\yng(1,1)} \left(p_{\yng(3)} \frac{\partial}{\partial p_{\yng(3)}} + 2p_{\yng(3,3)} \frac{\partial}{\partial p_{\yng(3,3)}} + 2p_{\yng(3,3,3)} \frac{\partial}{\partial p_{\yng(3,3,3)}} + p_{\yng(2,2,2)} \frac{\partial}{\partial p_{\yng(2,2,2)}} + p_{\yng(1,1,1)} \frac{\partial}{\partial p_{\yng(1,1,1)}} + \right.$$
$$\left. + p_{\yng(2,1)} \frac{\partial}{\partial p_{\yng(2,1)}} + 2p_{\yng(2,1,1)} \frac{\partial}{\partial p_{\yng(2,1,1)}} + p_{\yng(2,2)} \frac{\partial}{\partial p_{\yng(2,2)}} + p_{\yng(3,1)} \frac{\partial}{\partial p_{\yng(3,1)}} \right).$$

We also consider a family of 'twisted' versions of $X_{\lambda,\widetilde{\mathcal{C}}}$, defined as follows. For $m \in [1,n]$, recall that $\mu^{(m)}$ denotes the partition defined by $J_{\mu^{(m)}} := J_\mu - m \mod n$ (see Definition 11.6).

For $m \in [1,n]$ and Young diagrams λ, μ in $\mathcal{P}_{k,n}$, we set

$$c_\lambda^{(m)}(\mu) = c_{\lambda^{(m)}}(\mu^{(m)}) - c_{\lambda^{(m)}}(\emptyset^{(m)}). \tag{16.2}$$

Note also that, by definition, $c_\lambda^{(m)}(\emptyset) = 0$ for any m, λ. We also have that $c_\lambda^{(n)}(\mu) = c_\lambda(\mu)$ for any λ, μ.

For a Postnikov extended cluster $\widetilde{\mathcal{C}}$ as above, we have the regular vector field

$$X^{(m)}_{\lambda,\widetilde{\mathcal{C}}} := p_\lambda \sum_{p_\mu \in \widetilde{\mathcal{C}}} c_\lambda^{(m)}(\mu) p_\mu \frac{\partial}{\partial p_\mu} \tag{16.3}$$

on $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$. Note that $X^{(n)}_{\lambda,\widetilde{\mathcal{C}}} = X_{\lambda,\widetilde{\mathcal{C}}}$.

Let P_n be a regular polygon with vertices $1, 2, \ldots, n$ numbered clockwise. Then, in [52, §1], two k-subsets I, J of $[1,n]$ are said to be *weakly separated* if none of the chords between the vertices of P_n corresponding to $I \setminus J$ crosses any of the chords between the vertices corresponding to $J \setminus I$. Scott [84, Defn. 3] uses the term *non-crossing* and we shall use this terminology.

We recall the following result of Scott:

Theorem 16.2. *Let D be a Postnikov diagram of type (k,n). Then the collection of k-subsets labelling the alternating faces of D is an inclusion-maximal collection of pairwise non-crossing k-subsets of* $[1,n]$.

We will now show that the coefficients $c_\lambda^{(m)}(\mu)$ satisfy a certain additivity property on Postnikov diagrams, which will be very useful for understanding $X^{(m)}_{\lambda,\mathcal{C}}$. We first need some notation for the labels of the alternating faces around a given internal alternating face.

Definition 16.3. Let D be a Postnikov diagram. Let F be a non-boundary alternating face of D, labelled by the k-subset J_μ, with $\mu \in \mathcal{P}_{k,n}$. Consider the labels of the strands passing along the boundary of F, which are alternatingly anticlockwise and clockwise around F. Let a_1 be the minimal label amongst all labels of anticlockwise strands around F. Let c_1 be the label of the clockwise strand meeting a_1 where it enters the boundary of F. We can extend this labelling to all of the strands passing along the boundary of F to get a list

$$a_1, c_1, a_2, c_2, \ldots, a_r, c_r$$

of elements of $[1,n]$. These are the labels of all of the strands in the boundary of F starting from the minimal anticlockwise strand a_1 and going around F in clockwise order.

Consider next the faces adjacent to F. We denote the face meeting F in the crossing point of c_i and a_i by $F(i)$, and the face meeting F in the crossing point of a_i and c_{i-1} by $F'(i)$. Note that the strands on the boundary of F are oriented towards the intersection point where $F'(i)$ and F meet, and away from the intersection point where $F(i)$ and F meet. The labels of $F(i)$ and of $F'(i)$ are determined from the label J_μ of F as follows. The label of $F(i)$ is $(J_\mu \setminus \{a_i\}) \cup \{c_i\}$. The label of $F'(i)$ is $(J_\mu \setminus \{a_i\}) \cup \{c_{i-1}\}$.

Thus, the faces adjacent to F are, in order clockwise around F:

$$F'(1), F(1), \ldots, F'(r), F(r).$$

We denote the partitions corresponding to the k-subsets labelling these faces by:

$$\mu'(1), \mu(1), \mu'(2), \mu(2), \ldots, \mu'(r), \mu(r).$$

Thus $J_{\mu'(i)}$ labels $F'(i)$ and $J_{\mu(i)}$ labels $F(i)$ for all i. See Figure 34.

Proposition 16.4. *Suppose $\lambda \in \mathcal{P}_{k,n}$ is a partition such that J_λ is not a frozen variable. Fix a Postnikov diagram D such that D has an internal alternating face labelled J_λ. Let F be a different, non-boundary alternating face of D, labelled by the k-subset J_μ. Then, for $m \in [1,n]$, we have*

$$\sum_{i=1}^r c_\lambda^{(m)}(\mu(i)) = \sum_{i=1}^r c_\lambda^{(m)}(\mu'(i)).$$

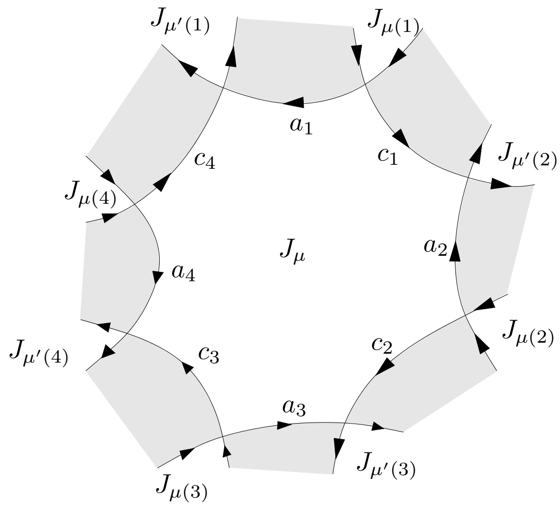


FIGURE 34. The part of a Postnikov diagram around an alternating face.

In particular, taking $m = n$, we have:

$$\sum_{i=1}^{r} c_\lambda(\mu(i)) = \sum_{i=1}^{r} c_\lambda(\mu'(i)).$$

Proof. We use the notation of Definition 16.3. We consider the strands along the boundary of the face F labelled J_μ. Note that:

$$J_{\mu(i)} = (J_\mu \setminus \{a_i\}) \cup \{c_i\};$$
$$J_{\mu'(i)} = (J_\mu \setminus \{a_i\}) \cup \{c_{i-1}\}.$$

Here we adopt the convention that the subscripts of the a_i and c_i are interpreted modulo r. Note that since $c_i \notin J_\mu$ and $a_j \in J_\mu$, we have $c_i \neq a_j$ for all i, j.

We consider first the untwisted case, $m = n$. We express $c_\lambda(\mu(i))$ in terms of $c_\lambda(\mu)$ and summands C_i and A_i associated to the strands c_i and a_i, respectively, which separate μ from $\mu(i)$. The C_i and A_i are defined in such a way that simultaneously $c_\lambda(\mu'(i)) = c_\lambda(\mu) + C_{i-1} + A_i$ (see the claim below).

For subsets I, I' of $[1, n]$ we write $I < I'$ to indicate that every element of I is less than every element of I' and for an element a we write $a < I$ (respectively, $I < a$) to denote $\{a\} < I$ (respectively, $I < \{a\}$). By Theorem 16.2 J_λ and J_μ are non-crossing. Since $J_\lambda \neq J_\mu$ there are four possible configurations for the subsets $J_\lambda \setminus J_\mu$ and $J_\mu \setminus J_\lambda$ in $[1, n]$:

(I) $J_\mu \setminus J_\lambda < J_\lambda \setminus J_\mu$;
(II) $J_\lambda \setminus J_\mu = K_1 \sqcup K_2$, $K_1, K_2 \neq \emptyset$ and $K_1 < J_\mu \setminus J_\lambda < K_2$;
(III) $J_\lambda \setminus J_\mu < J_\mu \setminus J_\lambda$;
(IV) $J_\mu \setminus J_\lambda = K_1 \sqcup K_2$, $K_1, K_2 \neq \emptyset$ and $K_1 < J_\lambda \setminus J_\mu < K_2$.

We display these in Figure 35.

In each case, we define integers C_i and A_i for $i = 1, 2, \ldots, r$ (again treating subscripts modulo r), as follows. Note that C_i depends only on λ, μ and c_i and similarly A_i depends on λ, μ and a_i.

Case I: If $J_\mu \setminus J_\lambda < J_\lambda \setminus J_\mu$, then set

$$C_i = \begin{cases} 0 & c_i \in J_\lambda; \\ 0 & c_i < J_\lambda \setminus J_\mu,\ c_i \notin J_\lambda; \\ 1 & c_i > J_\lambda \setminus J_\mu,\ c_i \notin J_\lambda; \end{cases} \quad \text{and} \quad A_i = \begin{cases} 0 & a_i \notin J_\lambda; \\ 1 & a_i < J_\mu \setminus J_\lambda,\ a_i \in J_\lambda; \\ 0 & a_i > J_\mu \setminus J_\lambda,\ a_i \in J_\lambda. \end{cases} \tag{16.4}$$

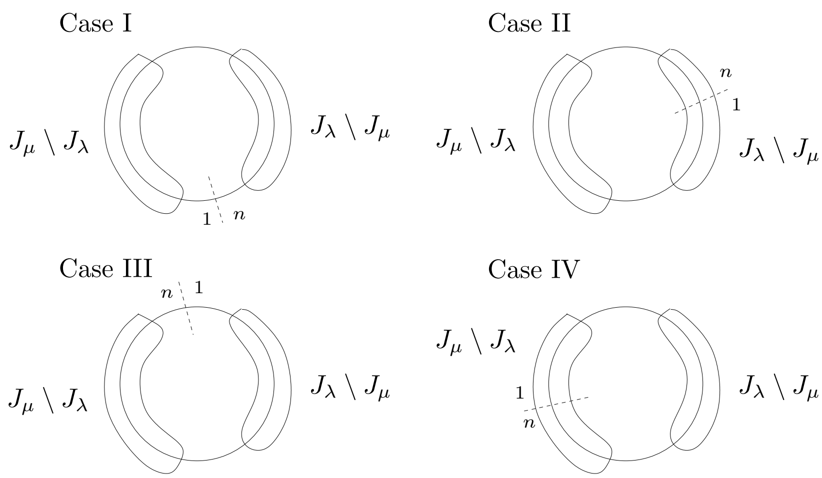


FIGURE 35. The four configurations of $J_\lambda \setminus J_\mu$ and $J_\mu \setminus J_\lambda$.

Case II: If $J_\lambda \setminus J_\mu = K_1 \sqcup K_2$ where K_1 and K_2 are nonempty and $K_1 < J_\mu \setminus J_\lambda < K_2$, then set

$$C_i = \begin{cases} 0 & c_i \notin J_\lambda; \\ -1 & c_i < J_\mu \setminus J_\lambda,\ c_i \in J_\lambda; \\ 0 & c_i > J_\mu \setminus J_\lambda,\ c_i \in J_\lambda; \end{cases} \quad \text{and} \quad A_i = \begin{cases} 0 & a_i \notin J_\lambda; \\ 1 & a_i < J_\mu \setminus J_\lambda,\ a_i \in J_\lambda; \\ 0 & a_i > J_\mu \setminus J_\lambda,\ a_i \in J_\lambda. \end{cases} \tag{16.5}$$

Case III: If $J_\lambda \setminus J_\mu < J_\mu \setminus J_\lambda$, then set

$$C_i = \begin{cases} 0 & c_i \in J_\lambda; \\ 0 & c_i < J_\lambda \setminus J_\mu,\ c_i \notin J_\lambda; \\ 1 & c_i > J_\lambda \setminus J_\mu,\ c_i \notin J_\lambda; \end{cases} \quad \text{and} \quad A_i = \begin{cases} -1 & a_i \notin J_\lambda; \\ 0 & a_i < J_\mu \setminus J_\lambda,\ a_i \in J_\lambda; \\ -1 & a_i > J_\mu \setminus J_\lambda,\ a_i \in J_\lambda. \end{cases} \tag{16.6}$$

Case IV: If $J_\mu \setminus J_\lambda = K_1 \sqcup K_2$ where K_1 and K_2 are non-empty and $K_1 < J_\lambda \setminus J_\mu < K_2$, then set

$$C_i = \begin{cases} 0 & c_i \in J_\lambda; \\ 0 & c_i < J_\lambda \setminus J_\mu,\ c_i \notin J_\lambda; \\ 1 & c_i > J_\lambda \setminus J_\mu,\ c_i \notin J_\lambda; \end{cases} \quad \text{and} \quad A_i = \begin{cases} 0 & a_i \in J_\lambda; \\ 0 & a_i < J_\lambda \setminus J_\mu,\ a_i \notin J_\lambda; \\ -1 & a_i > J_\lambda \setminus J_\mu,\ a_i \notin J_\lambda. \end{cases} \tag{16.7}$$

Claim:

(a) For $1 \le i \le r$, we have $c_\lambda(\mu(i)) - c_\lambda(\mu) = C_i + A_i$.
(b) For $1 \le i \le r$, we have $c_\lambda(\mu'(i)) - c_\lambda(\mu) = C_{i-1} + A_i$.

Proof of claim: The proof is case by case. Let $1 \le i \le r$ and set $j = i$ or $i - 1$ reduced modulo r. Set $\kappa = \mu(i)$ in the former case and $\kappa = \mu'(i)$ in the latter case. Then $J_\kappa = (J_\mu \setminus \{a_i\}) \cup \{c_j\}$.

Note that, since $c_j \notin J_\mu$ and $a_i \in J_\mu$, we can describe $J_\lambda \setminus J_\kappa$ in terms of $J_\lambda \setminus J_\mu$ and $J_\kappa \setminus J_\lambda$ in terms of $J_\mu \setminus J_\lambda$, as follows:

- If $c_j, a_i \in J_\lambda$, we have $J_\lambda \setminus J_\kappa = ((J_\lambda \setminus J_\mu) \setminus \{c_j\}) \cup \{a_i\}$ and $J_\kappa \setminus J_\lambda = J_\mu \setminus J_\lambda$.
- If $c_j \in J_\lambda, a_i \notin J_\lambda$, we have $J_\lambda \setminus J_\kappa = (J_\lambda \setminus J_\mu) \setminus \{c_j\}$ and $J_\kappa \setminus J_\lambda = (J_\mu \setminus J_\lambda) \setminus \{a_i\}$.
- If $c_j \notin J_\lambda$ and $a_i \in J_\lambda$, we have $J_\lambda \setminus J_\kappa = (J_\lambda \setminus J_\mu) \cup \{a_i\}$ and $J_\kappa \setminus J_\lambda = (J_\mu \setminus J_\lambda) \cup \{c_j\}$.
- If $c_j, a_i \notin J_\lambda$, we have $J_\lambda \setminus J_\kappa = J_\lambda \setminus J_\mu$ and $J_\kappa \setminus J_\lambda = ((J_\mu \setminus J_\lambda) \setminus \{a_i\}) \cup \{c_j\}$.

We consider each of the Cases (I)-(IV), from before, in turn. We divide each case into separate subcases depending on whether c_j, a_i lie in J_λ or not and also their position in relation to the subsets $J_\mu \setminus J_\lambda$ and $J_\lambda \setminus J_\mu$. We display the calculations as a table for each of the cases (I)–(IV). The claim then follows from the observation that the sum of the entries in the columns headed C_j and A_i coincides with the entry in the final column. Below the table for each case is a diagram illustrating the arrangement of the subsets $J_\lambda \setminus J_\mu$

and $J_\mu \setminus J_\lambda$ (note that these are not intervals, just contained in intervals in the arrangement shown, by the noncrossing property). The subset $[1, n]$ is drawn as a circle (with the numbering $1, 2, \ldots, n$ clockwise around the boundary). A dotted line cutting across the circle indicates the gap between 1 and n.

For example, in Case III, if $c_j, a_i \in J_\lambda$ and $a_i > J_\mu \setminus J_\lambda$, then we have $J_\lambda \setminus J_\mu < J_\mu \setminus J_\lambda$. We also have $J_\lambda \setminus J_\kappa = ((J_\lambda \setminus J_\mu) \setminus \{c_j\}) \cup \{a_i\}$ and $J_\kappa \setminus J_\lambda = J_\mu \setminus J_\lambda$. i.e. replacing J_μ with J_κ has the effect of removing c_j from $J_\lambda \setminus J_\mu$ and adding a_i. Since $a_i > J_\mu \setminus J_\lambda$, it can be seen from the definition that $c_\lambda(\kappa) = c_\lambda(\mu) - 1$, giving the entry -1 in the second row, final column of the table for Case III. From the definitions, $C_j = 0$ and $A_i = -1$ in this case, and we see that $C_j + A_i = c_\lambda(\kappa) - c_\lambda(\mu)$ as required. The arguments in the other cases are similar.

Occasionally we need to use the fact that the J_λ and J_κ are also noncrossing; this is why two of the subcases, as indicated in the tables, cannot occur. This is also used, for example, in Case III for the subcase $c_j \notin J_\lambda$, $a_i \in J_\lambda$, $c_j < J_\lambda \setminus J_\mu$, $a_i < J_\mu \setminus J_\lambda$. The noncrossing property implies that we must have $c_j < a_i$.

Case I: $J_\mu \setminus J_\lambda < J_\lambda \setminus J_\mu$

Membership in J_λ	Extra condition(s)	C_j	A_i	$c_\lambda(\kappa) - c_\lambda(\mu)$
$c_j, a_i \in J_\lambda$	$a_i < J_\mu \setminus J_\lambda$	0	1	1
	$a_i > J_\mu \setminus J_\lambda$	0	0	0
$c_j \in J_\lambda, a_i \notin J_\lambda$	none	0	0	0
$c_j \notin J_\lambda, a_i \in J_\lambda$	$c_j < J_\lambda \setminus J_\mu, a_i < J_\mu \setminus J_\lambda$	0	1	1
	$c_j < J_\lambda \setminus J_\mu, a_i > J_\mu \setminus J_\lambda$	0	0	0
	$c_j > J_\lambda \setminus J_\mu, a_i < J_\mu \setminus J_\lambda$	cannot occur		
	$c_j > J_\lambda \setminus J_\mu, a_i > J_\mu \setminus J_\lambda$	1	0	1
$c_j, a_i \notin J_\lambda$	$c_j < J_\lambda \setminus J_\mu$	0	0	0
	$c_j > J_\lambda \setminus J_\mu$	1	0	1

Case II: $J_\lambda \setminus J_\mu = K_1 \sqcup K_2$, $K_1, K_2 \neq \emptyset$ and $K_1 < J_\mu \setminus J_\lambda < K_2$

Membership in J_λ	Extra condition(s)	C_j	A_i	$c_\lambda(\kappa) - c_\lambda(\mu)$
$c_j, a_i \in J_\lambda$	$c_j < J_\mu \setminus J_\lambda, a_i < J_\mu \setminus J_\lambda$	-1	1	0
	$c_j < J_\mu \setminus J_\lambda, a_i > J_\mu \setminus J_\lambda$	-1	0	-1
	$c_j > J_\mu \setminus J_\lambda, a_i < J_\mu \setminus J_\lambda$	0	1	1
	$c_j > J_\mu \setminus J_\lambda, a_i > J_\mu \setminus J_\lambda$	0	0	0
$c_j \in J_\lambda, a_i \notin J_\lambda$	$c_j < J_\mu \setminus J_\lambda$	-1	0	-1
	$c_j > J_\mu \setminus J_\lambda$	0	0	0
$c_j \notin J_\lambda, a_i \in J_\lambda$	$a_i < J_\mu \setminus J_\lambda$	0	1	1
	$a_i > J_\mu \setminus J_\lambda$	0	0	0
$c_j, a_i \notin J_\lambda$	none	0	0	0

Case III: $J_\lambda \setminus J_\mu < J_\mu \setminus J_\lambda$

Membership in J_λ	Extra condition(s)	C_j	A_i	$c_\lambda(\kappa) - c_\lambda(\mu)$
$c_j, a_i \in J_\lambda$	$a_i < J_\mu \setminus J_\lambda$	0	0	0
	$a_i > J_\mu \setminus J_\lambda$	0	-1	-1
$c_j \in J_\lambda, a_i \notin J_\lambda$	none	0	-1	-1
$c_j \notin J_\lambda, a_i \in J_\lambda$	$c_j < J_\lambda \setminus J_\mu, a_i < J_\mu \setminus J_\lambda$	0	0	0
	$c_j < J_\lambda \setminus J_\mu, a_i > J_\mu \setminus J_\lambda$	cannot occur		
	$c_j > J_\lambda \setminus J_\mu, a_i < J_\mu \setminus J_\lambda$	1	0	1
	$c_j > J_\lambda \setminus J_\mu, a_i > J_\mu \setminus J_\lambda$	1	-1	0
$c_j, a_i \notin J_\lambda$	$c_j < J_\lambda \setminus J_\mu$	0	-1	-1
	$c_j > J_\lambda \setminus J_\mu$	1	-1	0

Case IV: $J_\mu \setminus J_\lambda = K_1 \sqcup K_2$, $K_1, K_2 \neq \emptyset$ and $K_1 < J_\lambda \setminus J_\mu < K_2$

Membership in J_λ	Extra condition(s)	C_j	A_i	$c_\lambda(\kappa) - c_\lambda(\mu)$
$c_j, a_i \in J_\lambda$	none	0	0	0
$c_j \in J_\lambda, a_i \notin J_\lambda$	$a_i < J_\lambda \setminus J_\mu$	0	0	0
	$a_i > J_\lambda \setminus J_\mu$	0	-1	-1
$c_j \notin J_\lambda, a_i \in J_\lambda$	$c_j < J_\lambda \setminus J_\mu$	0	0	0
	$c_j > J_\lambda \setminus J_\mu$	1	0	1
$c_j, a_i \notin J_\lambda$	$c_j < J_\lambda \setminus J_\mu, a_i < J_\lambda \setminus J_\mu$	0	0	0
	$c_j < J_\lambda \setminus J_\mu, a_i > J_\lambda \setminus J_\mu$	0	-1	-1
	$c_j > J_\lambda \setminus J_\mu, a_i < J_\lambda \setminus J_\mu$	1	0	1
	$c_j > J_\lambda \setminus J_\mu, a_i > J_\lambda \setminus J_\mu$	1	-1	0

By the claim we have

$$\sum_{i=1}^{r} c_\lambda(\mu(i)) = r c_\lambda(\mu) + \sum_{i=1}^{r} (C_i + A_i) = \sum_{i=1}^{r} c_\lambda(\mu'(i)). \tag{16.8}$$

This proves the Proposition in the case $n = m$.

For the general case, note that relabelling strand i as $i - m$ in D for each i, we obtain a new Postnikov diagram $D^{(m)}$ with a region labelled $J_{\lambda^{(m)}}$. Furthermore, the partitions corresponding to the k-subsets labelling the regions surrounding this region are $\mu'(1)^{(m)}, \mu(1)^{(m)}, \ldots, \mu'(r)^{(m)}, \mu(r)^{(m)}$. Hence, applying equation (16.8) with λ replaced by $\lambda^{(m)}$, we have, for $m \in [1, n]$:

$$\begin{aligned}\sum_{i=1}^{r} c_\lambda^{(m)}(\mu(i)) &= \sum_{i=1}^{r} \left(c_{\lambda^{(m)}}(\mu(i)^{(m)}) - c_{\lambda^{(m)}}(\emptyset^{(m)}) \right) \\ &= \sum_{i=1}^{r} \left(c_{\lambda^{(m)}}(\mu'(i)^{(m)}) - c_{\lambda^{(m)}}(\emptyset^{(m)}) \right) = \sum_{i=1}^{r} c_\lambda^{(m)}(\mu'(i)),\end{aligned}$$

and we are done. $\square$

Lemma 16.5. *Fix a Postnikov extended cluster $\widetilde{\mathcal{C}}$. Let $X_{\lambda,\widetilde{\mathcal{C}}}$ and $X^{(m)}_{\lambda,\widetilde{\mathcal{C}}}$ be the vector fields on $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$ defined in equations* (16.1) *and* (16.3)*. Then there is a regular vector field X_λ on $\check{\mathbb{X}}^\circ$ such that X_λ and $X_{\lambda,\widetilde{\mathcal{C}}}$ coincide on $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$. Similarly, for each $m \in [1, n]$, there is a regular vector field $X^{(m)}_\lambda$ on $\check{\mathbb{X}}^\circ$ such that $X^{(m)}_\lambda$ and $X^{(m)}_{\lambda,\widetilde{\mathcal{C}}}$ coincide on $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$.*

Proof. By [30, Lemma 2.3], the complement of $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}} \cup \bigcup_{p_\mu \in \widetilde{\mathcal{C}}} \check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}(\mu)}$ in $\check{\mathbb{X}}^\circ$ has codimension at least two. Hartog's Theorem says that a function that is regular on the complement of a codimension two subvariety of an algebraic variety X extends to a regular function on all of X. Hence, it suffices to prove that there is a regular extension of $X^{(m)}_{\lambda,\widetilde{\mathcal{C}}}$ to $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}} \cup \bigcup_{p_\mu \in \widetilde{\mathcal{C}}} \check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}(\mu)}$, where $\widetilde{\mathcal{C}}(\mu)$ denotes the extended cluster obtained from $\widetilde{\mathcal{C}}$ by mutating at p_μ. We consider first the case $m = n$.

Fix $\mu \neq \lambda$ such that $p_\mu \in \widetilde{\mathcal{C}}$. Let $\mu(i), \mu'(i)$, for $i = 1, 2, \ldots, r$ be as in Proposition 16.4. Then the mutation of $\widetilde{\mathcal{C}}$ at p_μ corresponds to the change of variables $\tilde{p}_\kappa = p_\kappa$ for $p_\kappa \in \widetilde{\mathcal{C}}$, $p_\kappa \neq p_\mu$, and

$$\tilde{p}_\mu = \frac{\prod_{i=1}^{r} p_{\mu(i)} + \prod_{i=1}^{r} p_{\mu'(i)}}{p_\mu}.$$

Consider the natural vector fields $p_\kappa \frac{\partial}{\partial p_\kappa}$ on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$ and the vector fields $\widetilde{p}_\kappa \frac{\partial}{\partial \widetilde{p}_\kappa}$ on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}(\mu)}$. On the intersection of these cluster tori we obtain:

$$\begin{aligned}
p_{\mu(i)}\frac{\partial}{\partial p_{\mu(i)}} &= \tilde{p}_{\mu(i)}\frac{\partial \tilde{p}_{\mu(i)}}{\partial p_{\mu(i)}}\frac{\partial}{\partial \tilde{p}_{\mu(i)}} + \tilde{p}_{\mu(i)}\frac{\partial \tilde{p}_{\mu}}{\partial p_{\mu(i)}}\frac{\partial}{\partial \tilde{p}_{\mu}} \\
&= \tilde{p}_{\mu(i)}\frac{\partial}{\partial \tilde{p}_{\mu(i)}} + \tilde{p}_{\mu(i)}\frac{\prod_{j=1,j\neq i}^r p_{\mu(j)}}{p_\mu}\frac{\partial}{\partial \tilde{p}_{\mu}} \\
&= \tilde{p}_{\mu(i)}\frac{\partial}{\partial \tilde{p}_{\mu(i)}} + \frac{\prod_{j=1}^r \tilde{p}_{\mu(j)}}{\prod_{j=1}^r \tilde{p}_{\mu(j)} + \prod_{j=1}^r \tilde{p}_{\mu'(j)}}\tilde{p}_\mu\frac{\partial}{\partial \tilde{p}_{\mu}}.
\end{aligned}$$

Similarly,

$$p_{\mu'(i)}\frac{\partial}{\partial p_{\mu'(i)}} = \tilde{p}_{\mu'(i)}\frac{\partial}{\partial \tilde{p}_{\mu'(i)}} + \frac{\prod_{j=1}^r \tilde{p}_{\mu'(j)}}{\prod_{j=1}^r \tilde{p}_{\mu(j)} + \prod_{j=1}^r \tilde{p}_{\mu'(j)}}\tilde{p}_\mu\frac{\partial}{\partial \tilde{p}_{\mu}}.$$

We also have:

$$p_\mu\frac{\partial}{\partial p_\mu} = p_\mu\frac{\partial \tilde{p}_\mu}{\partial p_\mu}\frac{\partial}{\partial \tilde{p}_\mu} = -\tilde{p}_\mu\frac{\partial}{\partial \tilde{p}_\mu}.$$

Hence, using Proposition 16.4, we have:

$$\begin{aligned}
p_\lambda \sum_{p_\kappa\in\widetilde{\mathcal{C}}} c_\lambda(\kappa)p_\kappa\frac{\partial}{\partial p_\kappa} &= \tilde{p}_\lambda \sum_{p_\kappa\in\widetilde{\mathcal{C}}, p_\kappa\neq p_\mu} c_\lambda(\kappa)\tilde{p}_\kappa\frac{\partial}{\partial \tilde{p}_\kappa} + \\
&\qquad \left(\frac{\sum_{i=1}^r c_\lambda(\mu(i))\prod_{j=1}^r \tilde{p}_{\mu(j)} + \sum_{i=1}^r c_\lambda(\mu'(i))\prod_{j=1}^r \tilde{p}_{\mu'(j)}}{\prod_{j=1}^r \tilde{p}_{\mu(j)} + \prod_{j=1}^r \tilde{p}_{\mu'(j)}} - c_\lambda(\mu)\right)\tilde{p}_\lambda\tilde{p}_\mu\frac{\partial}{\partial \tilde{p}_\mu} \\
&= \tilde{p}_\lambda \sum_{p_\kappa\in\widetilde{\mathcal{C}}, p_\kappa\neq p_\mu} c_\lambda(\kappa)\tilde{p}_\kappa\frac{\partial}{\partial \tilde{p}_\kappa} + \left(\left(\sum_{i=1}^r c_\lambda(\mu(i))\right) - c_\lambda(\mu)\right)\tilde{p}_\lambda\tilde{p}_\mu\frac{\partial}{\partial \tilde{p}_\mu}.
\end{aligned}$$

This is regular on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}(\mu)}$.

For the case $\mu = \lambda$, we have

$$p_\lambda p_{\lambda(i)}\frac{\partial}{\partial p_{\lambda(i)}} = p_\lambda\tilde{p}_{\lambda(i)}\frac{\partial}{\partial \tilde{p}_{\lambda(i)}} + \left(\prod_{j=1}^r \tilde{p}_{\lambda(j)}\right)\frac{\partial}{\partial \tilde{p}_\lambda},$$

and, similarly:

$$p_\lambda p_{\lambda'(i)}\frac{\partial}{\partial p_{\lambda'(i)}} = p_\lambda\tilde{p}_{\lambda'(i)}\frac{\partial}{\partial \tilde{p}_{\lambda'(i)}} + \left(\prod_{j=1}^r \tilde{p}_{\lambda'(j)}\right)\frac{\partial}{\partial \tilde{p}_\lambda}.$$

As before, we have

$$p_\lambda\frac{\partial}{\partial p_\lambda} = -\tilde{p}_\lambda\frac{\partial}{\partial \tilde{p}_\lambda}.$$

Hence, we have

$$p_\lambda \sum_{p_\kappa \in \widetilde{\mathcal{C}}} c_\lambda(\kappa) p_\kappa \frac{\partial}{\partial p_\kappa} = p_\lambda \sum_{\tilde{p}_\kappa \in \widetilde{\mathcal{C}}(\lambda), \tilde{p}_\kappa \neq \tilde{p}_\lambda} c_\lambda(\kappa) \tilde{p}_\kappa \frac{\partial}{\partial \tilde{p}_\kappa} +$$
$$\left(\sum_{i=1}^r c_\lambda(\lambda(i)) \prod_{j=1}^r \tilde{p}_{\lambda(j)} + \sum_{i=1}^r c_\lambda(\lambda'(i)) \prod_{j=1}^r \tilde{p}_{\lambda'(j)} - \tilde{p}_\lambda \right) \frac{\partial}{\partial \tilde{p}_\lambda}$$
$$= \frac{\left(\prod_{j=1}^r \tilde{p}_{\lambda(j)} + \prod_{j=1}^r \tilde{p}_{\lambda'(j)} \right)}{\tilde{p}_\lambda} \sum_{\tilde{p}_\kappa \in \widetilde{\mathcal{C}}(\lambda), \tilde{p}_\kappa \neq \tilde{p}_\lambda} c_\lambda(\kappa) \tilde{p}_\kappa \frac{\partial}{\partial \tilde{p}_\kappa} +$$
$$\left(\sum_{i=1}^r c_\lambda(\lambda(i)) \prod_{j=1}^r \tilde{p}_{\lambda(j)} + \sum_{i=1}^r c_\lambda(\lambda'(i)) \prod_{j=1}^r \tilde{p}_{\lambda'(j)} - \tilde{p}_\lambda \right) \frac{\partial}{\partial \tilde{p}_\lambda}.$$

This is regular on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}(\lambda)}$. Hence there is a regular extension of $X_{\lambda,\widetilde{\mathcal{C}}}$ to $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}} \cup \bigcup_{\mu \in \widetilde{\mathcal{C}}} \check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}(\mu)}$ as required.

A similar argument can be used for $X^{(m)}_{\lambda,\widetilde{\mathcal{C}}}$, using the additivity given by Proposition 16.4. □

Note that we have not shown that $X_\lambda^{(m)}$ is independent of the choice of initial extended cluster $\widetilde{\mathcal{C}}$. We expect this to hold, but we do not need it here. For $X_\lambda^{(m)}$, we shall always choose $\widetilde{\mathcal{C}}$ to be the extended cluster corresponding to the Postnikov diagram D_λ associated to λ (see Theorem 11.1).

We now fix a partition $\lambda \in \mathcal{P}_{k,n}$ and focus on the Postnikov diagram D_λ. This diagram contains a face $F(\lambda)$ labelled J_λ. Proposition 16.4 applies to every face F of D_λ except $F(\lambda)$, but we need a version of this proposition for the case $F = F(\lambda)$ also. We first introduce more notation.

Definition 16.6. Fix a partition $\lambda \in \mathcal{P}_{k,n}$. Set

$$K_\lambda = \{ i \in J_\lambda \,:\, i+1 \notin J_\lambda \}.$$

For $i \in K_\lambda$, let $\tau_i(\lambda)$ be the partition with the property that

$$J_{\tau_i(\lambda)} = (J_\lambda \setminus \{i\}) \cup \{i+1\}. \tag{16.9}$$

This is the partition which corresponds to adding a box to λ adjacent to the edge numbered i in the associated path (see Section 6.1).

If $F(\lambda)$ is internal in D_λ then in terms of the notation in Definition 16.3 we have

$$K_\lambda = \{a_1 < \cdots < a_r\},$$

and $\tau_{a_j}(\lambda) = \lambda'(j)$ for $j = 1, \ldots, r$, by the construction of D_λ in Section 11. In particular (16.9) reflects the fact that for $a_j = i$ the adjacent clockwise strand c_{j-1} around $F(\lambda)$ is $i+1$. For the other adjacent faces $\lambda(j)$ we note that

$$J_{\lambda(j)} = (J_\lambda \setminus \{a_j\}) \cup \{a_{j+1}+1\}. \tag{16.10}$$

since $c_j = a_{j+1}+1$ for $F(\lambda)$. If $F(\lambda)$ is on the boundary then $J_\lambda = L_i$ and $K_\lambda = \{i\}$ and the unique alternating face adjacent to $F(\lambda)$ is labelled with $(J_\lambda \setminus \{i\}) \cup \{i+1\}$, which equals $\widehat{L}_i$. The corresponding partition is $\tau_i(\lambda)$.

Definition 16.7. In the case $k \neq 1, n-1$, we have defined (see Definition 14.8) the faces F_i and F_i' to be the faces in D_λ adjacent to the unique crossing point P_i of strands i and $i+1$. If $k=1$ or $n-1$, there may be more than one such crossing point, but we may define F_i and F_i' in the same way: the definition does not depend on P_i.

The faces F_i and F_i' have labels I_i and I_i' respectively. Let $\kappa_{i,\lambda}$ and $\kappa'_{i,\lambda}$ be the corresponding partitions.

Our aim is to prove equation (10.4); i.e. to compute $X_\lambda^{(m)} W_q$. Thus we need to compute $X_\lambda \frac{p_{\widehat{L}_i}}{p_{L_i}}$ for each $i \in [1,n]$. We saw in Theorem 15.8 that

$$\frac{p_{\widehat{L}_i}}{p_{L_i}} = \frac{p_{I_i'}}{p_{I_i}} + \cdots$$

where $\frac{p_{I_i'}}{p_{I_i}}$ is the summand corresponding to M_i in the formula (12.2). In order to compute the action of $X_\lambda^{(m)}$ on $\frac{p_{I_i'}}{p_{I_i}}$, we need to know the value of $c_\lambda^{(m)}(\kappa_{i,\lambda}') - c_\lambda^{(m)}(\kappa_{i,\lambda})$ for each i. We first consider the case where $i \in K_\lambda$, observing the following (which follows from the construction of D_λ).

Remark 16.8. If $i \in K_\lambda$, the faces $F(\lambda)$ and F_i coincide. We have $\kappa_{i,\lambda} = \lambda$ and $\kappa_{i,\lambda}' = \tau_i(\lambda)$, since $I_i' = (I_i \setminus \{i\}) \cup \{i+1\}$. See the construction in Section 11.

Lemma 16.9. *Fix a partition $\lambda \in \mathcal{P}_{k,n}$, and let $D = D_\lambda$. Then, for $i \in K_\lambda$ and $m \in [1,n]$, we have*

$$c_\lambda^{(m)}(\tau_i(\lambda)) - c_\lambda^{(m)}(\lambda) = \begin{cases} 0, & \text{if } i = m; \\ 1, & \text{if } i \neq m. \end{cases}$$

Proof. Recall that, for any partition μ, we have $c_\mu^{(n)}(\mu) = c_\mu(\mu) = 0$. We first show that:

$$c_\lambda(\tau_i(\lambda)) = \begin{cases} 0, & \text{if } i = n; \\ 1, & \text{otherwise.} \end{cases} \tag{16.11}$$

For $i \in K_\lambda$, we have $J_\lambda \setminus J_{\tau_i(\lambda)} = \{i\}$ and $J_{\tau_i(\lambda)} \setminus J_\lambda = \{i+1\}$. If $i \neq n$, then $i+1 > i$, so $c_\lambda(\tau_i(\lambda)) = 1$. If $i = n$, then $i+1 = 1 < n = i$, so $c_\lambda(\tau_i(\lambda)) = 0$, and (16.11) is shown.

Using the definition (see equation (16.2)) of $c_\lambda^{(m)}$, we have that

$$\begin{aligned} c_\lambda^{(m)}(\tau_i(\lambda)) - c_\lambda^{(m)}(\lambda) &= c_{\lambda^{(m)}}\left(\tau_i(\lambda)^{(m)}\right) - c_{\lambda^{(m)}}\left(\lambda^{(m)}\right) \\ &= c_{\lambda^{(m)}}\left(\tau_{i-m}(\lambda^{(m)})\right), \end{aligned}$$

where the last equality follows from (16.9). The result then follows from (16.11) (replacing λ with $\lambda^{(m)}$). □

Using the definition of $X_\lambda^{(m)}$ (see equation (16.3) and Lemma 16.5) Lemma 16.9 implies that, for $i \in K_\lambda$,

$$X_\lambda^{(m)} \frac{p_{I_i'}}{p_{I_i}} = \begin{cases} 0, & i = m; \\ \dfrac{p_{I_i'}}{p_{I_i}} p_\lambda, & i \neq m. \end{cases} \tag{16.12}$$

In order to compute the action of $X_\lambda^{(m)}$ on the summands in the formula (10.4) corresponding to the matchings other than M_i (in the case $i \in K_\lambda$), we will need the following.

Proposition 16.10. *Assume that $F(\lambda)$ is an internal face of D_λ. Then, for any $m \in [1,n]$, we have:*

$$\sum_{j=1}^r c_\lambda^{(m)}(\lambda'(j)) - \sum_{j=1}^r c_\lambda^{(m)}(\lambda(j)) = 1.$$

Proof. We first prove that:

$$\sum_{j=1}^r c_\lambda(\lambda'(j)) - \sum_{j=1}^r c_\lambda(\lambda(j)) = 1. \tag{16.13}$$

Recall that

$$K_\lambda = \{a_1 < a_2 < \cdots < a_r\}.$$

We will interpret the subscripts j of the a_j modulo r. By our assumption on J_λ, we have $r > 1$.

By equation (16.11) in Lemma 16.9, we have

$$\sum_{j=1}^{r} c_\lambda\left(\lambda'(j)\right) = \sum_{j=1}^{r} c_\lambda\left(\tau_{a_j}(\lambda)\right) = \begin{cases} r, & \text{if } a_r \neq n; \\ r-1, & \text{if } a_r = n. \end{cases} \tag{16.14}$$

We will show that

$$\sum_{j=1}^{r} c_\lambda\left(\lambda(j)\right) = \begin{cases} r-1, & \text{if } a_r \neq n; \\ r-2, & \text{if } a_r = n; \end{cases} \tag{16.15}$$

from which (16.13) follows.

By (16.10), we have $J_\lambda \setminus J_{\lambda(j)} = \{a_j\}$ and $J_{\lambda(j)} \setminus J_\lambda = \{a_{j+1}+1\}$.

Suppose first that $a_r \neq n$. If $1 \leq j \leq r-1$ then $a_{j+1}+1 > a_j$, so $c_\lambda(\lambda(j)) = 1$. In the case $j = r$, we have

$$a_{r+1}+1 = a_1+1 < a_2 \leq a_r,$$

as $a_1+1 \notin K_\lambda$ and $r \geq 2$. Hence, $c_\lambda(\lambda(r)) = 0$. We obtain $\sum_{j=1}^{r} c_\lambda(\lambda(j)) = r-1$, as required.

We are left with the case $a_r = n$. If $1 \leq j \leq r-2$, then $a_{j+1}+1 > a_j$, so $c_\lambda(\lambda(j)) = 1$. We have, for the case $j = r-1$ (reducing modulo n as usual)

$$a_{(r-1)+1}+1 = a_r+1 = n+1 = 1 < a_{r-1},$$

since $r-1 \geq 1$ and $1 \notin K_\lambda$. Hence $c_\lambda(\lambda(r-1)) = 0$. For the case $j = r$, we have

$$a_{r+1}+1 = a_1+1 < a_2 \leq a_r = n,$$

so $a_{r+1}+1 < a_r$, and $c_\lambda(\lambda(r)) = 0$. We obtain $\sum_{i=1}^{r} c_\lambda(\lambda(j)) = r-2$.

We have now shown that (16.13) holds, which is the case $m = n$ of the proposition. For the general case, we have, using the definition of $c_\lambda^{(m)}$,

$$\sum_{j=1}^{r} c_\lambda^{(m)}\left(\lambda'(j)\right) - \sum_{j=1}^{r} c_\lambda^{(m)}(\lambda(j)) = \sum_{j=1}^{r} c_{\lambda^{(m)}}\left(\lambda'(j)^{(m)}\right) - \sum_{j=1}^{r} c_{\lambda^{(m)}}\left(\lambda(j)^{(m)}\right). \tag{16.16}$$

By the construction of D_λ (see Remark 11.7), the sets $\{\lambda(j)^{(m)} : j = 1,\ldots,r\}$ and $\{\lambda^{(m)}(j) : j = 1,\ldots,r\}$ coincide, and the sets $\{\lambda'(j)^{(m)} : j = 1,\ldots,r\}$ and $\{(\lambda')^{(m)}(j) : j = 1,\ldots,r\}$ coincide. The result follows. □

To obtain the analogue of equation (16.12) in the case $i \notin K_\lambda$, we need to compute $c_\lambda^{(m)}\left(\kappa'_{i,\lambda}\right) - c_\lambda^{(m)}(\kappa_{i,\lambda})$ for $i \notin K_\lambda$; this is covered by the following proposition.

Proposition 16.11. *Fix a partition $\lambda \in \mathcal{P}_{k,n}$. Let $D = D_\lambda$ be the Postnikov diagram constructed in Theorem 11.1. Suppose that $i \notin K_\lambda$. Then we have:*

$$c_\lambda^{(m)}\left(\kappa'_{i,\lambda}\right) - c_\lambda^{(m)}(\kappa_{i,\lambda}) = \begin{cases} -1, & i = m; \\ 0, & i \neq m. \end{cases}$$

Proof. We first consider the case $m = n$, i.e. we show that:

$$c_\lambda\left(\kappa'_{i,\lambda}\right) - c_\lambda(\kappa_{i,\lambda}) = \begin{cases} -1, & i = n; \\ 0, & i \neq n. \end{cases} \tag{16.17}$$

We divide the proof into three cases, depending on whether $i, i+1$ lie in J_λ or not (note that, since we assume $i \notin K_\lambda$, we cannot have the case $i \in J_\lambda, i+1 \notin J_\lambda$).

Case (a) $i, i+1 \notin J_\lambda$.

Then we have:

$$I'_i \setminus J_\lambda = ((I_i \setminus J_\lambda) \setminus \{i\}) \cup \{i+1\};$$
$$J_\lambda \setminus I'_i = J_\lambda \setminus I_i.$$

If $i \neq n$, then, since $i, i+1 \notin J_\lambda \setminus I_i$ and $i, i+1 \notin J_\lambda \setminus I'_i$, we have that $c_\lambda(\kappa_{i,\lambda}) = c_\lambda\left(\kappa'_{i,\lambda}\right)$, as required. We now consider the case $i = n$. Since I_n and J_λ are noncrossing and $n \in I_n \setminus J_\lambda$, we may write $I_n \setminus J_\lambda = K_1 \sqcup K_2$, where $K_1 < J_\lambda \setminus I_n < K_2$ and $n \in K_2$. Note that K_1 may be empty. We have $c_\lambda(\kappa_{n,\lambda}) = |K_2|$. Let $K'_1 = K_1 \cup \{1\}$ and $K'_2 = K_2 \setminus \{n\}$. Then we have $I'_n \setminus J_\lambda = K'_1 \sqcup K'_2$, with $K'_1 < J_\lambda \setminus I_n < K'_2$. Note that K'_1 is nonempty, but K'_2 may be empty. We then have $c_\lambda\left(\kappa'_{n,\lambda}\right) = |K'_2| = |K_2| - 1 = c_\lambda(\kappa_{n,\lambda}) - 1$ as required.

Case (b) $i \notin J_\lambda, i+1 \in J_\lambda$.

Then we have $i \in I_i \setminus J_\lambda$, $i+1 \in J_\lambda \setminus I_i$, and:

$$\begin{aligned} I'_i \setminus J_\lambda &= (I_i \setminus J_\lambda) \setminus \{i\}; \\ J_\lambda \setminus I'_i &= (J_\lambda \setminus I_i) \setminus \{i+1\}. \end{aligned}$$

Since I_i and J_λ are noncrossing, we must have one of the following three cases.

Case (b)(i): $J_\lambda \setminus I_i = K_1 \sqcup K_2$, $K_{,1}$ and K_2 are nonempty and $K_1 < I_i \setminus J_\lambda < K_2$. We have $c_\lambda(\kappa_{i,\lambda}) = |K_1|$. Since $i \in I_i \setminus J_\lambda$ and $i+1 \in J_\lambda \setminus I_i$, we must have $i+1 \in K_2$. Let $K'_2 = K_2 \setminus \{i+1\}$. Then we have $J_\lambda \setminus I'_i = K_1 \sqcup K'_2$, where $K_1 < I'_i \setminus J_\lambda < K'_2$. Note that K'_2 may be empty. We see that $c_\lambda\left(\kappa'_{i,\lambda}\right) = |K_1| = c_\lambda(\kappa_{i,\lambda})$, as required (note that $i < i+1$ in this case, so $i \neq n$).

Case (b)(ii): $J_\lambda \setminus I_i < I_i \setminus J_\lambda$. Then, since $i \in I_i \setminus J_\lambda$ and $i+1 \in J_\lambda \setminus I_i$, we have $i = n$. We have $J_\lambda \setminus I'_i < I'_i \setminus J_\lambda$, so $c_\lambda\left(\kappa'_{i,\lambda}\right) = c_\lambda(\kappa_{i,\lambda}) - 1$, as required.

Case (b)(iii): $I_i \setminus J_\lambda = K_1 \sqcup K_2$, K_1 is nonempty and $K_1 < J_\lambda \setminus I_i < K_2$. We have $c_\lambda(\kappa_{i,\lambda}) = |K_2|$. Since $i \in I_i \setminus J_\lambda$ and $i+1 \in J_\lambda \setminus I_i$ we have $i \in K_1$. Let $K'_1 = K_1 \setminus \{i\}$. Then we have $I'_i \setminus J_\lambda = K'_1 \sqcup K_2$, where $K'_1 < J_\lambda \setminus I'_i < K_2$. Note that K'_1 may be empty. We see that $c_\lambda\left(\kappa'_{i,\lambda}\right) = |K_2| = c_\lambda(\kappa_{i,\lambda})$ as required (note that $i < i+1$ in this case so $i \neq n$).

This completes case (b).

Case (c) $i, i+1 \in J_\lambda$. Then we have:

$$\begin{aligned} I'_i \setminus J_\lambda &= I_i \setminus J_\lambda; \\ J_\lambda \setminus I'_i &= ((J_\lambda \setminus I_i) \setminus \{i+1\}) \cup \{i\}. \end{aligned}$$

If $i \neq n$, then, since $i, i+1 \notin J_\lambda \setminus I_i$ and $i, i+1 \notin J_\lambda \setminus I'_i$, we have that $c_\lambda(\kappa_{i,\lambda}) = c_\lambda\left(\kappa'_{i,\lambda}\right)$, as required. We now consider the case $i = n$. Since I_n and J_λ are noncrossing and $1 \in J_\lambda \setminus I_n$, we may write $J_\lambda \setminus I_n = K_1 \sqcup K_2$, where $K_1 < I_n \setminus J_\lambda < K_2$ and $1 \in K_1$. Note that K_2 may be empty. We have $c_\lambda(\kappa_{n,\lambda}) = |K_1|$. Let $K'_1 = K_1 \setminus \{1\}$ and $K'_2 = K_2 \cup \{n\}$. We have $J_\lambda \setminus I'_i = K'_1 \sqcup K'_2$, where $K'_1 < I'_i \setminus J_\lambda < K'_2$. Note that K'_2 is nonempty, but K'_1 may be empty. We then have $c_\lambda\left(\kappa'_{n,\lambda}\right) = |K'_1| = |K_1| - 1 = c_\lambda(\kappa_{n,\lambda}) - 1$ as required. This completes case (c).

We have completed the proof of (16.17), and thus the proof of the statement required for $m = n$.

The result for arbitrary m follows from Remark 11.7 and (16.17) as follows, using the definition (16.2) of $c_\lambda^{(m)}$:

$$\begin{aligned} c_\lambda^{(m)}\left(\kappa'_{i,\lambda}\right) - c_\lambda^{(m)}(\kappa_{i,\lambda}) &= c_{\lambda^{(m)}}\left(\left(\kappa'_{i,\lambda}\right)^{(m)}\right) - c_{\lambda^{(m)}}\left(\kappa_{i,\lambda}^{(m)}\right) \\ &= c_{\lambda^{(m)}}\left(\kappa'_{i-m,\lambda^{(m)}}\right) - c_{\lambda^{(m)}}\left(\kappa_{i-m,\lambda^{(m)}}\right) \\ &= \begin{cases} -1, & i - m = n \mod n; \\ 0, & i - m \neq n \mod n; \end{cases} \\ &= \begin{cases} -1, & i = m; \\ 0, & i \neq m; \end{cases} \end{aligned}$$

as required. □

Proposition 16.11 implies that, for $i \notin K_\lambda$, we have

$$X_\lambda^{(m)} \frac{p_{I'_i}}{p_{I_i}} = \begin{cases} -\dfrac{p_{I'_i}}{p_{I_i}} p_\lambda, & i = m; \\ 0, & i \neq m. \end{cases} \tag{16.18}$$

17. Action of the vector field X_λ on W_q

Fix a partition $\lambda \in \mathcal{P}_{k,n}$. Our main aim in this section is to compute the action of $X_\lambda^{(m)}$ on W for each $m \in [1, n]$ (Theorem 17.3). We include the cases $k = 1, n-1$. Let $D = D_\lambda$ be the Postnikov diagram associated to λ constructed in Theorem 11.1. Recall that D has a face $F(\lambda)$ labelled J_λ. Let $\widetilde{\mathcal{C}}$ denote the Postnikov extended cluster associated to D.

We collect together the information we will need. Recall first that, on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$, the vector field $X_\lambda^{(m)}$ is given by the following formula (see (16.1)).

$$X_\lambda^{(m)} = p_\lambda \sum_{p_\mu \in \widetilde{\mathcal{C}}} c_\lambda^{(m)}(\mu) p_\mu \frac{\partial}{\partial p_\mu}. \tag{17.1}$$

Recall also that the superpotential W_q on $\check{\mathbb{X}}^\circ$ is given by:

$$W_q = \sum_{i=1}^{n} q^{\delta_{i,n}} \frac{p_{\widehat{L}_i}}{p_{L_i}}, \tag{17.2}$$

where, for $i \in [1, n]$, $L_i = [i-k+1, i]$ and $\widehat{L}_i = [i-k+1, i-1] \cup \{i+1\}$.

Let $D_1, \ldots, D_n$ be the Postnikov diagrams associated to D in Definition 12.1, with corresponding weighted dual bipartite graphs $G_1, G_2, \ldots, G_n$.

Let

$$p_{\widetilde{\mathcal{C}}} = \frac{p_{L_{i-1}} p_{L_{i+1}} \cdots p_{L_{i+k}}}{\prod_{p \in \mathcal{C}} p}.$$

Then, by Theorem 12.3, we have the following formula in $\mathbb{C}[\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}]$:

$$\frac{p_{\widehat{L}_i}}{p_{L_i}} = \sum_M p_{\widetilde{\mathcal{C}}} w_M, \tag{17.3}$$

where the sum is over all perfect matchings M of G_i and w_M denotes the matching monomial associated to a perfect matching M.

Suppose first that $k \neq 1, n-1$. Recall from Definition 14.8 that F_i is the alternating face which is to the left of strand i and to the right of strand $i+1$ and adjacent to the crossing point P_i of strands i and $i+1$. It has label I_i. Furthermore, F'_i is the alternating face adjacent to P_i on the other side of γ_i, i.e. to the right of strand i and to the left of strand $i+1$, with label I'_i. Recall also that:

$$K_\lambda = \{i \in J_\lambda \,:\, i+1 \notin J_\lambda\}.$$

Remark 17.1. If $i \in K_\lambda$ we have seen (Remark 16.8) that the faces $F(\lambda)$ and F_i coincide. If $i \notin K_\lambda$ then, since the label of every face of G_i^{in} contains i but not $i+1$, the face $F(\lambda)$ cannot be a face of G_i^{in}.

By Corollary 14.13 that the elementary components of G_i consist of G_i^{in} (see Definition 14.4) together with certain single edges on the remaining vertices. By Theorem 15.8, if F_i is an internal face, then the perfect matchings on G_i are obtained from an initial matching, M_i, in which F_i is flippable, namely by flipping F_i and then performing a sequence of flips (possibly empty) not involving F_i. The flips all take place in faces of G_i^{in}. If F_i is a boundary face, there is a unique perfect matching M_i on G_i.

If $k = 1$ or $n-1$, recall that F_i is defined in the same way as above (and doesn't depend on a choice of crossing point P_i of strands i and $i+1$). If $i \in K_\lambda$ then the faces $F(\lambda)$ and F_i coincide. For all i, we have that F_i is a boundary face and there is a unique matching M_i on G_i (see Remark 13.16). Each elementary component of G_i consists of a single edge e where e is an edge in M_i (see Corollary 14.13).

For any k we have, by Theorem 15.8:

$$p_{\widetilde{\mathcal{C}}} w_{M_i} = \frac{p_{I_i'}}{p_{I_i}} \tag{17.4}$$

for all $i \in [1, n]$.

We also have the following (see (16.12) and (16.18)):

$$X_\lambda^{(m)} \frac{p_{I_i'}}{p_{I_i}} = \begin{cases} 0, & i = m, i \in K_\lambda; \\ \dfrac{p_{I_i'}}{p_{I_i}} p_\lambda, & i \neq m, i \in K_\lambda; \\ -\dfrac{p_{I_i'}}{p_{I_i}} p_\lambda, & i = m, i \notin K_\lambda; \\ 0, & i \neq m, i \notin K_\lambda. \end{cases} \tag{17.5}$$

Lemma 17.2. *Let M, M' be perfect matchings on the bipartite graph G dual to a Postnikov diagram D, with M' obtained from M by flipping around an M-flippable face F_0 of G. Then*

$$\frac{w_{M'}}{w_M} = \frac{\prod_F p_F}{\prod_{F'} p_{F'}},$$

where the product in the numerator is over the faces F of G sharing an edge in M with F_0 and the product in the denominator is over the faces F' of G sharing an edge in M' with F_0.

Proof. This is easily seen to hold by considering the weights on the edges in G around F_0. □

We now have all the ingredients we need to prove the main result of this section.

Theorem 17.3. *Let λ be an arbitrary Young diagram in $\mathcal{P}_{k,n}$ and $m \in [1, n]$. Then we have:*

$$X_\lambda^{(m)} W_q = \left(\sum_\mu p_\mu + q \sum_\nu p_\nu \right) - q^{\delta_{mn}} \frac{p_{\widehat{L}_m}}{p_{L_m}} p_\lambda, \tag{17.6}$$

where μ, ν are exactly as in the quantum version of Monk's rule for $\sigma^\square *_q \sigma^\lambda$.

Proof. It suffices to check this on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$, since it is open dense in $\check{\mathbb{X}}^\circ$. Note that (17.6) can be rewritten as:

$$X_\lambda^{(m)} W_q = \left(\sum_{i \in K_\lambda} q^{\delta_{in}} p_{\tau_i(\lambda)} \right) - q^{\delta_{mn}} \frac{p_{\widehat{L}_m}}{p_{L_m}} p_\lambda. \tag{17.7}$$

As in Definition 16.6, we write

$$K_\lambda = \{a_1 < a_2 < \cdots < a_r\}.$$

We will show that

$$X_\lambda^{(m)} \frac{p_{\widehat{L}_i}}{p_{L_i}} = \begin{cases} p_{\tau_i(\lambda)} - \dfrac{p_{\widehat{L}_i}}{p_{L_i}} p_\lambda, & i = m, i \in K_\lambda; \\ p_{\tau_i(\lambda)}, & i \neq m, i \in K_\lambda; \\ -\dfrac{p_{\widehat{L}_i}}{p_{L_i}} p_\lambda, & i = m, i \notin K_\lambda; \\ 0, & i \neq m, i \notin K_\lambda. \end{cases} \tag{17.8}$$

The result follows from this, since then we have:

$$\begin{aligned} X_\lambda^{(m)} W_q &= X_\lambda \sum_{i=1}^n q^{\delta_{in}} \frac{p_{\widehat{L}_i}}{p_{L_i}} \\ &= \left(\sum_{i \in K_\lambda} q^{\delta_{in}} p_{\tau_i(\lambda)} \right) - q^{\delta_{mn}} \frac{p_{\widehat{L}_m}}{p_{L_m}} p_\lambda. \end{aligned}$$

We divide the proof into three cases: the first with F_i an internal face of D and $i \in K_\lambda$, the second with F_i an internal face of D and $i \notin K_\lambda$, and the third with F_i a boundary face of D.

Case I: Suppose that F_i is an internal face of D and $i \in K_\lambda$. By Remark 17.1, $F(\lambda) = F_i$, with label $I_i = J_\lambda$; by assumption this is not a boundary face of D. The face F_i' is labelled with $I_i' = J_{\tau_i(\lambda)}$. (see Definition 16.6). By (17.4) and (17.5) and, we have:

$$X_\lambda^{(m)}\left(p_{\widetilde{C}} w_{M_i}\right) = \begin{cases} 0, & i = m; \\ p_{\widetilde{C}} w_{M_i} p_\lambda, & i \neq m. \end{cases} \tag{17.9}$$

Let M_i' be the perfect matching obtained by flipping M_i at the face $F(\lambda) = F_i$. Then we have, by Lemma 17.2 and Proposition 16.10:

$$\begin{aligned} X_\lambda^{(m)}\left(\frac{w_{M_i'}}{w_{M_i}}\right) &= X_\lambda^{(m)}\left(\frac{\prod_{j=1}^r p_{\lambda(j)}}{\prod_{j=1}^r p_{\lambda'(j)}}\right) \\ &= \left(\sum_{j=1}^r c_\lambda^{(m)}(\lambda(j)) - \sum_{j=1}^r c_\lambda^{(m)}(\lambda'(j))\right) \frac{\prod_{j=1}^r p_{\lambda(j)}}{\prod_{j=1}^r p_{\lambda'(j)}} p_\lambda = -\frac{w_{M_i'}}{w_{M_i}} p_\lambda, \end{aligned} \tag{17.10}$$

using the definition of $X_\lambda^{(m)}$. Hence, by (17.9), (17.10) and the Leibniz rule,

$$X_\lambda^{(m)}\left(p_{\widetilde{C}} w_{M_i'}\right) = X_\lambda^{(m)}\left(p_{\widetilde{C}} w_{M_i} \frac{w_{M_i'}}{w_{M_i}}\right) = \begin{cases} -p_{\widetilde{C}} w_{M_i'} p_\lambda, & i = m; \\ 0, & i \neq m. \end{cases} \tag{17.11}$$

Let M_i'' be the perfect matching obtained from M_i' by flipping at some M_i'-flippable face not equal to $F(\lambda)$. By Lemma 17.2 and Proposition 16.4,

$$X_\lambda^{(m)}\left(\frac{w_{M_i''}}{w_{M_i'}}\right) = 0. \tag{17.12}$$

Hence, by (17.11), (17.12) and the Leibniz rule,

$$X_\lambda^{(m)}(p_{\widetilde{C}} w_{M_i''}) = \begin{cases} -p_{\widetilde{C}} w_{M_i''} p_\lambda, & i = m; \\ 0, & i \neq m. \end{cases}$$

By Theorem 15.8, any perfect matching on G_i can be reached from M_i by a sequence of face flips involving faces of G_i^{in} other than $F(\lambda)$. So, repeating the argument above and using (17.3), we obtain the following, where the sums are over all perfect matchings M of G_i.

$$\begin{aligned} X_\lambda^{(m)}\left(\frac{p_{\widehat{L}_i}}{p_{L_i}}\right) = X_\lambda^{(m)}\left(\sum_M p_{\widetilde{C}} w_M\right) &= \begin{cases} p_{\widetilde{C}} w_{M_i} p_\lambda - \sum_M p_{\widetilde{C}} w_M p_\lambda, & i = m; \\ p_{\widetilde{C}} w_{M_i} p_\lambda, & i \neq m; \end{cases} \\ &= \begin{cases} \dfrac{p_{I_i'}}{p_{I_i}} p_\lambda - \dfrac{p_{\widehat{L}_i}}{p_{L_i}} p_\lambda, & i = m; \\ \dfrac{p_{I_i'}}{p_{I_i}} p_\lambda, & i \neq m. \end{cases} \end{aligned}$$

Note that $p_{I_i} = p_\lambda$ and $p_{I_i'} = p_{\tau_i(\lambda)}$. Hence, we have:

$$X_\lambda^{(m)}\left(\frac{p_{\widehat{L}_i}}{p_{L_i}}\right) = \begin{cases} p_{\tau_i(\lambda)} - \dfrac{p_{\widehat{L}_i}}{p_{L_i}} p_\lambda, & i = m; \\ p_{\tau_i(\lambda)}, & i \neq m; \end{cases}$$

as required in this case.

Case II: Suppose that F_i is an internal face and $i \notin K_\lambda$.
Then, using (17.4) and (17.5), we have that:

$$X_\lambda^{(m)}\left(p_{\widetilde{C}} w_{M_i}\right) = \begin{cases} -p_{\widetilde{C}} w_{M_i} p_\lambda, & i = m; \\ 0, & i \neq m. \end{cases} \tag{17.13}$$

Since $i \notin K_\lambda$, $F(\lambda)$ is not a face of G_i^{in} by Remark 17.1. Hence all perfect matchings can be obtained from M_i by flips not involving $F(\lambda)$ by Theorem 15.8. If M_i' is obtained from M_i by the flip of a face in G_i^{in}, then, by the definition of $X_\lambda^{(m)}$ and applying Lemma 17.2 and Proposition 16.4:

$$X_\lambda^{(m)}\left(\frac{w_{M_i'}}{w_{M_i}}\right) = 0. \tag{17.14}$$

Hence, by (17.13), (17.14) and the Leibniz rule,

$$X_\lambda^{(m)}\left(p_{\widetilde{C}} w_{M_i'}\right) = \begin{cases} -p_{\widetilde{C}} w_{M_i'} p_\lambda, & i = m; \\ 0, & i \neq m. \end{cases}$$

Repeating this argument, we see that for all perfect matchings M of G_i,

$$X_\lambda^{(m)}\left(p_{\widetilde{C}} w_M\right) = \begin{cases} -p_{\widetilde{C}} w_M p_\lambda, & i = m; \\ 0, & i \neq m; \end{cases}$$

and therefore, using (17.3), we obtain the following, where the sum is over all perfect matchings M of G_i:

$$\begin{aligned} X_\lambda^{(m)}\left(\frac{p_{\widehat{L}_i}}{p_{L_i}}\right) &= \begin{cases} -\sum_M p_{\widetilde{C}} w_M p_\lambda, & i = m; \\ 0, & i \neq m; \end{cases} \\ &= \begin{cases} -\dfrac{p_{\widehat{L}_m}}{p_{L_m}} p_\lambda, & i = m; \\ 0, & i \neq m; \end{cases} \end{aligned}$$

as required in this case.
Case III: Suppose that F_i lies on the boundary of D. Then its label, $I_i = L_j = [j-k+1, j]$ for some $j \in [1, n]$. Since $i \in I_i$ and $i+1 \notin I_i$, we must have $j = i$ and $L_i = I_i$, so $\widehat{L}_i = I_i'$.

We consider the cases of (17.8). If $i \in K_\lambda$ then, by Remark 17.1, $p_{I_i} = p_\lambda$ and $p_{I_i'} = p_{\tau_i(\lambda)}$. If, in addition, $i = m$, then

$$p_{\tau_i(\lambda)} - \frac{p_{\widehat{L}_i}}{p_{L_i}} p_\lambda = p_{I_i'} - \frac{p_{I_i'}}{p_{I_i}} p_{I_i} = 0. \tag{17.15}$$

If $i \neq m$, then

$$p_{\tau_i(\lambda)} = p_{I_i'} = \frac{p_{I_i'}}{p_{I_i}} p_\lambda. \tag{17.16}$$

Also, if $i \notin K_\lambda$ and $i = m$, then

$$-\frac{p_{\widehat{L}_i}}{p_{L_i}} p_\lambda = -\frac{p_{I_i'}}{p_{I_i}} p_\lambda. \tag{17.17}$$

Equation (17.8) in this case now follows from (17.15), (17.16), (17.17) and (17.5). The proposition is proved. □

18. Completion of the proof of Theorem 4.1

In this section we complete the proof of Theorem 4.1. Namely, we will show that:

$$[q\frac{\partial W}{\partial q}p_\lambda\omega] = \sum_\mu [p_\mu\omega] + q\sum_\nu [p_\nu\omega]; \tag{18.1}$$

$$\frac{1}{z}[W_q p_\lambda\omega] = \frac{n}{z}\left(\sum_\mu [p_\mu\omega] + q\sum_\nu [p_\nu\omega]\right) - |\lambda|[p_\lambda\omega], \tag{18.2}$$

where μ, ν are exactly as in the quantum Monk's rule for $\sigma^\Box *_q \sigma^\lambda$.

We first of all note a corollary to Theorem 17.3.

Corollary 18.1. *Let λ be an arbitrary Young diagram in $\mathcal{P}_{k,n}$. Then we have:*
(a)

$$X_\lambda W_q = \left(\sum_\mu p_\mu + q\sum_\nu p_\nu\right) - q\frac{\partial W}{\partial q}p_\lambda;$$

(b)

$$\sum_{m=1}^n X_\lambda^{(m)} W_q = n\left(\sum_\mu p_\mu + q\sum_\nu p_\nu\right) - W_q p_\lambda,$$

where μ, ν are exactly as in the quantum Monk's rule for $\sigma^\Box *_q \sigma^\lambda$.

Proof. Part (a) is the case $m = n$ in Theorem 17.3. To see part (b) we add up the cases $m = 1, 2, \ldots, n$. $\Box$

Let ξ be a regular vector field on $\check{\mathbb{X}}^\circ$ and denote by $i_\xi\omega$ the insertion of ξ into ω. Then we obtain a relation

$$\left[d(i_\xi\omega) + \frac{1}{z}dW_q \wedge i_\xi\omega\right] = 0$$

in G^W by applying $d + \frac{1}{z}dW_q\wedge -$ to the $(n-1)$-form $i_\xi\omega$. Since $dW_q\wedge\omega = 0$ we have $dW_q\wedge i_\xi\omega = (i_\xi(dW_q))\omega$. Therefore the relation in G^W reads

$$[d(i_\xi\omega)] + \frac{1}{z}\left[(\xi\cdot W_q)\omega\right] = 0, \tag{18.3}$$

We compute the first term in (18.3) in the case $\xi = X_\lambda^{(m)}$. Recall that the m-th twist of the empty partition $\emptyset$, denoted by $\emptyset^{(m)}$, is equal to μ_{n-m} and corresponds to the k-subset $J_{n-m} = \{n-m+1, \ldots, n-m+k\}$ (interpreted cyclically modulo n).

Lemma 18.2. *Let $\lambda \in \mathcal{P}_{k,n}$. Then we have:*

$$\left[d\left(i_{X_\lambda^{(m)}}\omega\right)\right] = -c_{\lambda^{(m)}}(\emptyset^{(m)})[p_\lambda\omega]. \tag{18.4}$$

Proof. It suffices to check this on the cluster torus $\check{\mathbb{X}}^\circ_{\widetilde{\mathcal{C}}}$, where $\widetilde{\mathcal{C}}$ is the Postnikov extended cluster corresponding to D_λ. We have:

$$d\left(i_{X_\lambda^{(m)}}\omega\right) = \sum_{\mu\in\widetilde{\mathcal{C}}} d\left(i_{c_\lambda^{(m)}(\mu)p_\lambda p_\mu\frac{\partial}{\partial p_\mu}}\omega\right).$$

If $\mu \neq \lambda$ then

$$\begin{aligned} d\left(i_{c_\lambda^{(m)}(\mu)p_\lambda p_\mu\frac{\partial}{\partial p_\mu}}\omega\right) &= \pm d\left(c_\lambda^{(m)}(\mu)p_\lambda \bigwedge_{\varepsilon\in\widetilde{\mathcal{C}},\varepsilon\neq\mu}\frac{dp_\varepsilon}{p_\varepsilon}\right) \\ &= \pm c_\lambda^{(m)}(\mu)dp_\lambda \wedge \bigwedge_{\varepsilon\in\widetilde{\mathcal{C}},\varepsilon\neq\mu}\frac{dp_\varepsilon}{p_\varepsilon} = 0. \end{aligned}$$

Therefore the only non-zero summand is the one where $\mu=\lambda$. We may write ω in the extended cluster $\widetilde{\mathcal{C}}$ as $\omega=\pm\frac{dp_\lambda}{p_\lambda}\wedge\bigwedge_{\varepsilon\in\widetilde{\mathcal{C}},\varepsilon\neq\lambda}\frac{dp_\varepsilon}{p_\varepsilon}$. Then the $\mu=\lambda$ summand is

$$\begin{aligned}
d\left(c_\lambda^{(m)}(\lambda)p_\lambda i_{p_\lambda\frac{\partial}{\partial p_\lambda}}\omega\right) &= \pm c_\lambda^{(m)}(\lambda)dp_\lambda\wedge\bigwedge_{\varepsilon\in\widetilde{\mathcal{C}},\varepsilon\neq\lambda}\frac{dp_\varepsilon}{p_\varepsilon}\\
&= c_\lambda^{(m)}(\lambda)p_\lambda\omega\\
&= \left(c_{\lambda^{(m)}}(\lambda^{(m)})-c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)\right)p_\lambda\omega\\
&= -c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)p_\lambda\omega,
\end{aligned}$$

and we are done. □

Since $c_\lambda(\emptyset)=0$, it follows from Lemma 18.2 that

$$[d\left(i_{X_\lambda}\omega\right)]=0; \tag{18.5}$$

$$\left[d\left(i_{\sum_{m=1}^n X_\lambda^{(m)}}\right)\omega\right]=\left(-\sum_{m=1}^n c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)\right)[p_\lambda\omega]. \tag{18.6}$$

We need a simpler form for the coefficient in the second equation. This will be given Lemma 18.4 below. Recall that J_i is the k-subset corresponding to μ_i. The following statement follows immediately from the definitions.

Lemma 18.3. *Let $\lambda\in\mathcal{P}_{k,n}$. Then for $1\le i\le n$, we have*

$$c_\lambda(\mu_i)=\begin{cases}|[1,i]\cap J_\lambda|, & 1\le i\le n-k;\\ |[i+1,n]\setminus J_\lambda|, & n-k+1\le i\le n.\end{cases}$$

□

Lemma 18.4. *Let $\lambda\in\mathcal{P}_{k,n}$. Then*

$$\sum_{m=1}^n c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)=|\lambda|.$$

Proof. Firstly, we note that $c_\lambda^{(n)}(\emptyset)=0$. Recall also that $J_{\emptyset^{(m)}}=J_{n-m}$. If $1\le m\le k-1$, then $n-k+1\le n-m\le n-1$, so, by Lemma 18.3,

$$\begin{aligned}
c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right) &= |[n-m+1,n]\setminus J_{\lambda^{(m)}}|\\
&= |[n-m+1,n]\setminus(J_\lambda-m)|\\
&= |[1,m]\setminus J_\lambda|.
\end{aligned}$$

An element $j\in[1,k-1]\setminus J_\lambda$ contributes 1 to the term $[1,m]\setminus J_\lambda$ in the sum

$$\sum_{m=1}^{k-1}|[1,m]\setminus J_\lambda|$$

for all $m\ge j$, and zero otherwise. It follows that

$$\sum_{m=1}^{k-1}c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)=\sum_{j\in[1,k-1]\setminus J_\lambda}k-j. \tag{18.7}$$

We have

$$[1,n]\setminus J_\lambda=\{k-\lambda_1+1,\dots,k-\lambda_{n-k}+(n-k)\},$$

so

$$[1,k-1]\setminus J_\lambda=\{k-\lambda_1+1,\dots,k-\lambda_s+s\},$$

where s is maximal such that $\lambda_s > s$. Hence the sum in (18.7) can be rewritten as

$$\sum_{r=1}^{s} k - (k - \lambda_r + r) = \sum_{r=1}^{s} \lambda_r - r,$$

which is the number of boxes in λ strictly to the right of the leading diagonal.

If $k \leq m \leq n-1$, then $1 \leq n-m \leq n-k$ and, by Lemma 18.3,

$$\begin{aligned} c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right) &= |[1, n-m] \cap J_{\lambda^{(m)}}| \\ &= |[1, n-m] \cap (J_\lambda - m)| \\ &= |[m+1, n] \cap J_\lambda|. \end{aligned}$$

An element j in $[k+1, n] \cap J_\lambda$ contributes 1 to the term $|[m+1, n] \cap J_\lambda|$ in the sum

$$\sum_{m=k}^{n-1} |[m+1, n] \cap J_\lambda|$$

if $m \leq j-1$, and zero otherwise. It follows that

$$\sum_{m=k}^{n-1} c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right) = \sum_{j \in J_\lambda \cap [k+1, n]} j - k, \tag{18.8}$$

Let λ' be the transpose of λ. Then we have:

$$J_\lambda = \{k + \lambda'_1, k + \lambda'_2 - 1, \ldots, k + \lambda'_k - k + 1\}.$$

Hence,

$$[k+1, n] \cap J_\lambda = \{k + \lambda'_1, k + \lambda'_2 - 1, \ldots, k + \lambda'_u - u + 1\},$$

where u is maximal such that $\lambda'_u \geq u$. Therefore, the sum in (18.8) can be rewritten as

$$\sum_{r=1}^{u} (k + \lambda'_r - r + 1) - k = \sum_{\lambda'_r \geq r} \lambda'_r - r + 1,$$

which is the number of boxes in λ on or below the leading diagonal. Combining this with the above gives the claimed result. $\square$

By (18.6) and Lemma 18.4, we have:

$$\left[d\left(i_{\sum_{m=1}^{n} X_\lambda^{(m)}}\right)\omega\right] = -|\lambda|[p_\lambda \omega]. \tag{18.9}$$

Theorem 18.5. *Let $\lambda \in \mathcal{P}_{k,n}$. Then*

(a)

$$q\frac{\partial W}{\partial q}[p_\lambda \omega] = \sum_\mu [p_\mu \omega] + q \sum_\nu [p_\nu \omega], \text{ and}$$

(b)

$$\frac{1}{z}[W_q p_\lambda \omega] = \frac{n}{z}\left(\sum_\mu [p_\mu \omega] + q \sum_\nu [p_\nu \omega]\right) - |\lambda|[p_\lambda \omega],$$

where, in each case, μ, ν are exactly as in the quantum Monk's rule for $\sigma^\square *_q \sigma^\lambda$.

Proof. Part (a) follows from (18.3) in the case $\xi = X_\lambda$, using Corollary 18.1(a) and (18.5). Part (b) follows from (18.3) in the case $\xi = \sum_{m=1}^{n} X_\lambda^{(m)}$, using Corollary 18.1(b) and (18.9). $\square$

Proof of Theorem 4.1. By Theorem 18.5, equations (10.1) and (10.2) hold. This, together with Lemma 9.3, completes the proof of Theorem 4.1. $\square$

19. Background for mirror symmetry in the torus-equivariant setting

We now turn to the torus-equivariant mirror theorem, Theorem 5.5. We begin in Section 19.2 by reviewing the structure of the small equivariant quantum cohomology ring of a Grassmannian. We refer to [2] for background on equivariant cohomology, and [61] for relevant background on equivariant quantum cohomology. In Section 20 we recall the equivariant version of the superpotential (introduced for general G/P in [79]) and describe it in the case of the Grassmannian in terms of Plücker coordinates. The main ingredient to the proof of the mirror theorem, Theorem 5.5, is Theorem 21.1 proved in Section 21. Namely, in this section we work out the action on the equivariant superpotential of the vector fields X_λ constructed in Section 16.

Let us first fix our basic set-up regarding the torus and its action on X. Recall that $T^\vee$ denotes the maximal torus of diagonal matrices in $GL_n^\vee(\mathbb{C})$, the general linear group in the A-model. It naturally acts on $\mathbb{C}^n$ and on $X = Gr_{n-k}(\mathbb{C}^n)$. We denote the standard basis of $\mathbb{C}^n$ by $v_1, \dots, v_n$. The weight of the action of $T^\vee$ on the span of v_i is denoted by $\varepsilon_i^\vee \in \mathrm{Hom}(T^\vee, \mathbb{C}^*)$, and we have $\varepsilon^\vee = (\varepsilon_i^\vee) : T \xrightarrow{\sim} (\mathbb{C}^*)^n$. As usual we think of the lattice $X_*(T^\vee) = \mathrm{Hom}(T^\vee, \mathbb{C}^*)$ as embedded in the dual $(\mathfrak{h}^\vee)^*$ of the Lie algebra $\mathfrak{h}^\vee$ of $T^\vee$, and use additive notation for characters.

19.1. The equivariant cohomology ring of X.

Our conventions regarding the equivariant cohomology ring $H^*_{T^\vee}(X, \mathbb{C})$ of the Grassmannian X are as follows. First recall that the equivariant cohomology of X is a free module over the equivariant cohomology of a point. Moreover, the equivariant cohomology of a point is a polynomial ring

$$H^*_{T^\vee}(pt) = \mathbb{C}[x_1, \dots, x_n].$$

To be completely explicit, using the Borel construction and the isomorphism $\varepsilon^\vee : T^\vee \cong (\mathbb{C}^*)^n$ we have $H^*_{T^\vee}(pt) = H^*(BT^\vee) = H^*(\prod_{i=1}^n \mathbb{C}P^\infty)$, and our conventions are that x_i is the first Chern class of the line bundle coming from the $\mathcal{O}(1)$ of the i-th factor in $\prod_{i=1}^n \mathbb{C}P^\infty$. This class x_i is also the equivariant first Chern class of the one-dimensional representation $-\varepsilon_i^\vee$ of $T^\vee$, interpreted as an equivariant line bundle on the point. Therefore we have natural identifications

$$H^*_{T^\vee}(pt, \mathbb{C}) = \mathbb{C}[x_1, \dots, x_n] = S^\bullet\left((\mathfrak{h}^\vee)^*\right) = \mathbb{C}[\mathfrak{h}^\vee],$$

with $x_i = -\varepsilon_i^\vee$.

The Schubert basis in the equivariant setting is made up of equivariant fundamental classes of certain $T^\vee$-invariant Schubert varieties in X, which we need to choose explicitly as follows. Our Schubert varieties X^λ are obtained as as the closures of $B_+^\vee$-orbits in X. Recall that J_λ records the k horizontal steps in the Young diagram $\lambda \in \mathcal{P}_{k,n}$, compare Section 6.1. Let $\mathrm{Vert}(\lambda) = [1, n] \setminus J_\lambda$ and define $[v_\lambda] \in Gr_{n-k}(\mathbb{C}^n)$ by

$$[v_\lambda] = \langle v_j \mid j \in \mathrm{Vert}(\lambda) \rangle_{\mathbb{C}}.$$

We define

$$X^\lambda := \overline{B_+^\vee \cdot [v_\lambda]}.$$

With this definition, the Schubert variety denoted by X^λ has complex codimension $|\lambda|$, the number of boxes in λ. We denote by $\sigma^\lambda_{T^\vee}$ the associated equivariant fundamental class $[X^\lambda]_{T^\vee} \in H^{2|\lambda|}_{T^\vee}(X)$.

The equivariant version of Monk's rule involves the following linear combinations of equivariant parameters,

$$x_\lambda := \sum_{j \in \mathrm{Vert}(\lambda)} x_j. \tag{19.1}$$

Note that under the identification $x_i = -\varepsilon_i^\vee$ the x_λ are the *negatives* of the weights of the $(n-k)$-th fundamental representation of $GL_n^\vee(\mathbb{C})$. In particular $x_{\lambda_{max}} = x_1 + \ldots + x_{n-k}$ is the negative of the highest weight, and $x_\emptyset = x_{k+1} + x_{k+2} + \cdots + x_n$ the negative of the lowest weight. Here λ_{max} refers to the maximal Young diagram, the $(n-k) \times k$ rectangle.

We now consider three analogues of the 'hyperplane class' $\sigma^\square$ in the equivariant setting.

(1) We have the $T^\vee$-equivariant first Chern class of the equivariant line bundle $\mathcal{O}(1)$ coming from the Plücker embedding,

$$\zeta := c_1^{T^\vee}(\mathcal{O}(1)).$$

(2) In $H^2_{T^\vee}(X)$ we also have the Schubert class $\sigma^{\square}_{T^\vee}$ defined above, corresponding to the $B_+^\vee$-invariant Schubert divisor $X^{\square}$. One can check that

$$\sigma^{\square}_{T^\vee} = c_1^{T^\vee}(\mathcal{O}(1) \otimes L_{\varepsilon_{k+1}+\ldots+\varepsilon_n}) = c_1^{T^\vee}(\mathcal{O}(1)) - (x_{k+1} + \ldots + x_n) = \zeta - x_\emptyset.$$

(3) Finally, we have the alternative equivariant Schubert class $\tilde{\sigma}^{\square}_{T^\vee}$, which corresponds to the $B_-^\vee$-invariant Schubert divisor

$$\widetilde{X}^{\square} = \overline{B_-^\vee \cdot v_{\lambda_{submax}}},$$

where λ_{submax} denotes the Young diagram in $\mathcal{P}_{k,n}$ obtained by removing one box from the maximal Young diagram. This class is related to the other two choices by

$$\tilde{\sigma}^{\square}_{T^\vee} = \zeta - (x_1 + \ldots + x_{n-k}) = \zeta - x_{\lambda_{max}} = \sigma^{\square}_{T^\vee} + x_\emptyset - x_{\lambda_{max}}.$$

19.2. **The small equivariant quantum cohomology of X.** The equivariant quantum cohomology ring of X is denoted by $qH^*_{T^\vee}(X)$. It is defined by using $T^\vee$-equivariant versions of Gromov-Witten invariants [54] to specify a q-deformed cup product structure on $H^*_{T^\vee}(X, \mathbb{C}) \otimes \mathbb{C}[q]$; see also [36]. In the case of the Grassmannian X the structure of the ring $qH^*_{T^\vee}(X)$ was worked out by Mihalcea [61].

We now recall the equivariant version of the quantum Monk's rule [61, Section 1.1]. Expressed in our conventions, this states that quantum multiplication with the equivariant Chern class of $\mathcal{O}(1)$, is given by the formula

$$\zeta \star_{q,x} \sigma^\lambda_{T^\vee} = \sum_\mu \sigma^\mu_{T^\vee} + q \sum_\nu \sigma^\nu_{T^\vee} + x_\lambda \sigma^\lambda_{T^\vee}, \tag{19.2}$$

where the first two summands on the right hand side are as in the non-equivariant quantum Monk's rule and x_λ is given in Equation (19.1).

We will also need a special $T^\vee$-invariant anti-canonical divisor, X_{ac}. Note that we have $\mathbb{Z}/n\mathbb{Z}$-action on the Grassmannian X analogously to the one defined in Section 6.1 for the B-model. (Indeed X and $\check{\mathbb{X}}$ are isomorphic varieties). The divisor X_{ac} is the $\mathbb{Z}/n\mathbb{Z}$-orbit of the divisor $X^{\square}$. If we denote by $X^{\square}(i)$ the i-th translate of $X^{\square}$ under the $\mathbb{Z}/n\mathbb{Z}$-action then

$$X_{ac} = \bigcup_{i=1}^n X^{\square}(i). \tag{19.3}$$

Note that X_{ac} is the union of n distinct hyperplanes in X, including $X^{\square}$ (where $i = n$) and $\widetilde{X}^{\square}$ (where $i = n - k$). This is the Langlands dual version of the divisor D from (6.3).

The equivariant fundamental class $[X_{ac}]_{T^\vee}$ of X_{ac} is given by

$$[X_{ac}]_{T^\vee} = n\zeta - \sum_{j=1}^n (x_{j+1} + \ldots x_{j+n-k}) = n\zeta - (n-k)\left(\sum_{i=1}^n x_i\right)$$

where the indices are interpreted modulo n. Hence we have

$$[X_{ac}]_{T^\vee} \star_{q,x} \sigma^\lambda_{T^\vee} = n \sum_\mu \sigma^\mu_{T^\vee} + qn \sum_\nu \sigma^\nu_{T^\vee} + n x_\lambda \sigma^\lambda_{T^\vee} - (n-k)\left(\sum_{i=1}^n x_i\right) \sigma^\lambda_{T^\vee}, \tag{19.4}$$

where the terms in the first two summands are as in the non-equivariant quantum Monk's rule.

20. The $T^\vee$-equivariant version of the superpotential

By the equivariant superpotential of the target space X we mean a deformation of the usual superpotential to a (multi-valued) map involving the equivariant parameters, which encodes structures from the equivariant quantum cohomology of X. A torus-equivariant version of the superpotential for general type partial flag varieties was introduced in [79, Section 4], where it was denoted $\mathcal{F}_P + \ln(\phi)$, and was shown to recover the equivariant quantum cohomology rings in their presentation due to Dale Peterson [69]. In this section we express this equivariant superpotential, in the special case of the Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$, in terms of the Plücker coordinates on the mirror Grassmannian $\check{\mathbb{X}}^\circ$.

We may think of the superpotential $W : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \to \mathbb{C}$ as a section of a trivial line bundle $\mathbb{C}$ on $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$. The $T^\vee$-equivariant version of the superpotential will be a multi-valued section of the trivial vector bundle $\mathbb{C} \oplus \mathfrak{h}$ on $\check{\mathbb{X}}^\circ \times \mathbb{C}_q^*$, so a multi-valued map

$$W^{\mathrm{eq}} : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \xrightarrow{(W,\Phi)} \mathbb{C} \oplus \mathfrak{h}.$$

We give our new definition of W^{eq} in the Grassmannian setting first, followed by the original, more general definition of the equivariant superpotential for homogeneous spaces from [79]. Then we will demonstrate that the two are equivalent when the homogeneous space is a Grassmannian.

Definition 20.1. Recall that we have a natural identification of $\mathfrak{h}$ with $H^2_{T^\vee}(\{pt\})$ which sends $-\varepsilon_i^\vee$ to the equivariant parameter x_i, see Section 19.1. We define $W^{\mathrm{eq}} : \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* \dashrightarrow \mathbb{C} \oplus \mathfrak{h}$ by

$$W^{\mathrm{eq}} = W + \ln(q)(x_1 + \ldots + x_{n-k}) + \ln(p_{\mu_1})(x_2 - x_1) + \ln(p_{\mu_2})(x_3 - x_2) + \ldots + \ln(p_{\mu_{n-1}})(x_n - x_{n-1}), \tag{20.1}$$

where W is as in Definition 6.1 and we keep in mind that $p_{\mu_n} = p_\emptyset = 1$. Therefore $\ln(p_{\mu_n}) = 0$.

Note that we have $x_{i+1} - x_i = \alpha_i^\vee$ and $x_1 + \ldots + x_{n-k} = -\omega_{n-k}^\vee$, since the equivariant parameters are related to the usual basis of $\mathfrak{h}$ by $x_i = -\varepsilon_i^\vee$.

Recall the notations from Section 6.2. We have isomorphisms

$$\begin{array}{ccccc} \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* & \overset{\psi_L}{\longleftarrow} & B_- \cap U_+ \tilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+ & \overset{\psi_R}{\longrightarrow} & \mathcal{R} \times \mathbb{C}_q^*. \\ (Pb, \alpha_{n-k}(t)) & \leftarrow & b = u_1 t \dot{w}_P \dot{w}_0^{-1} u_2 & \mapsto & (b\dot{w}_0 B_-, \alpha_{n-k}(t)). \end{array}$$

Note that $B_- \cap U_+ \tilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+$ is a subset of the Borel subgroup B_- of $GL_n(\mathbb{C})$. Let π be the projection $B_- \to T$ sending an element of the Borel subgroup $B_- = TU_-$ onto its torus factor. We add this map to the diagram above, giving:

$$\begin{array}{ccccc} \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* & \overset{\psi_L}{\longleftarrow} & B_- \cap U_+ \tilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+ & \overset{\psi_R}{\longrightarrow} & \mathcal{R} \times \mathbb{C}_q^*. \\ & & \downarrow \pi & & \\ & & T & & \end{array}$$

The inverse to $\exp : \mathfrak{h} \to T$ defines a multivalued map

$$\ln_T : T \dashrightarrow \mathfrak{h}$$

Definition 20.2. [79, Section 4] The equivariant Lie-theoretic superpotential

$$\mathcal{F}^{\mathrm{eq}} : \mathcal{R} \times \mathbb{C}_q^* \dashrightarrow \mathbb{C} \oplus \mathfrak{h}$$

is defined by adding an $\mathfrak{h}$ component to the superpotential $\mathcal{F} : \mathcal{R} \times \mathbb{C}_q^* \longrightarrow \mathbb{C}$ from Definition 6.3 as follows. Consider the composition

$$\Phi = \ln_T \circ \pi \circ \psi_R^{-1} : \mathcal{R} \times \mathbb{C}_q^* \longrightarrow \mathfrak{h}.$$

Then

$$\mathcal{F}^{\mathrm{eq}} = \mathcal{F} + \Phi.$$

This definition is a slight variation of the definition of the equivariant superpotential from [79], the difference stemming from the fact that [79] considered the maximal torus of $PSL_n(\mathbb{C})$ whereas here T is the maximal torus of $GL_n(\mathbb{C})$. Composing with the map $\mathbb{C} \oplus \mathfrak{h} \to \mathbb{C} \oplus \mathfrak{h}_{PSL_n}$ defined by quotienting out the center of $\mathfrak{gl}_n$ recovers the original equivariant superpotential from [79, Section 4] associated to the action of the maximal torus of $PSL_n(\mathbb{C})$. (This equivariant superpotential is denoted $\mathcal{F}_P + \ln(\phi)$ in [79].)

The main goal of this section is to prove the following comparison result.

Proposition 20.3. *With the definitions as above, the following diagram commutes,*

$$\begin{array}{ccccc} \check{\mathbb{X}}^\circ \times \mathbb{C}_q^* & \xleftarrow{\psi_L} & B_- \cap U_+ \tilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U_+ & \xrightarrow{\psi_R} & \mathcal{R} \times \mathbb{C}_q^*. \\ \downarrow W^{\mathrm{eq}} & & & & \downarrow \mathcal{F}^{\mathrm{eq}} \\ \mathbb{C} \oplus \mathfrak{h} & & = & & \mathbb{C} \oplus \mathfrak{h}. \end{array}$$

Therefore the equivariant superpotentials $(\check{\mathbb{X}}^\circ, W_q^{\mathrm{eq}})$ and $(\mathcal{R}, \mathcal{F}_q^{\mathrm{eq}})$ are equivalent.

This proposition is an extension of Proposition 6.7. We begin with some remarks. Let $b = u_1 t \dot{w}_P \dot{w}_0^{-1} \in B^- \cap U_+ \tilde{T}^{W_P} \dot{w}_P \dot{w}_0^{-1} U^+$. Then, by Proposition 6.7, we have

$$\begin{aligned} \mathcal{F}^{eq}(\psi_R(b)) &= \mathcal{F}(\psi_R(b)) + \ln_T(\pi(b)) \\ &= W(\psi_L(b)) + \sum_{i=1}^n x_i \ln(b_{ii}), \end{aligned} \tag{20.2}$$

where b_{ii} denotes the ith diagonal entry of the matrix b. Looking at (20.1), we see that to prove Proposition 20.3, it is sufficient to show that the following holds, where $q = \alpha_{n-k}(t)$:

$$\begin{aligned} \sum_{i=1}^n x_i \ln(b_{ii}) = \ln(q)(x_1 + \ldots + x_{n-k}) + (x_2 - x_1)\ln(p_{\mu_{12}}(Pb)) + (x_3 - x_2)\ln(p_{\mu_{23}}(Pb)) + \ldots \\ + (x_n - x_{n-1})\ln(p_{\mu_{n-1,n}}(Pb)). \end{aligned} \tag{20.3}$$

Note that, since $u_1, u_2 \in U_+$ and $t \in \tilde{T}^{W_P}$, we have

$$\begin{aligned} \Delta_{J_n}^{[n-k+1,n]}(b) &= \Delta_{[1,k]}^{[n-k+1,n]}(u_1 t \dot{w}_P \dot{w}_0^{-1} u_2) \\ &= \Delta_{[1,k]}^{[n-k+1,n]}(u_1 t \dot{w}_P \dot{w}_0^{-1}) \\ &= \Delta_{[1,k]}^{[n-k+1,n]}(t \dot{w}_P \dot{w}_0^{-1}) = 1, \end{aligned}$$

where the last step is an easy calculation. Since our convention is that $p_\emptyset = 1$, it follows that, for $i = 1, \ldots, n-1$,

$$p_{\mu_i}(Pb) = \Delta_{J_i}^{[n-k+1,n]}(b).$$

Thus, (20.3) is equivalent to

$$\begin{aligned} \sum_{i=1}^n x_i \ln(b_{ii}) = \ln(q)(x_1 + \ldots + x_{n-k}) + (x_2 - x_1)\ln\left(\Delta_{J_1}^{[n-k+1,n]}(b)\right) + \\ + (x_3 - x_2)\ln(\Delta_{J_2}^{[n-k+1,n]}(b)) + \ldots + (x_n - x_{n-1})\ln\left(\Delta_{J_{n-1}}^{[n-k+1,n]}(b)\right). \end{aligned} \tag{20.4}$$

To prove (20.4), we will write the diagonal entries of b in terms of the minors $\Delta_{J_i}^{[n-k+1,n]}$ of b.

Lemma 20.4. *Let $b = u_+ t\dot{w}_P\dot{w}_0^{-1}u_2 \in B^- \cap U_+\tilde{T}^{W_P}\dot{w}_P\dot{w}_0^{-1}U_+$, with $q = \alpha_{n-k}(t)$. Then, for $1 \le i \le n$, we have the following:*

$$b_{ii} = \begin{cases} q\dfrac{1}{\Delta^{[n-k+1,n]}_{J_1}(b)}, & \text{if } i = 1; \\ q\dfrac{\Delta^{[n-k+1,n]}_{J_{i-1}}(b)}{\Delta^{[n-k+1,n]}_{J_i}(b)}, & \text{if } 2 \le i \le n-k; \\ \dfrac{\Delta^{[n-k+1,n]}_{J_{i-1}}(b)}{\Delta^{[n-k+1,n]}_{J_i}(b)}, & \text{if } n-k+1 \le i \le n-1; \\ \Delta^{[n-k+1,n]}_{J_{n-1}}(b), & \text{if } i = n. \end{cases}$$

Proof. Write $b = u_1 t\dot{w}_P w_0^{-1}u_2$, with $u_1, u_2 \in U_+$ and $t \in \tilde{T}^{W_P}$. Then $b^{-1} = u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}u_1^{-1}$. Hence,

$$(b^{-1})_{ii} = \frac{\Delta^{[1,i]}_{[1,i]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1})}{\Delta^{[1,i-1]}_{[1,i-1]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1})},$$

so

$$b_{ii} = \frac{\Delta^{[1,i-1]}_{[1,i-1]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1})}{\Delta^{[1,i]}_{[1,i]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1})}. \tag{20.5}$$

Recall that t is a diagonal matrix with

$$t_{ii} = \begin{cases} q, & \text{if } 1 \le i \le n-k; \\ 1, & \text{if } i+1 \le i \le n. \end{cases}$$

We claim that, for $1 \le i \le n$,

$$\Delta^{[1,i]}_{[1,i]}(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}) = \begin{cases} q^{-i}\Delta^{[n-k+1,n]}_{J_i}(b), & \text{if } 1 \le i \le n-k; \\ q^{-(n-k)}\Delta^{[n-k+1,n]}_{J_i}(b), & \text{if } n-k \le i \le n-1; \\ q^{-(n-k)}, & \text{if } i = n. \end{cases} \tag{20.6}$$

The result then follows from (20.5) and (20.6).

To prove the claim, we consider each of the three cases. We suppose first that $1 \le i \le n-k$. Then

$$\begin{aligned} \Delta^{[1,i]}_{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}\right) &= q^{-i}\Delta^{[1,i]}_{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}\right) \\ &= q^{-i}(-1)^{ik}\Delta^{[1,i]}_{[k+1,k+i]}(u_2^{-1}) \\ &= q^{-i}(-1)^{ik}(-1)^s\Delta^{[1,k]\cup[i+k+1,n]}_{[i+1,n]}(u_2), \end{aligned}$$

where $s = (1+2+\cdots+i)+((k+1)+(k+2)+\cdots+(k+i))$, using Jacobi's Theorem for the minors of an inverse matrix in the last step. Noting that s is congruent to ik mod 2 and that u_2 is upper unitriangular, we obtain:

$$\begin{aligned} \Delta^{[1,i]}_{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}\right) &= q^{-i}\Delta^{[1,k]}_{[i+1,i+k]}(u_2). \\ &= q^{-i}\Delta^{[n-k+1,n]}_{[i+1,i+k]}\left(t\dot{w}_P\dot{w}_0^{-1}u_2\right) \\ &= q^{-i}\Delta^{[n-k+1,n]}_{J_i}(b), \end{aligned}$$

as required in this case.

Next, suppose that $n-k+1 \le i \le n-1$. Then we have

$$\begin{aligned}\Delta_{[1,i]}^{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}\right) &= q^{-(n-k)}\Delta_{[1,i]}^{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}\right)\\ &= q^{-(n-k)}(-1)^{k(n-k)}\Delta_{[k+1,n]\cup[1,i+k-n]}^{[1,i]}\left(u_2^{-1}\right)\\ &= q^{-(n-k)}(-1)^{k(n-k)+(n-k)(i+k-n)}\Delta_{[1,i+k-n,]\cup[k+1,n]}^{[1,i]}\left(u_2^{-1}\right)\\ &= q^{-(n-k)}(-1)^{k(n-k)+(n-k)(i+k-n)}\Delta_{[k+1,n]}^{[i+k-n+1,i]}\left(u_2^{-1}\right),\end{aligned}$$

using in the last step the fact that u_2^{-1} is upper unitriangular. Applying Jacobi's theorem, we have:

$$\Delta_{[1,i]}^{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}\right) = q^{-(n-k)}(-1)^t\Delta_{[1,i+k-n]\cup[i+1,n]}^{[1,k]}(u_2),$$

where

$$\begin{aligned}t &= k(n-k)+(n-k)(i+k-n)+\sum_{j=1}^{k} j+\sum_{j=1}^{i+k-n} j+\sum_{j=1}^{n-i}(i+j)\\ &= k(n-k)+(n-k)(i+k-n)+\sum_{j=1}^{k} j+\sum_{j=n-i+1}^{k}(j-n+i)+\sum_{j=1}^{n-i}(i+j)\\ &= k(n-k)+(n-k)(i+k-n)-n(i+k-n)+\sum_{j=1}^{k} j+\sum_{j=1}^{k}(i+j)\\ &= k(n-k)+(n-k)(i+k-n)-n(i+k-n)+k(i+k+1)\\ &= 2kn-k(k-1),\end{aligned}$$

which is even. Hence

$$\begin{aligned}\Delta_{[1,i]}^{[1,i]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}\right) &= q^{-(n-k)}\Delta_{[1,i+k-n]\cup[i+1,n]}^{[1,k]}(u_2)\\ &= q^{-(n-k)}\Delta_{[1,i+k-n]\cup[i+1,n]}^{[n-k+1,n]}\left(\dot{w}_P\dot{w}_0^{-1}u_2\right)\\ &= q^{-(n-k)}\Delta_{[1,i+k-n]\cup[i+1,n]}^{[n-k+1,n]}\left(u_1t\dot{w}_P\dot{w}_0^{-1}u_2\right)\\ &= q^{-(n-k)}\Delta_{J_i}^{[n-k+1,n]}(b),\end{aligned}$$

as required in this case.

Finally, we consider the case $i=n$. We have:

$$\begin{aligned}\Delta_{[1,n]}^{[1,n]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}t^{-1}\right) &= q^{-(n-k)}\Delta_{[1,n]}^{[1,n]}\left(u_2^{-1}\dot{w}_0\dot{w}_P^{-1}\right)\\ &= q^{-(n-k)}(-1)^{k(n-k)}\Delta_{[k+1,n]\cup[1,k]}^{[1,n]}\left(u_2^{-1}\right)\\ &= q^{-(n-k)}(-1)^{k(n-k)}(-1)^{k(n-k)}\Delta_{[1,n]}^{[1,n]}\left(u_2^{-1}\right)\\ &= q^{-(n-k)},\end{aligned}$$

since u_2^{-1} is upper unitriangular. The result is shown. □

Proof of Proposition 20.3. By Lemma 20.4, we have:

$$\begin{aligned}\sum_{i=1}^{n} x_i\ln(b_{ii}) &= \ln(q)(x_1+x_2+\cdots+x_{n-k})-x_1\ln\left(\Delta_{J_1}^{[n-k+1,n]}(b)\right)\\ &\quad+x_n\ln\left(\Delta_{J_{n-1}}^{[n-k+1,n]}(b)\right)+\sum_{i=2}^{n-1}\left(x_i\ln\left(\Delta_{J_{i-1}}^{[n-k+1,n]}(b)\right)-x_i\ln\left(\Delta_{J_i}^{[n-k+1,n]}(b)\right)\right)\\ &= \ln(q)(x_1+x_2+\cdots+x_{n-k})+\sum_{i=1}^{n-1}\left((x_{i+1}-x_i)\ln\left(\Delta_{J_i}^{[n-k+1,n]}(b)\right)\right).\end{aligned}$$

Hence (20.4) holds, and we are done. □

21. Action of the vector field: $T^\vee$-equivariant case

In this section we prove the formulas needed to complete the proof of Theorem 5.5.

Theorem 21.1. *Let $\lambda \in \mathcal{P}_{k,n}$. Then we have:*

(a)

$$[q\frac{\partial W^{\mathrm{eq}}}{\partial q}p_\lambda\omega] = \left(\sum_\mu [p_\mu\omega] + q\sum_\nu [p_\nu\omega]\right) + x_\lambda[p_\lambda\omega],$$

and

(b)

$$-\frac{1}{z}[Wp_\lambda\omega] = |\lambda|[p_\lambda\omega] - \frac{n}{z}\left(\sum_\mu [p_\mu\omega] + q[p_\nu\omega]\right) - \frac{n}{z}x_\lambda[p_\lambda\omega] + \frac{(n-k)}{z}\left(\sum_{j=1}^n x_j\right)[p_\lambda\omega],$$

where, in each case, μ, ν are exactly as in the quantum Monk's rule for $\sigma^\square *_q \sigma^\lambda$.

To prove Theorem 21.1, we compute the action of the vector field X_λ on W_q^{eq}. Note that $W_q^{\mathrm{eq}} = W_q + \widetilde{W}_q^{\mathrm{eq}}$, where

$$\widetilde{W}_q^{\mathrm{eq}} = \ln(q)(x_1 + x_2 + \cdots + x_{n-k}) + \sum_{i=1}^{n-1}(x_{i+1} - x_i)\ln\left(p_{\mu_i}\right).$$

Recall that

$$x_\lambda = \sum_{i\in\mathrm{Vert}(\lambda)} x_i.$$

In particular, we have $x_{\lambda_{\max}} = x_1 + \cdots + x_{n-k}$.

Lemma 21.2. *Let $m \in [1,n]$. Then*

$$\sum_{i=1}^{n-1} c_\lambda^{(m)}(\mu_i)(x_{i+1} - x_i) = x_\lambda - (x_{m+1} + \cdots + x_{m+n-k}).$$

Proof. We interpret the subscripts of the μ_i modulo n, with representatives in $[1,n]$. We have:

$$\sum_{i=1}^{n-1} c_\lambda^{(m)}(\mu_i)(x_{i+1} - x_i) = \sum_{i=1}^{n}\left(c_\lambda^{(m)}(\mu_{i-1}) - c_\lambda^{(m)}(\mu_i)\right)x_i,$$

noting that $c_\lambda^{(m)}(\mu_n) = c_\lambda^{(m)}(\emptyset) = 0$. Thus the statement in the lemma is equivalent to the statement:

$$c_\lambda^{(m)}(\mu_{i-1}) - c_\lambda^{(m)}(\mu_i) = \begin{cases} 0, & i \in \mathrm{Vert}(\lambda),\ i \in [m+1, m+n-k]; \\ -1, & i \notin \mathrm{Vert}(\lambda),\ i \in [m+1, m+n-k]; \\ 1, & i \in \mathrm{Vert}(\lambda),\ i \notin [m+1, m+n-k]; \\ 0, & i \notin \mathrm{Vert}(\lambda),\ i \notin [m+1, m+n-k], \end{cases}$$

for all $i \in [1,n]$. We first show that this holds for $m = n$, i.e. that:

$$c_\lambda(\mu_{i-1}) - c_\lambda(\mu_i) = \begin{cases} 0, & i \in \mathrm{Vert}(\lambda),\ i \in [1, n-k]; \\ -1, & i \notin \mathrm{Vert}(\lambda),\ i \in [1, n-k]; \\ 1, & i \in \mathrm{Vert}(\lambda),\ i \notin [1, n-k]; \\ 0, & i \notin \mathrm{Vert}(\lambda),\ i \notin [1, n-k], \end{cases} \tag{21.1}$$

for all $i \in [1,n]$. Note that Lemma 18.3 can be restated as:

$$c_\lambda(\mu_i) = \begin{cases} [1,i] \setminus \mathrm{Vert}(\lambda), & 1 \le i \le n-k; \\ [i+1,n] \cap \mathrm{Vert}(\lambda), & n-k+1 \le i \le n. \end{cases} \tag{21.2}$$

For $i = 1$, we have $c_\lambda(\mu_n) - c_\lambda(\mu_1) = -c_\lambda(\mu_1)$ and the result follows from (21.2). For $2 \le i \le n-k$, it follows from (21.2) that $c_\lambda(\mu_{i-1}) - c_\lambda(\mu_i) = 0$ if $i \in \mathrm{Vert}(\lambda)$ and is equal to -1 if $i \notin \mathrm{Vert}(\lambda)$, giving the result in this case. The case $n-k+2 \le i \le n$ is similar, leaving the case $i = n-k+1$.

We have, using the fact that $|\mathrm{Vert}(\lambda)| = n-k$,

$$\begin{aligned} c_\lambda(\mu_{n-k}) - c_\lambda(\mu_{n-k+1}) &= |[1,n-k] \setminus \mathrm{Vert}(\lambda)| - |[n-k+2,n] \cap \mathrm{Vert}(\lambda)| \\ &= n-k - |[1,n-k] \cap \mathrm{Vert}(\lambda)| - |[n-k+2,n] \cap \mathrm{Vert}(\lambda)| \\ &= \begin{cases} n-k - |[1,n-k] \cap \mathrm{Vert}(\lambda)| - |[n-k+1,n] \cap \mathrm{Vert}(\lambda)| + 1, & n-k+1 \in \mathrm{Vert}(\lambda); \\ n-k - |[1,n-k] \cap \mathrm{Vert}(\lambda)| - |[n-k+1,n] \cap \mathrm{Vert}(\lambda)|, & n-k+1 \notin \mathrm{Vert}(\lambda); \end{cases} \\ &= \begin{cases} 1, & n-k+1 \in \mathrm{Vert}(\lambda); \\ 0, & n-k+1 \notin \mathrm{Vert}(\lambda); \end{cases} \end{aligned}$$

as required. For arbitrary $m \in [1,n]$, we have, recalling the definition of $c_\lambda^{(m)}$ (equation (16.2)) and using (21.1),

$$\begin{aligned} c_\lambda^{(m)}(\mu_{i-1}) - c_\lambda^{(m)}(\mu_i) &= c_{\lambda^{(m)}}\left(\mu_{i-1}^{(m)}\right) - c_{\lambda^{(m)}}\left(\mu_i^{(m)}\right) \\ &= \begin{cases} 0, & i-m \in \mathrm{Vert}(\lambda^{(m)}),\ i-m \in [1,n-k]; \\ -1, & i-m \notin \mathrm{Vert}(\lambda^{(m)}),\ i-m \in [1,n-k]; \\ 1, & i-m \in \mathrm{Vert}(\lambda^{(m)}),\ i-m \notin [1,n-k]; \\ 0, & i-m \notin \mathrm{Vert}(\lambda^{(m)}),\ i-m \notin [1,n-k]. \end{cases} \end{aligned} \tag{21.3}$$

The result then follows from (21.3), noting that $\mathrm{Vert}(\lambda^{(m)}) = \mathrm{Vert}(\lambda) - m$ (working mod n). □

Proposition 21.3. *Let $\lambda \in \mathcal{P}_{k,n}$. Then:*

$$X_\lambda^{(m)} \widetilde{W}_q^{\mathrm{eq}} = (x_\lambda - (x_{m+1} + \cdots + x_{m+n-k}))p_\lambda.$$

Proof. For any $i \in [1,n]$,

$$p_{\mu_i} \frac{\partial}{\partial p_{\mu_i}} \ln p_{\mu_i} = 1,$$

Hence, by Lemma 21.2,

$$X_\lambda^{(m)} \widetilde{W}_q^{\mathrm{eq}} = \sum_{i=1}^{n-1} c_\lambda^{(m)}(\mu_i) \alpha_i^\vee p_\lambda = (x_\lambda - (x_{m+1} + \cdots + x_{m+n-k}))p_\lambda,$$

□

Proposition 21.4. *Let $\lambda \in \mathcal{P}_{k,n}$ and $m \in [1,n]$. Then we have:*

$$\begin{aligned} \frac{1}{z} q^{\delta_{mn}} [\frac{p_{\widehat{L}_m}}{p_{L_m}} p_\lambda \omega] = \frac{1}{z} \left(\sum_\mu [p_\mu \omega] + q \sum_\nu [p_\nu \omega] \right) + \frac{1}{z}(x_\lambda - (x_{m+1} + \cdots + x_{m+n-k}))[p_\lambda \omega] \\ - c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)[p_\lambda \omega], \end{aligned}$$

Proof. Arguing as for equation (18.3), we have the following:

$$[d\,(i_\xi \omega)] + \frac{1}{z}\left[\left(\xi \cdot W_q^{\mathrm{eq}}\right)\omega\right] = 0, \tag{21.4}$$

for a regular vector field ξ on $\check{\mathbb{X}}^\circ$. We will apply this in the case $\xi = X_\lambda^{(m)}$, for each $m \in [1,n]$.

By Lemma 18.2, we have:

$$\left[d\left(i_{X_\lambda^{(m)}}\right)\omega\right] = -c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)[p_\lambda\omega], \tag{21.5}$$

giving the first term in (21.4).

For the second term, we first note that, by Theorem 17.3, we have:

$$X_\lambda^{(m)} W_q = \left(\sum_\mu p_\mu + q\sum_\nu p_\nu\right) - q^{\delta_{mn}}\frac{p_{\widehat{L}_m}}{p_{L_m}}p_\lambda. \tag{21.6}$$

By Proposition 21.3, we have:

$$X_\lambda^{(m)}\widetilde{W}_q^{\mathrm{eq}} = (x_\lambda - (x_{m+1}+\cdots+x_{m+n-k}))p_\lambda. \tag{21.7}$$

Combining (21.6) and (21.7), we obtain:

$$X_\lambda^{(m)} W^{\mathrm{eq}} = \left(\sum_\mu p_\mu + q\sum_\nu p_\nu\right) - q^{\delta_{mn}}\frac{p_{\widehat{L}_m}}{p_{L_m}}p_\lambda + (x_\lambda - (x_{m+1}+\cdots+x_{m+n-k}))p_\lambda. \tag{21.8}$$

Substituting (21.5) and (21.8) into (21.4), we obtain:

$$\frac{1}{z}q^{\delta_{mn}}\left[\frac{p_{\widehat{L}_m}}{p_{L_m}}p_\lambda\omega\right] = \frac{1}{z}\left(\sum_\mu[p_\mu\omega] + q\sum_\nu[p_\nu\omega]\right) + \frac{1}{z}(x_\lambda - (x_{m+1}+\cdots+x_{m+n-k}))[p_\lambda\omega] - c_{\lambda^{(m)}}\left(\emptyset^{(m)}\right)[p_\lambda\omega],$$

as required. □

We can now prove the following enhanced version of Proposition 5.1.

Proposition 21.5. *The Jacobi ring of $(\check{\mathbb{X}}^\circ, W^{\mathrm{eq}})$ is isomorphic to the equivariant quantum cohomology ring $qH^*_{T^\vee}(X,\mathbb{C})[q^{-1}]$ via an isomorphism which satisfies*

$$[p_\lambda] \mapsto \sigma^\lambda_{T^\vee}.$$

Moreover this isomorphism sends the summands of W to equivariant fundamental classes of $T^\vee$-invariant divisors $X^\square(i)$ from Section 19.2. Namely for $i \neq n-k$,

$$\left[\frac{p_{\widehat{\mu}_i}}{p_{\mu_i}}\right] \mapsto [X^\square(i)]_{T^\vee},$$

and for $i = n-k$

$$q\left[\frac{p_{\widehat{\mu}_{n-k}}}{p_{\mu_{n-k}}}\right] \mapsto [X^\square(n-k)]_{T^\vee} = \widetilde{\sigma}^\square_{T^\vee}$$

noting that $\widetilde{X}^\square$ is the $(n-k)$-th shift of $X^\square$.

Proof of Proposition 5.1. By the comparison result, Proposition 20.3, together with [79, Theorem 4.1] and Peterson's theory, see [79, Corollary 4.2], we know that the Jacobi ring of $(\check{\mathbb{X}}^\circ, W_q^{\mathrm{eq}})$ is isomorphic to the quantum cohomology $qH^*_{T^\vee}(X,\mathbb{C})[q^{-1}]$ via an isomorphism of graded (compare (8.11)) rings, which fixes the x_i. Moreover the image of p_λ is $\sigma^\lambda_{T^\vee}$ up to possible summands in the ideal generated by the equivariant parameters, by Proposition 9.2. Therefore the p_λ form an additive basis of the Jacobi ring as module over $\mathbb{C}[x_1,\dots,x_n,q,q^{-1}]$.

Consider (21.8) from above. Recall that $p_{L_k} = p_\emptyset = 1$, see (12.2). Setting $m = k$ we obtain the following relation in the Jacobi ring of W_q^{eq},

$$p_\square\, p_\lambda = \sum_\mu p_\mu + q\sum_\nu p_\nu + (x_\lambda - (x_{k+1}+\cdots+x_n))p_\lambda. \tag{21.9}$$

For example (assuming $n-k>1$), we have

$$(p_\square + (x_{k+1}+\cdots+x_n))p_\square = p_{\square\square} + p_{\substack{\square\\\square}} + x_\lambda p_\square,$$

in the Jacobi ring, which we can compare with the relation from the equivariant quantum Monk's rule,

$$\zeta \star_{q,x} \sigma^{\square}_{T^\vee} = \sigma^{\square\square}_{T^\vee} + \sigma^{\substack{\square\\\square}}_{T^\vee} + x_\lambda \sigma^{\square}_{T^\vee}.$$

It now follows from (21.9) by [61, Corollary 7.1] that the isomorphism from the Jacobi ring to quantum cohomology must take p_λ to $\sigma^\lambda_{T^\vee}$.

In particular $p_\square$ maps to $[X^\square]_{T^\vee}$. The equation (21.8) for $\lambda = \emptyset$ implies the identity in the Jacobi ring, if $m \neq n$,

$$\frac{p_{\widehat{L}_m}}{p_{L_m}} = p_\square + x_\emptyset - (x_{m+1} + \ldots + x_{m+n-k}),$$

and for $m = n$ the identity,

$$q\frac{p_{\widehat{L}_n}}{p_{L_n}} = p_\square + (x_{k+1} + \ldots + x_n) - (x_1 + \ldots + x_{n-k}).$$

Therefore under the isomorphism with quantum cohomology we have

$$\frac{p_{\widehat{L}_m}}{p_{L_m}} \mapsto \zeta - (x_{m+1} + \ldots + x_{m+n-k})$$

and

$$q\frac{p_{\widehat{L}_n}}{p_{L_n}} \;\mapsto\; \zeta - (x_1 + \ldots + x_{n-k}).$$

Since the $\mathbb{Z}/n\mathbb{Z}$-action on $X = Gr_{n-k}(\mathbb{C}^*)$ comes from the cyclic permutation of the basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$, and fixes ζ, the equivariant Chern class of $\mathcal{O}(1)$, we have that the fundamental class $[X^\square(m)]_{T^\vee}$ is related to

$$[X^\square]_{T^\vee} = \zeta - (x_{k+1} + \ldots + x_n)$$

by cyclic permutation of the equivariant parameters. Therefore

$$[X^\square(i)]_{T^\vee} = \zeta - (x_{k+i+1} + \ldots + x_{n+i})$$

with indices taken modulo n, and this agrees with the image of

$$\frac{p_{\widehat{\mu_i}}}{p_{\mu_i}} = \frac{p_{\widehat{L}_{k+i}}}{p_{L_{k+i}}}$$

respectively of

$$q\frac{p_{\widehat{\mu_{n-k}}}}{p_{\mu_{n-k}}} = q\frac{p_{\widehat{L}_n}}{p_{L_n}},$$

if $i = n - k$, which was to be proved. □

Finally, we complete the proof of Theorem 21.1.

Proof of Theorem 21.1. Putting $m = n$ in the statement in Proposition 21.4, we obtain:

$$q[\frac{p_{\widehat{L}_n}}{p_{L_n}} p_\lambda \omega] = \left(\sum_\mu [p_\mu \omega] + q\sum_\nu [p_\nu \omega]\right) + (x_\lambda - x_{\lambda_{\max}})[p_\lambda \omega].$$

We also have:

$$\begin{aligned} q\frac{\partial W^{\mathrm{eq}}}{\partial q} &= q\frac{\partial W}{\partial q} + x_{\lambda_{\max}} \\ &= q\frac{p_{\widehat{L}_n}}{p_{L_n}} + x_{\lambda_{\max}}, \end{aligned}$$

and part (a) of Theorem 21.1 follows.

For part (b), we use Lemma 18.4 and the sum of the cases $m = 1, \ldots, n$ in Proposition 21.4. □

Proof of Theorem 5.5. The free basis lemma, Lemma 9.3, also has an equivariant version. This is just obtained by replacing $\mathbb{C}[z^{\pm 1}, q^{\pm 1}]$ by $H^*_{T^\vee}(pt)[z^{\pm 1}, q^{\pm 1}]$, which does not affect the proof. Theorem 5.5 now follows from Theorem 21.1 and the equivariant free basis lemma. □

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