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# $L^2$ Properties of Lévy Generators on Compact Riemannian Manifolds

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## Abstract

We consider isotropic Lévy processes on a compact Riemannian manifold, obtained from an  $\mathbb{R}^d$ -valued Lévy process through rolling without slipping. We prove that the Feller semigroups associated with these processes extend to strongly continuous contraction semigroups on  $L^p$ , for  $1 \leq p < \infty$ . When  $p = 2$ , we show that these semigroups are self-adjoint. If, in addition, the motion has a non-trivial Brownian part, we prove that the generator has a discrete spectrum of eigenvalues and that the semigroup is trace-class.

**Keywords** Riemannian manifold · Frame bundle · Connection · Horizontal vector field · Lévy process · Lévy measure · Feller semigroup · Generator · Sobolev space · Transition density

**Mathematics Subject Classification 2010** 58J65 · 60G51 · 47D07 · 35P05 · 60H10 · 60J35

## 1 Introduction

The investigation of Lévy processes (essentially processes with stationary and independent increments) has become a rich theme in modern probability theory. These processes have a deep and interesting structure, there are wide-ranging applications, and they have been found to be a useful class of driving noises for stochastic differential equations. Usually Lévy processes are considered as having Euclidean space (or even the real line) as their state space, although Banach and Hilbert space valued

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processes are important for stochastic partial differential equations (see e.g. [22]). The most intensively studied class of nonlinear state spaces for Lévy processes has been Lie groups. Here, the notion of increment is defined using the group law and inverse operation, so if  $0 \leq s < t$ , where the increment of the process  $X$  in Euclidean space is  $X(t) - X(s)$ , on a Lie group we instead consider  $X(s)^{-1}X(t)$ . For a dedicated monograph treatment of such processes, see Liao [18].

Having considered Lévy processes on Lie groups, the next obvious step is to move to a Riemannian manifold. Indeed, arguably the most important Lévy process in Euclidean space is Brownian motion, and Brownian motion on manifolds is a very well developed theory in its own right (see e.g. Elworthy [10] and Hsu [15]). As its generator is the Laplace–Beltrami operator and transition density is the heat kernel, it is clear that this process is a natural expression of the geometry of the manifold. For development of a more general theory of Lévy processes, we are hampered by the fact that there is no obvious notion of increment on a manifold. When the space is a symmetric space, there is an alternative point of view pioneered by Gangolli [11, 12], who used spherical functions to establish an analogue of the Lévy–Khintchine formula. Using the fact that every symmetric space is a homogeneous space  $G/K$  of a Lie group  $G$ , quotiented by a compact subgroup  $K$ , Applebaum [1] was able to realise these processes as images of Lévy processes on  $G$  through the natural map (see [19] for more recent developments).

Motivated in part by the results of [1], and also the Eels–Elworthy construction of Brownian motion on a manifold by projection from the frame bundle, Applebaum and Estrade ([4]) constructed a process that they called a “Lévy process on a manifold” and proved that it was a Feller–Markov process. The generic such process has the structure of a Brownian motion that is interlaced with jumps along geodesics, which are controlled by an isotropic Lévy measure. It is now more than twenty years since this paper was published, and to the authors’ knowledge, there has been no further work carried out on such processes since that time, other than the ergodic theoretic analysis in Mohari [21]. In the current paper, we hope to start the process of resurrecting this neglected area.

Since every Lévy process on a manifold is a Feller process, there is an associated semigroup on the space of continuous functions vanishing at infinity, and the generator is the sum of the Laplace–Beltrami operator and an integral superposition of jumps along geodesics. Our first new result is to show that the semigroup also preserves the  $L^p$  space of the Riemannian volume measure. We are particularly interested in the case  $p = 2$ , and here, we show that the semigroup (and hence its generator) is self-adjoint. When there is a non-trivial Brownian motion component to the process, we find that the generator has a discrete spectrum of non-positive eigenvalues. Under the latter condition, we also prove that the semigroup is trace-class, and so the process has a transition density/heat kernel.

## 2 Lévy Processes on Manifolds

Let  $M$  be a compact, connected,  $d$ -dimensional Riemannian manifold of dimension  $d \geq 1$ , with Riemannian measure  $\mu$ . Let  $OM$  denote the corresponding bundle of

orthonormal frames, a principle  $O(d)$ -bundle over  $M$ . The Levi-Civita connection induces a decomposition

$$T_r OM \cong H_r OM \oplus V_r OM, \quad \forall r \in OM$$

of each tangent space of  $OM$  into vertical and horizontal subspaces, and, writing  $\pi : OM \rightarrow M$  for the projection map, we have that for each  $p \in M$  and  $r \in OM_p$ ,  $d\pi_r$  vanishes on  $V_r OM$  and defines a linear isomorphism  $H_r OM \cong T_p M$ , for each  $p \in M$  and  $r \in OM_p$ . Moreover,  $OM$  may be given the structure of a Riemannian manifold in such a way that  $d\pi_r : H_r OM \rightarrow T_p M$  is an isometric isomorphism.<sup>1</sup> The corresponding Riemannian measure  $\tilde{\mu}$  is sometimes referred as Liouville measure. Observe that

$$\mu = \tilde{\mu} \circ \pi^{-1}. \tag{2.1}$$

One can easily show that  $OM$  is compact, using the fact that both its base manifold and structure group are compact. As such, both  $\mu$  and  $\tilde{\mu}$  are finite measures.

The *canonical horizontal fields* consist of a family of vector fields  $\{H_x : x \in \mathbb{R}^d\}$  on  $OM$ , defined by

$$H_x(r) = r(x)^*, \quad \forall r \in OM, x \in \mathbb{R}^d,$$

where  $*$  denotes horizontal lift. Since  $OM$  is compact, these vector fields are complete; we write  $\text{Exp}(tH_x)$  for the associated flows of diffeomorphisms. These flows are related to the Riemann exponential map by

$$\pi(\text{Exp}(H_x)(r)) = \exp_p r(x), \quad \forall x \in \mathbb{R}^d, p \in M, r \in OM_p. \tag{2.2}$$

For standard basis vectors  $e_i$  of  $\mathbb{R}^d$ , we use the abbreviation  $H_{e_i} = H_i$ . The horizontal Laplacian

$$\Delta_H = \sum_{i=1}^d H_i^2,$$

on  $OM$  is related to the Laplace–Beltrami operator  $\Delta$  on  $M$  by

$$\Delta f(p) = \Delta_H(f \circ \pi)(r), \quad \forall f \in C^\infty(M), p \in M, r \in OM_p;$$

for more details see Hsu [15] Chapter 3.

Let  $Y$  be an  $\mathbb{R}^d$ -valued Lévy process. It is shown in Applebaum and Estrade [4] that the Marcus canonical SDE on  $OM$

$$dR(t) = \sum_{j=1}^d H_j(R(t-)) \diamond dY_j(t), \quad t \geq 0; \quad R(0) = r \text{ (a.s.)}, \tag{2.3}$$

<sup>1</sup> See Elworthy [10] Chapter III Section 4.

has a unique, càdlàg solution  $R$  that is a Feller process. As in [4], we will call processes obtained this way horizontal Lévy processes. We also impose the assumption from [4] that the Lévy process  $Y$  is isotropic, in that its law is  $O(d)$ -invariant. By Corollary 2.4.22 on page 128 of [2], the Lévy characteristics of  $Y$  then take the form  $(0, aI, \nu)$ , where  $a \geq 0$ , and  $\nu$  is  $O(d)$ -invariant. Then, by Theorem 3.1 of [4], the process  $X = \pi(R)$  obtained by projection onto the base manifold is also a Feller process. The infinitesimal generators of  $R$  and  $X$  are

$$\mathcal{L} = \frac{1}{2}a\Delta_H + \mathcal{L}_J \tag{2.4}$$

and

$$\mathcal{A} = \frac{1}{2}a\Delta + \mathcal{A}_J, \tag{2.5}$$

respectively, where the jump parts  $\mathcal{L}_J$  and  $\mathcal{A}_J$  are given by

$$\begin{aligned} \mathcal{L}_J f(r) &= \int_{\mathbb{R}^d \setminus \{0\}} \{f(\text{Exp}(H_x)(r)) - f(r) - \mathbf{1}_{|x| < 1} H_x f(r)\} \nu(dx), \\ &\forall f \in C^\infty(OM), r \in OM \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \mathcal{A}_J f(p) &= \int_{T_p M \setminus \{0\}} \{f(\exp_p y) - f(p) - \mathbf{1}_{|y| < 1} y f(p)\} \nu_p(dx), \\ &\forall f \in C^\infty(M), p \in M. \end{aligned}$$

Here, the family of Lévy measures  $\{\nu_p : p \in M\}$  act on each tangent space  $T_p M$  and are defined by  $\nu_p = \nu \circ r^{-1}$  for any frame  $r \in OM_p$ . The two generators also satisfy

$$\mathcal{A} f(p) = \mathcal{L}(f \circ \pi)(r), \quad \forall f \in C^\infty(M), p \in M, r \in OM_p.$$

Observe that since  $\nu$  is  $O(d)$ -invariant, the right hand side of (2.6) is invariant under the change of variable  $x \mapsto -x$ , and hence for all  $f \in C^\infty(OM)$  and  $r \in OM$ ,

$$\mathcal{L}_J f(r) = \int_{\mathbb{R}^d \setminus \{0\}} \{f(\text{Exp}(H_{-x})(r)) - f(r) + \mathbf{1}_{|x| < 1} H_x f(r)\} \nu(dx), \tag{2.7}$$

where we have used the fact that  $H_{-x} f = -H_x f$ . Summing (2.6) and (2.7) and dividing by two,

$$\mathcal{L}_J f(r) = \frac{1}{2} \int_{\mathbb{R}^d \setminus \{0\}} \{f(\text{Exp}(H_x)(r)) - 2f(r) + f(\text{Exp}(H_{-x})(r))\} \nu(dx) \tag{2.8}$$

for all  $f \in C^\infty(OM)$  and  $r \in OM$ . It follows that

$$\mathcal{A}_J f(p) = \frac{1}{2} \int_{T_p M \setminus \{0\}} [f(\exp_p y) - 2f(p) + f(\exp_p(-y))] \nu_p(dy) \quad (2.9)$$

for each  $f \in C^\infty(M)$  and  $p \in M$ . Note the analogous expression (5.4.16) in [3] for symmetric Lévy motion on a Lie group.

### 3 $L^p$ Properties of the Semigroup

We now turn our attention to the Feller semigroups associated with  $R$  and  $X$ . These consist of families of contraction operators  $(S_t, t \geq 0)$  and  $(T_t, t \geq 0)$  acting on the Banach spaces  $C(OM)$  and  $C(M)$  by

$$S_t f(r) = \mathbb{E}(f(R(t)) | R(0) = r), \quad \forall f \in C(OM), r \in OM, t \geq 0,$$

and

$$T_t f(p) = \mathbb{E}(f(X(t)) | X(0) = p), \quad \forall f \in C(M), p \in M, t \geq 0.$$

Since  $X = \pi(R)$ , it is immediate that

$$T_t f(p) = S_t(f \circ \pi)(r) \quad (3.1)$$

for all  $f \in C(M)$ ,  $p = \pi(r) \in M$ , and  $t \geq 0$ .

Any Riemannian manifold has a natural  $L^p$  structure arising from its Riemannian measure, and so we may consider the spaces  $L^p(OM) := L^p(OM, \tilde{\mu}, \mathbb{R})$  and  $L^p(M) := L^p(M, \mu, \mathbb{R})$  for  $1 \leq p \leq \infty$ . For  $p < \infty$ , we prove that  $(S_t, t \geq 0)$  and  $(T_t, t \geq 0)$  extend to strongly continuous contraction semigroups on these spaces, and that they are self-adjoint when  $p = 2$ .

We begin by considering the semigroup  $(S_t, t \geq 0)$  associated with the horizontal process  $R$ ; analogous results for  $(T_t, t \geq 0)$  are then obtained by projection down onto  $M$ .

**Theorem 3.1** *For all  $1 \leq p < \infty$ ,  $(S_t, t \geq 0)$  extends to a  $C_0$ -semigroup of contractions on  $L^p(OM)$ .*

**Proof** Let  $q_t(\cdot, \cdot)$  denote the transition measure of  $(S_t, t \geq 0)$ , so that

$$S_t f(r) = \int_{OM} f(u) q_t(r, du), \quad \forall t \geq 0, f \in C(OM), r \in OM.$$

The horizontal fields  $H_x$  are divergence free for all  $x \in \mathbb{R}^d$  (Proposition 4.1 of [20]), and so by Theorem 3.1 of [5],  $\tilde{\mu}$  is invariant for  $S_t$ , in the sense that

$$\int_{OM} (S_t f)(r) \tilde{\mu}(dr) = \int_{OM} f(r) \tilde{\mu}(dr), \quad \forall t \geq 0, f \in C(OM).$$

Therefore, by Jensen’s inequality,

$$\begin{aligned} \|S_t f\|_p^p &= \int_{OM} \left| \int_{OM} f(u)q_t(r, du) \right|^p \tilde{\mu}(dr) \leq \int_{OM} \int_{OM} |f(u)|^p q_t(r, du) \tilde{\mu}(dr) \\ &= \int_{OM} (S_t |f|^p)(r) \tilde{\mu}(dr) = \int_{OM} |f(r)|^p \tilde{\mu}(dr) = \|f\|_p^p \end{aligned}$$

for all  $t \geq 0$  and  $f \in C(OM)$ . Each  $S_t$  has domain  $C(OM)$ , which is a dense subspace of  $L^p(OM)$  for all  $1 \leq p < \infty$ . It follows that each  $S_t$  extends to a unique contraction defined on the whole of  $L^p(OM)$ , which we also denote by  $S_t$ . By continuity, the semigroup property

$$S_t S_s = S_{t+s}, \quad \forall s, t \geq 0$$

continues to hold in this larger domain. It remains to prove strong continuity, i.e. that

$$\lim_{t \rightarrow 0} \|S_t f - f\|_p = 0 \tag{3.2}$$

for all  $f \in L^p(OM)$ . By density of  $C(OM)$  in  $L^p(OM)$ , it is sufficient to verify this for  $f \in C(OM)$ . The map  $t \mapsto S_t$  is strongly continuous in  $C(OM)$ , and so  $\lim_{t \rightarrow 0} \|S_t f - f\|_\infty = 0$  for all  $f \in C(OM)$ . Since  $(OM, \mathcal{B}(OM), \tilde{\mu})$  is a finite measure space,

$$\|S_t f - f\|_p \leq \tilde{\mu}(OM)^{\frac{1}{p}} \|S_t f - f\|_\infty,$$

for all  $f \in C(OM)$ . Equation (3.2) now follows. □

**Remark 3.2** The final part of the above proof applies more generally, in that if  $X$  is a compact space equipped with a finite measure  $m$ , and if  $(P_t, t \geq 0)$  is a Feller semigroup on  $X$ , then  $(P_t, t \geq 0)$  is strongly continuous in  $L^p(X, m)$ .

Projection down onto the base manifold yields the following.

**Theorem 3.3** For all  $1 \leq p < \infty$ ,  $(T_t, t \geq 0)$  extends to a strongly continuous semigroup of contractions on  $L^p(M)$ .

**Proof** Let  $1 \leq p < \infty$ . By Eqs. (2.1) and (3.1), many of the conditions we must check follow from their analogues on the frame bundle. Indeed, for all  $f \in C^\infty(M)$  and  $t \geq 0$ , we have

$$\|T_t f\|_{L^p(M)}^p = \int_M |T_t f(p)|^p d\mu = \int_{OM} |S_t(f \circ \pi)(r)|^p d\tilde{\mu} = \|S_t(f \circ \pi)\|_{L^p(OM)}^p,$$

and so, using the fact that  $S_t$  is a contraction of  $L^p(OM)$ ,

$$\|T_t f\|_{L^p(M)} = \|S_t(f \circ \pi)\|_{L^p(OM)} \leq \|f \circ \pi\|_{L^p(OM)} = \|f\|_{L^p(M)}.$$

Hence,  $T_t$  extends to a contraction of  $L^p(M)$  for all  $t \geq 0$ . It is clear by continuity that the semigroup property continues to hold on this larger domain, as does Eq. (3.1). Strong continuity follows by Remark 3.2, or alternatively can be seen by the observation

$$\|T_t f - f\|_{L^p(M)} = \|S_t(f \circ \pi) - f \circ \pi\|_{L^p(OM)} \quad \forall f \in L^p(M).$$

Thus,  $(T_t, t \geq 0)$  extends to a contraction semigroup on  $L^p(M)$  for all  $1 \leq p < \infty$ .  $\square$

We continue to denote the generators of  $(S_t, t \geq 0)$  and  $(T_t, t \geq 0)$  by  $\mathcal{L}$  and  $\mathcal{A}$ , respectively. Note that by Lemma 6.1.14 of [9],  $\mathcal{L}$  and  $\mathcal{A}$  are both closed operators on  $L^p(OM)$ .

### 4 The Case $p = 2$

For the remainder of this paper, we focus on the case  $p = 2$ . Our aim in this section is to prove that the semigroups  $(S_t, t \geq 0)$  and  $(T_t, t \geq 0)$  are self-adjoint semigroups on  $L^2(OM)$  and  $L^2(M)$ , respectively. By a standard result from semigroup theory, it will follow that  $\mathcal{L}$  and  $\mathcal{A}$  are self-adjoint linear operators.

Let us first impose the assumption that the Lévy measure  $\nu$  is finite. In this case,  $\mathcal{A}_J$  is the generator of a compound Poisson process on  $M$  (see [4]).

**Lemma 4.1** *If  $\nu$  is finite, then  $\mathcal{L}$  is a self-adjoint operator on  $L^2(OM)$ .*

**Proof** Since  $\nu$  is finite,  $\mathcal{L}_J$  is a bounded linear operator on  $L^2(OM)$ , and so Eq. (2.8) extends by continuity to the whole of  $L^2(OM)$ . It follows that  $\mathcal{L}$  is a bounded perturbation of the horizontal Laplacian, and so its domain is  $\text{Dom}(\Delta_H)$ . Clearly,  $\mathcal{L}$  is symmetric on this domain.

Since  $\mathcal{L}$  is a closed, symmetric operator, by Theorem X.1 on page 136 of Reed and Simon [23], the spectrum  $\sigma(\mathcal{L})$  of  $\mathcal{L}$  is equal to one of the following:

1. The closed upper-half plane.
2. The closed lower-half plane.
3. The entire complex plane.
4. A subset of  $\mathbb{R}$ .

Moreover,  $\mathcal{L}$  is self-adjoint if and only if Case 4 holds. By Theorem 8.2.1 of [9],

$$\sigma(\mathcal{L}) \subseteq (-\infty, 0], \tag{4.1}$$

from which we see that Case 4 is the only option.  $\square$

We now drop the assumption that  $\nu$  is finite.

**Theorem 4.2**  *$(S_t, t \geq 0)$  and  $(T_t, t \geq 0)$  are self-adjoint semigroups of operators on  $L^2(OM)$  and  $L^2(M)$ , respectively.*



**Proof** We will find it convenient to rewrite the process  $R(t)$  (with initial condition  $R(0) = r$  (a.s.)) as the action of a stochastic flow  $\eta_t$  on the point  $r$ , as in [4]. Then, as shown in Section 4 of [4], there is a sequence  $(\eta_t^{(n)})$  of stochastic flows on  $OM$  such that each  $\eta_t^{(n)}$  is the flow of a horizontal Lévy process with finite Lévy measure, and

$$\lim_{n \rightarrow \infty} \eta_t^{(n)}(r) = \eta_t(r) \quad (\text{a.s.}),$$

for all  $r \in OM$  and  $t \geq 0$ .

Let  $(S_t^{(n)}, t \geq 0)$  be the transition semigroup corresponding to the flow  $(\eta_t^{(n)}, t \geq 0)$ , for each  $n \in \mathbb{N}$ . It is a standard result in semigroup theory that a semigroup of operators on a Hilbert space is self-adjoint if and only if its generator is self-adjoint.<sup>2</sup> By Lemma 4.1,  $(S_t^{(n)}, t \geq 0)$  is a self-adjoint semigroup on  $L^2(OM)$ , for all  $n \in \mathbb{N}$ .

By dominated convergence, for each  $f \in C(OM)$  and  $t \geq 0$ , we have

$$\lim_{n \rightarrow \infty} \|S_t f - S_t^{(n)} f\|_{L^2(OM)}^2 = \lim_{n \rightarrow \infty} \int_{OM} \left| \mathbb{E} \left( f(\eta_t(r)) - f(\eta_t^{(n)}(r)) \right) \right|^2 \tilde{\mu}(dr) = 0.$$

Then, by the density of  $C(OM)$  in  $L^2(OM)$ , and a standard  $\epsilon/3$  argument (using the fact that  $S_t^{(n)}$  is an  $L^2$ -contraction), we deduce that for all  $f \in L^2(OM)$ ,

$$\lim_{n \rightarrow \infty} \|S_t f - S_t^{(n)} f\|_{L^2(OM)} = 0$$

So,  $S_t$  is the strong limit of a sequence of bounded self-adjoint operators, and hence is itself self-adjoint. To see that  $(T_t, t \geq 0)$  is also self-adjoint, let  $t \geq 0$  and  $f, g \in L^2(M)$ , and observe that by (2.1) and (3.1),

$$\begin{aligned} \langle T_t f, g \rangle_{L^2(M)} &= \langle S_t(f \circ \pi), g \circ \pi \rangle_{L^2(OM)} = \langle f \circ \pi, S_t(g \circ \pi) \rangle_{L^2(OM)} \\ &= \langle f, T_t g \rangle_{L^2(M)}. \end{aligned}$$

□

By Theorem 4.6 of Davies [8],  $-\mathcal{L}$  and  $-\mathcal{A}$  are positive self-adjoint operators on  $L^2(OM)$  and  $L^2(M)$ , respectively.

### 5 Spectral Properties of the Generator

For this final section, we restrict attention to the case in which  $X$  has non-trivial Brownian part (that is, when  $a > 0$ ) and prove some spectral results that are already well-established for the case of Brownian motion and the Laplace–Beltrami operator  $\Delta$ . For example, it is well known that  $\Delta$  has a discrete spectrum of eigenvalues. Each eigenspace is finite dimensional, and the eigenvectors may be normalised so as to form an orthonormal basis of  $L^2(M)$  (see for example Lablée [17] Theorem 4.3.1).

<sup>2</sup> See for example Goldstein [13] page 31.

Moreover, such an eigenbasis  $(\psi_n)$  can be ordered so that the corresponding sequence of eigenvalues decreases to  $-\infty$ . For each  $n \in \mathbb{N}$ , write  $-\mu_n$  for the eigenvalue associated with  $\psi_n$ , so that the real sequence  $(\mu_n)$  satisfies

$$0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \rightarrow \infty \text{ as } n \rightarrow \infty. \tag{5.1}$$

We prove an analogous result for  $\mathcal{A}$ , using a generalisation of the approach used by Lablée [17].

**Theorem 5.1** *Let  $X$  be an isotropic Lévy process on  $M$  with non-trivial Brownian part. Then, its generator  $\mathcal{A}$  has a discrete spectrum*

$$\sigma(\mathcal{A}) = \{-\lambda_n : n \in \mathbb{N}\},$$

where  $(\lambda_n)$  is a sequence of real numbers satisfying

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty \text{ as } n \rightarrow \infty. \tag{5.2}$$

Moreover, each of the associated eigenspaces is finite dimensional, and there is a corresponding sequence  $(\phi_n)$  of eigenvectors that forms an orthonormal basis of  $L^2(M)$ .

**Remark 5.2** We will generally assume that (5.2) is listed with multiplicity, so that for all  $n \in \mathbb{N}$ ,  $-\lambda_n$  is the eigenvalue associated with  $\phi_n$ .

**Proof** Without loss of generality, assume that  $a = 2$  so that

$$\mathcal{A} = \Delta + \mathcal{A}_J, \tag{5.3}$$

where  $\mathcal{A}_J$  is given by (2.9). Both  $\mathcal{A}$  and  $\mathcal{A}_J$  are generators of self-adjoint contraction semigroups, and hence,  $-\mathcal{A}$  and  $-\mathcal{A}_J$  are positive, self-adjoint operators.<sup>3</sup> For  $f, g \in \text{Dom } \mathcal{A}$ , define

$$\langle f, g \rangle_{\mathcal{A}} = \langle f, g \rangle_2 - \langle \mathcal{A}f, g \rangle_2. \tag{5.4}$$

The operator  $I - \mathcal{A}$  is also positive and self-adjoint, and so by Theorem 11 of [7], there is a unique positive self-adjoint operator  $B$  such that  $B^2 = I - \mathcal{A}$ . By (4.1),  $I - \mathcal{A}$  is invertible, and hence  $B$  is injective. Moreover,

$$\langle f, g \rangle_{\mathcal{A}} = \langle Bf, Bg \rangle_2, \quad \forall f, g \in \text{Dom } \mathcal{A},$$

from which it is easy<sup>4</sup> to see that  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  defines an inner product on  $\text{Dom } \mathcal{A}$ .

Let  $V$  denote the completion of  $C^\infty(M)$  with respect to  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ . This is a ‘‘Lévy analogue’’ of the Sobolev space  $H^1(M)$  considered in Lablée [17] or Grigor’yan [14], where the completion is instead taken with respect to the Sobolev inner product

$$\langle f, g \rangle_{H^1} = \langle f, g \rangle_2 - \langle \Delta f, g \rangle_2, \quad \forall f, g \in C^\infty(M). \tag{5.5}$$

<sup>3</sup> See Theorem 4.6 on page 99 of Davies [8].

<sup>4</sup> Bilinearity and symmetry are clear, and positive-definiteness is immediate by injectivity of  $B$ .

In the case when  $M = \mathbb{R}^d$ , spaces of this type are discussed in Section 3.10 of Jacob [16], who refers to them as *anisotropic Sobolev spaces*. By (5.3), we have

$$\langle f, g \rangle_{\mathcal{A}} = \langle f, g \rangle_{H^1} - \langle \mathcal{A}_J f, g \rangle_2 \quad \forall f, g \in C^\infty(M),$$

and since  $-\mathcal{A}_J$  is a positive operator, it follows that  $\|f\|_{\mathcal{A}} \geq \|f\|_{H^1}$  for all  $f \in C^\infty(M)$ . Similarly, (5.5) implies  $\|f\|_{H^1} \geq \|f\|_2$  for all  $f \in C^\infty(M)$ . Hence,

$$V \subseteq H^1(M) \subseteq L^2(M),$$

and

$$\|f\|_2 \leq \|f\|_{H^1} \leq \|f\|_{\mathcal{A}}, \quad \forall f \in V. \quad (5.6)$$

In particular, the inclusion  $V \hookrightarrow H^1(M)$  is bounded. By Rellich's theorem<sup>5</sup>, the inclusion  $H^1(M) \hookrightarrow L^2(M)$  is compact, and hence so is the inclusion  $i : V \hookrightarrow L^2(M)$  (it is a composition of a compact operator with a bounded operator).

Let  $f \in L^2(M)$  and consider  $l \in V^*$  given by

$$l(g) = \langle f, g \rangle_2 \quad \forall g \in V.$$

For all  $g \in V$ , we have by the Cauchy–Schwarz inequality

$$|l(g)| \leq \|f\|_2 \|g\|_2 \leq \|f\|_2 \|g\|_{\mathcal{A}}.$$

Hence,

$$\|l\|_{V^*} \leq \|f\|_2, \quad (5.7)$$

where  $\|\cdot\|_{V^*}$  denotes the norm of  $V^*$ . By the Riesz representation theorem, there is a unique  $v_f \in V$  for which

$$\langle v_f, g \rangle_{\mathcal{A}} = l(g), \quad \forall g \in V.$$

Moreover,

$$\|v_f\|_{\mathcal{A}} = \sup_{g \in V \setminus \{0\}} \frac{|\langle v_f, g \rangle_{\mathcal{A}}|}{\|g\|_{\mathcal{A}}} = \|l\|_{V^*}.$$

Define  $T : L^2(M) \rightarrow V$  by  $Tf = v_f$  for all  $f \in L^2(M)$ . Then,

$$\langle Tf, g \rangle_{\mathcal{A}} = \langle f, g \rangle_2 \quad \forall f \in L^2(M), g \in V, \quad (5.8)$$

<sup>5</sup> See Lablée [17] page 68.

and  $T$  is bounded, since by (5.7),

$$\|Tf\|_{\mathcal{A}} = \|I\|_{V^*} \leq \|f\|_2,$$

for all  $f \in L^2(M)$ . By (5.6),

$$\|Tf\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}} \quad \forall f \in V,$$

and so  $T|_V$  is a bounded operator on  $V$ . We also have

$$T|_V = T \circ i,$$

and, since  $i$  is compact, so too is  $T|_V$ . By symmetry of inner products and Eq. (5.8),  $T$  is self-adjoint. Equation (5.8) also implies that  $T|_V$  is a positive operator, and that 0 is not an eigenvalue of  $T|_V$  (indeed, if  $Tf = 0$ , then  $\|f\|_2^2 = \langle Tf, f \rangle_{\mathcal{A}} = 0$ ).

By the Hilbert-Schmidt theorem<sup>6</sup>, the spectrum of  $T|_V$  consists of a sequence  $(\alpha_n)$  of positive eigenvalues that decrease to 0. Each eigenspace is finite dimensional, and the corresponding eigenvectors can be normalised so as to form an orthonormal basis  $(v_n)$  of  $(V, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ .

In fact, it is easy to see from the definition of  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  that

$$T = (I - \mathcal{A})^{-1}, \tag{5.9}$$

and hence, the spectrum of  $\mathcal{A}$  is just  $\{1 - \alpha_n^{-1} : n \in \mathbb{N}\}$ , with corresponding eigenvectors still given by the  $v_n$ . Moreover, we may scale these eigenvectors so that they are  $L^2$ -orthonormal. Indeed, for each  $n \in \mathbb{N}$ , let

$$\phi_n = \frac{1}{\sqrt{\alpha_n}} v_n.$$

Then, for all  $m, n \in \mathbb{N}$ ,

$$\langle \phi_n, \phi_m \rangle_2 = \frac{1}{\sqrt{\alpha_n \alpha_m}} \langle T v_n, v_m \rangle_{\mathcal{A}} = \sqrt{\frac{\alpha_n}{\alpha_m}} \langle v_n, v_m \rangle_{\mathcal{A}} = \delta_{m,n}.$$

By denseness of  $V$  in  $L^2(M)$ , the  $\phi_n$  form an orthonormal basis of  $L^2(M)$ .

Finally, let  $\lambda_n = \alpha_n^{-1} - 1$  for each  $n \in \mathbb{N}$ . Then,  $(\lambda_n)$  satisfies Eq. (5.2), since  $-\mathcal{A}$  is a positive operator, and  $(\alpha_n)$  is a positive sequence that decreases to 0.  $\square$

It is well known that the heat semigroup  $(K_t, t \geq 0)$  associated with Brownian motion on a compact manifold is trace-class and possesses an integral kernel. The final two results of this section extend this to the Lévy semigroup  $(T_t, t \geq 0)$ , subject to the assumption that  $a > 0$ .

<sup>6</sup> See Simon [24] Section 3.2.

**Theorem 5.3** *Let  $X$  be an isotropic Lévy process on  $M$  with non-trivial Brownian part. Then, the transition semigroup operator  $T_t$  is trace-class for all  $t > 0$ .*

**Proof** We again assume  $a = 2$ , so that  $\mathcal{A}$  has the form (5.3). The case for general  $a > 0$  is very similar.

Let  $(\lambda_n)$  and  $(\phi_n)$  be as in the statement of Theorem 5.1, and let  $(\mu_n)$  and  $(\psi_n)$  be the analogous sequences for  $\Delta$ , so that  $\psi_n$  is the  $n^{\text{th}}$  eigenvector of  $\Delta$ , with associated eigenvalue  $-\mu_n$ .

Let  $(K_t, t \geq 0)$  denote the heat semigroup associated with Brownian motion on  $M$ . This operator semigroup is known to possess many wonderful properties, including being trace-class. It follows that

$$\text{tr } K_t = \sum_{n=1}^{\infty} \langle K_t \psi_n, \psi_n \rangle = \sum_{n=1}^{\infty} e^{-t\mu_n} < \infty \tag{5.10}$$

for all  $t > 0$ . As an element of  $[0, \infty]$ , the trace of each  $T_t$  is given by

$$\text{tr } T_t = \sum_{n=1}^{\infty} \langle T_t \phi_n, \phi_n \rangle = \sum_{n=1}^{\infty} e^{-t\lambda_n}. \tag{5.11}$$

By the min-max principle for self-adjoint semibounded operators,<sup>7</sup> we have for all  $n \in \mathbb{N}$ ,

$$\lambda_n = - \sup_{f_1, \dots, f_{n-1} \in C^\infty(M)} \left[ \inf_{f \in \{f_1, \dots, f_{n-1}\}^\perp, \|f\|=1} \langle \mathcal{A} f, f \rangle \right],$$

and

$$\mu_n = - \sup_{f_1, \dots, f_{n-1} \in C^\infty(M)} \left[ \inf_{f \in \{f_1, \dots, f_{n-1}\}^\perp, \|f\|=1} \langle \Delta f, f \rangle \right].$$

As noted in the proof of Theorem 5.1, for all  $f \in C^\infty(M)$ ,

$$-\langle \mathcal{A} f, f \rangle \geq -\langle \Delta f, f \rangle \geq 0,$$

and hence  $\lambda_n \geq \mu_n$  for all  $n \in \mathbb{N}$ . But then,  $e^{-t\lambda_n} \leq e^{-t\mu_n}$  for all  $t > 0$  and  $n \in \mathbb{N}$ , and so, comparing (5.11) with (5.10),

$$\text{tr } T_t < \text{tr } K_t < \infty$$

for all  $t > 0$ . □

By Lemma 7.2.1 of Davies [9], we immediately obtain the following.

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<sup>7</sup> Simon [24] page 666.

**Corollary 5.4** *Let  $X$  be an isotropic Lévy process on  $M$  with non-trivial Brownian part. Then, its semigroup  $(T_t, t \geq 0)$  has a square-integrable kernel. That is, for all  $t > 0$  there is a map  $p_t \in L^2(M \times M)$  such that*

$$T_t f(x) = \int_M f(y) p_t(x, y) \mu(dy)$$

for all  $f \in L^2(M)$  and  $x \in M$ . Moreover, we have the following  $L^2$ -convergent expansion:

$$p_t(x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y), \quad \forall x, y \in M, t \geq 0.$$

It is natural to enquire as to whether similar results hold in the pure jump case when  $a = 0$  (perhaps under some further condition on  $\nu$ )? When  $M$  is a compact symmetric space, we can find an orthonormal basis of eigenfunctions that are common to the  $L^2$ -semigroups associated with all isotropic Lévy processes (see the results in section 5 of [6]). The key tool here, which enables a precise description of the spectrum of eigenvalues, is Gangolli's Lévy–Khinchine formula [11]. In the general case, as considered here, such methods are not available, and we are unable to make further progress at the present time.

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