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Hussein, MS, Lesnic, D orcid.org/0000-0003-3025-2770, Kamynin, VL et al. (1 more author) (2020) Direct and inverse source problems for degenerate parabolic equations. Journal of Inverse and Ill-Posed Problems, 28 (3). pp. 425-448. ISSN 0928-0219

https://doi.org/10.1515/jiip-2019-0046

Direct and inverse source problems for degenerate parabolic equations

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Abstract. Degenerate parabolic partial differential equations (PDEs) with vanishing or unbounded leading coefficient makes the PDE non-uniformly parabolic and new theories need to be developed in the context of practical applications of such rather unstudied mathematical models arising in porous media, population dynamics, financial mathematics, etc. With this new challenge in mind, this paper considers investigating newly formulated direct and inverse problems associated with non-uniform parabolic PDEs where the leading space- and time-dependent coefficient is allowed to vanish on a non-empty, but zero measure, kernel set. In the context of inverse analysis, we consider the linear but ill-posed identification of a space-dependent source from a time-integral observation of the weighted main dependent variable. For both this inverse source problem, as well as its corresponding direct formulation we rigorously investigate the question of well-posedness. We also give examples of inverse problems for which sufficient conditions guaranteeing the unique solvability are fulfilled, and present the results of numerical simulations. It is hoped that the analysis initiated in this study will open up new avenues for research in the field of direct and inverse problems for degenerate parabolic equations with applications.

Keywords: inverse source problem; degenerate parabolic equation; integral observation.
Mathematics Subject Classification: 35K20, 35R30.

1 Introduction

Degenerate parabolic equations/operators are manifestations of limiting diffusion processes with important practical applications in porous media, laminar flow, climate models, population genetics and financial mathematics [2, 4]. In those problems, the degeneracy occurs because the physical coefficient present in the partial differential equation (PDE) may vanish at certain points. As such, the degeneracy may occur in various ways, namely: (i) at the space boundary; (ii) at the initial time; or (iii) inside the space domain and possibly at various times.

Despite their fundamental importance and practical application, the literature on inverse degenerate problems for parabolic PDEs is rather recent and scarce, see [3, 4, 6, 7, 10, 13, 14, 15, 16]. Therefore, this paper is aimed at investigating both forward and inverse source problems associated to degenerate parabolic PDEs in which the degeneracy occurs in the leading diffusivity coefficient which is allowed to vanish on a zero measure subset of the space and time solution domain. In comparison with the non-degenerate parabolic PDEs, the above degeneracy makes the PDE non-uniformly parabolic and this causes the conditions and proof of the well-posedness of the direct problem different.

We investigate the questions of unique solvability and numerical solution of the linear direct and inverse source problems for the degenerate (in the sense to be defined below) parabolic equation

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - \mathrm{d}(x,t)u = p(x)g(x,t) + r(x,t),\ (x,t) \in Q := [0,l] \times [0,T], \quad (1.1)$$

with the initial and boundary conditions

$$u(x,0) = u_0(x),\ x \in [0,l]; \qquad u(0,t) = u(l,t) = 0,\ t \in [0,T]. \quad (1.2)$$

In the inverse problem, the additional information is given by the integral observation

$$\int_0^T u(x,t)\chi(t)\,dt = \varphi(x), \quad x \in [0,l], \quad (1.3)$$

which is more practically realistic/feasible and general than the specification of the temperature at the final time, namely, $u(x,T) = \Phi(x)$ for $x \in [0,l]$.

Remark 1. (i) We consider the homogeneous Dirichlet boundary conditions in (1.2), for simplicity. Our results can be also obtained for the non-homogeneous boundary conditions

$$u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t), \quad t \in [0,T]$$

under some natural assumptions on the functions $\mu_1(t)$ and $\mu_2(t)$. Other types of boundary conditions such as Neumann, Robin or mixed, can also be posed but the analysis would require substantial modifications due to the presence of degenerate conductivity coefficient in the flux normal derivative.

(ii) In the Definition 1 (of Section 2) of a generalized solution we require the function $u(x,t)$ to satisfy the equation (1.1) almost everywhere in Q, so in this sense there is no difference between the closed interval $[0,l] \times [0,T]$ and $(0,l) \times (0,T)$. On the other hand, our solution is a continuous function in $[0,l] \times [0,T]$ so, it is convenient to have $Q = [0,l] \times [0,T]$.

(iii) The right-hand of (1.1) represents a source depending on both space and time denoted by $f(x,t)$, see (2.1), in the direct problem; otherwise, in the inverse source problem there is the space-dependent component $p(x)$ that is unknown and has to be determined uniquely from the time integral space-dependent measurement (1.3). Inverse source problems for determining a time-dependent component from weighted mass/energy measurement in degenerate parabolic equations have been considered elsewhere [15]. In the left hand-side of (1.1), $\mathrm{d}(x,t)$ represents a reaction coefficient (physically, it can be a blood perfusion, radiative or heat transfer coefficient) and it is assumed to be known.

The main unknown dependent variable $u(x,t)$ may be a temperature, a pressure or a concentration. The functions a and b are known and they represent the conductivity/diffusivity and convection coefficient, respectively.

(iv) In (1.3), the weight function $\chi(t)$ can mimic a Dirac delta function, say $\delta(t-t^*)$, where $t^* \in (0,T]$, in which case the non-local measurement (1.3) becomes an instant measurement of u at $t = t^*$, namely, [25, 28]

$$u(x,t^*) = \varphi(x), \quad x \in [0,l]. \quad (1.4)$$

When the uniform parabolicity condition $\infty > a_1 = \text{const.} \geq a(x,t) \geq a_0 = \text{const.} > 0$ is relaxed, then the PDE (1.1) becomes degenerate (or singular) and the newly obtained object/concept, requires novel theories and approaches for solving both direct and inverse associated problems.

In this paper, we suppose that the leading coefficient $a(x,t)$ in (1.1) is bounded but is only non-negative (i.e., $0 \leqslant a(x,t) \leqslant a_1$ for $(x,t) \in Q$) and satisfy the condition $1/a \in L_q(Q)$ for some $q > 1$. Stronger degeneracies occurring in limiting processes in the subsurface such as clogging, where the porosity may vanish on a set of positive measure [1] are not considered herein. However, the rather arbitrary degeneracy in the coefficient $a(x,t)$ that is considered in this paper is more general than the previous concretely specified degeneracies at $t = 0$, [10, 13], or at $x = 0$, [2, 3], or at $x \in \{0, l\}$, [6, 7, 8].

Let us mention that the unique solvability of the inverse problem (1.1)–(1.3) with $a(x,t)$ strictly positive ($a(x,t) \geqslant a_0 = \text{const} > 0$) but unbounded was recently investigated in [26].

The paper is structured as follows. In Section 2, we study the direct problem given by (1.1) and (1.2) with known function $p \in L_\infty(0,l)$ in the right-hand side of (1.1). We find sufficient conditions for unique solvability and obtain some estimates of the solution $u(x,t)$. We then use these estimates to investigate the inverse problem (1.1)–(1.3). In Section 3, we give two types of sufficient conditions for the unique solvability of the inverse problem (1.1)–(1.3). In Section 4, we present some examples of inverse problems for which the results of Section 3 hold. In Sections 5 and 6, we present the numerical simulations of solving the considered direct and inverse problems, respectively.

In this paper, we use Lebesgue and Sobolev spaces with corresponding norms in the usual sense (see, for example, [19, 20]). By $C^{0,\sigma}(Q)$ with $\sigma \in (0,1)$, we will denote the Hölder space of continuous functions in Q with finite norm

$$|u|_{C^{0,\sigma}(Q)} = \max_Q |u(x,t)| + \sup_{\substack{(x_1,t_1),(x_2,t_2) \in Q \\ (x_1,t_1) \neq (x_2,t_2)}} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{|x_1 - x_2|^\sigma + |t_1 - t_2|^{\sigma/2}}.$$

For convenience, we denote the space $L_\infty(0,l)$ by E and the norm in the space $L_q(0,l)$ by $\|\cdot\|_q$ for $q \in [1,\infty]$. We recall the Poincaré-Steklov inequality, which, for $n = 1$ is in the form

$$\|z\|_2 \leqslant \frac{l}{\pi}\|z_x\|_2, \qquad \forall z \in \overset{\circ}{W}{}^1_2(0,l), \tag{1.5}$$

and the inequality

$$|\alpha\beta| \leqslant \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2, \qquad \forall \varepsilon > 0. \tag{1.6}$$

We denote $Q(0,\tau) = [0,l] \times [0,\tau]$, $0 < \tau \leqslant T$ and $Q(0,T) \equiv Q$.

2 Investigation of the direct problem

In this section, we consider the direct problem for the equation

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - \mathrm{d}(x,t)u = f(x,t), \quad (x,t) \in Q, \tag{2.1}$$

with initial and boundary conditions (1.2) and known function $f(x,t)$.

Definition 1. By a generalized solution of the direct problem given by (2.1) and (1.2) we mean a function

$$u \in C^{0,\sigma}(Q) \cap L_\infty(0,T;\overset{\circ}{W}{}_2^1(0,l)) \cap W_s^{2,1}(Q),\ s>1,\ \sigma\in(0,1), \tag{2.2}$$

which satisfies the equation (2.1) almost everywhere in Q and satisfies the conditions (1.2) in classical sense.

We establish sufficient conditions for the unique solvability of the direct problem given by (2.1) and (1.2) and obtain a series of estimates for the solution.

We assume that the functions appearing in the input data in (2.1) and (1.2) are measurable and satisfy the following conditions:

$$0 \leqslant a(x,t) \leqslant a_1,\ \left|\frac{a_x^2(x,t)}{a(x,t)}\right| \leqslant K_a^*,\ (x,t)\in Q;\quad \frac{1}{a}\in L_q(Q),\ q>1,\ \left\|\frac{1}{a}\right\|_{L_q(Q)} \leqslant a_2; \tag{A}$$

$$\left|\frac{b^2(x,t)}{a(x,t)}\right| \leqslant K_{b,a},\quad \left|\frac{\mathrm{d}^2(x,t)}{a(x,t)}\right| \leqslant K_{d,a},\quad (x,t)\in Q; \tag{B}$$

$$u_0 \in \overset{\circ}{W}{}_2^1(0,l),\quad \|u_0'\|_2 \leqslant M_0; \tag{C}$$

$$\frac{f^2}{a}\in L_1(Q),\quad \left\|\frac{f^2}{a}\right\|_{L_1(Q)} \leqslant K_{f,a}. \tag{D}$$

Here $a_1,\ a_2 = \text{const.} > 0$, and $K_{b,a},\ K_a^*,\ K_{d,a},\ M_0,\ K_{f,a} = \text{const.} \geqslant 0$.

Remark 2. Assumptions (A), (B) and (D) show the following:

(i) The coefficient $a(x,t)$ cannot have the power degeneracy with respect to x (see [3]). However, assumption (A) still includes a wide range of cases with time-dependent degeneracy, as further exemplified in subsection 3.3.

(ii) The degeneration of $a(x,t)$ must be consistent with the behavior of coefficients $b(x,t), d(x,t)$ and the right-hand side function $f(x,t)$.

2.1 Uniqueness of the solution of the direct problem

Theorem 1. *Let the conditions* (A) *and* (B) *hold. Then a generalized solution (in the sense of Definition 1) of the direct problem given by* (2.1) *and* (1.2) *is unique.*

Proof. Suppose that there are two solutions $u_1(x,t)$ and $u_2(x,t)$ of this problem. Set $v(x,t) = u_2(x,t) - u_1(x,t)$, then $v(x,t)$ is a solution of the homogeneous problem

$$v_t - a(x,t)v_{xx} - \sqrt{a(x,t)}\frac{b(x,t)}{\sqrt{a(x,t)}}v_x - \sqrt{a(x,t)}\frac{\mathrm{d}(x,t)}{\sqrt{a(x,t)}}v = 0,\quad (x,t)\in Q, \tag{2.3}$$

$$v(x,0)=0,\ x\in[0,l];\qquad v(0,t)=v(l,t)=0,\ t\in[0,T]. \tag{2.4}$$

Let us multiply (2.3) by $e^{-\lambda t}v$ (where $\lambda = \text{const.} > 0$ will be chosen below) and integrate the result over Q. Taking into account (2.4), after some manipulations, we obtain

$$
\begin{aligned}
&\frac{1}{2}e^{-\lambda T}\int_0^l v^2(x,T)\,dx+\frac{\lambda}{2}\int_Q e^{-\lambda t}v^2(x,t)\,dx\,dt+\int_Q e^{-\lambda t}a(x,t)v_x^2(x,t)\,dx\,dt\\
&\quad\leqslant \int_Q e^{-\lambda t/2}\sqrt{a}\,|v_x|\,e^{-\lambda t/2}\frac{|a_x|}{\sqrt{a}}\,|v|\,dx\,dt+\int_Q e^{-\lambda t/2}\sqrt{a}\,|v_x|\,e^{-\lambda t/2}\frac{|b|}{\sqrt{a}}\,|v|\,dx\,dt\\
&\qquad+\int_Q e^{-\lambda t}\sqrt{a}\,\frac{|\mathrm{d}|}{\sqrt{a}}\,v^2\,dx\,dt\leqslant\frac{1}{2}\int_Q e^{-\lambda t}\,\frac{a_x^2}{a}\,v^2\,dx\,dt+\frac{1}{2}\int_Q e^{-\lambda t}\,\frac{b^2}{a}\,v^2\,dx\,dt\\
&\qquad\qquad+\int_Q e^{-\lambda t}a\,v_x^2dx\,dt+\int_Q e^{-\lambda t}\sqrt{a}\,\frac{|\mathrm{d}|}{\sqrt{a}}\,v^2\,dx\,dt. \qquad (2.5)
\end{aligned}
$$

From (2.5) we obtain that

$$
\begin{aligned}
\frac{\lambda}{2}\int_Q e^{-\lambda t}v^2(x,t)dxdt &\leqslant \frac{1}{2}\int_Q e^{-\lambda t}\left(\frac{a_x^2+b^2}{a}\right)v^2dxdt+\int_Q e^{-\lambda t}\sqrt{a}\frac{|\mathrm{d}|}{\sqrt{a}}v^2dxdt\\
&\leqslant C\int_Q e^{-\lambda t}v^2dxdt. \qquad (2.6)
\end{aligned}
$$

Here $C=$ const. >0 does not depend on λ. To obtain the last inequality in (2.6) we have applied the assumptions (A) and (B). Setting $\lambda=4C$ we obtain from (2.6) that $v(x,t)\equiv 0$ in Q. Theorem 1 is proved. □

2.2 Existence of the solution of the direct problem

Now we prove the existence of the generalized solution to the direct problem given by (2.1) and (1.2) and derive a series of estimates for that solution. In these estimates, by C with index we will denote positive constants depending only on l, T, a_1, a_2, M_0, $K_{b,a}$, $K_{d,a}$ and $K_{f,a}$.

Theorem 2. *Let the assumptions (A)–(D) hold. Set*

$$
q^*=\frac{2q}{q+1},\qquad \lambda^*=3\Big(K_{b,a}+\frac{l^2}{\pi^2}K_{d,a}\Big). \qquad (2.7)
$$

Then there exists a generalized solution (in the sense of Definition 1) of the direct problem given by (2.1) and (1.2) with $s=q^*>1$, i.e. $u\in W^{2,1}_{q^*}(Q)$. Moreover, this solution satisfies the estimates:

$$
\sup_{0\leqslant t\leqslant T}\|u_x(\cdot,t)\|_2^2\leqslant e^{\lambda^*T}\big(\|u_0'\|_2^2+3\|f^2/a\|_{L_1(Q)}\big), \qquad (2.8)
$$

$$
\|au_{xx}^2\|_{L_1(Q)}\leqslant e^{\lambda^*T}\big(\|u_0'\|_2^2+3\|f^2/a\|_{L_1(Q)}\big), \qquad (2.9)
$$

$$
\|u_{xx}\|^2_{L_{q^*}(Q)}\leqslant a_2\,e^{\lambda^*T}\big(\|u_0'\|_2^2+3\|f^2/a\|_{L_1(Q)}\big), \qquad (2.10)
$$

$$
\|u_t\|^2_{L_2(Q)}+\Big\|\frac{u_t^2}{a}\Big\|_{L_1(Q)}\leqslant C_1, \qquad (2.11)
$$

$$
|u(x_1,t_1)-u(x_2,t_2)|\leqslant C_2\,|x_1-x_2|^{1/2}+C_3\,|t_1-t_2|^{1/6},\quad (x_1,t_1),\,(x_2,t_2)\in Q. \qquad (2.12)
$$

Proof. Let us put $a^n(x,t) := a(x,t) + 1/n$ for $n \in \mathbb{N}^*$, $h = 1/n$ and denote by $(b/\sqrt{a})^h$, $(\mathrm{d}/\sqrt{a})^h$, $(f/\sqrt{a})^h$ the mean functions for $b/\sqrt{a}$, $\mathrm{d}/\sqrt{a}$, $f/\sqrt{a}$, respectively (for their definition and properties see, for example, [21, p. 16]). Note that from the well-known properties of mean functions and in view of conditions (B) and (D) it follows that

$$\big\|(b/\sqrt{a})^h\big\|_{L_\infty(Q)} \leqslant \big\|b/\sqrt{a}\big\|_{L_\infty(Q)} \leqslant K_{b,a}^{1/2}, \quad \big\|(\mathrm{d}/\sqrt{a})^h\big\|_{L_\infty(Q)} \leqslant \big\|\mathrm{d}/\sqrt{a}\big\|_{L_\infty(Q)} \leqslant K_{\mathrm{d},a}^{1/2},$$

$$\big\|(f/\sqrt{a})^h\big\|_{L_1(Q)} \leqslant \big\|f/\sqrt{a}\big\|_{L_1(Q)} \leqslant K_{f,a}^{1/2}. \tag{2.13}$$

Let us consider in the rectangle Q the first initial boundary value problem for the equation

$$u_t^n - a^n(x,t)u_{xx}^n = \sqrt{a^n}\Big(\frac{b}{\sqrt{a}}\Big)^h u_x^n + \sqrt{a^n}\Big(\frac{\mathrm{d}}{\sqrt{a}}\Big)^h u^n + \sqrt{a^n}\Big(\frac{f}{\sqrt{a}}\Big)^h, \tag{2.14}$$

with initial and homogeneous Dirichlet boundary conditions (1.2). The equation (2.14) is uniformly parabolic, and so, by [19], the problem given by (2.14) and (1.2) has a unique solution $u^n \in C(0,T; \overset{\circ}{W}{}_2^1(0,l)) \cap W_2^{2,1}(Q)$.

Let us derive a series of estimates for solutions $u^n(x,t)$ uniform with respect to n. To do this, we multiply the equation (2.14) by $\exp(-\lambda^* t)u_{xx}^n$ and integrate the result over the rectangle $Q(0,\tau)$. Here $0 < \tau \leqslant T$, and the constant λ^* is defined in (2.7).

After some manipulations based on integration by parts, in view of (1.2) and (2.13) we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^l e^{-\lambda^*\tau}\,(u_x^n(x,\tau))^2\,dx &+ \int_{Q(0,\tau)} e^{-\lambda^* t}\,a^n(x,t)(u_{xx}^n)^2\,dx\,dt + \frac{\lambda^*}{2}\int_{Q(0,\tau)} e^{-\lambda^* t}\,(u_x^n)^2\,dx\,dt \\
&\leqslant \frac{1}{2}\|u_0'\|_2^2 + \sqrt{K_{b,a}}\int_{Q(0,\tau)} e^{-\lambda^* t/2}\,\sqrt{a^n}\,|u_{xx}^n|\,e^{-\lambda^* t/2}\,|u_x^n|\,dx\,dt \\
&\quad + \sqrt{K_{d,a}}\int_{Q(0,\tau)} e^{-\lambda^* t/2}\,\sqrt{a^n}\,|u_{xx}^n|\,e^{-\lambda^* t/2}\,|u^n|\,dx\,dt \\
&\quad + \int_{Q(0,\tau)} e^{-\lambda^* t/2}\,\sqrt{a^n}\,|u_{xx}^n|\,e^{-\lambda^* t/2}\,\Big|\Big(\frac{f}{\sqrt{a}}\Big)^h\Big|\,dx\,dt.
\end{aligned} \tag{2.15}$$

To obtain the inequality (2.15) we formally need the existence of the derivative u_{xt}^n. But this condition can easily be overcome by using the mean function with respect to t (see later on definition (3.32)).

To estimate the second, third and fourth terms in the right-hand side of (2.15) we apply the inequality (1.6), taking $\varepsilon = 1/3$ and $\alpha = e^{-\lambda^* t/2}\sqrt{a^n}|u_{xx}^n|$. As a result, also using (1.5) and the definition of λ^* in (2.7), we obtain

$$\begin{aligned}
\int_0^l e^{-\lambda^*\tau}(u_x^n(x,\tau))^2\,dx &+ \int_{Q(0,\tau)} e^{-\lambda^* t}a^n(x,t)(u_{xx}^n)^2\,dx\,dt + \lambda^*\int_{Q(0,\tau)} e^{-\lambda^* t}(u_x^n)^2\,dx\,dt \\
&\leqslant \|u_0'\|_2^2 + \lambda^*\int_{Q(0,\tau)} e^{-\lambda^* t}|u_x^n|^2\,dx\,dt + 3\int_{Q(0,\tau)} e^{-\lambda^* t}\Big|\Big(\frac{f}{\sqrt{a}}\Big)^h\Big|^2\,dx\,dt.
\end{aligned} \tag{2.16}$$

Cancelling the third term in the left-hand side with the second term in the right-hand side of (2.16) and using that $e^{-\lambda^* T} \leq e^{-\lambda^* t} \leq 1$, we obtain the following estimate:

$$\sup_{0\leqslant t\leqslant T} \|u_x^n(\cdot,t)\|_2^2 + \|a^n(u_{xx}^n)^2\|_{L_1(Q)} \leqslant e^{\lambda^* T}\|u_0'\|_2^2 + 3e^{\lambda^* T}\int_Q \Big|\Big(\frac{f}{\sqrt{a}}\Big)^h\Big|^2\,dx\,dt$$
$$\leqslant e^{\lambda^* T}\left(\|u_0'\|_2^2 + 3\big\|f^2/a\big\|_{L_1(Q)}\right), \quad (2.17)$$

which is uniform with respect to n.

Using the assumption (A) and the definition of q^* in (2.7), i.e. $\frac{1}{q^*} = \frac{1}{2q} + \frac{1}{2}$, after applying Hölder's inequality, we obtain the estimate

$$\|u_{xx}^n\|_{L_{q^*}(Q)}^2 = \left(\int_Q \Big(\frac{1}{\sqrt{a^n}}\Big)^{q^*}|\sqrt{a^n}u_{xx}^n|^{q^*}\,dx\,dt\right)^{2/q^*}$$
$$\leqslant \left\|\frac{1}{a^n}\right\|_{L_q(Q)}\int_Q a^n|u_{xx}^n|^2\,dx\,dt \leqslant a_2\|a^n(u_{xx}^n)^2\|_{L_1(Q)},$$

from which, taking into account (2.17), we get

$$\|u_{xx}^n\|_{L_{q^*}(Q)}^2 \leqslant a_2 e^{\lambda^* T}\left(\|u_0'\|_2^2 + 3\big\|f^2/a\big\|_{L_1(Q)}\right). \quad (2.18)$$

By virtue of (2.14), applying the already proved estimates (2.17) and (2.18), we obtain

$$\|u_t^n\|_{L_{q^*}(Q)} \leqslant (a_1+1)\|u_{xx}^n\|_{L_{q^*}(Q)} + \sqrt{(a_1+1)K_{b,a}}\,\|u_x^n\|_{L_{q^*}(Q)}$$
$$+\sqrt{(a_1+1)K_{d,a}}\,\|u^n\|_{L_{q^*}(Q)} + \sqrt{a_1+1}\,\|\Big(\frac{f}{\sqrt{a}}\Big)^h\|_{L_{q^*}(Q)} \leqslant C_1. \quad (2.19)$$

and

$$\|\frac{u_t^n}{\sqrt{a^n}}\|_{L_2(Q)} \leqslant \|\sqrt{a^n}\,u_{xx}^n\|_{L_2(Q)} + \sqrt{K_{b,a}}\,\|u_x^n\|_{L_2(Q)} + \sqrt{K_{d,a}}\,\|u^n\|_{L_2(Q)} + \|\Big(\frac{f}{\sqrt{a}}\Big)^h\|_{L_2(Q)} \leq \tilde{C}_1,$$

from which

$$\|\frac{(u_t^n)^2}{a^n}\|_{L_1(Q)} \leqslant \tilde{C}_1. \quad (2.20)$$

Finally, using (2.20) we have

$$\big\|u_t^n\big\|_{L_2(Q)}^2 = \int_Q a^n\,\frac{|u_t^n|^2}{a^n}\,dx\,dt \leqslant (a_1+1)\int_Q \frac{|u_t^n|^2}{a^n}\,dx\,dt \leqslant \tilde{C}_1(a_1+1). \quad (2.21)$$

Now let us estimate the Hölder norm of $u^n(t,x)$. From (2.17), for all (x_1,t), $(x_2,t) \in Q$ we have

$$|u^n(x_1,t) - u^n(x_2,t)| \leqslant \|u_x^n(\cdot,t)\|_2\,|x_1-x_2|^{1/2} \leqslant C_2|x_1-x_2|^{1/2}. \quad (2.22)$$

On the other hand, for all $t_1, t_2 \in [0,T]$, from (2.19) noting that $q^* < 2$, we have

$$\int_0^l |u^n(x,t_1) - u^n(x,t_2)|\,dx \leqslant \int_{t_1}^{t_2}\int_0^l |u_\tau^n(x,\tau)|\,dx\,d\tau$$
$$\leqslant l^{1/2}\,\|u_t^n\|_{L_2(Q)}\,|t_2-t_1|^{1/2} \leqslant C_*\,|t_2-t_1|^{1/2}.$$

and using the estimate from [20, p.79, estimate (2.9)], we obtain that for all (x,t_1), $(x,t_2) \in Q$

$$|u^n(x,t_2) - u^n(x,t_1)| \leqslant C_{**}\, \|u^n_x(\cdot,t_2) - u^n_x(\cdot,t_1)\|_2^{2/3} \cdot \|u^n(\cdot,t_2) - u^n(\cdot,t_1)\|_1^{1/3} \leqslant C_3\, |t_2 - t_1|^{1/6}. \quad (2.23)$$

Using the triangle inequality, from (2.22) and (2.23), we obtain that

$$|u^n(x_1,t_1) - u^n(x_2,t_2)| \leqslant C_2|x_1 - x_2|^{1/2} + C_3|t_2 - t_1|^{1/6}. \quad (2.24)$$

It follows from the estimates (2.17), (2.18), (2.21)–(2.23), that there exists a subsequence $n_k \to \infty$ and a function

$$u \in C(Q) \cap L_\infty(0,T; \overset{\circ}{W}{}^1_2(0,l)) \cap W^{2,1}_{q^*}(Q),\ u_t \in L_2(Q),$$

such that, as $k \to \infty$,

$$u^{n_k}(x,t) \rightrightarrows u(x,t) \quad \text{uniformly on } Q, \quad (2.25)$$

$$u^{n_k}_x(x,t) \to u_x(x,t) \quad \text{in the norm } L_{q^*}(Q) \text{ and } *\text{-weakly in } L_\infty(0,T; L_2(0,l)), \quad (2.26)$$

$$u^{n_k}_{xx}(x,t) \rightharpoonup u_{xx}(x,t) \quad \text{weakly in } L_{q^*}(Q), \quad (2.27)$$

$$u^{n_k}_t(x,t) \rightharpoonup u_t(x,t) \quad \text{weakly in } L_2(Q). \quad (2.28)$$

Let $\psi(x,t) \in C^\infty(Q)$ be a test function and denote $h_k = 1/n_k$. On the basis of (2.14), we write the integral identity

$$\int_Q \left[u^{n_k}_t - a^{n_k} u^{n_k}_{xx} - \sqrt{a^{n_k}} \Big(\frac{b}{\sqrt{a}}\Big)^{h_k} u^{n_k}_x - \sqrt{a^{n_k}} \Big(\frac{\mathrm{d}}{\sqrt{a}}\Big)^{h_k} u^{n_k} - \sqrt{a^{n_k}} \Big(\frac{f}{\sqrt{a}}\Big)^{h_k} \right] \psi(x,t)\, dx\, dt = 0. \quad (2.29)$$

It is easy to verify that from (2.25)–(2.28), we can pass to the limit in the identity (2.29), as $k \to \infty$, and obtain that the limit function $u(x,t)$ satisfies the equation (2.1) almost everywhere in Q, and for this function the estimates (2.8) – (2.12) hold. Note also that from assumptions (C) and (D) one can replace the term $\|u_0'\|_2^2 + 3\|f^2/a\|_{L_1(Q)}$ in (2.8)–(2.10) by $M_0^2 + 3K_{f,a}$.

Moreover, from (2.25), the function $u(x,t)$ satisfies the conditions (1.2) by continuity and hence this function is a generalized solution (in the sense of Definition 1) of the direct problem given by (2.1) and (1.2). Theorem 2 is proved. □

3 Investigation of the inverse problem

In this section, we assume that the the coefficient $b(x,t)$ in (1.1) has a special form

$$b(x,t) = a(x,t) b_1(x). \quad (3.1)$$

Then, we can rewrite the equation (1.1) as follows:

$$\rho(x,t) u_t - u_{xx} - b_1(x) u_x - \frac{\mathrm{d}(x,t)}{a(x,t)} u = p(x) \frac{g(x,t)}{a(x,t)} + \frac{r(x,t)}{a(x,t)}\,, \quad (3.2)$$

where $\rho(x,t) \equiv 1/a(x,t)$, and consider the inverse problem given by (3.2), (1.2) and (1.3).

In addition to the assumptions $(A)-(C)$ we suppose that

$$\left.\begin{array}{ll} \chi\in L_\infty(0,T),\quad b_1\in \mathrm{E}; & \mathrm{d}\chi/a,\ (\chi/a)_t\equiv(\rho\chi)_t\in L_1(0,T;\mathrm{E});\\ g/a,\ r/a\in L_1(0,T;\mathrm{E}); & g^2/a,\ r^2/a\in L_1(Q);\\ |\chi(t)|\leqslant K_\chi,\ \left|\int_0^T \frac{g}{a}\chi\,dt\right|\geqslant g_0>0, & \left|\frac{\chi(T)}{a(x,T)}\right|\leqslant a_3,\ \left|\frac{\chi(0)}{a(x,0)}\right|\leqslant a_4,\\ \left\|g^2/a\right\|_{L_1(Q)}\leqslant K_g^*,\ \left\|r^2/a\right\|_{L_1(Q)}\leqslant K_r^*; & \int_0^T \left\|\left(\frac{\chi}{a}\right)_t\right\|_\infty dt\leqslant K_{a,\chi};\ |b_1(x)|\leqslant K_b\,;\\ \int_0^T \left\|\frac{\mathrm{d}\chi}{a}\right\|_\infty dt\leqslant K_{\mathrm{d},\chi}; & \int_0^T \left\|\frac{r}{a}\right\|_\infty dt\leqslant K_r; \end{array}\right\}\qquad (E)$$

$$\varphi\in W_\infty^2(0,l),\ \varphi(0)=\varphi(l)=0;\quad |\varphi'(x)|\leqslant K_\varphi^*,\ |\varphi''(x)|\leqslant K_\varphi^{**}\quad \forall x\in[0,l]. \qquad (F)$$

Here K_g^*, g_0, $K_\chi=\text{const.}>0$, and K_b, K_χ, a_3, a_4, K_r^*, $K_{a,\chi}$, $K_{\mathrm{d},\chi}$, $K_r=\text{const.}\geqslant 0$.

Remark 3. In view of assumption (E), the constant $K_{b,a}$ in the definition of λ^* in (2.7) is equal to $K_b^2\,a_1$ so that now

$$\lambda^*=3\left(K_b^2\,a_1+\frac{l^2}{\pi^2}K_{d,a}\right). \qquad (3.3)$$

Definition 2. By a generalized solution of the problem (1.2), (1.3) and (3.2) we mean the pair of functions $\{u(x,t);p(x)\}$,

$$u\in C^{0,\sigma}(Q)\cap L_\infty(0,T;\overset{\circ}{W}{}_2^1(0,l))\cap W_s^{2,1}(Q),\ s>1,\ \sigma\in(0,1),\quad p\in L_\infty(0,l), \qquad (3.4)$$

which satisfies the equation (3.2) almost everywhere in Q and the function $u(x,t)$ satisfies the conditions (1.2) and (1.3) in classical sense.

In what follows, we will use the notation

$$G(x):=\int_0^T \frac{g(x,t)}{a(x,t)}\chi(t)\,dt,\quad R(x):=\int_0^T \frac{r(x,t)}{a(x,t)}\chi(t)\,dt. \qquad (3.5)$$

Remark 4. Let $p\in\mathrm{E}$ be a known function. Then by virtue of the assumptions (A)–(C), (E) and the Theorems 1 and 2 which were proved above, there exists a unique solution $u(x,t)$ of the direct problem given by (1.2) and (3.2). Moreover, $u\in W_{q^*}^{2,1}(Q)$, where q^* was defined in (2.7) and the estimates (2.8)–(2.12) with $f(x,t)=p(x)g(x,t)+r(x,t)$ hold.

3.1 The first variant of sufficient conditions for unique solvability of the inverse problem

We assume that the input data of the inverse problem given by (1.2), (1.3) and (3.2) satisfy the assumptions (A)– (C), (E), (F) and $q^*>1$ is defined in (2.7). Let us derive the operator equation for the unknown function $p(x)$.

Let pair of functions $\{u(x,t);p(x)\}$ be any generalized solution (in the sense of Definition 2) of the inverse problem given by (1.2), (1.3) and (3.2) with $s=q^*>1$, i.e. $u\in W_{q^*}^{2,1}(Q)$. Let us multiply equation (3.2) by $\chi(t)$ and integrate over the closed interval

$[0, T]$. Taking into account condition (1.3), notation (3.5) and integrating by parts using conditions (1.2), assumptions (E) and (F), we obtain the well-defined relation

$$p(x) = \frac{1}{G(x)} \left[\frac{\chi(T)}{a(x,T)} u(x,T) - \int_0^T \left(\frac{\mathrm{d}(x,t)}{a(x,t)} \chi(t) + \left(\frac{\chi(t)}{a(x,t)} \right)_t \right) u(x,t)\, dt \right] - b_0(x), \quad (3.6)$$

where

$$b_0(x) := \frac{1}{G(x)} \left[\varphi''(x) + b_1(x)\varphi'(x) + R(x) + \frac{\chi(0)}{a(x,0)} u_0(x) \right] \quad (3.7)$$

is a known function belonging to E. In view of this relation, let us introduce the operator $\mathcal{A} : \mathrm{E} \to \mathrm{E}$ defined by the right-hand side of (3.6) as

$$\mathcal{A}p = \frac{1}{G(x)} \Big[\frac{\chi(T)}{a(x,T)} u(x,T;p) - \int_0^T \left(\frac{\mathrm{d}(x,t)}{a(x,t)} \chi(t) + \left(\frac{\chi(t)}{a(x,t)} \right)_t \right) u(x,t;p)\, dt \Big] - b_0(x), \quad (3.8)$$

where $p(x)$ is an arbitrary function in E, and $u(x,t;p)$ is a solution of direct problem given by (3.2) and (1.2) with given $p(x)$ in the right-hand side of equation (3.2). Such a solution exists and is unique due to Remark 4. Then, the relation (3.6) can be written as the fix point equation

$$p = \mathcal{A}p. \quad (3.9)$$

Remark 5. In view of assumptions (A)–(C), (E), (F) and Theorems 1 and 2, the operator $\mathcal{A}$ is defined on the whole space E and its range belongs to the same space.

Lemma 1. *Let assumptions* (A)–(C), (E), (F) *hold. Then the operator equation* (3.9) *is equivalent to the inverse problem given by* (3.2), (1.2) *and* (1.3) *in the following sense. If pair* $\{u(x,t); p(x)\}$ *is a generalized solution of the inverse problem, then* $p(x)$ *satisfies* (3.9). *Conversely, if* $p \in \mathrm{E}$ *is a solution of operator equation* (3.9), *and* $u = u(x,t;p)$ *is a solution of direct problem given by* (3.2) *and* (1.2) *with this* p, *then the pair* $\{u(x,t); p(x)\}$ *is a generalized solution of inverse problem given by* (3.2), (1.2) *and* (1.3).

Proof. The first statement (necessity) has already been proved above when the relation (3.6) was derived.

Let us prove the second statement (sufficiency). Let $\hat{p} \in \mathrm{E}$ be a solution of the equation (3.9). Consider the function $\hat{u}(x,t)$ as a unique generalized solution of direct problem given by (3.2) and (1.2) with chosen function $p(x) = \hat{p}(x)$ on the right-hand side of equation (3.2). Set

$$\hat{\varphi}(x) \equiv \int_0^T \hat{u}(x,t)\chi(t)\, dt.$$

Then, by Theorem 2 we have $\hat{\varphi} \in W_2^2(0,l) \cap \overset{\circ}{W}{}_2^1(0,l)$. Repeating the arguments given above in the proof of (3.6), (in these arguments, it is sufficient to assume that $\hat{\varphi} \in W_2^2(0,l)$), we obtain the relation

$$G(x)\hat{p}(x) = \frac{\chi(T)}{a(x,T)} \hat{u}(x,T) - \int_0^T \Big(\frac{\mathrm{d}(x,t)}{a(x,t)} \chi(t) + \Big(\frac{\chi(t)}{a(x,t)} \Big)_t \Big) \hat{u}(x,t)\, dt - G(x)\hat{b}_0(x), \quad (3.10)$$

where $\hat{b}_0(x)$ has the same meaning as in (3.7) with φ replaced by $\hat{\varphi}$. In particular, (3.10) implies that $\hat{\varphi} \in W^2_\infty(0,l)$. Since $\hat{p}(x)$ is a solution of equation (3.9), then taking into account the definition of the operator $\mathcal{A}$ in (3.5) we see that the following relation also holds:

$$G(x)\hat{p}(x) \;=\; \frac{\chi(T)}{a(x,T)}\hat{u}(x,T) \;-\; \int_0^T \Big(\frac{\mathrm{d}(x,t)}{a(x,t)}\chi(t) \;+\; \Big(\frac{\chi(t)}{a(x,t)}\Big)_t\Big)\hat{u}(x,t)\,dt \;-\; G(x)b_0(x). \tag{3.11}$$

It follows from (3.10) and (3.11) that $b_0(x) = \hat{b}_0(x)$, or from (3.7)

$$(\hat{\varphi}-\varphi)''(x) + b_1(x)(\hat{\varphi}-\varphi)'(x) = 0, \quad x \in (0,l).$$

Moreover, from boundary conditions we have $\varphi(0) = \hat{\varphi}(0) = 0$, $\varphi(l) = \hat{\varphi}(l) = 0$. Therefore, $\varphi(x) = \hat{\varphi}(x)$ on $[0,l]$, and thus the pair of functions $\{\hat{u}(x,t);\hat{p}(x)\}$ is a generalized solution of the inverse problem given by (3.2), (1.2) and (1.3). Lemma 1 is proved. □

Theorem 3. *Let the assumptions (A)–(C), (E) and (F) hold, the number q^* defined in (2.7) and λ^* defined in (3.3). Suppose that*

$$\alpha_1 \equiv \frac{\sqrt{3lK_g^*}}{g_0}\exp(\lambda^* T/2)(a_3 + K_{d,\chi} + K_{a,\chi}) < 1. \tag{3.12}$$

Then there exists a generalized solution $\{u(x,t);p(x)\}$ of the inverse problem given by (3.2), (1.2) and (1.3), and $u \in W^{2,1}_{q^*}(Q)$. Moreover, such a solution is unique and the following estimates hold:

$$\|p\|_\infty \leqslant \frac{1}{(1-\alpha_1)g_0}\Big[l^{1/2}\exp(\lambda^* T/2)\big(a_3 + K_{a,\chi} + K_{d,\chi}\big)\big(\|u_0'\|_2^2 + 3K_r^*\big)^{1/2} + \\ + K_b K_\varphi^* + K_\varphi^{**} + K_\chi K_r + a_4\|u_0\|_\infty\Big], \tag{3.13}$$

$$\sup_{0\leqslant t\leqslant T}\|u_x(\cdot,t)\|_2^2 \leqslant \exp(\lambda^* T)\Big[\|u_0'\|_2^2 + 6K_g^*\|p\|_\infty + 6K_r^*\Big], \tag{3.14}$$

$$\|u_{xx}\|^2_{L_{q^*}(Q)} \leqslant a_2\exp(\lambda^* T)\Big[\|u_0'\|_2^2 + 6K_g^*\|p\|_\infty + 6K_r^*\Big], \tag{3.15}$$

$$\|u_t\|_{L_2(Q)} \leqslant C_4, \tag{3.16}$$

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant C_5|x_1-x_2|^{1/2} + C_6|t_1-t_2|^{1/6}, \quad \forall\,(x_1,t_1),\,(x_2,t_2) \in Q, \tag{3.17}$$

where C_4, C_5, C_6 depend only on l, T, α_1 and the constants entering into assumptions (A) – (C), (E) and (F).

Proof. Suppose that $p^{(1)}$, $p^{(2)} \in \mathrm{E}$, and

$$u^{(1)}(x,t) = u(x,t;p^{(1)}), \qquad u^{(2)}(x,t) = u(x,t;p^{(2)})$$

are solutions of corresponding direct problems given by (1.2) and (3.2). Set $v(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ and $y(x) = p^{(1)}(x) - p^{(2)}(x)$. Then, the pair $\{v(x,t);\, y(x)\}$ satisfies

$$\rho(x,t)v_t - v_{xx} - b_1(x)v_x - \frac{\mathrm{d}(x,t)}{a(x,t)}v = \frac{g(x,t)}{a(x,t)}y(x), \quad (x,t) \in Q, \tag{3.18}$$

$$v(x,0)=0,\quad x\in(0,l);\qquad v(0,t)=v(l,t)=0,\quad t\in[0,T]. \tag{3.19}$$

From the definition (3.8) and the assumption (E) we have

$$\|\mathcal{A}p^{(1)}-\mathcal{A}p^{(2)}\|_\infty \leqslant \frac{1}{g_0}\big[a_3+K_{d,\chi}+K_{a,\chi}\big]\cdot\|v\|_{L_\infty(Q)}. \tag{3.20}$$

On the other hand,

$$v(x,t)=\int_0^x v_z(z,t)\,dz, \quad \text{therefore} \quad \|v\|_{L_\infty(Q)}\leqslant l^{1/2}\sup_{0\leqslant t\leqslant T}\|v_x(\cdot,t)\|_2.$$

Since $v(x,t)$ satisfies (3.18) and (3.19), then we can apply the estimate (2.8), where $f(x,t)=y(x)g(x,t)$ and $v(x,0)=0$. As a result, we obtain

$$\begin{aligned}\|v\|_{L_\infty(Q)} &\leqslant \sqrt{3l}\exp(\lambda^*T/2)\Big(\int_0^T\int_0^l y^2(x)\,\frac{g^2(x,t)}{a(x,t)}\,dx\,dt\Big)^{1/2}\\ &\leqslant \sqrt{3lK_g^*}\,\exp(\lambda^*T/2)\big[\sup_{0\leqslant x\leqslant l}y^2(x)\big]^{1/2}=\sqrt{3lK_g^*}\,\exp(\lambda^*T/2)\cdot\|y\|_\infty.\end{aligned}$$

Substituting this estimate into (3.20), we obtain

$$\|\mathcal{A}p^{(1)}-\mathcal{A}p^{(2)}\|_\infty\leqslant\alpha_1\cdot\|p^{(1)}-p^{(2)}\|_\infty\,,$$

which from (3.12) implies that the operator $\mathcal{A}$ is a contraction and thus the operator equation (3.9) has a unique fix point, i.e. is uniquely solvable.

Therefore, in view of Lemma 1 there exists a unique generalized solution $\{u(x,t);p(x)\}$ of the inverse problem given by (1.2), (1.3) and (3.2).

Let us now prove the estimates (3.13)–(3.17). Denote by $z^0(x,t)\equiv u(x,t;0)$ the solution of the direct problem given by (3.2) and (1.2) with $p(x)\equiv 0$. Then, for $z^0(x,t)$ the estimate (2.8) with function $f(x,t)=r(x,t)$ is fulfilled, and therefore

$$\|z_x^0(\cdot,t)\|_2^2\leqslant\exp(\lambda^*T)\big(\|u_0'\|_2^2+3K_r^*\big). \tag{3.21}$$

Since in the operator equation (3.9) the operator $\mathcal{A}$ is a contraction, then it is convenient to solve it by the method of successive iterations. Let $p(x)\equiv 0$ be the zeroth approximation of the solution of (3.9). Then, from (3.8)

$$\mathcal{A}0=\frac{1}{G(x)}\left[\frac{\chi(T)}{a(x,T)}z^0(x,T)-\int_0^T\left(\frac{\mathrm{d}(x,t)}{a(x,t)}\chi(t)+\left(\frac{\chi(t)}{a(x,t)}\right)_t\right)z^0(x,t)\,dt\right]-b_0(x)$$

whence, in view of assumption (E), (3.7) and (3.21) we have

$$\begin{aligned}\|\mathcal{A}0\|_\infty &\leqslant\frac{1}{g_0}\big[a_3+K_{d,\chi}+K_{a,\chi}\big]l^{1/2}\cdot\sup_{0\leqslant t\leqslant T}\|z_x^0(t,\cdot)\|_2+\|b_0\|_\infty\\ &\leqslant\frac{1}{g_0}\big[a_3+K_{\mathrm{d},\chi}+K_{\rho,\chi}\big]l^{1/2}\cdot\exp(\lambda^*T/2)\cdot\big(\|u_0'\|_2^2+3K_r^*\big)^{1/2}\\ &\qquad+\frac{1}{g_0}\big[K_bK_\varphi^*+K_\varphi^{**}+K_\chi K_r+a_4\|u_0\|_\infty\big].\end{aligned} \tag{3.22}$$

Let $p_1(x)$ be the first approximation of the solution of equation (3.9) in the method of iterations. Then, using the well-known estimate for the error of the n-th approximation of the solution in the iterative method (see, for example, [22, p.43]), we obtain

$$\|p_1 - p\|_\infty \leqslant \frac{\alpha_1}{1-\alpha_1}\|\mathcal{A}0\|_\infty.$$

Then,

$$\|p\|_\infty \leqslant \|p_1 - p\|_\infty + \|p_1\|_\infty \leqslant \frac{\alpha_1}{1-\alpha_1}\|\mathcal{A}0\|_\infty + \|\mathcal{A}0\|_\infty \leqslant \frac{1}{1-\alpha_1}\|\mathcal{A}0\|_\infty,$$

and in the view of (3.22), we obtain (3.13). After this, estimates (3.14)–(3.17) are obtained by a direct consequence of the estimates (2.8)–(2.12) for $f(x,t) = g(x,t)p(x) + r(x,t)$. Theorem 3 is proved. □

3.2 The second variant of sufficient conditions for unique solvability of the inverse problem

For the sake of convenience, in the present subsection we make several changes in notation and fulfill some transformations of the original inverse problem in comparison with subsection 3.1. Herewith, we assume that (A)–(F) hold.

We again consider the problem of finding the pair $\{u(x,t);p(x)\}$ satisfying (3.2), (1.2) and (1.3). By virtue of assumption (E)

$$|\mathbf{l}(\frac{g}{a})(x)| \equiv \left|\int_0^T \frac{g(x,t)}{a(x,t)}\chi(t)\,dt\right| \equiv |G(x)| \geqslant g_0 > 0, \quad x \in [0,l].$$

Then, the operator of multiplication by the function $G(x)$ in the space E$=L_\infty(0,l)$ is continuously invertible. Performing the transformation of the right-hand side of equation (3.2) in the form

$$\frac{g(x,t)}{a(x,t)}p(x) \equiv \frac{g(x,t)}{a(x,t)G(x)}G(x)p(x) \equiv \frac{\tilde{g}(x,t)}{a(x,t)}\tilde{p}(x),$$

we reduce our problem given by (3.2), (1.2) and (1.3) to the equivalent problem of the same form, but for which the equality

$$\mathbf{l}(\frac{g}{a})(x) = G(x) \equiv 1, \quad x \in [0,l],$$

holds.

Thus, without loss of generality, we suppose in this subsection that for inverse problem given by (3.2), (1.2) and (1.3) we can take $G(x) \equiv 1$, while we retain the same notation for the unknown functions. Note that the definition of the solution will also not change.

Let us carry out one more simplification of the problem. For this purpose, we introduce the new unknown function $u(x,t) - z^0(x,t)$ for which we retain the notation $u(x,t)$ and obtain the inverse problem of finding the pair $\{u(x,t);p(x)\}$ satisfying

$$\rho(x,t)u_t - u_{xx} - b_1(x)u_x - \frac{\mathrm{d}(x,t)}{a(x,t)}u = \frac{g(x,t)}{a(x,t)}p(x), \quad (x,t) \in [0,l]\times[0,T], \tag{3.23}$$

$$u(x,0) = 0,\ x \in [0,l]; \qquad u(0,t) = u(l,t) = 0,\ t \in [0,T], \tag{3.24}$$

$$\int_0^T u(x,t)\chi(t)\,dt = \varphi(x) - \mathbf{l}(u^0)(x) \equiv \tilde{\varphi}(x), \quad x \in [0,l], \tag{3.25}$$

where $\tilde{\varphi} \in W^2_\infty(0,l) \cap \overset{\circ}{W}{}^1_2(0,l)$.

Further, in the present subsection we only investigate the problem (3.23)–(3.25). The results obtained for it, obviously, will also hold for the original inverse problem given by (3.2), (1.2) and (1.3). We remind the notation $\mathrm{E} \equiv L_\infty(0,l)$ and introduce the cone of non-negative functions on E:

$$\mathrm{E}_+ = \{p \in L_\infty(0,l) \mid p(x) \geqslant 0, \quad x \in [0,l]\}.$$

The cone E_+ is closed and reproducing (see, e.g., [17]). We define the linear operator $\mathcal{B}: \mathrm{E} \to \mathrm{E}$ by

$$\mathcal{B}p = \frac{\chi(T)}{a(x,T)} u(x,T;p) - \int_0^T \Big[\frac{\mathrm{d}(x,t)}{a(x,t)}\chi(t) + \Big(\frac{\chi(t)}{a(x,t)}\Big)_t \Big] u(x,t;p)\,dt, \tag{3.26}$$

where $u = u(x,t;p)$ is a solution of the direct problem (3.23) and (3.24) with chosen function $p \in \mathrm{E}$ in the right-hand side of (3.23). By virtue of Theorem 1 and the estimates for the direct problem from Theorem 2, the operator $\mathcal{B}$ is defined on the whole space E and is bounded.

Denote by $\mathcal{L}(\mathrm{E})$ the set of bounded linear operators from E into E. Then, we can write $\mathcal{B} \in \mathcal{L}(\mathrm{E})$. Let us consider the operator equation of the second kind in the space E:

$$(I - \mathcal{B})p = \psi, \tag{3.27}$$

where I is the identity operator and $\psi(x) = -\tilde{\varphi}''(x) - b_1(x)\tilde{\varphi}'(x)$, so that $\psi \in \mathrm{E}$ in view of assumptions (B)–(F). Just as in subsection 3.1 we prove the following lemma (see Lemma 1).

Lemma 2. *Let assumptions (A)–(C), (E), (F) hold. Then the inverse problem given by* (3.23)–(3.25) *is equivalent to the operator equation* (3.27).

Let us require the following additional conditions on the input data functions:

$$\chi(t) > 0,\ a.e.\ t \in [0,T]; \qquad g(x,t) \geqslant 0, \quad \Big(\frac{\chi(t)}{a(x,t)}\Big)_t + \frac{\mathrm{d}(x,t)}{a(x,t)}\chi(t) \leqslant 0,\ a.e.\ (x,t) \in Q. \tag{3.28}$$

In the lemmas below we establish a number of properties for the operator $\mathcal{B}$ from which the solvability of the equation (3.27) will follow.

Lemma 3. *Let assumptions (A)–(C), (E), (F) hold, as well as* (3.28)*. Then $\mathcal{B}\mathrm{E}_+ \subseteq \mathrm{E}_+$.*

Proof. First, we establish the inequality $u(x,t;p) \geqslant 0$ in Q for any $p \in \mathrm{E}_+$. For the case of uniformly parabolic equations with smooth coefficients this inequality is a simple consequence of the weak maximum principle. Let us carry out the corresponding arguments in our case. For this purpose, as in the proof of Theorem 2, we consider the sequence of direct problems with smooth coefficients

$$\begin{cases} u_t^{n,h} - a^{n,h}u_{xx}^{n,h} - \sqrt{a^{n,h}}\big(\frac{b}{\sqrt{a}}\big)^h u_x^{n,h} - \sqrt{a^{n,h}}\big(\frac{\mathrm{d}}{\sqrt{a}}\big)^h u^{n,h} = \sqrt{a^{n,h}}\big(\frac{gp}{\sqrt{a}}\big)^h \geqslant 0, \quad (x,t) \in Q, \\ u^{n,h}(x,0) = 0,\ x \in [0,l], \qquad u^{n,h}(0,t) = u^{n,h}(l,t) = 0,\ t \in [0,T], \end{cases}$$

where $a^{n,h}(x,t) = a^h(x,t) + 1/n$ and a^h, $(b/\sqrt{a})^h$, $(\mathrm{d}/\sqrt{a})^h$, $(gp/\sqrt{a})^h$ are the mean functions for a, $b/\sqrt{a}$, $\mathrm{d}/\sqrt{a}$, $gp/\sqrt{a}$, respectively.

From the classical maximum principle we have that $u^{n,h}(x,t) \geqslant 0$ in Q. For the functions $u^{n,h}(x,t)$ when n is fixed, the estimates in the norm of $W^{2,1}_{q^*}(Q)$ uniform with respect to h are valid. It is well known (see [19]) that $u^{n,h}(x,t) \to u^n(x,t)$ in $W^{2,1}_{q^*}(Q)$ as $h \to 0$, and $u^n(x,t) \geqslant 0$ are the solutions of the corresponding "limit" problems

$$\begin{cases} u^n_t - a^n(x,t)u^n_{xx} - \sqrt{a^n}\left(\frac{b}{\sqrt{a}}\right)u^n_x - \sqrt{a^n}\left(\frac{\mathrm{d}}{\sqrt{a}}\right)u^n = \sqrt{a^n}\left(\frac{gp}{\sqrt{a}}\right) \geqslant 0, & (x,t) \in Q, \\ u^n(x,0) = 0, \ x \in [0,l], \qquad u^n(0,t) = u^n(l,t) = 0, \ t \in [0,T], \end{cases}$$

with $a^n(x,t) = a(x,t)+1/n$. For $u^n(x,t)$, the estimates (2.17)–(2.23) hold. As in the proof of Theorem 2 it is established that there exists a subsequence $u^{n_k}(x,t)$, that converges uniformly in Q to the function $u(x,t)$, which is a generalized solution of the problem (3.23) and (3.24). Hence, we have that $u(x,t) \geqslant 0$ in Q. Then, from the conditions (3.28) we obtain that $\mathcal{B}p \in \mathrm{E}_+$ for any $p \in \mathrm{E}_+$. Lemma 3 is proved. □

Let us prove now that the operator $\mathcal{B}$ is a compact operator in E. To do this, we represent $\mathcal{B}$ in the form

$$\mathcal{B}p = \mathcal{B}_1 p + \mathcal{B}_2 p, \quad \text{where} \quad \mathcal{B}_1 p = \frac{\chi(T)}{a(x,T)} u(x,T;p) \quad \text{and}$$

$$\mathcal{B}_2 p = -\int_0^T \left[\left(\frac{\chi(t)}{a(x,t)} \right)_t + \frac{\mathrm{d}(x,t)}{a(x,t)} \chi(t) \right] u(x,t;p)\, dt \equiv \int_0^T \omega(x,t) u(x,t;p)\, dt,$$

and use the following estimates and the properties of the solutions for the direct problem (3.23) and (3.24):

$$u \in C(Q), \qquad \|u(\cdot,t)\|_\infty \leqslant \mathrm{const} \cdot \|p\|_\infty \quad \forall\, t \in [0,T], \tag{3.29}$$

$$\|u_x(\cdot,t)\|_2 \leqslant \mathrm{const} \cdot \|p\|_\infty \quad \forall\, t \in [0,T], \tag{3.30}$$

which were established in the proof of Theorem 2.

Lemma 4. *Let assumptions (A)–(F) hold. Then, the operator $\mathcal{B} \in \mathcal{L}(\mathrm{E})$ is a compact operator.*

Proof. Let us prove that each of the operators $\mathcal{B}_1$ and $\mathcal{B}_2$ is a compact operator in E. From (3.29) and (3.30), the operator $\mathcal{B}_1 \in \mathcal{L}(\mathrm{E})$ is compact, since it is a product of an operator bounded in the space E with the compact operator $\widetilde{\mathcal{B}}_1 p = u(x,T;p)$. Indeed, from (3.30) the operator $\widetilde{\mathcal{B}}_1$ transforms each bounded set of the space E to a set bounded in $C^{1/2}[0,l]$, which is compact in E.

Now consider the operator $\mathcal{B}_2$. Taking into account assumptions (B)–(E), we can note that the weight function

$$\omega(x,t) \equiv -\left(\frac{\chi(t)}{a(x,t)} \right)_t - \frac{\mathrm{d}(x,t)}{a(x,t)} \chi(t) \in L_1(0,T;\mathrm{E}).$$

We extend $\omega(x,t)$ by zero for $t > T$ and introduce its mean function with respect to t given by

$$\omega^h(x,t) = \frac{1}{h} \int_t^{t+h} \omega(x,\tau)\, d\tau. \tag{3.31}$$

Then

$$\lim_{h\searrow 0}\int_0^T \|\omega(\cdot,t)-\omega^h(\cdot,t)\|_\infty\,dt=0,$$

and

$$\begin{aligned}\|\omega(\cdot,t)-\omega^h(\cdot,t)\|_\infty &= \frac{1}{h}\Big\|\int_t^{t+h}(\omega(\cdot,t)-\omega(\cdot,\tau))\,d\tau\Big\|_\infty\\ &\leqslant \frac{1}{h}\int_t^{t+h}\|\omega(\cdot,t)-\omega(\cdot,\tau)\|_\infty\,d\tau=\frac{1}{h}\int_0^h\|\omega(\cdot,t+z)-\omega(\cdot,t)\|_\infty\,dz.\end{aligned}$$

Integrating over $t\in[0,T]$, we have

$$\int_0^T \|\omega(\cdot,t)-\omega^h(\cdot,t)\|_\infty\,dt\leqslant \frac{1}{h}\int_0^h dz\int_0^T\|\omega(\cdot,t+z)-\omega(\cdot,t)\|_\infty\,dt\leqslant C\varepsilon.$$

Here, we take into account that, from the continuity in mean, one has (see, for example, [24])

$$\int_0^T\|\omega(\cdot,t+z)-\omega(\cdot,t)\|_\infty\,dt\leqslant C\varepsilon \quad \text{as} \quad |z|<h_0.$$

Define the operator $\mathcal{B}_2^h : E\to E$ by

$$\mathcal{B}_2^h p=\int_0^T \omega^h(x,t)u(x,t;p)\,dt. \tag{3.32}$$

Since $\omega^h\in C(0,T;\mathrm{E})$ and $u\in C(0,T;\mathrm{E})$, it is obvious that $\mathcal{B}_2^h\in\mathcal{L}(\mathrm{E})$. From (3.29) and (3.30), the operator $\mathcal{R}^h p\equiv\omega^h(x,t)u(x,t;p)$ is compact in E for any $t\in[0,T]$, since it is a product of a bounded operator with a compact operator. Therefore, the operator $\mathcal{B}_2^h$ is also compact as it is the limit of Riemann integral sums in the norm of $\mathcal{L}(\mathrm{E})$.

We estimate the difference (using (3.29))

$$\begin{aligned}\|\mathcal{B}_2 p-\mathcal{B}_2^h p\|_\infty &= \Big\|\int_0^T\big(\omega(x,t)-\omega^h(x,t)\big)u(x,t;p)\,dt\Big\|_\infty\\ &\leqslant \max_{[0,T]}\|u(\cdot,t)\|_\infty\cdot\int_0^T\big\|\omega(\cdot,t)-\omega^h(\cdot,t)\big\|_\infty\,dt\\ &\leqslant \mathrm{const}\cdot\|p\|_\infty\cdot\int_0^T\big\|\omega(\cdot,t)-\omega^h(\cdot,t)\big\|_\infty\,dt\to 0,\ \text{as}\ h\searrow 0.\end{aligned}$$

Therefore, $\mathcal{B}_2^h\to\mathcal{B}_2$ in $\mathcal{L}(\mathrm{E})$, and hence $\mathcal{B}$ is a compact operator in E (see, e.g., [22]). Lemma 4 is proved. □

Remark 6. From Lemmas 2 and 4 we obtain that the inverse problem given by (3.23)–(3.25) has the Fredholm property, since it is equivalent to the linear operator equation (3.27) of second kind with compact operator $\mathcal{B}$. Such a result is well-known for inverse problems (of finding the unknown source term from the additional condition (1.3)) in the case of uniformly parabolic non-degenerate equations (see, for example, [27]).

Lemma 5. *Let assumptions (A)–(F) and* (3.28) *hold. Then, the spectral radius* $r_0(\mathcal{B})$ *of the operator* $\mathcal{B}$ *is less then one, i.e.* $r_0(\mathcal{B})<1$.

Proof. Suppose that $r_0(\mathcal{B}) \geqslant 1$. In view of Lemmas 3 and 4, the operator $\mathcal{B}$ is non-negative and compact, and the cone E_+ is closed and reproducing. Then, by the theorem of Krein–Rutman(see [18]) the number $\lambda = r_0(\mathcal{B})$ is an eigenvalue of the operator $\mathcal{B}$ with positive eigenvector, that is, $\exists\, p_0 \neq 0 : \mathcal{B}p_0 = r_0(\mathcal{B})p_0$, and $p_0 \in \mathrm{E}_+$.

Consider the function $v^0 = v^0(x,t;p_0) \geqslant 0$ as a solution of the direct problem given by (3.23) and (3.24) with $p = p_0 \in \mathrm{E}_+$. Then, from (3.27)

$$0 \geq p_0 - r_0(\mathcal{B})p_0 = p_0 - \mathcal{B}p_0 = -b_1(x)\mathbf{l}(v^0)_x - \mathbf{l}(v^0)_{xx}, \quad x \in (0,l), \quad \mathbf{l}(v^0)(0) = \mathbf{l}(v^0)(l) = 0.$$

From the weak maximum principle for elliptic equations (see, for example, [9]) it follows that $\mathbf{l}(v^0)(x) \leqslant 0$ in $[0,l]$. From Lemma 3 we have $\mathbf{l}(v^0)(x) \geqslant 0$ in $[0,l]$, and therefore $\mathbf{l}(v^0)(x) \equiv 0$ in $[0,l]$. Since $\chi(t) > 0$ for a.e. $t \in [0,T]$, this is possible only if $v^0(x,t) = 0$ a.e. in Q. Substituting $v^0 = 0$ in (3.23), we obtain $g(x,t)p_0(x) = 0$ a. e. in Q, hence

$$\left(\int_0^T \frac{g(x,t)}{a(x,t)} \chi(t)\, dt \right) p_0(x) \equiv p_0(x) = 0 \quad \text{a. e. in} \quad [0,l].$$

This contradicts the fact that $p_0(x)$ is an eigenvector of the operator $\mathcal{B}$. Lemma 5 is proved. □

As a corollary of Lemmas 2–5, we obtain a theorem on the unique solvability of the inverse problem given by (3.23)–(3.25), and thus for the inverse problem given by (3.2), (1.2) and (1.3).

Theorem 4. *Let assumptions (A)–(F) hold, as well as* (3.28). *Suppose that the function*

$$\tilde{\varphi} \equiv \varphi - \mathbf{l}(u^0) \in W^2_\infty(0,l) \cap \overset{\circ}{W}{}^1_2(0,l).$$

Then, there exists a generalized solution $\{u(x,t);p(x)\}$ *of the inverse problem given by* (3.2), (1.2) *and* (1.3) *with* $u \in W^{2,1}_{q^*}(Q)$. *Moreover, such a solution is unique and the estimate of stability*

$$\|p\|_\infty \leqslant C \cdot \|\psi\|_\infty\,, \tag{3.33}$$

holds, where $\psi(x) = -\tilde{\varphi}''(x) - b_1(x)\tilde{\varphi}'(x)$.

Proof. From Lemma 5 we have that $r_0(\mathcal{B}) < 1$ and therefore the operator equation (3.27) of the second kind has a unique solution $p \in \mathrm{E}$ and the stability estimate (3.33) holds. Then, by Lemma 2, the inverse problem given by (3.23)–(3.25) has a generalized solution which is unique. The estimates for u follows from (3.33) and the estimates proved in Theorem 2. Theorem 4 is proved. □

3.3 Some examples

In this subsection, we present some examples of inverse problems for which the above-proved theorems hold.

Example 1. Consider the inverse problem

$$u_t - t^\beta(x+1)u_{xx} = p(x)t^\beta(x+1), \quad (x,t) \in [0,l] \times [0,T], \tag{3.34}$$

$$u(x,0) = u_0(x), \quad x \in [0,l]; \qquad u(0,t) = u(l,t) = 0, \quad t \in [0,T], \tag{3.35}$$

$$\int_0^T u(x,t)\, t(T-t)\, dt = \varphi(x), \quad x \in [0,l], \tag{3.36}$$

with $u_0 \in \overset{\circ}{W}{}_2^1(0,l)$, $\varphi \in W_\infty^2(0,l)$, $\varphi(0) = \varphi(l) = 0$ and $\beta \in (0,1)$. According to (1.1), in (3.34) we have $a(x,t) = t^\beta(x+1)$, $b(x,t) = \mathrm{d}(x,t) = r(x,t) = 0$, $g(x,t) = t^\beta(x+1)$ and according to (1.3), in (3.36) we have $\chi(t) = t(T-t)$. One can also observe that the degeneracy occurs at the initial time $t = 0$, as in many of the works [10, 11, 12, 13].

Obviously, assumptions (A)–(C), (E) and (F) are satisfied. It is easy to calculate that for this inverse problem we can take

$$g_0 = \frac{T^3}{6},\ K_g^* = \frac{1}{2(\beta+1)} T^{\beta+1}(l^2 + 2l),\ \lambda^* = 0,$$
$$a_3 = 0,\ K_{d,\chi} = K_{d,a} = K_b = 0,\ K_{a,\chi} = 2\,T^{2-\beta}.$$

Then, the constant α_1 in (3.12) is equal to

$$\alpha_1 = 6\sqrt{\frac{6}{\beta+1}} \cdot \frac{l(2+l)^{1/2}}{T^{1/2+\beta/2}}.$$

Hence, the condition (3.12) is certainly valid for sufficiently small l (T is fixed), or, on the opposite, if T is sufficiently large (l is fixed). Therefore, in both cases we can apply Theorem 3 to the inverse problem given by (3.34)–(3.36) and thus the generalized solution of this problem exists and is unique.

Example 2. Consider the inverse problem for the equation

$$u_t - t^\beta(x+1)\, u_{xx} + t^{\beta/2}(x+1)^{\beta/2}\, u = p(x) t^\beta(x+1), \quad (x,t) \in Q, \tag{3.37}$$

with initial and boundary conditions (3.35), and additional condition

$$\int_0^T u(x,t)\, t\, dt = \varphi(x), \quad x \in [0,l], \tag{3.38}$$

with $u_0 \in \overset{\circ}{W}{}_2^1(0,l)$, $\varphi \in W_\infty^2(0,l)$, $\varphi(0) = \varphi(l) = 0$ and $\beta \in (0,1)$. Here, $a(x,t) = t^\beta(x+1)$, $b(x,t) = r(x,t) = 0$, $\mathrm{d}(x,t) = -t^{\beta/2}(x+1)^{\beta/2}$, $g(x,t) = t^\beta(x+1)$ and $\chi(t) = t$.

Then the assumptions (A)–(C), (E), (F) are obviously fulfilled. By direct calculations we can take

$$K_{d,a} = 1, \quad g_0 = \frac{T^2}{2}, \quad K_g^* = \frac{l(l+2)T^{\beta+1}}{2(\beta+1)}, \quad \lambda^* = \frac{3l^2}{\pi^2},$$
$$a_3 = T^{1-\beta}, \quad K_{a,\chi} = T^{1-\beta}, \quad K_{d,\chi} = \frac{2\,T^{2-\beta/2}}{4-\beta}, \quad K_b = 0.$$

Then, the constant α in (3.12) is equal to

$$\alpha_1 = \sqrt{\frac{6}{\beta+1}} \cdot l(l+2)^{1/2} \exp\left(\frac{3l^2 T}{2\pi^2}\right)\left(2\,T^{(3-\beta)/2} + \frac{2\,T^{5/2}}{4-\beta}\right).$$

Hence, the condition (3.12) is certainly valid if T is fixed and l is sufficiently small. In this case we can also apply Theorem 3 to the inverse problem given by (3.35), (3.37) and (3.38) and thus the generalized solution of this problem exists and is unique.

Now we give an example of the application of Theorem 4.

Example 3. Consider the inverse problem of finding the pair $\{u(x,t);p(x)\}$ satisfying

$$u_t - t^\beta a_1(x)u_{xx} - t^\beta b_1(x)u_x + t^{\beta/2}d_1(x)u = t^\beta g_1(x)p(x), \quad (x,t) \in Q, \tag{3.39}$$

with initial and boundary conditions (3.35) and additional condition

$$\int_0^T u(x,t)\, t^\beta\, (T-t)\, dt = \varphi(x), \quad x \in [0,l], \tag{3.40}$$

with $u_0 \in \overset{\circ}{W}{}_2^1(0,l)$, $\varphi \in W_\infty^2(0,l)$, $\varphi(0) = \varphi(l) = 0$, $\beta \in (0,1)$, and the arbitrary functions a_1, b_1, d_1, $g_1 \in L_\infty(0,l)$, satisfying the assumption (B) and assumptions

$$0 < a_0 \leqslant a_1(x), \quad 0 \leqslant d_1(x), \quad |g_1(x)| \geqslant \tilde{g}_0 > 0, \quad x \in [0,l].$$

Hence, the assumptions (A)–(F) hold. Since in this example $\chi(t) = t^\beta\,(T-t)$, $a(x,t) = t^\beta a_1(x)$, $d(x,t) = -t^{\beta/2} d_1(x)$, then the inequality

$$\left(\frac{\chi(t)}{a(x,t)}\right)_t + \frac{d(x,t)\chi(t)}{a(x,t)} \leqslant 0, \quad (x,t) \in Q \quad \Leftrightarrow \quad -1 - d_1(x)\, t^{\beta/2}\,(T-t) \leqslant 0$$

is also satisfied. So, by Theorem 4 the solution of the inverse problem given by (3.35), (3.39) and (3.40) exists and is unique for any $T > 0$, $l > 0$.

4 Numerical solution of the direct problem

In this section, we consider the direct (forward) initial value problem given by equations (1.1) and (1.2) when the coefficients $a(x,t)$, $b(x,t)$, $\mathrm{d}(x,t)$, $g(x,t)$, $r(x,t)$ and $p(x)$ are given and the dependent variable $u(x,t)$ is the solution to be determined.

The discrete form of the direct problem is as follows. Taking the positive integer numbers M and N, the solution domain $Q_T = [0,l] \times [0,T]$ is divided by a $M \times N$ mesh with spatial step size $\Delta x = l/M$ in x-direction and the time step size $\Delta t = T/N$. The solution at the node (i,j) is denoted by $u_{i,j} := u(x_i, t_j)$, where $x_i = i\Delta x$, $t_j = j\Delta t$, $a_{i,j} := a(x_i,t_j)$, $b_{i,j} := b(x_i,t_j)$, $d_{i,j} := \mathrm{d}(x_i,t_j)$, $g_{i,j} := g(x_i,t_j)$, $r_{i,j} := r(x_i,t_j)$ and $p_i := p(x_i)$ for $i = \overline{0,M}$ and $j = \overline{0,N}$.

Using the Crank-Nicolson finite difference method (FDM) we approximate (1.1) and (1.2) by

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left(\mathcal{G}_{i,j} + \mathcal{G}_{i,j+1}\right), \quad i = \overline{1,(M-1)},\ j = \overline{0,(N-1)}, \tag{4.1}$$

$$u_{i,0} = u_0(x_i), \quad i = \overline{0,M}, \tag{4.2}$$

$$u_{0,j} = 0, \quad u_{M,j} = 0, \quad j = \overline{0,N}, \tag{4.3}$$

where

$$\mathcal{G}_{i,j} = a_{i,j}\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + b_{i,j}\frac{u_{i+1,j} - u_{i-1,j}}{2(\Delta x)} + d_{i,j}u_{i,j} + p_i g_{i,j} + r_{i,j},$$
$$i = \overline{1,(M-1)},\ j = \overline{0,N}. \tag{4.4}$$

Collecting the terms alike, (4.1) can be rewritten as

$$
\begin{aligned}
&- A_{i,j+1}u_{i-1,j+1} + (1 + B_{i,j+1})u_{i,j+1} - C_{i,j+1}u_{i+1,j+1}\\
&= A_{i,j}u_{i-1,j} + (1 - B_{i,j})u_{i,j} + C_{i,j}u_{i+1,j} + \frac{\Delta t}{2}p_i(g_{i,j} + g_{i,j+1}) + \frac{\Delta t}{2}\left(r_{i,j} + r_{i,j+1}\right), \quad (4.5)
\end{aligned}
$$

for $i = \overline{1,(M-1)}$, $j = \overline{0,(N-1)}$, where

$$
A_{i,j} = \frac{(\Delta t)a_{i,j}}{2(\Delta x)^2} - \frac{b_{i,j}(\Delta t)}{4(\Delta x)},\ B_{i,j} = \frac{(\Delta t)a_{i,j}}{(\Delta x)^2} - \frac{\Delta t}{2}d_{i,j},\ C_{i,j} = \frac{(\Delta t)a_{i,j}}{2(\Delta x)^2} + \frac{b_{i,j}(\Delta t)}{4(\Delta x)}.
$$

At each time step t_{j+1}, for $j = \overline{0,(N-1)}$, using the homogenous Dirichlet boundary conditions (4.3), the above difference equation can be reformulated as a $(M-1)\times(M-1)$ system of linear equations of the form

$$
D\mathbf{u}_{j+1} = E\mathbf{u}_j + \mathbf{b} \tag{4.6}
$$

$$
D = \begin{pmatrix}
1+B_{1,j+1} & -C_{1,j+1} & 0 & \cdots & 0 & 0 & 0\\
-A_{2,j+1} & 1+B_{2,j+1} & -C_{2,j+1} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & -A_{M-2,j+1} & 1+B_{M-2,j+1} & -C_{M-2,j+1}\\
0 & 0 & 0 & \cdots & 0 & -A_{M-1,j+1} & 1+B_{M-1,j+1}
\end{pmatrix},
$$

$$
E = \begin{pmatrix}
1-B_{1,j} & C_{1,j} & 0 & \cdots & 0 & 0 & 0\\
A_{2,j} & 1-B_{2,j} & C_{2,j} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & A_{M-2,j} & 1-B_{M-2,j} & C_{M-2,j}\\
0 & 0 & 0 & \cdots & 0 & A_{M-1,j} & 1-B_{M-1,j}
\end{pmatrix},
$$

and

$$
\mathbf{b} = \frac{\Delta t}{2}\begin{pmatrix}
p_1(g_{1,j} + g_{1,j+1}) + (r_{1,j} + r_{1,j+1})\\
p_2(g_{2,j} + g_{2,j+1}) + (r_{2,j} + r_{2,j+1})\\
\vdots\\
p_{M-2}(g_{M-2,j} + g_{M-2,j+1}) + (r_{M-2,j} + r_{M-2,j+1})\\
p_{M-1}(g_{M-1,j} + g_{M-1,j+1}) + (r_{M-1,j} + r_{M-1,j+1})
\end{pmatrix}.
$$

The numerical solutions for the desired output (1.3) is calculated using the trapezoidal rule,

$$
\varphi(x_i) = \int_0^T \chi(t)u(x_i,t)dt = \frac{\Delta t}{2}\left(\chi(0)u_{i,0} + \chi(T)u_{i,N} + 2\sum_{j=1}^{N-1}\chi(t_j)u_{i,j}\right),\ i = \overline{0,M}, \tag{4.7}
$$

As an example, consider the direct (forward) problem (1.1) and (1.2) with $T = l = 1$, $\beta = 1/2$ and

$$
\begin{aligned}
&a(x,t) = t^\beta(x+1), \quad b(x,t) = d(x,t) = 0, \quad u(x,0) = u_0(x) = 0,\\
&g(x,t) = t^\beta(x+1), \quad r(x,t) = x(1-x) + 2t^{\beta+1}(x+1),
\end{aligned} \tag{4.8}
$$

and

$$p(x) = 0. \quad (4.9)$$

The exact solution is given by

$$u(x,t) = tx(1-x). \quad (4.10)$$

For $\chi(t) = t(1-t)$, the function in (1.3) is given by

$$\varphi(x) = \int_0^T \chi(t)u(x,t)dt = \frac{1}{12}x(1-x). \quad (4.11)$$

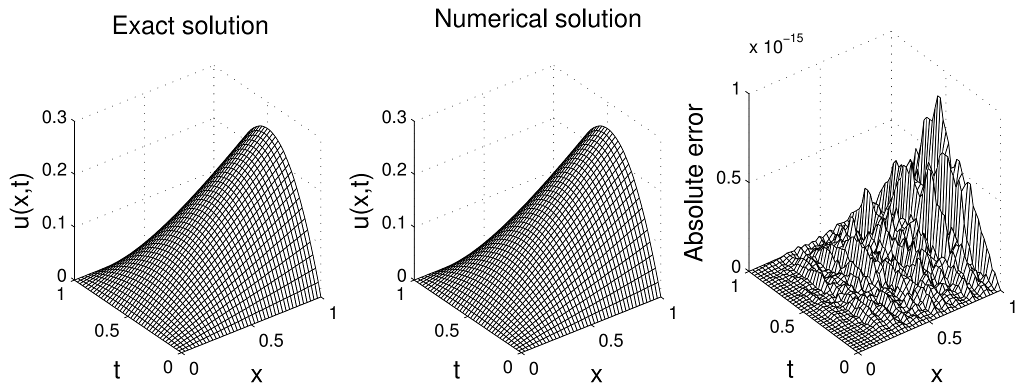

Figure 1: Exact and numerical solutions for $u(x,t)$ and the absolute error for the direct problem obtained with $M = N = 40$.

From Figure 1 it can be seen that there is an excellent agreement (absolute error of $O(10^{-14})$) between the numerical solution and exact solution (4.10) for $u(x,t)$.

5 Numerical solution of the inverse problem

For the numerical solution of the inverse problem, we employ the Tikhonov regularization based on minimizing the functional

$$F(p) = \left\| \int_0^T u(\cdot,t)\chi(t)dt - \varphi(\cdot) \right\|^2_{L_2(0,l)} + \lambda\|p\|^2_{L_2(0,l)}, \quad (5.1)$$

where u satisfies (1.1)–(1.3) for given p, and $\lambda \geq 0$ is a regularization parameter to be prescribed. A discrete form of (5.1) is

$$F(\mathbf{p}) \;=\; \sum_{i=1}^{M-1} \left[\int_0^T u(x_i,t)\chi(t)dt - \varphi(x_i) \right]^2 + \lambda \sum_{i=1}^{M-1} p_i^2. \quad (5.2)$$

The minimization of F subject to simple bounds on $\mathbf{p} = (p_i)_{i=\overline{1,(M-1)}}$ is accomplished using the *lsqnonlin* routine from MATLAB optimization toolbox which does not require supplying (by the user) the gradient of (5.1), [23]. This routine tries to find the minimum point for the sum of squares of function stating from an initial guess and is based of the Trust-Region-Reflection algorithm (TRR), [5].

In the numerical computation, we take the parameters of the routine *lsqnonlin* as follows:

- Number of variables $M = N = 40$.
- Maximum number of iterations $= 10^2 \times$ (number of variables).
- Maximum number of objective function evaluations $= 10^3 \times$(number of variables).
- Solution and object function tolerances $= 10^{-15}$.
- The lower and upper bounds on the components of the vector $\mathbf{p}$ are -10^2 and 10^2, respectively.

The inverse problem given by (1.1)–(1.3) is solved subject to both exact and noisy measurement (1.3), which is numerically simulated as

$$\varphi^\epsilon(x_i) = \varphi(x_i) + \epsilon_i, \quad i = \overline{1, (M-1)}, \tag{5.3}$$

where ϵ_i are random variables generated from a Gaussian normal distribution with mean zero and standard deviation $\sigma = \delta \times \|\varphi\|_{L_\infty(0,l)}$, where δ represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\underline{\epsilon} = (\epsilon_i)_{i=\overline{1,(M-1)}} = normrnd(0, \sigma, M-1)$. In the case of noisy data (5.3), we replace $\varphi(x_i)$ by $\varphi^\epsilon(x_i)$ in (5.1).

6 Numerical results and discussion

Throughout this section we take $T = l = 1$, $\beta = 1/2$ and $M = N = 40$ (unless otherwise specified).

6.1 Test 1

Consider the degenerate inverse problem (1.1) – (1.3) with the input data (4.8) and (4.11). The exact solution is given by (4.9) and (4.10). We take the initial guess for the unknown function $p(x)$ as $p^0(x) = x(1-x)$, $x \in (0, 1)$, which ensures the continuity at the endpoints $x \in \{0, 1\}$.

We consider first the case of no noise introduced in the input data (1.3), i.e. $\delta = 0$. Figure 2 shows the objective function (5.2) without regularization, i.e. $\lambda = 0$, as a function of the number of iterations. From this figure it can be seen the rapid decreasing convergence of the objective function (5.2) to take a very low value of order $O(10^{-19})$ in only 7 iterations. The corresponding numerical results for $p(x)$ illustrated in Figure 3 show a very good agreement (absolute error of $O(10^{-4})$) with the exact solution $p(x) = 0$ given by (4.9).

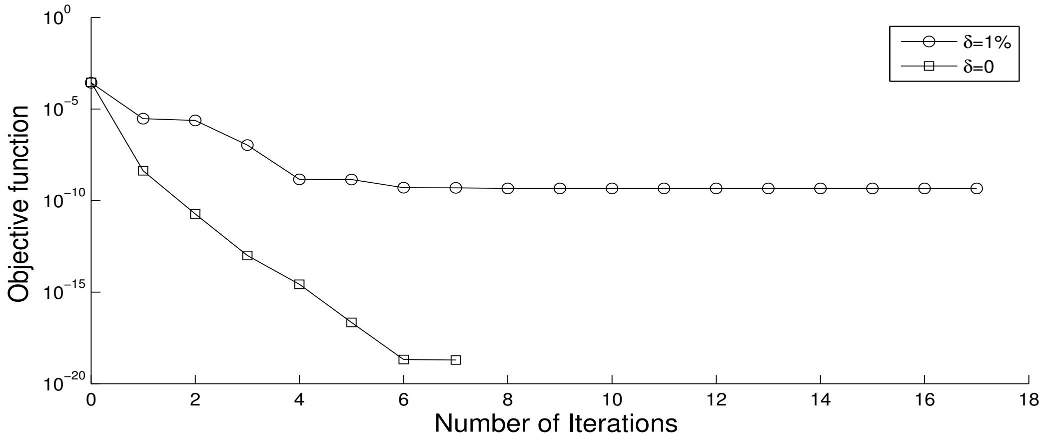

Figure 2: The objective function (5.2) without regularization, as a function of the number of iterations, for test 1, $\delta \in \{0, 1\%\}$ noise.

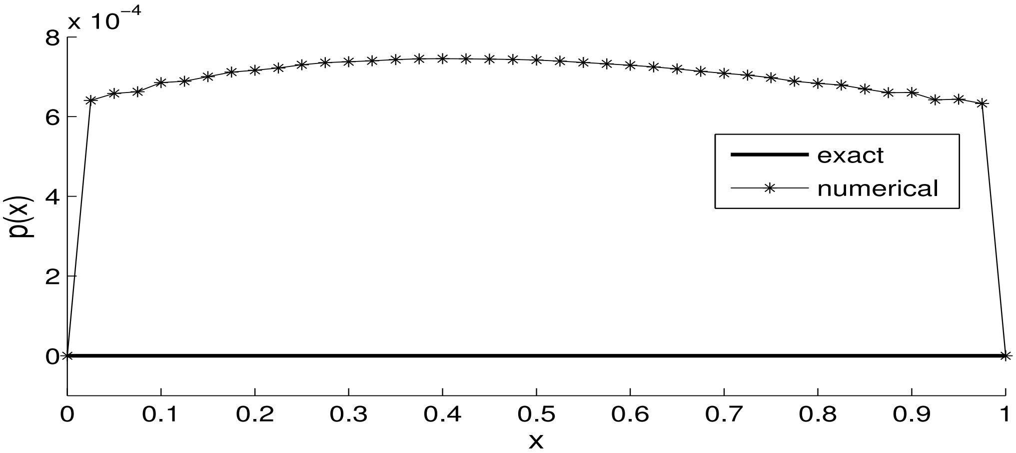

Figure 3: Exact and numerical solutions for $p(x)$, for test 1, no noise, without regularization.

Next, we perturb the measured data (1.3) with $\delta = 1\%$ noise, as in equation (5.3). The convergence of the unregularized objective function (5.2) is illustrated in Figure 2. Figure 4 presents the numerical reconstruction for $p(x)$ when no regularization employed, i.e. $\lambda = 0$, and an unstable behaviour it can be clearly seen. This is to be expected since the inverse source problem under investigation is ill-posed and small errors in the input data lead to dramatic errors in output solution. Therefore, regularization should be applied in order to retrieve stability. We utilise Tikhonov's regularization method by adding the penalty term $\lambda \|p\|^2_{L_2(0,l)}$ to the ordinary least-squares functional, as given in (5.2). The numerical results for $p(x)$ obtained using various regularization parameters λ are shown in Figure 5 and Table 1. From this figure and table it can be seen that, compared to the highly unstable solution of Figure 4, the values of λ between 10^{-3} to 10^{-1} produce stable and reasonably accurate numerical results. Of course, a more rigorous choice of the regularization parameter can be based on the Morozov's discrepancy principle [29].

Table 1: The number of iterations, the number of function evaluations, the objective function (5.2) at the final iteration, and the root mean square error $rmse(p)$, for various regularization parameters, for test 1, $\delta = 1\%$ noise.

$\delta = 1\%$	$\lambda = 0$	$\lambda = 10^{-3}$	$\lambda = 10^{-2}$	$\lambda = 10^{-1}$
No. of iterations	17	18	13	21
No. of function evaluations	756	798	588	924
Objective function at final iteration	4.5E-10	3.3E-6	3.4E-6	3.4E-6
$rmse(p)$	6.763	1.1E-3	1.3E-4	1.3E-5

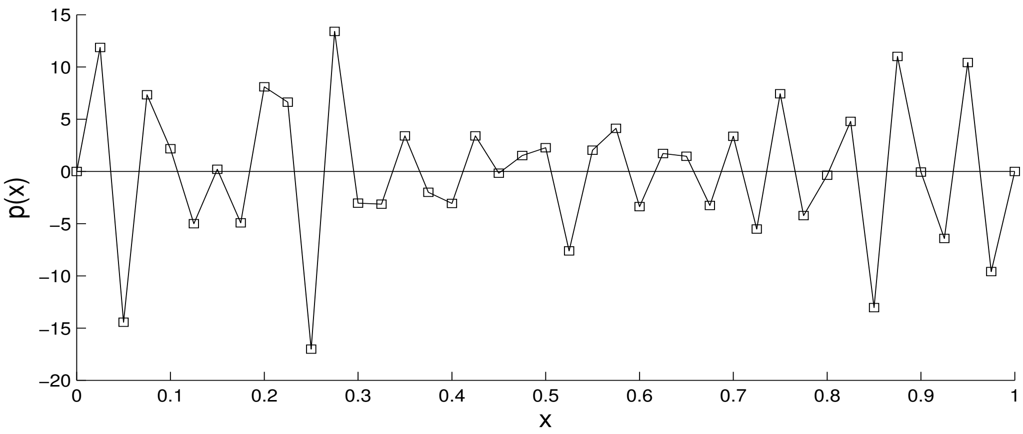

Figure 4: Exact and (unstable) numerical solutions for $p(x)$, for test 1, $\delta = 1\%$ noise, without regularization.

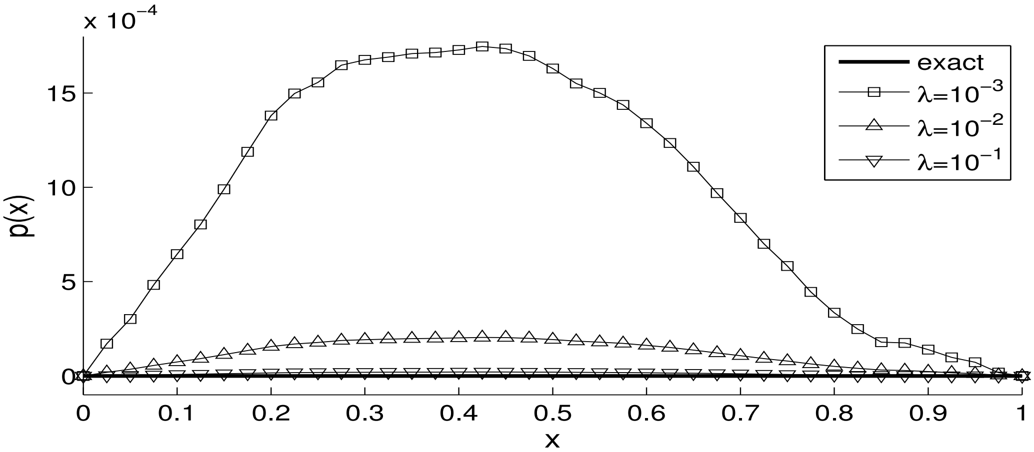

Figure 5: Exact and (stable) numerical solutions for $p(x)$ for test 1, $\delta = 1\%$ noise, for various regularization parameters $\lambda \in \{10^{-3}, 10^{-2}, 10^{-1}\}$.

6.2 Test 2

We consider the degenerate inverse problem given by equations (3.35), (3.39) and (3.40), with the following input data:

$$\beta = 1/2, \quad a_1(x) = b_1(x) = d_1(x) = g_1(x) = 1,$$
$$u(x,0) = u_0(x) = x(1-x), \quad \chi(t) = t^{1/2}(1-t). \tag{6.1}$$

We take the piecewise-smooth function

$$p(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1/2, \\ 1-x & \text{if} \quad 1/2 \le x \le 1, \end{cases} \tag{6.2}$$

as the analytical source to be retrieved. Since in this case there is no analytical solution available for $u(x,t)$, the data (1.3) is numerically simulated by solving the direct problem (1.1) and (1.2) with the input data (6.1) and (6.2), and employ expression (4.7). Although not illustrated, it is reported that independence on the FDM mesh is rapidly achieved.

Next, in the inverse problem we take 40 values of φ from the curve with $M = N = 80$ at equally space points and run the inverse problem (3.35), (3.39) and (3.40) with $M = N = 40$ (different mesh size in order to avoid an inverse crime) and the initial guess $p^0(x) = 0$. Figures 6–8 of test 2 have analogy to Figures 2–5 of test 1 and similar conclusions can be deduced.

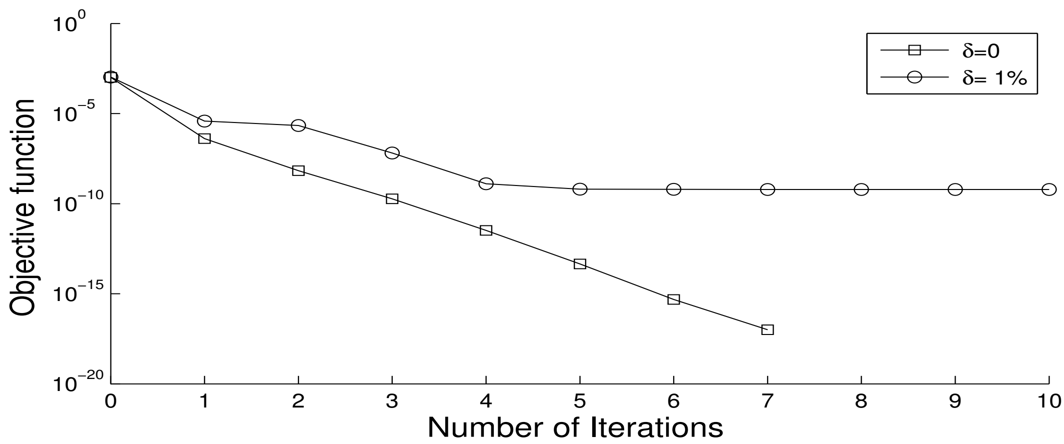

Figure 6: The objective function (5.2) without regularization, as a function of the number of iterations for test 2, $\delta \in \{0, 1\%\}$ noise.

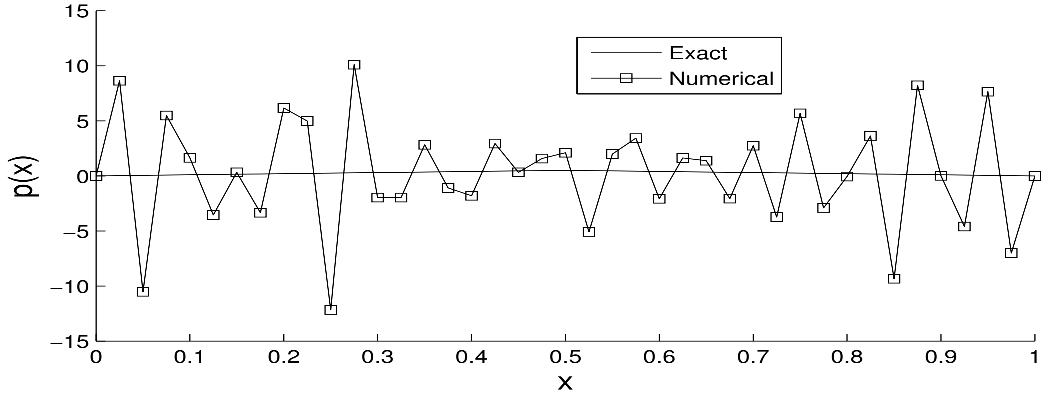

Figure 7: Exact and (unstable) numerical solutions for $p(x)$ for test 2, $\delta = 1\%$ noise, without regularization.

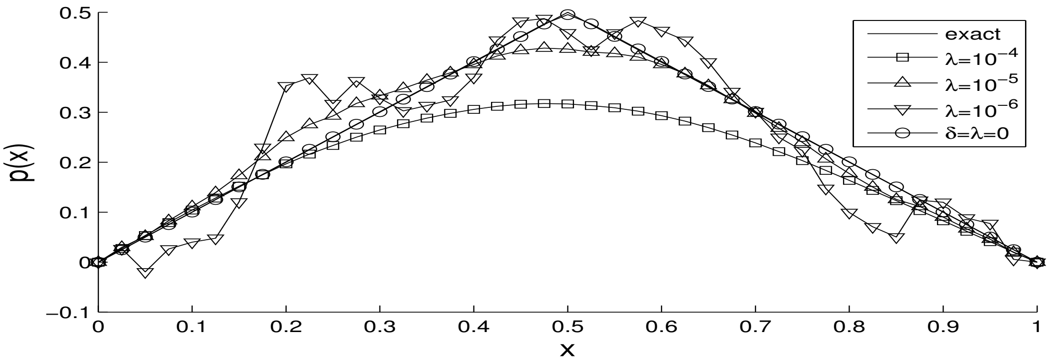

Figure 8: Exact and (stable) numerical solutions for $p(x)$ for test 2, $\delta = 1\%$ noise, and various regularization parameters $\lambda \in \{10^{-4}, 10^{-5}, 10^{-6}\}$. The numerical solution for $\delta = 0$ no noise obtained without regularization, $\lambda = 0$, shown with (-○-) overlaps the analytical solution (6.2) shown with (—).

6.3 Test 3

Consider now the degenerate inverse problem given by equations (3.35), (3.39) and (3.40), with the input data (6.1), as in test 2, but in this case we take the additional data (1.3) given by

$$\varphi(x) = x(1 - x). \tag{6.3}$$

This test is unusual and different from the previous two test problems in the sense that no analytical solution is available for both $u(x,t)$ and $p(x)$. However, from the analysis of Example 3 in Section 3.3, via the unique solvability Theorem 4, we are ensured that the solution pair $\{u(x,t), p(x)\}$ exists and is unique.

We run the inverse solver with the initial guess $p^0(x) = 1 + 2x$ and $M = N = 40$.

In the case of no noise and no regularization, Figure 9 shows the numerical solution for $p(x)$ at various iteration numbers, as it evolves from the initial guess to the converged minimizer solution of the functional (5.2) with $\lambda = 0$, within 11 iterations.

Next, we add $\delta = 1\%$ in the input data (6.3). In case of no regularization, the minimization of the unregularized functional (5.2) with $\lambda = 0$ produces a highly unstable solution of $O(10^2)$, which is not illustrated. Figure 10 shows the numerical results for

$p(x)$ obtained with various values of the regularization parameter $\lambda \in \{10^{-7}, 10^{-6}, 10^{-5}\}$. From this figure it can be seen that by imposing regularization with λ between 10^{-6} to 10^{-5} produces stable numerical solutions, which are in reasonable good agreement with the numerical solution obtained in the case of no noise.

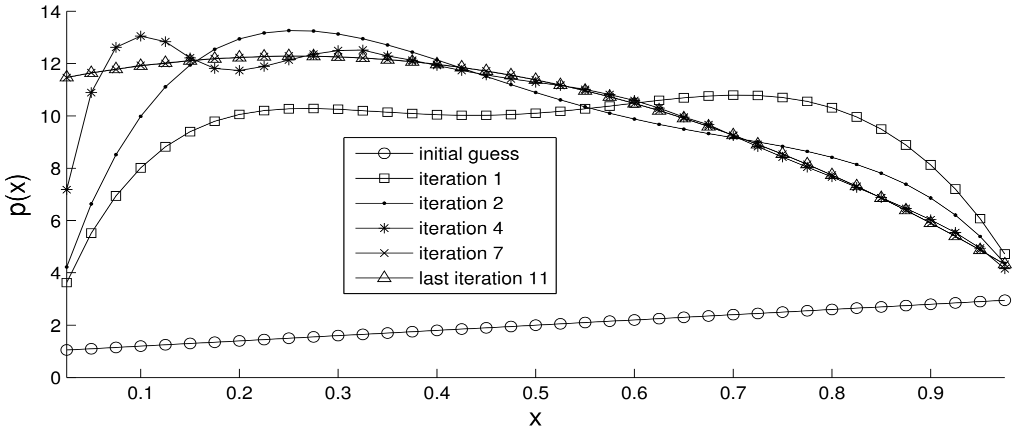

Figure 9: Numerical solutions for $p(x)$ at various iteration numbers, for test 3, no noise, without regularization.

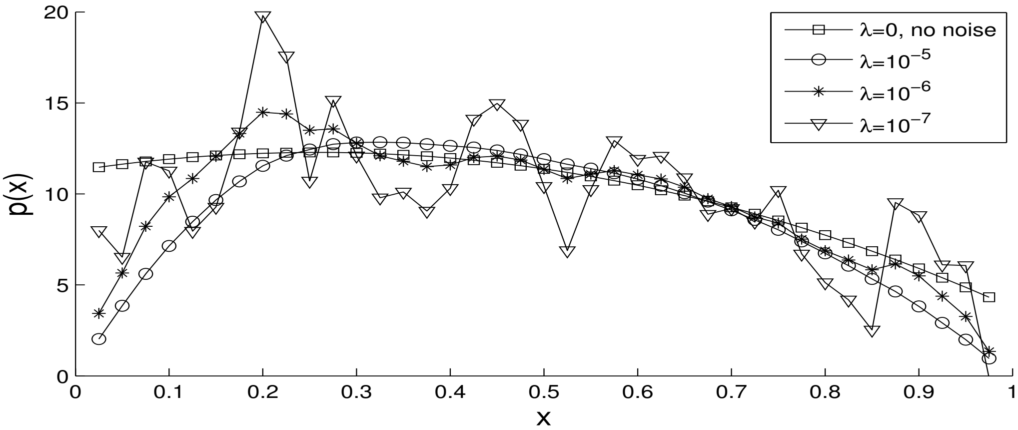

Figure 10: Numerical solutions for $p(x)$, for test 3, $\delta = 1\%$ noise, and various regularization parameters $\lambda \in \{0, 10^{-7}, 10^{-6}, 10^{-5}\}$.

7 Conclusions

This paper has analysed direct and inverse space-dependent source problems in degenerate PDEs which are non-uniformly parabolic. Two variants of easy-to-check sufficient conditions for the unique solvability of the problem have been considered. Stability estimates have also been established. Furthermore, numerical results have been provided to illustrate the accuracy and stability of the numerical reconstructions for a wide range of typical test examples in one-dimension. Higher dimensions, as well as higher order fourth-order Euler-Bernoulli beam-type degenerate equations can be the subject of further investigations.

Acknowledgements

V.L. Kamynin and A.B. Kostin were partially supported by the Programme of Competitiveness Increase of the National Research Nuclear University MEPhI (Moscow Engineering Physics Institute); contract no 02.a03.21.0005, 27.08.2013. D. Lesnic would like to acknowledge some small financial support received from the EPSRC funded research network on inverse problems EP/P005985/1 for a week research visit of Professor V.L. Kamynin to Leeds in July 2017.

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