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Fuzzy Differential Equations for Nonlinear System Modeling With Bernstein Neural Networks

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ABSTRACT With the fuzzy set theory, the uncertainty of nonlinear systems can be modeled using fuzzy differential equations. The solutions of these equations are the model output, but they are very difficult to obtain. In this paper, we first transform fuzzy differential equations into four ordinary differential equations. Then, we construct neural models with the structure of these ordinary differential equations. Theory analysis and simulation results show that these new models are effective for modeling uncertain nonlinear systems.

INDEX TERMS Fuzzy equation, nonlinear system modeling, neural networks.

I. INTRODUCTION

Since the uncertainty in parameters can be transformed into fuzzy set theory [56], fuzzy set and fuzzy system theory are good tools to address uncertainty systems. Fuzzy models are applied to a large class of uncertain nonlinear systems, for example Takagi-Sugeno fuzzy model [52]. When the parameters of an equation are changeable in the manner of a fuzzy set, this equation becomes a fuzzy equation [13]. When the parameters or the states of the differential equations are uncertain, they can be modeled with fuzzy differential equations (FDEs).

Many FDEs use fuzzy numbers as the coefficients of the differential equations to describe the uncertainties [21]. The applications of these FDEs are connected with nonlinear modeling and control [29]–[32]. Another type of FDE uses fuzzy variables to express the uncertainties. Studies on the solutions of FDEs are applied to chaotic analysis, quantum systems, and engineering problems, such as those in civil engineering. The basic idea of the fuzzy derivative was first introduced in [16] and was extended in [19]. The linear first-order equation is the simplest FDE. By generalizing the differentiability, [8] gave an analytical solution. In [35], the first-order FDE with periodic boundary conditions was analyzed. Then, higher order linear FDEs were studied. In [6], the analytical solutions of the second-order FDE were obtained. The analytical solutions of third-order linear FDEs are found in [26], while [12] and [5] proposed analytical approaches to resolve nth-order linear FDEs.

It is more difficult to solve nonlinear FDEs. Using the interval-valued method, [50] examined the basis of solutions for nonlinear FDEs with generalized differentiability. [44] used periodic boundary and Hukuhara differentiability for impulsive FDEs. [22] suggested some suitable criteria to fuzzify the crisp solutions. [38] used two-point fuzzy boundary values for FDEs. [25] used homeotypic analysis technique for FDEs. However, all of above analytical methods for the solutions of FDEs are very difficult, especially for nonlinear FDEs.

Numerical solutions of FDEs have been discussed recently by many scientists. The numerical solutions of first-order FDEs were proposed in [49] with an iterative technique. [3] used Laplace transformation for second-order FDEs. By extending classical fuzzy set theory, [28] obtained a numerical solution for an FDE. The predictor-corrector approach was applied in [4]. The Euler numerical technique was used in [42], [53], and [43] to solve FDEs. Other numerical techniques, such as the Nystrom approach [34], Taylor method [1] and Runge-Kutta approach [45], can also be applied to solve FDEs. However, the approximation accuracy of these numerical calculations are normally lower [39].

The solutions of FDEs are uniformly continuous and inside compact sets [11]. Neural networks can provide good estimations for the solutions of FDEs. [2] showed that the solution of an ordinary differential equation (ODE) can be approximated by a neural network. [24] discussed the differences between the exact solution and approximation solutions of ODEs. [40] and [55] applied neural approximations of ODEs to dynamic systems. [41] used the B-splines neural network to estimate the solutions of nonlinear ODEs. [37] applied dynamic neural networks to approximate first-order ODEs.

However, there are few studies that use neural networks to solve FDEs. [20] suggested a static neural network to solve FDEs. Since the structure of these neural networks in the above mentioned works are not suitable for FDEs, the approximation accuracy is poor.

In this paper, we apply a new model, the Bernstein neural network, which uses the properties of the Bernstein polynomial, to FDEs. The Bernstein polynomial has good uniform approximation abilities for continuous functions [18]. It also has innumerable drawing properties, homogeneous shape-sustaining approximation, bona fide estimation, and low boundary bias. A very important property of the Bernstein polynomial is that it generates a smooth estimation for equal distance knots [17]. This property is suitable for FDE approximation.

We use two types of neural networks, static and dynamic models, to approximate the solutions of the FDEs. These numerical methods use the generalized differentiability of FDEs. The solutions of the FDEs are then substituted into four ODEs, and the corresponding Bernstein neural networks are applied. Finally, we use some real examples to show the effectiveness of our approximation methods with the Bernstein neural networks.

II. FUZZY DIFFERENTIAL EQUATION FOR UNCERTAIN NONLINEAR SYSTEM MODELING

Consider the following controlled unknown nonlinear system

$$\dot{x} = f_1(x_1, u, t) \tag{1}$$

where $f_1(x_1, u)$ is the unknown vector function, $x_1 \in \Re^n$ is an internal state vector, and $u \in \Re^m$ is the input vector.

In this paper, we use the following FDE to model the uncertain nonlinear system (1),

$$\frac{d}{dt}x = f(x, u) \tag{2}$$

where $x \in \Re^n$ is the fuzzy variable that corresponds to the state x_1 in (1), $f(t, x)$ is a fuzzy vector function that relates to $f_1(x_1, u)$, and $\frac{d}{dt}x$ is the fuzzy derivative. Here, the uncertainties of the nonlinear system (1) are in the sense of fuzzy logic. They are defined as follows.

Definition 1: If x is: 1) normal, there exists $\zeta_0 \in \mathbb{R}$ in such a manner that $x(\zeta_0) = 1$; 2) convex, $x(\lambda\zeta + (1-\lambda)\zeta) \geq \min\{x(\zeta), x(\xi)\}$, $\forall\zeta, \xi \in \mathbb{R}, \forall\lambda \in [0, 1]$; 3) upper semi-continuous on $\mathbb{R}$, $x(\zeta) \leq x(\zeta_0) + \varepsilon$, $\forall\zeta \in N(\zeta_0)$, $\forall\zeta_0 \in \mathbb{R}$, $\forall\varepsilon > 0$, $N(\zeta_0)$ is a neighborhood; or 4) $x^+ = \{\zeta \in \mathbb{R}, x(\zeta) > 0\}$ is compact, then x is a fuzzy variable, and the fuzzy set is defined as E, $x \in E : \mathbb{R} \to [0, 1]$.

The fuzzy variable x can also be represented as

$$x = A\left(\underline{x}, \bar{x}\right) \tag{3}$$

where $\underline{x}$ is the lower-bound variable, $\bar{x}$ is the upper-bound variable, and A is a continuous function. The membership functions are utilized to implicate the fuzzy variable x. The best known membership functions are the triangular function

$$x(\zeta) = F(a, b, c) = \begin{cases} \dfrac{\zeta - a}{b - a} & a \leq \zeta \leq b \\ \dfrac{c - \zeta}{c - b} & b \leq \zeta \leq c \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and trapezoidal function

$$x(\zeta) = F(a, b, c, d) = \begin{cases} \dfrac{\zeta - a}{b - a} & a \leq \zeta \leq b \\ \dfrac{d - \zeta}{d - c} & c \leq \zeta \leq d \\ 1 & b \leq \zeta \leq c \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

For the crisp variable, the fuzzy variable x possess three essential operations: $\oplus$, $\ominus$ and $\odot$, which signify sum, subtract, and multiply, respectively.

The fuzzy variable x that contains the dimension of ζ is dependent on the membership functions, where (4) includes three variables and (5) includes four variables. To demonstrate the consistency of operations, the application initially lies within the $\alpha-$level operation of the fuzzy number.

A fuzzy number x associates with a real value with α-level as

$$[x]^\alpha = \{a \in \mathbb{R} : x(a) \geq \alpha\} \tag{6}$$

where $0 < \alpha \leq 1, x \in E$.

If $x, y \in E$, $\lambda \in \mathbb{R}$, the fuzzy operations are as follows:
Sum,

$$[x \oplus y]^\alpha = [x]^\alpha + [y]^\alpha = [\underline{x}^\alpha + \underline{y}^\alpha, \bar{x}^\alpha + \bar{y}^\alpha] \tag{7}$$

subtract,

$$[x \ominus y]^\alpha = [x]^\alpha - [y]^\alpha = [\underline{x}^\alpha - \underline{y}^\alpha, \bar{x}^\alpha - \bar{y}^\alpha] \tag{8}$$

or multiply,

$$\underline{z}^\alpha \leq [x \odot y]^\alpha \leq \bar{z}^\alpha \text{ or } [x \odot y]^\alpha = A\left(\underline{z}^\alpha, \bar{z}^\alpha\right) \tag{9}$$

where $\underline{z}^\alpha = \underline{x}^\alpha \underline{v}^1 + \underline{x}^1\underline{y}^\alpha - \underline{x}^1\underline{y}^1$, $\bar{z}^\alpha = \bar{x}^\alpha\bar{y}^1 + \bar{x}^1\bar{y}^\alpha - \bar{x}^1\bar{y}^1$, and $\alpha \in [0, 1]$.

Therefore, $[x]^0 = x^+ = \{\zeta \in \mathbb{R}, x(\zeta) > 0\}$. Since $\alpha \in [0, 1]$, $[x]^\alpha$ is a bounded interval such that $\underline{x}^\alpha \leq [x]^\alpha \leq \bar{x}^\alpha$. The α-level of x in the midst of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ is given as

$$[x]^\alpha = A\left(\underline{x}^\alpha, \bar{x}^\alpha\right) \tag{10}$$

Definition 2: The fuzzy derivative of f at x_0 is expressed as

$$\frac{d}{dt}f(x_0) = \lim_{h\to 0}\frac{1}{h}[f(x_0 + h) \ominus_{gH} f(x_0)] \tag{11}$$

where $\ominus_{gH}$ is the Hukuhara difference [9], defined by

$$x \ominus_{gH} y = z \Longleftrightarrow \begin{cases} 1)\ x = y \oplus z \\ 2)\ y = x \oplus (-1)z \end{cases} \tag{12}$$

The $\alpha-$level of the fuzzy derivative is

$$f(x, \alpha) = [\underline{f}(x, \alpha), \bar{f}(x, \alpha)]$$

where $x \in E$ for each $\alpha \in [0, 1]$.

If we apply the $\alpha-$level (10) to $f(x,u)$ in (2)

$$[x \ominus_{gH} y]^\alpha = [\min\{\underline{x}^\alpha - \underline{y}^\alpha, \bar{x}^\alpha - \bar{y}^\alpha\}, \\ \max\{\underline{x}^\alpha - \underline{y}^\alpha, \bar{x}^\alpha - \bar{y}^\alpha\}]$$

Then, we obtain two functions: $\underline{f}\left[u, \underline{x}(\zeta,\alpha), \bar{x}(\zeta,\alpha)\right]$ and $\bar{f}\left[u, \underline{x}(\zeta,\alpha), \bar{x}(\zeta,\alpha)\right]$. Thus, the fuzzy differential equation (2) can be equivalent to the following four ordinary differential equations (ODE)

$$1)\begin{cases} \frac{d}{dt}\underline{x} = \underline{f}\left[u, \underline{x}(\zeta,\alpha), \bar{x}(\zeta,\alpha)\right] \\ \frac{d}{dt}\bar{x} = \bar{f}\left[u, \underline{x}(\zeta,\alpha), \bar{x}(\zeta,\alpha)\right] \end{cases}$$
$$2)\begin{cases} \frac{d}{dt}\underline{x} = \bar{f}\left[u, \underline{x}(\zeta,\alpha), \bar{x}(\zeta,\alpha)\right] \\ \frac{d}{dt}\bar{x} = \underline{f}\left[u, \underline{x}(\zeta,\alpha), \bar{x}(\zeta,\alpha)\right] \end{cases} \quad (13)$$

The fuzzy model of (1) can be regarded as four ordinary differential equations (13).

In this paper, we use the fuzzy differential equation (2) to model the uncertain nonlinear system (1), such that the output of the plant x can follow the plant output x_1,

$$\min_f \|x - x_1\| \quad (14)$$

This modeling object can be considered as finding $\bar{f}$ and $\underline{f}$ in the fuzzy equations of (13), or as fining the solutions of these models. It is impossible to obtain analytical solutions, but in this paper, we use neural networks to approximate them, as shown in Figure 1.

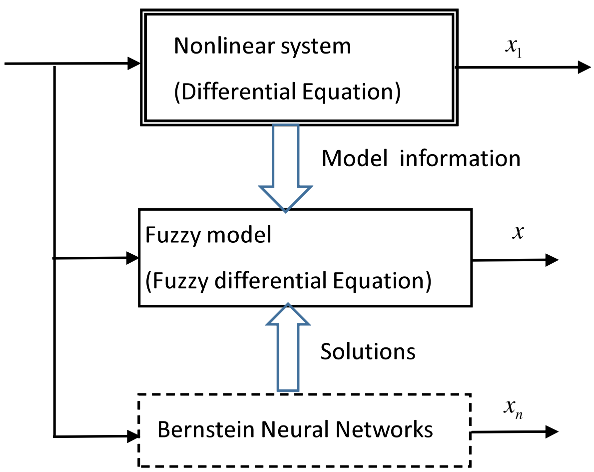

FIGURE 1. Nonlinear system modeling with fuzzy differential equations.

The following theorems give theoretical support to nonlinear system modeling via fuzzy differential equations.

Theorem 1: If the fuzzy function f and its derivative $\frac{\partial f}{\partial x}$ are on the rectangle $[-p,p] \times [-q,q]$, where $p,q \in E$, E is a fuzzy set and there exists an unique fuzzy solution for the following fuzzy differential equation

$$\frac{d}{dt}x = f(t,x), \quad x(t_0) = x_0 \quad (15)$$

for all $t \in (-b,b)$, $b \le p$

Proof: We utilize Picard's iteration technique [10] to develop a sequence of fuzzy functions $\varphi_n(t)$ as

$$\varphi_{n+1}(t) = \varphi_0 \oplus \int_0^t f(s,\varphi_n(s))ds$$
$$= \varphi_0 \ominus_H (-1)\int_0^t f(s,\varphi_n(s))ds$$

We first validate that $\varphi_n(t)$ is continuous and prevails for all n. Obviously, if $\varphi_n(t)$ prevails, then $\varphi_{n+1}(t)$ also prevails as

$$\varphi_{n+1}(t) = \varphi_0 \oplus \int_0^t f(s,\varphi_n(s))ds$$
$$= \varphi_0 \ominus_H (-1)\int_0^t f(s,\varphi_n(s))ds$$

Since f is continuous, there exists $N \in E$ such that $|f(t,x)| \le N$ for all $t \in [-p,p]$, as well as all $x \in [-q,q]$. If we set $t \in [-b,b]$ for $b \le \min(q/N,p)$, then it is possible

$$\|\varphi_{n+1} \ominus \varphi_0\| = \|\int_0^t f(s,\varphi_n(s))ds\| \le N|t| \le Nb \le q$$

This validates that $\varphi_{n+1}(t)$ acquires values in $[-q,q]$. Because

$$\varphi_n(t) = \sum_{k=1}^{n}(\varphi_n(t) \ominus \varphi_{n-1}(t))$$

for any $\gamma < 1$, we select $t \in (-b,b)$ such that $|\ \varphi_k(t) \ominus \varphi_{k-1}(t)\ | \le \gamma^k$ for all k. This signifies that there exists $\gamma < 1$ [33]

$$|\ \varphi_k(t) \ominus \varphi_{k-1}(t)\ | \le \gamma^k$$

From the mean value theorem [48],

$$\varphi_k(t) \ominus \varphi_{k-1}(t) = \int_0^t [f(s,\varphi_{k-1}(s)) \ominus f(s,\varphi_{k-2}(s))]ds$$

Applying the mean value theorem to the fuzzy function $h(x) = f(s,x)$ at the two points $\varphi_{k-1}(s)$ and $\varphi_{k-2}(s)$,

$$h(\varphi_{k-1}(s)) \ominus h(\varphi_{k-2}(s)) = h'(\psi_k(s))(\varphi_{k-1}(s)) \ominus \varphi_{k-2}(s))$$

Taking into consideration $h'(x) = \frac{\partial f}{\partial x}$, we obtain

$$\varphi_k(t) \ominus \varphi_{k-1}(t) = \int_0^t \frac{\partial f}{\partial x}(s,\psi_k(s))(\varphi_{k-1}(s) \ominus \varphi_{k-2}(s))ds \quad (16)$$

Because $|\ \varphi_{k-1}(s) \ominus \varphi_{k-2}(s)\ | \le \gamma^{k-1}$ for $s \le t$, and $b < \gamma/N$, by substituting the above relation in (16) and bounding $\frac{\partial f}{\partial x}$ by N we have ,

$$|\ \varphi_k(t)) \ominus \varphi_{k-1}(t)\ | \le \int_0^t N\gamma^{k-1}ds = Nt\gamma^{k-1} \le Nb\gamma^{k-1}$$

To validate that x is continuous, it is necessary to show that for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\ t_2 - t_1\ | < \delta$

implies $| \varphi(t_2) \ominus \varphi(t_1) | < \epsilon$. For notation convenience, we suppose that $t_1 < t_2$. It follows that

$$\begin{aligned}\varphi(t_2) \ominus \varphi(t_1) &= \lim_{n\to\infty} \varphi_n(t_2) \ominus \lim_{n\to\infty} \varphi_n(t_1) \\ &= \lim_{n\to\infty}(\varphi_n(t_2) \ominus \varphi_n(t_1)) \\ &= \lim_{n\to\infty} \int_{t_1}^{t_2} f(s, \varphi_n(s))ds\end{aligned}$$

There exists N such that $| f(s, x) | \leq N$. Hence

$$| \varphi(t_2) \ominus \varphi(t_1) | \leq \int_{t_1}^{t_2} Nds = N | t_2 - t_1 | \leq N\delta$$

Thus, by selecting $\delta < \epsilon/N$ it is observed that $| \varphi(t_2) \ominus \varphi(t_1) | < \epsilon$. So $\lim_{n\to\infty} \varphi_n(t)$ prevails for all t.

Now we demonstrate that $\lim_{n\to\infty} \varphi_n(t)$ is continuous. Since

$$\begin{aligned}\varphi(t) = \lim_{n\to\infty} \varphi_n(t) &= \lim_{n\to\infty} \int_0^t f(s, \varphi_{n-1}(s))ds \\ &= \int_0^t \lim_{n\to\infty} f(s, \varphi_{n-1}(s))ds \\ &= \int_0^t f(s, \lim_{n\to\infty} \varphi_{n-1}(s))ds\end{aligned}$$

where the last step (moving the limit inside the function) follows from the fact that f is continuous in each variable. Hence it is clear that

$$\varphi(t) = \int_0^t f(s, \varphi(s))ds$$

because all functions are continuous,

$$\frac{d}{dt}\varphi = f(s, \varphi(t))$$

If there exists another solution $\phi(t)$,

$$\varphi(t) \ominus \phi(t) = \int_0^t (f(s, \varphi(t)) \ominus f(s, \phi(t)))ds$$

Since the two functions are different, there exists $\epsilon > 0$ and $| \varphi(t) \ominus \phi(t) | > \epsilon$. We define

$$m = \max_{0\leq t\leq b} | \varphi(t) \ominus \phi(t) |$$

N is the bound for $\frac{\partial f}{\partial x}$. Utilizing the mean value theorem,

$$| \varphi(t) \ominus \phi(t) | \leq \int_0^t N | \varphi(t) - \phi(t) | ds \leq N | t | m \leq Nbm$$

If we select $b < \epsilon/2mN$, it signifies that for all $t < b$, $| \varphi(t) - \phi(t) | < \epsilon/2$, indicating that the least difference is ϵ. Therefore, there exists a unique fuzzy solution. ∎

Theorem 2: If the following fuzzy differential equation

$$\frac{d}{dt}x = f(t, x) \tag{17}$$

where $f \in \bar{J}_{ab}$ and $\bar{J}_{ab}$ is the set of linear strongly bounded operators, for every operators f there exists a function $\tau \in L([a, b]; E_+)$ such that $|f(\nu)(t)| \leq \tau(t)\|\nu\|_G$, $t \in [a, b]$ and $\nu \in G([a, b]; E)$, and there prevail $f_0, f_1 \in \varphi_{ab}$, where φ_{ab} is a set of linear operators $f \in \bar{J}_{ab}$ from the set $G([a, b]; E_+)$ to the set $L([a, b]; E_+)$, such that

$$\begin{aligned}&|\underline{f}(t, \underline{\nu}, \overline{\nu}) + \underline{f}_1(t, \underline{\nu}, \overline{\nu})| \leq \underline{f}_0(t, |\underline{\nu}|, |\overline{\nu}|), \quad t \in [a, b] \\ &|\overline{f}(t, \underline{\nu}, \overline{\nu}) + \overline{f}_1(t, \underline{\nu}, \overline{\nu})| \leq \overline{f}_0(t, |\underline{\nu}|, |\overline{\nu}|), \quad t \in [a, b]\end{aligned} \tag{18}$$

then (15) has an unique solution.

Proof: If x is a solution of (17) and $-\frac{1}{2}f_1 \in J_{ab}(a)$,

$$\frac{d}{dt}\beta = -\frac{1}{2}f_1(t, \beta) \oplus f_0(t, |x|) \oplus \frac{1}{2}f_1(t, |x|) \tag{19}$$

contains a unique solution β. Moreover, as $f_0, f_1 \in \varphi_{ab}$

$$\begin{aligned}&\underline{\beta}(t) \geq 0, \quad t \in [a, b] \\ &\overline{\beta}(t) \geq 0, \quad t \in [a, b]\end{aligned} \tag{20}$$

According to (18) and the condition $f_1 \in \varphi_{ab}$, from (19) we have

$$\begin{aligned}&\frac{d}{dt}\underline{\beta} \geq -\frac{1}{2}\underline{f}_1(t, \underline{\beta}, \overline{\beta}) + \underline{f}(t, \underline{x}, \overline{x}) + \frac{1}{2}\underline{f}_1(t, \underline{x}, \overline{x}) \\ &\frac{d}{dt}\overline{\beta} \geq -\frac{1}{2}\overline{f}_1(t, \underline{\beta}, \overline{\beta}) + \overline{f}(t, \underline{x}, \overline{x}) + \frac{1}{2}\overline{f}_1(t, \underline{x}, \overline{x})\end{aligned}$$

thus $t \in [a, b]$

$$\begin{aligned}&\frac{d}{dt}(-\underline{\beta}) \leq -\frac{1}{2}\underline{f}_1(t, -\underline{\beta}, -\overline{\beta}) + \underline{k}(t, \underline{x}, \overline{x}) + \frac{1}{2}\underline{k}_1(t, \underline{x}, \overline{x}) \\ &\frac{d}{dt}(-\overline{\beta}) \leq -\frac{1}{2}\overline{f}_1(t, -\underline{\beta}, -\overline{\beta}) + \overline{f}(t, \underline{x}, \overline{x}) + \frac{1}{2}\overline{f}_1(t, \underline{x}, \overline{x})\end{aligned}$$

The last two inequalities are due to the presumption $-\frac{1}{2}f_1 \in J_{ab}(a)$

$$\begin{aligned}&|\underline{x}(t)| \leq \underline{\beta}(t) \quad t \in [a, b] \\ &|\overline{x}(t)| \leq \overline{\beta}(t) \quad t \in [a, b]\end{aligned} \tag{21}$$

According to (21) and the conditions $f_0, f_1 \in \varphi_{ab}$, (19) results in

$$\begin{aligned}&\frac{d}{dt}\underline{\beta} \leq \underline{f}_0(t, \underline{\beta}, \overline{\beta}), \quad t \in [a, b] \\ &\frac{d}{dt}\overline{\beta} \leq f_0(t, \underline{\beta}, \overline{\beta}), \quad t \in [a, b]\end{aligned}$$

As $f_0 \in J_{ab}(a)$, the last inequality with $\beta(a) = 0$ yields $\underline{\beta}(t) \leq 0$ and $\overline{\beta}(t) \leq 0$ for $t \in [a, b]$. (20) implies $\beta \equiv 0$. Thus, based on (21) we have $x \equiv 0$. ∎

In fact, the nonlinear system can be modeled by the neural network directly. However, this data-driven black box identification method does not use the model information. Conversely, the fuzzy differential equation uses the model information of the nonlinear system, such as the brief form of the differential equation.

III. SOLVING FUZZY DIFFERENTIAL EQUATION WITH NEURAL NETWORKS

In general, it is difficult to solve the four equations (13) or (2). In this paper, we use a special neural network, the Bernstein neural network, to approximate the solutions of the fuzzy differential equation (2).

The Bernstein neural network use the following Bernstein polynomial,

$$B(x_1, x_2) = \sum_{i=0}^{N} \sum_{j=0}^{M} (Ni) \binom{M}{j} W_{i,j} x_{1i} (T - x_{1i})^{N-i} x_{2j} (1 - x_{2j})^{M-j} \quad (22)$$

where $\binom{N}{i} = \frac{N!}{i!(N-i)!}$, $\binom{M}{j} = \frac{M!}{j!(M-j)!}$, $W_{i,j}$ is the coefficient.

This two-variables polynomial can be regarded as a neural network, which has two inputs x_{1i} and x_{2j}, and one output y,

$$y = \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_i \gamma_j W_{i,j} x_{1i} (T - x_{1i})^{N-i} x_{2j} (1 - x_{2j})^{M-j} \quad (23)$$

where $\lambda_i = \binom{N}{i}$, $\gamma_j = \binom{M}{j}$.

Because the Bernstein neural network (23) has similar forms as (13), we use the Bernstein neural network (23) to approximate the solutions of four ODEs in (13).

If x_1 and x_2 in the Bernstein polynomial are defined as the time interval t and the α-level, respectively, the solution of (2) in the form of the Bernstein neural network is

$$x_m(t, \alpha) = x_m(0, \alpha) \oplus t \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_i \gamma_j W_{i,j} t_i (T - t_i)^{N-i} \times \alpha_j (1 - \alpha_j)^{M-j} \quad (24)$$

where $x_m(0, \alpha)$ is the initial condition of the solution.

Thus, the derivative of (23) is

$$\begin{aligned} &1) \begin{cases} \frac{d}{dt} \underline{x}_m = C_1 + C_2 \\ \frac{d}{dt} \bar{x}_m = D_1 + D_2 \end{cases} \\ &2) \begin{cases} \frac{d}{dt} \underline{x}_m = C_1 + C_2 \\ \frac{d}{dt} \bar{x}_m = D_1 + D_2 \end{cases} \end{aligned} \quad (25)$$

where $t \in [0, T]$, $\alpha \in [0, 1]$, $t_k = kh$, $h = \frac{T}{k}$, $k = 1, \ldots, N$, $\alpha_j = jh_1$, $h_1 = \frac{1}{M}$, $j = 1, \ldots, M$,

$$C_1 = \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_i \gamma_j \underline{W}_{i,j} t_i (T - t_i)^{N-i} \alpha_j (1 - \alpha_j)^{M-j}$$

$$D_1 = \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_i \gamma_j \overline{W}_{i,j} t_i (T - t_i)^{N-i} \alpha_j (1 - \alpha_j)^{M-j}$$

$$C_2 = t_k \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_i \gamma_j \underline{W}_{i,j} [i t_{i-1,j} (T - t_i)^{N-i} - (N - i)\, t_{i,j} (T - t_i)^{N-i-1}] \alpha_j^i (1 - \alpha_j)^{M-j}$$

$$D_2 = t_k \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_i \gamma_j \overline{W}_{i,j} [i t_{i-1,j} (T - t_i)^{N-i} - (N - i)\, t_{i,j} (T - t_i)^{N-i-1}] \alpha_j^i (1 - \alpha_j)^{M-j}$$

The above equations can be regarded as the neural network form, as shown in Figure 2. The output is

$$N(t, \alpha) = \sum_{i=0}^{N} \sum_{j=0}^{M} (a_{i,j} \lambda_i f_i^1(t) f_i^2(t) \gamma_j g_j^1(\alpha) g_j^2(\alpha)))$$

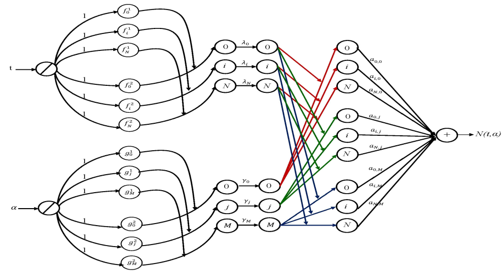

FIGURE 2. Bernstein neural network.

where $f_i^1 = t^i$, $f_i^2 = (T - t)^{N-i}$, $\lambda_i = \frac{1}{T^N} \binom{N}{i}$, $g_j^1 = \alpha^j$, $g_j^2 = (1 - \alpha)^{M-j}$, and $\gamma_j = \binom{M}{j}$.

We define the training errors between (25) and (13) as

$$\begin{aligned} &1) \begin{cases} \underline{e}_1 = C_1 + C_2 - \underline{f} \\ \bar{e}_1 = D_1 + D_2 - \bar{f} \end{cases} \\ &2) \begin{cases} \underline{e}_2 = C_1 + C_2 - \bar{f} \\ \bar{e}_2 = D_1 + D_2 - \underline{f} \end{cases} \end{aligned} \quad (26)$$

The standard back-propagation learning algorithm is utilized to update the weights with the above training errors

$$\underline{W}_{i,j}(k+1) = \underline{W}_{i,j}(k) - \eta_1 \left(\frac{\partial \underline{e}_1^2}{\partial \underline{W}_{i,j}} + \frac{\partial \bar{e}_1^2}{\partial \underline{W}_{i,j}} \right)$$

$$\overline{W}_{i,j}(k+1) = \overline{W}_{i,j}(k) - \eta_2 \left(\frac{\partial \underline{e}_2^2}{\partial \overline{W}_{i,j}} + \frac{\partial \bar{e}_2^2}{\partial \overline{W}_{i,j}} \right) \quad (27)$$

where η_1 and η_2 are positive learning rates.

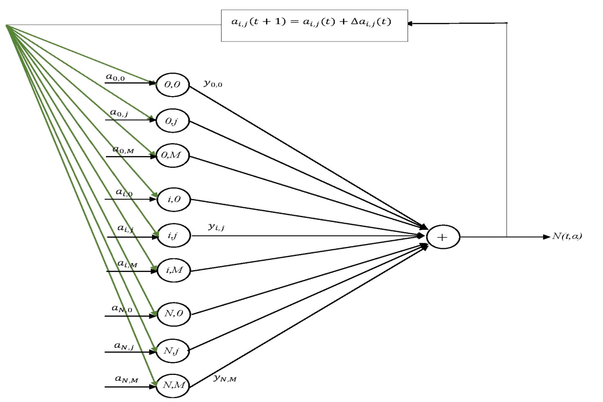

FIGURE 3. A dynamic Bernstein neural network.

The momentum terms, $\gamma \Delta \underline{W}_{i,j}(k-1)$ and $\gamma \Delta \bar{W}_{i,j}(k-1)$, can be added to stabilize the training process. The above Bernstein neural network can be converted to a recurrent (dynamic) form, as shown in Figure 3. The dynamic Bernstein neural network is

$$\begin{cases} \frac{d}{dt} \underline{x}_m(t, \alpha) = \underline{P}(t, \alpha) A(\underline{x}_m(t, \alpha), \bar{x}_m(t, \alpha)) + \underline{Q}(t, \alpha) \\ \frac{d}{dt} \bar{x}_m(t, \alpha) = \overline{P}(t, \alpha) A(\underline{x}_m(t, \alpha), \bar{x}_m(t, \alpha)) + \overline{Q}(t, \alpha) \end{cases} \quad (28)$$

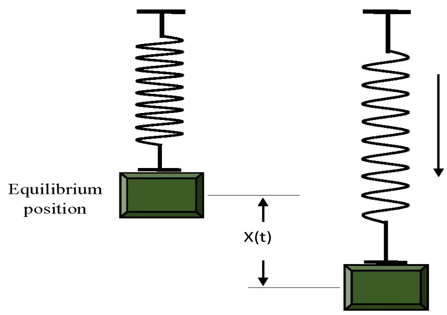

FIGURE 4. Vibration mass.

Obviously, this dynamic network has the form of

$$f(t,x) = P(t)x + Q(t)$$

and it is closed to (2). The training algorithm is similar to (27) and only the training errors are changed as

$$1)\begin{cases}\underline{e}_1 = C_1 + C_2 - \underline{P}A(\underline{x}_m, \bar{x}_m) - \underline{Q} \\ \bar{e}_1 = D_1 + D_2 - \bar{P}A(\underline{x}_m, \bar{x}_m) - \bar{Q}\end{cases}$$

$$2)\begin{cases}\underline{e}_2 = C_1 + C_2 - \bar{P}A(\underline{x}_m, \bar{x}_m) - \bar{Q} \\ \bar{e}_2 = D_1 + D_2 - \underline{P}A(\underline{x}_m, \bar{x}_m) - \underline{Q}\end{cases} \quad (29)$$

IV. APPLICATIONS

In this section, we use several real applications to show how to use fuzzy differential equations (FDEs) and Bernstein neural networks (BNNs) to model the nonlinear systems.

Example 1: The vibration mass system shown in Figure 4 can be modeled by the ordinary differential equation (ODE),

$$\frac{d}{dt}v(t) = \frac{k}{m}x(t), \quad v(t) = \frac{d}{dt}x(t) \quad (30)$$

where the spring constant $k = 1$, and the mass m is changeable in fuzzy number (0.75, 1.125). The ODE (30) becomes the FDE, and $x(t)$ becomes a fuzzy variable. If the initial position is $x(0) = (0.75+0.25\alpha, 1.125-0.125\alpha)$, $\alpha \in [0, 1]$, then the exact solutions of (30) are [27]

$$x(t,\alpha) = \left[(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t\right] \quad (31)$$

where $t \in [0, 1]$. We use the static Bernstein neural network (24), SNN, to approximate the solution (31)

$$\begin{cases}\underline{x}_m(t,\alpha) = (0.75 + 0.25\alpha) \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\underline{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j} \\ \bar{x}_m(t,\alpha) = (1.125 - 0.125\alpha) \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\bar{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j}\end{cases}$$

We also use the dynamic Bernstein neural network (28), DNN, to approximate the solutions. The learning rates are $\eta = 0.01$ and $\gamma = 0.01$. To compare our results, we use the other two popular methods: Max-Min Euler method and Average Euler method [53]. The results are compared in Table 1. Corresponding solution plots are shown in Figure 5.

TABLE 1. Approximation errors.

α	SNN	DNN	Max-Min Euler	Average Euler
0	[0.0601,0.1098]	[0.0207,0.0601]	[0.0934,0.1401]	[0.2054,0.4390]
0.2	[0.0658,0.1067]	[0.0241,0.0612]	[0.0996,0.1370]	[0.1394,0.3761]
0.8	[0.0791,0.0891]	[0.0328,0.0499]	[0.1183,0.1276]	[0.0586,0.1874]
1	[0.0921,0.0921]	[0.0534,0.0534]	[0.1246,0.1246]	[0.1246,0.1246]

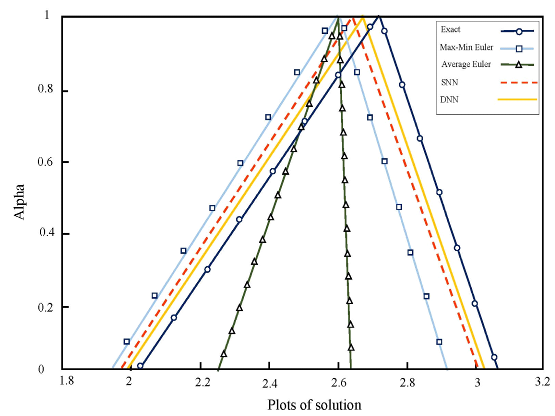

FIGURE 5. Comparison plot of SNN, DNN, Max-Min Euler, Average Euler and the exact solution.

Example 2: The heat treatment system in welding can be modeled as [14]:

$$\frac{d}{dt}x(t) = 3Ax^2(t) \quad (32)$$

where the transfer area A is uncertain as $A = (1+\alpha, 3-\alpha)$, $\alpha \in [0, 1]$. Therefore, (32) is a fuzzy differential equation. The initial condition is $x(0) = (0.5\sqrt{\alpha}, 0.2\sqrt{1-\alpha}+0.5)$. The static Bernstein neural network (24) has the form of

$$\begin{cases}\underline{x}_m(t,\alpha) = 0.5\sqrt{\alpha} \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\underline{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j} \\ \bar{x}_m(t,\alpha) = 0.2\sqrt{1-\alpha} + 0.5 \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\bar{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j}\end{cases}$$

With the learning rates $\eta = 0.002$ and $\gamma = 0.002$. The approximation results are shown in Table 2.

TABLE 2. Approximation errors of BNN.

α	SNN	DNN
0	[0.0511,0.0754]	[0.0224,0.0381]
0.1	[0.0402,0.0623]	[0.0203,0.0362]
0.8	[0.0373,0.0509]	[0.0157,0.0362]
0.9	[0.0401,0.0635]	[0.0202,0.0408]
1	[0.0394,0.0394]	[0.0167,0.0167]

Example 3: A tank system is shown in Figure 6. Assume $I = t + 1$ to be inflow disturbances of the tank, which generates vibration in liquid level x, where $R = 1$ is the flow

TABLE 3. Solutions of different method.

α	SNN	DNN	Neural network
0	[0.0387, 0.0884]	[0.0101, 0.0398]	[0.0701, 0.1012]
0.2	[0.0451, 0.0841]	[0.0225, 0.0575]	[0.0771, 0.1207]
0.8	[0.0544, 0.0635]	[0.0144, 0.0289]	[0.0649, 0.0812]
1	[0.0554, 0.0554]	[0.0311, 0.0311]	[0.0901, 0.0901]

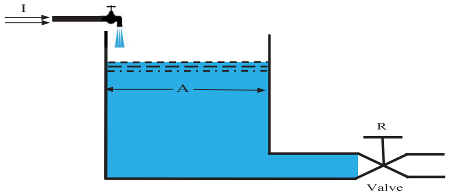

FIGURE 6. Liquid tank system.

obstruction that can be curbed using the valve. $A = 1$ is the cross section of the tank. The liquid level can be described as [51],

$$\frac{d}{dt}x(t) = -\frac{1}{AR}x(t) + \frac{I}{A} \tag{33}$$

The initial condition is $x(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$. The static Bernstein neural network (24) has the form of

$$\begin{cases} \underline{x}_m(t,\alpha) = (0.96 + 0.04\alpha) \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\underline{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j} \\ \overline{x}_m(t,\alpha) = (1.01 - 0.01\alpha) \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\bar{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j} \end{cases}$$

where $t \in [0, 1]$. We also use the dynamic Bernstein neural network (28) to approximate the solutions. To compare our results, we use the other generalization of the neural network method [20]. The comparison results are shown in Table 4. The specifications quoted here are $\eta = 0.001$ and $\gamma = 0.001$. Corresponding errors are shown in Figure 7.

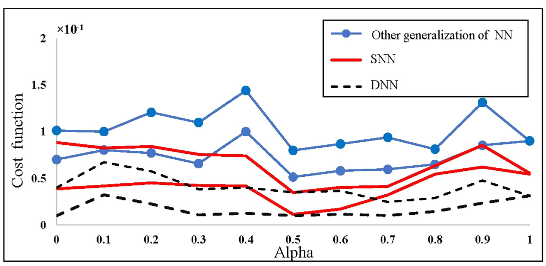

FIGURE 7. Errors between the exact solution and the approximations.

Example 4: A tank with a heating system is shown in Figure 8, where $R = 0.5$ and the thermal capacitance is considered to be $C = 2$. The temperature is x. The model is [46],

$$\frac{d}{dt}x(t) = -\frac{1}{RC}x(t) \tag{34}$$

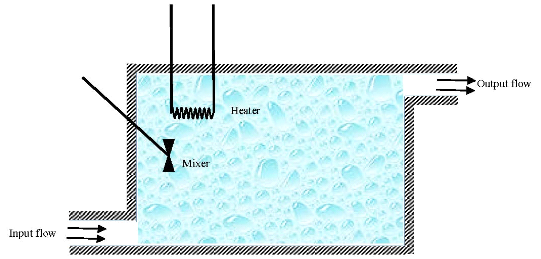

FIGURE 8. Thermal system.

where $t \in [0, 1]$ and x is the amount of sinking in each moment. If the initial position is $u(0) = (\alpha - 1, 1 - \alpha)$ and $\alpha \in [0, 1]$, then the exact solutions of the fuzzy differential equation (34) are

$$x(t,\alpha) = [(\alpha - 1)e^t, (1-\alpha)e^t] \tag{35}$$

We use the static Bernstein neural network (24) to approximate the solution (35)

$$\begin{cases} \underline{x}_m(t,\alpha) = (\alpha - 1) \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\underline{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j} \\ \overline{x}_m(t,\alpha) = (1 - \alpha) \\ \quad +t\sum_{i=0}^{N}\sum_{j=0}^{M}\lambda_i\gamma_j\bar{W}_{i,j}t_i(T-t_i)^{N-i}\alpha_j(1-\alpha_j)^{M-j} \end{cases}$$

where $\eta = 0.001$ and $\gamma = 0.001$. We also use the dynamic Bernstein neural network (28) to approximated the solutions. The errors related to SNN and DNN are illustrated in Table 4.

TABLE 4. NN approximation errors.

α	SNN	DNN
0	[0.0407, 0.0604]	[0.0184, 0.0317]
0.1	[0.0351, 0.0578]	[0.0151, 0.0305]
0.2	[0.0334, 0.0523]	[0.0111, 0.0284]
0.8	[0.0282, 0.0417]	[0.0104, 0.0301]
0.9	[0.0253, 0.0501]	[0.0102, 0.0313]
1	[0.0323, 0.0323]	[0.0112, 0.0112]

TABLE 5. Different NNs.

τ	α=0.2,n=10	α=0.2,n=15	α=0.2,n=20
100	[0.0687, 0.1087]	[0.0585, 0.0901]	[0.0487, 0.0831]
300	[0.0404, 0.0814]	[0.0392, 0.0789]	[0.0334, 0.0613]
τ	α=0.5,n=10	α=0.5,n=15	α=0.5,n=20
100	[0.0545, 0.0957]	[0.0416, 0.0852]	[0.0352, 0.0683]
300	[0.0390, 0.0611]	[0.0291, 0.0581]	[0.0267, 0.0411]
τ	α=0.8,n=10	α=0.8,n=15	α=0.8,n=20
100	[0.0389, 0.0855]	[0.0311, 0.0748]	[0.0219, 0.0533]
300	[0.0308, 0.0552]	[0.0206, 0.0498]	[0.0192, 0.0317]

For different number of learning steps $\tau = 100$, $\tau = 200$, and $\tau = 300$, and hidden neurons $n = 10$, $n = 15$, and $n = 20$, the results are shown in Table 5 and Figure 9.

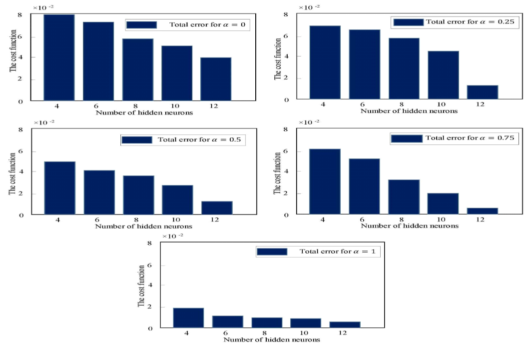

FIGURE 9. Different neural elements for $\tau = 300$.

Both static neural network and dynamic neural network are suitable for solving the fuzzy differential equation. The leaning process of the dynamic Bernstein neural network (28) is faster than the static Bernstein neural network (24). The robustness of (24) is better than (28), because the weights of the dynamic Bernstein neural network are difficult to converge.

V. CONCLUSIONS

In this paper, we use fuzzy differential equations (FDEs) to model unknown nonlinear systems. The existence conditions of FDEs are given. Since the solutions of the fuzzy differential equations are difficult to obtain, we use static and dynamic Bernstein neural networks to approximate the solutions. We first transform the FDEs into four ODEs with Hukuhara differentiability. Then, we construct neural models with the structure of the ODEs. With a modified backpropagation method for the fuzzy variables, the neural networks are trained. Some real examples are given to show the effectiveness of our methods. Future works will involve the application of these methods to fuzzy partial differential equations.

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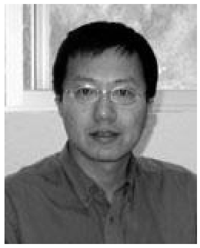

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