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Decompositions of some Specht modules I

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Abstract

We give a decomposition, as a direct sum of indecomposable modules, of several types of Specht modules in characteristic 2. These include the Specht modules labelled by hooks, whose decomposability was considered by Murphy, [15]. Since the main arguments are essentially no more difficult for Hecke algebras at parameter $q = -1$, we proceed in this generality.

1 Introduction

Let K be a field and r a positive integer. We write $\mathrm{Sym}(r)$ for the symmetric group of degree r. For each partition λ of r we have the Specht module $\mathrm{Sp}(\lambda)$ and for each composition α of r the permutation module $M(\alpha)$. The Specht module $\mathrm{Sp}(\lambda)$ may be viewed as a submodule of $M(\lambda)$. James proved, [12, 13.17], that unless the characteristic of K is 2 and λ is 2-singular, the space of homomorphisms $\mathrm{Hom}_{\mathrm{Sym}(r)}(\mathrm{Sp}(\lambda), M(\lambda))$ is one dimensional. It follows that $\mathrm{Sp}(\lambda)$ has one dimensional endomorphism algebra and in particular that $\mathrm{Sp}(\lambda)$ is indecomposable (unless K has characteristic 2 and λ is 2-singular).

We now suppose K has characteristic 2. Then, for λ a 2-singular partition, the Specht module $\mathrm{Sp}(\lambda)$ may certainly decompose but in general neither a criterion for decomposability nor the nature of a decomposition as a direct sum of indecomposable components is known. The first example of such a module, discovered by James, [13], is the Specht module $\mathrm{Sp}(5, 1, 1)$ for the symmetric group $\mathrm{Sym}(7)$. Some years later G. Murphy generalised James' example and in [15] gave a criterion for the decomposability of Specht modules labelled by hook partitions, i.e. partitions of the form $\lambda = (a, 1^b)$. More recently, Dodge and Fayers found in [4] some new decomposable Specht modules for partitions of the form $\lambda = (a, 3, 1^b)$.

In the more general context of the Hecke algebras $\mathrm{Hec}(r)$ with parameter $q \neq -1$, Dipper and James showed in [3] that the corresponding Specht modules are indecomposable. Recently Speyer generalised Murphy's criterion regarding the decomposability of Specht modules labeled by hook partitions for Hecke algebras with $q = -1$, see [17].

Our motivation comes from the 2-modular representation theory of symmetric groups, which is covered by taking q equal to -1, and hence 1, in a field of characteristic 2. We here obtain many new families of decomposable Specht modules for Hecke algebras at parameter $q = -1$ and describe explicitly their indecomposable components. More precisely, we give a decomposition of the Specht modules $\mathrm{Sp}(a, m-1, m-2, \ldots, 2, 1^b)$, with $a \geq m$, $b \geq 1$ and $a-m$ even and b odd. Moreover, we show that there is no uniform bound on the number of indecomposable summands in such a decomposition. We also point out that the decomposition of $\mathrm{Sp}(a, m-1, m-2, \ldots, 2, 1^b)$ lays the foundations for the discovery of many other families of decomposable Specht modules. In fact, using this approach we describe decompositions of Specht modules of the form $\mathrm{Sp}(a, 3, 1^b)$ which do not appear in the list produced by Dodge and Fayers. More results in this direction will appear in a follow up paper, [10].

Our method is to compare the situation with an analogous problem for certain modules for the general linear groups and apply the Schur functor, as in [7, Section 2.1]. The key feature which we are able to exploit at $q = -1$ is that the Schur functor on a tensor product of symmetric and exterior powers of the natural module is the same as on a tensor product of the corresponding symmetric powers only (see Section 6).

Section 2 is devoted to preliminaries on rational and polynomial representations, the associated combinatorics, and connections with the Hecke algebra. In Section 3 we give an explicit description of certain weight multiplicities in certain simple polynomial modules. This is used in Section 4 to decompose certain tensor products of symmetric powers of the natural module from which the corresponding decomposition for certain q permutation modules is deduced. In Section 5 we use some operators on the ring of symmetric polynomials to identify certain induced modules as block components of a certain tensor product of symmetric and exterior powers of the natural module. In Section 6 we use this information to obtain, via the Schur functor, our main result on the decomposition of Specht modules labelled by a family of partitions which include the hook partitions. In Section 7 we illustrate that our methods may also be used beyond these cases by decomposing Specht modules labelled by certain partitions of the form $(a, 3, 1^b)$.

2 Preliminaries

2.1 Combinatorics

The standard reference for the polynomial representations of general linear groups is the monograph [11]. Though we work in the quantised context this reference is appropriate as the combinatorics is essentially the same and so we adopt the notation of [11] wherever convenient. Further details may also be found in the monograph [7], which treats the quantised case.

We begin by introducing some of the associated combinatorics. By a partition we mean an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\lambda_j = 0$ for j sufficiently large. If n is a positive integer such that $\lambda_j = 0$ for $j > n$ we identify λ with the finite sequence $(\lambda_1, \ldots, \lambda_n)$. The length $\mathrm{len}(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is 0 if $\lambda = 0$ and is the positive integer n such that $\lambda_n \neq 0$, $\lambda_{n+1} = 0$, if $\lambda \neq 0$. For a partition λ, we denote by λ' the transpose partition of λ. We define the degree of $\lambda = (\lambda_1, \lambda_2, \ldots)$ by $\deg(\lambda) = \lambda_1 + \lambda_2 + \cdots$.

We fix a positive integer n. We set $X(n) = \mathbb{Z}^n$. There is a natural partial order on $X(n)$. For $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in X(n)$, we write $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $i = 1, 2, \ldots, n-1$ and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$. We shall use the standard $\mathbb{Z}$-basis $\epsilon(1), \ldots, \epsilon(n)$ of $X(n)$, where $\epsilon(i) = (0, \ldots, 1, \ldots, 0)$ (with 1 in the ith position), for $1 \leq i \leq n$. We write $\omega(i)$ for the element $\epsilon(1) + \cdots + \epsilon(i)$ of $X(n)$, for $1 \leq i \leq n$.

We write $X^+(n)$ for the set of dominant n-tuples of integers, i.e., the set of elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$. We write $\Lambda(n)$ for the set of n-tuples of nonnegative integers and $\Lambda^+(n)$ for the set of partitions into at most n-parts, i.e., $\Lambda^+(n) = X^+(n) \bigcap \Lambda(n)$. We shall sometimes refer to elements of $\Lambda(n)$ as polynomial weights and to elements of $\Lambda^+(n)$ as polynomial dominant weights. For a nonnegative integer r we write $\Lambda^+(n, r)$ for the set of partitions of r into at most n parts, i.e., the set of elements of $\Lambda^+(n)$ of degree r.

We fix a positive integer l. We write $X_1(n)$ for the set of l-restricted partitions into at most n parts, i.e., the set of elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n)$ such that $0 \leq \lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n < l$.

A dominant weight $\lambda \in X^+(n)$ has a unique expression $\lambda = \lambda^0 + l\bar{\lambda}$ with $\lambda^0 \in X_1(n)$, $\bar{\lambda} \in X^+(n)$, moreover if $\lambda \in \Lambda^+(n)$ then $\bar{\lambda} \in \Lambda^+(n)$. We call this the standard expansion for λ. It will be used a great deal in what follows.

2.2 Rational Modules and Polynomial Modules

Let K be a field. If V, W are vector spaces over K, we write $V \otimes W$ for the tensor product $V \otimes_K W$. We shall be working with the representation theory of quantum groups over K. By the category of quantum groups over K we understand the opposite category of the category of Hopf algebras over

K. Less formally we shall use the expression "G is a quantum group" to indicate that we have in mind a Hopf algebra over K which we denote $K[G]$ and call the coordinate algebra of G. We say that $\phi : G \to H$ is a morphism of quantum groups over K to indicate that we have in mind a morphism of Hopf algebras over K, from $K[H]$ to $K[G]$, denoted $\phi^\sharp$ and called the co-morphism of ϕ. We will say H is a quantum subgroup of the quantum group G, over K, to indicate that H is a quantum group with coordinate algebra $K[H] = K[G]/I$, for some Hopf ideal I of $K[G]$, which we call the defining ideal of H. The inclusion morphism $i : H \to G$ is the morphism of quantum groups whose co-morphism $i^\sharp : K[G] \to K[H] = K[G]/I$ is the natural map.

Let G be a quantum group over K. The category of left (resp. right) G-modules is the the category of right (resp. left) $K[G]$-comodules. We write $\mathrm{Mod}(G)$ for the category of left G-modules and $\mathrm{mod}(G)$ for the category of finite dimensional left G-modules. We shall also call a G-module a rational G-module (by analogy with the representation theory of algebraic groups). A G-module will mean a left G-module unless indicated otherwise. For a finite dimensional G-module V and a non-negative integer d we write $V^{\otimes d}$ for the d-fold tensor product $V \otimes \cdots \otimes V$ and we write $V^{(d)}$ for the direct sum $V \oplus \cdots \oplus V$ of d copies of V.

Let V be a finite dimensional G-module with structure map $\tau : V \to V \otimes K[G]$. The coefficient space $\mathrm{cf}(V)$ of V is the subspace of $K[G]$ spanned by the "coefficient elements" f_{ij}, $1 \le i, j \le m$, defined with respect to a basis $v_1, \dots, v_m$ of V, by the equations

$$\tau(v_i) = \sum_{j=1}^{m} v_j \otimes f_{ji}$$

for $1 \le i \le m$. The coefficient space $\mathrm{cf}(V)$ is independent of the choice of basis and is a subcoalgebra of $K[G]$.

We fix a positive integer n. We shall be working with $G(n)$, the quantum general linear group of degree n, as in [7]. We fix a non-zero element q of K. We have a K-bialgebra $A(n)$ given by generators c_{ij}, $1 \le i, j \le n$, subject to certain relations (depending on q) as in [7, 0.20]. The comultiplication map $\delta : A(n) \to A(n) \otimes A(n)$ satisfies $\delta(c_{ij}) = \sum_{r=1}^{n} c_{ir} \otimes c_{rj}$ and the augmentation map $\epsilon : A(n) \to K$ satisfies $\epsilon(c_{ij}) = \delta_{ij}$ (the Kronecker delta), for $1 \le i, j \le n$. The elements c_{ij} will be called the coordinate elements and we define the determinant element

$$d_n = \sum_{\pi \in \mathrm{Sym}(n)} \mathrm{sgn}(\pi) c_{1,\pi(1)} \dots c_{n,\pi(n)}.$$

Here $\mathrm{sgn}(\pi)$ denotes the sign of the permutation π. We form the Ore localisation $A(n)_{d_n}$. The comultiplication map $A(n) \to A(n) \otimes A(n)$ and augmentation map $A(n) \to K$ extend uniquely to K-algebra maps $A(n)_{d_n} \to$

$A(n)_{d_n} \otimes A(n)_{d_n}$ and $A(n)_{d_n} \to K$, giving $A(n)_{d_n}$ the structure of a Hopf algebra. By the quantum general linear group $G(n)$ we mean the quantum group over K with coordinate algebra $K[G(n)] = A(n)_{d_n}$.

We write $T(n)$ for the quantum subgroup of $G(n)$ with defining ideal generated by all c_{ij} with $1 \leq i, j \leq n$, $i \neq j$. We write $B(n)$ for the quantum subgroup of $G(n)$ with defining ideal generated by all c_{ij} with $1 \leq i < j \leq n$. We call $T(n)$ a maximal torus and $B(n)$ a Borel subgroup of $G(n)$ (by analogy with the classical case).

We now assign to a finite dimension rational $T(n)$-module its formal character. We form the integral group ring $\mathbb{Z}X(n)$. This has $\mathbb{Z}$-basis of formal exponentials e^λ, which multiply according to the rule $e^\lambda e^\mu = e^{\lambda+\mu}$, $\lambda, \mu \in X(n)$. For $1 \leq i \leq n$ we define $\bar{c}_{ii} = c_{ii} + I_{T(n)} \in K[T(n)]$, where $I_{T(n)}$ is the defining ideal of the quantum subgroup $T(n)$ of $G(n)$. Note that $\bar{c}_{11} \ldots \bar{c}_{nn} = d_n + I_{T(n)}$, in particular each $\bar{c}_{ii}$ is invertible in $K[T(n)]$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ we define $\bar{c}^\lambda = \bar{c}_{11}^{\lambda_1} \ldots \bar{c}_{nn}^{\lambda_n}$. The elements $\bar{c}^\lambda$, $\lambda \in X(n)$, are group-like and form a K-basis of $K[T(n)]$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$, we write K_λ for K regarded as a (one dimensional) $T(n)$-module with structure map $\tau : K_\lambda \to K_\lambda \otimes K[T(n)]$ given by $\tau(v) = v \otimes \bar{c}^\lambda$, $v \in K_\lambda$. For a finite dimensional rational $T(n)$-module V with structure map $\tau : V \to V \otimes K[T(n)]$ and $\lambda \in X(n)$ we have the weight space

$$V^\lambda = \{v \in V \mid \tau(v) = v \otimes \bar{c}^\lambda\}.$$

Moreover, we have the weight space decomposition $V = \bigoplus_{\lambda \in X(n)} V^\lambda$. We say that $\lambda \in X(n)$ is a weight of V if $V^\lambda \neq 0$. The dimension of a finite dimensional vector space V over K will be denoted by $\dim V$. The character $\operatorname{ch} V$ of a finite dimensional rational $T(n)$-module V is the element of $\mathbb{Z}X(n)$ defined by $\operatorname{ch} V = \sum_{\lambda \in X(n)} \dim V^\lambda e^\lambda$.

For each $\lambda \in X^+(n)$ there is an irreducible rational $G(n)$-module $L_n(\lambda)$ which has unique highest weight λ, and λ occurs as a weight with multiplicity one. The modules $L_n(\lambda)$, $\lambda \in X^+(n)$, form a complete set of pairwise non-isomorphic irreducible rational G-modules. For a finite dimensional rational $G(n)$-module V and $\lambda \in X^+(n)$ we write $[V : L_n(\lambda)]$ for the multiplicity of $L_n(\lambda)$ as a composition factor of V.

We write D_n for the one dimensional $G(n)$-module corresponding to the determinant. Thus D_n has structure map $\tau : D_n \to D_n \otimes K[G]$, given by $\tau(v) = v \otimes d_n$, for $v \in D_n$. We have $D_n = L_n(\omega(n)) = L_n(1, 1, \ldots, 1)$. We write E_n for the natural $G(n)$-module. Thus E_n has basis $e_1, \ldots, e_n$, and the structure map $\tau : E_n \to E_n \otimes K[G(n)]$ is given by $\tau(e_i) = \sum_{j=1}^n e_j \otimes c_{ji}$. We also have that $E_n = L_n(1, 0, \ldots, 0)$.

A finite dimensional $G(n)$-module V is called polynomial if $\operatorname{cf}(V) \leq A(n)$. The modules $L_n(\lambda)$, $\lambda \in \Lambda^+(n)$, form a complete set of pairwise non-isomorphic irreducible polynomial $G(n)$-modules. We have a grading $A(n) = \bigoplus_{r=0}^\infty A(n, r)$ in such a way that each c_{ij} has degree 1. Moreover each $A(n,r)$ is a finite dimensional subcoalgebra of $A(n)$. The dual algebra $S(n,r)$ is known as the q-Schur algebra. A finite dimensional $G(n)$-module V is polynomial of degree r if $\mathrm{cf}(V) \leq A(n,r)$. We write $\mathrm{pol}(n)$ (resp. $\mathrm{pol}(n,r)$) for the full subcategory of $\mathrm{mod}(G(n))$ whose objects are the polynomial modules (resp. the modules which are polynomial of degree r).

An arbitrary finite dimensional polynomial $G(n)$-module V may be written uniquely as $V = \bigoplus_{r=0}^{\infty} V(r)$ in such a way that $V(r)$ is polynomial of degree r, for $r \geq 0$. Let $r \geq 0$. The modules $L_n(\lambda)$, $\lambda \in \Lambda^+(n,r)$, form a complete set of pairwise non-isomorphic irreducible polynomial $G(n)$-modules which are polynomial of degree r. We write $\mathrm{mod}(S)$ for the category of left modules for a finite dimensional K-algebra S. The category $\mathrm{pol}(n,r)$ is naturally equivalent to the category $\mathrm{mod}(S(n,r))$.

We shall also need modules induced from $B(n)$ to $G(n)$. (For details of the induction functor $\mathrm{Mod}(B(n)) \to \mathrm{Mod}(G(n))$ see, for example, [6].) For $\lambda \in X(n)$ there is a unique (up to isomorphism) one dimensional $B(n)$-module whose restriction to $T(n)$ is K_λ. We also denote this module by K_λ. The induced module $\mathrm{ind}_{B(n)}^{G(n)} K_\lambda$ is non-zero if and only if $\lambda \in X^+(n)$. For $\lambda \in X^+(n)$ we set $\nabla_n(\lambda) = \mathrm{ind}_{B(n)}^{G(n)} K_\lambda$. Then $\nabla_n(\lambda)$ is finite dimensional and its character is given by Weyl's character formula and it is the Schur symmetric polynomial corresponding λ for $\lambda \in \Lambda^+(n)$. The $G(n)$-module socle of $\nabla_n(\lambda)$ is $L_n(\lambda)$. The module $\nabla_n(\lambda)$ has unique highest weight λ and this weight occurs with multiplicity one.

A filtration $0 = V_0 \leq V_1 \leq \cdots \leq V_r = V$ of a finite dimensional rational $G(n)$-module V is said to be *good* if for each $1 \leq i \leq r$ the quotient V_i/V_{i-1} is either zero or isomorphic to $\nabla_n(\lambda^i)$ for some $\lambda^i \in X^+(n)$. For a rational $G(n)$-module V admitting a good filtration for each $\lambda \in X^+(n)$, the multiplicity $|\{1 \leq i \leq r \mid V_i/V_{i-1} \cong \nabla_n(\lambda)\}|$ is independent of the choice of the good filtration, and will be denoted $(V : \nabla_n(\lambda))$.

For $\lambda, \mu \in X^+(n)$ we have $\mathrm{Ext}^1_{G(n)}(\nabla_n(\lambda), \nabla_n(\mu)) = 0$ unless $\lambda > \mu$. Given Kempf's Vanishing Theorem, [7, Theorem 3.4], this follows exactly as in the classical case, e.g., [5, Lemma 3.2.1], (or the original source [2, Corollary (3.2)]). It follows that if V has a good filtration $0 = V_0 \leq V_1 \leq \cdots \leq V_t = V$ with sections $V_i/V_{i-1} \cong \nabla_n(\lambda_i)$, $1 \leq i \leq t$, and $\mu_1, \ldots, \mu_t$ is a reordering of the $\lambda_1, \ldots, \lambda_t$ such that $\mu_i < \mu_j$ implies that $i < j$ then there is a good filtration $0 = V_0' < V_1' < \cdots < V_t' = V$ with $V_i'/V_{i-1}' \cong \nabla_n(\mu_i)$, for $1 \leq i \leq t$.

Similarly it will be of great practical use to know that
$\mathrm{Ext}^1_{G(n)}(\nabla_n(\lambda), \nabla_n(\mu)) = 0$ when λ and μ belong to different blocks. Here the relationship with cores of partitions diagrams will be crucial for us. For a partition λ we denote by $[\lambda]$ the corresponding partition diagram (as in [11]). The l-core of $[\lambda]$ is the diagram obtained by removing skew l-hooks, as in [12]. If $\lambda, \mu \in \Lambda^+(n,r)$ and $[\lambda]$ and $[\mu]$ have different l-cores then the simple modules $L_n(\lambda)$ and $L_n(\mu)$ belong to different blocks and it follows in

particular that $\operatorname{Ext}^i_{S(n,r)}(\nabla(\lambda), \nabla(\mu)) = 0$, for all $i \geq 0$. A precise description of the blocks of the q-Schur algebras was found by Cox, see [1, Theorem 5.3]. For a polynomial $G(n)$-module V and an l-core γ we mean by the expression, the component of V corresponding to γ, the sum of all block components for blocks consisting of partitions with core γ.

For $\lambda \in \Lambda^+(n, r)$ we write $I_n(\lambda)$ for the injective envelope of $L_n(\lambda)$ in the category of polynomial modules. Then $I_n(\lambda)$ is a finite dimensional module which is polynomial of degree r. Moreover, the module $I_n(\lambda)$ has a good filtration and we have the reciprocity formula $(I_n(\lambda) : \nabla_n(\mu)) = [\nabla_n(\mu) : L_n(\lambda)]$ see e.g., [6, Section 4, (6)].

2.3 The Frobenius Morphism

It will be important for us to make a comparison with the classical case $q = 1$. In this case we will write $\dot{G}(n)$ for $G(n)$ and write x_{ij} for the coordinate element c_{ij}, $1 \leq i, j \leq n$. Also, we write $\dot{L}_n(\lambda)$ for the $\dot{G}(n)$-module $L_n(\lambda)$, $\lambda \in X^+(n)$, and write $\dot{E}_n$ for E_n.

We return to the general situation. If q is not a root of unity, or if K has characteristic 0 and $q = 1$ then all $G(n)$-modules are completely reducible, see e.g., [6, Section 4, (8)]. We therefore assume, from now on that q, is a root of unity and that if K has characteristic 0 then $q \neq 1$. We denote by l the smallest positive integer such that $1 + q + \cdots + q^{l-1} = 0$.

We have a morphism of Hopf algebras $\theta : K[\dot{G}(n)] \to K[G(n)]$ given by $\theta(x_{ij}) = c^l_{ij}$, for $1 \leq i, j \leq n$. We write $F : G(n) \to \dot{G}(n)$ for the morphism of quantum groups such that $F^\sharp = \theta$. Given a $\dot{G}(n)$-module V we write V^F for the corresponding $G(n)$-module. Thus, V^F as a vector space is V and if the $\dot{G}(n)$-module V has structure map $\tau : V \to V \otimes K[\dot{G}(n)]$ then V^F has structure map $(\mathrm{id}_V \otimes F) \circ \tau : V^F \to V^F \otimes K[G(n)]$, where $\mathrm{id}_V : V \to V$ is the identity map on the vector space V.

For an element $\phi = \sum_{\xi \in X(n)} a_\xi e^\xi$ of $\mathbb{Z}X(n)$ we write ϕ^F for the element $\sum_{\xi \in X(n)} a_\xi e^{l\xi}$. For a finite dimensional $\dot{G}(n)$-module V we have $\operatorname{ch} V^F = (\operatorname{ch} V)^F$. Moreover, we have the following relationship between the irreducible modules for $G(n)$ and $\dot{G}(n)$, see [7, Section 3.2, (5)].

Steinberg's Tensor Product Theorem *For $\lambda^0 \in X_1(n)$ and $\bar{\lambda} \in X^+(n)$ we have*

$$L_n(\lambda^0 + l\bar{\lambda}) \cong L_n(\lambda^0) \otimes \dot{L}_n(\bar{\lambda})^F.$$

Usually we shall abbreviate the quantum groups $G(n)$, $B(n)$, $T(n)$ to G, B, T and $\dot{G}(n)$ to $\dot{G}$. Likewise, we usually abbreviate the modules $L_n(\lambda)$, $\nabla_n(\lambda)$, $I_n(\lambda)$ and $\dot{L}_n(\lambda)$ to $L(\lambda)$, $\nabla(\lambda)$, $I(\lambda)$ and $\dot{L}(\lambda)$, for $\lambda \in \Lambda^+(n)$, and abbreviate the modules E_n and D_n to E and D.

2.4 A truncation functor

Let N, n be positive integers with $N \geq n$. We identify $G(n)$ with the quantum subgroup of $G(N)$ whose defining ideal is generated by all $c_{ii} - 1$, $n < i \leq N$, and all c_{ij} with $1 \leq i \neq j \leq N$ and $i > n$ or $j > n$. We have an exact functor (the truncation functor) $d_{N,n} : \mathrm{pol}(N) \to \mathrm{pol}(n)$ taking $V \in \mathrm{pol}(N)$ to the $G(n)$ submodule $\bigoplus_{\alpha \in \Lambda(n)} V^\alpha$ of V and taking a morphism of polynomial modules $V \to V'$ to its restriction $d_{N,n}(V) \to d_{N,n}(V')$. For a discussion of this functor at the level of modules for Schur algebras in the classical case see [11, Section 6.5].

By [7, Section 4.2], the functor $d_{N,n}$ has the following properties:
(i) for polynomial $G(N)$-modules X, Y we have $d_{N,n}(X \otimes Y) = d_{N,n}(X) \otimes d_{N,n}(Y)$;
(ii) for $\lambda \in \Lambda^+(N, r)$ and $X_\lambda = L_N(\lambda)$ or $\nabla_N(\lambda)$ then $d_{N,n}(X_\lambda) \neq 0$ if and only if $\lambda \in \Lambda^+(n, r)$;
(iii) for $\lambda \in \Lambda^+(n, r)$, $d_{N,n}(L_N(\lambda)) = L_n(\lambda)$ and $d_{N,n}(\nabla_N(\lambda)) = \nabla_n(\lambda)$.

Let $\lambda \in \Lambda^+(N, r)$, for some $r \geq 0$, $\alpha \in \Lambda(N, r)$ and $\lambda_i = \alpha_i = 0$, for $n < i \leq N$. We identify λ and α with elements of $\Lambda^+(n, r)$ and $\Lambda(n, r)$ in the obvious way. It follows that $\dim L_N(\lambda)^\alpha = \dim L_n(\lambda)^\alpha$.

2.5 Connections with the Hecke algebras

We now record some connections with representations of Hecke algebra of type A. We fix a positive integer r. We write $\mathrm{len}(\pi)$ for the length of a permutation π. The Hecke algebra $\mathrm{Hec}(r)$ is the K-algebra with basis T_w, $w \in \mathrm{Sym}(r)$, and multiplication satisfying

$$\begin{aligned} &T_w T_{w'} = T_{ww'}, \qquad \text{if } \mathrm{len}(ww') = \mathrm{len}(w) + \mathrm{len}(w'), \text{and} \\ &(T_s + 1)(T_s - q) = 0 \end{aligned}$$

for $w, w' \in \mathrm{Sym}(r)$ and a basic transposition $s \in \mathrm{Sym}(r)$. For brevity we will denote the Hecke algebra $\mathrm{Hec}(r)$ by $H(r)$. We write K (resp. K_{sgn}) for the one dimensional $H(r)$-module on which each basic transposition T_s acts as multiplication by q (resp. -1).

For λ a partition of degree r we denote by $\mathrm{Sp}(\lambda)$ the corresponding (Dipper-James) Specht module and by $Y(\lambda)$ the corresponding Young module. For $\alpha \in \Lambda(n, r)$ we write $M(\alpha)$ for the permutation module corresponding to α.

Let $n \geq r$. We have the Schur functor $f : \mathrm{mod}(S(n, r)) \to \mathrm{mod}(H(r))$, see [7, Section 2.1]. By [7, Sections 4.4 and 4.5] the functor f has the following properties:
(i) f is exact;
(ii) for $\lambda \in \Lambda^+(n, r)$ we have $f(\nabla(\lambda)) = \mathrm{Sp}(\lambda)$;

(iii) for $\lambda \in \Lambda^+(n,r)$ we have $f(I(\lambda)) = Y(\lambda)$.

For a finite string of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_m)$ we have the polynomial $G(n)$-modules

$$S^\alpha E = S^{\alpha_1} E \otimes \cdots \otimes S^{\alpha_m} E$$

and

$$\bigwedge^\alpha E = \bigwedge^{\alpha_1} E \otimes \cdots \otimes \bigwedge^{\alpha_m} E.$$

For $\alpha \in \Lambda(n,r)$ we write $H(\alpha)$ for the subalgebra $H(\alpha_1) \otimes \cdots \otimes H(\alpha_n)$ of $H(r)$. By [7, Section 2.1, (20)] we have:
(iv) $f(S^\alpha E) = H(r) \otimes_{H(\alpha)} K = M(\alpha)$;
(v) $f(\Lambda^\alpha E) = H(r) \otimes_{H(\alpha)} K_{\mathrm{sgn}}$.

3 Some weight space multiplicities

We shall need some information about weight spaces of simple modules. Our considerations reduce to the case $n = 2$ so we recall the weight space multiplicities in this case. First consider the group $\dot{G}(2)$.

For a prime p and a non-negative integer r we have the base p expansion $r = \sum_{i=0}^{\infty} p^i r_i$ (where $0 \le r_i < p$ for all i and $r_i = 0$ for i large), or just $r = \sum_{i=0}^{N} p^i r_i$, if $r < p^{N+1}$.

Definition 3.1. *Let r, b be integers with $r \ge 0$ and $p \ge 0$. We shall say that the pair (r,b) is p-special if*
(i) $p = 0$, $-r \le b \le r$ and $r - b$ is even.
(ii) p is a prime, r has base p expansion $r = \sum_{i=0}^{\infty} p^i r_i$ and there exists an expression $b = \sum_{i=0}^{\infty} p^i t_i$ with $-r_i \le t_i \le r_i$ and $r_i - t_i$ even for all $i \ge 0$.

Lemma 3.2. *For $(a,b) \in \Lambda^+(2,r)$ and $(c,d) \in \Lambda(2,r)$, we have*

$$\dim \dot{L}(a,b)^{(c,d)} = \begin{cases} 1, & \text{if } (a-b, c-d) \text{ is } p\text{-special;} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. All non-zero weights spaces of $\dot{\nabla}(\lambda)$, and hence $\dot{L}(\lambda)$ have dimension 1, for $\lambda \in \Lambda^+(2)$. If K has characteristic zero then $\dot{L}(\lambda) = \dot{\nabla}(\lambda)$, for $\lambda \in \Lambda^+(2)$. By restricting to the group scheme SL_2 over K, for example, we see that for $\lambda = (a,b)$, $\mu = (c,d)$ the weight space $\dot{L}(\lambda)^\mu$ is non-zero is and only if λ and μ have the same degree, and $(a-b, c-d)$ is 0-special. Similarly, if K has characteristic $p > 0$, then we have $\dot{L}_2(\lambda) = \dot{\nabla}_2(\lambda)$ if $a - b < p$. It follows from the usual form of Steinberg's tensor product theorem, for the group scheme SL_2 over K (see e.g. [14, II,3.17]) that $\dot{L}(\lambda)^\mu \ne 0$ if and only if λ and μ have the same degree and $(a-b, c-d)$ is p-special. $\square$

Remark 3.3. *Recall that if $\lambda \in \Lambda^+(n,r)$, $\alpha \in \Lambda(n,r)$ and $L(\lambda)^\alpha \neq 0$ then $\alpha \leq \lambda$. Moreover if λ has length m and $\alpha_i = 0$ for $i > m$ then*

$$\dim L(\lambda)^\alpha = \dim L_m(\lambda)^\alpha = \dim(D_m \otimes L(\lambda - \omega(m)))^\alpha = \dim L(\lambda - \omega(m))^{\alpha - \omega(m)}.$$

Remark 3.4. *We note that if $\lambda = (a,b) \in \Lambda^+(2)$ is l-restricted then $L(\lambda) = \nabla(\lambda)$, e.g. by block considerations, and hence the weights of $L(\lambda)$ are $(a,b), (a-1,b+1), \ldots, (b,a)$.*

For a positive integer m we write $\delta(m)$ for the partition $(m, m-1, \ldots, 1)$ (of length m) and $\sigma(m)$ for $(l-1)\delta_m$.

Proposition 3.5. *Assume that $n \geq 2$ and $\mu \in \Lambda^+(n)$. Let $m \geq 1$ and $\tau = (u,v)$ with $u, v \geq 0$.*
(i) $\dim L(\sigma(m) + l\mu)^{\sigma(m)+l\tau} = \dim L(\sigma(1) + l\mu)^{\sigma(1)+l\tau}$.
(ii) If $\mu = (c,d)$ and has the same degree as τ, then

$$\dim L(\sigma(m) + l\mu)^{\sigma(m)+l\tau} = \dim \dot{L}(\mu)^\tau = \begin{cases} 1, & \textit{if } (c-d, u-v) \textit{ is } p\textit{-special;} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. (i) First notice that if $L(\sigma(m) + l\mu)^{\sigma(m)+l\tau} \neq 0$ then $\sigma(m) + l\mu \geq \sigma(m) + l\tau$ and so $\mu \geq \tau$. Hence we may assume that $\mathrm{len}(\mu) \leq 2$.
For $m \geq 2$ we have $L(\sigma(m) + l\mu)^{\sigma(m)+l\tau} = L(\sigma(m-1) + l\mu)^{\sigma(m-1)+l\tau}$ by Remark 3.3. Now the result follows immediately by induction on m.
(ii) By (i) we have that $\dim L(\sigma(m) + l\mu)^{\sigma(m)+l\tau} = \dim L(\sigma(1) + l\mu)^{\sigma(1)+l\tau}$. Now $L(\sigma(1) + l\mu) = L(\sigma(1)) \otimes \dot{L}(\mu)^F$. We consider $L(\sigma(1) + l\mu)$ as a $G(2)$-module. By Remark 3.4 a weight of $L(\sigma(1)) \otimes \dot{L}(\mu)^F$ has the form $\alpha + l\beta$, where

$$\alpha \in \{(l-1, 0), (l-2, 1), \ldots, (0, l-1)\}$$

and β is a weight of $\dot{L}(\mu)$ and all non-zero weight spaces are one dimensional. The dimension of the $\sigma(1) + l\tau$ weight space is the number of solutions $\alpha + l\beta$. Since $\sigma(1) + l\tau$ then first entry is -1 modulo l and the only possibility for α is $(l-1, 0)$. Hence we have $\dim L(\sigma(1) + l\mu)^{\sigma(1)+l\tau} = \dim \dot{L}(\mu)^\tau$, and hence the result by Lemma 3.2. □

4 Decompositions of some polynomial modules

We use the weight space calculations of the previous section to give a decomposition of a certain tensor product of symmetric powers.

Suppose that $n \geq r$. Then, for $\alpha \in \Lambda(n,r)$, the module $S^\alpha E$ is injective and we have

$$S^\alpha E = \bigoplus_{\lambda \in \Lambda^+(n,r)} I(\lambda)^{(d_{\lambda\alpha})} \tag{1}$$

where $d_{\lambda\alpha} = \dim L(\lambda)^\alpha$, [7, Section 2.1 (8)].

Applying the Schur functor to (1), for any $\alpha \in \Lambda(n,r)$, we get

$$M(\alpha) = \bigoplus_{\lambda \in \Lambda^+(n,r)} Y(\lambda)^{(d_{\lambda\alpha})}.$$

Proposition 4.1. *Assume $n \geq m \geq 2$. Let $\mu \in \Lambda^+(n)$ and $\tau = (u,v)$ with $u, v \geq 0$. The component of*

$$S^{\sigma(m)+l\tau}E$$

corresponding to the core $\sigma(m)$ is

$$\bigoplus_{\mu} I(\sigma(m) + l\mu)$$

where the sum is over all partitions $\mu = (c,d)$ such that, $c + d = u + v$ and $(c - d, u - v)$ is p-special.

Proof. We note that if, for $\nu \in \Lambda^+(n)$, the module $I(\nu)$ occurs in $S^{\sigma(m)+l\tau}E$ then, by (1) above, $L(\nu)^{\sigma(m)+l\tau} \neq 0$ and so $\nu \geq \sigma(m) + l\tau$. Hence ν has length at most m. It follows that if ν has core $\sigma(m)$ then ν has the form $\sigma(m) + l\mu$, for some $\mu \in \Lambda^+(n)$. The result now follows from (1) above and Proposition 3.5.

□

Applying the Schur functor we then obtain the following.

Corollary 4.2. *Let $\mu \in \Lambda^+(n)$. Let $m \geq 2$ and $\tau = (u,v)$ with $u, v \geq 0$. The component*

$$M(\sigma(m) + l\tau)$$

corresponding to the core $\sigma(m)$ is

$$\bigoplus_{\mu} Y(\sigma(m) + l\mu)$$

where the sum is over all partitions $\mu = (c,d)$ such that $c + d = u + v$ and $(c - d, u - v)$ is p-special.

5 Adapted partitions and symmetric polynomials

For $\lambda \in \Lambda^+(n)$ we write $s(\lambda)$ for the Schur symmetric function corresponding to λ. The elements $s(\lambda)$ of $\mathbb{Z}X(n)$, as λ varies over partitions with at most n parts, form a $\mathbb{Z}$-basis of the ring of symmetric functions $\mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{Sym}(n)}$.

Let $\gamma \in \Lambda^+(n)$ be an l-core. For a polynomial $G(n)$-module V we write $V(\gamma)$ for the component of V corresponding to γ. We write C^*_γ for the endomorphism of the ring of symmetric functions such that

$$C^*_\gamma(s(\lambda)) = \begin{cases} s(\lambda), & \text{if } \lambda \text{ has core } \gamma; \\ 0, & \text{otherwise} \end{cases}$$

for $\lambda \in \Lambda^+(n)$.

Lemma 5.1. *Let $\gamma \in \Lambda^+(n)$ be an l-core. For a finite dimensional polynomial module V we have*

$$\operatorname{ch} V(\gamma) = C^*_\gamma(\operatorname{ch} V).$$

Proof. Since both sides are additive on short exact sequences of $G(n)$-modules, it is enough to check for a set of polynomial modules that generate the Grothendieck group of finite dimensional polynomial $G(n)$-modules. Hence it is enough to check for $V = \nabla(\lambda)$, $\lambda \in \Lambda^+(n)$, and for these modules it is clear from the definition. □

From now on we restrict attention to the case $l = 2$. The cores available are the staircase partitions $\sigma(m) = (m, m-1, \ldots, 1)$, $m \geq 0$ (where $\sigma(0) = 0$). We need to keep track of the part of a symmetric function corresponding to such a partition. To this end we introduce the following notion.

Definition 5.2. *Let m be a non-negative integer and $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. We say that λ is m-adapted if $\lambda_i > m - i$, for all $i \geq 1$ with $\lambda_i > 0$.*

We write C_m for the additive endomorphism of the ring of symmetric functions in n variables such that

$$C_m(s(\lambda)) = \begin{cases} s(\lambda), & \text{if } \lambda \text{ is } m\text{-adapted}; \\ 0, & \text{otherwise} \end{cases}$$

for $\lambda \in \Lambda^+(n)$.

Lemma 5.3. *For a symmetric polynomial g (in n variables) and $0 \leq r \leq n$ we have*

$$C_{m+1}(gs(1^r)) = C_{m+1}(C_m(g)s(1^r)).$$

Proof. It is enough to check this for $g = s(\lambda)$ for a partition λ, with at most n parts. By Pieri's Formula, [16, 5.17], we have

$$s(\lambda)s(1^r) = \sum_\mu s(\mu)$$

where the sum is over all partitions μ with at most n parts, whose Young diagram may be obtained from the Young diagram of λ by adding a box in each of r distinct rows. Hence we have $\mu_i \leq \lambda_i + 1$ for each μ appearing in the above sum.

If λ is not m adapted then $C_m(s(\lambda)) = 0$. Moreover, in this case, we have $\lambda_i \leq m - i$ for some i, and so, for μ appearing in the above sum we have $\mu_i \leq \lambda_i + 1 \leq (m+1) - i$. Hence, μ is not $(m+1)$-adapted and $C_{m+1}(s(\mu)) = 0$. Therefore we get $C_{m+1}(s(\lambda)s(1^r)) = 0$.

Suppose now that λ is m-adapted. Thus we have

$$C_{m+1}(C_m(s(\lambda))s(1^r)) = C_{m+1}(s(\lambda)s(1^r))$$

and we are done. $\square$

Proposition 5.4. *Let $m \geq 2$, $a \geq m$ and $b \geq m-1$. Then, for all n sufficiently large, we have*

$$\begin{aligned}&C_m(s(a)s(1^b)s(1^{m-2})\dots s(1))\\&\quad = s(a+1, m-1, \dots, 2, 1^{b-m+1}) + s(a, m-1, \dots, 2, 1^{b-m+2}).\end{aligned}$$

Proof. We argue by induction on m. First suppose $m = 2$. Then

$$s(a)s(1^b) = s(a+1, 1^{b-1}) + s(a, 1^b)$$

by Pieri's Formula. Both $(a+1, 1^{b-1})$ and $(a, 1^b)$ are 2-adapted, so the result holds in this case.

Now suppose that $m \geq 3$ and the result holds for $m-1$. By Lemma 5.3 and the induction hypothesis we have

$$\begin{aligned}&C_m((s(a)s(1^b)s(1^{m-3})\dots s(1))s(1^{m-2}))\\&\quad = C_m(C_{m-1}(s(a)s(1^b)s(1^{m-3})\dots s(1))s(1^{m-2}))\\&\quad = C_m(s(a+1, m-2, \dots, 2, 1^{b-m+2})s(1^{m-2}))\\&\qquad + C_m(s(a, m-2, \dots, 2, 1^{b-m+3})s(1^{m-2})).\end{aligned}$$

But now, again by Pieri's Formula, for $\lambda = (a+1, m-2, \dots, 2, 1^{b-m+2})$ we have

$$s(\lambda)s(1^{m-2}) = \sum_{\mu} s(\mu)$$

where the sum is over all partitions whose diagram is obtained by adding a box to $m-2$ rows of the Young diagram of λ. But such a diagram is not m-adapted unless the boxes are added to rows 2 up to $m-1$ and in this case we have $\mu = (a+1, m-1, \dots, 2, 1^{b-m+1})$. Hence we obtain

$$C_m(s(a+1, m-2, \dots, 2, 1^{b-m+2})s(1^{m-2})) = s(a+1, m-1, \dots, 2, 1^{b-m+1})$$

and similarly

$$C_m(s(a, m-2, \dots, 2, 1^{b-m+3})s(1^{m-2})) = s(a, m-1, \dots, 2, 1^{b-m+2})$$

so we are done. $\square$

We write C_m^* for $C_{\sigma(m)}^*$. We note that for a symmetric polynomial g contributions to $C_m^*(g)$ can only come from Schur polynomials $s(\lambda)$ with λ an m-adapted partition. So we have $C_m^*(g) = C_m^*(C_m(g))$. Using Proposition 5.4 it is now easy to verify the following.

Corollary 5.5. *Let $m \geq 2$, $a \geq m$ and $b \geq m-1$. Then we have, for all n sufficiently large,*

$$
\begin{aligned}
&C_m^*(s(a)s(1^b)s(1^{m-2})\dots s(1)) \\
&= \begin{cases} s(a+1, m-1, \dots, 2, 1^{b-m+1}), & \textit{if } a-m \textit{ is odd and } b-m \textit{ is even}; \\ s(a, m-1, \dots, 2, 1^{b-m+2}), & \textit{if } a-m \textit{ is even } b-m \textit{ is odd}; \\ 0, & \textit{otherwise.} \end{cases}
\end{aligned}
$$

Now the module

$$S^aE \otimes \textstyle\bigwedge^b E \otimes \bigwedge^{m-2} E \otimes \cdots \otimes \bigwedge^2 E \otimes E$$

has a good filtration, e.g. by [6, Section 4, (3)]. Interpreting Corollary 5.5 in terms of $G(n)$-modules we obtain the following.

Corollary 5.6. *Assume that $m \geq 2$. Let $a \geq m$ and $b \geq m-1$. For all n sufficiently large, the component of the module*

$$S^aE \otimes \textstyle\bigwedge^b E \otimes \bigwedge^{m-2} E \otimes \cdots \otimes \bigwedge^2 E \otimes E$$

corresponding to the core $\sigma(m)$ is:

$$
\begin{array}{ll}
\nabla(a+1, m-1, \dots, 2, 1^{b-m+1}), & \textit{if } a-m \textit{ is odd and } b-m \textit{ is even}; \\
\nabla(a, m-1, \dots, 2, 1^{b-m+2}), & \textit{if } a-m \textit{ is even } b-m \textit{ is odd}; \\
0, & \textit{otherwise.}
\end{array}
$$

We specialise to the following.

Corollary 5.7. *Assume that $m \geq 2$. Let $a \geq m$ and $b \geq m-1$ with $a-m$ even and $b-m$ odd. Then, for all n sufficiently large, the component of the module*

$$S^aE \otimes \textstyle\bigwedge^b E \otimes \bigwedge^{m-2} E \otimes \cdots \otimes \bigwedge^2 E \otimes E$$

corresponding to the core $\sigma(m)$ is

$$\nabla(a, m-1, \dots, 2, 1^{b-m+2}).$$

6 Decomposable Specht modules

We continue to assume that $q = -1$ and so $l = 2$. Let $n, r \geq 0$ with $n \geq r$. Recall that for $\alpha \in \Lambda(n,r)$ we write $H(\alpha)$ for the subalgebra $H(\alpha_1) \otimes \cdots \otimes H(\alpha_n)$ of $H(r)$ and that we have

$$f(S^\alpha E) = H(r) \otimes_{H(\alpha)} K$$

$$f(\Lambda^\alpha E) = H(r) \otimes_{H(\alpha)} K_{\mathrm{sgn}}$$

For finite strings of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_a)$ and $\beta = (\beta_1, \ldots, \beta_b)$ we write $(\alpha \,|\, \beta)$ for the concatenation $(\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b)$. Assume that $a + b \leq n$ and that $\deg(\alpha) = r_1$, $\deg(\beta) = r_2$ with $r = r_1 + r_2$. Then it follows that

$$f(S^\alpha E \otimes \textstyle\bigwedge^\beta E) = H(r) \otimes_{H(\alpha \,|\, \beta)} (K \otimes K_{\mathrm{sgn}}).$$

But since $q = -1$ we have that $K_{\mathrm{sgn}} \cong K$, and so

$$f(S^\alpha E \otimes \textstyle\bigwedge^\beta E) = H(r) \otimes_{H(\alpha \,|\, \beta)} K = M(\alpha \,|\, \beta).$$

Applying the Schur functor to Corollary 5.7 yields the following result.

Corollary 6.1. *Assume that $m \geq 2$. Let $a \geq m$ and $b \geq m-1$ with $a - m$ even and $b - m$ odd. Then the block component of the module $M(a, b, m-2, \ldots, 2, 1)$ corresponding to the core $\sigma(m)$ is*

$$\mathrm{Sp}(a, m-1, \ldots, 2, 1^{b-m+2}).$$

Comparing now Corollary 4.2 with Corollary 6.1 we get our main result.

Theorem 6.2. *Assume that $m \geq 2$. Let $a \geq m$ and $b \geq m-1$ with $a - m = 2u$ even and $b - m = 2v - 1$ odd. Then we have*

$$\mathrm{Sp}(a, m-1, \ldots, 2, 1^{b-m+2}) = \bigoplus_{\mu} Y(\sigma(m) + 2\mu)$$

where the sum is over all partitions $\mu = (c, d)$ such that $c + d = u + v$ and $(c - d, u - v)$ is p-special.

We give now an example of such a decomposition to point out that there is no uniform bound on the number of indecomposable summands.

Example 6.3. *Assume that K has characteristic 2. By Theorem 6.2 and using a simple inductive argument it follows that for $k \geq 1$ we have*

$$\mathrm{Sp}(2^k + 2, 1^{2^k - 1}) = \bigoplus_{j=1}^{k} Y(2^k + 2^j, 2^k - 2^j + 1).$$

Remark 6.4. *We note that in fact we already have a supply of cases in which* $\mathrm{Sp}(\lambda)$ *is a Young module, namely when* λ *is, in the terminology of [9], a Young partition, [9, Proposition 3.2 (i)]. However, in these cases the* $\mathrm{Sp}(\lambda)$ *is the indecomposable Young module* $Y(\lambda)$*. So these partitions are not of interest from the point of view of our current investigation.*

7 Further Remarks

7.1 Hook partitions

As an immediate application of Theorem 6.2 for $m = 2$, one obtains a decomposition for the Specht module $\mathrm{Sp}(a, 1^b)$ where a is even and b is odd. We point out here that our method gives also a decomposition of $\mathrm{Sp}(a, 1^b)$ when a is odd and b is even. In fact, we have the following result.

Proposition 7.1.1. *Let* $a, b \geq 1$ *and assume that* a *and* b *have different parity. Then we have the following decompositions.*
(i) For a *even and* b *odd with* $a = 2 + 2u$ *and* $b = 2v + 1$ *we have*

$$\mathrm{Sp}(a, 1^b) = \bigoplus_{\mu} Y(\sigma(2) + 2\mu)$$

where the sum is over all partitions $\mu = (c, d)$*, such that* $c + d = u + v$ *and* $(c - d, u - v)$ *is* p*-special.*
(ii) For a *odd and* b *even with* $a = 2u + 1$ *and* $b = 2v$ *we have*

$$\mathrm{Sp}(a, 1^b) = \bigoplus_{\mu} Y(\sigma(1) + 2\mu)$$

where the sum is over all partitions $\mu = (c, d)$*, such that* $c + d = u + v$ *and* $(c - d, u - v)$ *is* p*-special.*

Proof. (i) This follows directly from Theorem 6.2 with $m = 2$.
(ii) We consider the tensor product $S^a E \otimes \bigwedge^b E$. This module decomposes as

$$S^a E \otimes \bigwedge\nolimits^b E = \nabla(a + 1, 1^{b-1}) \oplus \nabla(a, 1^b), \tag{$\dagger$}$$

(since the 2-cores $\sigma(2)$ of $(a + 1, 1^{b-1})$ and $\sigma(1)$ of $(a, 1^b)$ are different).

Applying the Schur functor to ($\dagger$) we get, as in Section 6,

$$M(a, b) = \mathrm{Sp}(a + 1, 1^{b-1}) \oplus \mathrm{Sp}(a, 1^b).$$

Hence, the block component of $M(a, b)$ corresponding to the core $\sigma(1)$ is $\mathrm{Sp}(a, 1^b)$.

It is easy to see, as in Corollary 4.2, that the block component of $M(a,b)$ corresponding to the core $\sigma(1)$ is

$$\bigoplus_{\mu} Y(\sigma(1)+2\mu)$$

where the sum is over all partitions $\mu=(c,d)$, such that $c+d=u+v$ and $(c-d,u-v)$ is p-special. Thus we get the desired decomposition of $\mathrm{Sp}(a,1^b)$. $\square$

7.2 More Decomposable Specht Modules

We point out here that the argument giving the decomposition of the Specht modules described in Theorem 6.2, lays the foundations for the discovery of other families of decomposable Specht modules. We describe such a case here and take our considerations further in [10]. For simplicity we assume that K has characteristic 2 throughout this subsection, so that the Hecke algebra is the group algebra of the corresponding symmetric group. The Hecke algebras analogues of the results of this section will appear in a much more general form in [10].

Since K has characteristic 2 there is no need from now on to distinguish between the irreducible modules $L(\lambda)$ and $\dot{L}(\lambda)$. We will need the following lemma describing the dimension of some weight spaces for certain irreducible $G(n)$-modules.

Lemma 7.2.1. *Let $n\geq 3$. Let $a,b\geq 2$ with $a=2u+2=4w+2\equiv 2$ (mod 4) and $b=2v+1$ odd. Let μ be a two part partition with standard expansion $\mu=\mu^0+2\bar{\mu}$. Then*

$$\dim L(\sigma(2)+2\mu)^{(a,b,2)} = \begin{cases} 2\dim L(\mu)^{(u+1,v)}+\dim L(2\bar{\mu})^{(u-2,v)}, & \text{if } \mu^0=\sigma(2); \\ 2\dim L(\mu)^{(u+1,v)}, & \text{otherwise.} \end{cases}$$

Proof. Since $(a,b,2)$ has at most 2 parts the weight space multiplicity $\dim L(\sigma(2)+2\mu)^{(a,b,2)}$ is determined by the case $n=3$. The weights for the $G(3)$-module $L(\sigma(2))$ are

$$(2,1,0),(2,0,1),(0,2,1),(0,1,2),(1,0,2),(1,2,0),(1,1,1) \qquad (*)$$

(with $(1,1,1)$ appearing with multiplicity 3 and all other weights with multiplicity 1) e.g. by considering the Schur function $s(2,1,0)$. Since a is even and b is odd the only weights in this list that can contribute to the $(a,b,2)$ weight space of $L(\sigma(2)+2\mu)=L(\sigma(2))\otimes L(\mu)^F$ are $(2,1,0)$ and $(0,1,2)$. In these cases we can have $(a,b,2)=(2,1,0)+2\rho$ and $(a,b,2)=(0,1,2)+2\xi$, with $\rho=(2w,v,1)$ and $\xi=(2w+1,v,0)$. Therefore we get

$$\dim L(\sigma(2)+2\mu)^{(a,b,2)} = \dim L(\mu)^{(2w,v,1)} + \dim L(\mu)^{(2w+1,v)}. \qquad (\dagger)$$

The possibilities for μ^0 are $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$ and we consider these in turn.

Case (i) : $\mu^0 = (0,0)$.
The we have $L(\mu) = L(\bar{\mu})^F$, which only has even weights divisible by 2 so that

$$\dim L(\mu)^{(2w,v,1)} = \dim L(\mu)^{(2w+1,v)} = 0$$

and the assertion of the Lemma holds.

Case (ii): $\mu^0 = (1,0)$.
The weights of $L(1,0,0)$ are $(1,0,0),(0,1,0),(0,0,1)$ (with multiplicity 1) and Steinberg's tensor product theorem implies that any weight of $L(\mu)$ has precisely one odd entry. We may assume that $v = 2z$ is even (for otherwise the weight space multiplicity given by the statement of the lemma and by $(\dagger)$ is 0). Now $(\dagger)$ and Steinberg's tensor product theorem gives

$$\dim L(\sigma(2)+2\mu)^{(a,b,2)} = \dim L(\bar{\mu})^{(w,z)} + \dim L(\bar{\mu})^{(w,z)} = 2\dim L(\mu)^{(2w+1,v)}.$$

Case (iii): $\mu^0 = (1,1)$.
Similar to (ii).

Case (iv): $\mu^0 = (2,1)$.
If v is odd then, by (*), there are no weights of $L(\mu^0)$ which have precisely two odd entries. Steinberg's tensor product theorem then implies that

$$\dim L(\mu)^{(2w,v,1)} = \dim L(\mu)^{(2w+1,v)} = L(\bar{\mu})^{(2w-2,v)} = 0$$

and the assertion holds. Thus we assume $v = 2z$ is even. Then $(*)$ and Steinberg's tensor product theorem gives

$$\dim L(\mu)^{(2w,v,1)} = \dim L(\bar{\mu})^{(w-1,z)} + \dim L(\bar{\mu})^{(w,z-1)}$$

and

$$\dim L(\mu)^{(2w+1,v)} = \dim L(\bar{\mu})^{(w,z-1)}.$$

Hence from $(\dagger)$ we have

$$\begin{aligned}\dim L(\sigma(2)+2\mu)^{(a,b,2)} &= 2\dim L(\bar{\mu})^{(w,z-1)} + \dim L(\bar{\mu})^{(w-1,z)} \\ &= 2\dim L(\mu)^{(2w+1,v)} + \dim L(2\bar{\mu})^{(2w-2,v).}\end{aligned}$$

(We have assumed $w > 0$. If $w = 0$ the calculation gives
$\dim L(\sigma(2)+2\mu)^{(a,b,2)} = 2\dim L(\mu)^{(1,v)}$ and the desired result still holds since $(-2,0)$ is not a weight of a polynomial module so $L(2\bar{\mu})^{(-2,0)} = 0$.)

This completes the analysis of all the cases and concludes the proof. $\square$

Proposition 7.2.2. *Let $a \geq 4, b \geq 3$ with $a = 2u + 2 \equiv 2$ (mod 4) even and $b = 2v + 1$ odd. Then*

$$\mathrm{Sp}(a, 3, 1^{b-1}) = \bigoplus_{\mu} Y(\sigma(2) + 2\mu) \ \oplus \ \bigoplus_{\rho} Y(\sigma(2) + 2\rho)$$

where the sums are over all partitions μ and ρ such that:
$\mu = (\mu_1, \mu_2, 1)$, $\mu_1 + \mu_2 = u + v$ and $(\mu_1 - \mu_2, u - v)$ is 2-special; and $\rho = (\rho_1, \rho_2)$ with $\rho = \sigma(2) + 2\bar{\rho}$, $2(\bar{\rho}_1 + \bar{\rho}_2) = u + v - 2$ and $(2(\bar{\rho}_1 - \bar{\rho}_2), u - v - 2)$ is 2-special.

Proof. We consider the $G(n)$-module $S^a E \otimes \bigwedge^b E \otimes S^2 E$, for n sufficiently large. This has the following decomposition

$$S^a E \otimes \bigwedge^b E \otimes S^2 E = (\nabla(a+1, 1^{b-1}) \otimes S^2 E) \oplus (\nabla(a, 1^b) \otimes S^2 E)$$

(since the 2-cores $\sigma(1)$ of $(a+1, 1^{b-1})$ and $\sigma(2)$ of $(a, 1^b)$ are different).

Now using Pieri's Formula we can easily see that the component of $\nabla(a+1, 1^{b-1}) \otimes S^2 E$ corresponding to the core $\sigma(2)$ is just $\nabla(a+2, 1^b)$.

Let V be the component of the module $\nabla(a, 1^b) \otimes S^2 E$ corresponding to the core $\sigma(2)$. Then V has a good filtration and again by Pieri's Formula we see that V actually fits in the short exact sequence

$$0 \to \nabla(a, 3, 1^{b-1}) \to V \to \nabla(a+2, 1^b) \to 0.$$

By [7, Section 4.2 (17)], one has that

$$\mathrm{Ext}^1_{G(n)}(\nabla(a+2, 1^b), \nabla(a, 3, 1^{b-1})) = \mathrm{Ext}^1_{G(2)}(\nabla(a+1), \nabla(a-1, 2))$$

and since $a \equiv 2$ (mod 4), we have that $\mathrm{Ext}^1_{G(2)}(\nabla(a+1), \nabla(a-1, 2)) = 0$, see for e.g. [8, Corollary 5.12]. Therefore, $V = \nabla(a, 3, 1^{b-1}) \oplus \nabla(a+2, 1^b)$.

Hence, the block component of $S^a E \otimes \bigwedge^b E \otimes S^2 E$ corresponding to the core $\sigma(2)$ is

$$\nabla(a, 3, 1^{b-1}) \oplus \nabla(a+2, 1^b) \oplus \nabla(a+2, 1^b).$$

Now as in Sections 4 and 6, applying the Schur functor to the $G(n)$-module $S^a E \otimes \bigwedge^b E \otimes S^2 E$ and projecting onto the to the block component of $M(a, b, 2)$ corresponding to the core $\sigma(2)$, we get

$$\mathrm{Sp}(a, 3, 1^{b-1}) \oplus \mathrm{Sp}(a+2, 1^b) \oplus \mathrm{Sp}(a+2, 1^b) = \bigoplus_{\mu} Y(\sigma(2) + 2\mu)^{(d_\mu)}$$

where the sum is over all partitions $\mu = (\mu_1, \mu_2, \mu_3)$ with
$d_\mu = \dim L(\sigma(2) + 2\mu)^{(a,b,2)} \neq 0$.

Assume first that $\mu_3 \neq 0$. If $L(\sigma(2)+2\mu)^{(a,b,2)} \neq 0$, then $(a,b,2) \leq \sigma(2)+2\mu$ and so we must have $\mu_3 = 1$. Thus

$$\begin{aligned}\dim L(\sigma(2)+2\mu)^{(a,b,2)} &= \dim L(\sigma(2)+2(\mu-\omega(3)))^{(a-2,b-2)} \\ &= \dim L(2(\mu-\omega(3)))^{(a-4,b-3)}.\end{aligned}$$

Now

$$\dim L(2(\mu-\omega(3)))^{(a-4,b-3)} = \dim L(\mu_1-1,\mu_2-1)^{(u-1,v-1)}$$

and by Lemma 3.2 this dimension is 1 if $(\mu_1-\mu_2, u-v)$ is 2-special and 0 otherwise.

Assume now that $\mu_3 = 0$ and so $\mu = (\mu_1,\mu_2)$. Then Lemmas 7.2.1 and 3.2 give the dimension of $L(\sigma(2)+2(\mu_1,\mu_2))^{(a,b,2)}$.

Putting these two points together we conclude that the direct sum $\mathrm{Sp}(a,3,1^{b-1}) \oplus \mathrm{Sp}(a+2,1^b) \oplus \mathrm{Sp}(a+2,1^b)$, decomposes as

$$\bigoplus_\mu Y(\sigma(2)+2\mu) \;\oplus\; \bigoplus_\nu Y(\sigma(2)+2\nu)^{(2)} \;\oplus\; \bigoplus_\rho Y(\sigma(2)+2\rho), \qquad (1)$$

where the sums are over all partitions μ, ν and ρ respectively such that,
$\mu = (\mu_1,\mu_2,1)$, $\mu_1+\mu_2 = u+v$ and $(\mu_1-\mu_2, u-v)$ is 2-special;
$\nu = (\nu_1,\nu_2)$, $\nu_1+\nu_2 = u+v+1$ and $(\nu_1-\nu_2, u-v+1)$ is 2-special;
$\rho = (\rho_1,\rho_2)$ with $\rho = \sigma(2)+2\bar\rho$, $2(\bar\rho_1+\bar\rho_2) = u+v-2$ and $(2(\bar\rho_1-\bar\rho_2), u-v-2)$ is 2-special.

On the other hand, by Theorem 6.2, $\mathrm{Sp}(a+2,1^b)$ decomposes as

$$\bigoplus_\nu Y(\sigma(2)+2\nu), \qquad (2)$$

where the sum is over all partitions $\nu = (\nu_1,\nu_2)$ such that $\nu_1+\nu_2 = u+v+1$ and $(\nu_1-\nu_2, u-v+1)$ is 2-special.

Comparing now the decompositions given by (1) and (2) we get

$$\mathrm{Sp}(a,3,1^{b-1}) = \bigoplus_\mu Y(\sigma(2)+2\mu) \;\oplus\; \bigoplus_\rho Y(\sigma(2)+2\rho)$$

where the sums run over all partitions μ and ρ as described in the statement of the proposition. □

Remark 7.2.3. Proposition 7.2.2 gives many new decomposable Specht modules of the form $\mathrm{Sp}(a,3,1^b)$ which do not appear in the list of Dodge and Fayers, [4, Theorem 3.1 and Corollary 3.2]. For instance we have the following example.

Example 7.2.4. *For $a = 14$ and $b = 9$ we have*

$$\mathrm{Sp}(14,3,1^8) = Y(14,9,2) \oplus Y(18,5,2) \oplus Y(14,11).$$

Of course we can now obtain new decomposable Specht modules by considering the linear duals of the modules appearing above. More precisely, by [12, Theorem 8.15], in characteristic 2, $\mathrm{Sp}(\lambda)^* = \mathrm{Sp}(\lambda')$, where $\mathrm{Sp}(\lambda)^*$ is the linear dual of $\mathrm{Sp}(\lambda)$ and λ' is the transpose of the partition λ. Moreover the Young modules are self-dual. So by considering the linear duals of the modules appearing in Proposition 7.2.2 we get the following result.

Corollary 7.2.5. *Let $a \geq 4, b \geq 3$ with $a = 2u + 2 \equiv 2$ (mod 4) and $b = 2v + 1$ odd. Then*

$$\mathrm{Sp}(b+1, 2, 2, 1^{a-3}) = \bigoplus_{\mu} Y(\sigma(2) + 2\mu) \ \oplus \ \bigoplus_{\rho} Y(\sigma(2) + 2\rho)$$

where the sums are over all partitions μ and ρ such that,
$\mu = (\mu_1, \mu_2, 1)$, $\mu_1 + \mu_2 = u + v$ and $(\mu_1 - \mu_2, u - v)$ is 2-special; and
$\rho = (\rho_1, \rho_2)$ with $\rho = \sigma(2)+2\bar{\rho}$, $2(\bar{\rho}_1+\bar{\rho}_2) = u+v-2$ and $(2(\bar{\rho}_1-\bar{\rho}_2), u-v-2)$ is 2-special.

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