**Book Section:**
Rathjen, M orcid.org/0000-0003-1699-4778 and Tupailo, S (2020) On the Strength of the Uniform Fixed Point Principle in Intuitionistic Explicit Mathematics. In: Kahle, R and Rathjen, M, (eds.) The Legacy of Kurt Schütte. Springer , pp. 377-399. ISBN 978-3-030-49423-0

https://doi.org/10.1007/978-3-030-49424-7_19

# On the Strength of the Uniform Fixed Point Principle in Intuitionistic Explicit Mathematics

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**Abstract**

The paper is concerned with a line of research that plumbs the scope of constructive theories. The object of investigation here is Feferman's intuitionistic theory of explicit mathematics augmented by the monotone fixed point principle which asserts that every monotone operation on classifications (Feferman's notion of set) possesses a least fixed point. To be more precise, the new axiom not merely postulates the existence of a least solution, but, by adjoining a new functional constant to the language, it is ensured that a fixed point is uniformly presentable as a function of the monotone operation.

The strength of the classical non-uniform version, **MID**, was investigated in [GRS97] whereas that of the uniform version was determined in [Ra98, Ra99] and shown to be that of subsystems of second order arithmetic based on $\mathbf{\Pi}^1_2$-comprehension. This involved a rendering of $\mathbf{\Pi}^1_2$-comprehension in terms of fixed points of non-monotonic $\Pi^1_1$-operators and a proof-theoretic interpretation of the latter in specific operator theories that can be interpreted in explicit mathematics with the uniform monotone fixed point principle.

The intent of the current paper is to show that the same strength obtains when the underlying logic is taken to be intuitionistic logic.

## 1 Introduction

This paper continues research (cf. [Fef82, BFPS, Tak89, GRS97, Ra96, Ra98, Ra99, Ra02, Tu04]) addressing the status of monotone inductive definitions in the general constructive setting of Feferman's explicit mathematics [Fef75, Fef79], called $\mathbf{T_0}$. It has a strong bearing on the problem of determining the limits of what is constructively justifiable that was of great interest to logicians ever since the 1960s (cf. [Kr63]). The question of the strength of systems of explicit mathematics with fixed point principles **MID** and **UMID** was raised by Feferman in [Fef82]; we quote:

> *What is the strength of* $\mathbf{T_0}+\mathbf{MID}$*? [...] I have tried, but did not succeed, to extend my interpretation of* $\mathbf{T_0}$ *in* $\Sigma^1_2-AC+BI$ *to include the statement* **MID**. *The theory* $\mathbf{T_0}+\mathbf{MID}$ *includes all constructive formulations of iteration of monotone inductive definitions of which I am aware, while* $\mathbf{T_0}$ *(in its IG axiom) is based squarely on the general iteration of accessibility inductive definitions. Thus it would be of great interest for the present subject to settle the relationship between these theories.* (p. 88)

We are particularly interested in the intuitionistic strength of the axiom $\mathbf{UMID}_{\mathbb{N}}$ which postulates the existence of a least fixed point for any monotone operation $f$ on subsets of the natural numbers, where a least solution $\mathbf{lfp}(f)$ is presented as a function of the operation by adjoining a new constant **lfp** to the language of $\mathbf{T_0}$. To relate the state of the art in these matters we shall need some terminology. Below we shall distinguish between the classical and the intuitionistic version of a theory by appending the superscript $c$ and $i$, respectively. For a system $S$ of explicit mathematics we denote by $S\restriction$ the version wherein the induction principles for the natural numbers and for inductive generation are restricted to sets. $\mathbf{IND}_{\mathbb{N}}$ stands for the schema of induction on natural numbers for arbitrary formulas of the language of explicit

mathematics. $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})_0$ denotes the subsystem of second order arithmetic (based on classical logic) with $\mathbf{\Pi^1_2}$-comprehension but with induction restricted to sets, whereas $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})$ also contains the full schema of induction on $\mathbb{N}$.
The papers [Ra98, Ra99] yielded the following results:

**Theorem 1.1** *(i)* $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})_0$ *and* $\mathbf{T^c_0}\!\restriction + \mathbf{UMID}_{\mathbb{N}}$ *have the same proof-theoretic strength.*

*(ii)* $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})$ *and* $\mathbf{T^c_0}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbb{N}}$ *have the same proof-theoretic strength.*

The first result about $\mathbf{UMID}_{\mathbb{N}}$ on the basis of intuitionistic explicit mathematics was obtained by the second author in [Tu04].

**Theorem 1.2** $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})_0$ *and* $\mathbf{T^i_0}\!\restriction + \mathbf{UMID}_{\mathbb{N}}$ *have the same proof-theoretic strength.*

[Tu04] uses a characterization of $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})_0$ via a classical $\mu$-calculus (a theory which extends the concept of an inductive definition), dubbed $\mathbf{ACA}_0(\mathcal{L}^\mu)$, given by Möllerfeld [Mö02] and then proceeds to show that $\mathbf{ACA}_0(\mathcal{L}^\mu)$ can be interpreted in its intuitionistic version, $\mathbf{ACA}^i_0(\mathcal{L}^\mu)$, by means of a double negation translation. Finally, as the latter theory is readily interpretable in $\mathbf{T^i_0}\!\restriction + \mathbf{UMID}_{\mathbb{N}}$, the proof-theoretic equivalence stated in Theorem 1.2 follows in view of Theorem 1.1.
The proof of [Tu04], however, does not readily generalize to $\mathbf{T^i_0}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbb{N}}$ and extensions by further induction principles. The main reason for this is that adding induction principles such as induction on natural numbers for all formulas to $\mathbf{ACA}_0(\mathcal{L}^\mu)$ only slightly increases the strength of the theory.[1] It is suggested by the results of [Ra98] that in order to arrive at a $\mu$-calculus of the strength of $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})$ one has to allow for transfinite nestings of the $\mu$-operator for any ordinal $\alpha < \varepsilon_0$. By engineering a double negation translation in a similar vein as in [Tu04], we will be able to conclude the following result.

**Theorem 1.3** *(i)* $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})_0$ *and* $\mathbf{T^i_0}\!\restriction + \mathbf{UMID}_{\mathbb{N}}$ *have the same proof-theoretic strength.*

*(ii)* $(\mathbf{\Pi^1_2}\text{-}\mathbf{CA})$ *and* $\mathbf{T^i_0}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbb{N}}$ *have the same proof-theoretic strength.*

Through Theorem 1.3 we get another proof of Theorem 1.2 (which also does not hinge upon [Mö02]).
Finally, it's worth mentioning that the same results could be obtained by subjecting the operator theories $\mathbf{T}^{\mathbf{OP}}_{<\omega}$ and $\mathbf{T}^{\mathbf{OP}}_{<\varepsilon_0}$ to a double negation interpretation. Moreover, this translation works for extensions of Theorem 1.3(ii) where one allows for transfinite nestings of the $\mu$-operator as long as the ordinals come from a primitive recursive ordinal representation system.

# 2 Fixed point theories

We consider different frameworks for expressing the existence of fixed point of operators.

## 2.1 The $\mu$-calculus

The $\mu$-calculus extends the concept of an inductive definition. It is basically an algebra of monotone functions over the power class of the domain of a first order structure (or over a complete lattice), whose basic constructors are first order definable operators, functional composition and least and greatest fixed point operators. The $\mu$-calculus arose from numerous works of logicians and computer scientists. It originated with Scott and DeBakker [SD69] and was developed by Hitchcock and Park [HP73], Park [Pa70], Kozen [K83], Pratt [Pr81], and others (see [AN01]). The $\mu$-calculus is used in verification of computer programs and provides a tool box for modelling a variety of phenomena, from finite automata to alternating automata on infinite trees and infinite games with finitely presentable winning conditions. Here we will be interested in the $\mu$-calculus over the natural numbers. The $\mu$-definable sets over the natural numbers were first described by Lubarsky [Lu93]. He determined their complexity in the constructible hierarchy and showed that their ordinal ranks in that hierarchy can reach rather large countable ordinals. In the following we denote by $\mathbf{ACA}_0(\mathcal{L}^\mu)$ an axiomatic theory whose language is an extension of that of the classical $\mu$-calculus over $\mathbb{N}$,

[1] In actuality, adding induction on natural numbers for all formulas does not increase the proof-theoretic strength at all.

$\mathcal{L}^\mu$ (see [Lu93]), by set quantifiers and comprehension for first-order properties. This version was formalized in [Mö02]. The letters "*ACA*" stand for "arithmetic comprehension axiom" and the subscript 0 indicates that the induction principle on natural numbers holds for sets rather than arbitrary classes. The name "$\mathbf{ACA}_0(\mathcal{L}^\mu)$" for this theory is somewhat misleading as its comprehension axioms allow for the formation of non-arithmetic sets. However, we will stick to this notation for 'historical' reasons.

**Definition 2.1** The language of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ builds on the language of Peano arithmetic, **PA**. It has variables $x, y, z, \ldots, X, Y, Z, \ldots$ ranging over numbers and sets of numbers, respectively. The terms of **PA** will be referred to as number terms. *Number terms*, *set terms* and *formulas* of the language $\mathcal{L}^\mu$ are defined as follows.

1. The terms of **PA** are *number terms* of $\mathcal{L}^\mu$.
2. Set variables are *set terms.*
3. $\perp$ is a *formula.*
4. If $s$ and $t$ are *number terms* then $s = t$ is a *formula.*
5. If $s$ is a *number term* and $S$ is a *set term* then $s \in S$ is a *formula.*
6. If $\varphi_0$ and $\varphi_1$ are *formulas* then $\varphi_0 \wedge \varphi_1$, $\varphi_0 \vee \varphi_1$ and $\varphi_0 \to \varphi_1$ are *formulas.*
7. If $\psi$ is a *formula* then $\forall x\psi$ and $\exists x\psi$ are *formulas.*
8. If $\psi$ is a *formula* then $\forall X\psi$ and $\exists X\psi$ are *formulas.*
9. If $\varphi$ is an $X$-positive first-order *formula* then $\mu x X.\varphi$ is a *set term.*

In the definition above we call a formula *first-order* if it does not contain set quantifiers $\exists X, \forall X$. For $X$ a set variable an expression $\mathfrak{E}$ is said to be *$X$-positive ($X$-negative)* if every occurrence of $X$ in $\mathfrak{E}$ is positive (negative). In classical logic we can restrict ourselves to the connectives $\neg, \wedge, \vee$ and then $X$ is positive in a formula $\varphi$ if every occurrence of $X$ in $\varphi$ is in the scope of an even number of negations. But as we shall also be concerned with the intuitionistic $\mu$-calculus, we define this notion inductively as follows:
(1) $X$ is *$X$-positive*; (2) $Y$ is both *$X$-positive* and *$X$-negative* if $Y$ is a set variable different from $X$; (3) $\perp$ and $s = t$ are also both *$X$-positive* and *$X$-negative*; (4) $s \in S$ is *$X$-positive (-negative)* iff $S$ is; (5) polarity does not change with $\wedge$, $\vee$, quantifiers and the $\mu$-symbol; (6) and, finally, $\varphi_0 \to \varphi_1$ is *$X$-positive (-negative)* iff $\varphi_0$ is *$X$-negative (-positive)* and $\varphi_1$ is *$X$-positive (-negative).*
For set terms $S, T$, $S \subseteq T$ is the formula $\forall x(x \in S \to x \in T)$.

**Definition 2.2** The axioms of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ are the following:

1. The axioms of **PA**.
2. (Induction) $\forall X\, \big(0 \in X \ \wedge\ \forall u(u \in X \to u + 1 \in X) \ \to\ \forall u\, u \in X\big)$.
3. (First-order comprehension) $\exists Z \forall x[x \in Z \leftrightarrow \varphi(x)]$ for every first-order formula $\varphi$ in which the set variable $Z$ does not appear free.
4. (Least fixed point axiom)
$$\forall x[x \in P \leftrightarrow \varphi(x, P)] \ \wedge\ \forall Y\big[\forall x\big(\varphi(x, Y) \to x \in Y\big) \to P \subseteq Y\big] \tag{1}$$
where $P$ stands for the set term $\mu x X.\varphi$.

$\mathbf{ACA}_0(\mathcal{L}^\mu)$ is based on classical logic. The system with the underlying logic changed to intuitionistic logic will be denoted by $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$.
The theories with the full induction scheme **IND** will be denoted by $\mathbf{ACA}(\mathcal{L}^\mu)$ and $\mathbf{ACA}^i(\mathcal{L}^\mu)$, respectively. **IND** is the scheme
$$\psi(0) \ \wedge\ \forall x[\psi(x) \to \psi(x + 1)] \ \to\ \forall x\psi(x)$$
for all formulas $\psi$.

That $X$ is positive (negative) in $\psi$ will be notated by $\psi(X^+)$ ($\psi(X^-)$). Positivity is a guarantor of monotonicity, while negativity guarantees anti-monotonicity.

**Lemma 2.3** *For every $X$-positive formulas $\psi(X^+)$ and and every $X$-negative formula $\theta(X^-)$ of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ we have:*

*(i)* $\mathbf{ACA}_0^i(\mathcal{L}^\mu) \vdash \forall X \forall Y\, [X \subseteq Y \ \wedge\ \psi(X) \ \rightarrow\ \psi(Y)]$.

*(ii)* $\mathbf{ACA}_0^i(\mathcal{L}^\mu) \vdash \forall X \forall Y\, [X \subseteq Y \ \wedge\ \theta(Y) \ \rightarrow\ \theta(X)]$.

**Proof:** Use induction on the complexity of the formulas. □

At first blush, the $\mu$-calculus appears to be innocent enough. Though a first order formula $\varphi(X^+, x)$ may contain complicated $\mu$-terms, it might seem that these act solely as parameters and therefore one could obtain $\mu x X.\varphi(X^+, x)$ via an ordinary first order arithmetic inductive definition in these parameters, so that all the $\mu$-definable sets would turn out to be sets recursive in finite iterations of the hyperjump. But this is far from being true. The $\mu$-calculus allows for nestings of least fixed point operators. Better yet, there can be feedback. This provides the major difficulty in understanding the expressive power of $\mathcal{L}^\mu$. To illustrate the complexity of nested set terms in $\mathcal{L}^\mu$, let $\theta(X^+, Y^-, Z^+, W^-)$ be a first order formula of $\mathcal{L}^\mu$. Then the following are set terms: $\mu z Z.\theta$, $\mu y Y.\, w \notin \mu z Z.\theta$, $\mu x X.\mu y Y.\, w \notin \mu z Z.\theta$, $\mu w W.\mu x X.\mu y Y.\, w \notin \mu z Z.\theta$.
In the $\mu$-calculus one can also define the *greatest fixed point* constructor $\nu$: If $\varphi(X^+, x)$ is first order, $\nu x X.\varphi(X^+, x)$ is $\{u \mid u \notin \mu x X.\neg\varphi(\neg X, x)\}$. The appropriate measure for the complexity of $\mu$-terms was determined by Lubarsky [Lu93]. $\mu$ and $\nu$ can be viewed as higher order quantifiers giving rise to complexity classes $\Sigma_n^\mu$ and $\Pi_n^\mu$ of $\mathcal{L}^\mu$ formulas which measure the alternations of $\mu$ and $\nu$.
The pivotal proof-theoretic connection between $\mathbf{ACA}_0(\mathcal{L}^\mu)$ and $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$ was established by Tupailo.

**Theorem 2.4** (Tupailo) $\mathbf{ACA}_0(\mathcal{L}^\mu)$ *can be interpreted in* $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$ *via a double negation translation.*

**Proof:** [Tu04]. □

## 2.2 Fragments of second order arithmetic

The proof-theoretic strength of theories is commonly calibrated using standard theories and their canonical fragments. In classical set theory this linear line of consistency strengths is couched in terms of large cardinal axioms while for weaker theories the line of reference systems traditionally consist in second order arithmetic and its fragments, owing to Hilbert's and Bernays' [HB38] observation that large chunks of mathematics can already be formalized in second order arithmetic.

**Definition 2.5** The language $\mathcal{L}_2$ of second-order arithmetic contains number variables $x, y, z, u, \ldots$, set variables $X, Y, Z, U, \ldots$ (ranging over subsets of $\mathbb{N}$), the constant 0, function symbols $Suc, +, \cdot$, and relation symbols $=, <, \in$. *Suc* stands for the successor function. *Terms* are built up as usual. For $n \in \mathbb{N}$, let $\bar{n}$ be the canonical term denoting $n$. Formulae are built from the prime formulae $s = t$, $s < t$, and $s \in X$ using $\wedge, \vee, \neg, \forall x, \exists x, \forall X$ and $\exists X$ where $s, t$ are terms. Note that equality in $\mathcal{L}_2$ is only a relation on numbers. However, equality of sets will be considered a defined notion, namely $X = Y$ if and only if $\forall x[x \in X \ \leftrightarrow\ x \in Y]$. As per usual, number quantifiers are called bounded if they occur in the context $\forall x(x < s \rightarrow \ldots)$ or $\exists x(x < s \wedge \ldots)$ for a term $s$ which does not contain $x$. The $\mathbf{\Sigma}_0^0$-formulae are those formulae in which all quantifiers are bounded number quantifiers. For $k > 0$, $\mathbf{\Sigma}_k^0$-formulae are formulae of the form $\exists x_1 \forall x_2 \ldots Q x_k \phi$, where $\phi$ is $\mathbf{\Sigma}_0^0$; $\mathbf{\Pi}_k^0$-formulae are those of the form $\forall x_1 \exists x_2 \ldots Q x_k \phi$. The union of all $\mathbf{\Pi}_k^0$- and $\mathbf{\Sigma}_k^0$-formulae for all $k \in \mathbb{N}$ is the class of *arithmetical* or $\mathbf{\Pi}_\infty^0$*-formulae.* The $\mathbf{\Sigma}_k^1$-formulae ($\mathbf{\Pi}_k^1$-formulae) are the formulae $\exists X_1 \forall X_2 \ldots Q X_k \phi$ (resp. $\forall X_1 \exists X_2 \ldots Q x_k \phi$) for arithmetical $\phi$.
The basic axioms in all theories of second-order arithmetic are the defining axioms of $0, 1, +, \cdot, <$ and the *induction axiom*

$$\forall X(0 \in X \wedge \forall x(x \in X \rightarrow x + 1 \in X) \rightarrow \forall x(x \in X)),$$

respectively the *scheme of induction*

$$\mathbf{IND} \qquad \phi(0) \wedge \forall x(\phi(x) \to \phi(x+1)) \to \forall x \phi(x),$$

where $\phi$ is an arbitrary $\mathcal{L}_2$-formula. We consider the axiom scheme of $\mathcal{C}$-*comprehension* for formula classes $\mathcal{C}$ which is given by

$$\mathcal{C}\text{-}\mathbf{CA} \qquad \exists X \forall u(u \in X \leftrightarrow \phi(u))$$

for all formulae $\phi \in \mathcal{C}$ in which $X$ does not occur.
For each axiom scheme $\mathbf{Ax}$ we denote by $(\mathbf{Ax})$ the theory consisting of the basic arithmetical axioms, the scheme $\mathbf{\Pi}^0_\infty$-$\mathbf{CA}$, the scheme of induction and the scheme $\mathbf{Ax}$. If we replace the scheme of induction by the induction axiom, we denote the resulting theory by $(\mathbf{Ax})_0$. An example for these notations is the theory $(\mathbf{\Pi}^1_1\text{-}\mathbf{CA})$ which contains the induction scheme, whereas $(\mathbf{\Pi}^1_1\text{-}\mathbf{CA})_0$ only contains the induction axiom in addition to the comprehension scheme for $\mathbf{\Pi}^1_1$-formulae.
In the framework of these theories one can introduce defined symbols for all primitive recursive functions. Especially, let $\langle , \rangle : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a primitive recursive and bijective pairing function. The $x^{th}$ section of $U$ is defined by $U_x := \{y : \langle x, y \rangle \in U\}$. Observe that a set $U$ is uniquely determined by its sections on account of $\langle , \rangle$'s bijectivity. Any set $R$ gives rise to a binary relation $\prec_R$ defined by $y \prec_R x := \langle y, x \rangle \in R$. Using the foregoing coding, we can formulate the schema of *Bar induction*

$$\mathbf{BI} \qquad \forall X \big[ \mathbf{WF}(\prec_X) \ \wedge \ \forall u \big( \forall v \prec_X u \phi(v) \to \phi(u) \big) \ \to \ \forall u \phi(u) \big]$$

for all formulae $\phi$, where $\mathbf{WF}(\prec_X)$ expresses that $\prec_X$ is well-founded, i.e., $\mathbf{WF}(\prec_X)$ stands for the formula $\forall Y \, \big[ \forall u \big[ (\forall v \prec_X u \; v \in Y) \ \to \ u \in Y \big] \ \to \ \forall u \; u \in Y \big]$.
For a collection of formulas, $\mathcal{F}$, we also formulate the axiom of choice for these formulas:

$$\mathcal{F}\text{-}\mathbf{AC} \qquad \forall x \exists Y \, F(x, Y) \to \exists Y \forall x \, F(x, Y_x),$$

where $F(x, X)$ belongs to $\mathcal{F}$.

**Definition 2.6** A binary relation $\prec$ on $\mathbb{N}$ is said to be a *prewellordering* if $\prec$ is well-founded and transitive and satisfies

$$\forall x, y \, [\, x \prec y \ \vee \ y \prec x \ \vee \ x \equiv_\prec y \,],$$

where $x \equiv_\prec y$ signifies $\forall u \, \big( [u \prec x \leftrightarrow u \prec y] \ \wedge \ [x \prec u \leftrightarrow y \prec u] \big)$.
Given two prewellorderings $\lhd$ and $\prec$, we say that a function $f : \mathbb{N} \to \mathbb{N}$ *embeds* $\lhd$ *into* $\prec$ if $\forall xy \, (\, y \lhd x \ \to \ f(y) \prec f(x) \,)$ and $\forall xz \, [\, z \prec f(x) \ \to \ \exists y (\, y \lhd x \ \wedge \ f(y) \equiv_\prec z \,)]$.
Note that $(\mathbf{\Pi}^1_1 - \mathbf{CA})_0$ suffices to prove that there exists a function $f$ such that $f$ embeds $\lhd$ into $\prec$ or $f$ embeds $\prec$ into $\lhd$.
We use the abbreviations $\mathbf{PWO}(\prec)$ to express that $\prec$ is a prewellordering. By field$(\prec)$ we mean the set $\{x : \exists y \, (x \prec y \ \vee \ y \prec x)\}$. For a set $V$, let field$(V)$ = field$(\prec_V)$.
$\prec$ is a *wellordering* (written $\mathbf{WO}(\prec)$) if $\mathbf{PWO}(\prec)$ and $\forall xy \in \text{field}(\prec) \, [x \prec y \ \vee \ y \prec x \ \vee \ x = y]$. $\prec$ is a *wellordering of* $\mathbb{N}$ if $\mathbf{WO}(\prec)$ and field$(\prec) = \mathbb{N}$.

**Definition 2.7** Let $\mathcal{F}$ be a collection of $\mathcal{L}_2$-formulae. The principle that any operator which is describable via an $\mathcal{F}$-formula inductively defines a set, $\mathcal{F}$-$\mathbf{Fix}$, is expressed by the schema

$$\forall X \exists! Y \, \phi(X, Y) \ \to \ \exists V \exists Z \exists U \, \big[ \mathbf{PWO}(V) \ \wedge \ \forall x \, \phi(Z_{Vx}, Z_x) \ \wedge \ \phi(\bigcup_x Z_x, U) \ \wedge \ U \subseteq \bigcup_x Z_x \, \big]$$

where $\phi$ belongs to $\mathcal{F}$.
Note that $\bigcup_x Z_x$ is uniquely determined by $\phi$, that is if

$$\mathbf{PWO}(\bar{V}) \ \wedge \ \forall x \, \phi(\bar{Z}_{\bar{V}x}, \bar{Z}_x) \ \wedge \ \phi(\bigcup_x \bar{Z}_x, \bar{U}) \ \wedge \ \bar{U} \subseteq \bigcup_x \bar{Z}_x, \tag{2}$$

then $\bigcup_x Z_x = \bigcup_x \bar{Z}_x$. To see this assume that $f$ embeds $\prec_V$ into $\prec_{\bar{V}}$. By induction on $\prec_V$ one then verifies that $\forall x Z_x = \bar{Z}_{f(x)}$. The latter implies the assertion. As to a fragment of second order arithmetic in which the previous proof can be carried out, one needs provability of comparability of prewellorderings; thus, e.g. $(\mathbf{\Pi}^1_1\text{-}\mathbf{CA})_0$ suffices.
We shall denote $\bigcup_x Z_x$ by $\mathbf{I}^\infty_\phi$. By $\prec_\phi$ we shall refer to an arbitrary choice of prewellordering $\prec_{\bar{V}}$ satisfying (2).

The crucial result linking $\Pi^1_2$-comprehension and and the schema $\mathcal{F}$-**Fix** is the following.

**Theorem 2.8** (Rathjen)

*(i)* $(\mathbf{\Sigma^1_2}\text{-}\mathbf{AC})_0 + \mathbf{\Pi^1_1}\text{-}\mathbf{Fix} = (\mathbf{\Pi^1_2}\text{-}\mathbf{CA})_0$.

*(ii)* $(\mathbf{\Sigma^1_2}\text{-}\mathbf{AC}) + \mathbf{\Pi^1_1}\text{-}\mathbf{Fix} = (\mathbf{\Pi^1_2}\text{-}\mathbf{CA})$.

**Proof.** [Ra98] Theorem 3.15 and Corollary 3.16. □

## 2.3 The theories $\mathcal{M}_{<\gamma}$

To begin with, we fix an ordinal notation system OT. For this paper it will be sufficient to assume that OT is a standard notation system for $\varepsilon_0$. $<$ will refer to the primitive recursive "less than" relation which comes with OT, and $\alpha < \varepsilon_0$ will mean $\gamma \in$ OT. The language $\mathcal{L}_\mu$, extending the language $\mathcal{L}_2$ of 2nd order arithmetic, is described below. It is a generalization of the language of $\mu$-calculus as presented in [Mö02, Section 1b] and [Tu04, Section 1]. *Number terms*, *set terms* and *formulas* of the language $\mathcal{L}_\mu$ are defined as follows.

**Definition 2.9** *1. Number terms of $\mathcal{L}_2$ are* number terms *of $\mathcal{L}_\mu$.*

*2. Set variables are* set terms.

*3. $\perp$ is a* formula.

*4. If $s$ and $t$ are* number terms *then $s = t$ is a* formula.

*5. If $s$ is a* number term *and $S$ is a* set term *then $s \in S$ is a* formula.

*6. If $\varphi_0$ and $\varphi_1$ are* formulas *then $\varphi_0 \wedge \varphi_1$, $\varphi_0 \vee \varphi_1$ and $\varphi_0 \to \varphi_1$ are* formulas.

*7. If $\psi$ is a* formula *then $\forall x\psi$ and $\exists x\psi$ are* formulas.

*8. If $\psi$ is a* formula *then $\forall X\psi$ and $\exists X\psi$ are* formulas.

*9. If $\varphi$ is a first-order (i.e. not containing set quantifiers)* formula *then $\mu xyXY.\varphi$ is a* set term.

Fix a **limit ordinal** $\gamma \le \varepsilon_0$ for the remainder of this article. $\mathcal{M}_{<\gamma}$ is based on classical logic. The basic axioms of $\mathcal{M}_{<\gamma}$ are those of $\mathbf{ACA_0}(\mathcal{L}_\mu)$, i.e. $\mathbf{ACA_0}$ extended to the language $\mathcal{L}_\mu$. The main additional axiom scheme $\mathbf{LFP}_\gamma[\varphi[x, y, X, Y], T]$ will govern the $\mu$-term $T := \mu xyXY.\varphi$; its formulation requires some preparation. The basic idea is to state it as

$$\forall y{<}\alpha \forall Y \text{"}\varphi[x, y, X, Y] \text{ is monotone in } X\text{"} \to \forall y{<}\alpha\, \mathrm{LFP}_{x,X}[\varphi[x, y, X, T_{<y}], T_y] \tag{3}$$

(for abbreviations see below), for every first-order formula $\varphi$ and every $\alpha < \gamma$, but for technical reasons we formulate it in a slightly different way. In our formulation, in order to claim $\forall y{<}\alpha\, \mathrm{LFP}_{x,X}[\varphi[x, y, X, T_{<y}], T_y]$, we require monotonicity not only of the formula $\varphi$, but of all formulas $\chi$ s.t. $T$ "depends on" a $\mu$-term $\mu xyXY.\chi$. Exact definitions are given next.

**Definition 2.10** *For every formula $\psi$ and every $\mu$-term $T := \mu xyXY.\varphi$, assuming that all bound variables in them are renamed so as to avoid collisions, we define finite sets $\mathcal{M}(\psi)$ and $\mathcal{M}(T)$ of the form $\{\langle \ell_i, \mu x_i y_i X_i Y_i.\varphi_i\rangle \mid \ldots\}$, where $\ell_i$ is a finite list of variables, inductively as follows:*

$$\mathcal{M}(\psi) := \begin{cases} \emptyset & \text{if } \psi \text{ is } \perp,\ s = t \text{ or } s \in X; \\ \mathcal{M}(\psi_0) \cup \mathcal{M}(\psi_1) & \text{if } \psi \text{ is } \psi_0 \circ \psi_1 \text{ and } \circ \in \{\wedge, \vee, \to\}; \\ \{\langle (\ell_i, x), T_i\rangle \mid \langle \ell_i, T_i\rangle \in \mathcal{M}(\chi[x])\} & \text{if } \psi \text{ is } Qx\chi[x] \text{ and } Q \in \{\forall, \exists\}; \\ \mathcal{M}(\chi[X]) & \text{if } \psi \text{ is } QX\chi[X] \text{ and } Q \in \{\forall, \exists\}; \\ \{\langle \emptyset, \mu xyXY.\varphi\rangle\} \ \cup\ \{\langle (\ell_i, x, y, X, Y), T_i\rangle \mid \langle \ell_i, T_i\rangle \in \mathcal{M}(\varphi)\} & \text{if } \psi \text{ is } s \in \mu xyXY.\varphi. \end{cases}$$

*Finally we define $\mathcal{M}(T) := \mathcal{M}(0 \in T)$.*

Now $\mathbf{LFP}_\alpha[\varphi, T]$ is defined as an axiom scheme

$$\mathrm{Mon}(\mathcal{M}(T), \alpha) \to \forall y{<}\alpha\, \mathrm{LFP}_{x,X}[\varphi[x, y, X, T_{<y}], T_y], \tag{4}$$

for every first-order formula $\varphi$ and every $\alpha < \gamma$, where we adopt the following abbreviations:
$\mathrm{Mon}(\mathcal{M}(T), \alpha)$ stands for the conjunction of all formulas

$$\forall \vec{z_i} \forall \vec{Z_i} \forall y_i{<}\alpha \forall Y_i \text{"}\varphi_i[x_i, y_i, X_i, Y_i] \text{ is monotone in } X_i\text{"},$$

where $\langle(\vec{z_i}, \vec{Z_i}), \mu x_i y_i X_i Y_i.\varphi_i\rangle \in \mathcal{M}(T)$.

| | | |
|---|---|---|
| $X \subseteq Y$ | for | $\forall x\,(x \in X \to x \in Y)$, |
| $\mathrm{LFP}_{x,X}[\psi[x, X], Z]$ | for | $\forall x\,(x \in Z \leftrightarrow \psi[x, Z]) \;\wedge\; \forall U(\forall z(\psi[z, U] \to z \in U) \to Z \subseteq U)$, |
| $x \in Z_y$ | for | $(y, x) \in Z$, |
| $x \in Z_{<y}$ | for | $(x)_0 < y \wedge x \in Z$, |
| $(\cdot, \cdot), (\cdot)_0, (\cdot)_1$ | for | the usual pairing and unpairing operations on natural numbers. |

Spelling out the formulas the abbreviations $\mathrm{Mon}(\mathcal{M}(T), \alpha)$ and $\mathbf{LFP}_\alpha[\varphi, T]$ stand for, we have that $\mathrm{Mon}(\mathcal{M}(T), \alpha)$ is the conjunction of the formulas

$$\forall \vec{z_i} \forall \vec{Z_i} \forall y_i{<}\alpha \forall Y_i \forall X_i' \forall X_i''\; (X_i' \subseteq X_i'' \to (\varphi_i[x_i, y_i, X', Y_i] \to \varphi_i[x_i, y_i, X_i'', Y_i])) \tag{5}$$

where $\langle(\vec{z_i}, \vec{Z_i}), \mu x_i y_i X_i Y_i.\varphi_i\rangle \in \mathcal{M}(T)$; and $\mathbf{LFP}_\alpha[\varphi, T]$ is the formula

$$\begin{aligned}\mathrm{Mon}(\mathcal{M}(T), \alpha) \quad \to \quad & \forall y{<}\alpha\,(\forall x\,(x \in T_y \leftrightarrow \varphi[x, y, T_y, T_{<y}]) \\ & \wedge \forall U(\forall z(\varphi[z, y, U, T_{<y}] \to z \in U) \to T_y \subseteq U)).\end{aligned} \tag{6}$$

In addition to the above, $\mathcal{M}_{<\gamma}$ contains the scheme of transfinite induction $\mathbf{TI}_\alpha$, for every formula $\psi[x] \in \mathcal{L}_\mu$ and every $\alpha < \gamma$:

$$\forall x(\forall y{<}x\, \psi[y] \to \psi[x]) \to \forall x{<}\alpha\, \psi[x]. \tag{7}$$

**Lemma 2.11** *Let $\varphi(x, X)$ be a first order formula of $\mathcal{M}_{<\gamma}$ (which usually contains other free variables) such that $\varphi(x, X)$ is $\mathcal{M}_{<\gamma}$-provably monotone in $x, X$, i.e.,*

$$\mathcal{M}_{<\gamma} \vdash \forall x \forall X \forall Z\,[X \subseteq Z \wedge \varphi(x, X) \;\to\; \varphi(x, Z)].$$

*Let the first-order formula $\varphi_{st}(x, X)$ be defined by*

$$\begin{aligned}\varphi_{st}(x, X) \quad :=& \quad \exists u, v\,\big[\, x = (u, v) \;\wedge\; \varphi(u, \{z : (z, u) \in X\}) \;\wedge \\ & \neg\varphi(v, \{z : \neg\varphi(u, \{w : (w, z) \in X\})\})\big].\end{aligned} \tag{8}$$

*Then $\varphi_{st}(x, X)$ is provably monotone in $\mathcal{M}_{<\gamma}$ with respect to $x, X$. Letting $x <_\varphi y$ stand for $(x, y) \in \mu x X.\varphi_{st}(x, X)$, we get*

$$\mathcal{M}_{<\gamma} \quad \vdash \quad \mathrm{WF}(<_\varphi) \;\wedge\; \forall u\,[u \in \mu x X.\varphi(x, X) \;\leftrightarrow\; \varphi(u, \{v :\, v <_\varphi u\})]. \tag{9}$$

**Proof.**The monotonicity of $\varphi_{st}(x, X)$ follows from that of $\varphi(x, X)$. (9) is proved in [Ra96], section 3 and stated in [Ra96], Corollary 3.3. The proof presented in [Ra96], though, is formally carried out in a system of explicit mathematics with an extra axiom asserting that every monotone operation on sets has a least fixed point. However, one easily checks that that proof carries over to $\mathcal{M}_{<\gamma}$. □

**Proposition 2.12** *To every first-order formula $\theta(x, X)$ of $\mathcal{M}_{<\gamma}$ and variables $x, X$ we can effectively assign a first-order formula $\Upsilon(x, X)$ of $\mathcal{M}_{<\gamma}$ with the same free variables such that $\Upsilon(x, X)$ is $\mathcal{M}_{<\gamma}$-provably monotone with respect to $x, X$, i.e.,*

$$\mathcal{M}_{<\gamma} \vdash \forall x \forall X \forall Z\,[X \subseteq Z \wedge \Upsilon(x, X) \;\to\; \Upsilon(x, Z)]. \tag{10}$$

*Moreover, setting*

$$
\begin{aligned}
\Upsilon^u &:= \{y : \Upsilon(y, \{v : v <_\Upsilon u\})\},\\
\Theta^u &:= \{a : (0,a,a) \in \Upsilon^u\},\\
\Theta^{<u} &:= \bigcup_{y <_\Upsilon u} \Theta^y := \{b : \exists y <_\Upsilon u\, b \in \Theta^y\}\\
\Theta^\infty &:= \bigcup_u \Theta^u := \{z : \exists u\, z \in \Theta^u\}\\
\Gamma_\theta(X) &:= \{x : \theta(x,X)\},
\end{aligned}
$$

*$\mathcal{M}_{<\gamma}$ proves that*

$$\Theta^u \;=\; \Gamma_\theta(\Theta^{<u}) \cup \Theta^{<u}, \tag{11}$$
$$\Gamma_\theta(\Theta^\infty) \;\subseteq\; \Theta^\infty. \tag{12}$$

*In particular, $\Theta^\infty$ is first-order definable in the language of $\mathcal{M}_{<\gamma}$ and $\mathcal{M}_{<\gamma}$ proves that $\Theta^\infty$ is a set and that $\Theta^\infty$ arises by iterating the operator $\Gamma_\theta$ along the stage comparison prewellordering $<_\Upsilon$ of $\mu xX.\Upsilon(x,X)$. Moreover, $\Theta^\infty$ is closed under $\Gamma_\theta$. In other words, $\Theta^\infty$ is the set inductively defined by the operator $\Gamma_\theta$.*

**Proof.** The details of the definition of $\Upsilon$ can be found in [Ra96] Definition 4.3. More precisely, one has to substitute $\Gamma_\theta$ for the operator $\Theta$ in [Ra96] Definition 4.3 and then define $\Upsilon(x,X)$ by $x \in \Upsilon(X)$, where the latter $\Upsilon$ denotes the operator defined in [Ra96] Definition 4.3. The statement we want to prove is [Ra96] Theorem 4.1 except that we have to replace the theory $\mathbf{T_0}\restriction +\mathbf{MID}$ by $\mathcal{M}_{<\gamma}$. Upon nearer inspection of the proof of [Ra96] Theorem 4.1, one sees that in works in $\mathcal{M}_{<\gamma}$ as well. □

**Definition 2.13** For every first order formula $\theta(x,X)$ of $\mathcal{M}_{<\gamma}$ and variables $x, X$ we notate the first order definable set $\Theta^\infty$ of Proposition 2.12 by $\mu_\nu xX.\theta(x,X)$.

**Proposition 2.14** *To every first order formula $\theta(x,y,X,Y)$ of $\mathcal{M}_{<\gamma}$, $\delta < \gamma$, and variables $x,y,X,Y$ we can assign a first order definable set $S$ such that*

$$\mathcal{M}_{<\gamma} \;\;\vdash\;\; \forall\alpha \le \delta\, S_\alpha \;=\; \nu_\mu xX.\,\theta(x,\alpha,X,S_{<\alpha}) \tag{13}$$

*where $S_\alpha := \{u : (\alpha,u) \in S\}$ and $S_{<\alpha} := \{(\beta,v) \in S : \beta < \alpha\}$.*

**Proof.** Let $\Upsilon(x,y,X,Y)$ be the formula of Proposition 2.12 assigned to $\theta$ and the variables $x, X$. Then $\Upsilon(x,y,X,Y)$ is $\mathcal{M}_{<\gamma}$-provably monotone with respect to $x,X$. Let $\Upsilon_{st}(x,y,X,Y)$ be the formula introduced in Lemma 2.11 that inductively defines the stage comparison relation on $\mu xX.\Upsilon(x,y,X,Y)$. This formula is also $\mathcal{M}_{<\gamma}$-provably monotone with respect to $x,X$. Let

$$
\begin{aligned}
\tilde{Y}(\alpha) &:= \{(\beta,(z)_1) : z \in Y_\beta \;\wedge\; (z)_0 = 1 \;\wedge\; \beta < \alpha\}\\
\tilde{\Upsilon}(x,\alpha,X,Y) &:= \exists u,v\,\big(x = (u,v) \;\wedge\; \big([u = 0 \;\wedge\; \Upsilon_{st}(v,\alpha,\{y : (0,y) \in X\},\tilde{Y}(\alpha))]\\
&\qquad \vee\; [u = 1 \;\wedge\; \exists w\,\Upsilon((0,v,v),\alpha,\{z : (0,(z,w)) \in X\},\tilde{Y}(\alpha))]\big)\big).
\end{aligned}
$$

Note that $\tilde{\Upsilon}(x,\alpha,X,Y)$ is monotone (provably so in $\mathcal{M}_{<\gamma}$) with respect to $x,X$. Thus $T := \mu xyXY.\tilde{\Upsilon}$ is a term of $\mathcal{M}_{<\gamma}$. Finally put

$$S \;\;:=\;\; \{(\alpha,w) : (\alpha,(1,w)) \in T\}.$$

We shall now prove (13). Letting $Y = T_{<\alpha}$ we have

$$
\begin{aligned}
\tilde{Y}(\alpha) &= \{(\beta,(z)_1) : z \in T_\beta \;\wedge\; (z)_0 = 1 \;\wedge\; \beta < \alpha\}\\
&= S_{<\alpha}.
\end{aligned}
$$

Let

$$\begin{array}{rcl} A & := & \mu x X.\Upsilon_{st}(x,\alpha,X,S_{<\alpha}), \\ B & := & \{v:\,\exists w\,\Upsilon((0,v,v),\alpha,\{z:\,(z,w)\in A\},S_{<\alpha})\}, \\ C & := & \{(0,y):\,y\in A\}\,\cup\,\{(1,z):\,z\in B\}. \end{array}$$

As $\forall v\,[v\in A\leftrightarrow\Upsilon_{st}(v,\alpha,A,S_{<\alpha})]$ we then have

$$\begin{array}{rcl} \tilde{\Upsilon}(x,\alpha,C,T_{<\alpha}) & \leftrightarrow & \exists u,v\,\big(x=(u,v)\,\wedge\,\big([u=0\,\wedge\,\Upsilon_{st}(v,\alpha,A,S_{<\alpha})] \\ & & \quad\vee\,[u=1\,\wedge\,\exists w\,\Upsilon((0,v,v),\alpha,\{z:\,(z,w)\in A\},S_{<\alpha})]\big)\big) \\ & \leftrightarrow & \exists u,v\,(\,x=(u,v)\,\wedge\,([u=0\wedge v\in A]\,\vee\,[u=1\wedge v\in B])) \\ & \leftrightarrow & x\in C, \end{array} \tag{14}$$

so that

$$T_\alpha \quad\subseteq\quad C. \tag{15}$$

From $\forall x\,[x\in T_\alpha\leftrightarrow\tilde{\Upsilon}(x,\alpha,T_\alpha,T_{<\alpha})]$ it follows that

$$\forall v\,[(0,v)\in T_\alpha\leftrightarrow\Upsilon_{st}(v,\alpha,\{y:\,(0,y)\in T_\alpha\},S_{<\alpha})]$$

and hence

$$A\subseteq\{y:\,(0,y)\in T_\alpha\}. \tag{16}$$

(15) and (16) together yield $\{y:\,(0,y)\in T_\alpha\}=A$. Moreover, as

$$\forall v\,[(1,v)\in T_\alpha\leftrightarrow\exists w\,\Upsilon((0,v,v),\alpha,\{z:\,(z,w)\in\{y:\,(0,y)\in T_\alpha\},S_{<\alpha})]$$

we also get $B=\{v:\,(1,v)\in T_\alpha\}$ and consequently arrive at

$$T_\alpha \quad=\quad C \tag{17}$$

$$S_\alpha \quad=\quad B. \tag{18}$$

In view of Proposition 2.12 this entails that

$$S_\alpha\;=\;\mu_\nu xX.\,\theta(x,\alpha,X,S_{<\alpha}).$$

□

Next, we want to show that $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})_0$ and $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})$ can be reduced to $\mathcal{M}_{<\omega}$ and $\mathcal{M}_{<\varepsilon_0}$, respectively. Here we shall draw on [Ra98]. It follows from [Ra98] Theorem 3.15, Corollary 3.16, Corollary 4.25 and Corollary 4.29 that $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})_0$ and $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})$ can be reduced to certain operator theories $\mathbf{T}^{\mathbf{OP}}_{<\omega}$ and $\mathbf{T}^{\mathbf{OP}}_{<\varepsilon_0}$, respectively. More precisely, we have

**Theorem 2.15** *(i)* $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})_0$ *and* $\mathbf{T}^{\mathbf{OP}}_{<\omega}$ *prove the same* $\Pi^1_3$ *sentences.*
*(ii)* $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})$ *and* $\mathbf{T}^{\mathbf{OP}}_{<\varepsilon_0}$ *prove the same* $\Pi^1_3$ *sentences.*

Thus it suffices to show that $\mathbf{T}^{\mathbf{OP}}_{<\omega}$ and $\mathbf{T}^{\mathbf{OP}}_{<\varepsilon_0}$ can be interpreted in $\mathcal{M}_{<\omega}$ and $\mathcal{M}_{<\varepsilon_0}$, respectively. The foregoing theories are based on axioms for finite and transfinite iterations of certain operators. $\mathbf{T}^{\mathbf{OP}}_{<\omega}$ is the the theory $(\mathbf{\Pi}^0_\infty-\mathbf{CA})_0$ augmented by the axioms

$$\mathbf{OP}_n \qquad \forall\vec{X}\forall Y\exists Z\,\Phi^{\vec{X}}_n(Y)=Z$$

for all $n$.

$\mathbf{T}^{\mathbf{OP}}_{<\varepsilon_0}$ be the theory $(\mathbf{\Pi}^0_\infty - \mathbf{CA})_0$ augmented by the axioms

$$\mathbf{OP}_\alpha \qquad \forall \vec{X} \forall Y \exists Z \, \Phi^{\vec{X}}_\alpha(Y) = Z$$

for all $\alpha < \varepsilon_0$.
For a detailed account of the syntax and axioms of $\mathbf{T}^{\mathbf{OP}}_{<\omega}$ and $\mathbf{T}^{\mathbf{OP}}_{<\varepsilon_0}$ we refer to [Ra98]. The crucial observation is that the operators $\Phi^{\vec{X}}_n$ can be defined in $\mathcal{M}_{<\omega}$ and the operators $\Phi^{\vec{X}}_\alpha$ can be defined in $\mathcal{M}_{<\varepsilon_0}$, using Proposition 2.14. As a result we can conclude the following theorem.

**Theorem 2.16** *(i) $\mathcal{M}_{<\omega}$ has the same proof-theoretic strength as $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})_0$.*

*(ii) $\mathcal{M}_{<\varepsilon_0}$ has the same proof-theoretic strength as $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})$.*

**Proof**: In light of the foregoing remarks we only need to show that $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})_0$ and $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})$ can accommodate $\mathcal{M}_\omega$ and $\mathcal{M}_{\varepsilon_0}$, respectively. This follows from Theorem 2.8, i.e., [Ra98] Theorem 3.15 and Corollary 3.16. □

# 3 Double-negation translation

Fix $\gamma \leq \varepsilon_0$. Let $\mathcal{M}^i_{<\gamma}$ result from $\mathcal{M}_{<\gamma}$ by changing the logic from classical to intuitionistic. In this section, following the method of [Tu04, Section 1], we will prove that $\mathcal{M}^i_{<\gamma}$ has the same strength as $\mathcal{M}_{<\gamma}$.

**Definition 3.1** (Negative, completely negative, $\varphi^N$)
*A formula $\varphi$ is* negative *iff occurrences of every atom, $\vee$, $\exists x$ or $\exists X$ not in the scope of the $\mu$-symbol in $\varphi$ are negated.*
*For any $\varphi$ by $\varphi^N$ we define the formula obtained from $\varphi$ by putting $\neg\neg$ in front of every atom, $\vee$, $\exists x$ or $\exists X$ in $\varphi$ not in the scope of $\mu$.*
*An expression is* completely negative *iff all occurrences of atoms, $\vee$, $\exists x$ and $\exists X$ in it, including those in the scope of $\mu$, are negated.*

Note that $\varphi^N$ is negative and $\mathcal{M}(\varphi) = \mathcal{M}(\varphi^N)$ for every formula $\varphi$.

**Definition 3.2** (Complete negation operation $N(e)$)
*For any expression $e$ we define $N(e)$ recursively as follows:*

1. *$N(e) := e$ if $e$ is $\perp$, a number term or a set variable.*
2. *$N$ commutes with $=$, $\in$ and logical connectives.*
3. *$N(\mu xyXY.\varphi) := \mu xyXY.(N(\varphi))^N$.*

Note that if $e$ is a $\mu$-term then $N(e)$ is completely negative.

**Lemma 3.3** *For any $\mu$-term $T$ we have*

$$\mathcal{M}(N(T)) = \{\langle \ell_i, \mu x_i y_i X_i Y_i.(N(\varphi_i))^N \rangle \mid \langle \ell_i, \mu x_i y_i X_i Y_i.\varphi_i \rangle \in \mathcal{M}(T)\}.$$

**Proof.** By induction on $T$. □

The calculus $\mathcal{M}^N_{<\gamma}$ is the same as $\mathcal{M}_{<\gamma}$, with the only difference that $\mathbf{LFP}_\alpha[\varphi[x,y,X,Y],T]$ is replaced by an axiom $\mathbf{LFP}^N_\alpha[\varphi[x,y,X,Y],T]$, where $T$ is $\mu xyXY.\varphi$, which is

$$\begin{aligned} \mathrm{Mon}(\mathcal{M}(N(T),\alpha) \quad \to \quad & \forall x\,(x \in (N(T))_y \leftrightarrow N(\varphi)[x,y,(N(T))_y,(N(T))_{<y}]) \wedge \\ & \forall U(\forall z(N(\varphi)[z,y,U,(N(T))_{<y}] \to z \in U) \to (N(T))_y \subseteq U), \end{aligned} \tag{19}$$

where $\mathrm{Mon}(\mathcal{M}(N(T),\alpha)$ signifies the conjunction of all formulas

$$\forall \vec{z_i}\forall \vec{Z_i}\forall y_i{<}\alpha\forall Y_i\forall X_i'\forall X_i''\,(X_i' \subseteq X_i'' \to (N(\varphi_i[x_i,y_i,X',Y_i]) \to N(\varphi_i[x_i,y_i,X_i'',Y_i])))$$

with $\langle(\vec{z_i},\vec{Z_i}),\mu x_i y_i X_i Y_i.\varphi_i\rangle \in \mathcal{M}(T)$.

**Lemma 3.4** *$\mathcal{M}_{<\gamma}$ can be interpreted in $\mathcal{M}^N_{<\gamma}$.*

**Proof.** Given a derivation $\mathsf{d}$ in $\mathcal{M}_{<\gamma}$, replace every formula $\varphi$ in $\mathsf{d}$ by $N(\varphi)$ in order to obtain a derivation in $\mathcal{M}^N_{<\gamma}$. The only little thing to check is that $N\left(\mathbf{LFP}_\gamma[\varphi[x,y,X,Y],\mu xyXY.\varphi]\right)$ is of the form (19), but this is straightforward. $\square$

For a set $Z$ by $\overline{Z}$ we denote the set $\{x\,|\,\neg x \in Z\}$, which exists by Arithmetical Comprehension. Below we use the standard notation $Y \doteq Z$ to mean $\forall x(x \in Y \leftrightarrow x \in Z)$.

**Lemma 3.5** (Extensionality Lemma)
*For every $\alpha < \gamma$ $\mathcal{M}^i_{<\gamma}$ proves that if $Z_1 \doteq Z_2$ then:*
*(a) if $\mathrm{Mon}(\mathcal{M}(\varphi[Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(\varphi[Z_2]),\alpha)$ then $\varphi[Z_1] \leftrightarrow \varphi[Z_2]$, for every formula $\varphi$;*
*(b) for a term $T[Z] := \mu xyXY.\varphi[x,y,X,Y,Z]$ if*

$$\mathrm{Mon}(\mathcal{M}(T[Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(T[Z_2]),\alpha)$$

*then $\forall y{<}\alpha(\mu xyXY.\varphi[Z_1])_y \doteq (\mu xyXY.\varphi[Z_2])_y$, for every first-order formula $\varphi$.*

**Proof** proceeds by induction on the buildup of an expression $e[Z]$. Below we use IH as an abbreviation for "induction hypothesis". The assertion is obvious when $e$ is an elementary atom, i.e. an atom not of the form $s \in S$ where $S$ is a $\mu$-term. The induction step for logical connectives is also straightforward, we consider $\to$ and $\exists z$ for illustration.
Assume $\varphi$ is $\varphi_0 \to \varphi_1$. From $\mathrm{Mon}(\mathcal{M}(\varphi[Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(\varphi[Z_2]),\alpha)$ we get
$\mathrm{Mon}(\mathcal{M}(\varphi_i[Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(\varphi_i[Z_2]),\alpha)$ for both $i = 0,1$. By IH $\varphi_i[Z_1] \leftrightarrow \varphi_i[Z_2]$ for both $i = 0,1$, thus yielding $\varphi[Z_1] \leftrightarrow \varphi[Z_2]$.
Assume $\varphi$ is $\exists z\psi[z]$. From $\mathrm{Mon}(\mathcal{M}(\varphi[Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(\varphi[Z_2]),\alpha)$ we get
$\mathrm{Mon}(\mathcal{M}(\psi[z,Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(\psi[z,Z_2]),\alpha)$, for any $z$. By IH $\psi[z,Z_1] \leftrightarrow \psi[z,Z_2]$, for any $z$, thus yielding $\varphi[Z_1] \leftrightarrow \varphi[Z_2]$. The induction step for (b) requires subsidiary transfinite induction on $y$. So assuming $\mathrm{Mon}(\mathcal{M}(T[Z_1]),\alpha) \wedge \mathrm{Mon}(\mathcal{M}(T[Z_2]),\alpha)$, $\varphi[x,y,X,Y,Z_1] \leftrightarrow \varphi[x,y,X,Y,Z_2]$, $y < \alpha$, $(T[Z_1])_z \doteq (T[Z_2])_z$ for all $z < y$ and $\varphi$ be first-order, it remains to show $(T[Z_1])_y := (\mu xyXY.\varphi[x,y,X,Y,Z_1])_y \doteq (\mu xyXY.\varphi[x,y,X,Y,Z_2])_y =: (T[Z_2])_y$. From (6) we have

$$\begin{aligned}&\forall x\,(x \in (T[Z_1])_y \leftrightarrow \varphi[x,y,(T[Z_1])_y,(T[Z_1])_{<y},Z_1]) \;\wedge \\ &\forall U(\forall z(\varphi[z,y,U,(T[Z_1])_{<y},Z_1] \to z \in U) \to (T[Z_1])_y \subseteq U)\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\forall x\,(x \in (T[Z_2])_y \leftrightarrow \varphi[x,y,(T[Z_2])_y,(T[Z_2])_{<y},Z_2]) \;\wedge \\ &\forall U(\forall z(\varphi[z,y,U,(T[Z_2])_{<y},Z_2] \to z \in U) \to (T[Z_2])_y \subseteq U).\end{aligned} \tag{21}$$

From the first conjunct of (20), by the main IH, we have

$$\forall x\,(x \in (T[Z_1])_y \leftrightarrow \varphi[x,y,(T[Z_1])_y,(T[Z_1])_{<y},Z_2]);$$

applying the subsiduary IH and then the main IH again we arrive at

$$\forall x\,(x \in (T[Z_1])_y \leftrightarrow \varphi[x,y,(T[Z_1])_y,(T[Z_2])_{<y},Z_2]).$$

From the second conjunct of (21), taking $U := (T[Z_1])_y$, we obtain $(T[Z_2])_y \subseteq (T[Z_1])_y$. Similarly we get $(T[Z_1])_y \subseteq (T[Z_2])_y$. Together this gives $(T[Z_1])_y \doteq (T[Z_2])_y$. $\square$

**Lemma 3.6** *$\mathcal{M}^N_{<\gamma}$ can be interpreted in $\mathcal{M}^i_{<\gamma}$.*

**Proof.** Apply the double-negation translation $(\cdot)^N$. Classical logic goes into intuitionistic. It's easily checked (and well-known) that translations of all axioms of $\mathbf{ACA_0}(\mathcal{L}_\mu)$ are derivable intuitionistically from the axioms of $\mathbf{ACA_0}(\mathcal{L}_\mu)$. Transfinite induction (7) is also no problem, since $<$ is a recursive relation and $\neg\neg z_1 < z_2$ can be equivalently replaced by $z_1 < z_2$. So we need only derive in $\mathcal{M}^i_{<\gamma}$ the formula $\Big(\mathbf{LFP}^N_\gamma[\varphi[x,y,X,Y],T]\Big)^N$ $(T := \mu xyXY.\varphi)$, i.e.

$$\begin{aligned}&\textstyle\bigwedge_{\langle(\vec{z_i},\vec{Z_i}),\mu x_i y_i X_i Y_i.\varphi_i\rangle\in\mathcal{M}(T)} \forall\vec{z_i}\forall\vec{Z_i}\forall y_i \overline{\overline{<}}\alpha\forall Y_i\forall X_i'\forall X_i''\\ &\qquad\Big(\overline{\overline{X_i'}}\subseteq\overline{\overline{X_i''}}\to\Big(\big(N(\varphi_i[x_i,y_i,X',Y_i])\big)^N\to\big(N(\varphi_i[x_i,y_i,X_i'',Y_i])\big)^N\Big)\Big)\to\\ &\forall y\overline{\overline{<}}\alpha\Big(\forall x\,(x\in\overline{\overline{N(T)}}_y\leftrightarrow(N(\varphi))^N[x,y,(N(T))_y,(N(T))_{<y}])\wedge\\ &\qquad\forall U(\forall z((N(\varphi))^N[z,y,U,(N(T))_{<y}]\to z\in\overline{\overline{U}})\to\overline{\overline{N(T)}}_y\subseteq\overline{\overline{U}})\Big),\end{aligned}\tag{22}$$

where by $z_1\overline{\overline{<}}z_2$ we denote $\neg\neg z_1 < z_2$. As remarked above, $\overline{\overline{<}}$ can be equivalently replaced by $<$; also, by Lemma 3.3 the premise of the implication (22) can be equivalently replaced by

$$\begin{aligned}&\textstyle\bigwedge_{\langle(\vec{z_i},\vec{Z_i}),\mu x_i y_i X_i Y_i.\varphi_i\rangle\in\mathcal{M}(N(T))} \forall\vec{z_i}\forall\vec{Z_i}\forall y_i<\alpha\forall Y_i\forall X_i'\forall X_i''\\ &\qquad\Big(\overline{\overline{X_i'}}\subseteq\overline{\overline{X_i''}}\to(\varphi_i[x_i,y_i,X',Y_i]\to\varphi_i[x_i,y_i,X_i'',Y_i])\Big).\end{aligned}$$

We will derive now in $\mathcal{M}^i_{<\gamma}$ the formula

$$\begin{aligned}&\textstyle\bigwedge_{\langle(\vec{z_i},\vec{Z_i}),\mu x_i y_i X_i Y_i.\varphi_i\rangle\in\mathcal{M}(T)} \forall\vec{z_i}\forall\vec{Z_i}\forall y_i<\alpha\forall Y_i\forall X_i'\forall X_i''\\ &\qquad\Big(X_i'\subseteq X_i''\to\Big(\big(N(\varphi_i[x_i,y_i,X',Y_i])\big)^N\to\big(N(\varphi_i[x_i,y_i,X_i'',Y_i])\big)^N\Big)\Big)\to\\ &\forall y<\alpha\Big(\forall x\,(x\in\overline{\overline{N(T)}}_y\leftrightarrow(N(\varphi))^N[x,y,(N(T))_y,(N(T))_{<y}])\wedge\\ &\qquad\forall U(\forall z((N(\varphi))^N[z,y,U,(N(T))_{<y}]\to z\in\overline{\overline{U}})\to\overline{\overline{N(T)}}_y\subseteq\overline{\overline{U}})\Big),\end{aligned}\tag{23}$$

which is stronger than (22) and will imply the latter by intuitionistic logic. So assume

$$\begin{aligned}&\textstyle\bigwedge_{\langle(\vec{z_i},\vec{Z_i}),\mu x_i y_i X_i Y_i.\varphi_i\rangle\in\mathcal{M}(N(T))} \forall\vec{z_i}\forall\vec{Z_i}\forall y_i<\alpha\forall Y_i\forall X_i'\forall X_i''\\ &\qquad\big(X_i'\subseteq X_i''\to(\varphi_i[x_i,y_i,X',Y_i]\to\varphi_i[x_i,y_i,X_i'',Y_i])\big);\end{aligned}$$

this means $\mathrm{Mon}(\mathcal{M}(N(T)),\alpha)$. By the $\mathbf{LFP}_\gamma[(N(\varphi))^N,N(T)]$ axiom of $\mathcal{M}^i_{<\gamma}$ we get now $\forall y<\alpha\,\mathrm{LFP}_{x,X}[(N(\varphi))^N[x,y,X,(N(T))_{<y}],(N(T))_y]$, i.e.

$$\begin{aligned}&\forall y<\alpha\Big(\forall x\,(x\in N(T)_y\leftrightarrow(N(\varphi))^N[x,y,(N(T))_y,(N(T))_{<y}])\wedge\\ &\qquad\forall U(\forall z((N(\varphi))^N[z,y,U,(N(T))_{<y}]\to z\in U)\to N(T)_y\subseteq U)\Big).\end{aligned}\tag{24}$$

Fixing $y<\alpha$, the first conjunct in the conclusion of (23) is derived by prenexing $\neg\neg$ to the first conjunct in (24). For the second conjunct, given $U$, assume $\forall z\big((N(\varphi))^N[z,y,U,(N(T))_{<y}]\to z\in\overline{\overline{U}}\big)$. For any $z$, we have $\mathrm{Mon}(\mathcal{M}((N(\varphi))^N[z,y,U,(N(T))_{<y}]),\alpha)$. Since the formula $(N(\varphi))^N$ is completely negative, we can use the fact $\overline{U}\doteq\overline{\overline{\overline{U}}}$ and Lemma 3.5 to prove by induction $\mathrm{Mon}(\mathcal{M}((N(\varphi))^N[z,y,\overline{\overline{U}},(N(T))_{<y}]),\alpha)$. Now using again complete negativeness, the fact $\overline{U}\doteq\overline{\overline{\overline{U}}}$ and Lemma 3.5(a), we obtain $(N(\varphi))^N[z,y,U,(N(T))_{<y}]\leftrightarrow(N(\varphi))^N[z,y,\overline{\overline{U}},(N(T))_{<y}]$, for any $z$. Therefore we can conclude $\forall z\big((N(\varphi))^N[z,y,\overline{\overline{U}},(N(T))_{<y}]\to z\in\overline{\overline{U}}\big)$. By the second conjunct of (24) we obtain $(N(T))_y\subseteq\overline{\overline{U}}$, which intuitionistically implies $\overline{\overline{N(T)}}_y\subseteq\overline{\overline{U}}$. □

**Theorem 3.7** *$\mathcal{M}^i_{<\gamma}$ has the strength of $\mathcal{M}_{<\gamma}$.*

**Proof.** This follows now from Lemmata 3.4 and 3.6. □

# 4 Embedding into intuitionistic Explicit Mathematics

In this section we will give embeddings of systems $\mathcal{M}^i_\omega$ and $\mathcal{M}^i_{\varepsilon_0}$ into theories of intuitionistic Explicit Mathematics $\mathbf{EETJ}\!\upharpoonright + \mathbf{UMID_N}$ and $\mathbf{EETJ} + \mathbf{UMID_N}$, respectively. Together with Theorems 2.16 and 3.7 above and Theorem 6.1 of [Ra99] about classical $\mathbf{EETJ}^c\!\upharpoonright + \mathbf{UMID_N}$ and $\mathbf{EETJ}^c + \mathbf{UMID_N}$ this will prove the following

**Theorem 4.1** *(a)* $\mathbf{T_0}\!\upharpoonright + \mathbf{UMID_N}$ *has exactly the strength of* $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})_0$*;*
*(b)* $\mathbf{T_0}\!\upharpoonright + \mathbf{Ind_N} + \mathbf{UMID_N}$ *has exactly the strength of* $(\mathbf{\Pi}^1_2\text{-}\mathbf{CA})$*.*

The proof of part (a) of this theorem is another way to the main result of [Tu04]; part (b) is a new result.

## 4.1 Explicit Mathematics: a reminder

**Language $\mathcal{L}_{\mathrm{EM}}$.** All theories of Explicit Mathematics, considered in this paper, are formulated in a two-sorted language, containing variables for operations (individuals) and names, along with operation constants. Names are thought of as a special kind of operations, coding types (sets) of operations. We use **variables** $a, b, c, \dots$ as ranging over operations, and $A, B, C, \dots$ as ranging over names. The main **constants** of $\mathcal{L}_{\mathrm{EM}}$ are the following: combinators $\mathsf{k}$, $\mathsf{s}$, pairing $\mathsf{p}$ and projections $\mathsf{p_0}$, $\mathsf{p_1}$, zero 0, successor $\mathsf{s_N}$ and predecessor $\mathsf{p_N}$, distinction by cases on natural numbers $\mathsf{d_N}$, join $\mathsf{j}$ and inductive generation $\mathsf{i}$. Additionally we have the following 9 **constants** called *name generators*: $\mathsf{nat}$, $\mathsf{id}$, $\mathsf{inv}$, $\mathsf{emp}$, $\mathsf{and}$, $\mathsf{or}$, $\mathsf{imp}$, $\mathsf{all}$, $\mathsf{ex}$. **Terms** are built from variables and constants by the following application clause: if $s$ and $t$ are *terms* then $s \cdot t$ (also written as $st$) is a *term*, so that the *application* function symbol $\cdot$ accepts arguments of both sorts and returns an operation. In writing terms, parentheses are thought of as associated to the left. **Atomic formulas** are $\bot$ (falsity), $s = t$ ($s$ coincides with $t$) and $s \,\varepsilon\, t$ ($s$ belongs to the type named by $t$, $s$ is classified under $t$), where $s$ and $t$ are terms. **Formulas** are built from atomic formulas by $\wedge, \vee, \rightarrow$ and two kinds of quantifiers, over operations and over names, e.g. $\forall a, \exists a, \forall A, \exists A$. Finally, **expression** is a term or a formula. We use the following standard abbreviations:
$\neg F \;:\Leftrightarrow\; F \rightarrow \bot$;
$F_0 \leftrightarrow F_1 \;:\Leftrightarrow\; (F_0 \rightarrow F_1) \wedge (F_1 \rightarrow F_0)$;
$t\!\downarrow \;:\Leftrightarrow\; \exists x(t = x)$;
$\mathcal{N}[t] \;:\Leftrightarrow\; \exists A(t = A)$;
$t \doteq \{s[x_1, \dots, x_n] \mid F[x_1, \dots, x_n]\} \;:\Leftrightarrow\; \mathcal{N}[t] \wedge \forall x(x \,\varepsilon\, t \leftrightarrow \exists x_1 \dots \exists x_n(x = s[x_1, \dots, x_n] \wedge F[x_1, \dots, x_n]))$;
$s \simeq t \;:\Leftrightarrow\; (s\!\downarrow \vee\, t\!\downarrow) \rightarrow s = t$;
$s \subseteq t \;:\Leftrightarrow\; \forall x \,\varepsilon\, s(x \,\varepsilon\, t)$; $s \doteq t \;:\Leftrightarrow\; s \,\dot\subseteq\, t \wedge t \,\dot\subseteq\, s$;
$t'$ for $\mathsf{s_N} \cdot t$; 1 for $0'$; $st$ for $s \cdot t$; $t(s_1, \dots, s_n)$ for $(\dots(ts_1)\dots s_n)$; $\langle s, t\rangle$ for $\mathsf{p}st$; $s \neq t$ for $\neg s = t$, etc.
**Logic.** Intuitionistic 2-sorted logic of partial terms with equality.
**Axioms.** The axioms are divided in several groups, according to their nature.

**I. Applicative axioms.** These axioms formalise that operations form a partial combinatory algebra, that we have pairing and projections, usual closure conditions on natural numbers, as well as definition by numerical cases: (1) $\mathsf{k}ab = a$; (2) $\mathsf{s}ab\!\downarrow \wedge\, \mathsf{s}abc \simeq ac(bc)$; (3) $\mathsf{p}ab\!\downarrow \wedge\, \mathsf{p_0}a\!\downarrow \wedge\, \mathsf{p_1}a\!\downarrow \wedge\, \mathsf{p_0}(\mathsf{p}ab) = a \wedge \mathsf{p_1}(\mathsf{p}ab) = b$; (4) $0 \,\varepsilon\, \mathsf{nat} \wedge \forall x \,\varepsilon\, \mathsf{nat}(\mathsf{s_N}x \,\varepsilon\, \mathsf{nat})$; (5) $\forall x \,\varepsilon\, \mathsf{nat}(\mathsf{s_N}x \neq 0 \wedge \mathsf{p_N}(\mathsf{s_N}x) = x)$; (6) $\forall x \,\varepsilon\, \mathsf{nat}(x \neq 0 \rightarrow \mathsf{p_N}x \,\varepsilon\, \mathsf{nat} \wedge \mathsf{s_N}(\mathsf{p_N}x) = x)$; (7) $a \,\varepsilon\, \mathsf{nat} \wedge b \,\varepsilon\, \mathsf{nat} \rightarrow (a = b \rightarrow \mathsf{d_N}xyab = x) \wedge (a \neq b \rightarrow \mathsf{d_N}xyab = y)$.
**II. Induction on nat.** $F[0] \wedge \forall x(F[x] \rightarrow F[\mathsf{s_N}x]) \rightarrow \forall x \,\varepsilon\, \mathsf{nat} F[x]$,
for every formula $F$.
The following lemmata 4.2 and 4.3 are provable using only applicative axioms I (see, for example, [Fef79]).

**Lemma 4.2** $\lambda$-abstraction
*For every term $t[x]$ there exists a term $\lambda x.t[x]$ such that $\lambda x.t[x]\!\downarrow$ and for every term $s$*

$$s\!\downarrow \rightarrow (\lambda x.t[x])s \simeq t[s]).$$

**Lemma 4.3** Recursion Theorem
*There exists a closed term* $\mathsf{rec}$ *such that*

$$\mathsf{rec}f\!\downarrow \wedge\, \mathsf{rec}fx \simeq f(\mathsf{rec}f)x.$$

**III. Explicit representation.** This axiom states that each name is an operation: $\exists x(x = A)$.

**IV. Elementary comprehension (ECA).** These axiomatise *name generators*: (1) $\mathcal{N}[\mathsf{nat}]$; (2) $\mathcal{N}[\mathsf{id}] \wedge \forall x(x\varepsilon\mathsf{id} \leftrightarrow x = \langle \mathsf{p_0}x, \mathsf{p_1}x\rangle \wedge \mathsf{p_0}x = \mathsf{p_1}x)$; (3) $\mathcal{N}[\mathsf{inv}(f,A)] \wedge \forall x(x\varepsilon\mathsf{inv}(f,A) \leftrightarrow fx\varepsilon A)$; (4) $\mathcal{N}[\mathsf{emp}] \wedge \forall x(x\varepsilon\mathsf{emp} \leftrightarrow \bot)$; (5) $\mathcal{N}[\mathsf{and}(A,B)] \wedge \forall x(x\,\varepsilon\,\mathsf{and}(A,B) \leftrightarrow x\,\varepsilon\,A \wedge x\,\varepsilon\,B)$; (6) $\mathcal{N}[\mathsf{or}(A,B)] \wedge \forall x(x\,\varepsilon\,\mathsf{or}(A,B) \leftrightarrow x\,\varepsilon\,A \vee x\,\varepsilon\,B)$; (7) $\mathcal{N}[\mathsf{imp}(A,B)] \wedge \forall x(x\,\varepsilon\,\mathsf{imp}(A,B) \leftrightarrow x\,\varepsilon\,A \to x\,\varepsilon\,B)$; (8) $\mathcal{N}[\mathsf{all}A] \wedge \forall x(x\,\varepsilon\,\mathsf{all}A \leftrightarrow \forall y(\langle x,y\rangle\,\varepsilon\,A))$; (9) $\mathcal{N}[\mathsf{ex}A] \wedge \forall x(x\,\varepsilon\,\mathsf{ex}A \leftrightarrow \exists y(\langle x,y\rangle\,\varepsilon\,A))$.

**Definition 4.4** Elementary formula
*A formula is* elementary *iff it's constructed from $\bot$, $s = t$ and $t\,\varepsilon\,A$ by means of $\wedge, \vee, \to, \forall x, \exists x$ only. (No occurrences of $t\,\varepsilon\,s$ with $s$ not a name variable and name quantifiers are allowed.)*

The following lemma reduces Elementary Comprehension to a finite number of its instances; its proof requires only axioms I, III and IV (see [Tu03, L.1.4]).

**Lemma 4.5 ECA**
*If a formula $F := F[x;\bar{a};\bar{A}]$ is elementary then there exists a term $\mathsf{t}_F^x$ such that $\mathrm{FV}(\mathsf{t}_F^x) = \mathrm{FV}(F) \setminus \{x\}$ and*

$$\mathcal{N}[\mathsf{t}_F^x] \wedge \forall x(x\,\varepsilon\,\mathsf{t}_F^x \leftrightarrow F).$$

**V. Join (J).** This axiom states that if $f$ is an operation from a type named by A, each value of which is a name, then $\mathsf{j}(A,f)$ names a disjoint union of all $fx$ for $x\,\varepsilon\,A$:

$$\forall x\,\varepsilon\,A\mathcal{N}[fx] \to \big(\mathcal{N}[\mathsf{j}(A,f)] \ \wedge\ \forall z(z\,\varepsilon\,\mathsf{j}(A,f) \leftrightarrow z = \langle \mathsf{p_0}z, \mathsf{p_1}z\rangle \wedge \mathsf{p_0}z\,\varepsilon\,A \wedge \mathsf{p_1}z\,\varepsilon\,f(\mathsf{p_0}z))\big)\,.$$

**VI. Inductive Generation (IG).** The first part of this axiom states that $\mathsf{i}(A,B)$ names a wellfounded part of a type named by A along an ordering named by B; the second part allows induction over that type for an arbitrary formula:

$$\begin{aligned}&\mathcal{N}[\mathsf{i}(A,B)] \ \wedge\ \forall x\,\varepsilon\,A(\forall y(\langle y,x\rangle\,\varepsilon\,B \to y\,\varepsilon\,\mathsf{i}(A,B)) \to x\,\varepsilon\,\mathsf{i}(A,B))\\ &\wedge\ (\forall x\,\varepsilon\,A(\forall y(\langle y,x\rangle\,\varepsilon\,B \to F[y]) \to F[x]) \to \forall x\,\varepsilon\,\mathsf{i}(A,B)F[x]),\end{aligned}$$

where $F$ is an arbitrary formula.

The theory $\mathsf{App}$ is the one containing only applicative axioms I; **EON** has axioms I–II. The theory **EONN** has axioms of the groups I–III. **EET** is **EONN** + **ECA**, **EETJ** is **EET** + **J** and $\mathbf{T_0}$ is **EETJ** + **IG**. By $\mathbf{T}{\restriction}$ we mean a version of a theory $\mathbf{T}$ where both induction on natural numbers II and inductive generation VI are restricted to formulas $F := x \in C$. By $\mathcal{L}_{\mathrm{EM},\mathsf{lfp}}$ we denote the language $\mathcal{L}_{\mathrm{EM}}$ extended by an operation constant $\mathsf{lfp}$. For the statement of **UMID** and $\mathbf{UMID_N}$ principles see e.g. [Ra02, Section 2.2]. We repeat these definitions here:

| | | |
|---|---|---|
| $\mathbf{Clop}[f,A]$ | means | $\forall X \subseteq A \exists Y \subseteq A\ fX = Y$; |
| $\mathbf{Mon}[f,A]$ | -"- | $\forall X \subseteq A \forall Y \subseteq A\ (X \subseteq Y \to fX \subseteq fY)$; |
| $\mathbf{Lfp}[Y,f,A]$ | -"- | $fY \subseteq Y \wedge Y \subseteq A \wedge \forall X \subseteq A\,(fX \subseteq X \to Y \subseteq X)$; |
| $\mathbf{UMID}_A$ | -"- | $\forall f\,\big(\mathbf{Clop}[f,A] \wedge \mathbf{Mon}[f,A] \to \mathbf{Lfp}[\mathsf{lfp}f,f,A]\big)$. |

Now, **UMID** is the principle $\mathbf{UMID_V}$, where $\mathbf{V} \doteq \{x \mid x = x\}$ is (a name of) the universal type, and $\mathbf{UMID_N}$ is $\mathbf{UMID}_{\mathsf{nat}}$.

$\mathbf{Mon}[f,A]$ above means that the operation $f$ is *monotone on A*. Plain "*f monotone*" means that $f$ is monotone on $\mathbf{V}$.

**Lemma 4.6** *Define* $\mathbf{Lfp}'[Y,f,A]$ *to be* $\mathbf{Lfp}[Y,f,A] \wedge Y \subseteq fY$, $\mathbf{UMID}'_A$ *to be* $\forall f\,\big(\mathbf{Clop}[f,A] \wedge \mathbf{Mon}[f,A] \to \mathbf{Lfp}'[\mathsf{lfp}f,f,A]\big)$. *Then, on the basis of intuitionistic logic,* $\mathbf{UMID}_A \leftrightarrow \mathbf{UMID}'_A$.

**Proof.** The direction $\leftarrow$ is obvious. For $\to$, assume $\mathbf{UMID}_A$, $\mathbf{Clop}[f,A]$ and $\mathbf{Mon}[f,A]$. This implies $\mathsf{lfp}f \subseteq A$ and $f(\mathsf{lfp}f) \subseteq \mathsf{lfp}f$. By monotonicity of $f$ we have $f(f(\mathsf{lfp}f)) \subseteq f(\mathsf{lfp}f)$. But this yields $\mathsf{lfp}f \subseteq f(\mathsf{lfp}f)$ by the remaining part of the $\mathbf{Lfp}[\mathsf{lfp}f,f,A]$ assertion. □

For every number term $t$ of $\mathcal{L}_\mu$ one defines in the standard way its translation $t^{\mathrm{EM}}$ into the language $\mathcal{L}_{\mathrm{EM}}$.

### 4.2 Embeddings of $\mathcal{M}^i_\omega$ and $\mathcal{M}^i_{\varepsilon_0}$

In this subsection we start with proving Theorem 4.1(a), by showing how to translate a derivation in $\mathcal{M}^i_\omega$ into a derivation in $\mathbf{EETJ}{\restriction} + \mathbf{UMID_N}$. Changes necessary to upgrade the argument to $\mathcal{M}^i_{\varepsilon_0}$ (Theorem 4.1(b)) will be indicated in the end. First we note that every proof in $\mathcal{M}^i_\omega$ is a proof in $\mathcal{M}^i_n$ for some natural number $n$; from now on we fix this $n$. In Explicit Mathematics, we reason here in $\mathbf{EETJ}{\restriction} + \mathbf{UMID_N}$.

**Lemma 4.7** *There is an operation* $\mathsf{J}$ *s.t. if* $\forall y < n \forall Y \subseteq \mathsf{nat}\,(\mathbf{Clop}[fyY, \mathsf{nat}] \wedge \mathbf{Mon}[fyY, \mathsf{nat}])$ *then the following holds:*
*(a)* $\exists Z \subseteq \mathsf{nat}\,(Z = \mathsf{J}f)$;
*(b) for every* $y < n$ $(\mathsf{J}f)_y \doteq \mathsf{lfp}(fy \bigoplus_{z<y} (\mathsf{J}f)_z)$, *where* $A_i := \{x \,\varepsilon\, \mathsf{nat} \,|\, (i, x) \,\varepsilon\, A\}$,
$\bigoplus_{z<y} A_z := \{(z, x) \,|\, z < y \wedge (z, x) \,\varepsilon\, A\}$, $(\cdot,\cdot)$ *and* $<$ *denote (translations of) appropriate operations/relations on natural numbers.*

**Proof.** Assume $\forall y < n \forall Y \subseteq \mathsf{nat}\,(\mathbf{Clop}[fyY, \mathsf{nat}] \wedge \mathbf{Mon}[fyY, \mathsf{nat}])$. By the $\mathbf{UMID_N}$ axiom we can define types $\mathsf{J}^y$, for all $y < n$, to satisfy $\mathsf{J}^y = \mathsf{lfp}(fy \bigoplus_{z<y} \mathsf{J}^z)$. Finally we put $\mathsf{J}f := \bigoplus_{z<n} \mathsf{J}^z$. □

**Definition 4.8** $(\mathsf{t}^x_\varphi,\, f^{x,X}_\varphi)$
*For every first-order formula* $\varphi$ *and a variable* $x$ *of* $\mathcal{L}_\mu$ *we define a term* $\mathsf{t}^x_\varphi$ *of* $\mathcal{L}_{\mathrm{EM},\mathsf{lfp}}$ *by recursion on* $\varphi$ *in the following way:*

$$\mathsf{t}^x_\varphi := \begin{cases} \mathsf{emp} & \textit{if } \varphi \textit{ is } \bot; \\ \mathsf{inv}(\lambda x.\langle s^{\mathrm{EM}}[x], \mathsf{t}^{\mathrm{EM}}[x]\rangle, \mathsf{id}) & \textit{if } \varphi \textit{ is } s[x] = t[x]; \\ \mathsf{inv}(\lambda x.s^{\mathrm{EM}}[x], X) & \textit{if } \varphi \textit{ is } s[x] \in X; \\ \mathsf{and}(\mathsf{t}^x_{\varphi_0[x]}, \mathsf{t}^x_{\varphi_1[x]}) & \textit{if } \varphi \textit{ is } \varphi_0[x] \wedge \varphi_1[x]; \\ \mathsf{or}(\mathsf{t}^x_{\varphi_0[x]}, \mathsf{t}^x_{\varphi_1[x]}) & \textit{if } \varphi \textit{ is } \varphi_0[x] \vee \varphi_1[x]; \\ \mathsf{imp}(\mathsf{t}^x_{\varphi_0[x]}, \mathsf{t}^x_{\varphi_1[x]}) & \textit{if } \varphi \textit{ is } \varphi_0[x] \to \varphi_1[x]; \\ \mathsf{t}^x_{\forall y \varepsilon \mathsf{nat}(\langle y,x\rangle \varepsilon \mathsf{j}(\mathsf{nat}, \lambda y.\mathsf{t}^x_{\psi[x,y]}))} & \textit{if } \varphi \textit{ is } \forall y \psi[x,y]; \\ \mathsf{t}^x_{\exists y \varepsilon \mathsf{nat}(\langle y,x\rangle \varepsilon \mathsf{j}(\mathsf{nat}, \lambda y.\mathsf{t}^x_{\psi[x,y]}))} & \textit{if } \varphi \textit{ is } \exists y \psi[x,y]; \\ \mathsf{inv}(\lambda x.\langle x, s^{\mathrm{EM}}[x]\rangle, \mathsf{j}(\mathsf{nat}, \lambda x.\mathsf{J}(f[x]))), & \\ \textit{where } f[x] := \lambda z \lambda Z \lambda Y.\mathsf{and}(\mathsf{t}^y_{\psi[x,y,z,Y,Z]}, \mathsf{nat}) & \textit{if } \varphi \textit{ is } s[x] \in \mu yzYZ.\psi[x,y,z,Y,Z], \end{cases}$$

*where in the quantifier clauses for an elementary formula* $\eta[x]$ *of* $\mathcal{L}_{\mathrm{EM},\mathsf{lfp}}$ $\mathsf{t}^x_{\eta[x]}$ *is the standard term s.t.* $\mathsf{t}^x_{\eta[x]} :\doteq \{x \,|\, \eta[x]\}$ *(see Lemma 4.5). The operation* $f^{x,X}_\varphi$ *is now defined as* $\lambda X.\mathsf{and}(\mathsf{t}^x_\varphi, \mathsf{nat})$.

**Definition 4.9** $(\varphi^{\mathrm{EM}})$
*For every formula* $\varphi$ *of* $\mathcal{L}_\mu$ *we define its translation* $\varphi^{\mathrm{EM}}$ *of* $\mathcal{L}_{\mathrm{EM},\mathsf{lfp}}$ *by recursion on* $\varphi$ *in the following way:*

$$\varphi^{\mathrm{EM}} := \begin{cases} \bot & \textit{if } \varphi \textit{ is } \bot; \\ s^{\mathrm{EM}} = t^{\mathrm{EM}} & \textit{if } \varphi \textit{ is } s = t; \\ s^{\mathrm{EM}} \in X & \textit{if } \varphi \textit{ is } s \in X; \\ \varphi_0^{\mathrm{EM}} \wedge \varphi_1^{\mathrm{EM}} & \textit{if } \varphi \textit{ is } \varphi_0 \wedge \varphi_1; \\ \varphi_0^{\mathrm{EM}} \vee \varphi_1^{\mathrm{EM}} & \textit{if } \varphi \textit{ is } \varphi_0 \vee \varphi_1; \\ \varphi_0^{\mathrm{EM}} \to \varphi_1^{\mathrm{EM}} & \textit{if } \varphi \textit{ is } \varphi_0 \to \varphi_1; \\ \forall x \,\varepsilon\, \mathsf{nat}\, \psi^{\mathrm{EM}}[x] & \textit{if } \varphi \textit{ is } \forall x \psi[x]; \\ \exists x \,\varepsilon\, \mathsf{nat}\, \psi^{\mathrm{EM}}[x] & \textit{if } \varphi \textit{ is } \exists x \psi[x]; \\ \forall X \subseteq \mathsf{nat}\, \psi^{\mathrm{EM}}[X] & \textit{if } \varphi \textit{ is } \forall X \psi[X]; \\ \exists X \subseteq \mathsf{nat}\, \psi^{\mathrm{EM}}[X] & \textit{if } \varphi \textit{ is } \exists X \psi[X]; \\ s^{\mathrm{EM}} \,\varepsilon\, \mathsf{J}(\lambda y \lambda Y.f^{x,X}_\psi) & \textit{if } \varphi \textit{ is } s \in \mu xyXY.\psi[x,y,X,Y]. \end{cases}$$

*In short, this translation leaves everything intact, except it relativizes quantifiers to (subtypes of)* $\mathsf{nat}$ *and replaces* $\mu$ *by iterated* $\mathsf{lfp}$ *applied to appropriate operations.*

**Lemma 4.10** *For a term $T := \mu xyXY.\varphi \in \mathcal{L}_\mu$ if $\big(\mathrm{Mon}(\mathcal{M}(T), n)\big)^{\mathrm{EM}}$ then the following holds:*
*(a) $\mathsf{t}^x_\varphi$ is a name, i.e. $\exists Z(Z = \mathsf{t}^x_\varphi)$;*
*(b) $\forall y{<}n \forall Y{\subseteq}\mathsf{nat}\,(\mathbf{Clop}(f^{x,X}_\varphi, \mathsf{nat}) \wedge \mathbf{Mon}(f^{x,X}_\varphi, \mathsf{nat}))$.*

**Proof.** (a) and (b) are proved simultaneously by induction on $T$. By IH we may assume that the assertion holds for all $\mu$-terms $\mu x_i y_i X_i Y_i.\chi$ occurring in $\varphi$ but not inside its $\mu$-terms. Assuming $\big(\mathrm{Mon}(\mathcal{M}(T), n)\big)^{\mathrm{EM}}$, we therefore have $\forall y_i < n \forall Y_i \subseteq \mathsf{nat} \forall X_i \subseteq \mathsf{nat}\, \exists Z_i(Z_i = \mathsf{t}^{x_i}_{\chi_i}) \;\wedge\; \forall y_i < n \forall Y_i \subseteq \mathsf{nat}\,(\mathbf{Clop}(f^{x_i,X_i}_{\chi_i}, \mathsf{nat}) \wedge \mathbf{Mon}(f^{x_i,X_i}_{\chi_i}, \mathsf{nat}))$, for all $i$, and for all $x \,\varepsilon\, \mathsf{nat}$, $y{<}n$, $X \subseteq \mathsf{nat}$ and $Y \subseteq \mathsf{nat}$. By the $\mathbf{LFP}_n[\chi_i, \mu x_i y_i X_i Y_i.\chi_i]$ axioms and Definition 4.8 this proves $\exists Z\,(Z = \mathsf{t}^x_\varphi)$, i.e. (a), for any $y < n$, $X \subseteq \mathsf{nat}$ and $Y \subseteq \mathsf{nat}$. Now another use of $\big(\mathrm{Mon}(\mathcal{M}(T), n)\big)^{\mathrm{EM}}$ and Definition 4.8 gives us (b). □

**Lemma 4.11** *If $\varphi[x, y, X, Y]$ is first-order then* $\mathbf{EETJ}{\upharpoonright} + \mathbf{UMID_N}$ *proves*
$\forall y{<}n \forall Y{\subseteq}\mathsf{nat} \forall x \,\varepsilon\, \mathsf{nat} \forall X{\subseteq}\mathsf{nat}\,\big(\varphi^{\mathrm{EM}}[x, y, X, Y] \leftrightarrow x \,\varepsilon\, (\lambda y \lambda Y. f^{x,X}_\varphi) y Y X\big)$.

**Proof.** This follows immediately from Definitions 4.8 and 4.9. □

**Lemma 4.12** *$\mathcal{M}^i_n$ can be interpreted in* $\mathbf{EETJ}{\upharpoonright} + \mathbf{UMID_N}$.

**Proof.** This is straightforward now. Interpretations of all axioms, except the $\mathbf{LFP}_n$ axiom (4) and $\mathbf{TI}_n$, are standard. In particular, Arithmetical Comprehension requires Elementary Comprehension of **EM**, and restricted induction calls for the restricted induction on $\mathsf{nat}$. $(\mathbf{TI}_n)^{\mathrm{EM}}$ is derivable by logic. Now we turn to the $\mathbf{LFP}_n$ axiom. Applying Definition 4.9, in Explicit Mathematics this turns into

$$\big(\mathrm{Mon}(\mathcal{M}(T), n)\big)^{\mathrm{EM}} \to \big(\forall y{<}n\, \mathrm{LFP}_{x,X}[\varphi[x, y, X, T_{<y}], T_y]\big)^{\mathrm{EM}},$$

where $T := \mu xyXY.\varphi[x, y, X, Y]$. So assume $\big(\mathrm{Mon}(\mathcal{M}(T), n)\big)^{\mathrm{EM}}$. We are left with the task to prove

$$\forall y{<}n \Big( \big(\forall x\,(x \in T_y \leftrightarrow \varphi[x, y, T_y, T_{<y}])\big)^{\mathrm{EM}} \;\wedge\; \big(\forall U(\forall z(\varphi[z, y, U, T_{<y}] \to z \in U) \to T_y \subseteq U)\big)^{\mathrm{EM}} \Big),$$

i.e.

$$\forall y{<}n\,\big(\forall x \,\varepsilon\, \mathsf{nat}\,(x \,\varepsilon\, \mathsf{T}_y \leftrightarrow \varphi^{\mathrm{EM}}[x, y, \mathsf{T}_y, \mathsf{T}_{<y}]) \;\wedge\; \forall U{\subseteq}\mathsf{nat}(\forall z \,\varepsilon\, \mathsf{nat}(\varphi^{\mathrm{EM}}[z, y, U, \mathsf{T}_{<y}] \to z \,\varepsilon\, U) \to \mathsf{T}_y \subseteq U)\big),$$

where $\mathsf{T} := \mathsf{J}(\lambda y \lambda Y. f^{x,X}_\varphi)$. By Lemma 4.11 this is equivalent to

$$\forall y{<}n\,\big(\forall x \,\varepsilon\, \mathsf{nat}\,(x \,\varepsilon\, \mathsf{T}_y \leftrightarrow x \,\varepsilon\, (\lambda y \lambda Y. f^{x,X}_\varphi) y \mathsf{T}_{<y} \mathsf{T}_y) \wedge \forall U{\subseteq}\mathsf{nat}(\forall z \,\varepsilon\, \mathsf{nat}(z \,\varepsilon\, (\lambda y \lambda Y. f^{x,X}_\varphi) y \mathsf{T}_{<y} U \to z \,\varepsilon\, U) \to \mathsf{T}_y \subseteq U)\big),$$

which follows from $\mathbf{UMID_N}$ axiom by Lemmata 4.10, 4.7 and 4.6. □

This completes embedding of $\mathcal{M}^i_\omega$ into $\mathbf{EETJ}{\upharpoonright} + \mathbf{UMID_N}$, and thereby the proof of Theorem 4.1(a). For Theorem 4.1(b), we proceed in exactly the same way, with the following small changes: (1) From the very beginning, we observe that a derivation in $\mathcal{M}^i_{\varepsilon_0}$ is a derivation in $\mathcal{M}^i_\beta$ for some $\beta < \varepsilon_0$. We use this $\beta$ in place of $n < \omega$ from the case of $\mathcal{M}^i_\omega$. (2) In the proof of Lemma 4.7 we need to refer to Recursion Theorem 4.3 and transfinite induction for $\beta$. Namely, existence of an operation $\tilde{\mathsf{J}}$ s.t. $(\tilde{\mathsf{J}} f)_y \simeq \mathsf{lfp}(fy \bigoplus_{z<y} (\tilde{\mathsf{J}} f)_z)$ follows from Recursion Theorem, but, taking as before $\mathsf{J} f := \bigoplus_{y<\beta} \tilde{\mathsf{J}} f$, in order to prove $\exists Z{\subseteq}\mathsf{nat}\,(Z = \mathsf{J} f)$ and $\forall y{<}\beta (\mathsf{J} f)_y \doteq \mathsf{lfp}(fy \bigoplus_{z<y} (\mathsf{J} f)_z)$ we need to use $\mathbf{TI}_\beta$. (3) In the beginning of proof of Lemma 4.12, we observe that $(\mathbf{TI}_\beta)^{\mathrm{EM}}$ is a theorem of **EETJ**.

**Acknowledgment.** The authors were supported in this work by United Kingdom Engineering and Physical Sciences Research Council Grant GR/R 15856/01. The bulk of the research was carried out in 2005 at the Ohio State University. As regards the completion of this paper, the first author is grateful for support by the John Templeton Foundation ("A new dawn of intuitionism: mathematical and philosophical advances," ID 60842).[2]

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[2]The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

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