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# Study of the Hopf functional equation for turbulence: Duhamel principle and dynamical scaling

Koji Ohkitani\*

*School of Mathematics and Statistics, University of Sheffield,  
Hicks Building, Hounsfield Road, Sheffield S3 7RH, U.K.*

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## Abstract

We consider a formulation for the Hopf functional differential equation which governs statistical solutions of the Navier-Stokes equations. By introducing an exponential operator with a functional derivative, we recast the Hopf equation as an integro-differential functional equation by the Duhamel principle. On this basis we introduce a successive approximation to the Hopf equation. As an illustration we take the Burgers equation and carry out the approximations to the leading order.

Scale-invariance of the statistical Navier-Stokes equations in  $d$ -dimensions is formulated and contrasted with that of the deterministic Navier-Stokes equations. For the statistical Navier-Stokes equations, critical scale-invariance is achieved for the characteristic functional of the  $d$ -th derivative of the vector potential in  $d$ -dimensions. The deterministic equations corresponding to this choice of the dependent variable acquire the linear Fokker-Planck operator under dynamic scaling. In three dimensions it is the vorticity gradient that behaves like a fundamental solution (more precisely, source-type solution) of deterministic Navier-Stokes equations in the long-time limit.

Physical applications of these ideas include study of a self-similar decaying profile of fluid flows. Moreover, we reveal typical physical properties in the late stage evolution by combining statistical scale-invariance and the source-type solution. This yields an asymptotic form of the Hopf functional in the long-time limit, improving the well-known Hopf-Titt solution. In particular, we present analyses for the Burgers equations to illustrate the main ideas and indicate a similar analysis for the Navier-Stokes equations.

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\*Electronic address: [K.Ohkitani@sheffield.ac.uk](mailto:K.Ohkitani@sheffield.ac.uk)

## I. INTRODUCTION

The problem of Navier-Stokes turbulence remains a major challenge in theoretical physics and mathematics. In particular deriving the statistical properties of solutions to the Navier-Stokes equations (i.e. the governing equations) in a purely deductive manner has been regarded as a difficult task. On the other hand there are attempts to describe statistical properties of turbulence on the basis of approximations relying on physical ideas. In this paper we consider and revisit a formulation from first principles.

Roughly speaking, there are two different ways in writing down equations that govern statistical solutions of the Navier-Stokes equations. One method uses the characteristic functional of the velocity field as the basic variable, which is the Fourier transform of the probability measure of the velocity and its governing equation is called the Hopf equation. The other method deals directly with the probability measure of the velocity field and the corresponding Liouville equation is called the Hopf-Foias equation, see e.g. [1–9]. See also [10–20] for related works and [21–38] for mathematical works.

It is fair to say that methods of solving the Hopf equation are at the moment underdeveloped. There are at least two different approaches to determine the Hopf functional. One is to try solving the functional equation as is, taking realizability into account. This approach faces formidable difficulty. The other one starts from deterministic solutions of a nonlinear PDE and after taking average we can try determine a functional form of the Hopf functional at least asymptotically. While the latter approach is available only when the deterministic PDE is explicitly solvable, it does give a hint as to how the functional actually behaves. As our understanding of the Hopf equation is limited at the moment, it makes sense to combine the both approaches.

In view of physical applications, we recall self-similar solutions often reveal typical properties of nonlinear problems in fluid mechanics [39]. To exemplify our approach, here we first consider the Burgers equations and indicate extensions to handle the Navier-Stokes equations. The Burgers equations on their own footing appear as a physical model for compressible fluid motion, see e.g. [40]. We show in particular *how the source-type solution (to be defined below) determines the Hopf functional for the Burgers equations in its long-time limit through self-similarity, when we choose the dependent variable suitably.* This generalizes the well-known expression of the Hopf-Titt solution the final period decay, which totally

neglects the nonlinear terms. It should be noted an explicit form of the source-type solutions for the multi-dimensional Burgers equations are obtained as a by-product, which have not been reported before. We will clarify which ideas carry over to the Navier-Stokes equations in two and three dimensions.

The purpose of this paper is two-fold. First, we will recast the Hopf functional differential equation into an integral equation by introducing a kind of Duhamel principle and thereby yielding a successive approximations systematically. In so doing we will make use of symbolic manipulations. Second, we will clarify the concept of scale-invariance for the statistical Navier-Stokes equations by extending a previous work by Rosen [41] and combine it with the source-type solutions of the deterministic equations.

The rest of this paper is organised as follows. In Section II, after reviewing its scaling property, we convert the Hopf equation for the 3D Navier-Stokes equations into an integral equation with the use of an exponential operator. We introduce a successive approximation on this basis. In section III, dynamic scaling property for 1D statistical Burgers equation is described. In section IV, dynamic scaling property for  $d$ -dimensional statistical Navier-Stokes equations is described. In Section V, the implications of the source-type solutions on the late-stage behavior of the Hopf functional are discussed. Section VI is devoted to a summary and outlook. In Appendix A, a formal derivation of the action of the exponential operator is stated. In Appendix B the leading-order approximation is presented for the Burgers equation. In Appendix C, an error estimate of the successive approximation is derived. Finally, Appendix D recalls the self-similar solution of the Burgers equation.

## II. HOPF EQUATION FOR THE BURGERS EQUATION

There are many publications on the Hopf equations, for example [42–49]. We will revisit this equation from a fundamental viewpoint. To illustrate the basic idea, we mainly consider the Burgers equation for simplicity. Statistical solutions of this equation have been studied in many works, such as [50–58].

The 1D Burgers equation written in standard notations

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad (1)$$

satisfies invariance under the following set of (static) scaling transformations

$$x \rightarrow \lambda x, t \rightarrow \lambda^2 t, u \rightarrow \lambda^{-1} u,$$

where  $\lambda(> 0)$  is an arbitrary parameter. Hence,

*Property 1: if  $u(x, t)$  is a solution to (1), so is  $\lambda u(\lambda x, \lambda^2 t)$ .*

The characteristic functional for the velocity is defined by

$$\Phi[\theta(x), t] = \left\langle \exp \left( i \int_{-\infty}^{\infty} u(x, t) \theta(x) dx \right) \right\rangle,$$

where  $\langle \rangle$  denotes an ensemble average taken with respect to an initial velocity distribution.

It satisfies Hopf functional differential equation (FDE, hereafter)

$$\frac{\partial \Phi}{\partial t} = L\Phi, \quad (2)$$

where [76]

$$L\Phi \equiv \frac{i}{2} \int \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \Phi}{\delta \theta(x)^2} dx + \nu \int \theta(x) \frac{\partial^2}{\partial x^2} \frac{\delta \Phi}{\delta \theta(x)} dx.$$

The Hopf functional  $\Phi$  satisfies some realizability conditions. It is required that

$$\Phi[\theta(x)]|_{\theta(x)=0} = 1$$

and positive-definiteness

$$\sum_{k=1}^n \sum_{l=1}^n \Phi[\theta_k(x) - \theta_l(x)] c_k c_l^* = \left\langle \left| \sum_{k=1}^n c_k \exp \left( i \int u(x, t) \theta(x) dx \right) \right|^2 \right\rangle \geq 0,$$

hold for  $n = 1, 2, 3, \dots$ . We are interested in solutions of (2) that satisfy those conditions at any time  $t \geq 0$ . Even though it is a linear equation, no general method for its solutions is known.

Scale-invariance of the Hopf equation for the 3D Navier-Stokes equations has been studied in [41]. For the statistical solutions of the 1D Burgers equations, the corresponding argument goes as follows. Under the following set of transformations

$$\theta(x) \rightarrow \theta(\lambda x), u(x) \rightarrow \lambda u(\lambda x),$$

where  $\lambda(> 0)$  is an arbitrary parameter, the Hopf equation becomes

$$\frac{\partial \Phi}{\partial t} = \lambda^2 L\Phi[\theta(\lambda x), t].$$

This can be made invariant by scaling the time variable as  $t \rightarrow \lambda^{-2} t$ . Hence,

*Property 2: if  $\Phi[\theta(x), t]$  is a solution to (2), so is  $\Phi[\theta(\lambda x), \lambda^{-2}t]$ .*

In particular, let us consider the heat diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

by ignoring the nonlinear term of the Burgers equation. The corresponding FDE reads

$$\frac{\partial \Phi}{\partial t} = \nu \int \theta(x) \frac{\partial^2}{\partial x^2} \frac{\delta \Phi}{\delta \theta(x)} dx,$$

which can be solved explicitly [1, 2] as

$$\Phi[\theta(x), t] = \Phi_0 \left[ \frac{1}{\sqrt{4\pi\nu t}} \int \exp\left(-\frac{(x-y)^2}{4\nu t}\right) \theta(y) dy \right] \equiv \Phi_0[\exp(\nu t \Delta) \theta], \quad (3)$$

where  $\Delta = \frac{\partial^2}{\partial x^2}$  denotes the Laplacian. This is called the Hopf-Titt solution and it follows from the self-dual property of  $g_t$

$$(g_t * u_0, \theta) = (u_0, g_t * \theta), \quad (4)$$

where  $g_t(x) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{x^2}{4\nu t}\right)$  denotes the heat kernel and  $(f, g) = \int f(x)g(x)dx$  an inner-product. In this case, the characteristic functional does not essentially change its form, rather its argument develops following a heat flow. This method of solutions may be regarded as a FDE version of method of characteristics, e.g. [59–61].

We now consider an operator  $D$ , defined formally by

$$D \equiv \int dx \theta(x) \frac{\partial^2}{\partial x^2} \frac{\delta}{\delta \theta(x)} \quad (5)$$

and write above solution heuristically (symbolically) as follows

$$\Phi[\theta(x), t] = \exp\left(\nu t \int dx \theta(x) \frac{\partial^2}{\partial x^2} \frac{\delta}{\delta \theta(x)}\right) \Phi_0[\theta] \equiv \exp(\nu t D) \Phi_0[\theta].$$

In other words we define  $D$  by its action on  $\Phi[\theta]$  as follows

$$\exp(\nu t D) \Phi[\theta] \equiv \Phi[\exp(\nu t \Delta) \theta]. \quad (6)$$

Note that (5) is a purely symbolic notation whose meaning is given by (6).

We may regard the operator on the left-hand side as a functional version of “shift operator” on the basis of the above solution [77]. In other words, we turn a particular solution (3) into a definition of the new operator  $D$ . Its meaning is to update all the arguments,  $\theta(x)$  in this case, in the operand functional by convoluting the heat kernel.

With this understanding, we can now recast the Hopf equation as follows

$$\exp(\nu t D) \frac{\partial}{\partial t} \exp(-\nu t D) \Phi[\theta, t] = \frac{i}{2} \int \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \Phi}{\delta \theta(x)^2} dx. \quad (7)$$

This allows us to convert the Hopf equation into an integral equation by a straightforward application of the Duhamel principle

$$\Phi[\theta(x), t] = \exp(\nu t D) \Phi_0[\theta] + \frac{i}{2} \int_0^t \exp(\nu(t-s)D) \int \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \Phi}{\delta \theta(x)^2} [\theta(x), s] dx ds. \quad (8)$$

It may be in order to compare (8) with the integral form of deterministic Navier-Stokes equations.[75] Defining yet another operator  $G$  by

$$G \equiv \frac{i}{2} \int_0^t ds \exp(\nu(t-s)D) \int dx \theta(x) \frac{\partial}{\partial x} \frac{\delta^2}{\delta \theta(x)^2},$$

the integral form of the Hopf equation can be written

$$\Phi = \tilde{\Phi} + G\Phi,$$

where  $\tilde{\Phi} = \Phi_0[\exp(\nu t \Delta)\theta]$  is the Hopf functional for the heat flow. See Appendix A for its formal derivation. It is noted that the operator  $G$  depends on time  $t$ , while its dependence is suppressed for simplicity. In passing we note that (8) resembles the Lipmann-Schwinger equation in scattering theory in quantum mechanics.

We are in a position to introduce a successive approximation

$$\Phi_{n+1} = \tilde{\Phi} + G\Phi_n \quad (n = 0, 1, 2, \dots)$$

to derive a Neumann series for the Hopf functional

$$\Phi = (I - G)^{-1} \tilde{\Phi} = (I + G + G^2 + G^3 + \dots) \tilde{\Phi}.$$

The zero-th order approximation is given by the above (3). The first order approximation is given by

$$\Phi \approx (I + G) \tilde{\Phi},$$

which is reminiscent of the Born approximation in scattering theory.

In order to apply this approximation in practice, numerical evaluation of the above integral would be needed. Here we restrict attention to leading order analysis to see how

the successive approximations look like and what conditions are required to assure convergence of the successive approximations. Taking the initial Hopf functional of the following Gaussian form

$$\Phi_0[\theta] = \exp\left(-\frac{1}{2} \iint Q(x', x'') \theta(x') \theta(x'') dx' dx''\right),$$

where  $Q(x', x'') = \langle u(x', 0)u(x'', 0) \rangle$  denotes the initial velocity correlation function, we find

$$\begin{aligned} G\tilde{\Phi} &= -\frac{i}{2} \int_0^t ds \left\{ \frac{-1}{4\pi\nu s} \iiint dx dx' dx'' e^{\nu(t-s)\Delta} \theta(x) Q(x', x'') \frac{x' + x'' - 2x}{2\nu s} e^{-\frac{(x'-x)^2}{4\nu s} - \frac{(x''-x)^2}{4\nu s}} \right. \\ &\quad + \frac{1}{2\pi\nu s} \int dx e^{\nu(t-s)\Delta} \theta(x) \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu s}} e^{\nu t \Delta} \theta(x'') \\ &\quad \left. \times \iint dx' dx'' Q(x', x'') \frac{x' - x}{2\nu s} e^{-\frac{(x'-x)^2}{4\nu s}} e^{\nu t \Delta} \theta(x'') \right\} \Phi_0[e^{\nu t \Delta} \theta], \end{aligned} \quad (9)$$

where we have denoted a function of  $x''$   $e^{\nu t \Delta} \theta(x'') = \frac{1}{\sqrt{4\pi\nu t}} \int \exp\left(-\frac{(x''-y)^2}{4\nu t}\right) \theta(y) dy$ , See Appendix B for the derivations.

It is in order to derive a sufficient condition for the convergence of the above successive approximation. Define the operator norm of  $G$  by

$$\|G\| \equiv \sup_{\tilde{\Phi}} \frac{\|G\tilde{\Phi}\|}{\|\tilde{\Phi}\|}.$$

By a standard argument [62, 63], provided that  $\|G\| < 1$  we have

$$\begin{aligned} \|(I - G)^{-1} - (I + G + \dots + G^{n-1})\| &\leq \sum_{m=n}^{\infty} \|G\|^m \\ &= \|G\|^n (1 - \|G\|)^{-1} \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Because

$$\|G\tilde{\Phi}\| \leq \frac{\|\tilde{\Phi}\|}{2} \left\| \int_0^t ds \int e^{\nu(t-s)\Delta} \theta(x) \frac{\partial}{\partial x} \left\{ e^{\nu s(\Delta' + \Delta'')} Q(x', x'') - \left( \int dx'' e^{\nu s \Delta'} Q(x', x'') e^{\nu t \Delta} \theta(x'') \right)^2 \right\} dx \right\|,$$

we find that

$$\frac{1}{2} \left\| \int_0^t ds \int e^{\nu(t-s)\Delta} \theta(x) \frac{\partial}{\partial x} \left\{ e^{\nu s(\Delta' + \Delta'')} Q(x', x'') - \left( \int dx'' e^{\nu s \Delta'} Q(x', x'') e^{\nu t \Delta} \theta(x'') \right)^2 \right\} dx \right\| < 1$$

is sufficient for the convergence. It is shown in Appendix C that the condition is satisfied, for example, if

$$\frac{C}{\nu} \left\| \frac{\partial \theta}{\partial x} \right\|_{L^1} \|Q(x, y)\|_{L^1(\mathbb{R}^2)} (1 + \|Q(x, y)\|_{L^1(\mathbb{R}^2)} \|\theta\|_{L^\infty}^2) < 1, \quad (10)$$

where  $C$  is a non-dimensional constant. In particular, it is satisfied when  $\nu$  is large.



### III. DYNAMIC SCALING FOR THE HOPF EQUATION FOR BURGERS EQUATION

In [41], scale-invariance of the 3D Navier-Stokes equations have been discussed and used to study a self-similar decaying process of turbulence. We present a variant of its argument adapted to one-spatial dimension.

#### A. Dynamic scaling: deterministic version

We will be interested in a decaying process of Burgers 'turbulence'. We consider solutions with forward self-similarity, where the relevant scaling parameter is  $\sqrt{2a(t+t_*)}$  with a constant  $a(> 0)$ .

We generalize static scaling transformations using a dynamically-rescaled time variable. Applying a set of dynamic scaling transformations

$$\xi = \frac{x}{\sqrt{2a(t+t_*)}}, \quad \tau = \frac{1}{2a} \log \frac{t+t_*}{t_*}$$

$$u(x, t) = \frac{1}{\sqrt{2a(t+t_*)}} U(\xi, \tau)$$

to the Burgers equation with  $2at_* = 1$ , we obtain

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} = a \frac{\partial}{\partial \xi} (\xi U) + \nu \frac{\partial^2 U}{\partial \xi^2}. \quad (11)$$

Note that the dynamically-scaled Burgers equation, when linearized, coincides with a linear Fokker-Planck equation, that is, a Fokker-Planck equation with a linear drift term, associated with the Ornstein-Uhlenbeck process. It should be noted that the associated stochastic process reaches stationarity while Brownian motion (i.e. the Wiener process) does not. Note the distinction in behaviors of  $D$  and  $D^*$  below. It has a conservative term in the form  $a \partial_\xi (\xi U)$ , rather than the drift term  $a \xi \partial_\xi U$  only. It is known that solutions to (11) converges to a steady solution as  $\tau \rightarrow \infty$ . See Section V and Appendix D for the steady solution.

#### B. Dynamic scaling: statistical version

In principle there are two ways to derive the Hopf equation for the dynamically-scaled Burgers equation. We will show that they actually lead to the same result. To avoid prolif-

eration of notations, we use the same symbol  $\Phi$  for the Hopf functional for the dynamically-scaled equations, which can be distinguished by the argument.

**Method 1:** We first apply dynamic scaling and then move onto a statistical description.

By (11) it is straightforward to check the characteristic functional of the velocity

$$\Phi[\theta(\xi), \tau] = \left\langle \exp \left( i \int_{-\infty}^{\infty} U(\xi, \tau) \theta(\xi) d\xi \right) \right\rangle$$

satisfies

$$\begin{aligned} \frac{\partial \Phi}{\partial \tau} &= \frac{i}{2} \int \theta(\xi) \frac{\partial}{\partial \xi} \frac{\delta^2 \Phi}{\delta \theta(\xi)^2} d\xi + \nu \int \theta(\xi) \frac{\partial^2}{\partial \xi^2} \frac{\delta \Phi}{\delta \theta(\xi)} d\xi + a \int \theta(\xi) \frac{\partial}{\partial \xi} \left( \xi \frac{\delta \Phi}{\delta \theta(\xi)} \right) d\xi \\ &= \frac{i}{2} \int \theta(\xi) \frac{\partial}{\partial \xi} \frac{\delta^2 \Phi}{\delta \theta(\xi)^2} d\xi + \nu \int \frac{\partial^2 \theta}{\partial \xi^2} \frac{\delta \Phi}{\delta \theta(\xi)} d\xi - a \int \frac{\delta \Phi}{\delta \theta(\xi)} \xi \frac{\partial \theta}{\partial \xi} d\xi, \end{aligned} \quad (12)$$

that is,

$$\frac{\partial \Phi}{\partial \tau} = \frac{i}{2} \int \theta(\xi) \frac{\partial}{\partial \xi} \frac{\delta^2 \Phi}{\delta \theta(\xi)^2} d\xi + \int \theta(\xi) \left( \nu \frac{\partial^2}{\partial \xi^2} + a \frac{\partial}{\partial \xi} \xi \right) \frac{\delta \Phi}{\delta \theta(\xi)} d\xi. \quad (13)$$

**Method 2:** We first consider a statistical description and then apply dynamic scaling.

We write

$$\Phi[\theta(x), t] = \Psi[\zeta(y), \tau],$$

where  $\zeta(y) \equiv \theta(\sqrt{2a(t+t_*)}x)$ ,  $y = \sqrt{2a(t+t_*)}x$ . By  $\frac{\partial \zeta}{\partial t} = \frac{1}{2(t+t_*)}y \frac{\partial \theta}{\partial y}$  we find

$$\frac{\partial \Phi}{\partial t} = \int \frac{\delta \Psi}{\delta \zeta} \frac{\partial \theta}{\partial t} dy + \frac{\partial \Psi}{\partial \tau} \frac{\partial \tau}{\partial t} = \int \frac{\delta \Psi}{\delta \zeta} \frac{1}{2(t+t_*)} y \frac{\partial \theta}{\partial y} dy + \frac{\partial \Psi}{\partial \tau} \frac{1}{2a(t+t_*)}.$$

By the scaling of  $\Phi$ , the right-hand side equals  $\frac{1}{\lambda^2}L$ , that is,

$$= \frac{1}{2a(t+t_*)} \left( \frac{i}{2} \int \zeta(y) \frac{\partial}{\partial y} \frac{\delta^2 \Psi}{\delta \zeta(y)^2} dy + \nu \int \zeta(y) \frac{\partial^2}{\partial y^2} \frac{\delta \Psi}{\delta \zeta(y)} dy \right).$$

Hence we obtain

$$\frac{\partial \Psi}{\partial \tau} + a \int \frac{\delta \Psi}{\delta \zeta} y \frac{\partial \zeta}{\partial y} dy = \frac{i}{2} \int \zeta(y) \frac{\partial}{\partial y} \frac{\delta^2 \Psi}{\delta \zeta(y)^2} dy + \nu \int \zeta(y) \frac{\partial^2}{\partial y^2} \frac{\delta \Psi}{\delta \zeta(y)} dy,$$

which matches the previous expression (13) by rewriting  $\zeta(y) \rightarrow \theta(\xi)$ .

### C. Operator $D^*$ for the modified heat kernel

Consider a linearization of (11), the linear Fokker-Planck equation,

$$\frac{\partial V}{\partial \tau} = a \frac{\partial}{\partial \xi} (\xi V) + \nu \frac{\partial^2 V}{\partial \xi^2}.$$

Its solution is given by

$$V(\xi, \tau) = \tilde{g}_\tau * V_0 \equiv \left( \frac{a}{2\pi\nu(1 - e^{-2a\tau})} \right)^{1/2} \int_{\mathbb{R}^1} e^{a\tau} V_0(e^{a\tau}y) \exp\left(-\frac{a}{2\nu} \frac{(\xi - y)^2}{1 - e^{-2a\tau}}\right) dy,$$

where  $\tilde{g}_\tau$  denotes a modified heat kernel and  $*$  convolution as defined above. It is convenient to define an operator  $D^*$  and write symbolically the solution to the above Fokker-Planck solution as

$$V(\xi, \tau) = \exp(\nu\tau D^*) V_0 = \exp(\tau \{\nu\Delta + a\partial_\xi(\xi\cdot)\}) V_0.$$

In the limit of  $\tau \rightarrow \infty$  we have

$$V(\xi, \tau) \rightarrow \exp\left(\frac{\nu}{2a} D\right) V_0(\xi) \Big|_{V_0=M\delta} = M \sqrt{\frac{a}{2\pi\nu}} \exp\left(-\frac{a}{2\nu} \xi^2\right),$$

where  $M = \int V_0(\eta) d\eta$ . Here we have made use of the formula

$$\frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \rightarrow M\delta(x); \text{ as } \epsilon \rightarrow 0,$$

which holds for any localised function  $f$  with  $M = \int f(x) dx$ .

#### D. Duhamel principle for the scaled Hopf equation

Replacing  $\nu\Delta$  with  $\nu\Delta + a\partial_\xi(\xi\cdot)$  in (7) we can write the dynamically-scaled version of the Hopf equation as

$$\exp(\nu\tau D^*) \frac{\partial}{\partial\tau} \exp(-\nu\tau D^*) \Phi[\theta(\xi), \tau] = \frac{i}{2} \int \theta(\xi) \frac{\partial}{\partial\xi} \frac{\delta^2\Phi}{\delta\theta(\xi)^2} d\xi,$$

from which we find

$$\Phi[\theta(\xi), \tau] = \exp(\nu\tau D^*) \Phi_0[\theta(\xi)] + \frac{i}{2} \int_0^\tau \exp(\nu(\tau - s) D^*) \int \theta(\xi) \frac{\partial}{\partial\xi} \frac{\delta^2\Phi}{\delta\theta(\xi)^2} [\theta(\xi), s] d\xi ds. \quad (14)$$

We are interested in studying the long-time limit  $\Phi[\theta(\xi)] = \lim_{\tau \rightarrow \infty} \Phi[\theta(\xi), \tau]$ , but the evaluation of the second term on the right-hand side faces difficulty. In Section V we will show how we may obtain an asymptotic expression  $\Phi[\theta(\xi)]$ , working directly from the definition of Hopf functional.

For the scaled-version of the Hopf equation for the linearized equation, that is, the first term on the right-hand side of (14), we have

$$\exp(\nu\tau D^*) \Phi_0[\theta(\xi)] = \langle \exp(i(\tilde{g}_\tau * V_0, \theta)) \rangle.$$

Unlike the original heat kernel, self-duality  $(\tilde{g}_\tau * V_0, \theta) = (V_0, \tilde{g}_\tau * \theta)$  does *not* hold in general for the modified kernel  $\tilde{g}_\tau$ . It holds when  $V_0$  is an homogeneous function of degree -1, that is,  $\lambda V_0(\lambda x) = V_0(x)$  for all  $\lambda > 0$  (just like the Dirac delta function).

#### IV. DYNAMIC SCALING FOR HOPF EQUATION FOR NAVIER-STOKES EQUATIONS

Dynamic scaling can be applied to the Hopf equation for the Navier-Stokes equations without any problem. Scaling properties of the Hopf equation has been discussed in [41], which we generalize here into  $d$ -spatial dimensions. We will show that if a critical condition in the statistical solution is achieved, the Hopf functional takes the simplest form of self-similarity. This is crucial in improving the Hopf-Titt solution with a “near-Gaussian” solution associated with the heat-kernel. The final form would be more complicated with other choices of dependent variables.

##### A. Dynamic scaling (deterministic version)

For the incompressible Navier-Stokes equations written in standard notations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \nu \Delta \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{15}$$

static scale-invariance implies the following

*Property 1’*: if  $\mathbf{u}(\mathbf{x}, t)$  is a solution to (15), so is  $\lambda \mathbf{u}(\lambda \mathbf{x}, \lambda^2 t)$ .

By applying the dynamic scaling transforms

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \frac{1}{\sqrt{2a(t+t_*)}} \mathbf{U}(\boldsymbol{\xi}, \tau), \\ \boldsymbol{\xi} &= \frac{\mathbf{x}}{\sqrt{2a(t+t_*)}}, \quad \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a} \log \frac{t+t_*}{t_*}, \end{aligned}$$

we obtain the scaled version of the Navier-Stokes equations, also known as the Leray equations

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial \tau} + \mathbf{U} \cdot \nabla_{\boldsymbol{\xi}} \mathbf{U} &= -\nabla_{\boldsymbol{\xi}} P + \nu \Delta_{\boldsymbol{\xi}} \mathbf{U} + a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \mathbf{U} + \mathbf{U}), \\ \nabla_{\boldsymbol{\xi}} \cdot \mathbf{U} &= 0. \end{aligned}$$

## B. Dynamic scaling (statistical version)

The characteristic functional of the velocity

$$\Phi[\boldsymbol{\theta}, t] = \left\langle \exp \left( i \int \mathbf{u}(\mathbf{x}, t) \cdot \boldsymbol{\theta}(\mathbf{x}) d\mathbf{x} \right) \right\rangle$$

satisfies

$$\Phi[\boldsymbol{\theta}, t] = \Phi[\boldsymbol{\theta}^\perp, t], \quad \Phi[\boldsymbol{\theta}, t]^* = \Phi[-\boldsymbol{\theta}, t], \quad |\Phi[\boldsymbol{\theta}, t]| \leq 1,$$

where  $\boldsymbol{\theta}^\perp$  denotes a solenoidal projection of  $\boldsymbol{\theta}$ .

The Hopf equation can be written

$$\frac{\partial \Phi}{\partial t} = L\Phi, \tag{16}$$

where

$$L \equiv i \int d\mathbf{x} \theta_j^\perp(\mathbf{x}) \frac{\partial}{\partial x_k} \frac{\delta^2}{\delta \theta_j(\mathbf{x}) \delta \theta_k(\mathbf{x})} + \nu \int d\mathbf{x} \theta_j(\mathbf{x}) \Delta \frac{\delta}{\delta \theta_j(\mathbf{x})}.$$

Note that a set of dilation transforms  $\boldsymbol{\theta}(\mathbf{x}) \rightarrow \lambda^{d-1} \boldsymbol{\theta}(\lambda \mathbf{x})$ ,  $\mathbf{u}(\mathbf{x}) \rightarrow \lambda \mathbf{u}(\lambda \mathbf{x})$  leave  $\int \mathbf{u} \cdot \boldsymbol{\theta} d\mathbf{x}$  invariant. Also note that functional derivatives transform as

$$\frac{\delta}{\delta \theta_j(\mathbf{x})} \rightarrow \lambda \frac{\delta}{\delta \theta_j(\lambda \mathbf{x})}$$

to ensure

$$\left[ \frac{\delta}{\delta \theta_j(\mathbf{x})}, \theta_k(\mathbf{x}') \right] = \delta_{jk} \delta(\mathbf{x} - \mathbf{x}'),$$

where  $\delta(\cdot)$  denotes the Dirac delta function. On the other hand, the operator  $L$  transforms as

$$\begin{aligned} L &\rightarrow i \lambda^{d+1} \int d\mathbf{x} \theta_j^\perp(\lambda \mathbf{x}) \frac{\partial}{\partial x_k} \frac{\delta^2}{\delta \theta_j(\lambda \mathbf{x}) \delta \theta_k(\lambda \mathbf{x})} - \nu \lambda^d \int d\mathbf{x} \theta_j(\lambda \mathbf{x}) \Delta \frac{\delta}{\delta \theta_j(\lambda \mathbf{x})} \\ &= \lambda^2 L, \end{aligned}$$

where the last line follows by  $\mathbf{x} \rightarrow \lambda^{-1} \mathbf{x}$ . The Hopf equation then takes the form

$$\left( \frac{\partial}{\partial t} - \lambda^2 L \right) \Phi [\lambda^{d-1} \boldsymbol{\theta}(\lambda \mathbf{x}), t] = 0,$$

or

$$\left( \frac{\partial}{\partial t} - L \right) \Phi [\lambda^{d-1} \boldsymbol{\theta}(\lambda \mathbf{x}), \lambda^{-2} t] = 0,$$

after rescaling  $t$ . Hence we conclude that

*Property 2': if  $\Phi[\boldsymbol{\theta}(\mathbf{x}), t]$  is a solution to (16), so is  $\Phi[\lambda^{d-1} \boldsymbol{\theta}(\lambda \mathbf{x}), \lambda^{-2} t]$ .*

It is important to observe that, unlike the deterministic case, the invariance property depends on the spatial dimension  $d$ . In particular, statistics of the velocity field attains criticality when and only when  $d = 1$  in the sense that the argument of  $\Phi[\cdot]$  takes a simplified form as  $\theta(t^{1/2}x)$ . It is noted that with the choice of velocity,  $\theta(x)$  of the characteristic functional for the 1D Burgers equation acquires the same physical dimension as that of  $1/\nu$ . We may contrast the situation with the deterministic counterpart of invariance *Property 1'* for the Navier-Stokes equations, which is *not* critical with the use of the velocity variable. We would have used the vector potential to achieve criticality for the deterministic Navier-Stokes equations [64].

In *Property 2'*, if we consider the Hopf functional of the vorticity, we have  $\Phi[\lambda^{d-2}\boldsymbol{\theta}(\lambda\mathbf{x}), \lambda^{-2}t]$  for the scaled functional. Likewise, if we consider the Hopf functional of the vorticity gradient, we have  $\Phi[\lambda^{d-3}\boldsymbol{\theta}(\lambda\mathbf{x}), \lambda^{-2}t]$  for the scaled functional. Thus, by choosing the variable suitably we can make the prefactor of  $\boldsymbol{\theta}(\lambda\mathbf{x})$  vanish, achieving statistical criticality, which renders the analysis of the Hopf equation the simplest possible one. It is noted that for with the choice of vorticity gradient,  $\boldsymbol{\theta}(\mathbf{x})$  of the characteristic functional for the 3D Navier-Stokes equations acquires the same physical dimension as that of  $1/\nu$ .

To be more specific, we derive dynamically-scaled Hopf equation following the two different ways as above.

**Method 1:** We apply dynamic scaling to the Navier-Stokes equations and move onto the statistical description.

$$\frac{\partial\Phi}{\partial\tau} = i \int \theta_j^\perp(\boldsymbol{\xi}) \frac{\partial}{\partial\xi_k} \frac{\delta^2\Phi}{\delta\theta_j(\boldsymbol{\xi})\delta\theta_k(\boldsymbol{\xi})} d\boldsymbol{\xi} + \nu \int \theta_j(\boldsymbol{\xi}) \Delta_\xi \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} d\boldsymbol{\xi} + a \int \theta_j(\boldsymbol{\xi}) (\boldsymbol{\xi} \cdot \nabla + 1) \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} d\boldsymbol{\xi}, \quad (17)$$

or

$$\frac{\partial\Phi}{\partial\tau} = i \int \theta_j^\perp(\boldsymbol{\xi}) \frac{\partial}{\partial\xi_k} \frac{\delta^2\Phi}{\delta\theta_j(\boldsymbol{\xi})\delta\theta_k(\boldsymbol{\xi})} d\boldsymbol{\xi} + \nu \int \theta_j(\boldsymbol{\xi}) \Delta_\xi \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} d\boldsymbol{\xi} - a \int \frac{\delta\Phi}{\delta\theta_j(\boldsymbol{\xi})} (\boldsymbol{\xi} \cdot \nabla + d - 1) \theta_j(\boldsymbol{\xi}) d\boldsymbol{\xi}. \quad (18)$$

It should be noted that when  $d = 1$  the final term simplifies (that is, reduces to the drift term only).

**Method 2:** We start with the statistical formulation and apply dynamic scaling to it. We apply the following set of transformations

$$\Phi = \Psi[\boldsymbol{\zeta}, \tau],$$

$$\boldsymbol{\zeta} = (2a(t + t_*))^{d-1} \boldsymbol{\theta} \left( \sqrt{2a(t + t_*)} \mathbf{x} \right), \quad \tau = \frac{1}{2a} \log \frac{t + t_*}{t_*}$$

to the Hopf equation. Setting  $\mathbf{y} = \sqrt{2a(t + t_*)} \mathbf{x}$ , we have

$$\frac{\partial \zeta_j}{\partial t} = \frac{1}{2(t + t_*)} \left( (d-1)\zeta_j + y_k \frac{\partial \zeta_j}{\partial y_k} \right)$$

and

$$\frac{\partial \Phi}{\partial t} = \int \frac{\delta \Phi}{\delta \zeta_j} \frac{\partial \zeta_j}{\partial t} d\mathbf{y} + \frac{\partial \Phi}{\partial \tau} \frac{\partial \tau}{\partial t} = \int \frac{\delta \Phi}{\delta \zeta_j} \frac{1}{2(t + t_*)} \left( (d-1)\zeta_j + y_k \frac{\partial \zeta_j}{\partial y_k} \right) d\mathbf{y} + \frac{\partial \Phi}{\partial \tau} \frac{1}{2a(t + t_*)},$$

which equals

$$= \frac{1}{2a(t + t_*)} \int \left( i\zeta_j^\perp \frac{\partial}{\partial y_k} \frac{\delta^2 \Phi}{\delta \zeta_j(\mathbf{y}) \delta \zeta_k(\mathbf{y})} + \nu \zeta_j \Delta \frac{\delta \Phi}{\delta \zeta_j(\mathbf{y})} \right) d\mathbf{y}$$

by virtue of a substitution  $L \rightarrow \frac{1}{\lambda^2} L$ . Hence we find

$$\frac{\partial \Phi}{\partial \tau} + a \int \frac{\delta \Phi}{\delta \zeta_j} \left( (d-1)\zeta_j + y_k \frac{\partial \zeta_j}{\partial y_k} \right) d\mathbf{y} = \int \left( i\zeta_j^\perp \frac{\partial}{\partial y_k} \frac{\delta^2 \Phi}{\delta \zeta_j(\mathbf{y}) \delta \zeta_k(\mathbf{y})} + \nu \zeta_j \Delta \frac{\delta \Phi}{\delta \zeta_j(\mathbf{y})} \right) d\mathbf{y},$$

which agrees with the above result (18) when a replacement  $\boldsymbol{\zeta}(\mathbf{y}) \rightarrow \boldsymbol{\theta}(\boldsymbol{\xi})$  is made.

We make a set of two observations. (1) For the Hopf equation, critical scale-invariance is attained when we use a characteristic functional of the  $d$ -th derivatives of the vector potential. (2) For the corresponding scaled Navier-Stokes equations (i.e. the Leray equations), the dissipative term takes the form of the linear Fokker-Planck operator. This in principle provides a method of deriving long-time asymptotics of the Hopf functional for the Navier-Stokes equations, just as for the Burgers equations (see below).

## V. SOURCE-TYPE SOLUTIONS AND THEIR IMPLICATION ON HOPF FUNCTIONALS

To illustrate physical applications of the ideas developed here, we revisit the Burgers equations in one and two spatial dimensions. A source-type solution[78] is a forward self-similar solution, which starts from the Dirac delta function (in a variable attaining statistical criticality) and ends in a near-Gaussian universal profile associated with the heat-kernel. We will show how it is crucial to choose a dependent variable to have the source-type solutions in the analysis of statistical solutions and the role they play in the determination of Hopf functionals in the late stage. Source-type solutions for the Burgers equations have been studied extensively [65–68], where their existence has been established in multi-dimensions, but an explicit functional form has been given only in one dimension.

### A. One-dimensional Burgers equation

We will find a steady solution of (11) by taking the long time limit  $\tau \rightarrow \infty$ . By the Cole-Hopf transform, the scaled velocity can be written

$$U(\xi, \tau) = -2\nu \frac{\partial_\xi \int_{\mathbb{R}^1} \psi_0(\lambda\eta) \exp\left(-\frac{a}{2\nu} \frac{(\xi - \eta)^2}{1 - e^{-2a\tau}}\right) d\eta}{\int_{\mathbb{R}^1} \psi_0(\lambda\eta) \exp\left(-\frac{a}{2\nu} \frac{(\xi - \eta)^2}{1 - e^{-2a\tau}}\right) d\eta},$$

where  $\psi_0$  is the initial velocity potential. The numerator can be written by taking the derivative under the integral sign and integration by parts

$$\int_{\mathbb{R}^1} \lambda \psi_0'(\lambda\eta) \exp\left(-\frac{a}{2\nu} \frac{(\xi - \eta)^2}{1 - e^{-2a\tau}}\right) d\eta.$$

Now, noting  $\lambda = e^{a\tau}$ ,

$$\lambda \psi_0'(\lambda\eta) \rightarrow K \delta(\eta) \text{ as } \tau \rightarrow \infty,$$

where  $K = \int_{-\infty}^{\infty} U_0(\xi) d\xi = \int_{-\infty}^{\infty} \psi_0'(\xi) d\xi = \psi_0(\infty) - \psi_0(-\infty)$ . Note that  $U_0(\xi)$  denotes the initial (localised) velocity, not to be confused with  $U(0)$ . The numerator tends to

$$K \exp\left(-\frac{a}{2\nu} \xi^2\right),$$

whereas the denominator tends to its indefinite integral. Hence  $U(\xi) = \lim_{\tau \rightarrow \infty} U(\xi, \tau)$  is given by

$$U(\xi) = -2\nu \frac{K \exp\left(-\frac{a}{2\nu} \xi^2\right)}{C + K \int_0^\xi \exp\left(-\frac{a}{2\nu} \eta^2\right) d\eta},$$

where  $C$  is a constant. Fixing  $C$  as  $U(0) = -2\nu K/C$ , we find

$$U(\xi) = \frac{U(0) \exp\left(-\frac{a}{2\nu} \xi^2\right)}{1 - \frac{U(0)}{2\nu} \int_0^\xi \exp\left(-\frac{a}{2\nu} \eta^2\right) d\eta} = -2\nu \frac{\partial}{\partial \xi} \log \left\{ 1 - \frac{U(0)}{2\nu} \int_0^\xi \exp\left(-\frac{a}{2\nu} \eta^2\right) d\eta \right\}.$$

See Appendix D for an alternative derivation.

In the literature [65–67] it is often written  $U(0) = F(K)$  for some function  $F$ , but we need to be more specific here. Defining  $M = \int_{-\infty}^{\infty} U(\xi) d\xi$ , it can be readily computed that

$$U(0) = \sqrt{\frac{8a\nu}{\pi}} \tanh \frac{M}{4\nu}.$$

In fact, we know that  $M = K$  because of the conservation of the  $L^1$ -norm of the velocity under time-evolution of the Burgers equation. This fact will be used below.



## B. Higher-dimensional Burgers equations

Let us consider the two-dimensional case. Again using the Cole-Hopf transform, the scaled velocity reads

$$U_i(\boldsymbol{\xi}, \tau) = -2\nu \frac{\partial_{\xi_i} \int_{\mathbb{R}^2} \psi_0(\lambda \boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu} \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|^2}{1 - e^{-2a\tau}}\right) d\boldsymbol{\eta}}{\int_{\mathbb{R}^2} \psi_0(\lambda \boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu} \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|^2}{1 - e^{-2a\tau}}\right) d\boldsymbol{\eta}}.$$

We know that  $\lim_{\tau \rightarrow \infty} U_i(\boldsymbol{\xi}, \tau)$  exists, however it is impossible to tell what it is from the expression above. The numerator can be written

$$\int_{\mathbb{R}^2} \lambda \partial_i \psi_0(\lambda \boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu} \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|^2}{1 - e^{-2a\tau}}\right) d\boldsymbol{\eta},$$

where  $\partial_i = \frac{\partial}{\partial(\lambda \eta_i)}$  denotes differentiation with respect to the argument. It is important to realize that, in two-dimensions, to make use of

$$\lambda^2 f(\lambda \boldsymbol{\eta}) \rightarrow K \delta(\boldsymbol{\eta}) \text{ as } \tau \rightarrow \infty$$

for localised  $f$  with  $K = \int f d\boldsymbol{\eta}$ , we must take higher derivatives

$$\frac{\partial^2}{\partial \xi_i \partial \xi_j} \int_{\mathbb{R}^2} \psi_0(\lambda \boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu} \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|^2}{1 - e^{-2a\tau}}\right) d\boldsymbol{\eta}.$$

It then has the definite limit

$$\int_{\mathbb{R}^2} \lambda^2 \partial_i \partial_j \psi_0(\lambda \boldsymbol{\eta}) \exp\left(-\frac{a}{2\nu} \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|^2}{1 - e^{-2a\tau}}\right) d\boldsymbol{\eta} \rightarrow M_{ij} \exp\left(-\frac{a|\boldsymbol{\xi}|^2}{2\nu}\right),$$

as  $\tau \rightarrow \infty$ , where  $M_{ij} = \int_{\mathbb{R}^2} \frac{\partial U_i}{\partial \xi_j} d\boldsymbol{\eta}$  ( $i, j = 1, 2$ ) denote constants. This is why we ought to choose the second derivatives of the velocity potential to achieve criticality.

The argument above identifies a choice of the velocity potential, e.g.

$$\phi = \log\left(1 - \frac{M_{12}}{2\nu} \int_0^{\xi_1} \int_0^{\xi_2} \exp\left(-\frac{a}{2\nu}(\xi^2 + \eta^2)\right) d\xi d\eta\right),$$

giving rise to  $\frac{\partial U_1}{\partial \xi_2} = -2\nu \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2}$ . To summarise, the source-type solutions can be written using the velocity gradient, rather than the velocity, as

$$\begin{aligned} \frac{\partial U_1}{\partial \xi_2} &= \frac{M_{12}}{(1 - R(\xi_1, \xi_2))^2} \exp\left(-\frac{a}{2\nu}(\xi_1^2 + \xi_2^2)\right) \\ &= -2\nu \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \log(1 - R(\xi_1, \xi_2)), \end{aligned}$$

where

$$R(\xi_1, \xi_2) = \frac{M_{12}}{2\nu} \int_0^{\xi_1} \exp\left(-\frac{a\xi^2}{2\nu}\right) d\xi \int_0^{\xi_2} \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta.$$

Observe that the velocity gradient has a near-Gaussian form, slightly moderated by  $R(\xi_1, \xi_2)$  which is small because of the late stage of evolution.

The idea can be extended to any dimensions, noting that we have in  $d$ -dimensions

$$\lambda^d f(\lambda\boldsymbol{\eta}) \rightarrow K\delta(\boldsymbol{\eta}) \text{ as } \tau \rightarrow \infty$$

with  $K = \int f d\boldsymbol{\eta}$ . This explains why we need to choose the  $d$ -th derivative of the vector (or tensor) potential to achieve criticality.

### C. Late stage behavior of the Hopf functional for the Burgers equation

We consider an initial-value problem of the Hopf equation (2) for the Burgers and its scaled counterpart (13). Needless to mention, the Burgers equation and the scaled Burgers equation describe the same initial-value problem. Likewise, the Hopf equation for the Burgers equation and its scaled counterpart describes the same statistical initial-value problem, with a common initial probability measure of the velocity field. To realize this, it helps to recall that the exponent of the characteristic functional, in abridged notations, reads

$$\int_{-\infty}^{\infty} u(x, t)\theta(x)dx = \int_{-\infty}^{\infty} \frac{1}{\lambda(t)} U\left(\frac{x}{\lambda(t)}, \frac{t}{\lambda(t)^2}\right) \theta(x)dx = \int_{-\infty}^{\infty} U(\xi, \tau)\zeta(\xi)d\xi,$$

where  $x = \lambda\xi, t = \lambda^2\tau, \zeta(\xi) = \theta(\lambda\xi)$ .

Once a source-type solution is obtained we can determine an asymptotic form of the Hopf functional in the late stage. Let us illustrate how this works using the 1D Burgers equation.

We recall that by substituting the Cole-Hopf solution

$$u(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left(-\frac{1}{2\nu} \int^y u_0(x')dx' - \frac{(x-y)^2}{4\nu t}\right) dy}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\nu} \int^y u_0(x')dx' - \frac{(x-y)^2}{4\nu t}\right) dy}$$

into the definition of Hopf functional

$$\Phi[\theta(x), t] = \left\langle \exp\left(i \int_{-\infty}^{\infty} u(x, t)\theta(x)dx\right) \right\rangle,$$

we can in principle obtain the time-dependent Hopf functional. However, the above expression is obscure in that we must carry out infinite-dimensional (functional) integration

because  $\langle \dots \rangle = \int \dots d\mu(u_0)$  denotes an average over initial velocity, which is generally a formidable task.

We will show that if we focus on the late stage of the decaying process, we can obtain an asymptotic form of the Hopf functional up to a quadrature. This is done by combining the dynamically-scaled Hopf equation and the source-type solution in the late stage. In taking the limit of  $\tau \rightarrow \infty$  almost all the information of the initial data will be lost through the emergence of the delta function; the information will be squeezed into that of  $K$  alone:

$$\Phi[\theta(\xi)] = \int \exp \left( i \int_{-\infty}^{\infty} \frac{\sqrt{\frac{8a\nu}{\pi}} \tanh \frac{K}{4\nu} \exp \left( -\frac{a}{2\nu} \xi^2 \right)}{1 - \frac{1}{2\nu} \sqrt{\frac{8a\nu}{\pi}} \tanh \frac{K}{4\nu} \int_0^\xi \exp \left( -\frac{a}{2\nu} \eta^2 \right) d\eta} \theta(\xi) d\xi \right) d\mu(K), \quad (19)$$

where  $d\mu(K) = P(K)dK$  denotes the probability measure of the  $L^1$ -norm of the initial velocity, with  $P(K)$  its density. Note that the above average is just a definite integral (a quadrature), whose meaning is clear and that the denominator never hits zero, because

$$\frac{1}{2\nu} \sqrt{\frac{8a\nu}{\pi}} \tanh \frac{K}{4\nu} \cdot \int_0^\xi \exp \left( -\frac{a}{2\nu} \eta^2 \right) d\eta < \frac{1}{2\nu} \sqrt{\frac{8a\nu}{\pi}} \frac{1}{2} \sqrt{\frac{2\pi\nu}{a}} = 1.$$

Note also that the above expression generalizes the Hopf-Titt solution. For small  $|K/\nu|$ , in fact we have

$$\Phi[\theta(\xi)] \approx \int \exp \left( iK \int_{-\infty}^{\infty} \tilde{g}(\xi) \theta(\xi) d\xi \right) d\mu(K),$$

where  $\tilde{g}(\xi) = \sqrt{\frac{a}{2\pi\nu}} \exp \left( -\frac{a}{2\nu} \xi^2 \right)$ . Because we have a source-type solution as  $\tau \rightarrow \infty$  with  $U_0 = K\delta(\cdot)$ , the exponent can be written[79]

$$i \int_{-\infty}^{\infty} (\tilde{g} * U_0)(\xi) \theta(\xi) d\xi = i \int_{-\infty}^{\infty} U_0(\xi) (\tilde{g} * \theta)(\xi) d\xi = i \exp \left( \frac{\nu}{2a} D \right) \int_{-\infty}^{\infty} U_0(\xi) \theta(\xi) d\xi.$$

Hence, we find

$$\Phi[\theta(\xi)] \approx \exp \left( \frac{\nu}{2a} D \right) \Phi_0[\theta(\xi)],$$

which in the original variables corresponds to

$$\Phi[\theta(x)] \approx \exp(\nu t D) \Phi_0[\theta(x)].$$

This is nothing but the Hopf-Titt solution. Thus the expression (19) is an improvement of the Hopf-Titt solution.

equations	deterministic criticality achieved with	statistical criticality achieved with	source-type solutions
1D Burgers	velocity potential $\phi$	velocity $u$	known explicitly
$n$ -D Burgers	velocity potential $\phi$	$\partial^n \phi$	existence known, made explicit here
2D Navier-Stokes	stream function $\psi$	vorticity $\omega$	known explicitly as the Burgers vortex
3D Navier-Stokes	vector potential $\boldsymbol{\psi}$	vorticity gradient $\nabla \times \boldsymbol{\omega}$	existence known, explicit form unknown

TABLE I: Self-similar (source-type) solutions

#### D. Source-type solutions for the Navier-Stokes equations

Let us clarify how the results obtained for the Burgers equations can carry over to the Navier-Stokes equations. As confirmed in Section IV, the derivation of the Hopf equation for the dynamically-scaled Navier-Stokes equations stand parallel to that of the Burgers equations. It is also known that there exist self-similar solutions of the 3D Navier-Stokes equations for small data [70–72]. See also e.g. [73, 74] for recent developments. A self-similar profile is known to exist, but its functional form is unknown.

A connection of the late-stage behavior of the Hopf functional to the source-type solution with a suitably chosen dependent variable also holds, if smoothness of solutions is assumed for the 3D Navier-Stokes equations. It should be noted that this connection itself is valid, even though the universal profile is not known explicitly. See Table I for a comparison of studies on source-type solutions of the Burgers and Navier-Stokes equations. No methods of exact linearization is known for the Navier-Stokes equations, but the existence of forward self-similar solutions and the conservation of its  $L^1$ -norm of the suitably chosen unknown suffice to apply the current methods to the the Navier-Stokes equations.

The source-type solution for the 2D Navier-Stokes equations is known explicitly as the Burgers vortex[80]

$$\Omega(\boldsymbol{\xi}) = \frac{a\Gamma}{2\pi\nu} \exp\left(-\frac{a|\boldsymbol{\xi}|^2}{2\nu}\right),$$

where  $\Gamma = \int_{\mathbb{R}^2} \Omega(\boldsymbol{\xi}) d\boldsymbol{\xi}$  denotes a circulation invariant. Following the current approach, the late-stage Hopf functional is clearly given by

$$\Phi[\theta(\boldsymbol{\xi})] = \int \exp\left(i\frac{a\Gamma}{2\pi\nu} \int_{-\infty}^{\infty} \exp\left(-\frac{a}{2\nu}\xi^2\right) \theta(\boldsymbol{\xi}) d\boldsymbol{\xi}\right) d\mu(\Gamma).$$

However, the Burgers vortex solves not only the linearised equation, but it also solves the

fully-nonlinear equations accidentally, because of its radial-symmetry (dependent on  $|\boldsymbol{\xi}|$  only). For this reason the above Hopf functional is precisely equivalent to, and not an improvement of the Hopf-Titt solution for the 2D Navier-Stokes equations.

For the 3D Navier-Stokes equations the existence of self-similar solutions is known (in velocity) in a number of function spaces. The corresponding vorticity gradient gives the source-type solution implicitly. With this variable it is near-Gaussian and we have the scaled vorticity curl

$$\mathbf{X}(\boldsymbol{\xi}) \text{ close to } \left[ \exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right) \right]',$$

where the effects of nonlinear terms and incompressibility should be taken into account. We can in principle write

$$\mathbf{X}(\boldsymbol{\xi}) = \mathbf{F} \left[ \exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right); \mathbf{K} \right],$$

where  $\mathbf{F}$  denotes a *near-identity* non-local functional and  $\mathbf{K} = \int_{\mathbb{R}^3} \mathbf{X}(\boldsymbol{\xi}) d\boldsymbol{\xi}$  an invariant. With this understanding the late-stage Hopf functional can be written

$$\Phi[\boldsymbol{\theta}(\boldsymbol{\xi})] = \int \exp\left(i \int_{-\infty}^{\infty} \mathbf{F} \left[ \exp\left(-\frac{a}{2\nu}|\boldsymbol{\xi}|^2\right); \mathbf{K} \right] \boldsymbol{\theta}(\boldsymbol{\xi}) d\boldsymbol{\xi}\right) d\mu(\mathbf{K}).$$

This motivates a determination of the self-similar profile  $\mathbf{X}(\boldsymbol{\xi})$ , at least approximately.

## VI. SUMMARY AND OUTLOOK

We have studied basic issues of statistical solutions of the Navier-Stokes equations. After presenting the main ideas using the Burgers equations, we have shown how those ideas carry over to the Navier-Stokes equations.

First, we have introduced the exponential operator  $G$ , which enables us to write the Hopf equations as an integral equation (in time). By applying the Duhamel principle on this basis, a successive approximation is formulated. The leading order approximation is presented for the Burgers equation for an illustrative purpose. This can in principle be extended to the Navier-Stokes equations, while the algebra involved would be lengthy because of the incompressibility condition.

Second, we have seen that the suitable choice of dependent variables, that achieve critical scale-invariance of statistical solutions of the Burgers and Navier-Stokes equations, depends on spatial dimensions. It should be noted that this recognition holds valid for the Navier-Stokes equations for which exact linearization is not available. It may be in order

to summarize statistical criticality, comparing to more conventional dependent variables for the 3D Navier-Stokes equations (Table II).

variables	general forms	scale-invariant forms
vector potential $\psi$	$\Phi[\lambda^d \boldsymbol{\theta}(\lambda \mathbf{x}), \lambda^{-2} t]$	$\Phi[t^{\frac{d}{2}} \boldsymbol{\theta}(t^{1/2} \mathbf{x})]$
velocity $\mathbf{u}$	$\Phi[\lambda^{d-1} \boldsymbol{\theta}(\lambda \mathbf{x}), \lambda^{-2} t]$	$\Phi[t^{\frac{d-1}{2}} \boldsymbol{\theta}(t^{1/2} \mathbf{x})]$
vorticity $\boldsymbol{\omega}$	$\Phi[\lambda^{d-2} \boldsymbol{\theta}(\lambda \mathbf{x}), \lambda^{-2} t]$	$\Phi[t^{\frac{d-2}{2}} \boldsymbol{\theta}(t^{1/2} \mathbf{x})]$
vorticity gradient $\nabla \times \boldsymbol{\omega}$	$\Phi[\lambda^{d-3} \boldsymbol{\theta}(\lambda \mathbf{x}), \lambda^{-2} t]$	$\Phi[t^{\frac{d-3}{2}} \boldsymbol{\theta}(t^{1/2} \mathbf{x})]$

TABLE II: Characteristic functionals for the scaled Navier-Stokes equations. To attain criticality, the temporal prefactor in front of  $\boldsymbol{\theta}(\cdot)$  must be absent, i.e.  $t^0$  in the scale-invariant form.

Furthermore, we have seen that when we choose a dependent variable satisfying statistical criticality, the variable will take a near-Gaussian form in the long time limit for the corresponding deterministic problem. On this basis, the late-stage evolution of statistical solutions of the Burgers and Navier-Stokes equations can be determined most conveniently by using the source-type solutions of the deterministic problem. We have employed the Cole-Hopf transform for the Burgers equations for illustration. The method itself, however, can also be applied to the Navier-Stokes equations, where self-similar solutions of the 3D Navier-Stokes equations are known to exist, but not explicitly.

In two dimensions it is the vorticity whose statistics satisfies statistical criticality. In fact, the late evolution of the 2D Navier-Stokes solutions is dominated by a collection of Burgers vortices (source-type solutions), e.g. [69], which are steady solutions of the two-dimensional linear Fokker-Planck equation. The late-stage asymptotics for the Hopf functional is equivalent to the Hopf-Titt solution.

In three dimensions it is the vorticity gradient whose statistics satisfies a critical condition. We can still argue that the late-stage asymptotics for the Hopf functional is given modulo a near-identity functional of the Gaussian function. It is of interest to determine their functional form, at least approximately, as a 3D analogue of the Burgers vortex in two dimensions. This is left for future study.

Finally, we note fact that criticality for the statistical equations is achieved with a dynamical variable whose dissipative term is given by the linear Fokker-Planck operator, is closely related to the self-adjoint property (duality) of the dissipative operators in the long-time

limit.

## APPENDIX A: FORMAL DERIVATION OF FORMULA FOR OPERATOR $G$

We give a formal derivation on the action of the operator  $G$ .

$$\Phi[\theta(x), t] = \exp(\nu t D) \Phi_0[\theta] - \frac{i}{2} \int_0^t \exp(\nu(t-s)D) \int \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \Phi}{\delta \theta(x)^2}[\theta(x), s] dx ds.$$

where

$$D \equiv \int dx \theta(x) \frac{\partial^2}{\partial x^2} \frac{\delta}{\delta \theta(x)}.$$

*Proof*

Consider the Liouville equation (also known as the Hopf-Foias equation)

$$\frac{d}{dt} \int e^{i(\theta, u(t))} d\mu(u_0) = i \int (\theta, \Delta u(t)) e^{i(\theta, u(t))} d\mu(u_0) - \frac{i}{2} \int (\theta, \partial_x u(t)^2) e^{i(\theta, u(t))} d\mu(u_0),$$

where

$$(\theta, u(t)) = \int_{-\infty}^{\infty} \theta(x) u(x, t) dx.$$

Setting  $\theta(t) = e^{(T-t)\Delta} \theta_0$ , we write

$$\begin{aligned} \frac{d}{dt} \int e^{i(\theta(t), u(t))} d\mu(u_0) &= i \int [(\dot{\theta}(t), u(t)) + (\theta(t), \dot{u}(t))] e^{i(\theta(t), u(t))} d\mu(u_0) \\ &= -\frac{i}{2} \int e^{i(\theta(t), u(t))} (\theta, \partial_x u(t)^2) d\mu(u_0). \end{aligned}$$

Integrating with respect to time in  $0 \leq t \leq T$ , we get

$$\int e^{i(\theta(T), u(T))} d\mu(u_0) = \int e^{i(\theta(0), u(0))} d\mu(u_0) - \frac{i}{2} \int_0^T ds \int e^{i(\theta(s), u(s))} (\theta(s), \partial_x u(s)^2) d\mu(u_0),$$

that is,

$$\int e^{i(\theta_0, u(T))} d\mu(u_0) = \int e^{i(e^{T\Delta} \theta_0, u(0))} d\mu(u_0) - \frac{i}{2} \int_0^T ds \int e^{i(e^{(T-s)\Delta} \theta_0, u(s))} (e^{(T-s)\Delta} \theta_0, \partial_x u(s)^2) d\mu(u_0),$$

Replacing  $T \rightarrow t$  and  $\theta_0 \rightarrow \theta$ , we find

$$\int e^{i(\theta, u(t))} d\mu(u_0) = \int e^{i(e^{t\Delta} \theta, u(0))} d\mu(u_0) - \frac{i}{2} \int_0^t ds \int e^{i(e^{(t-s)\Delta} \theta, u(s))} (e^{(t-s)\Delta} \theta, \partial_x u(s)^2) d\mu(u_0),$$

Now, the left-hand side is  $\Phi[\theta, t]$ , the first term on the right-hand side is  $\Phi(e^{t\Delta} \theta)$ . Noting that in the second term on the right-hand side

$$\partial_x u(s)^2 \Phi = -\partial_x \frac{\delta^2 \Phi}{\delta \theta^2},$$

we obtain the desired result by noting that the first term on the right-hand side equals  $\int e^{i(\theta, e^{t\Delta} u(0))} d\mu(u_0)$  by self-adjointness of the heat semigroup.  $\square$

## APPENDIX B: LEADING ORDER APPROXIMATION

The zero-th order functional is

$$\begin{aligned}
\tilde{\Phi}[\theta] &= \Phi_0[\exp(\nu t \Delta) \theta] \\
&= \exp\left(-\frac{1}{2} \iint dx' dx'' Q(x', x'') \frac{1}{\sqrt{4\pi\nu t}} \int e^{-\frac{(x'-y')^2}{4\nu t}} \theta(y') dy' \frac{1}{\sqrt{4\pi\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy''\right) \\
&= \exp\left(-\frac{1}{8\pi\nu t} \iint dx' dx'' Q(x', x'') \int e^{-\frac{(x'-y')^2}{4\nu t}} \theta(y') dy' \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy''\right). \quad (\text{B1})
\end{aligned}$$

It is straightforward to compute

$$\frac{\delta \tilde{\Phi}}{\delta \theta(x)} = -\frac{1}{4\pi\nu t} \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy'' \times \tilde{\Phi}, \quad (\text{B2})$$

$$\begin{aligned}
\frac{\delta^2 \tilde{\Phi}}{\delta \theta(x)^2} &= -\frac{1}{4\pi\nu t} \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu t} - \frac{(x''-x)^2}{4\nu t}} \times \tilde{\Phi} \\
&+ \left(\frac{1}{4\pi\nu t} \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy''\right)^2 \times \tilde{\Phi}, \quad (\text{B3})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{\delta^2 \tilde{\Phi}}{\delta \theta(x)^2} &= -\frac{1}{4\pi\nu t} \iint dx' dx'' Q(x', x'') \frac{x' + x'' - 2x}{2\nu t} e^{-\frac{(x'-x)^2}{4\nu t} - \frac{(x''-x)^2}{4\nu t}} \times \tilde{\Phi} \\
&+ 2 \left(\frac{1}{4\pi\nu t} \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy''\right) \\
&\times \left(\frac{1}{4\pi\nu t} \iint dx' dx'' Q(x', x'') \frac{x' - x}{2\nu t} e^{-\frac{(x'-x)^2}{4\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy''\right) \times \tilde{\Phi} \quad (\text{B4})
\end{aligned}$$

Hence we find

$$\begin{aligned}
\int dx \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \tilde{\Phi}}{\delta \theta(x)^2}(t) &= \frac{-1}{4\pi\nu t} \iiint dx dx' dx'' \theta(x) Q(x', x'') \frac{x' + x'' - 2x}{2\nu t} e^{-\frac{(x'-x)^2}{4\nu t} - \frac{(x''-x)^2}{4\nu t}} \times \tilde{\Phi} \\
&+ \frac{2}{(4\pi\nu t)^2} \int dx \theta(x) \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy'' \\
&\times \iint dx' dx'' Q(x', x'') \frac{x' - x}{2\nu t} e^{-\frac{(x'-x)^2}{4\nu t}} \int e^{-\frac{(x''-y'')^2}{4\nu t}} \theta(y'') dy'' \times \tilde{\Phi} \\
&= \frac{-1}{4\pi\nu t} \iiint dx dx' dx'' \theta(x) Q(x', x'') \frac{x' + x'' - 2x}{2\nu t} e^{-\frac{(x'-x)^2}{4\nu t} - \frac{(x''-x)^2}{4\nu t}} \times \Phi_0[e^{\nu t \Delta} \theta] \\
&+ \frac{1}{2\pi\nu t} \int dx \theta(x) \iint dx' dx'' Q(x', x'') e^{-\frac{(x'-x)^2}{4\nu t}} e^{\nu t \Delta} \theta(x'') \\
&\times \iint dx' dx'' Q(x', x'') \frac{x' - x}{2\nu t} e^{-\frac{(x'-x)^2}{4\nu t}} e^{\nu t \Delta} \theta(x'') \times \Phi_0[e^{\nu t \Delta} \theta],
\end{aligned}$$

which for convenience we may write in a compact form

$$= -\Phi_0[e^{\nu t \Delta} \theta] \int dx \theta(x) \frac{\partial}{\partial x} \left\{ e^{\nu t(\Delta' + \Delta'')} Q(x', x'') - \left( \int dx'' e^{\nu t \Delta'} Q(x', x'') e^{\nu t \Delta} \theta(x'') \right)^2 \right\}.$$



Here  $e^{\nu t(\Delta'+\Delta'')}Q(x', x'')$  and  $e^{\nu t\Delta'}Q(x', x'')$  denote functions of  $x$  in the relevant arguments, that is,

$$e^{\nu t(\Delta'+\Delta'')}Q(x', x'') = \frac{1}{4\pi\nu t} \iint \exp\left(-\frac{(y-x)^2 + (z-x)^2}{4\nu t}\right) Q(y, z) dy dz,$$

$$e^{\nu t\Delta'}Q(x', x'') = \frac{1}{\sqrt{4\pi\nu t}} \int \exp\left(-\frac{(y-x)^2}{4\nu t}\right) Q(y, x'') dy.$$

Thus we find

$$\begin{aligned} G\tilde{\Phi} &= \frac{i}{2} \int_0^t ds e^{\nu(t-s)D} \int dx \theta(x) \frac{\partial}{\partial x} \frac{\delta^2 \tilde{\Phi}}{\delta\theta(x)^2}(s) \\ &= -\frac{i}{2} \Phi_0[e^{\nu t\Delta}\theta] \int_0^t ds \int dx e^{\nu(t-s)\Delta}\theta(x) \\ &\quad \times \frac{\partial}{\partial x} \left\{ e^{\nu s(\Delta'+\Delta'')}Q(x', x'') - \left( \int dx'' e^{\nu s\Delta'}Q(x', x'') e^{\nu t\Delta}\theta(x'') \right)^2 \right\}. \end{aligned}$$

Spelling out more explicitly, we obtain (9).

### APPENDIX C: DERIVATION OF THE ERROR ESTIMATE

We shall estimate

$$I \equiv \left\| \int_0^t ds \int e^{\nu(t-s)\Delta} \frac{\partial\theta}{\partial x} \left\{ e^{\nu s(\Delta'+\Delta'')}Q(x', x'') - \left( \int dx'' e^{\nu s\Delta'}Q(x', x'') e^{\nu t\Delta}\theta(x'') \right)^2 \right\} dx \right\|.$$

We first divide it in two parts to consider

$$\begin{aligned} I_1 &= \int e^{\nu(t-s)\Delta} \frac{\partial\theta}{\partial x} \cdot e^{\nu s(\Delta'+\Delta'')}Q(x', x'') dx, \\ I_2 &= \int e^{\nu(t-s)\Delta} \frac{\partial\theta}{\partial x} \cdot \left( \int e^{\nu s\Delta'}Q(x', x'') e^{\nu t\Delta}\theta(x'') dx'' \right)^2 dx. \end{aligned}$$

By Hölder inequality

$$\int |f(x)g(x)| dx \leq \|f\|_{L^p} \|g\|_{L^q}, \text{ for } \frac{1}{p} + \frac{1}{q} = 1,$$

and an estimate for the heat-kernel, e.g. [72],

$$\|e^{\nu t\Delta}\theta\|_{L^p} \leq \frac{1}{(4\pi\nu t)^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}} \|\theta\|_{L^q},$$

we bound  $I_1$  as

$$I_1 \leq \frac{C_1}{\{\nu(t-s)\}^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}(\nu s)^{\frac{1}{2}(\frac{1}{u}-\frac{1}{q})}} \left\| \frac{\partial \theta}{\partial x} \right\|_{L^q} \left\| \|Q(x, y)\|_{L^r(dx)} \|L^u(dy)\right\|,$$

where  $1 \leq q \leq p \leq \infty, 1 \leq r \leq q \leq \infty, 1 \leq u \leq r \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$  and  $C_1$  is a non-dimensional constant. Taking  $p = \infty, q = 1, r = u = 1$  and noting  $\int_0^t \frac{ds}{\sqrt{s(t-s)}} = \pi (< \infty)$ , we find

$$I_1 \leq \frac{C}{\nu} \left\| \frac{\partial \theta}{\partial x} \right\|_{L^1} \|Q(x, y)\|_{L^1(\mathbb{R}^2)},$$

where  $C (= \pi C_1)$  is another constant. Similarly, we bound  $I_2$  as

$$I_2 \leq \frac{C_2}{\{\nu(t-s)\}^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}} \frac{1}{(\nu s)^{\frac{1}{2}(\frac{1}{r}-\frac{1}{2q})}} \frac{1}{(\nu t)^{\frac{1}{2}(\frac{1}{w}-\frac{1}{v})}} \left\| \frac{\partial \theta}{\partial x} \right\|_{L^q} \left\| \|Q(x, y)\|_{L^r(dx)} \|L^u(dy)\| \|\theta\|_{L^w}^2 \right\|,$$

where  $1 \leq q \leq p \leq \infty, 1 \leq r \leq 2q \leq \infty, 1 \leq w \leq v \leq \infty, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{u} + \frac{1}{v} = 1$  and  $C_2$  is a constant. Choosing, for example  $r = u = 1, p = \infty, q = 1, v = \infty, w = \infty$ , we find

$$I_2 \leq \frac{C'}{\nu} \left\| \frac{\partial \theta}{\partial x} \right\|_{L^1} \|Q(x, y)\|_{L^1(\mathbb{R}^2)}^2 \|\theta\|_{L^\infty}^2,$$

with  $C' (= \pi C_2)$ . Combining those two results and writing  $C = \max(C, C')$ , we obtain (10). It is readily checked that those bounds are non-dimensional noting the physical dimension of  $\theta$  is the same as that of  $\nu$  for the 1D Burgers equation.

#### APPENDIX D: SELF-SIMILAR DECAY

In the case of forward self-similarity, we take the length scale as  $\lambda(t) = \sqrt{2a(t+t_*)}$  to write the Leray equation of the form

$$U \frac{\partial U}{\partial \xi} = \nu \frac{\partial^2 U}{\partial \xi^2} + a \left( \xi \frac{\partial U}{\partial \xi} + U \right).$$

This can be exactly solved as follows.

Upon integration we find, after taking a constant to be zero,

$$\frac{dU}{d\xi} = \frac{1}{2\nu} U(U - 2a\xi).$$

By  $V = 1/U$  in this Bernoulli equation, we have

$$\frac{dV}{d\xi} = -\frac{1}{2\nu} (1 - 2a\xi V),$$

which is a linear inhomogeneous equation. Solving it we find

$$V(\xi) = e^{a\xi^2/(2\nu)} \left( V(0) - \frac{1}{2\nu} \int_0^\xi e^{-a\eta^2/(2\nu)} d\eta \right)$$

or

$$U(\xi) = \frac{e^{-a\xi^2/(2\nu)}}{U(0)^{-1} - \frac{1}{2\nu} \int_0^\xi e^{-a\eta^2/(2\nu)} d\eta}.$$

Note that by making  $U(0)$  small enough such that  $U(0) < \sqrt{\frac{8a\nu}{\pi}}$ , we can make the denominator to be non-zero for all  $\xi$ . This means that the particular self-similar solution is valid for small initial data.

In passing we comment on backward self-similar solutions. By reversing the sign of  $a$ , a possibility of blowup can be studied using backward self-similarity with the length-scale  $\sqrt{2a(t_* - t)}$ . (Of course, it is known there is no blowup.). In this case, the steady equation is

$$U \frac{\partial U}{\partial \xi} + a \left( \xi \frac{\partial U}{\partial \xi} + U \right) = \nu \frac{\partial^2 U}{\partial \xi^2},$$

whose smooth solution is  $U \equiv 0$  only. The solution is nonetheless obtained as

$$U(\xi) = \frac{e^{a\xi^2/(2\nu)}}{U(0)^{-1} - \frac{1}{2\nu} \int_0^\xi e^{a\eta^2/(2\nu)} d\eta}.$$

Note that  $U(\xi) \rightarrow \frac{1}{|\xi|}$  as  $\xi \rightarrow \pm\infty$ . More importantly, this has a singular point (a pole) somewhere, say at  $\xi = \xi_*$ , at which

$$U(0)^{-1} - \int_0^{\xi_*} e^{a\eta^2/(2\nu)} d\eta = 0.$$

Hence,  $U(\xi) \propto 1/(\xi - \xi_*)$  near there and is non-integrable  $U \notin L^1$ .

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- [75] Using the heat kernel

$$g_t = \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\nu t}\right),$$



we apply the Duhamel principle to the Navier-Stokes equations to obtain

$$\mathbf{u}(t) = g_t * \mathbf{u}_0 - \int_0^t g_{t-s} * \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u})(\cdot, s) ds,$$

where  $\mathbb{P} = \mathbf{I} - \nabla \Delta^{-1} \nabla \cdot$  denotes solenoidal projection and  $*$  convolution.

- [76] Strictly speaking, the second-order functional derivative should be understood as a kind of trace formula

$$\frac{\delta^2 \Phi}{\delta \theta(x) \delta \theta(x)} = \lim_{y \rightarrow x} \frac{\delta^2 \Phi}{\delta \theta(x) \delta \theta(y)}$$

by regarding

$$\frac{\delta^2 \Phi}{\delta \theta(x) \delta \theta(x)} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}$$

as a differential form in a distributional sense [23].

- [77] Compare with  $\exp\left(a \frac{d}{dx}\right) f(x) = f(x+a)$ , where  $f(x)$  a smooth function and  $a$  a constant.
- [78] It is a counterpart to a fundamental solution to linear problems, e.g. the heat equation.
- [79] This means that the self-dual property of  $\tilde{g}_\tau$  in scaled variables is restored after taking the  $\tau \rightarrow \infty$  limit.
- [80] In the original Burgers vortex the linear straining has been introduced to represent stretching by other vortices. Here the linear strain term appears as a result of dynamic scaling. The equations are identical.