Article:

Gil Molto, M. orcid.org/0000-0001-6247-1228, Poyago-Theotoky, J., Rodrigues-Neto, J.A. et al. (1 more author) (2020) Mixed oligopoly, cost-reducing research and development, and privatisation. European Journal of Operational Research, 283 (3). pp. 1094-1106. ISSN 0377-2217

https://doi.org/10.1016/j.ejor.2019.11.071

Mixed oligopoly, cost-reducing research and development, and privatisation[☆,☆☆]

Maria José Gil-Moltó[a], Joanna Poyago-Theotoky[b,c,*], José A. Rodrigues-Neto[d], Vasileios Zikos[e]

[a] *Department of Economics, The University of Sheffield, 9 Mappin Street, Sheffield S1 4DT, UK*

[b] *Department of Economics and Finance, La Trobe University, Melbourne, Victoria 3086, Australia*

[c] *ICRE8 Research Centre, Artemidos 6 & Epidavrou, Maroussi, Athens, Greece*

[d] *Australian National University, Research School of Economics, H. W. Arndt Building 25A, Canberra ACT – 2601, Australia*

[e] *Faculty of Economics, Chulalongkorn University, 254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand*

Abstract

We develop a mixed oligopoly model to examine the role of R&D subsidies and evaluate the welfare effects of privatisation. In solving the oligopoly model we propose a novel use of aggregative games techniques. Our analysis reveals that privatisation reduces the optimal R&D subsidy. Furthermore, privatisation improves social welfare but only when the number of firms is sufficiently large. Implementing solely a subsidy to R&D does not lead to a 'privatisation neutrality theorem' or 'irrelevance result'.

Keywords: Game theory, mixed oligopoly, aggregative games techniques, privatisation, R&D.
JEL: C72, L13, L32, L33, O38.

[☆] We thank the Editor, Steffen Rebennack, four anonymous reviewers, Ben Chen, Rogerio Mazali, Marcella Scrimitore, James Taylor, Yoshihiro Tomaru, Ishii Yasunori as well as participants at WEAI 2019 for helpful comments and suggestions. The paper was written during a visit by Poyago-Theotoky to the Department of Economics at the University of Zürich. We thank the University of Zürich for their hospitality.

[☆☆] This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declarations of interest: none.

[*] Corresponding author

Email addresses: m.j.gil-molto@sheffield.ac.uk (Maria José Gil-Moltó), j.poyago.theotoky@gmail.com (Joanna Poyago-Theotoky), jarnwi@yahoo.com (José A. Rodrigues-Neto), vasileios.z@chula.ac.th (Vasileios Zikos)

1. Introduction

Over the last decades, there have been arguments in favour of and against the privatisation of public firms. In what follows we use the term 'public' firm to refer to a firm that is state-owned, in line with the majority of the literature on mixed oligopoly. This debate occurs in many countries. At the same time, there is a substantial academic literature on the controversial topic of whether one should privatise or nationalise firms. If so, which ones? Many western countries have privatised large public or state firms in many sectors once considered strategic for their economies. Has productivity improved? What are the consequences of privatisation or nationalisation of firms?

Existing empirical evidence cites the importance of public funding towards research and development (R&D) (Hart (1998); Katz (2001)) and the substantial presence of public firms in innovative industries such as healthcare (Aanestad et al. (2003)), bioagriculture (Oehmke (2001)) as well as energy and transportation.

In the field of operations research there have been contributions to mixed oligopoly, notably Kalashnikov et al. (2009) and Kalashnikov et al. (2011) who explore the concept of conjectural variations equilibrium (CVE), albeit in the absence of any R&D and/or subsidies. Further, Kalashnikov et al. (2010) introduce the issue of leadership in the context of a domestic public firm competing with a foreign firm in the absence of R&D and privatisation concerns, while Duan (2013) examines efficiency-enhancing privatisation in a mixed olipopoly and Lee and Tomaru (2017) explore the first-best allocation in the presence of R&D and output subsidies. In line with the empirical interest in R&D competition in mixed markets, the theoretical literature in economics analysing this issue is also developing; for example, Delbono and Denicolò (1993); Poyago-Theotoky (1998); Matsumura and Matsushima (2004); Ishibashi and Matsumura (2006); Heywood and Ye (2009); Gil-Moltó et al. (2011); Kesavayuth and Zikos (2013); Haruna and Goel (2015).

New technologies improve production processes, reducing costs or improving quality. Are there special considerations to be made when privatisation occurs in sectors associated with new technologies? How much R&D activities are undertaken by public and private firms? Should these R&D activities be subsidised and to what extent? Does the answer depend on the existence of a public firm? In this paper, we study the use of a uniform R&D subsidy in the context of a mixed oligopoly and evaluate the effects of privatisation. There is a unique homogeneous, perfectly divisible good. Even though the final good produced may be a traditional one, more R&D activities lead to less expensive, or 'lean', production, as the marginal cost of each firm is a decreasing function of its level of R&D. In this sense R&D activities lead to more efficient production. We consider two market structures: in the first, n private firms and one public firm produce and sell the good to a population of consumers (the mixed oligopoly). In the second structure, $n+1$ private firms operate (the private oligopoly). The basic idea is to compare the outcomes of these two structures to understand the consequences of privatising the public firm.

The choice of policy instrument (R&D subsidies) in our paper is motivated by a number of reasons.

Firstly, there exists ample evidence of the use of R&D subsidies (either directly allocated through grants or via tax credits) from all around the world. In fact, according to OECD figures, 10% to 20% of business R&D expenditure is funded by the government in most OECD countries (OECD (2018)). Examples of R&D subsidy programmes can be found in countries such as Argentina, Germany, Israel and the US (OECD (2018)), Finland (Einiö (2014)), Korea (Cin et al. (2017)) or China (Aggarwal and Evenett (2012); Boeing (2016)), to cite just a few. Secondly, R&D subsidies appear to be less controversial policy instruments than output subsidies, both from a political and a legal point of view. R&D subsidies have often been seen by policy-makers as a tool to enable sustained growth, particularly but not exclusively in times of crises (Hud and Hussinger (2015)). Moreover, international organisations such as the International Monetary Fund (IMF henceforth) and the World Trade Organization (WTO henceforth) have traditionally viewed output subsidies rather unfavourably. For example, the IMF has routinely advocated the scrapping of production subsidies in sectors such as farming, energy and transportation in many countries. In contrast, R&D subsidies are presented under a more positive light, as a way to encourage R&D investments and innovation by businesses (IMF, 2016). With regard to the WTO and its rules, it is worth noting that R&D subsidies were classified as 'non-actionable' (that is, they could not be challenged under WTO rules) up until 1999, in contrast with output subsidies which were never subject to this exemption (Avi-Yonah and Vallespinos (2018)). Interestingly, despite the fact that this differential treatment ceased to exist in 1999, R&D subsidies have still been the subject of very little WTO enforcement (Shin and Lee (2013)).

Our paper also contributes to a broader literature strand within operations research studying a variety of investment problems, such as R&D, advertising, manufacturing and capacity investments (e.g., Fruchter (1999), Grishagin et al. (2001), Murto et al. (2004), Boonman et al. (2015)), in the context of an oligopolistic market structure, where there is a finite, typically small, number of players interacting strategically.

We develop a model of cost-reducing R&D where a public firm competes with a multitude of private firms. Our results show that apart from addressing the market failures arising from the R&D activity, the use of R&D subsidies mitigates the inefficiency in the distribution of production costs which is likely to arise in a mixed market. This yields an insight into the economic effects of an R&D subsidy and suggests that it may serve, at least partially, the same purpose as an output subsidy. In this way, we extend and further substantially develop Gil-Moltó et al. (2011) who examined the case of a mixed duopoly and found that the optimal R&D subsidy is always higher relative to the private duopoly, it is increasing in the degree of spillovers while privatising the public firm is welfare reducing.

We find that if there are less than four private firms in the mixed oligopoly, the socially optimal R&D subsidy is positive. However, if there are more than four private firms, a suitable tax on R&D activities implements the maximal welfare (cf. Leahy and Neary (1997)). A policy implication of this result is that for an effective design of R&D subsidisation programmes, careful consideration of the number of firms in a given industry is required. A positive R&D subsidy may not lead to desirable effects on welfare for industries with a relatively large number of firms. Under the socially optimal R&D subsidy, the public firm produces more than any private firm. The socially optimal R&D subsidy leads to more total R&D than the laissez-faire policy. However, the public firm can have different levels of R&D and production than private firms. This is inefficient, as it does not lead to the equalisation of per firm output. There is an inefficient distribution of production costs. Privatization of the public firm can lead to a smaller level of total R&D in highly concentrated industries, even when the socially optimal level of R&D subsidy is implemented. Conversely, privatisation may increase aggregate R&D and welfare provided that the number of private firms is sufficiently large. We note that the latter contrasts with the various 'irrelevance results' or 'privatisation neutrality theorems' obtained in previous contributions that consider output subsidies in the absence of R&D competition, but it nevertheless complies with the seminal contribution of De Fraja and Delbono (1989) in the absence of any form of subsidisation policy. Cato and Matsumura (2013) in a related paper consider free-entry and privatisation in the absence of R&D, and find that the 'privatisation neutrality theorem' holds only when both an output subsidy and an entry-fee are used by the government. The desirability or not of privatization continues to be a lively research topic (e.g., see recent contributions by Haraguchi et al. (2018), Haraguchi and Matsumura (2018), Lin and Matsumura (2012), Matsumura and Shimizu (2010) and Matsumura and Okamura (2015)). The result we obtain here adds to this literature by showing that, when firms engage in cost-reducing R&D, privatisation improves welfare when the number of firms is relatively large. In terms of policy implications, this suggests that even if R&D subsidies are employed, privatisation may be socially desirable as long as the number of private firms is sufficiently large.

The paper is organised as follows. In section 2 we describe the basic model, and in 3 we provide details of the methodological approach followed to solve the games we analyse. In section 4 we find the unique subgame perfect equilibrium of the mixed oligopoly game, in section 5 we do the same for the private oligopoly, while in section 6 we briefly consider the first-best. In section 7 we present the main results and compare the outcomes of the two market arrangements, and in section 8 we offer some concluding remarks.

2. The Model

Consider an industry consisting of a public welfare-maximising firm and n identical private profit-maximising firms (let $n \geq 1$ be an integer) producing a homogeneous good. Firms are indexed by j: the set of all firms is $J = \{0, 1, 2, \cdots, n\}$, with $J^P = \{1, 2, \cdots, n\}$ representing all private firms (where the superscript (or, subscript later on) P stands for private) and $j = 0$ indicating the public firm. The demand function is linear and given by $Q = a - P$, for $0 \leq P \leq a$, and $Q = 0$, for $P > a$, where P denotes price. Aggregate output is $Q = \sum_{j=0}^{n} q_j$, $q_j \geq 0$ denotes the quantity produced by each firm $j \in J$. All firms engage in cost-reducing (process) R&D and there are no spillovers. The exclusion of spillovers allows us to concentrate purely on the effect of competition, and therefore provides a sharp characterisation of the circumstances under which privatisation is socially desirable. The role of spillovers is examined in detail but only for the case of a duopoly in Gil-Moltó et al. (2011). The total cost function of each firm C_j, is decomposed into assembly cost, $\widehat{C}_j$, and R&D cost, R_j: $C_j(q_j, x_j) = \widehat{C}_j(q_j, x_j) + R_j(x_j)$, where $x_j \geq 0$ is the R&D level (cost reduction) of each firm $j \in J$. The assembly cost of firm j is $\widehat{C}_j(q_j, x_j) = (c - x_j)q_j + q_j^2$, when $0 \leq x_j \leq c$, and $\widehat{C}_j(q_j, x_j) = q_j^2$, if $x_j > c$, and $a > c > 0$. This cost function is in line with Gil-Moltó et al. (2011) and provides a distinct characterisation of the role of cost reducing R&D, in that R&D reduces the intercept of marginal cost in a uniform manner, while unit costs are increasing. The presence of the quadratic term is standard in the mixed oligopoly literature and rules out the possibility of a public monopoly by introducing diminishing returns in production. Further, this assumption reflects that the public firm and the private firms are ex-ante equally efficient. According to White (2002), this assumption can be qualified in several ways. For instance, as there is mixed evidence concerning the relative efficiency of public and private firms the assumption of identical and symmetric technologies seem quite reasonable. Moreover, public firms that survive for a substantial time period may be considered at least as efficient as their private rivals. Noting that marginal costs are increasing, this leads to a higher cost for the public firm after production decisions have been made. Furthermore, with linear costs, if the public firm was more efficient it would serve the entire market, and if it was too inefficient this would leave room for government intervention for either privatising or shutting down the public firm (White (2002)). As we intend to examine strategic interactions between firms in a mixed oligopoly and the welfare effects of privatising the public firm, considering monopolistic public firms or private markets would not be relevant for our purposes. We also make the standard assumption that R&D spending is subject to diminishing returns, $R_j(x_j) = x_j^2$, for all $x_j \geq 0$.

Suppose that the government subsidizes R&D in the form of $S(x_j) = sx_j$ where s denotes the (per unit) subsidy to R&D level (similar results obtain if the subsidy is provided on the R&D expenditure/input instead); all firms, i.e., the public and the private firms, receive a uniform subsidy. Therefore government expenditure to finance the R&D subsidies is $\sum_{j=0}^{n} S(x_j)$.

We develop the analysis as a multi-stage game with complete information and use subgame perfection as the solution concept. Initially, the game has two stages: in the first stage, all firms simultaneously choose their amounts of R&D. Every firm observes the profile of R&D levels, $x = (x_0, x_1, x_2, \cdots, x_n)$. After observing x, in the second stage firms simultaneously choose quantities, q_j.

A firm's profit function is given by

$$F_j(q_j, x_j) = Pq_j - C_j(q_j, x_j) + S(x_j). \tag{1}$$

The public firm maximises social welfare W, defined as the sum of consumer surplus and producer surplus net of R&D subsidies, by choosing x_0 and q_0

$$W = \int_{Q=0}^{Q^*} (a - Q - P^*)dQ + \sum_{j=0}^{n}[F_j - S(x_j)]. \tag{2}$$

Initially, the common level of R&D subsidy, s, is exogenous. Next, in a modified game, we let the government/regulator set the uniform R&D subsidy, s, at an initial stage, stage 0. In this situation, the timing of the game is as follows: In stage zero, the government/regulator commits to a subsidy on R&D level so as to maximise welfare. In stage one, firms make their R&D decisions. In stage two, firms play a Cournot game. The following technical assumption is made to ensure the existence of a Subgame Perfect Nash Equilibrium (SPNE henceforth), where all firms have positive output and R&D. This condition is always satisfied when the subsidy s is maximising social welfare.

Assumption 1. *The adjusted R&D subsidy, defined as* $\frac{s}{a-c}$*, satisfies the following condition:* $\frac{s}{a-c} < \frac{(8n+67)}{3n(n+6)}$.

This condition ensures that the public firm generates a non-negative amount of R&D, $x_0 \geq 0$, and produces positive quantity, $q_0 > 0$.

3. Methodology - Explaining the Use of Aggregative Games

In terms of methodology, to solve the models we use game theory. In particular, to solve the various subgames, we make a novel methodological contribution in that we use aggregative games techniques, a group of procedures studied and developed by Cornes and Hartley (2000, 2003, 2010), Jensen (2010), Anderson et al. (2013), Cornes (2016) and Nocke and Schutz (2018), among others. The sequential games in the sequel have two or three stages. As usual in finite horizon games, we use backward induction and apply aggregative games techniques. By using these techniques, it is possible to obtain fully analytical, closed form solutions for all variables. This is possible because aggregative games destroy the usual 'curse of dimensionality', allowing us to find the equilibrium with any arbitrary number of players.

How do aggregative games usually work? For each player in a game (or subgame), one calculates the so-called *replacement function*. These replacement functions are somewhat similar to the usual best-reply functions for players in classical game theory. However, a replacement function for player j establishes the optimal action for player j as a function of an *aggregator* (i.e., an aggregate action). This aggregator includes the action of player j herself, not only the actions of other players. After one has obtained the replacement functions of all players, with the action of player j on the left-hand side, one then adds up all of these equations (the sum of the left-hand sides is equal to the sum of the right-hand sides) to find the aggregate action on the left-hand side. The right-hand side will have a function of the aggregator. Next, one solves this equation for the aggregator, establishing a formula for the aggregator in terms of exogenous variables only. Once the aggregator is known, it is possible to substitute it back into the right-hand side of each replacement equation to find the equilibrium actions of players in terms of exogenous variables only.

In the present paper, the technique of aggregative games is used as described above to solve the quantities-stage of the private oligopoly subgame. However, the mixed oligopoly quantities-subgame is more complicated and the technique needs to be adapted. It is necessary to solve a linear system with two variables. First, we define an aggregator which comprises the sum of the quantities of private firms only. We aim at finding one equation linking this (quantity) aggregator and the quantity chosen by the public firm. This equation derives from aggregating the replacement equations of all private firms. The first-order condition of the profit maximization problem of each private firm generates, after some algebra, the replacement equation of this firm, as usual, writing the optimal action (quantity) of private firm j in terms of our aggregator and the action of the public firm. We also compute the first-order condition for the welfare maximization problem of the public firm, leading to another equation linking the same two variables, the quantity chosen by the public firm and the aggregator of the quantities chosen by all private firms. In this way it is possible to find a 2 x 2 linear system. Solving this linear system, we find the quantity produced by the public firm and the aggregate quantity of all private firms. Once we find the aggregator for the quantities of all private firms, we substitute this back into the replacement equations of the private firms to derive the individual quantities chosen by the private firms. Of course, at this stage we have to consider any given R&D actions in the previous stage; that is, R&D actions are assumed to be fixed/given at this point. After we have the formulas for quantities in terms of R&D (previous) actions, then we plug them back into the relevant payoff functions. The payoffs are now written in terms of R&D actions and exogenous variables only (no quantities appear in these formulas). Then, we move backwards to the stage of R&D choices. The solution is similar. Again, we use two variables, one for the aggregate R&D of private firms only, and another variable for the R&D choice of the public firm. Again, we find first-order conditions leading to two equations forming a linear system. All of this assumes a given (fixed) subsidy rate. Finally, in the game where the subsidy is also a strategic variable, we move backwards one more period/stage to calculate the socially optimal subsidy. Each subgame uses a different aggregator. But instead of having to solve complex systems with many equations and many variables, the equilibrium of each one of these stages is computed by solving a system of two equations and two variables. This is true regardless of the number of players, breaking the curse of dimensionality.

In more detail, and anticipating the results in the next sections, in Lemma 1 we compute the quantities that all firms will choose as functions of R&D amounts in the mixed oligopoly. This Lemma needs the aggregator of quantities, $Q = \sum_{i=0}^{n} q_i$, the sum of all quantities, including the quantity chosen by the public firm. The previous stage of the backwards induction argument computes the R&D amounts of firms. This is done in Lemma 2 and Proposition 1, with the help of aggregator X, which adds the R&D amount of all private firms, $X = \sum_{i=1}^{n} x_i$. These results use the 'trick' of defining the aggregator X without the amount of R&D of the public firm. We also use an auxiliary variable B to make the computations manageable. It is also necessary to use the aggregator Q in the expression that finds the derivative of the welfare. The proof of Proposition 1 requires two aggregators, solving linear systems, and some changes of variables, all of which make it a complex, non-straightforward proof. Propositions 2 and 3 have long, yet straightforward proofs. The case of the private oligopoly is simpler than the mixed oligopoly, but still requires the use of the same aggregators X and Q. The main steps to solve the private oligopoly model are Lemma 3 and Proposition 4.

The next sections provide the detailed steps followed to solve the models and the analysis of the results we obtain.

4. Mixed Oligopoly

In the last stage, firms choose quantities, the R&D subsidy has already been determined and the R&D investments have already been made. Under Cournot competition, the public firm acts simultaneously with the private firms by choosing outputs to maximise their respective objectives. The unique Nash equilibrium of this stage game is characterised in the next two Lemmas. In what follows, we define $\xi \equiv a - c > 0$ (a measure of market size).

Lemma 1. *(*EQUILIBRIUM QUANTITIES-QUANTITY STAGE SUBGAME*) Given R&D levels, the respective equilibrium quantities of the public firm, aggregate, and private firms are:*

$$q_0^* = \frac{(n+3)x_0 + 3\xi - \sum_{i=1}^n x_i}{2n+9} \quad (3)$$

$$Q^* = \frac{3x_0 + (2n+3)\xi + 2\sum_{i=1}^n x_i}{2n+9} \quad (4)$$

$$q_j^* = \frac{6\xi - 3x_0 + (2n+9)x_j - 2\sum_{i=1}^n x_i}{3(2n+9)}, j \in J^P. \quad (5)$$

Proof. To compute the Nash equilibrium, we find the optimal action of the public firm, q_0, and the optimal actions of private firms, q_j, for $j \in \{1, 2, \cdots, n\}$. Then, we solve the linear system of two equations, where the variables are q_0 and the aggregate quantity Q. This solution procedure uses aggregative games techniques. (See Cornes and Hartley (2000, 2003, 2010) and Cornes (2016) for further details on aggregative games.) Once q_0 and Q are established, we substitute these values into the replacement equations in order to compute the quantities produced by the private firms, q_j, for $j \in \{1, 2, \cdots, n\}$.

The public firm chooses q_0 to maximise social welfare, W. The aggregate quantity is $Q^* = q_0 + \sum_{j=1}^n q_j$. Then, for every $j \in \{0, 1, 2, \cdots, n\}$, $\frac{dQ^*}{dq_j} = 1$. From (2) we obtain

$$W = \frac{-(Q^*)^2}{2} + \xi Q^* + \sum_{j=0}^n x_j q_j - \sum_{j=0}^n q_j^2 - \sum_{j=0}^n x_j^2. \quad (6)$$

Taking the first and second partial derivatives of (6) with respect to q_j, for $j \in \{0, 1, 2, \cdots, n\}$, we obtain: $\frac{\partial W}{\partial q_j} = -Q^* + \xi + x_j - 2q_j$, and $\frac{\partial^2 W}{\partial q_j^2} = -1 - 2 = -3 < 0$. Then, $\frac{\partial W}{\partial q_0} = -Q^* + \xi + x_0 - 2q_0$ and the associated first-order condition $-3q_0 + x_0 + \xi - \sum_{j=1}^n q_j = 0$, yields:

$$2q_0 + Q^* = x_0 + \xi \quad (7)$$

$$q_0^* = \frac{x_0 + \xi}{3} - \frac{1}{3}\sum_{j=1}^n q_j. \quad (8)$$

Using (2) again, we obtain:

$$W = \frac{(Q^*)^2}{2} + (\xi - Q^*)\sum_{j=0}^{n} q_j + \sum_{j=0}^{n} x_j q_j - \sum_{j=0}^{n} q_j^2 - \sum_{j=0}^{n} x_j^2 \tag{9}$$

Define the auxiliary variables $\beta = \xi - \sum_{j=1}^{n} q_j$ and $\gamma = \sum_{j=1}^{n} x_j q_j - \sum_{j=1}^{n} q_j^2 - \sum_{j=1}^{n} x_j^2$. Then

$$2W = -3q_0^2 - 2x_0^2 + 2x_0 q_0 + 2\beta q_0 + \left(\sum_{j=1}^{n} q_j\right)^2 + 2\beta \sum_{j=1}^{n} q_j + 2\gamma. \tag{10}$$

As $\sum_{j=1}^{n} q_j = Q - q_0$, (10) becomes $2W = -2q_0^2 - 2x_0^2 + 2x_0 q_0 + Q^2 - 2Qq_0 + 2\beta Q + 2\gamma$. Substituting $\beta = \xi - (Q - q_0) = \xi + q_0 - Q$ into the above results in the important equation below:

$$2W = 2q_0^2 - 2x_0^2 + 2x_0 q_0 - Q^2 + 2\xi Q + 2\gamma. \tag{11}$$

Each private firm $j \in J^P$ solves its profit maximisation problem: $q_j^* = \underset{q_j \geq 0}{\arg\max}\, F_j$. Let $Q_{-j} = Q - q_j$ denote the sum of the quantities produced by all firms but j. Firm j's profit, F_j, is therefore expressed as

$$F_j = (\xi + x_j - Q_{-j} - 2q_j)q_j - x_j^2 + s x_j \tag{12}$$

and its marginal profit is $\frac{dF_j}{dq_j} = \xi + x_j - Q_{-j} - 4q_j$. Firm j produces a positive quantity if and only if $Q_{-j} < \xi + x_j$. If this condition holds, then there is an interior solution and the equation $dF_j/dq_j = 0$ determines the optimal quantity for firm j. The second-order condition holds, $d^2F_j/dq_j^2 = -4 < 0$. The first-order condition of the profit maximisation problem of each private firm $j \in J^P$ is:

$$\xi + x_j - Q = 3q_j. \tag{13}$$

Equation (13) is the *replacement function* associated with the private firms in this stage. Adding the equations $Q + 3q_i = \xi + x_i$, for all private firms $i \in \{1, 2, \cdots, n\}$ leads to:

$$-3q_0 + (n+3)Q = n\xi + \sum_{i=1}^{n} x_i. \tag{14}$$

Consider next the system of equations (7) and (14) in the variables q_0 and Q:

$$\begin{cases} 2q_0 + Q = x_0 + \xi \\ -3q_0 + (n+3)Q = n\xi + \sum_{i=1}^{n} x_i. \end{cases}$$

The solution to this system is expressions (3) and (4). Now, substitute $Q = Q^*$ from equation (4) into the replacement equation (13) for a private firm $j \in J^P$ to obtain (5). This concludes the proof. ■

Lemma 2. (Equilibrium profits-Quantity Stage Game) *The last-stage equilibrium profit for* *private firm* j*,* $j \in J^P$*, is given by:*

$$F_j^* = 2(q_j^*)^2 - x_j^2 + sx_j$$

and for the public firm by:

$$F_0^* = (q_0^*)^2 - x_0^2 + sx_0.$$

Proof. Assuming all private firms produce positive quantities and $Q^* \leq a$, the equilibrium price is $P^* = a - Q^* = \frac{-3x_0+6a+(2n+3)c-2\sum_{i=1}^n x_i}{2n+9}$. Following some algebraic manipulation, we obtain:

$$P^* - c - q_j^* + x_j = 2\left(\frac{6\xi - 3x_0 + (2n+9)x_j - 2\sum_{i=1}^n x_i}{3(2n+9)}\right) = 2q_j^*.$$

The equilibrium profit of each private firm $j \in J^P$, is calculated by equation (12) as:

$$\begin{aligned} F_j^* &= (P^* - c - q_j^* + x_j)q_j^* - x_j^2 + sx_j \\ &= 2(q_j^*)^2 - x_j^2 + sx_j \\ &= 2\left(\frac{6\xi - 3x_0 + (2n+9)x_j - 2\sum_{i=1}^n x_i}{3(2n+9)}\right)^2 - x_j^2 + sx_j. \end{aligned} \tag{15}$$

The public firm's profit is:

$$\begin{aligned} F_0 &= [a - Q]q_0 - [(c - x_0)q_0 + q_0^2] - x_0^2 + sx_0 \\ &= (\xi + x_0 - Q - q_0)q_0 - x_0^2 + sx_0. \end{aligned} \tag{16}$$

By equation (7), the quantity produced by the public firm satisfies the following condition:

$$q_0 = \xi + x_0 - Q - q_0.$$

Substituting this last equation into equation (16) yields:

$$F_0 = q_0^2 - x_0^2 + sx_0.$$

This concludes the proof. ■

In the second stage, firms choose R&D simultaneously given the subsidy s. Unlike the case of quantity-setting stage, here it is analytically convenient to define the aggregate of R&D levels for the private firms only, so that total R&D of all $n+1$ firms in the mixed oligopoly is $X + x_0$. The next result establishes the equilibrium R&D levels.

Proposition 1. *(*EQUILIBRIUM R&D*) For a given R&D subsidy* s*, the equilibrium R&D functions are:*

$$x_0^*(s) = \frac{-3n(n+6)s + (8n+67)\xi}{\Theta}, \tag{17}$$

$$X^*(s) = \frac{3n(6n^2+56n+135)s + 12n(2n+7)\xi}{2\Theta}, \tag{18}$$

$$x_P^*(s) = \frac{X^*(s)}{n} = \frac{3(6n^2+56n+135)s + 12(2n+7)\xi}{2\Theta}, \tag{19}$$

where $\Theta = 18n^2 + 148n + 335 > 0$.

Proof. In order to compute the second-stage equilibrium R&D levels, we first look at the optimisation problem of private firms. We thus establish the relevant replacement equation (see (22) below). Adding these equations, we find the aggregate R&D level for the private firms, X, in terms of x_0. This is summarised in equation (23) below. Then, we look at the behaviour of the public firm, x_0, which is a function of X. We obtain equation (24) below. We thus obtain a system of equations, (23) and (24), in the variables x_0 and X. By solving this linear system, we find the equilibrium values of x_0 and X. Substituting these equilibrium values in the replacement equation (22), we find the equilibrium amounts of R&D of private firms.

In detail, we proceed as follows: From Lemma 2, profits are written as in (15). Let $\lambda = (2n+7)/(2n+9)$ and $\eta_j = \left[6\xi - 3x_0 - 2\sum_{i\neq j} x_i\right]/(2n+9)$, for every $j \in J^P$, where the symbol $\sum_{i\neq j}$ represents the sum over all $i \in \{1, 2, \cdots, n\} - \{j\}$. Rearranging the profit equation yields:

$$F_j^* = \frac{2}{9}(\lambda x_j + \eta_j)^2 - x_j^2 + s x_j,$$

The auxiliary variable η_j does not depend on x_j; it depends only on R&D levels of others, x_i, for $i \neq j$. The first and second derivatives of the profit of private firm j with respect to the amount of R&D are: $\frac{dF_j^*}{dx_j} = \frac{4\lambda}{9}(\lambda x_j + \eta_j) - 2x_j + s = \left(\frac{4\lambda^2}{9} - 2\right)x_j + \frac{4\lambda}{9}\eta_j + s$, and $\frac{d^2F_j^*}{dx_j^2} = \frac{4\lambda^2}{9} - 2 < 0$, where the last inequality comes from noting that $0 < \lambda < 1$, so that $0 < 4\lambda^2/9 < 1$. The second-order condition holds. Then the unique maximum is the solution of the first-order condition $dF_j^*/dx_j = 0$:

$$x_j = \frac{1}{18 - 4\lambda^2}\left[4\lambda\eta_j + 9s\right].$$

Substituting back the auxiliary variables λ and η_j yields:

$$x_j = \frac{24(2n+7)\xi - (24n+84)x_0 - (16n+56)\sum_{i\neq j} x_i + 9s(2n+9)^2}{56n^2 + 536n + 1262}. \tag{20}$$

Let $X = \sum_{i=1}^n x_i$, hence $\sum_{i\neq j} x_i = X - x_j$. Therefore, from (20):

$$(56n^2 + 536n + 1262)x_j = -(24n+84)x_0 - (16n+56)X + (16n+56)x_j + 24(2n+7)\xi + 9s(2n+9)^2, \tag{21}$$

and we can decompose $56n^2 + 536n + 1262 = (56n^2 + 520n + 1206) + (16n + 56)$. Define the auxiliary variable $B = [24(2n+7)\xi + 9s(2n+9)^2]/2$. We can then rewrite (21) to obtain the *replacement* equations below:

$$(28n^2 + 260n + 603)x_j = -(12n+42)x_0 - (8n+28)X + B. \tag{22}$$

Adding the replacement equations (22), for all private firms $j \in \{1, 2, \cdots, n\}$ leads to:

$$(36n^2 + 288n + 603)X + (12n^2 + 42n)x_0 = nB. \tag{23}$$

Changes in any x_j, for $j \in J$, may generate a change in the quantities chosen in the second stage. From Lemma 1 we have $\frac{dQ^*}{dx_0} = \frac{3}{2n+9}$, $\frac{dq_j^*}{dx_0} = \frac{-1}{2n+9}$ and $\frac{dq_0^*}{dx_0} = \frac{n+3}{2n+9}$. Recall that $X = \sum_{j=1}^n x_j$ and $\gamma = \sum_{j=1}^n x_j q_j - \sum_{j=1}^n q_j^2 - \sum_{j=1}^n x_j^2$. Notice that neither $\sum_{j=1}^n x_j^2$ nor $dq_j^*/dx_0 = -1/(2n+9)$ depend on j. Thus

$$\frac{d\gamma}{dx_0} = \sum_{j=1}^n x_j \frac{dq_j^*}{dx_0} - \sum_{j=1}^n 2q_j^* \frac{dq_j^*}{dx_0} = \frac{dq_j^*}{dx_0}\left[\sum_{j=1}^n x_j - 2\sum_{j=1}^n q_j^*\right] = \frac{-1}{2n+9}[X - 2(Q^* - q_0^*)] = \frac{2Q^* - 2q_0^* - X}{2n+9}.$$

Social welfare satisfies the previously derived equation (11). Taking the derivative with respect to x_0 of both sides of (11) and substituting the expressions for dQ^*/dx_0, dq_0^*/dx_0 and $d\gamma/dx_0$ after some manipulation leads to

$$\frac{dW}{dx_0} = \frac{q_0 - (3n+15)x_0 - Q^* + 3\xi - X}{2n+9},$$

and the associated first-order condition is simply $q_0^* - (3n+15)x_0 + 3\xi - X = Q^*$. Substituting (3) and (4) into this first-order condition and simplifying, results in

$$(6n^2 + 56n + 135)x_0 + (2n + 12)X = (4n + 27)\xi. \tag{24}$$

We then solve the system of replacement equations (23) and (24) in x_0 and X:

$$\begin{cases} (12n^2 + 42n)x_0 + (36n^2 + 288n + 603)X = nB \\ (6n^2 + 56n + 135)x_0 + (2n + 12)X = (4n + 27)\xi. \end{cases}$$

Tedious but straightforward calculations lead to the solution as given by equations (17) and (18). Substituting the values of x_0 and X into the replacement equation (22), yields the R&D output of private firms, as given by equation (19). This completes the proof. ■

Some consequences of this result are as follows: As can be seen from (17) and (19), the subsidy exerts a positive effect on the R&D level of a private firm, whereas the reverse holds for the public firm. The total amount of R&D generated by the public and private firms together is:

$$x_0^* + X^* = \frac{4.5sn(2n^2 + 18n + 41) + (12n^2 + 50n + 67)\xi}{18n^2 + 148n + 335}.$$

This implies that similarly to an output subsidy (see White (1996); Poyago-Theotoky (2001)), a subsidy to R&D has a cost redistribution effect.

Using the results from Proposition 1 we then obtain the associated quantities, price, aggregate output and profits for the private firms.

Proposition 2. *(*Equilibrium Quantities*) For a given R&D subsidy s, the second-stage equilibrium* *quantities are given by:*

$$q_0^*(s) = \frac{2(25n + 134)\xi - 3n(4n + 19)s}{2\Theta}, \tag{25}$$

$$Q^*(s) = \frac{2(9n^2 + 53n + 67)\xi + 3n(3n + 13)s}{\Theta}, \tag{26}$$

$$q_P^*(s) = \frac{(2n + 9)(18\xi + 15s)}{2\Theta}, \tag{27}$$

where $\Theta = 18n^2 + 148n + 335 > 0$.

Proof. Substituting (17) and (18) into (3) results in $q_0^*(s) = \frac{3\xi+(n+3)x_0(s)^*-X^*(s)}{2n+9}$ which, after some algebraic manipulation, yields (25). Next, substituting (17) and (18) into (4) yields $Q^*(s) = \frac{(2n+3)\xi+3x_0(s)^*+2X^*(s)}{2n+9}$ and simplifying results in (26). Subtracting q_0^* from Q^* results in the aggregate quantity produced by all private firms:

$$Q^*(s) - q_0^*(s) = \frac{(18n^2 + 81n)\xi + 3sn(5n + 22.5)}{\Theta} \tag{28}$$

where $\Theta = 18n^2 + 148n + 335$. In the symmetric equilibrium, each private firm produces $q_P^* = (Q^* - q_0^*)/n$; using (28) results in (27). ■

Profit of each private firm denoted F_P^* is

$$F_P^*(s) = 2(q_P^*(s))^2 - (x_P^*(s))^2 + sx_P^*(s) \tag{29}$$

and profit for the public firm is,

$$F_0^*(s) = (q_0^*(s))^2 - (x_0^*(s))^2 + sx_0^*(s). \tag{30}$$

Using (2), we rewrite welfare as follows

$$\begin{aligned} W =& \frac{(Q^*)^2}{2} + \sum_{j=0}^{n} [(a - c - Q^*)q_j + x_j q_j - q_j^2 - x_j^2] \\ =& \frac{-(Q^*)^2}{2} + \xi Q^* + \sum_{j=0}^{n} x_j q_j - \sum_{j=0}^{n} q_j^2 - \sum_{j=0}^{n} x_j^2. \end{aligned} \tag{31}$$

which, following some further manipulation, becomes

$$W = \frac{-(Q^*)^2}{2} + \xi Q^* + [x_0^* q_0^* - (x_0^*)^2 - (q_0^*)^2] + n[x_P^* q_P^* - (x_P^*)^2 - (q_P^*)^2] \tag{32}$$

where, in the interest of clarity, we have suppressed the dependence of the various variables on s.

We then modify the game by adding an initial stage, stage 0, where the government/regulator sets the R&D subsidy to maximise social welfare, as given by (32) above. Before deciding on R&D and then quantities, firms observe the subsidy choice of the regulator, who anticipates how firms will react to the subsidy choice. Using the results from Propositions 1 and 2 and performing the maximisation of

W with respect to s, we obtain the optimal R&D subsidy in the mixed oligopoly, denoted by s^*, which we shall later compare to the optimal subsidy obtained in the private oligopoly, denoted by s^{**} .

Proposition 3. *(*Optimal R&D Subsidy and SPNE Equilibrium*) In the mixed oligopoly, the socially optimal R&D subsidy is:*

$$s^* = \frac{2(4-n)\xi}{3(2n+7)(n+5)}. \tag{33}$$

Under the socially optimal subsidy, s^*, the equilibrium quantities for the public and private firms are

$$q_0^*|_{s=s^*} = \frac{(3n+14)\xi}{(2n+7)(n+5)},\ q_P^*|_{s=s^*} = \frac{(2n+9)\xi}{(2n+7)(n+5)},$$

the aggregate quantity is

$$Q^*|_{s=s^*} = \frac{2(n^2+6n+7)\xi}{(2n+7)(n+5)},$$

R&D levels (cost-reduction) are

$$x_0^* = \frac{(n+7)\xi}{(2n+7)(n+5)},\ x_P^* = \frac{(n+6)\xi}{(2n+7)(n+5)}$$

profits for the public and private firms are

$$F_0^*|_{s=s^*} = \frac{(22n^2+204n+497)\xi^2}{3(2n+7)^2(n+5)^2},\ F_P^*|_{s=s^*} = \frac{19n^2+176n+426}{3(2n+7)^2(n+5)^2}\xi^2$$

and welfare is given by

$$W^*|_{s=s^*} = \frac{(6n^4+93n^3+502n^2+1086n+791)\xi^2}{3(2n+7)^2(n+5)^2}.$$

Proof. Welfare is given by expression (32). Taking the derivative with respect to the subsidy, s, yields

$$\frac{dW}{ds} = -Q^*\frac{dQ^*}{ds} + \xi\frac{dQ^*}{ds} + \frac{d[x_0^*q_0^* - (x_0^*)^2 - (q_0^*)^2]}{ds} + n\frac{d[x_P^*q_P^* - (x_P^*)^2 - (q_P^*)^2]}{ds}.$$

and the associated first-order condition, $dW/ds = 0$, is

$$\xi\frac{dQ^*}{ds} + \frac{d[x_0^*q_0^* - (x_0^*)^2 - (q_0^*)^2]}{ds} = -n\frac{d[x_P^*q_P^* - (x_P^*)^2 - (q_P^*)^2]}{ds} + Q^*\frac{dQ^*}{ds}. \tag{34}$$

Next we calculate the various terms contained in the above first-order condition. Recall that q_0^* is given by (25), while x_0^* comes from (17). To find $\frac{d}{ds}[x_0^*q_0^* - (x_0^*)^2 - (q_0^*)^2]$ we compute

$$[x_0^*q_0^* - (x_0^*)^2 - (q_0^*)^2]\Theta^2 = [(8n+67)\xi - 3sn(n+6)][(17n+67)\xi - 1.5sn(2n+7)] - [(25n+134)\xi - 1.5sn(4n+19)]^2.$$

Then we take the derivative with respect to s to find

$$\frac{d[x_0^* q_0^* - (x_0^*)^2 - (q_0^*)^2]}{ds} = \frac{1}{\Theta^2}\Big[- 1.5n(2n+7)[(8n+67)\xi - 3sn(n+6)]$$
$$- 3n(n+6)[(17n+67)\xi - 1.5sn(2n+7)] + 3n(4n+19)[(25n+134)\xi - 1.5sn(4n+19)]\Big].$$

After some further algebraic manipulation we obtain

$$\frac{d[x_0^* q_0^* - (x_0^*)^2 - (q_0^*)^2]}{ds} = \frac{9n\left[(50n^2+498n+1273)\xi - sn(12n^2+114n+277)\right]}{2\Theta^2}.$$

Using (25), (28) and (27),

$$\frac{dQ^*}{ds} = \frac{3n(3n+13)}{\Theta}, \frac{d(Q^* - q_0^*)}{ds} = \frac{3n(5n+22.5)}{\Theta}, \frac{dq_P^*}{ds} = \frac{7.5(2n+9)}{\Theta}.$$

By equations (19) and (27), we obtain

$$x_P^* q_P^* = \frac{[1.5s(6n^2+56n+135) + \xi(12n+42)](2n+9)(9\xi + 7.5s)}{\Theta^2}$$

and:

$$(x_P^*)^2 + (q_P^*)^2 = \frac{[1.5s(6n^2+56n+135) + \xi(12n+42)]^2}{\Theta^2} + \frac{(2n+9)^2(9\xi+7.5s)^2}{\Theta^2}.$$

Then,

$$[x_P^* q_P^* - (x_P^*)^2 - (q_P^*)^2]\Theta^2 = 3[s(3n^2+23n+45) - \xi(2n+13)](2n+9)(9\xi+7.5s) - [1.5s(6n^2+56n+135) + \xi(12n+42)]^2.$$

Differentiating with respect to s yields:

$$\frac{d[x_P^* q_P^* - (x_P^*)^2 - (q_P^*)^2]}{ds} = \frac{-27[s(12n^4+204n^3+1342n^2+4050n+4725) + \xi(4n^3+66n^2+362n+645)]}{2\Theta^2}.$$

Substituting the equilibrium variables into the first-order condition (the second-order condition is satisfied) (34), it becomes

$$\frac{3n(3n+13)\xi}{\Theta} + \frac{9n\left[(50n^2+498n+1273)\xi - sn(12n^2+114n+277)\right]}{2\Theta^2}$$
$$= \frac{27n[s(12n^4+204n^3+1342n^2+4050n+4725) + \xi(4n^3+66n^2+362n+645)]}{2\Theta^2}$$
$$+ \frac{3n(3n+13)\left[2(9n^2+53n+67)\xi + 3sn(3n+13)\right]}{\Theta^2}.$$

Multiplying both sides of this last equation by $2\Theta^2/(3n)$ and simplifying yields

$$2(n-4)\xi = 3s(2n+7)(n+5)$$

the solution of which is the socially optimal subsidy, s^* given by (33). Substituting (33) into (25), (27) and (26), after some manipulation, we find the the SPNE quantities. Similarly, using (17), (19) and (33) we obtain equilibrium R&D, and then substituting the SPNE quantities and R&D levels into

(29) and (30), we find the SPNE profits. From (9), $W = (a - P^*)Q^* - \frac{(Q^*)^2}{2} + \sum_{j=0}^{n}[F_j - S(x_j)] = \frac{(Q^*)^2}{2} + F_0 + \sum_{j=1}^{n} F_j - sx_0 - s\sum_{j=1}^{n} x_j$ which, at $s = s^*$

$$W^*|_{s=s^*} = \frac{(Q^*|_{s=s^*})^2}{2} + F_0^*|_{s=s^*} + nF_P^*|_{s=s^*} - ns^*x_P^*|_{s=s^*}$$

and evaluated using (33) and the other SPNE values calculated above becomes

$$\begin{aligned} W^*|_{s=s^*} &= \frac{6(n^2+6n+7)^2 + (22n^2+204n+497) + n(19n^2+176n+426) - n(8-2n)(n+6)}{3(2n+7)^2(n+5)^2}\xi^2 \\ &= \frac{(6n^4+93n^3+502n^2+1086n+791)\xi^2}{3(2n+7)^2(n+5)^2}. \end{aligned}$$

This completes the proof. ■

These SPNE solutions are summarised in Table 1. Clearly, the public firm produces more output, generates more R&D and obtains a higher profit than a private firm ($q_0^* > q_P^*, x_0^* > x_P^*$ and $F_0^* > F_P^*$).

These results extend and confirm the results obtained by Gil-Moltó et al. (2011) for the case of a duopoly. Moreover, apart from providing fully analytical results for the $n+1$ firms oligopoly, we uncover new insights regarding the optimal R&D subsidy. Notice that s^* is positive for $1 \leq n < 4$, zero for $n = 4$, but negative for $n > 4$. The implication of this result is that R&D should be subsidised if there are less than four firms in the market (one public firm and three private firms) and taxed otherwise. This result is reminiscent of Leahy and Neary (1997), who provided a similar result in the context of a private oligopoly: they concluded that R&D should be taxed in the absence of spillovers (see also Lee and Tomaru (2017)).

Actual policy and practice tend to provide tax concessions (in the form of R&D tax credits) or subsidies rather than taxes to R&D. However, we can explain this somewhat surprising result by considering the market failures in operation: In the absence of any policy, there is underproduction in aggregate relative to the social optimum (market failure due to market power), inefficiency in the allocation of production costs across firms as well as inefficiencies related to the amount of R&D performed by firms. In the second-best world we are describing, the regulator has only one instrument, the R&D subsidy, to address these market failures. Therefore, when there are relatively few firms in the market ($n < 4$), the R&D subsidy operates primarily to redress underproduction. In contrast, when there are relatively many firms in the market ($n > 4$), there is too much R&D in aggregate, hence the need for a negative R&D subsidy (R&D tax). The result in Proposition 3 complements and extends the various results obtained in the context of a duopoly where typically the R&D subsidy is positive; in other words, by considering an oligopoly we find that an R&D subsidy makes sense only in relatively concentrated markets.

A particular case occurs when there are no R&D subsidies. Setting $s = 0$ in the results obtained above we then have the following:

$q_0^* = \frac{(3n+14)\xi}{(2n+7)(n+5)}$	$q_P^* = \frac{(2n+9)\xi}{(2n+7)(n+5)}$
$x_0^* = \frac{(n+7)\xi}{(2n+7)(n+5)}$	$x_P^* = \frac{(n+6)\xi}{(2n+7)(n+5)}$
$F_0^* = \frac{(22n^2+204n+497)\xi^2}{3(2n+7)^2(n+5)^2}$	$F_P^* = \frac{(19n^2+176n+426)}{3(2n+7)^2(n+5)^2}\xi^2$
$Q^* = \frac{2(n^2+6n+7)\xi}{(2n+7)(n+5)}$	$W^* = \frac{(6n^4+93n^3+502n^2+1086n+791)\xi^2}{3(2n+7)^2(n+5)^2}$

Table 1: SPNE Solutions in the Mixed Oligopoly

Corollary 1. *In the absence of subsidies to R&D, from Proposition 1 we obtain:* $x_0^*(0) = \frac{(8n+67)\xi}{\Theta}$, $X^*(0) = \frac{6n(2n+7)\xi}{\Theta}$, $x_P^*(0) = \frac{X^*(0)}{n} = \frac{6(2n+7)\xi}{\Theta}$, *where* $\Theta = 18n^2 + 148n + 335 > 0$. *Both* $x_P^*(0)$ *and* $x_0^*(0)$ *are decreasing with the number of private firms,* n. *Moreover,* $x_P^*(0) > x_0^*(0)$ *if and only if* $n > 6.25$.

Corollary 2. *In the absence of R&D subsidies, from Proposition 2, we obtain that the public firm produces more than a private firm, regardless of* n. *For every* $n \geq 1$*:*

$$q_0^*(0) = \frac{(25n+134)\xi}{18n^2+148n+335} > \frac{(18n+81)\xi}{18n^2+148n+335} = q_P^*(0).$$

Corollary 3 (Social Welfare under Zero Subsidy). *In the absence of R&D subsidies, in the unique subgame perfect equilibrium of the mixed oligopoly game, social welfare is:*

$$W^*(0) = \frac{(162n^4 + 2412n^3 + 13\,415n^2 + 31\,190n + 22\,445)\xi^2}{(18n^2+148n+335)^2}.$$

Notice that in the special case of a mixed duopoly, $n = 1$, Corollaries 1 and 2 imply that $x_P^*(0)|_{n=1} = 18\xi/167$, $x_0^*(0) = 25\xi/167$, and $q_0^*(0)|_{n=1} = \dfrac{53\xi}{167} > \dfrac{33\xi}{167} = q_P^*(0)|_{n=1}$ respectively, which is the exact result obtained by Gil-Moltó et al. (2011) when spillovers are absent, and similarly for welfare and profits.

5. Private Oligopoly

The industry now consists of $(n+1)$ profit-maximising (private) firms, indexed by j. Let $J = \{0, 1, 2, \cdots, n\}$ be the set of all firms. All other aspects remain the same as in the previous section. In the last stage firms compete by setting quantities. The following proposition summarises.

Lemma 3. *(*Equilibrium of Last Stage Subgame*) Given R&D levels, the last-stage equilibrium quantities in the private oligopoly are:*

$$q_j^{**} = \frac{3\xi + (n+4)x_j - \sum_{i=0}^n x_i}{3(n+4)}, \tag{35}$$

$$Q^{**} = \frac{(n+1)\xi}{n+4} + \frac{\sum_{i=0}^n x_i}{n+4}. \tag{36}$$

Profit per firm is:

$$F_j^{**} = \frac{2\left[3\xi + (n+4)x_j - \sum_{i=0}^n x_i\right]^2 - 9(n+4)^2 x_j^2 + 9(n+4)^2 s x_j}{9(n+4)^2}. \tag{37}$$

Proof. Each firm $j \in J$ solves its profit maximisation problem $q_j^* = \arg\max_{q_j \geq 0} F_j$. For every profile q_{-j} of quantities from its opponents, firm j chooses its quantity $q_j \geq 0$ to maximise its individual profit. Let $Q_{-j} = Q - q_j$. Firm j's profit, F_j is written as

$$F_j = (\xi + x_j - Q_{-j} - q_j)q_j - q_j^2 - x_j^2 + s x_j \tag{38}$$

and marginal profit is $\frac{dF_j}{dq_j} = \xi + x_j - Q_{-j} - 4q_j$. Firm j produces a positive quantity if and only if $\xi + x_j - Q_{-j} > 0$, or equivalently, $Q_{-j} < \xi + x_j$. The first-order condition of the profit maximisation problem of each private firm $j \in J$, gives the associated *replacement* function

$$3q_j = \xi + x_j - Q. \tag{39}$$

Adding the equations $Q + 3q_i = \xi + x_i$, for all $i \in J$ leads to $(n+4)Q = (n+1)\xi + \sum_{i=0}^{n} x_i$. The solution to this is the equilibrium aggregate quantity, Q^{**}, in terms of $x = (x_0, x_1, \cdots, x_n)$, and given by equation (36) in the main text. Substituting Q^{**} into the replacement equation (39) results in

$$q_j^{**} = \frac{3\xi + x_j(n+4)}{3(n+4)} - \frac{\sum_{i=0}^n x_i}{3(n+4)}.$$

Assuming all private firms produce positive quantities and $Q^{**} \leq a$, then, by (36), the equilibrium price is $P^{**} = \frac{3a + (n+1)c - \sum_{i=0}^n x_i}{n+4}$. The equilibrium profit is

$$F_j^{**} = (\xi + x_j - Q^{**})q_j^{**} - (q_j^{**})^2 - x_j^2 + s x_j = (P^{**} - c + x_j)q_j^{**} - (q_j^{**})^2 - x_j^2 + s x_j.$$

By (36) and (35)

$$P^{**} - c + x_j = \frac{3a + (n+1)c - \sum_{i=0}^n x_i}{n+4} - c + x_j = 3q_j^{**}$$

and equilibrium profit of each firm $j \in J$, is

$$F_j^{**} = 2(q_j^{**})^2 - x_j^2 + s x_j. \tag{40}$$

Substituting equation (35) into this last equation leads to

$$F_j^{**} = \frac{2\left[3\xi + (n+4)x_j - \sum_{i=0}^{n} x_i\right]^2 - 9(n+4)^2 x_j^2 + 9(n+4)^2 s x_j}{9(n+4)^2}.$$

This completes the proof. ■

In the second stage, firms choose R&D levels. The next result establishes equilibrium R&D and quantities.

Proposition 4. *(*Equilibrium - R&D Stage*) Equilibrium solutions in the R&D stage are given by:*

$$x_P^{**}(s) = \frac{2(n+3)\xi + 1.5(n+4)^2 s}{\Delta} \tag{41}$$

$$q_P^{**}(s) = \frac{3(n+4)\xi + 1.5(n+4)s}{\Delta} \tag{42}$$

$$Q^{**}(s) = \frac{1.5(n+1)(n+4)(2\xi + s)}{\Delta} \tag{43}$$

where $\Delta = 3n^2 + 22n + 42 > 0$.

Proof. Let $\widehat{X} = \sum_{i=0}^{n} x_i$ be the total expenditure on R&D. From (40) and expanding leads to

$$9(n+4)^2 F_j^{**} = 2\widehat{X}^2 - 12\xi\widehat{X} + 18\xi^2 - 7(n+4)^2 x_j^2 + 12\xi(n+4)x_j - 4(n+4)\widehat{X}x_j + 9(n+4)^2 s x_j.$$

Because $\widehat{X} = \sum_{i=0}^{n} x_i$, then $\frac{d\widehat{X}}{dx_j} = 1$. Then

$$9(n+4)^2 \frac{dF_j^{**}}{dx_j} = -4(n+3)\widehat{X} - 2(n+4)(7n+30)x_j + 12\xi(n+3) + 9(n+4)^2 s.$$

The first-order condition $dF_j^{*}/dx_j = 0$ generates the equilibrium (the second-order condition is satisfied) solution, denoted x_j^{**}:

$$4(n+3)\widehat{X} + 2(n+4)(7n+30)x_j = 12\xi(n+3) + 9s(n+4)^2. \tag{44}$$

Adding the above replacement equation for every $j \in J$ yields

$$\left[2(n+3)(n+1) + (n+4)(7n+30)\right]\widehat{X} = 6\xi(n+1)(n+3) + 4.5s(n+1)(n+4)^2,$$

and factorising the left-hand side results in

$$3(3n^2 + 22n + 42)\widehat{X} = 6\xi(n+1)(n+3) + 4.5s(n+1)(n+4)^2.$$

Solving for $\widehat{X}$:

$$\widehat{X} = \left[\frac{2(n+3)\xi + 1.5(n+4)^2 s}{3n^2 + 22n + 42}\right](n+1). \tag{45}$$

Plugging this back into the replacement equation (44) yields

$$\frac{24\xi(n+1)(n+3)^2 + 18s(n+1)(n+3)(n+4)^2}{3(3n^2+22n+42)} + 2(n+4)(7n+30)x_j = 12\xi(n+3) + 9s(n+4)^2$$

and solving for x_j, after some manipulation, results in

$$x_j^{**} = \frac{2(n+3)\xi + 1.5(n+4)^2 s}{\Delta},$$

where $\Delta = 3n^2 + 22n + 42 > 0$. This is the common amount of R&D for all (private) firms, denoted $x_P^{**}(s)$. Plugging (45) and (41) back into equation (35) results in

$$q_P^{**}(s) = \frac{3(n+4)\xi + 1.5(n+4)s}{\Delta}.$$

and

$$Q^{**}(s) = \frac{1.5(n+1)(n+4)(2\xi + s)}{\Delta}.$$

This completes the proof. ■

Equilibrium profit per firm, F_P^{**}, is given by: $F_P^{**}(s) = 2(q_P^{**}(s))^2 - (x_P^{**}(s))^2 + s x_P^{**}(s)$. Then, using (2), we express welfare as:

$$W = \frac{-Q^2}{2} + \xi Q + \sum_{j=0}^{n} x_j q_j - \sum_{j=0}^{n} q_j^2 - \sum_{j=0}^{n} x_j^2. \tag{46}$$

or,

$$W^{**} = \frac{-(Q^{**})^2}{2} + \xi Q^{**} + (n+1)[x_P^{**} q_P^{**} - (x_P^{**})^2 - (q_P^{**})^2]. \tag{47}$$

Alternatively, social welfare can be expressed as

$$W = (a - P^*)Q^* - \frac{(Q^*)^2}{2} + \sum_{j=0}^{n}[F_j - S(x_j)] = \frac{(Q^*)^2}{2} + \sum_{j=0}^{n} F_j - s\sum_{j=0}^{n} x_j,$$

which in the private oligopoly becomes:

$$W^{**} = \frac{(Q^{**})^2}{2} + (n+1)F_P^{**} - s(n+1)x_P^{**}.$$

Next, we move to the first stage and the determination of the optimal R&D subsidy by the regulator.

Proposition 5. *(*Optimal R&D Subsidy in the Private Oligopoly*) In the private oligopoly, the optimal R&D subsidy is given by*

$$s^{**} = \frac{2\xi(3-n)}{3(n+3)(2n+9)}. \tag{48}$$

Under the socially optimal subsidy, s^{**}, the equilibrium quantity for the private firms is

$$q_P^{**}|_{s=s^{**}} = \frac{2(n+4)\xi}{(n+3)(2n+9)}$$

and the SPNE aggregate quantity is

$$(n+1)q_P^{**}|_{s=s^{**}} = \frac{2(n+1)(n+4)\xi}{(n+3)(2n+9)}.$$

R&D levels (cost-reduction) are

$$x_P^{**}|_{s=s^{**}} = \frac{(n+5)\xi}{(n+3)(2n+9)},$$

profits per firm are

$$F_P^{**}|_{s=s^{**}} = \frac{(19n^2+158n+339)\xi^2}{3(n+3)^2(2n+9)^2}$$

and welfare is given by

$$W^{**}|_{s=s^{**}} = \frac{3(n+1)(n+5)}{(n+3)(2n+9)}\xi^2.$$

Proof. From the welfare expression (47)

$$\frac{dW^{**}}{ds} = (\xi - Q^{**})\frac{dQ^{**}}{ds} + (n+1)\frac{d}{ds}\left(x_P^{**}q_P^{**} - (x_P^{**})^2 - (q_P^{**})^2\right),$$

therefore the associated first-order condition is

$$\frac{dW^{**}}{ds} = 0 \quad \Leftrightarrow \quad (\xi - Q^{**})\frac{dQ^{**}}{ds} = -(n+1)\frac{d}{ds}\left(x_P^{**}q_P^{**} - (x_P^{**})^2 - (q_P^{**})^2\right).$$

Then

$$\frac{d}{ds}\left(x_P^{**}q_P^{**} - (x_P^{**})^2 - (q_P^{**})^2\right) = (x_P^{**} - 2q_P^{**})\frac{dq_P^{**}}{ds} + (q_P^{**} - 2x_P^{**})\frac{dx_P^{**}}{ds}, \tag{49}$$

Next, by equations (41) and (42)

$$q_P^{**} - x_P^{**} = \frac{(n+6)\xi - 1.5s(n+4)(n+3)}{\Delta},$$

$$x_P^{**} - 2q_P^{**} = \frac{-2\xi(2n+9) + 1.5s(n+4)(n+2)}{\Delta},$$

$$q_P^{**} - 2x_P^{**} = \frac{-1.5(n+4)(2n+7)s - n\xi}{\Delta},$$

$$\frac{dq_P^{**}}{ds} = \frac{1.5(n+4)}{\Delta},$$

and

$$\frac{dx_P^{**}}{ds} = \frac{1.5(n+4)^2}{\Delta}.$$

Substituting these expressions into (49) results in

$$\frac{d}{ds}\left(x_P^{**}q_P^{**} - (x_P^{**})^2 - (q_P^{**})^2\right) = \frac{-1.5(n+4)[\xi(n^2+8n+18)+3s(n+4)(n^2+7n+13)]}{\Delta^2}. \tag{50}$$

From expression (43)

$$\frac{dQ^{**}}{ds} = \frac{1.5(n+1)(n+4)}{\Delta}. \tag{51}$$

Substituting (43), (50) and (51) into (5) and simplifying leads to

$$\xi(3-n) = 1.5s(n+3)(2n+9),$$

the solution of which is

$$s^{**} = \frac{2\xi(3-n)}{3(n+3)(2n+9)}.$$

Using this in (42), (41) as well as the profit and welfare expressions, after some manipulation, yields the SPNE equilibrium values. This completes the proof. ■

The socially optimal R&D subsidy, s^{**}, is positive when $n < 3$, zero when $n = 3$, and negative (R&D is taxed) when $n > 3$. The intuition for this result is similar to the one provided for the mixed oligopoly, the main difference being that here there is no inefficiency associated with the allocation of production costs across firms. Next, by using the optimal subsidy (48) we obtain the SPNE solutions for the private oligopoly, summarised in Table 2.

Finally, in line with the previous section, the following result relates to the case of no subsidy, $s = 0$.

Corollary 4 (Equilibrium of Private Oligopoly with Zero Subsidy). *Suppose that there is no subsidy on R&D activities, $s = 0$. Then, the equilibrium is:*

$$\begin{aligned} x_P(0) &= \frac{2(n+3)\xi}{3n^2+22n+42} \\ q_P(0) &= \frac{3(n+4)\xi}{3n^2+22n+42} \\ Q(0) &= \frac{3(n+1)(n+4)\xi}{3n^2+22n+42} \end{aligned}$$

and social welfare is:

$$W(0) = \frac{(n+1)(9n^3+109n^2+456n+648)\xi^2}{2(3n^2+22n+42)^2}.$$

$q_P^{**} = \frac{2(n+4)\xi}{(n+3)(2n+9)}$
$x_P^{**} = \frac{(n+5)\xi}{(n+3)(2n+9)}$
$F_P^{**} = \frac{(19n^2+158n+339)\xi^2}{3(n+3)^2(2n+9)^2}$
$Q^{**} = \frac{2(n+1)(n+4)\xi}{(n+3)(2n+9)}$
$W^{**} = \frac{(n+1)(n+5)\xi^2}{(n+3)(2n+9)}.$

Table 2: SPNE solutions in the private oligopoly

6. First-Best (Socially managed industry)

In this section, we briefly consider the first-best where a social planner is managing the whole industry so that she instructs all $n+1$ firms to choose R&D and output simultaneously, $\widehat{x}$ and $\widehat{q}$ respectively, to maximise welfare (cf. Dasgupta and Stiglitz (1980)). From equation (2):

$$\begin{aligned} W &= \frac{-(Q)^2}{2} + \xi Q + \sum_{j=0}^{n} x_j q_j - \sum_{j=0}^{n} q_j^2 - \sum_{j=0}^{n} x_j^2 \\ &= \frac{-(n+1)^2}{2}\widehat{q}^2 + \xi(n+1)\widehat{q} + (n+1)\widehat{x}\widehat{q} - (n+1)\widehat{q}^2 - (n+1)\widehat{x}^2 \\ &= (n+1)\left[\widehat{x}\widehat{q} - \frac{(n+3)}{2}\widehat{q}^2 - \widehat{x}^2 + \xi\widehat{q}\right]. \end{aligned}$$

In order to maximize W, it suffices to choose $\widehat{q}$ and $\widehat{x}$ that maximize

$$\frac{W}{n+1} = \widehat{x}\widehat{q} - \frac{(n+3)}{2}\widehat{q}^2 - \widehat{x}^2 + \xi\widehat{q}.$$

The relevant first-order conditions yield, $\frac{\partial}{\partial \widehat{q}}\left[\frac{W}{n+1}\right] = \widehat{x} - (n+3)\widehat{q} + \xi$ and $\frac{\partial}{\partial \widehat{q}}\left[\frac{W}{n+1}\right] = \widehat{x} - (n+3)\widehat{q} + \xi$, and it is clear from these that the second-order conditions for a maximum are satisfied, $\det H_1 < 0$ and $\det H_2 > 0$, where H is the Hessian matrix:

$$H = \begin{bmatrix} -(n+3) & 1 \\ 1 & -2 \end{bmatrix}.$$

Thus, solving the first-order conditions results in the following unique solution for the first-best:

$$\widehat{q} = \frac{2\xi}{2n+5} > 0, \tag{52}$$

$$\widehat{x} = \frac{\xi}{2n+5} > 0. \tag{53}$$

7. The Main Results - Comparing the Mixed and Private Oligopolies

In this section, we compare the two market configurations and provide some tentative policy remarks with respect to privatising the public firm. We capture privatisation in the simplest possible way: the private oligopoly is equivalent to a setup where the public firm maximises its own profit, i.e. it is acting like any other private firm. Comparing the results obtained in the two previous sections, we can state the following results.

Proposition 6. *(*OPTIMAL R&D SUBSIDIES*) The optimal R&D subsidy in the mixed oligopoly is always greater than in the private oligopoly, $s^* > s^{**}$.*

Proof. Using the optimal subsidies in the mixed and private oligopolies, (33) and (48) respectively,

$$\begin{aligned} s^* &> s^{**} \implies \\ (4-n)(n+3)(2n+9) &> (3-n)(2n+7)(n+5) \implies \\ 4n^2+17n+3 &> 0. \end{aligned}$$

This last inequality holds for all $n \in Z_+$. This completes the proof. ■

The intuition behind Proposition 6 follows: In the case of a private oligopoly, two sources of market failure exist: (i) imperfect competition, which leads to underproduction (and hence, allocative inefficiency), and (ii) the R&D undervaluation effect whereby private firms do not take into account the increases in consumers' welfare as a consequence of the investment in R&D (as consumers' welfare does not enter into firms' objective function). This combination will result in per-firm under-investment in R&D. In the case of a mixed oligopoly a further source of market failure exists: the different nature of firms (public or private) in the market and the associated observation that, as a result, production (assembly) costs are inefficiently distributed, given decreasing returns. Hence, the regulator subsidises more heavily a mixed market. This, in turn, explains why the optimal subsidy may fall with privatisation (see below).

Proposition 7. *(*EFFECT OF *n) The optimal R&D subsidies, s^*, s^{**}, whenever positive, are decreasing in n.*

Proof. Obvious, hence omitted. ■

The next proposition compares total R&D level, quantity, and profits under the mixed and the private oligopolies: we are comparing the two cases (mixed oligopoly versus private oligopoly) when the R&D subsidy is set at the optimal level, i.e. the level that maximizes welfare (in each case).

Proposition 8. *(*AGGREGATE COMPARISONS*) Given the optimal R&D subsidies in the respective market configurations, s^* and s^{**}:*

(i) Total R&D level in the private oligopoly is higher than in the mixed oligopoly if $n \geq 4$: $(n+1)x_P^{**} > x_0^* + nx_P^*$. For $n < 4$ the reverse holds.

(ii) The aggregate output in the mixed oligopoly exceeds total output in the private oligopoly, $Q^* > Q^{**}$.

(iii) Total profits in the private oligopoly exceed total profits in the mixed one: $(n+1)F_P^{**} > F_0^* + nF_P^*$.

Proof.

(i) Using the relevant SPNE values from Tables 1 and 2, $x_0^* + nx_P^* > (n+1)x_P^{**}$ if and only if

$$\begin{aligned}
\frac{(n^2+7n+7)\xi}{(2n+7)(n+5)} &> \frac{(n+1)(n+5)\xi}{(n+3)(2n+9)} \implies \\
(n^2+7n+7)(n+3)(2n+9) &> (n+1)(n+5)(2n+7)(n+5) \implies \\
-n^2 - n + 14 &> 0.
\end{aligned}$$

This last inequality holds for all $n \leq 3$. When $n \geq 4$, $x_0^* + nx_P^* < (n+1)x_P^{**}$.

(ii) Using the relevant SPNE values from Tables 1 and 2, $Q^* > Q^{**}$ if and only if

$$\begin{aligned}
\frac{2(n^2+6n+7)\xi}{(2n+7)(n+5)} &> \frac{2(n+1)(n+4)\xi}{(n+3)(2n+9)} \implies \\
(n^2+6n+7)(n+3)(2n+9) &> (n+1)(n+4)(2n+7)(n+5) \implies \\
3n^2 + 24n + 49 &> 0.
\end{aligned}$$

It is clear that this last inequality holds for all $n \in \{1, 2, \cdots\}$.

(iii) Using the SPNE values for the profits from Tables 1 and 2, after some algebraic manipulation we obtain

$$(F_0^* + nF_P^*) - (n+1)F_P^{**} < -\frac{\xi^2 \Psi}{3(3+n)^2(5+n)^2(7+2n)^2(9+2n)^2} < 0$$

where $\Psi > 0 \ \forall n$ and $\Psi = 68n^6 + 1448n^5 + 12583n^4 + 56329n^3 + 133543n^2 + 150395n + 52962$.

This completes the proof. ■

The following remark may be useful in explaining the above results (the proof is straightforward and omitted).

Remark 1. *(i)* $x_0^* > x_P^{**}$*, (ii)* $x_P^* < x_P^{**}$

(iii) $q_0^* > q_P^{**}$*, (iv)* $q_P^* < q_P^{**}$

(v) $F_0^* > F_P^{**}$ *if and only if* $n \geq 4$*, (vi)* $F_P^* < F_P^{**}$*.*

With regard to proposition 8, it is relevant to note that the public firm will tend to reduce its R&D investment more than a private firm as n increases (leading therefore to higher levels of total R&D level in the private oligopoly than in the mixed one when there are are 'not too few' firms ($n > 4$)). It turns out, however, that the public firm's behaviour will not impact total output quantity in the same way and output will be always higher in the mixed oligopoly than in the private one. Regarding equilibrium profits, the underproduction problem is more serious in the private oligopoly as a result of the lower intensity of competition (in the absence of the pressure exerted by the public firm). This leads to higher oligopoly rents and allocative inefficiency.

Comparing the first-best with the results obtained for the mixed and private oligopolies reveals the following:

Proposition 9. *(*Comparing with First-best*) For every n:*

$$q_P^*|_{s=s^*} < q_P^{**}|_{s=s^{**}} < \widehat{q} < q_0^*|_{s=s^*},$$

$$x_P^*|_{s=s^*} < x_P^{**}|_{s=s^{**}} < \widehat{x} < x_0^*|_{s=s^*}.$$

Proof. Straightforward, hence omitted. ■

In the mixed oligopoly, the public firm receives the optimal subsidy, produces more quantity and more R&D relative to the first-best while the private firms produce less output and R&D. Similarly, in the case of the private oligopoly, the private firms generate less output and less R&D than the first-best. Private firms always produce and have R&D amount below the first-best due to the market failures associated with imperfect competition and the undervaluation effect. As there is only one policy instrument to remedy these, i.e., the subsidy to R&D, both s^* and s^{**} can only go so far. Hence, to compensate for this shortage of quantity and R&D, the public firm produces more output, $q_0^*|_{s=s^*}$, and more R&D, $x_0^*|_{s=s^*}$, well above the first-best, $\widehat{q}$ and $\widehat{x}$.

The next proposition contains a welfare assessment of privatisation in this context and is largely a consequence of Proposition 8.

Proposition 10. *When policy takes the form of an optimal subsidy to R&D, privatisation enhances total welfare, i.e.,* $W^{**} > W^*$*, only for* $n > 4$ *.*

Proof. Using the relevant SPNE values from Tables 1 and 2, $W^{**} > W^{*}$ if and only if:

$$\frac{(n+1)(n+5)\xi^2}{(n+3)(2n+9)} > \frac{(6n^4+93n^3+502n^2+1086n+791)\xi^2}{3(2n+7)^2(n+5)^2}$$

Simplifying yields, $10n^4+99n^3+104n^2-1287n-2982 > 0$. This inequality holds for all $n \geq 4$. Clearly, $W^{**} < W^{*}$ if and only if $n < 4$. This completes the proof. ■

The intuition for the above proposition follows: First, we note that privatisation typically improves productive efficiency. The reason is that in the move from the mixed to the private oligopoly, the inefficiency in the distribution of production costs vanishes. Furthermore, R&D subsidies also aim at reducing productive inefficiency in the mixed market, in a similar way to an output subsidy. A mixed marked has the advantage of a relevant R&D investment carried out by the public firm, which also improves private firms' productive efficiency. However, this higher R&D investment by the public firm reduces the R&D of the private firms. As long as n is relatively small ($n < 4$), the first positive effect on welfare overcomes the negative effects. When the market is relatively large ($n > 4$), a privatised market has the advantage of achieving higher welfare than a mixed marked due to the higher aggregate R&D investment by all private firms, which eliminates the productive inefficiencies characterising the mixed market. The result of Proposition 10 contrasts with White (1996), who finds that providing an output subsidy to firms in the mixed or the private oligopoly does not change welfare (an 'irrelevance result') in the context of a model without R&D investments. However, it complies with the established conclusion by De Fraja and Delbono (1989), obtained in the absence of subsidies and/or R&D activities, according to which privatisation improves welfare in a market with a relatively large number of private firms. It also provides a link to the result of Leahy and Neary (1997) of taxing R&D in private oligopolies with no spillovers, as well as Haraguchi and Matsumura (2018).

8. Concluding Remarks

In this paper, we aim to fill a gap in the literature on mixed oligopoly and privatisation by introducing R&D activity and R&D subsidies. We find that when both markets have the same number of firms, R&D should be more heavily subsidised if the industry is a mixed oligopoly than if it is a purely private oligopoly. Similarly to an output subsidy, a subsidy to R&D can address the inefficient distribution of production costs. However, in contrast to the 'irrelevance results' of privatisation when output subsidies are provided and there is no R&D competition, we find that privatisation with R&D subsidisation is welfare enhancing if and only if the number of firms in the industry is 'sufficiently large'. Further, under the same conditions, privatisation yields an increase in the aggregate R&D levels. Surprisingly, in markets with a small number of private firms, privatisation is likely to result in a loss of social surplus and reductions in the R&D activity.

The present model can be seen as a building block for the analysis of more general cases, and therefore it could support several extensions. We have assumed that the public firm maximises social welfare with consumer and producer surplus carrying the same weight. Indeed, one potential extension of the present work would be to allow for a more general class of objectives for the public firm that may favour business over consumer interests, and vice versa. Within this class of objectives, examining partial privatization as an alternative would be a first step, following the seminal contribution of Matsumura (1998). In this case, it may not be possible to obtain fully analytical results. However, if there are n private firms and one partially-privatised public firm, the resulting actions of the partially public firm would likely be convex combinations of the corresponding actions of the public and private firms in the mixed oligopoly model. Alternatively, one could consider the effect of having more than one public firm, in line with Haraguchi and Matsumura (2016). For example, one could envisage k public firms and $(n + 1 - k)$ private firms. When the number of firms increases, but the proportion of public to private firms remains constant, then all firms produce less. Since the total cost is quadratic, there are gains to welfare in both the mixed oligopoly market and the private oligopoly. Fixing the total number of firms, as more firms are nationalized and become public, the total quantity produced by public firms is more efficiently split between the k public firms. This increases social welfare. Hence, we conjecture that the cutoff number of firms that specifies the most efficient market configuration moves upwards as we compare the private oligopoly with a mixed oligopoly with multiple public firms. For fixed $(n+1)$ (total number of firms), as k grows, it becomes even harder for the private oligopoly to be more efficient than the mixed oligopoly with k public firms. We leave these extensions for future research.

In future work, it would also be worthwhile to consider the situation where the regulator has two policy instruments at its disposal, a subsidy to R&D and an output subsidy, and thus complement the work of Cato and Matsumura (2013) and Lee and Tomaru (2017). We suspect that in this instance, one should be able to recover the first-best and hence restore an ‘irrelevance result’. Another extension could consider the case of free-entry with potentially different fixed entry costs, along the lines of Haraguchi and Matsumura (2018). In future work, it would also be worthwhile to model more explicitly the public firm’s management behaviour in order to analyse further the welfare effects of privatisation.

References

Aanestad, P. D. M., Mørk, B. E., Grisot, M., and Hanseth, O. (2003). Knowledge as a barrier to learning: a case study from medical R&D. In *in 4th European Conference on Organisational Knowledge, Learning and Capabilities. 2003. IESE Business School*. Citeseer.

Aggarwal, V. K. and Evenett, S. J. (2012). Industrial policy choice during the crisis era. *Oxford Review of Economic Policy*, 28(2):261–283.

Anderson, S. P., Erkal, N., and Piccinin, D. (2013). *Aggregate oligopoly games with entry*. Department of Economics, University of Melbourne. Research Paper 1175.

Avi-Yonah, R. S. and Vallespinos, M. (2018). The elephant always forgets: Us tax reform and the wto. *U of Michigan Law & Econ Research Paper*, (18-006).

Boeing, P. (2016). The allocation and effectiveness of china's r&d subsidies-evidence from listed firms. *Research policy*, 45(9):1774–1789.

Boonman, H. J., Hagspiel, V., and Kort, P. M. (2015). Dedicated vs product flexible production technology: Strategic capacity investment choice. *European Journal of Operational Research*, 244(1):141–152.

Cato, S. and Matsumura, T. (2013). Long-run effects of tax policies in a mixed market. *FinanzArchiv: Public Finance Analysis*, 69(2):215–240.

Cin, B. C., Kim, Y. J., and Vonortas, N. S. (2017). The impact of public r&d subsidy on small firm productivity: evidence from korean smes. *Small Business Economics*, 48(2):345–360.

Cornes, R. (2016). Aggregative environmental games. *Environmental and Resource Economics*, 63:339 – 365.

Cornes, R. and Hartley, R. (2000). *Joint production games and share functions*. Department of Economics, University of Nottingham. DP 00/23.

Cornes, R. and Hartley, R. (2003). Risk aversion, heterogeneity and contests. *Public Choice*, 117(1-2):1–25.

Cornes, R. and Hartley, R. (2010). Well-behaved aggregative games. mimeo.

Dasgupta, P. and Stiglitz, J. (1980). Industrial structure and the nature of innovative activity. *The Economic Journal*, 90(358):266–293.

De Fraja, G. and Delbono, F. (1989). Alternative strategies of a public enterprise in oligopoly. *Oxford Economic Papers*, 41(2):302–311.

Delbono, F. and Denicolò, V. (1993). Regulating innovative activity: the role of a public firm. *International Journal of Industrial Organization*, 11(1):35–48.

Duan, L. (2013). Mixed oligopoly model for enterprise privatization without subsidies. *International Journal of Management Science and Engineering Management*, 8(2):79–83.

Einiö, E. (2014). R&d subsidies and company performance: Evidence from geographic variation in government funding based on the erdf population-density rule. *Review of Economics and Statistics*, 96(4):710–728.

Fruchter, G. E. (1999). The many-player advertising game. *Management Science*, 45(11):1609–1611.

Gil-Moltó, M. J., Poyago-Theotoky, J., and Zikos, V. (2011). R&D subsidies, spillovers, and privatization in mixed markets. *Southern Economic Journal*, 78(1):233–255.

Grishagin, V. A., Sergeyev, Y. D., and Silipo, D. (2001). Firms' R&D decisions under incomplete information. *European Journal of Operational Research*, 129(2):414–433.

Haraguchi, J. and Matsumura, T. (2016). Cournot–bertrand comparison in a mixed oligopoly. *Journal of Economics*, 117(2):117–136.

Haraguchi, J. and Matsumura, T. (2018). Multiple Long-Run Equilibria in a Free-Entry Mixed Oligopoly. MPRA Paper 86704, University Library of Munich, Germany.

Haraguchi, J., Matsumura, T., and Yoshida, S. (2018). Competitive pressure from neighboring markets and optimal privatization policy. *Japan and The World Economy*, 46:1–8.

Hart, D. (1998). US technology policy: New tools for new times. *NIRA Review, Summer*, pages 3–6.

Haruna, S. and Goel, R. K. (2015). R&D strategy in international mixed duopoly with research spillovers. *Australian Economic Papers*, 54(2):88–103.

Heywood, J. S. and Ye, G. (2009). Partial privatization in a mixed duopoly with an R&D rivalry. *Bulletin of Economic Research*, 61(2):165–178.

Hud, M. and Hussinger, K. (2015). The impact of r&d subsidies during the crisis. *Research policy*, 44(10):1844–1855.

Ishibashi, I. and Matsumura, T. (2006). R&D competition between public and private sectors. *European Economic Review*, 50(6):1347–1366.

Jensen, M. K. (2010). Aggregative games and best-reply potentials. *Economic Theory*, 43(1):45–66.

Kalashnikov, V., Cordero, A. E., and Kalashnikov, V. (2010). Cournot and Stackelberg equilibrium in mixed duopoly models. *Optimization*, 59(5):689–706.

Kalashnikov, V., Kemfert, C., and Kalashnikov, V. (2009). Conjectural variations equilibrium in a mixed duopoly. *European Journal of Operational Research*, 192(3):717–729.

Kalashnikov, V. V., Bulavsky, V. A., Kalashnykova, N. I., and Castillo, F. J. (2011). Mixed oligopoly with consistent conjectures. *European Journal of Operational Research*, 210(3):729–735.

Katz, J. (2001). Structural reforms and technological behaviour: The sources and nature of technological change in latin america in the 1990s. *Research Policy*, 30(1):1–19.

Kesavayuth, D. and Zikos, V. (2013). R&D versus output subsidies in mixed markets. *Economics Letters*, 118(2):293–296.

Leahy, D. and Neary, J. P. (1997). Public policy towards R&D in oligopolistic industries. *The American Economic Review*, 87(4):642–662.

Lee, S. and Tomaru, Y. (2017). Output and R&D subsidies in a mixed oligopoly. *Operations Research Letters*, 45(3):238–241.

Lin, M. H. and Matsumura, T. (2012). Presence of foreign investors in privatized firms and privatization policy. *Journal of Economics*, 107(1):71–80.

Matsumura, T. (1998). Partial privatization in mixed duopoly. *Journal of Public Economics*, 70(3):473–483.

Matsumura, T. and Matsushima, N. (2004). Endogenous cost differentials between public and private enterprises: a mixed duopoly approach. *Economica*, 71(284):671–688.

Matsumura, T. and Okamura, M. (2015). Competition and privatization policies revisited: the payoff interdependence approach. *Journal of Economics*, 116(2):137–150.

Matsumura, T. and Shimizu, D. (2010). Privatization waves. *The Manchester School*, 78(6):609–625.

Murto, P., Näsäkkälä, E., and Keppo, J. (2004). Timing of investments in oligopoly under uncertainty: A framework for numerical analysis. *European Journal of Operational Research*, 157(2):486–500.

Nocke, V. and Schutz, N. (2018). Multiproduct-firm oligopoly: An aggregative games approach. *Econometrica*, 86(2):523–557.

OECD, O. (2018). Science, technology and innovation outlook 2018. *Adapting to Technological and Societal Disruption.*

Oehmke, J. F. (2001). Biotechnology R&D races, industry structure, and public and private sector research orientation. *AgBioForum*, 4:105–114.

Poyago-Theotoky, J. (1998). R&D competition in a mixed duopoly under uncertainty and easy imitation. *Journal of Comparative Economics*, 26(3):415–428.

Poyago-Theotoky, J. (2001). Mixed oligopoly, subsidization and the order of firms' moves: an irrelevance result. *Economics Bulletin*, 12(3):1–5.

Shin, W. and Lee, W. (2013). Legality of r&d subsidies and its policy framework under the world trading system: The case of civil aircraft disputes. *STI Policy Review*, 4.

White, M. D. (1996). Mixed oligopoly, privatization and subsidization. *Economics letters*, 53(2):189–195.

White, M. D. (2002). Political manipulation of a public firm's objective function. *Journal of Economic Behavior & Organization*, 49(4):487–499.