The Sequential Dominance Argument for the Independence Axiom of Expected Utility Theory

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Independence is the condition that, if X is preferred to Y, then a lottery between X and Z is preferred to a lottery between Y and Z given the same probability of Z. Is it rationality required that one's preferences conform to Independence? The main objection to this requirement is that it would rule out the alleged rationality of Allais and Ellsberg Preferences. In this paper, I put forward a sequential dominance argument with fairly weak assumptions for a variant of Independence (called Independence for Constant Prospects), which shows that Allais and Ellsberg Preferences are irrational. Hence this influential objection (that is, the alleged rationality of Allais and Ellsberg Preferences) can be rebutted. I also put forward a number of sequential dominance arguments that various versions of Independence are requirements of rationality. One of these arguments is based on very minimal assumptions, but the arguments for the versions of Independence which are strong enough to serve in the standard axiomatization of Expected Utility Theory need notably stronger assumptions.

Consider the prospect of either getting a trip to Freedonia or getting a trip to Sylvania, depending on a coin toss. Compare this first prospect with a second prospect, which is just like the first except that you also get some extra travel money in case you get the Freedonia trip. Other things being equal, you prefer getting the extra money. Since the second prospect is the same as the first except that one outcome has been replaced by a preferred outcome with the same probability, the second prospect should be preferred to the first. This is the basic thought behind Independence, which—along with Completeness, Continuity, and Transitivity—is one of the standard axioms of Expected Utility Theory.

But is Independence a requirement of rationality? That is, is it rationally required that one's preferences conform to Independence? The usual defence of this requirement takes the form of a *sequential dominance argument*, that is, an argument showing that anyone who violates this alleged requirement would, in some sequential situation, be forced to act against their own preference. In this paper, I shall argue that different versions of Independence differ significantly in their support for Expected Utility Theory and in what assumptions are needed to defend their status as requirements of rationality with the help of sequential dominance arguments.

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Let XpY be a prospect consisting in a lottery between X and Y such that X occurs with probability p and Y occurs with probability $1 - p$. In XpY, outcomes X and Y are also prospects, which are either lotteries themselves or *final outcomes*, that is, outcomes that are final in the sense that they involve no further risk or uncertainty.[1] The most straightforward version of Independence can be stated as follows:

> *Independence (the biconditional weak-preference version)*
> For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, X is at least as preferred as Y if and only if XpZ is at least as preferred as YpZ.[2]

Still, the standard axiomatization of *Expected Utility Theory* (the theory that prospects are preferred in accordance with an expected-utility function) makes do with a weaker version of Independence, namely,

> *Independence (the strong strict-preference version)*
> For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, if X is preferred to Y, then XpZ is preferred to YpZ.[3]

The strong strict-preference version of Independence together with the following conditions are necessary and sufficient for Expected Utility Theory:[4]

> *Completeness*
> For all prospects X and Y, either X is at least as preferred as Y or Y is at least as preferred as X.[5]
>
> *Continuity*
> For all X, Y, and Z, if X is preferred to Y and Y is preferred to Z, then there are probabilities $0 < p < 1$ and $0 < q < 1$ such that XpZ is preferred to Y and Y is preferred to XqZ.[6]

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[1] We may also wish to allow that prospects could be future choices, rather than just lotteries or final outcomes. But, for the arguments in this paper, this complication isn't necessary, since we shall be concerned with prospects of following plans rather than prospects of individual choices that lead to further choices.

[2] Rubin 1949, p. 2. For a historical account of Independence, see Fishburn and Wakker 1995.

[3] Jensen 1967, p. 173.

[4] Jensen 1967, pp. 172–182. See Fishburn 1970, pp. 111–115, 1982, pp. 12–20 and Hammond 1998, pp. 152–164 for other versions of the proof.

[5] von Neumann and Morgenstern 1944, pp. 26–27 and Jensen 1967, p. 173.

[6] von Neumann and Morgenstern 1944, pp. 26–27, Blackwell and Girshick 1954, p. 106, and Jensen 1967, p. 173.

Transitivity
For all prospects X, Y, and Z, if X is at least as preferred as Y and Y is at least as preferred as Z, then X is at least as preferred as Z.[7]

An implication of this standard axiomatization is that, if these four conditions are requirements of rationality, then it is rationally required to prefer prospects in accordance with an expected-utility function.

The standard objection to the idea that Independence is a requirement of rationality is that the most straightforward version of Independence conflicts with some seemingly rational preferences, namely, Allais and Ellsberg Preferences. These preferences also conflict with the following variation of Independence:

Independence for Constant Prospects (the weak strict-preference version)
For all prospects X, Y, U, and V and probabilities p such that $0 < p < 1$, if XpU is preferred to YpU, then YpV is not preferred to XpV.[8]

This condition, however, can be shown to be a requirement of rationality with the help of a sequential dominance argument with fairly weak assumptions (§2). Accordingly, this argument shows that Allais and Ellsberg Preferences are irrational. And, thereby, it rebuts several recent decision theories that try to account for the alleged rationality of Allais and Ellsberg Preferences.[9]

Furthermore, there is a sequential dominance argument, with even weaker assumptions, that the following version of Independence is a requirement of rationality (§3):

Independence (the weak strict-preference version)
For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, if X is preferred to Y, then YpZ is not preferred to XpZ.

This version of Independence is too weak to characterize Expected Utility Theory together with Completeness, Continuity, and Transitivity (§4). Still, there is a way to extend this argument for the weak strict-preference version so that it also works for the *strong* strict-preference version. This extended argument, however, requires notably stronger assumptions (§5). But, given these assumptions, one can also show that the biconditional weak-preference version of Independence is a requirement of rationality (§6).

1. The Logical Relationship between these Versions of Independence

Before we go on, it may help to clear up the logical relationships between these different versions of Independence. The weak strict-preference version is

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[7] von Neumann and Morgenstern 1944, pp. 26–27 and Jensen 1967, p. 171.
[8] McClennen 1990, p. 45.
[9] For example, Buchak 2013, p. 71 and Bradley 2017, pp. 171–177.

logically weaker than the others. Violations of the weak strict-preference version of Independence can only be of the following kind, where p is a probability such that $0 < p < 1$:

(1) A is preferred to B, and BpC is preferred to ApC.

The strong strict-preference version is somewhat stronger. In addition to preferences of the kind in (1), violations of the strong strict-preference version can also be of the following kinds:

(2) A is preferred to B, and ApC is equally preferred as BpC.

(3) A is preferred to B, and there is a preferential gap between ApC and BpC.

The biconditional weak-preference version of Independence is stronger still. In addition to preferences of the kinds in (1)–(3), violations of the biconditional weak-preference version can also be of the following kinds:

(4) A is equally preferred as B, and ApC is preferred to BpC.

(5) A is equally preferred as B, and there is a preferential gap between ApC and BpC.

(6)   There is a preferential gap between A and B, and ApC is preferred to BpC.

(7)   There is a preferential gap between A and B, and ApC is equally preferred as BpC.

As we shall see, the argument against the rationality of the preferences in (2) and (3) needs stronger assumptions than the argument against the rationality of the preferences in (1). But the argument against the rationality of the preferences in (4)–(7) needs no more assumptions than the argument against the rationality of the preferences in (2) and (3).

2. Allais, Ellsberg, and Independence for Constant Prospects

The two most prominent challenges to Independence are the Allais Paradox (first put forward by Maurice Allais) and the Ellsberg Paradox (first put forward by Daniel Ellsberg). These paradoxes are direct challenges to the biconditional weak-preference version of Independence, but they are also direct challenges to the following, logically weaker, requirement:

> *Independence for Constant Prospects (the weak strict-preference version)*
> For all prospects X, Y, U, and V and probabilities p such that $0 < p < 1$, if XpU is preferred to YpU, then YpV is not preferred to XpV.

Violations of this variant of Independence can only be of the following kind:

(8) ApC is preferred to BpC, and BpD is preferred to ApD,

where p is a probability such that $0 < p < 1$. As we shall see, the Allais Paradox and the Ellsberg Paradox both feature seemingly rational preferences of this kind.

The Allais Paradox involves four gambles: In Allais Gamble 1, one gets \$1 M for certain; in Allais Gamble 2, there is a 10 % probability of getting \$5 M, an 89 % probability of getting \$1 M, and a 1 % probability of getting nothing; in Allais Gamble 3, there is an 11 % probability of getting \$1 M and an 89 % probability of getting nothing; and, in Allais Gamble 4, there is a 10 % probability of getting \$5 M and a 90 % probability of getting nothing:[10]

	Probability		
	1 %	10 %	89 %
Allais Gamble 1	\$1 M	\$1 M	\$1 M
Allais Gamble 2	\$0	\$5 M	\$1 M
Allais Gamble 3	\$1 M	\$1 M	\$0
Allais Gamble 4	\$0	\$5 M	\$0

Many people have the following preferences, which we can call *Allais Preferences*:

(9)   Allais Gamble 1 is preferred to Allais Gamble 2, and Allais Gamble 4 is preferred to Allais Gamble 3.

To see that Allais Preferences violate the weak strict-preference version of Independence for Constant Prospects, let A, B, C, and D be the following prospects:

	Probability	
	1/11	10/11
A	\$1 M	\$1 M
B	\$0	\$5 M
C	\$1 M	\$1 M
D	\$0	\$0

Then, if we let p be 11/100, Allais Gamble 1 is equivalent to ApC, Allais Gamble 2 is equivalent to BpC, Allais Gamble 3 is equivalent to ApD, and Allais Gamble 4 is equivalent to BpD. So (9) can be stated as

(8) ApC is preferred to BpC, and BpD is preferred to ApD.

Hence Allais Preferences violate the weak strict-preference version of Independence for Constant Prospects.

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[10] Allais 1953, p. 527; 1979, p. 89. In Allais's original version, the prizes were 100 million and 500 million francs.

The Ellsberg Paradox features an urn containing 30 red balls and 60 balls that are either black or yellow. The proportion of black to yellow balls is unknown. A ball will be drawn at random from the urn. Consider the following gambles: *Ellsberg Gamble 1* pays \$100 if the ball is red, otherwise nothing; *Ellsberg Gamble 2* pays \$100 if the ball is black, otherwise nothing; *Ellsberg Gamble 3* pays \$100 if the ball is red or yellow, otherwise nothing; and *Ellsberg Gamble 4* pays \$100 if the ball is black or yellow, otherwise nothing:[11]

	30	60	
	Red	Black	Yellow
Ellsberg Gamble 1	\$100	\$0	\$0
Ellsberg Gamble 2	\$0	\$100	\$0
Ellsberg Gamble 3	\$100	\$0	\$100
Ellsberg Gamble 4	\$0	\$100	\$100

Many people have the following preferences, which we can call *Ellsberg Preferences*:

(10) Ellsberg Gamble 1 is preferred to Ellsberg Gamble 2, and Ellsberg Gamble 4 is preferred to Ellsberg Gamble 3.

Ellsberg Preferences violate the weak strict-preference version of Independence for Constant Prospects. To see this, let p be the unknown probability of *the ball's being either red or black*, and let A, B, C, and D now be the following prospects:

	Probability	
	$\dfrac{1}{3p}$	$1 - \dfrac{1}{3p}$
A	\$100	\$0
B	\$0	\$100
C	\$0	\$0
D	\$100	\$100

We then have that Ellsberg Gamble 1 is equivalent to ApC, Ellsberg Gamble 2 is equivalent to BpC, Ellsberg Gamble 3 is equivalent to ApD, and Ellsberg Gamble 4 is equivalent to BpD. So (10) can be stated as

(8) ApC is preferred to BpC, and BpD is preferred to ApD.

Hence—just like Allais Preferences—Ellsberg Preferences violate the weak strict-preference version of Independence for Constant Prospects.

We have that both Allais and Ellsberg Preferences entail preferences of the kind in (8) and, therefore, that they both violate the weak strict-preference

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[11] Ellsberg 1961, pp. 653–654.

version of Independence for Constant Prospects. We also have that Allais and Ellsberg Preferences violate the biconditional weak-preference version of Independence, since that condition entails the weak strict-preference version of Independence for Constant Prospects. So, if Allais or Ellsberg Preferences are rationally permissible, the biconditional weak-preference version of Independence cannot be a requirement of rationality.

(Neither Allais nor Ellsberg Preferences, however, violate the strong or the weak strict-preference version of Independence. Still, if we assume that—in addition to having the preferences in (8)—one also prefers one of A and B to the other, then we do get a violation of both the strong and the weak strict-preference version of Independence.[12] But having Allais or Ellsberg Preferences doesn't commit one to having this additional preference. If one is indifferent between A and B, there will only be a violation of the weak or the strong strict-preference version of Independence in combination with certain other conditions.)

As we have seen, the seemingly rational Allais and Ellsberg Preferences violate the weak strict-preference version of Independence for Constant Prospects. Can we defend this condition's status as a requirement of rationality from these alleged counter-examples? We can. Any preferences that violate the weak strict-preference version of Independence for Constant Prospects—that is, preferences of the kind in (8)—can be shown to be irrational with the help of a sequential dominance argument. This argument assumes four requirements of rationality. The first is

> *Continuity of Strict Preference*
> For all prospects X and Y, if X is preferred to Y, then there is a prospect X^- that is just like X except that each final outcome in X has been replaced with an equally probable yet less preferred final outcome and X^- is preferred to Y.

The idea is that, if X is strictly preferred to Y, then X is preferred to Y *with some margin*. So X should still be preferred to Y if X were soured by an arbitrarily small amount.

From (8) and Continuity of Strict Preference, we get that there are prospects A^-pC^- and B^-pD^- that are just like ApC and BpD respectively except that each final outcome in ApC and BpD has been replaced with an equally probable yet less preferred outcome and
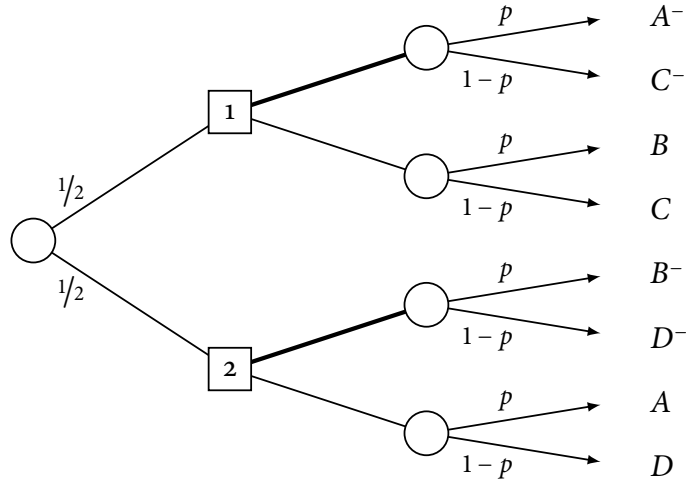
(11) A^-pC^- is preferred to BpC, and B^-pD^- is preferred to ApD.

Now, consider the following decision tree:[13]

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[12] See Rabinowicz 1995, pp. 588–589.

[13] This is a generalization of an argument in Raiffa 1968, pp. 83–85. See also Raiffa's (1961, p. 694) earlier argument, which uses similar reasoning but doesn't involve any dominance violation.

Case 1

The squares represent choice nodes where one has a choice between the paths forward. The circles represent chance nodes where chance determines the path forward, and the numbers next to these paths represent their probability given that the chance node is reached. The thick lines represent the choices one would make at the choice nodes if one were guided by the preferences in (11).

At choice node 1, one has a choice between going up, the prospect of which is $A^- p C^-$, and going down, the prospect of which is BpC. And, at choice node 2, one has a choice between going up, the prospect of which is $B^- p D^-$, and going down, the prospect of which is ApD.

Let a plan at a node n be a specification of what to choose at each choice node that can be reached from n. Let us say that one follows a plan at node n' if and only if, for each choice node n'' that can be reached from n', one would choose in accordance with that plan if one were to face n''. Moreover, let us say that one *intentionally* follows a plan at node n' if and only if one follows the plan at n' and, for all nodes n'' such that n'' can be reached from n', if one were to face n'', one would either form or have formed at n'' an intention to choose in accordance with the plan at every choice node that can be reached from each of n' and n''.[14] Finally, let us say that a plan is available at a node n if and only if the plan can be intentionally followed at n.

The second principle we shall assume to be a requirement of rationality is

> *The Principle of Prospect Guidance*
> For all reachable nodes n (that is, the current node and nodes that can be reached from that node), if one were to face n and there were two alternative plans P' and P'' available at n such that the prospect of following P' were preferred to the prospect of following P'', then one would not follow P''.[15]

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[14] Carlson (2003, pp. 182–183) proposes a similar account of performability.

[15] It may be objected that this requirement is too strong, because even preferences that

The idea behind this requirement is that, if one were to violate the Principle of Prospect Guidance, one would freely act against one's own interests, which seems irrational.

In Case 1, we have, from (11) and the Principle of Prospect Guidance, that one wouldn't go down at any of the choice nodes. Hence one would go up at each of choice nodes 1 and 2. At the initial chance node, two of the available plans are (i) to go up at both choice nodes and (ii) to go down at both choice nodes. Consider the prospects of following these plans at the initial chance node—letting E^- be the prospect of following the plan to go up at both choice nodes and E be the prospect of following the plan to go down at both choice nodes:

	Probability			
	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{1-p}{2}$	$\frac{1-p}{2}$
E^-	A^-	B^-	C^-	D^-
E	A	B	C	D

Here, E seems preferable to E^-, since for each final outcome of E^- there is a corresponding equally likely final outcome of E which is preferred. This idea is captured by the following dominance principle, which we shall assume is a requirement of rationality:

> *The Weak Principle of Equiprobable Dominance*
> For all prospects X and Y, if there is a one-to-one mapping of the final outcomes of prospect X to the final outcomes of prospect Y where each final outcome of Y is paired with a preferred final outcome in X with the same probability, then X is preferred to Y.

This requirement should be acceptable even if one is risk-averse.[16] In terms of risk, the dominated prospect must be less preferable than the dominating prospect. For every potential undesired outcome of the dominating prospect, the dominated prospect has a corresponding outcome with the same probability which is even less preferred. The probability of getting an undesired outcome must be at least as high in the dominated prospect as in the dominating prospect. In any compelling violation of Independence for Constant

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conform to Expected Utility Theory could force a violation of this requirement in some situations where there are infinitely many alternatives. To avoid this problem, one could restrict the principle to situations where the number of alternatives is finite. See Nozick 1963, p. 89 and Gustafsson 2013, p. 464. It's unclear, however, whether it's physically possible that an agent could ever face a choice between infinitely many alternatives; for a discussion, see Pruss 2018, pp. 107–108.

[16] For example, Buchak (2013, pp. 37–38), who defends Allais-preferences and risk-aversion, accepts the Strong Principle of Stochastic Dominance, which is a stronger requirement than the Weak Principle of Equiprobable Dominance. See note 24.

Prospects, no individual preference between two prospects violates the Weak Principle of Equiprobable Dominance. For example, none of the following preferences violate the Weak Principle of Equiprobable Dominance: (i) Allais Gamble 1 is preferred to Allais Gamble 2, (ii) Allais Gamble 4 is preferred to Allais Gamble 3, (iii) Ellsberg Gamble 1 is preferred to Ellsberg Gamble 2, and (iv) Ellsberg Gamble 4 is preferred to Ellsberg Gamble 3. Hence the Weak Principle of Equiprobable Dominance does not assume the point at issue against Allais and Ellsberg Preferences.

From (11) and the Weak Principle of Equiprobable Dominance, we have

(12) E is preferred to E^-.

Hence, at the initial chance node, the prospect of the plan to go down at each choice node (that is, E) is preferred to the prospect of the plan to go up at each choice node (that is, E^-). Given (12), the Principle of Prospect Guidance requires that, at the initial chance node, one wouldn't follow the plan to go up at each choice node.[17] Yet, as we saw earlier, the Principle of Prospect Guidance also requires that one wouldn't go down at any of the choice nodes, given (11). The upshot is that, if one has preferences of the kind in (8), one is forced to violate the Principle of Prospect Guidance in this type of case.[18]

The fourth principle we shall assume is a requirement of rationality is

> *The Principle of Preferential Invulnerability*
> If one has a certain set of preferences, then there is no possible (synchronic or dynamic) situation where having these preferences forces one to violate a requirement of rationality.[19]

Given that this principle is a requirement of rationality, rational preferences cannot lead to any conflicts with any requirements of rationality in any possible situation. In Case 1, as we have seen, the preferences in (8) force one to violate

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[17] It may be objected that there's no choice between plans at the initial node, since it's a chance node. Note, however, that plans concern not only present choices but also upcoming choices, and there are upcoming choices at the initial node. Moreover, if we really were worried about this objection, we could add an earlier choice node with a choice between getting to face Case 1 and getting the dominated prospect E^-. Then there would be an initial choice between plans. Yet, given Allais Preferences, one would still end up with E^- rather than E and hence violate the Principle of Prospect Guidance. (This reply also applies to similar worries about Case 2, where one could add an initial choice between getting to face Case 2 and getting the dominated prospect ApC).

[18] If we further assume that E^- is just like E except that one has less money (some money has been given to an exploiter), then Case 1 is a money pump against the preferences in (8). One ends up with E^- by following the plan to go up in both choice node even though one could have ended up with E by following the plan to go down in both choice nodes. Hence one freely gives away money.

[19] One may wish to restrict the Principle of Preferential Invulnerability to situations where the number of alternatives is finite in order to avoid situations where even preferences that conform to Expected Utility Theory could give rise to rational dilemmas. See note 15.

the Principle of Prospect Guidance, which (we have assumed) is a requirement of rationality. So then the Principle of Preferential Invulnerability yields that the preferences in (8) are irrational.

We can, changing what needs to be changed, run the same argument against any preferences of the kind in (8). Since all violations of the weak strict-preference version of Independence for Constant Prospects are of the same kind as the preferences in (8), we have that all violations of this condition are irrational. Hence we have an argument that the weak strict-preference version of Independence for Constant Prospects is a requirement of rationality. And this argument is based on the following requirements of rationality:

- Continuity of Strict Preference
- The Weak Principle of Equiprobable Dominance
- The Principle of Prospect Guidance
- The Principle of Preferential Invulnerability

It follows that Allais and Ellsberg Preferences are irrational, since those preferences violate the weak strict-preference version of Independence for Constant Prospects. So we can rebut the main objection to the biconditional weak-preference version of Independence. Nevertheless, it doesn't follow that the biconditional weak-preference version is a rational requirement, because that condition is logically stronger than the weak strict-preference version of Independence for Constant Prospects.[20]

3. The Weak Strict-Preference Version of Independence

Having rebutted the alleged rationality of Allais and Ellsberg Preferences, let us explore whether there are any compelling *positive* arguments that Independence is a requirement of rationality. We begin with the weakest version, namely,

> *Independence (the weak strict-preference version)*
> For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, if X is preferred to Y, then YpZ is not preferred to XpZ.

This version of Independence can be shown to be a requirement of rationality with the help of a sequential dominance argument with even weaker assumptions than those we relied on in the argument for Independence for Constant Prospects.

Let p be a probability such that $0 < p < 1$, and suppose that one violates the weak strict-preference version of Independence by having the following preferences:
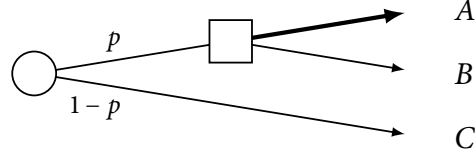
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[20] As we shall see in §4, there is a theory that violates the biconditional weak-preference version of Independence even though it satisfies Completeness, Transitivity, Continuity, and the weak strict-preference version of Independence for Constant Prospects.

(1)   *A* is preferred to *B*, and *BpC* is preferred to *ApC*.

And consider the following decision tree:[21]

Case 2

Here, the thick line represents the choice one would make at the choice node if one were guided by the preferences in (1).

In this case, there are two available plans at the chance node: The first plan is to go up if one were to reach the choice node. The second plan is to go down if one were to reach the choice node. If one follows either of the these plans and one has the preferences in (1), then one violates the Principle of Prospect Guidance. Given (1), we have that the Principle of Prospect Guidance requires that one wouldn't follow the up plan at the chance node, since the prospect of the down plan (that is, *BpC*) is preferred to the prospect of the up plan (that is,

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[21] Hammond 1988a, pp. 43, 45. Hammond (1988b, pp. 292–293) puts forward a more compli-cated argument with a further initial choice. Hammond's argument relies on Continuity for Strict Preferences, which isn't needed for the argument put forward here. Still, with a variation of this approach, we can create a money pump against preferences of the kind in (1). Suppose that you have the preferences in (1). From (1) and Continuity of Strict Preference, we have

(I) $B^- pC^-$ is preferred to *ApC*.

We further assume that $B^- pC^-$ is like *BpC* except that you have less money (you have given some money to an exploiter). Now, consider

Case 2*
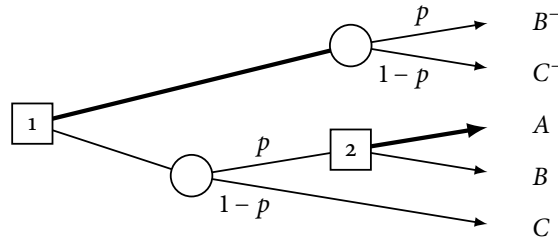
Here, the thick lines represents the choices you would make at the choice nodes if you were guided by backward induction and the preferences in (1) and (I). Since you prefer *A* to *B*, you would go up at choice node 2. Using backward induction, you take this prediction into account at choice node 1. At choice node 1, the prospect of going down is then *ApC* and the prospect of going up is $B^- pC^-$. From (I), we then have that you prefer the prospect of going up to the prospect of going down at choice node 1. So you go up at choice node 1. But then you end up with $B^- pC^-$ when you could have had *BpC* if you had followed the plan to go down at each choice node. Hence you have freely given away money. Moreover, if we assume that both chance nodes depend on the same event, we get that the prospect of going up at choice node 1 is statewise dominated by the prospect of following the plan to go down at each choice node.

ApC). Given (1), we also have that the Principle of Prospect Guidance requires that one wouldn't follow the down plan at the chance node, because doing so involves following the down plan at the *choice* node, which violates the Principle of Prospect Dominance. Following the down plan at the choice node violates the Principle of Prospect Guidance, because, at that node, the prospect of the up plan (that is, *A*) is preferred to the prospect of the down plan (that is, *B*). We have that, if one has the preferences in (1), then one is forced to violate the Principle of Prospect Guidance in Case 2.

Assuming that the Principle of Prospect Guidance is a requirement of rationality, we then have, by the Principle of Preferential Invulnerability, that the preferences in (1) are irrational. Hence we have a sequential dominance argument with very minimal assumptions against preferences of the kind in (1).

Since all violations of the weak strict-preference version of Independence are of the same kind as those in (1), we have a compelling sequential dominance argument that the weak strict-preference version of Independence is a rational requirement, and this argument is merely based on the following requirements of rationality:

- The Principle of Prospect Guidance

- The Principle of Preferential Invulnerability

Still, axiomatizations of Expected Utility Theory typically rely on a stronger version of Independence, like the strong strict-preference version.

4. The Weak Strict-Preference Version Isn't Strong Enough

As mentioned earlier, Expected Utility Theory can be axiomatized by Completeness, Transitivity, Continuity, and the strong strict-preference version of Independence. Can we strengthen this standard axiomatization so that it relies on the *weak* strict-preference version of Independence rather than the strong one? We cannot. Likewise, we cannot replace the strong strict-preference version of Independence with the weak strict-preference version of Independence for Constant Prospects in the axiomatization. We shall prove these negative claims with a counter-example. Consider

> *Cancelling Utility Theory*
> There are three mutually exclusive kinds of final outcomes: GOOD outcomes, CANCELLING outcomes, and NEUTRAL outcomes. Let $G(X)$ be the probability of a GOOD final outcome in X. Let $C(X)$ be the probability of a CANCELLING final outcome in X. And let $V(X)$ be $\max\{G(X)-C(X), 0\}$. That is, $V(X)$ is equal to $G(X)-C(X)$ if $G(X) > C(X)$; otherwise $V(X)$ is equal to 0. Prospect X is at least as preferred as prospect Y if and only if $V(X) \geq V(Y)$.

A CANCELLING outcome should not be thought of as a bad outcome; the probability of a CANCELLING outcome does not make a prospect overall bad, it just cancels out an equal probability of a GOOD outcome.

Clearly, Cancelling Utility Theory satisfies Completeness and Transitivity. To see that Cancelling Utility Theory satisfies the weak strict-preference version of Independence, note that, if YpZ is preferred to XpZ, then

$$V(YpZ) = \max\left\{p\big(G(Y) - C(Y)\big) + (1 - p)\big(G(Z) - C(Z)\big), 0\right\}$$

must be greater than

$$V(XpZ) = \max\left\{p\big(G(X) - C(X)\big) + (1 - p)\big(G(Z) - C(Z)\big), 0\right\}.$$

This could only happen if $G(Y) - C(Y)$ is greater than $G(X) - C(X)$. But, if $G(Y) - C(Y)$ is greater than $G(X) - C(X)$, then X is not preferred to Y. We have that Cancelling Utility Theory satisfies the weak strict-preference version of Independence.

Likewise, we have that Cancelling Utility Theory satisfies the weak strict-preference version of Independence for Constant Prospects. By analogous reasoning, we have that, if XpU is preferred to YpU, then $G(Y) - C(Y)$ is greater than $G(X) - C(X)$ and that, if YpV is preferred to XpV, then $G(X) - C(X)$ is greater than $G(Y) - C(Y)$. Since $G(X) - C(X)$ cannot be greater than $G(Y) - C(Y)$ if $G(Y) - C(Y)$ is greater than $G(X) - C(X)$, we have that, if XpU is preferred to YpU, then YpV is not preferred to XpV. Therefore, Cancelling Utility Theory satisfies the weak strict-preference version of Independence for Constant Prospects.

To see that Cancelling Utility Theory also satisfies Continuity, suppose that X is preferred to Y and Y is preferred to Z. Then, given a probability p less than 1 but arbitrarily close to 1, $V(XpZ)$ will be arbitrarily close to $V(X)$ and hence greater than $V(Y)$, so XpZ is preferred to Y. And, given a probability q greater than 0 but arbitrarily close to 0, $V(XqZ)$ will be arbitrarily close to $V(Z)$ and hence lesser than $V(Y)$, so Y is preferred to XqZ. So we have that Cancelling Utility Theory satisfies Continuity.

Finally, to see that Cancelling Utility Theory *violates* the strong strict-preference version of Independence, suppose that X is a GOOD final outcome, that Y is a NEUTRAL final outcome, and that Z is a CANCELLING final outcome. Then, with $p = 1/2$, we have that X is preferred to Y but XpZ is equally preferred as YpZ. Therefore, Cancelling Utility Theory violates the strong strict-preference version of Independence. And, since Expected Utility Theory *satisfies* the strong strict-preference version of Independence, we have that Cancelling Utility Theory is not a version of Expected Utility Theory.

Of course, Cancelling Utility Theory is an implausible theory. Its purpose

here is merely to illustrate that we do need the strong strict-preference version of Independence in the standard axiomatization of Expected Utility Theory. Neither the weak strict-preference version of Independence nor the weak strict-preference version of Independence for Constant Prospects is strong enough.

5. The Strong Strict-Preference Version of Independence

So let us turn to

> *Independence (the strong strict-preference version)*
> For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, if X is preferred to Y, then XpZ is preferred to YpZ.

The good news is that there is a sequential dominance argument that this version of Independence is a requirement of rationality; the bad news is that the argument requires notably stronger assumptions than the argument for the weak strict-preference version. In order to show that the strong strict-preference version is a requirement of rationality, it's not enough to show that preferences of the kind in (1) are irrational. We also need to show the irrationality of violations of the following kinds, where (like before) p is a probability such that $0 < p < 1$:

(2) A is preferred to B, and ApC is equally preferred as BpC.

(3) A is preferred to B, and there is a preferential gap between ApC and BpC.

The sequential dominance argument in §3 doesn't work against the preferences in (2) and (3), because with these preferences it's no longer clear that it's irrational to choose A over B at the choice node in Case 2. Preferences of the kind in (3) could be ruled out if we assume that Completeness is a requirement of rationality.[22] The preferences in (2) are more challenging. These preferences violate the strong strict-preference version of Independence, but they do not violate any of the other standard axioms of Expected Utility Theory.[23] And, since the biconditional weak-preference version of Independence is logically stronger than the strong strict-preference version, the preferences in (2) violate that version too. Hence, to have a cogent argument that these versions of Independence are requirements of rationality, we must show that the preferences in (2) are irrational.

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[22] A problem with relying on Completeness in a general defence of Independence by sequential dominance arguments is that it seems like it cannot be shown with the help of sequential dominance arguments that Completeness is a requirement of rationality; see Gustafsson 2016, pp. 54–66.

[23] As we saw in §4, Cancelling Utility Theory satisfies Completeness, Continuity, Transitivity, and the weak strict-preference version of Independence. To see that Cancelling Utility Theory violates the strong strict-preference version of Independence, note that, with $p = 1/2$, Cancelling Utility Theory yields the preferences in (2) if A is a GOOD final outcome, B is a NEUTRAL final outcome, and C is a CANCELLING final outcome.

To establish the irrationality of preferences of the kind in (2), we shall assume that the following dominance principle is a requirement of rationality:

> *The Strong Principle of Equiprobable Dominance*
> For all prospects X and Y, if there is a one-to-one mapping of the final outcomes of prospect X to the final outcomes of prospect Y where each final outcome of Y is paired with an at least as preferred final outcome in X with the same probability and one final outcome in Y is paired with a more preferred final outcome in X, then X is preferred to Y.

Just like the Weak Principle of Equiprobable Dominance, this requirement should be acceptable even if one is risk-averse. The probability of getting an undesired outcome must be at least as high in the dominated prospect as in the dominating prospect.[24] In any compelling violation of Independence, the individual preferences do not violate the Strong Principle of Equiprobable Dominance.

We shall show that preferences of the kind in (1) can be derived from preferences of the kind in (2), given that Continuity of Strict Preference, the Strong Principle of Equiprobable Dominance, and Transitivity are requirements of rationality.

From (2) and Continuity of Strict Preference, we have that there is a prospect A^- that is just like A except that each final outcome in A has been replaced with an equally probable yet less preferred final outcome and

(13) A^- is preferred to B.

From the Strong Principle of Equiprobable Dominance, we have

(14) ApC is preferred to A^-pC.

Then—from (2), (14), and Transitivity—we have

(15) BpC is preferred to A^-pC.

Finally, from (13) and (15), we have

(16) A^- is preferred to B, and BpC is preferred to A^-pC.

We have derived preferences of the same kind as those in (1). Since preferences of that kind can be shown to be irrational by the sequential dominance argument in §3, we can show that preferences of the kind in (2) are irrational. The argument in §3 relies on the Principle of Prospect Guidance and the Principle of Preferential Invulnerability. Hence we have a sequential dominance argument that the strong strict-preference version of Independence is a requirement of rationality. This argument is based on the following requirements of rationality:

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[24] Buchak (2013, pp. 37–38), who defends Allais-preferences and risk-aversion, accepts the Strong Principle of Stochastic Dominance, which is a stronger requirement than the Strong Principle of Equiprobable Dominance. The Strong Principle of Equiprobable Dominance is a special case of the Strong Principle of Stochastic Dominance.

- Completeness
- Continuity of Strict Preference
- The Strong Principle of Equiprobable Dominance
- Transitivity
- The Principle of Prospect Guidance
- The Principle of Preferential Invulnerability

These assumptions are notably stronger than those needed in the argument for the weak strict-preference version of Independence for Constant Prospects, because we additionally assume that Completeness, Transitivity, and the Strong (rather than the Weak) Principle of Equiprobable Dominance are requirements of rationality. And these assumptions are much stronger than those needed in the argument for the weak strict-preference version of Independence, since that argument only needs the Principle of Prospect Guidance and the Principle of Preferential Invulnerability.

6. The Biconditional Weak-Preference Version of Independence

Finally, let us turn to

> *Independence (the biconditional weak-preference version)*
> For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, X is at least as preferred as Y if and only if XpZ is at least as preferred as YpZ.

With the same assumptions we relied on in the argument that the strong strict-preference version is a requirement of rationality, we can also show that the biconditional weak-preference version is a requirement of rationality.

In addition to preferences of the kind in (1)–(3) which we have already shown are irrational (with the arguments in §3 and §5), violations of the biconditional weak-preference version of Independence can also be of the following kinds, where again p is a probability such that $0 < p < 1$:

(4) A is equally preferred as B, and ApC is preferred to BpC.

(5) A is equally preferred as B, and there is a preferential gap between ApC and BpC.

(6)   There is a preferential gap between A and B, and ApC is preferred to BpC.

(7)   There is a preferential gap between A and B, and ApC is equally preferred as BpC.

Three of these violations—namely, (5), (6), and (7)—can be ruled out if we, like before, assume that Completeness is a requirement of rationality. So, to finish the argument for the biconditional weak-preference version, we only need to show that preferences of kind in (4) are irrational.

From (4) and Continuity of Strict Preferences, we have that there is a prospect A^-pC^- that is just like ApC except that each final outcome in ApC has been replaced with an equally probable yet less preferred final outcome and

(17) A^-pC^- is preferred to BpC.

From the Strong (or the Weak) Principle of Equiprobable Dominance, we have

(18) A is preferred to A^-.

And—from (4), (18), and Transitivity—we have

(19) B is preferred to A^-.

From the Strong Principle of Equiprobable Dominance, we have

(20) A^-pC is preferred to A^-pC^-.

Then—from (17), (20), and Transitivity—we have

(21) A^-pC is preferred BpC.

Finally, from (19) and (21), we have

(22) B is preferred to A^-, and A^-pC is preferred BpC.

We have, once more, derived preferences of the same kind as those in (1). And, since such preferences can be shown to be irrational by the sequential dominance argument in §3, we can show that preferences of the kind in (4) are irrational.

The sequential dominance argument in §3 relies on the Principle of Prospect Guidance and the Principle of Preferential Invulnerability. Hence we have a sequential dominance argument that the biconditional weak-preference version of Independence is a requirement of rationality, and this argument is based on the following requirements of rationality:[25]

- Completeness

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[25] This argument also supports that the following, logically weaker, version of Independence is a requirement of rationality:

> *Independence (the strong equal-preference version)*
> For all prospects X, Y, and Z and probabilities p such that $0 < p < 1$, if X is equally preferred as Y, then XpZ is equally preferred as YpZ.

This version was proposed by Marschak (1950, pp. 120–121) and Nash (1950, p. 156). Violations of the strong equal-preference version of Independence can only be of the kind in (4) and (5).

- Continuity of Strict Preference
- The Strong Principle of Equiprobable Dominance
- Transitivity
- The Principle of Prospect Guidance
- The Principle of Preferential Invulnerability

So the argument that the biconditional weak-preference version is a requirement of rationality is based on the same assumptions as the argument for the strong strict-preference version.

7. Summary

There is, as we saw in §3, a sequential dominance argument that the weak strict-preference version of Independence is a requirement of rationality, and this argument is based on the following requirements of rationality:

- The Principle of Prospect Guidance
- The Principle of Preferential Invulnerability

Even though this argument has very minimal assumptions, it's of limited interest since it doesn't rule out Allais or Ellsberg Preferences and it's too weak for the standard axiomatization of Expected-Utility Theory, as we saw in §4.

Nevertheless, with just slightly stronger assumptions, we can show that Allais and Ellsberg Preferences are irrational. As we saw in §2, there is a sequential dominance argument that the weak strict-preference version of Independence for Constant Prospects is a requirement of rationality, and this argument based on the following requirements of rationality:

- Continuity of Strict Preference
- The Weak Principle of Equiprobable Dominance
- The Principle of Prospect Guidance
- The Principle of Preferential Invulnerability

Since the weak strict-preference version of Independence for Constant Prospects rules out Allais and Ellsberg Preferences, this argument shows that Allais and Ellsberg Preferences are irrational. But, as we saw in §4, the weak strict-preference version of Independence for Constant Prospects is too weak to replace the Independence condition in the standard axiomatization of Expected Utility Theory.

The standard axiomatization needs the strong strict-preference version or the biconditional weak-preference version of Independence. There are, as we saw in §5 and §6, sequential dominance arguments that these conditions

are requirements of rationality. These arguments, however, are based on the following requirements of rationality:

- Completeness

- Continuity of Strict Preference

- The Strong Principle of Equiprobable Dominance

- Transitivity

- The Principle of Prospect Guidance

- The Principle of Preferential Invulnerability

Hence a drawback of these arguments is that they require notably stronger assumptions than the previous arguments.

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The main objection to the view that Independence is a requirement of rationality is the alleged rationality of Allais and Ellsberg Preferences. This influential objection can be rebutted with the help of a sequential dominance argument with fairly weak assumptions. And the versions of Independence which are strong enough to serve in the standard axiomatization of Expected Utility Theory can also be shown to be requirements of rationality with the help of sequential dominance arguments, but these arguments require notably stronger assumptions.

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