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Delayed point control of a reaction-diffusion PDE under discrete-time point measurements

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Abstract

We consider stabilization problem for reaction-diffusion PDEs with point actuations subject to a known constant delay. The point measurements are sampled in time and transmitted through a communication network with a time-varying delay. To compensate the input delay, we construct an observer for the future value of the state. Using a time-varying observer gain, we ensure that the estimation error vanishes exponentially with a desired decay rate if the delays and sampling intervals are small enough while the number of sensors is large enough. The convergence conditions are obtained using a Lyapunov–Krasovskii functional, which leads to linear matrix inequalities (LMIs). We design output-feedback point controllers in the presence of input delays using the above observer. The boundary controller is constructed using the backstepping transformation, which leads to a target system containing the exponentially decaying estimation error. The in-domain point controller is designed and analysed using an improved Wirtinger-based inequality. We show that both controllers can guarantee the exponential stability of the closed-loop system with an arbitrary decay rate smaller than that of the observer's estimation error.

Key words: Distributed parameter systems; Boundary control; Point control; Input delay; Networked control systems

1 Introduction

Networked control systems (NCSs) are composed of spatially distributed sensors, controllers, and actuators connected through a shared communication network. Such systems have become widespread due to great advantages they bring, such as long distance control, reduced system wiring, low cost, increased system agility, ease of reconfiguration, diagnosis, and maintenance [1,2]. The main theoretical challenges caused by networked architecture are data sampling and transmission delays, which have been extensively studied for finite-dimensional plants. In particular, predictors, originally proposed for continuous-time measurements [3–5], have been extended to discrete-time measurements for both static [6–9] and dynamic feedback [10–12].

Another way to compensate the input delay is to use an observer that predicts the future value of the state [13]. Such observer is a copy of the plant shifted in time with a correcting term that is proportional to the difference between the last available measurement and correspondingly delayed observer output. The stability analysis consists in proving the observer's robustness with respect to measurement delays. This idea can be used to analyse chain observers [14–17] and sequential predictors [18–21]. In [22], a time-varying injection gain was introduced in such an observer to improve its exponential convergence under delayed measurements. In [23], time-varying observer/controller gains were used to increase the period of a sampled-data system.

A constant input delay can be compensated in a reaction-diffusion system by representing it as a PDE–PDE cascade [24], which is analysed using the backstepping transformation [25,26]. However, this method is hard to combine with data sampling. In [27–29], some qualitative stability results are provided for sampled-data infinite-dimensional systems in general form. The same problem can be studied using Galerkin's method (see, e.g., [30–32] and references therein), which idea is to approximate the PDE by a finite-dimensional system capturing the dominant dynamics of the PDE. A drawback of such approach is the inherent loss of process information due to truncation before the controller design. Thus, it is difficult to guarantee the stability/performance of the original system.

Sampled-data observers under point measurements that enter the observer dynamics through shape functions were introduced for heat equations in [33]. The stability analy-

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sis of the error equation was provided using the time-delay approach to sampled-data and network-based control (see Chapter 7 of [34] and the references therein). In [33,35–38], Lyapunov–Krasovskii functionals were used to obtain LMI-based quantitative stability conditions for sampled-data control of parabolic PDEs. These works consider control applied through distributed shape functions. So far, it is not clear whether this direct Lyapunov–Krasovskii approach can be extended to state-feedback boundary or point control, since such control is represented by an unbounded operator (see Remark 2).

Some qualitative stability results for sampled-data state-feedback boundary control have been recently obtained in [39]. The analysis is based on the Fourier method and Input-to-State Stability ideas of [40]. Robustness of boundary stabilization with respect to input delay may probably be studied in a manner similar to [39] leading to qualitative results. Quantitative conditions for delayed (boundary or in-domain) point control under sampled in time and space measurements are missing. Such conditions are important for practical implementation of point control.

In this paper, we introduce an observer-based design for delayed boundary and in-domain point control of a reaction-diffusion PDE under the discrete-time point measurements. Inspired by the ideas of [13], we construct an observer whose correction term is the difference between the currently available measurement and an artificially delayed observer's output. This artificial delay essentially transforms the observer into a predictor and allows to compensate the input delay. By introducing a time-varying injection gain [22] and performing the stability analysis in a manner similar to [33,37], we show that the estimation error exponentially vanishes with any desired decay rate if the delays and time-sampling intervals are small enough while the number of sensors is large enough. Such an observer allows to eliminate the constant input delay.

We show that the above observer can be efficiently used for boundary and in-domain point control subject to input delay. First, the boundary control is studied using the backstepping transformation [25,26], which leads to a target system containing the exponentially decaying estimation error. Then, the point controllers represented by the Dirac delta functions are studied using the results of [41,42], which we improve by deriving a more precise Wirtinger-based inequality (Lemma 2). This allows to guarantee exponential stability with fewer actuators. We show that both the boundary control and the point control with large enough number of actuations guarantee the exponential stability of the closed-loop system with an arbitrary decay rate smaller than that of the observer's estimation error. Preliminary results on the boundary control have been published in [43].

1.1 Preliminaries

The following lemmas will be used in the proofs of the main results.

Lemma 1 (Wirtinger inequality [44]) *For $f\in\mathcal{H}^1(a,b)$,*

$$\|f\|^2_{L^2}\le\frac{(b-a)^2}{\pi^2}\|f'\|^2_{L^2}\qquad \text{if } f(a)=f(b)=0,$$
$$\|f\|^2_{L^2}\le\frac{4(b-a)^2}{\pi^2}\|f'\|^2_{L^2}\qquad \text{if } f(a)=0 \text{ or } f(b)=0.$$

Lemma 2 *For $f\in\mathcal{H}^1(a_1,a_2)$, $\nu>1$, and $i=1,2$,*

$$\|f\|^2_{L^2(a_1,a_2)}\le\nu(a_2-a_1)f^2(a_i)+\frac{4(a_2-a_1)^2\nu}{\pi^2(\nu-1)}\|f'\|^2_{L^2(a_1,a_2)}$$

Proof. From Wirtinger's inequality (Lemma 1),
$\int_{a_1}^{a_2}(f(x)-f(a_i))^2\,dx\le\frac{4(a_2-a_1)^2}{\pi^2}\int_{a_1}^{a_2}f_x^2(x)\,dx.$
Since $2f(x)f(a_i)\le\frac{1}{\nu}f^2(x)+\nu f^2(a_i)$, we obtain

$$\begin{aligned}&\int_{a_1}^{a_2}f^2(x)\,dx+\int_{a_1}^{a_2}f^2(a_i)\,dx\\ &\le\frac{4(a_2-a_1)^2}{\pi^2}\int_{a_1}^{a_2}f_x^2(x)\,dx+2\int_{a_1}^{a_2}f(x)f(a_i)\,dx\\ &\le\frac{4(a_2-a_1)^2}{\pi^2}\int_{a_1}^{a_2}f_x^2(x)\,dx+\frac{1}{\nu}\int_{a_1}^{a_2}f^2(x)\,dx+\nu\int_{a_1}^{a_2}f^2(a_i)\,dx.\end{aligned}$$

Reorganizing the terms, we prove the lemma. ∎

Remark 1 *In [41,42] the inequality*

$$\|f\|^2_{L^2(a_1,a_2)}\le2(a_2-a_1)f^2(a_i)+2(a_2-a_1)^2\|f'\|^2_{L^2(a_1,a_2)}$$

was used to study point control of PDEs. By employing Wirtinger's inequality, Lemma 2 provides tighter estimate: for $\nu=2$ it takes the form

$$\|f\|^2_{L^2(a_1,a_2)}\le2(a_2-a_1)f^2(a_i)+\frac{8(a_2-a_1)^2}{\pi^2}\|f'\|^2_{L^2(a_1,a_2)}$$

with the last term more than two times smaller than in [41,42].

Lemma 3 *For any $0\le\underline{a}<\overline{a}$, $0\le\underline{b}<\overline{b}$,*

$$\underline{a}+\underline{b}<K\le\overline{a}+\overline{b}\ \Leftrightarrow\ \exists\,\mu\in(0,1):\ \begin{cases}\underline{a}<\mu K\le\overline{a},\\ \underline{b}<(1-\mu)K\le\overline{b}.\end{cases}$$

Proof. The relation $\underline{a}+\underline{b}<K\le\overline{a}+\overline{b}$ implies $K-\overline{b}\le\overline{a}$ and $\underline{a}<K-\underline{b}$. Therefore,
$\exists\,\mu\colon\ \mu K\in(\underline{a},\overline{a}]\cap[K-\overline{b},K-\underline{b})\ne\varnothing.$
Since $0\le\underline{a}<\mu K<K-\underline{b}\le K$, we have $\mu\in(0,1)$. ∎

2 Boundary control

We consider the system schematically presented in Fig. 1. The plant is governed by the reaction-diffusion PDE

$$\begin{aligned}&z_t(x,t)=z_{xx}(x,t)+az(x,t),\\ &d_Lz(0,t)+(1-d_L)z_x(0,t)=0,\\ &d_Rz(1,t)+(1-d_R)z_x(1,t)=u(t-r),\end{aligned}\tag{1}$$

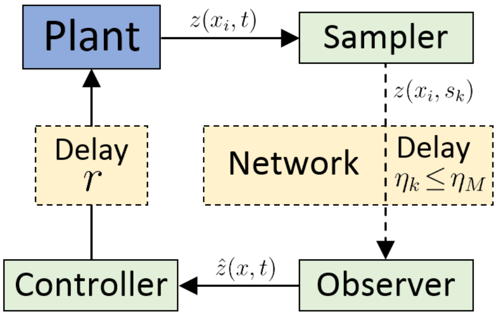

Fig. 1. System representation

where $z\colon [0,1]\times[0,\infty)\to\mathbb{R}$ is the state, u is the boundary control, and $r \geq 0$ is a known constant delay. Each constant $d_L, d_R \in \{0,1\}$ sets either the Dirichlet or the Neumann boundary condition. If $u(t)=0$, the plant is unstable if the reaction coefficient a is large enough.

Remark 2 *The robustness analysis of the boundary control with respect to the input delay is essentially more difficult than the one for distributed control as considered in [33,37]. To illustrate this, consider* (1) *with* $d_L = d_R = 1$ *and the state-feedback backstepping-based controller [25]. It can be shown that the backstepping transformation leads to the target system with the boundary delay*

$$\begin{aligned}
&w_t(x,t) = w_{xx}(x,t),\\
&w(0,t) = 0,\\
&w(1,t) = -\textstyle\int_0^1 l(1,y)[w(y,t)-w(y,t-r)]\,dy,
\end{aligned}$$

where l *is the kernel of the inverse transformation [25, Chapter 4]. It appears that finding an appropriate Lyapunov functional for delay-dependent stability in the case of boundary delay is an extremely difficult problem. Even Lyapunov-based ISS analysis in the case of boundary disturbances (which is the first step towards delay-dependent stability analysis) is problematic, since the disturbance is multiplied by an unbounded operator [40]. To avoid these difficulties, we compensate the input delay using an appropriate observer.*

We assume that N in-domain sensors provide point measurements of the state, which are sampled in time and transmitted through a network with a time-varying delay. That is, the values $z(x_i, s_k)$ are available at time t_k, where

$$\begin{aligned}
&0 \leq x_1 < x_2 < \ldots < x_N \leq 1,\\
&0 = s_0 < s_1 < \ldots, \quad s_{k+1}-s_k \leq h, \quad \lim s_k = \infty,\\
&t_k = s_k + \eta_k, \quad \eta_k \in [0,\eta_M]\colon\ t_k \leq t_{k+1}.
\end{aligned} \tag{2}$$

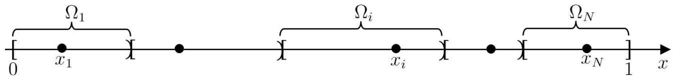

Fig. 2. Partition of $[0,1]$ with point measurements at x_i

2.1 *Observer/predictor construction*

We construct an observer to estimate the *future* value of the state: $\hat{z}(x,t) \approx z(x,t+r)$,

$$\begin{aligned}
&\hat{z}_t(x,t) = \hat{z}_{xx}(x,t) + a\hat{z}(x,t) + Le^{-\alpha_o(t+r-s_k)}\times\\
&\qquad \textstyle\sum_{i=1}^N b_i(x)[\hat{z}(x_i, s_k - r) - z(x_i, s_k)],\\
&\qquad\qquad t \in [t_k, t_{k+1}),\ k = 0,1,2,\ldots\\
&d_L\hat{z}(0,t) + (1-d_L)\hat{z}_x(0,t) = 0,\\
&d_R\hat{z}(1,t) + (1-d_R)\hat{z}_x(1,t) = u(t),\\
&\hat{z}(\cdot,t) = 0, \quad t \leq t_0.
\end{aligned} \tag{3}$$

The observer (3) is obtained by shifting the plant (1) in time by r and introducing a correction term. The time-varying injection gain $Le^{-\alpha_o(t+r-s_k)}$ will allow to guarantee that the observation error decays with the rate α_o [22]. As in [33], the shape functions $b_i \in L^2(0,1)$ are given by

$$\begin{cases} b_i(x) = 1, & x \in \Omega_i,\\ b_i(x) = 0, & x \notin \Omega_i, \end{cases} \tag{4}$$

where $\{\Omega_i\}$ is a partition of $[0,1]$ such that [1] $x_i \in \Omega_i$ (Fig. 2).

Due to (1), (3), the observation/prediction error $\bar{z}(x,t) = \hat{z}(x,t-r) - z(x,t)$ satisfies (if $u(t)=0$ for $t < t_0$)

$$\begin{aligned}
&\bar{z}_t = \bar{z}_{xx} + a\bar{z}, \quad t \in [0, t_0 + r),\\
&\bar{z}_t = \bar{z}_{xx} + a\bar{z} + Le^{-\alpha_o(t-s_k)}\textstyle\sum_{i=1}^N b_i(x)\bar{z}(x_i,s_k)\\
&\qquad\qquad t \in [t_k + r, t_{k+1}+r),\ k = 0,1,2,\ldots\\
&d_L\bar{z}(0,t) + (1-d_L)\bar{z}_x(0,t) = 0,\\
&d_R\bar{z}(1,t) + (1-d_R)\bar{z}_x(1,t) = 0,\\
&\bar{z}(\cdot,0) = -z(\cdot,0).
\end{aligned} \tag{5}$$

Now we study the well-posedness of (5) for the initial conditions $\bar{z}(\cdot,0) \in X$, where

$$X = \{w \in \mathcal{H}^1(0,1) \mid d_L w(0) = 0,\ d_R w(1) = 0\}$$

is the state space with the $\mathcal{H}^1$-norm. The system (5) can be presented in the form

$$\begin{aligned}
&\dot{\zeta}(t) + \mathcal{A}\zeta(t) = 0, \qquad t \in [0, t_0+r),\\
&\dot{\zeta}(t) + \mathcal{A}\zeta(t) = f_k(t), \quad t \in [t_k + r, t_{k+1}+r),\ k = 0,1,2,\ldots
\end{aligned} \tag{6}$$

[1] It is reasonable to choose $\{\Omega_i\}$ that minimizes $\max_i |\Omega_i|$

where $\zeta(t) = \bar{z}(\cdot, t)$ and

$$\mathcal{A}: D(\mathcal{A}) \to L^2(0,1)$$
$$\mathcal{A}w = -w'' - aw$$

is a linear operator on the Hilbert space

$$D(\mathcal{A}) = \left\{ w \in \mathcal{H}^2(0,1) \;\middle|\; \begin{array}{l} d_L w(0) + (1-d_L)w'(0) = 0 \\ d_R w(1) + (1-d_R)w'(1) = 0 \end{array} \right\}$$

with the inner product $(u,v)_{D(\mathcal{A})} = (Au, Av)_{L^2}$. The functions $f_k \in L^2(t_k + r, t_{k+1} + r; L^2(0,1))$ are given by

$$f_k(t) = Le^{-\alpha_o(t-s_k)} \sum_{i=1}^{N} b_i(\cdot)[\zeta(s_k)](x_i).$$

Note that each function f_k can be viewed as inhomogeneity, since $[\zeta(s_k)](x_i)$ are fixed values for $t \in [t_k + r, t_{k+1} + r)$.

A *strong solution* of (6) on $[0, T]$ is a function

$$\zeta \in L^2(0,T; D(\mathcal{A})) \cap C([0,T]; X), \tag{7}$$

such that $\dot{\zeta} \in L^2(0,T; L^2(0,1))$ and (6) holds almost everywhere on $[0,T]$.

The eigenfunctions of the Sturm-Liouville operator $\mathcal{A}$ form a complete orthonormal basis of $L^2(0,1)$ [45, Theorem 7.5.7]. Therefore, in a manner similar to the proof of [46, Theorem 7.7], one can show that (6) has a unique strong solution on $[0, t_0 + r]$ and on every $[t_k + r, t_{k+1} + r]$ for the initial conditions $\zeta(0) \in X$, $\zeta(t_k + r) \in X$. Taking the endpoint value of the solution on $[t_{k-1} + r, t_k + r]$ as the initial condition for the solution on $[t_k + r, t_{k+1} + r]$, we obtain the strong solution on $[0, \infty)$ for the initial condition $\zeta(0) = \bar{z}(\cdot, 0) \in X$.

Proposition 1 *For positive α_o, α_1 let there exist a scalar G and positive scalars S_i, R_i, p_i with $i = 1, 2$, such that*[2]

$$\Phi < 0, \quad \alpha_1 p_2 \le 2p_1, \quad \begin{bmatrix} R_2 & G \\ G & R_2 \end{bmatrix} \ge 0,$$

with $\Phi = \{\Phi_{ij}\}$ being a symmetric matrix composed from

$$\begin{aligned}
\Phi_{11} &= -R_1 e^{-\alpha_1 r} + S_1 + 2p_1(a + \alpha_o) + \alpha_1 \\
&\qquad - \pi^2(2p_1 - \alpha_1 p_2)\tfrac{\max\{d_L, d_R\}}{4 - 3d_L d_R}, \\
\Phi_{12} &= 1 - p_1 + p_2(a + \alpha_o), \\
\Phi_{13} &= R_1 e^{-\alpha_1 r}, \\
\Phi_{14} &= \Phi_{16} = p_1 L, \\
\Phi_{22} &= -2p_2 + r^2 R_1 + (h + \eta_M)^2 R_2, \\
\Phi_{24} &= \Phi_{26} = p_2 L, \\
\Phi_{33} &= -(R_1 + S_1 - S_2)e^{-\alpha_1 r} - R_2 e^{-\alpha_1 \tau_M}, \\
\Phi_{34} &= \Phi_{45} = (R_2 - G)e^{-\alpha_1 \tau_M}, \\
\Phi_{35} &= G e^{-\alpha_1 \tau_M}, \\
\Phi_{44} &= -2(R_2 - G)e^{-\alpha_1 \tau_M} - \alpha_1, \\
\Phi_{55} &= -(R_2 + S_2)e^{-\alpha_1 \tau_M}, \\
\Phi_{66} &= -\tfrac{\alpha_1 p_2 \pi^2}{4 \max_i |\Omega_i|^2},
\end{aligned}$$

where $\tau_M = h + \eta_M + r$. Then the system (5) *is exponentially stable with the decay rate α_o, i.e.,*

$$\|\bar{z}(\cdot, t)\|_{\mathcal{H}^1} \le \bar{C} e^{-\alpha_o t} \|\bar{z}(\cdot, 0)\|_{\mathcal{H}^1}, \quad t \ge 0 \tag{8}$$

for some $\bar{C} > 0$. Moreover,

$$\|\sigma(\cdot, t)\|_{L^2} \le C_\sigma e^{-\alpha_o t} \|z(\cdot, 0)\|_{\mathcal{H}^1}, \quad t \ge 0 \tag{9}$$

for some $C_\sigma > 0$, where

$$\sigma(x, t) = \textstyle\sum_{i=1}^N b_i(x) \bar{z}(x_i, t), \quad x \in [0,1], \ t \ge 0. \tag{10}$$

Proof is given in Appendix A.

Remark 3 *Using the standard arguments for time-delay systems [34], one can show that the LMIs of Proposition 1 are feasible for any given α_o and appropriate L if the delays r, η_M, sampling h, and the maximum subdomain length $\max_i |\Omega_i|$ are small enough (i.e., the number of sensors N is large enough).*

Remark 4 *We consider synchronously sampled measurements $z(x_i, s_k)$ because the proof of Proposition 1 uses the Halanay inequality (see* (A.8)*). The proof can be modified to cope with asynchronous sampling but this will lead to quite restrictive stability conditions (see Remark 1 of [37]).*

2.2 Boundary controller synthesis

A boundary controller for (1) is constructed based on the estimation $\hat{z}$ using the backstepping transformation [25,26]

$$w(x, t) = \hat{z}(x, t) - \int_0^x k(x, y)\hat{z}(y, t)\, dy, \tag{11}$$

where $k(x, y)$ is the solution of

$$\begin{aligned}
&k_{xx}(x, y) - k_{yy}(x, y) = \lambda k(x, y), \\
&k(x, x) = -\tfrac{\lambda}{2} x, \\
&d_L k(x, 0) + (1 - d_L) k_y(x, 0) = 0
\end{aligned} \tag{12}$$

[2] MATLAB codes for solving the LMIs are available at `https://github.com/AntonSelivanov/Aut18`

with some $\lambda \in \mathbb{R}$. Such kernel $k(x, y)$ exists and is bounded for any λ (see, e.g., [26, Theorem 2.1]). Let

$$\begin{aligned} &u(t) = \int_0^1 k(1, y)\hat{z}(y, t)\, dy && \text{if } d_R = 1, \\ &u(t) = k(1, 1)\hat{z}(1, t) + \int_0^1 k_x(1, y)\hat{z}(y, t)\, dy && \text{if } d_R = 0 \end{aligned} \tag{13}$$

for $t \geq t_0$ and $u(t) = 0$ for $t < t_0$. Then, performing calculations similar to those in [26, Chapter 2.2], we have

$$\begin{aligned} &w_t(x, t) = w_{xx}(x, t) - (\lambda - a)w(x, t) + v(x, t), \\ &d_L w(0, t) + (1 - d_L)w_x(0, t) = 0, \\ &d_R w(1, t) + (1 - d_R)w_x(1, t) = 0, \\ &w(\cdot, t_0) = 0 \end{aligned} \tag{14}$$

for $t \geq t_0$, where

$$v(x, t) = Le^{-\alpha_o(t+r-s_k)} \times \left[\sigma(x, s_k) - \int_0^x k(x, y)\sigma(y, s_k)\, dy\right],\ t \in [t_k, t_{k+1})$$

with $\sigma(x, t)$ defined in (10). The proof of well-posedness of (14) is similar to that of (5). Since (11) is invertible, this implies the well-posedness of (3) and, consequently, of (1) (since $z(x, t) = \hat{z}(x, t - r) - \bar{z}(x, t)$).

Proposition 2 *Under the assumptions of Proposition 1, if*

$$\lambda > \alpha_c + a - \frac{\max\{d_L, d_R\}\pi^2}{4 - 3d_L d_R + \pi^2} \tag{15}$$

with $\alpha_c > 0$, then the solutions of the system (14) *satisfy*

$$\|w(\cdot, t)\|_{\mathcal{H}^1} \leq C_w e^{-\min\{\alpha_o, \alpha_c\}t}\|z(\cdot, 0)\|_{\mathcal{H}^1}, \quad t \geq t_0 \tag{16}$$

with some $C_w > 0$.

Proof is given in Appendix B.

Corollary 1 *If the assumptions of Proposition 1 are satisfied, the observer-based boundary controller* (3), (12), (13) *with λ satisfying* (15) *exponentially stabilizes the system* (1) *with the decay rate $\min\{\alpha_o, \alpha_c\}$, i.e.,*

$$\|z(\cdot, t)\|_{\mathcal{H}^1} \leq C_z e^{-\min\{\alpha_o, \alpha_c\}t}\|z(\cdot, 0)\|_{\mathcal{H}^1}, \quad t \geq 0 \tag{17}$$

with some $C_z > 0$.

Proof. The transformation (11) has an inverse, which is bounded in $\mathcal{H}^1$ norm (see, e.g., [26]). Therefore, there exists a constant $\tilde{C}$ such that

$$\|\hat{z}(\cdot, t)\|_{\mathcal{H}^1} \leq \tilde{C}\|w(\cdot, t)\|_{\mathcal{H}^1} \overset{(16)}{\leq} \tilde{C}C_w e^{-\min\{\alpha_o, \alpha_c\}t}\|z(\cdot, 0)\|_{\mathcal{H}^1}$$

for $t \geq t_0$. Since $z(x, t) = \hat{z}(x, t - r) - \bar{z}(x, t)$, the latter and (8) imply (17). ∎

Remark 5 *One can achieve an arbitrary decay rate in* (17) *if the delays and time-sampling intervals are small enough while the number of sensors is large enough. This follows from Remark 3 and the solvability of* (12) *for any λ satisfying* (15).

3 Point control

In this section, we study point control modelled by the Dirac delta function. We consider the system schematically presented in Fig. 1 with the plant governed by

$$\begin{aligned} &z_t(x, t) = z_{xx}(x, t) + az(x, t) + \textstyle\sum_{j=1}^M \delta(x - \bar{x}_j)u_j(t - r), \\ &d_L z(0, t) + (1 - d_L)z_x(0, t) = 0, \\ &d_R z(1, t) + (1 - d_R)z_x(1, t) = 0, \end{aligned} \tag{18}$$

where $z\colon [0, 1] \times [0, \infty) \to \mathbb{R}$ is the state, $\delta(x)$ is the Dirac delta function representing point actuations, $r \geq 0$ is a known constant delay, and u_j are the control signals applied at $0 \leq \bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_M \leq 1$. Note that $\bar{x}_1 = 0$ or $\bar{x}_M = 1$ model boundary actuations. Each constant $d_L, d_R \in \{0, 1\}$ sets either the Dirichlet or the Neumann boundary condition (if $d_L = 1$ then $\bar{x}_1 \neq 0$, if $d_R = 1$ then $\bar{x}_N \neq 1$).

Like in Section 2, the values of $z(x_i, s_k)$ are available to the observer at time t_k, where x_i, s_k, and t_k satisfy (2). Note that x_i and $\bar{x}_j$ are not related, that is, the sensors and actuators are not necessarily collocated.

We construct an observer similar to (3), which estimates the future value of the state: $\hat{z}(x, t) \approx z(x, t + r)$,

$$\begin{aligned} &\hat{z}_t(x, t) = \hat{z}_{xx}(x, t) + a\hat{z}(x, t) + \textstyle\sum_{j=1}^M \delta(x - \bar{x}_j)u_j(t) \\ &\quad + Le^{-\alpha_o(t+r-s_k)} \textstyle\sum_{i=1}^N b_i(x)[\hat{z}(x_i, s_k - r) - z(x_i, s_k)], \\ &\qquad t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \ldots \\ &d_L \hat{z}(0, t) + (1 - d_L)\hat{z}_x(0, t) = 0, \\ &d_R \hat{z}(1, t) + (1 - d_R)\hat{z}_x(1, t) = 0, \\ &\hat{z}(\cdot, t) = 0, \quad t \leq t_0 \end{aligned} \tag{19}$$

with the shape functions $b_i \in L^2(0, 1)$ given in (4). The control signals are chosen as

$$u_j(t) = -K_j \hat{z}(\bar{x}_j, t), \quad j \in 1{:}M. \tag{20}$$

In view of (18)–(20), the observation/prediction error $\bar{z}(x, t) = \hat{z}(x, t - r) - z(x, t)$ satisfies (5). Therefore, (8) and (9) hold under the assumptions of Proposition 1.

The solutions of (19), (20) should be understood in the weak sense. Namely, define the state space

$$X = \left\{w \in \mathcal{H}^1(0, 1) \,|\, d_L w(0) = 0,\ d_R w(1) = 0\right\}$$

with the $\mathcal{H}^1$-norm. Let X^* be its dual space. A *weak solution* of (19), (20) on $[t_0, T]$ is a function

$$\hat{z} \in L^2(t_0, T; X) \cap C([t_0, T]; L^2(0, 1)), \tag{21}$$

such that $\hat{z}_t \in L^2(t_0, T; X^*)$ and

$$\begin{aligned} &\tfrac{d}{dt} \int_0^1 \hat{z}(\xi, t)\varphi(\xi)\, d\xi = -\int_0^1 \hat{z}_\xi(\xi, t)\varphi_\xi(\xi)\, d\xi \\ &\quad + a\int_0^1 \hat{z}(\xi, t)\varphi(\xi)\, d\xi - \textstyle\sum_{j=1}^M K_j \hat{z}(\bar{x}_j, t)\varphi(\bar{x}_j) \\ &\qquad + \int_0^1 g(\xi, t)\varphi(\xi)\, d\xi \end{aligned} \tag{22}$$

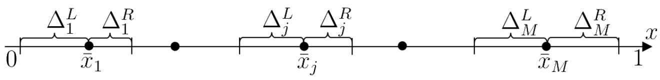

Fig. 3. Partition of $[0,1]$ for point controllers

for any $\varphi \in X$ and almost all $t \in [t_0, T]$, where $g \in L^{\infty}_{loc}(t_0, T; L^2(0,1))$ is given by

$$g(\xi,t) = Le^{-\alpha_o(t+r-s_k)} \sum_{i=1}^{N} b_i(\xi)\bar{z}(x_i, s_k), \quad t \in [t_k, t_{k+1}).$$

Here, $\bar{z}$ is the strong solution of (5), therefore, g is a well-defined inhomogeneity.

The condition (22) is motivated by the integration-by-parts formula. Using the standard Galerkin approximation procedure (see, e.g., [42]), one can show that (19), (20) has a unique weak solution on $[t_0, \infty)$ for any initial conditions $\hat{z}(\cdot, t_0) \in L^2(0,1)$, including $\hat{z}(\cdot, t_0) = 0$ required in (19).

Proposition 3 *Under the assumptions of Proposition 1, if*

$$\begin{aligned} &2(a+\alpha_c)\max_{j\in 1:M} |\Delta_j^L|^2 < \tfrac{\pi^2}{4}, \\ &2(a+\alpha_c)\max_{j\in 1:M} |\Delta_j^R|^2 < \tfrac{\pi^2}{4}, \end{aligned} \tag{23}$$

where $\{\Delta_j^L, \Delta_j^R\}$ *is a partition*[3] *of* $[0,1]$ *depicted in Fig. 3, then the solutions of the system* (19) *under the controllers* (20) *with*

$$\begin{aligned} &K_j \in \left(2(a+\alpha_c)|\Delta_j^L|, \tfrac{\pi^2}{4|\Delta_j^L|}\right] && \text{if } |\Delta_j^R| = 0, \\ &K_j \in \left(2(a+\alpha_c)|\Delta_j^R|, \tfrac{\pi^2}{4|\Delta_j^R|}\right] && \text{if } |\Delta_j^L| = 0, \\ &K_j \in \left(2(a+\alpha_c)\left(|\Delta_j^L|+|\Delta_j^R|\right), \tfrac{\pi^2}{4}\left(\tfrac{1}{|\Delta_j^L|}+\tfrac{1}{|\Delta_j^R|}\right)\right] && \text{otherwise,} \end{aligned} \tag{24}$$

satisfy

$$\|\hat{z}(\cdot,t)\|_{L^2} \le \hat{C}e^{-\min\{\alpha_o,\alpha_c\}t}\|z(\cdot,0)\|_{\mathcal{H}^1}, \quad t \ge t_0 \tag{25}$$

with some $\hat{C} > 0$.

Proof is given in Appendix C.

The strong solution of (5) is also its weak solution. Since $z(x,t) = \hat{z}(x,t-r) - \bar{z}(x,t)$, there exists a weak solution of the closed-loop system (18)–(20). Using (8), (25) and the representation $z(x,t) = \hat{z}(x,t-r) - \bar{z}(x,t)$, we obtain the following corollary.

Corollary 2 *If the assumptions of Proposition 1 are satisfied and* (23) *is true, then the observer-based point controller* (19), (20), (24) *exponentially stabilizes the system* (18) *with the decay rate* $\min\{\alpha_o, \alpha_c\}$, *i.e.,*

$$\|z(\cdot,t)\|_{L^2} \le C_z e^{-\min\{\alpha_o,\alpha_c\}t}\|z(\cdot,0)\|_{\mathcal{H}^1}, \quad t \ge 0 \tag{26}$$

with some $C_z > 0$.

[3] In view of (23), it is reasonable to choose $|\Delta_j^R| = |\Delta_{j+1}^L|$

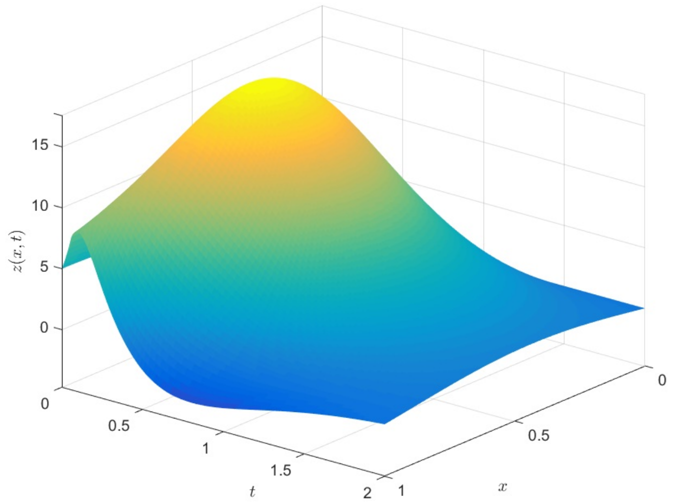

Fig. 4. Boundary control: The state $z(x,t)$

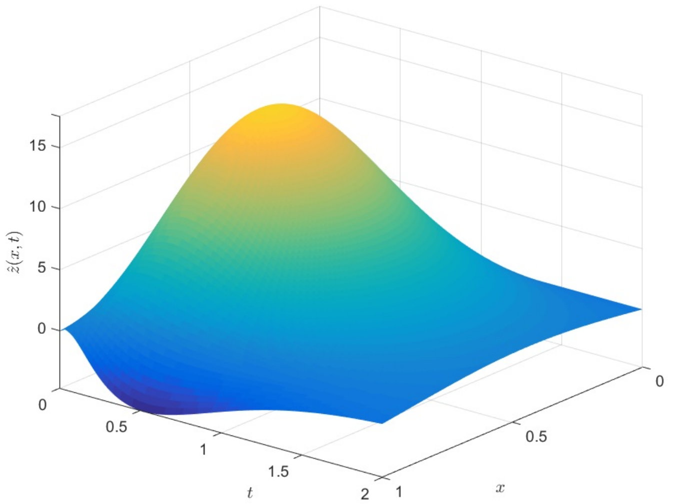

Fig. 5. Boundary control: The estimation/prediction $\hat{z}(x,t)$

Remark 6 *Similarly to the boundary control case, one can achieve an arbitrary decay rate in* (26) *if the delays and time-sampling intervals are small enough while the number of sensors and point actuators is large enough. This follows from Remark 3 and the feasibility of* (23) *for small enough subdomain lengths, i.e., for large enough number of actuation points.*

4 Example

4.1 Boundary control

Consider the plant (1) with $a = 10$, $r = 0.05$, $d_L = 1$, $d_R = 0$, which is unstable if $u(t-r) = 0$. Assume that there are $N = 10$ in-domain sensors transmitting point measurements at $x_i = \frac{2i-1}{2N}$, $i \in 1\!:\!N$ (the centres of $\Omega_i = [\frac{i-1}{N}, \frac{i}{N})$) with the sampling period $h = 0.01$ and time-varying network delay $\eta_k \le \eta_M = 0.01$. The conditions

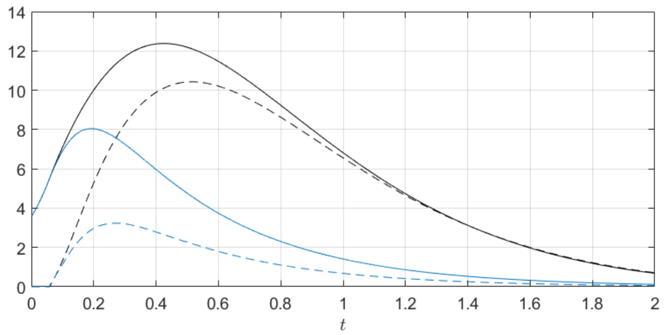

Fig. 6. $\|z(\cdot,t)\|_{L^2}$ (solid line) and $\|\hat{z}(\cdot,t-r)\|_{L^2}$ (dashed line) under boundary (black line) and point (blue line) control

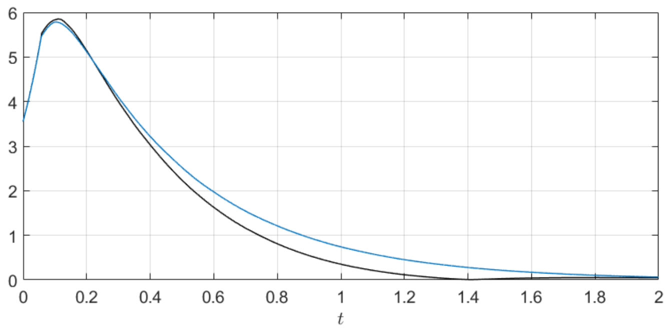

Fig. 7. The error $\|z(\cdot,t)-\hat{z}(\cdot,t-r)\|_{L^2}$ under boundary (black line) and point (blue line) control

of Proposition 1 are satisfied with $L=-10$, $\alpha_o=0.5$, $\alpha_1=1$. Therefore, the observer (3) provides a prediction of the state that converges with the rate α_o. Taking $\alpha_c=0.5$, we derive the boundary controller (13) with

$$k(1,1)=-\frac{\lambda}{2},\quad k_x(1,y)=-\lambda y\frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2},$$

where $\lambda=a+\alpha_o-\pi^2/(4+\pi^2)+10^{-5}$ and I_2 is the Modified Bessel Function. Corollary 1 guarantees exponential stability of the plant with the decay rate $\min\{\alpha_o,\alpha_c\}=0.5$.

The numerical simulations were performed with

$$z(x,0)=5\sin\left(\frac{\pi x}{2}\right)$$

and randomly chosen $\eta_k\in[0,0.01]$ such that $t_k\le t_{k+1}$. The results are presented in Figs. 4–7.

4.2 Point control

Consider the plant (18) and the observer (19) with the same parameters as in Section 4.1. The conditions (23) are satisfied with $\alpha_c=0.5$ if the two point actuators (20) are located at $\bar{x}_1=0.25$, $\bar{x}_2=0.75$ and the partition of $[0,1]$ is chosen to be uniform, i.e., $\Delta_j^L=\Delta_j^R=0.25$ for $j=1,2$. Then Corollary 2 guarantees the exponential stability of the closed-loop system (18)–(20) with

$$K_j=\frac{\pi^2}{4}\left(\frac{1}{|\Delta_j^L|}+\frac{1}{|\Delta_j^R|}\right)=2\pi^2,\quad j=1,2.$$

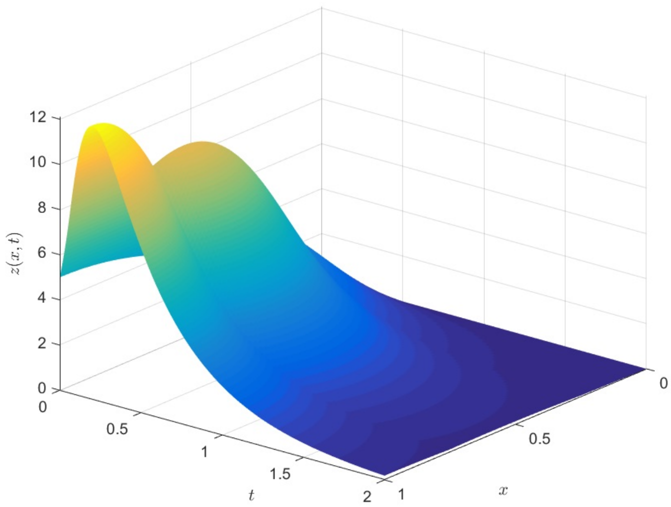

Fig. 8. Point control: The state $z(x,t)$

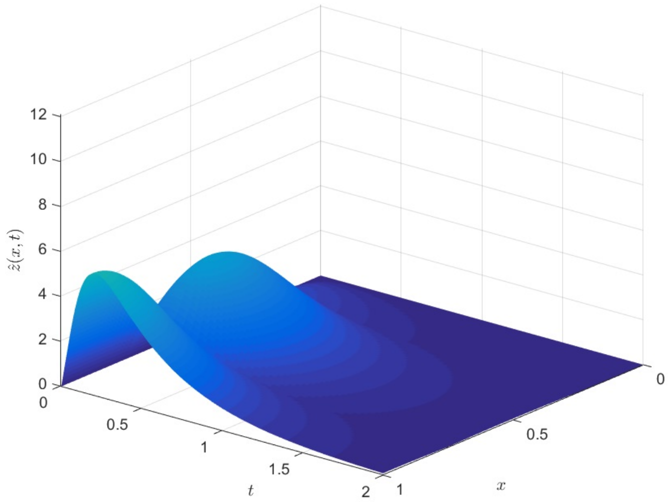

Fig. 9. Point control: The estimation/prediction $\hat{z}(x,t)$

The numerical simulations were performed with the same initial conditions as in Section 4.1 and randomly chosen $\eta_k\in[0,0.01]$ such that $t_k\le t_{k+1}$. The results are presented in Figs. 6–9. In Fig. 6 one can see that the point control with two actuators leads to smaller norm overshoot than the boundary control, which requires only one actuator.

5 Conclusion

Delayed boundary and in-domain point controllers for a reaction-diffusion PDE under the discrete-time point measurements were designed by employing observers that estimate the future value of the state. Quantitative LMI-based conditions were provided for the number of point measurements/actuations and the maximum delays and time-sampling intervals that preserve the stability of the closed-loop system. The results can be extended to smooth time-varying delays as considered in [21] and to sequential predictors. A challenging direction for the future research may be sampled-data implementation of the presented controllers.

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A Proof of Proposition 1

Let $\zeta(x,t) = e^{\alpha_o t}\bar{z}(x,t)$. For $t \geq t_0 + r$, (5) implies

$$\begin{aligned}
&\zeta_t = \zeta_{xx} + (a+\alpha_o)\zeta + L\textstyle\sum_{i=1}^N b_i(x)\zeta(x_i, t-\tau(t)),\\
&d_L\zeta(0,t) + (1-d_L)\zeta_x(0,t) = 0,\\
&d_R\zeta(1,t) + (1-d_R)\zeta_x(1,t) = 0,
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
&\tau(t) = t - s_k, \quad t \in [t_k + r, t_{k+1} + r),\ k = 0,1,2,\dots\\
&r \leq \tau(t) \leq \tau_M = r + h + \eta_M.
\end{aligned}$$

Consider the Lyapunov–Krasovskii functional

$$V_\zeta = V_1 + V_2 + V_{S1} + V_{R1} + V_{S2} + V_{R2}, \tag{A.2}$$

where

$$\begin{aligned}
&V_1 = \int_0^1 \zeta^2(x,t)\,dx,\\
&V_2 = p_2\int_0^1 \zeta_x^2(x,t)\,dx,\\
&V_{S1} = S_1\int_0^1\int_{t-r}^t e^{-\alpha_1(t-s)}\zeta^2(x,s)\,ds\,dx,\\
&V_{R1} = rR_1\int_0^1\int_{-r}^0\int_{t+\theta}^t e^{-\alpha_1(t-s)}\zeta_s^2(x,s)\,ds\,d\theta\,dx,\\
&V_{S2} = S_2\int_0^1\int_{t-\tau_M}^{t-r} e^{-\alpha_1(t-s)}\zeta^2(x,s)\,ds\,dx,\\
&V_{R2} = (h+\eta_M)R_2\int_0^1\int_{-\tau_M}^{-r}\int_{t+\theta}^t e^{-\alpha_1(t-s)}\zeta_s^2(x,s)\,ds\,d\theta\,dx.
\end{aligned}$$

Similarly to [47], we formally set $\zeta(\cdot,t) = \zeta(\cdot,0)$ for $t<0$ so that V_ζ is defined on [4] $[t_0 + r - \tau_M, \infty)$. Note that for the strong solution (7), the functional V_ζ is well-defined and continuous. For $t \geq t_0 + r$,

$$\begin{aligned}
&\dot V_1 + \alpha_1 V_1 = 2\textstyle\int_0^1 \zeta\zeta_t + \alpha_1\int_0^1\zeta^2,\\
&\dot V_2 + \alpha_1 V_2 = 2p_2\textstyle\int_0^1 \zeta_x\zeta_{xt} + \alpha_1 p_2\int_0^1\zeta_x^2,\\
&\dot V_{S1} + \alpha_1 V_{S1} = S_1\textstyle\int_0^1\zeta^2 - S_1e^{-\alpha_1 r}\int_0^1\zeta^2(x,t-r)\,dx,\\
&\dot V_{S2} + \alpha_1 V_{S2} = S_2e^{-\alpha_1 r}\textstyle\int_0^1\zeta^2(x,t-r)\,dx\\
&\qquad\qquad -S_2e^{-\alpha_1\tau_M}\textstyle\int_0^1\zeta^2(x,t-\tau_M)\,dx.
\end{aligned}$$

Using Jensen's inequality [48, Proposition B.8],

$$\begin{aligned}
&\dot V_{R1} + \alpha_1 V_{R1} =\\
&r^2R_1\int_0^1\zeta_t^2(x,t)\,dx - rR_1\int_0^1\int_{t-r}^t e^{-\alpha_1(t-s)}\zeta_s^2(x,s)\,ds\,dx\\
&\leq r^2R_1\int_0^1\zeta_t^2(x,t)\,dx - R_1e^{-\alpha_1 r}\int_0^1\big(\zeta(x,t)-\zeta(x,t-r)\big)^2dx.
\end{aligned}$$

Jensen's inequality and reciprocally convex approach [49, Theorem 1] allow to obtain [5]

$$\begin{aligned}
&\dot V_{R2} + \alpha_1 V_{R2} \leq (h+\eta_M)^2R_2\textstyle\int_0^1\zeta_t^2(x,t)\,dx - e^{-\alpha_1\tau_M}\times\\
&\int_0^1\begin{bmatrix}\zeta(x,t-r)-\zeta(x,t-\tau(t))\\ \zeta(x,t-\tau(t))-\zeta(x,t-\tau_M)\end{bmatrix}^T\begin{bmatrix}R_2 & G\\ G & R_2\end{bmatrix}\begin{bmatrix}\zeta(x,t-r)-\zeta(x,t-\tau(t))\\ \zeta(x,t-\tau(t))-\zeta(x,t-\tau_M)\end{bmatrix}dx.
\end{aligned}$$

Instead of replacing ζ_t with the right-hand side of (A.1), we employ the descriptor method [50]. Namely, (A.1) implies

$$\begin{aligned}
0 = 2\textstyle\int_0^1 &\big[p_1\zeta(x,t) + p_2\zeta_t(x,t)\big]\big[-\zeta_t(x,t) + \zeta_{xx}(x,t)\\
&+(a+\alpha_o)\zeta(x,t) + L\textstyle\sum_{i=1}^N b_i(x)\zeta(x_i,t-\tau(t))\big]\,dx,
\end{aligned}$$

[4] This is required for (A.8) to be meaningful

[5] Similar calculation is given in [37, (A.1)] in more detail

which right-hand side will be added to $\dot{V}_\zeta$. Denoting

$$\kappa(x,t)=\zeta(x_i,t)-\zeta(x,t), \quad x\in\Omega_i,\ i\in 1{:}N \tag{A.3}$$

and using (4), the latter can be rewritten as

$$\begin{aligned}0=2\textstyle\sum_{i=1}^N\int_{\Omega_i}[p_1\zeta+p_2\zeta_t][-\zeta_t+\zeta_{xx}+(a+\alpha_o)\zeta\\ +L\kappa(x,t-\tau(t))+L\zeta(x,t-\tau(t))]\,dx.\end{aligned} \tag{A.4}$$

Integrating by parts and taking into account the boundary conditions with $d_L, d_R\in\{0,1\}$, we obtain

$$\begin{aligned}&2p_1\textstyle\sum_{i=1}^N\int_{\Omega_i}\zeta\zeta_{xx}=2p_1\int_0^1\zeta\zeta_{xx}\\ &\quad=2p_1\zeta\zeta_x\big|_0^1-2p_1\int_0^1(\zeta_x)^2=-2p_1\textstyle\sum_{i=1}^N\int_{\Omega_i}\zeta_x^2,\\ &2p_2\textstyle\sum_{i=1}^N\int_{\Omega_i}\zeta_t\zeta_{xx}=2p_2\int_0^1\zeta_t\zeta_{xx}\\ &\quad=2p_2\zeta_t\zeta_x\big|_0^1-2p_2\int_0^1\zeta_{xt}\zeta_x=-2p_2\int_0^1\zeta_{xt}\zeta_x=-\dot{V}_2.\end{aligned} \tag{A.5}$$

Since $\alpha_1p_2\le 2p_1$, Wirtinger's inequality (Lemma 1) implies

$$\begin{aligned}0\le(2p_1-\alpha_1p_2)\max\{d_L,d_R\}\times\\ \left[\int_0^1\zeta_x^2(x,t)\,dx-\tfrac{\pi^2}{4-3d_Ld_R}\int_0^1\zeta^2(x,t)\,dx\right].\end{aligned} \tag{A.6}$$

Denote $[x_i^L,x_i^R)=\Omega_i$. Since $\kappa(x_i,t)=0$ and $\kappa_x=-\zeta_x$,

$$\begin{aligned}\int_{\Omega_i}\kappa^2=\int_{x_i^L}^{x_i}\kappa^2+\int_{x_i}^{x_i^R}\kappa^2\overset{\text{Lem.1}}{\le}\tfrac{4|\Omega_i|^2}{\pi^2}\left[\int_{x_i^L}^{x_i}\zeta_x^2+\int_{x_i}^{x_i^R}\zeta_x^2\right]\\ \le\tfrac{4\max_i|\Omega_i|^2}{\pi^2}\int_{\Omega_i}\zeta_x^2.\end{aligned} \tag{A.7}$$

Therefore, for any $\alpha_2>0$,

$$\begin{aligned}&-\alpha_2\sup_{\theta\in[t-\tau_M,t]}V_\zeta(\theta)\le-\alpha_2V_\zeta(t-\tau(t))\\ &\le-\alpha_2\sum_{i=1}^N\int_{\Omega_i}\zeta^2(x,t-\tau(t))dx-\alpha_2p_2\sum_{i=1}^N\int_{\Omega_i}\zeta_x^2(x,t-\tau(t))dx\\ &\le-\alpha_2\sum_{i=1}^N\int_{\Omega_i}\zeta^2(x,t-\tau(t))dx\\ &\qquad-\frac{\alpha_2p_2\pi^2}{4\max_i|\Omega_i|^2}\sum_{i=1}^N\int_{\Omega_i}\kappa^2(x,t-\tau(t))dx.\end{aligned}$$

Consider the matrix Ψ that coincides with Φ except for

$$\begin{aligned}&\Psi_{44}=-2(R_2-G)e^{-\alpha_1\tau_M}-\alpha_2,\\ &\Psi_{66}=-\tfrac{\alpha_2p_2\pi^2}{4\max_i|\Omega_i|^2}.\end{aligned}$$

Since $\Phi<0$ is a strict inequality, $\Psi<0$ holds for large enough $\alpha_2<\alpha_1$. By adding the right-hand sides of (A.4), (A.6) to $\dot{V}_\zeta$ and using (A.5), we obtain

$$\begin{aligned}&\dot{V}_\zeta+\alpha_1V_\zeta-\alpha_2\sup\nolimits_{\theta\in[t-\tau_M,t]}V_\zeta(\theta)\\ &\quad\le\textstyle\sum_{i=1}^N\int_{\Omega_i}\psi^T(x,t)\Psi\psi(x,t)\,dx\\ &\qquad-(1-\max\{d_L,d_R\})(2p_1-\alpha_1p_2)\|\zeta_x(\cdot,t)\|_{L^2}^2\end{aligned}$$

with $\psi(x,t)=\operatorname{col}\{\zeta,\zeta_t,\zeta(x,t-r),\zeta(x,t-\tau(t)),\zeta(x,t-\tau_M),\kappa(x,t-\tau(t))\}$. Since $\Psi<0$ and $2p_1\ge\alpha_1p_2$,

$$\dot{V}_\zeta(t)\le-\alpha_1V_\zeta(t)+\alpha_2\sup_{\theta\in[t-\tau_M,t]}V_\zeta(\theta),\quad t\ge t_0+r.$$

The Halanay inequality [34, Lemma 4.2] implies

$$V_\zeta(t)\le e^{-\bar{\alpha}(t-t_0-r)}\sup_{\theta\in[t_0+r-\tau_M,t_0+r]}V_\zeta(\theta),\quad t\ge t_0+r, \tag{A.8}$$

where $\bar{\alpha}$ is the unique positive solution of $\bar{\alpha}=\alpha_1-\alpha_2e^{\bar{\alpha}\tau_M}$.

For $t\in[0,t_0+r)$, (5) implies (A.1) with $L=0$. Then calculations similar to the above imply $\dot{V}_\zeta(t)\le\delta V_\zeta(t)$ for $t\in[0,t_0+r)$ with large enough δ. Therefore,

$$V_\zeta(t)\le e^{\delta t}V_\zeta(0)\le e^{\delta(t_0+r)}V_\zeta(0),\quad t\in[0,t_0+r].$$

Moreover, since we set $\zeta(\cdot,t)=\zeta(\cdot,0)$ for $t<0$,

$$V_\zeta(t)=V_\zeta(0),\quad t\in[t_0+r-\tau_M,0].$$

Consequently,

$$\sup_{\theta\in[t_0+r-\tau_M,t_0+r]}V_\zeta(\theta)\le e^{\delta(t_0+r)}V_\zeta(0)\le C_V\|\zeta(\cdot,0)\|_{\mathcal{H}^1}^2$$

for some $C_V>0$. Recalling that $\zeta(x,t)=e^{\alpha_ot}\bar{z}(x,t)$, the latter and (A.8) yield

$$\begin{aligned}\|\bar{z}(\cdot,t)\|_{\mathcal{H}^1}^2&=e^{-2\alpha_ot}\|\zeta(\cdot,t)\|_{\mathcal{H}^1}^2\le\tfrac{e^{-2\alpha_ot}}{\min\{1,p_2\}}V_\zeta(t)\\ &\le\bar{C}^2e^{-2\alpha_ot}\|\zeta(\cdot,0)\|_{\mathcal{H}^1}^2=\bar{C}^2e^{-2\alpha_ot}\|\bar{z}(\cdot,0)\|_{\mathcal{H}^1}^2\end{aligned}$$

for $t\ge0$ with some $\bar{C}>0$. This proves (8).

Using the notation (A.3), $b_i(x)\zeta(x_i,t)=b_i(x)(\zeta(x,t)+\kappa(x,t))$ for any $x\in[0,1]$. Therefore,

$$\begin{aligned}&e^{2\alpha_ot}\int_0^1\sigma^2=\int_0^1\left(\textstyle\sum_{i=1}^Nb_i(x)\zeta(x_i,t)\right)^2dx\\ &\quad=\int_0^1(\zeta(x,t)+\kappa(x,t))^2\left(\textstyle\sum_{i=1}^Nb_i(x)\right)^2dx\\ &\quad\le2\int_0^1\kappa^2+2\int_0^1\zeta^2\overset{(A.7)}{\le}2\max\left\{1,\tfrac{4\max_i|\Omega_i|^2}{p_2\pi^2}\right\}V_\zeta(t)\\ &\quad\le C_\sigma^2\|\bar{z}(\cdot,0)\|_{\mathcal{H}^1}^2=C_\sigma^2\|z(\cdot,0)\|_{\mathcal{H}^1}^2\end{aligned}$$

for $t\ge0$ with some $C_\sigma>0$. This proves (9).

B Proof of Proposition 2

Consider $V_w=V_{w1}+V_{w2}$ with

$$V_{w1}=\int_0^1w^2(x,t)\,dx,\quad V_{w2}=\int_0^1w_x^2(x,t)\,dx.$$

We have

$$\dot{V}_{w1}=2\int_0^1ww_{xx}-2(\lambda-a)\int_0^1w^2+2\int_0^1wv.$$

Since

$$\begin{aligned}&2\int_0^1ww_{xx}=-2\int_0^1w_x^2 &&\text{(integration by parts)}\\ &2\int_0^1wv\le2\mu\int_0^1w^2+\tfrac{1}{2\mu}\int_0^1v^2 &&\text{(Young's inequality)}\end{aligned}$$

with an arbitrary $\mu>0$, we obtain

$$\dot{V}_{w1}\le-2\int_0^1w_x^2-2(\lambda-a-\mu)\int_0^1w^2+\tfrac{1}{2\mu}\int_0^1v^2.$$

Using integration by parts, we have

$$\begin{aligned}\dot{V}_{w2}&=2\int_0^1w_xw_{xt}=-2\int_0^1w_{xx}w_t\\ &=-2\int_0^1w_{xx}^2+2(\lambda-a)\int_0^1w_{xx}w-2\int_0^1w_{xx}v.\end{aligned}$$

Since

$2(\lambda - a)\int_0^1 w_{xx}w = -2(\lambda - a)\int_0^1 w_x^2 \qquad \text{(int. by parts)},$

$-2\int_0^1 w_{xx}v \le 2\int_0^1 w_{xx}^2 + \frac{1}{2}\int_0^1 v^2 \qquad \text{(Young's inequality)},$

we obtain

$\dot V_{w2} \le -2(\lambda - a)\int_0^1 w_x^2 + \frac{1}{2}\int_0^1 v^2.$

Summing up, for any $\mu > 0$

$$\dot V_w + 2\alpha_c V_w \le -2(1 + \lambda - a - \alpha_c)\|w_x\|^2_{L_2} - 2(\lambda - a - \alpha_c - \mu)\|w\|^2_{L_2} + \left(\frac{1}{2\mu} + \frac{1}{2}\right)\int_0^1 v^2.$$

The condition (15) yields $1 + \lambda - a - \alpha_c > 0$. Then, using

$-\|w_x\|^2_{L_2} \overset{\text{Lem.1}}{\le} -\frac{\max\{d_L, d_R\}\pi^2}{4 - 3d_L d_R}\|w\|^2_{L_2},$

and (15), for small enough $\mu > 0$, we obtain

$\dot V_w \le -2\alpha_c V_w + \left(\frac{1}{2\mu} + \frac{1}{2}\right)\int_0^1 v^2.$

Since $k(x, y)$ is bounded, there exists $C_v > 0$ such that

$$\int_0^1 v^2(x,t)\,dx \le C_v e^{-2\alpha_o(t - s_k)}\|\sigma(\cdot, s_k)\|^2_{L^2} \overset{(9)}{\le} C_v C_\sigma^2 e^{-2\alpha_o t}\|z(\cdot, 0)\|^2_{\mathcal{H}^1}.$$

Summing up,

$\dot V_w(t) \le -2\alpha_c V_w(t) + \left(\frac{1}{2\mu} + \frac{1}{2}\right) C_v C_\sigma^2 e^{-2\alpha_o t}\|z(\cdot, 0)\|^2_{\mathcal{H}^1}.$

If $\alpha_c \ne \alpha_o$, the comparison principle implies (16) (note that $V_w(t_0) = 0$). If (15) holds for $\alpha_c = \alpha_o$, it remains true for slightly larger $\alpha_c' > \alpha_c$, implying (16) for α_c.

C Proof of Proposition 3

Consider the Lyapunov functional $\hat V = \int_0^1 \hat z^2(x,t)\,dx$, which is well-defined and continuous for the weak solution (21). Using (22) with $\varphi(\xi) = \hat z(\xi, t)$, we calculate the derivative $\dot{\hat V}$ along (19), (20):

$\dot{\hat V} = -2\int_0^1 \hat z_x^2 + 2a\int_0^1 \hat z^2 - 2\sum_{j=1}^M K_j \hat z^2(\bar x_j, t) + 2\int_0^1 \hat z\hat v,$

where

$\hat v(x,t) = L e^{-\alpha_o(t + r - s_k)}\sigma(x, s_k), \quad t \in [t_k, t_{k+1})$

with σ defined in (10). For $\epsilon > 0$ such that

$2(a + \alpha_c + \epsilon)(|\Delta_j^L| + |\Delta_j^R|) < K_j, \quad \forall j \in 1:M$

(it exists due to (24)), Young's inequality implies

$2\int_0^1 \hat z\hat v \le 2\epsilon\int_0^1 \hat z^2 + \frac{1}{2\epsilon}\int_0^1 \hat v^2.$

Thus,

$$\dot{\hat V} + 2\alpha_c\hat V \le -2\int_0^1 \hat z_x^2 + 2(a + \alpha_c + \epsilon)\int_0^1 \hat z^2 - 2\sum_{j=1}^M K_j \hat z^2(\bar x_j, t) + \frac{1}{2\epsilon}\int_0^1 \hat v^2.$$

Let $|\Delta_j^L||\Delta_j^R| \ne 0$. Using Lemma 2 (with $\nu = 2$ for simplicity), for any $\mu_j \in (0,1)$ we have

$$\begin{aligned} -2K_j\hat z^2(\bar x_j, t) &= -2\mu_j K_j \hat z^2(\bar x_j, t) - 2(1 - \mu_j)K_j \hat z^2(\bar x_j, t) \\ &\le -\frac{\mu_j K_j}{|\Delta_j^L|}\int_{\Delta_j^L}\hat z^2 + \mu_j K_j \frac{8|\Delta_j^L|}{\pi^2}\int_{\Delta_j^L}\hat z_x^2 \\ &\quad - \frac{(1-\mu_j)K_j}{|\Delta_j^R|}\int_{\Delta_j^R}\hat z^2 + (1 - \mu_j)K_j\frac{8|\Delta_j^R|}{\pi^2}\int_{\Delta_j^R}\hat z_x^2, \end{aligned}$$

which leads to

$$\begin{aligned} \dot{\hat V} + 2\alpha_c\hat V &\le \sum_{j=1}^M\left(-2 + \mu_j K_j\frac{8|\Delta_j^L|}{\pi^2}\right)\int_{\Delta_j^L}\hat z_x^2 \\ &+ \sum_{j=1}^M\left(-2 + (1 - \mu_j)K_j\frac{8|\Delta_j^R|}{\pi^2}\right)\int_{\Delta_j^R}\hat z_x^2 \\ &+ \sum_{j=1}^M\left(2(a + \alpha_c + \epsilon) - \frac{\mu_j K_j}{\Delta_j^L}\right)\int_{\Delta_j^L}\hat z^2 \\ &+ \sum_{j=1}^M\left(2(a + \alpha_c + \epsilon) - \frac{(1-\mu_j)K_j}{\Delta_j^R}\right)\int_{\Delta_j^R}\hat z^2 \\ &+ \frac{1}{2\epsilon}\int_0^1 \hat v^2 \end{aligned}$$

In view of (23) and (24), Lemma 3 with

$\underline{a} = 2(a + \alpha_c + \epsilon)\Delta_j^L, \qquad \overline{a} = \frac{\pi^2}{4\Delta_j^L},$

$\underline{b} = 2(a + \alpha_c + \epsilon)\Delta_j^R, \qquad \overline{b} = \frac{\pi^2}{4\Delta_j^R}$

guarantees the existence of $\mu_j \in (0,1)$ such that

$\dot{\hat V} + 2\alpha_c\hat V \le \frac{1}{2\epsilon}\int_0^1 \hat v^2.$

The calculations are similar if $|\Delta_j^L| = 0$ (with $\mu_j = 0$) or $|\Delta_j^R| = 0$ (with $\mu_j = 1$). Using the definition of $\hat v$,

$$\int_0^1 \hat v^2 \le L^2 e^{-2\alpha_o(t + r - s_k)}\|\sigma(\cdot, s_k)\|^2_{L^2} \overset{(9)}{\le} e^{-2\alpha_o t}C\|z(\cdot, 0)\|^2_{\mathcal{H}^1}$$

with $C = L^2 e^{-2\alpha_o r}C_\sigma^2$. Therefore,

$\dot{\hat V} \le -2\alpha_c\hat V + e^{-2\alpha_o t}\frac{C}{2\epsilon}\|z(\cdot, 0)\|^2_{\mathcal{H}^1}.$

If $\alpha_c \ne \alpha_o$, the comparison principle implies (25) (note that $\hat V(t_0) = 0$). If the conditions of Proposition 3 hold for $\alpha_c = \alpha_o$, they remain true for slightly larger $\alpha_c' > \alpha_c$, implying (25) for α_c.