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Improved sampled-data implementation of derivative-dependent control

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Abstract: We consider an LTI system of relative degree two that can be stabilized using the output and its derivative. The derivative is approximated using a finite difference, what leads to a time-delayed feedback. This feedback is analyzed using a Lyapunov-Krasovskii functional that compensates the derivative approximation error presented in an integral form. We show that if the derivative-dependent control exponentially stabilizes the system, then one can use consecutively sampled measurements to approximate the derivative and this approximation will preserve the stability if the sampling period is small enough. We provide linear matrix inequalities that allow to find admissible sampling period and can be used for robustness analysis with respect to system uncertainties. The results are demonstrated by two examples: 2D uncertain system and the Furuta pendulum.

1. INTRODUCTION

Control laws that depend on the output derivative are used to stabilize LTI systems of relative degree two. To estimate the derivative, which can hardly be measured directly, one can use the finite difference: $\dot{y} \approx (y(t) - y(t-h))/h$. Such approximation leads to time-delayed feedback that preserves the stability if the delay $h > 0$ is small enough (French et al. (2009)). For a given h , the delay-induced stability can be checked using frequency-domain techniques (Niculescu and Michiels (2004); Kharitonov et al. (2005); Ramírez et al. (2016)) or complete Lyapunov-Krasovskii functionals (Gu et al. (2003); Kharitonov (2012); Egorov (2016)), which give necessary and sufficient conditions.

The delay-induced stability can be also studied using linear matrix inequalities (LMIs) (Gu (1997); Seuret and Gouaisbaut (2013, 2015)). The advantage of LMIs is that, though being conservative, they allow for performance and robustness analysis, can cope with certain types of nonlinearities (Fridman (2014)), and can deal with stochastic perturbations (Fridman and Shaikhet (2016, 2017)). Simple and yet efficient LMIs for the delay-induced stability were obtained in Fridman and Shaikhet (2016, 2017). The key idea was to use the Taylor's expansion of the delayed terms with the remainders in the integral form that are compensated by appropriate terms in the Lyapunov-Krasovskii functional. Compared to Gu (1997); Seuret and Gouaisbaut (2013, 2015), the resulting LMIs have a lower order, contain less decision variables, and were proved to be feasible for small delays if the derivative-dependent feedback stabilizes the system.

LMIs can be used to study sampled-data implementation of stabilizing controllers with artificial delays. This has been done in Liu and Fridman (2012) via discretized Lyapunov functionals with a Wirtinger-based term and in Seuret and Briat (2015) by employing impulsive system representation and looped Lyapunov functionals. The

high-order LMIs obtained in Liu and Fridman (2012) and Seuret and Briat (2015) contain many decision variables, which make them hard to solve numerically. Using the ideas of Fridman and Shaikhet (2016, 2017), simple LMIs for sampled-data delay-induced stabilization were derived in Selivanov and Fridman (2018). These conditions were proved to be feasible for a small enough sampling period if the continuous-time derivative-dependent feedback stabilizes the system.

In this paper, we improve the results of Selivanov and Fridman (2018). Namely, we show that one can always take *consecutively* sampled measurements to approximate the output derivative while Selivanov and Fridman (2018) required *distant* measurements (cf. (7) and (18)). This novelty allows to use less memory when one uses time-delays to implement derivative-dependent feedback. Such improvement is achieved by representing the errors due to sampling in a different way: the errors used to be multiplied by sampling-dependent gains but now they are multiplied by constant gains (see Remark 3). We provide linear matrix inequalities that allow to find admissible sampling period and can be used for robustness analysis with respect to system uncertainties. The results are demonstrated by two examples: 2D uncertain system and the Furuta pendulum.

Auxiliary lemmas

Lemma 1. (Wirtinger's inequality). Let $f: [a, b] \rightarrow \mathbb{R}^n$ be an absolutely continuous function with a square integrable first derivative such that $f(a) = 0$ or $f(b) = 0$. Then for any $0 \leq W \in \mathbb{R}^{n \times n}$,

$$\int_a^b f^T(t) W f(t) dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \dot{f}^T(t) W \dot{f}(t) dt.$$

Proof is given in Liu et al. (2010).

Lemma 2. (Jensen's inequality). Let $\rho: [a, b] \rightarrow [0, \infty)$ and $f: [a, b] \rightarrow \mathbb{R}^n$ be such that the integration concerned is well-defined. Then for any $0 < Q \in \mathbb{R}^{n \times n}$,

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$$\begin{aligned} & \left[\int_a^b \rho(s) f(s) ds \right]^T Q \left[\int_a^b \rho(s) f(s) ds \right] \\ & \leq \int_a^b \rho(s) ds \int_a^b \rho(s) f^T(s) Q f(s) ds. \end{aligned}$$

Proof is given in Solomon and Fridman (2013).

2. DERIVATIVE IMPLEMENTATION USING SAMPLED-DATA CONTROL

Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^l, \quad (1)$$

of relative degree two, i.e.,

$$CB = 0, \quad CAB \neq 0. \quad (2)$$

For such systems, it is common to look for a stabilizing controller in the form

$$u = \bar{K}_0 y + \bar{K}_1 \dot{y}, \quad \bar{K}_0, \bar{K}_1 \in \mathbb{R}^{m \times l} \quad (3)$$

Remark 1. Appropriate \bar{K}_0 and \bar{K}_1 can always be found if (1) is a square ($m = l$) minimum-phase system with $\det CAB \neq 0$ (Ilchmann and Sangwin (2004)).

The controller (3) depends on the output derivative, which is hard to measure directly. Instead, the derivative can be approximated by the finite-difference

$$\dot{y}(t) \approx y_1(t) = \frac{y(t) - y(t-h)}{h}, \quad h > 0. \quad (4)$$

This approximation leads to the delay-dependent control

$$u(t) = \bar{K}_0 y(t) + \bar{K}_1 y_1(t) = K_0 y(t) + K_1 y(t-h), \quad (5)$$

where $y(t) = 0$ for $t < 0$ and

$$K_0 = \bar{K}_0 + \frac{1}{h} \bar{K}_1, \quad K_1 = -\frac{1}{h} \bar{K}_1. \quad (6)$$

If (1) is stable under (3), it can be stabilized by (5) with a small enough delay $h > 0$ (French et al. (2009)).

In this paper, we assume that only sampled in time measurements $y(t_k)$ are available to the controller, where $k \in \mathbb{N}_0$ and $t_k = kh$ are the sampling instants with a sampling period $h > 0$. The derivative-dependent controller (3) is implemented as the sampled-data controller (cf. (5))

$$u(t) = \bar{K}_0 y(t_k) + \bar{K}_1 y_1(t_k) = K_0 y(t_k) + K_1 y(t_{k-1}), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0 \quad (7)$$

with $y(t_{-1}) = 0$ and K_i from (6).

We will show that the sample-data controller (7) stabilizes the system (1), if (3) stabilizes (1) and the sampling period $h > 0$ is small enough. Moreover, we will derive LMIs that allow to find appropriate h .

First, we present the estimation error $\dot{y}(t) - y_1(t)$ in a convenient integral form.

Lemma 3. If $y \in C^1$ and \dot{y} is absolutely continuous, then y_1 defined in (4) satisfies

$$y_1(t) = \dot{y}(t) + \int_{t-h}^t \frac{t-h-s}{h} \ddot{y}(s) ds. \quad (8)$$

Proof. Taylor's expansion with the remainder in the integral form gives

$$y(t-h) = y(t) - \dot{y}(t)h - \int_{t-h}^t (t-h-s) \ddot{y}(s) ds.$$

Reorganizing the terms, we obtain

$$y_1(t) = \frac{y(t) - y(t-h)}{h} = \dot{y}(t) + \int_{t-h}^t \frac{t-h-s}{h} \ddot{y}(s) ds. \quad \blacksquare$$

Since

$$\dot{y} \stackrel{(1)}{=} C[Ax + Bu] \stackrel{(2)}{=} CAx, \quad (9)$$

we have $\ddot{y} = CA\dot{x}$, which is piecewise-continuous. Therefore, \dot{y} is absolutely continuous. For $t \in [t_k, t_{k+1})$, we present the sampled signals as

$$\begin{aligned} y(t_k) &= y(t) - \int_{t_k}^t \dot{y}(s) ds, \\ y_1(t_k) &= y_1(t) - \int_{t_k}^t \dot{y}_1(s) ds \\ &\stackrel{(8)}{=} \dot{y}(t) + \int_{t-h}^t \frac{t-h-s}{h} \ddot{y}(s) ds - \int_{t_k}^t \dot{y}_1(s) ds. \end{aligned} \quad (10)$$

Substituting (10) into (7), we obtain

$$u(t) = \bar{K}_0 y(t) + \bar{K}_1 \dot{y}(t) + \delta_0(t) + \delta_1(t) + \kappa(t), \quad (11)$$

where

$$\begin{aligned} \delta_0(t) &= -\bar{K}_0 \int_{t_k}^t \dot{y}(s) ds, \\ \delta_1(t) &= -\bar{K}_1 \int_{t_k}^t \dot{y}_1(s) ds, \\ \kappa(t) &= \bar{K}_1 \int_{t-h}^t \frac{t-h-s}{h} \ddot{y}(s) ds. \end{aligned}$$

Then the closed-loop system (1), (11) takes the form

$$\dot{x} = Dx + B[\delta_0(t) + \delta_1(t) + \kappa(t)], \quad (12)$$

where $D = A + B\bar{K}_0 C + B\bar{K}_1 CA$. The system (1), (3) is equivalent to $\dot{x} = Dx$. Therefore, if (1), (3) is stable, then D is Hurwitz. The theorem below guarantees that the errors δ_0 , δ_1 , and κ do not ruin the stability of (12).

Theorem 1. Consider an LTI system (1) of relative degree two, i.e., satisfying (2).

- (i) The sampled-data feedback (7) with a sampling period $h > 0$ and controller gains (6) exponentially stabilizes (1) if there exist

$$0 < P \in \mathbb{R}^{n \times n}, \quad P_2, P_3 \in \mathbb{R}^{n \times n}, \\ 0 < W_0 \in \mathbb{R}^{m \times m}, \quad 0 < W_1 \in \mathbb{R}^{m \times m}, \quad 0 < R_1 \in \mathbb{R}^{m \times m}$$

such that¹

$$N = \begin{bmatrix} N_{11} & N_{12} & P_2^T B & P_2^T B & P_2^T B \\ * & N_{22} & P_3^T B & P_3^T B & P_3^T B \\ * & * & -\frac{\pi^2}{4} W_0 & 0 & 0 \\ * & * & * & -\frac{\pi^2}{4} W_1 & 0 \\ * & * & * & * & -R_1 \end{bmatrix} < 0,$$

where

$$\begin{aligned} N_{11} &= D^T P_2 + P_2^T D, \\ N_{12} &= P - P_2^T + D^T P_3, \\ N_{22} &= -P_3 - P_3^T + h^2 [\bar{K}_0 C]^T W_0 [\bar{K}_0 C] \\ &\quad + h^2 [\bar{K}_1 CA]^T (W_1 + \frac{1}{4} R_1) [\bar{K}_1 CA] \end{aligned}$$

with $D = A + B\bar{K}_0 C + B\bar{K}_1 CA$.

- (ii) If the derivative-dependent feedback (3) with controller gains $\bar{K}_0, \bar{K}_1 \in \mathbb{R}^{m \times l}$ stabilizes (1), then there exists a sufficiently small sampling period $h > 0$ such that the sampled-data feedback (7) with the controller gains (6) stabilizes (1).

¹ MATLAB codes for solving the LMIs are available at <https://github.com/AntonSelivanov/ROCOND18>

Proof. (i) For $t \geq h$ consider the functional

$$V = V_0 + V_{\delta_0} + V_{\delta_1} + V_{y_1} + V_{\kappa},$$

where

$$V_0 = x^T P x,$$

$$V_{\delta_0} = h^2 \int_{t_k}^t [\bar{K}_0 \dot{y}(s)]^T W_0 [\bar{K}_0 \dot{y}(s)] ds - \frac{\pi^2}{4} \int_{t_k}^t \delta_0^T(s) W_0 \delta_0(s) ds, \quad t \in [t_k, t_{k+1}),$$

$$V_{\delta_1} = h^2 \int_{t_k}^t [\bar{K}_1 \dot{y}_1(s)]^T W_1 [\bar{K}_1 \dot{y}_1(s)] ds - \frac{\pi^2}{4} \int_{t_k}^t \delta_1^T(s) W_1 \delta_1(s) ds, \quad t \in [t_k, t_{k+1}),$$

$$V_{y_1} = h \int_{t-h}^t (s-t+h) [\bar{K}_1 \ddot{y}(s)]^T W_1 [\bar{K}_1 \ddot{y}(s)] ds,$$

$$V_{\kappa} = \int_{t-h}^t \frac{(t-h-s)^2}{4} [\bar{K}_1 \ddot{y}(s)]^T R_1 [\bar{K}_1 \ddot{y}(s)] ds.$$

Since $\dot{\delta}_0(t) = -\bar{K}_0 \dot{y}(t)$ and $\delta_0(t_k) = 0$, Lemma 1 implies $V_{\delta_0} \geq 0$. Similarly, $V_{\delta_1} \geq 0$. Therefore, $V \geq 0$. Calculating the derivatives, we obtain

$$\begin{aligned} \dot{V}_0 &= 2x^T P \dot{x}, \\ \dot{V}_{\delta_0} &= h^2 [\bar{K}_0 \dot{y}]^T W_0 [\bar{K}_0 \dot{y}] - \frac{\pi^2}{4} \delta_0^T W_0 \delta_0, \\ \dot{V}_{\delta_1} &= h^2 [\bar{K}_1 \dot{y}_1]^T W_1 [\bar{K}_1 \dot{y}_1] - \frac{\pi^2}{4} \delta_1^T W_1 \delta_1. \end{aligned} \quad (13)$$

The functional V_{y_1} is introduced to compensate the term $h^2 [\bar{K}_1 \dot{y}_1]^T W_1 [\bar{K}_1 \dot{y}_1]$ in the above expression. We have

$$\begin{aligned} \dot{V}_{y_1} &= h^2 [\bar{K}_1 \ddot{y}]^T W_1 [\bar{K}_1 \ddot{y}] - h \int_{t-h}^t [\bar{K}_1 \ddot{y}(s)]^T W_1 [\bar{K}_1 \ddot{y}(s)] ds \\ &\stackrel{\text{Lem.2}}{\leq} h^2 [\bar{K}_1 \ddot{y}]^T W_1 [\bar{K}_1 \ddot{y}] - \int_{t-h}^t [\bar{K}_1 \ddot{y}(s)]^T W_1 [\bar{K}_1 \ddot{y}(s)] ds. \end{aligned}$$

Differentiating (4), we obtain

$$\dot{y}_1 = \frac{1}{h} \int_{t-h}^t \ddot{y}(s) ds.$$

The latter and (9) lead to

$$\begin{aligned} \dot{V}_{y_1} &\leq h^2 [\bar{K}_1 C A \dot{x}]^T W_1 [\bar{K}_1 C A \dot{x}] \\ &\quad - h^2 [\bar{K}_1 \dot{y}_1]^T W_1 [\bar{K}_1 \dot{y}_1]. \end{aligned} \quad (14)$$

The term $-h^2 [\bar{K}_1 \dot{y}_1]^T W_1 [\bar{K}_1 \dot{y}_1]$ will cancel the first term of \dot{V}_{δ_1} . Since (9) implies $\ddot{y} = C A \dot{x}$, we have

$$\begin{aligned} \dot{V}_{\kappa} &= \frac{h^2}{4} [\bar{K}_1 \ddot{y}(t)]^T R_1 [\bar{K}_1 \ddot{y}(t)] - \int_{t-h}^t \frac{s-t+h}{2} [\bar{K}_1 \ddot{y}(s)]^T R_1 [\bar{K}_1 \ddot{y}(s)] ds \\ &\stackrel{\text{Lem.2}}{\leq} \frac{h^2}{4} [\bar{K}_1 C A \dot{x}]^T R_1 [\bar{K}_1 C A \dot{x}] - \kappa^T R_1 \kappa. \end{aligned} \quad (15)$$

Instead of substituting (12) for \dot{x} , we will use the following descriptor representation of the system (12):

$$0 = 2[x^T P_2^T + \dot{x}^T P_3^T] [-\dot{x} + D x + B \delta_0 + B \delta_1 + B \kappa]. \quad (16)$$

Summing up the right-hand sides of (13), (14), (15), and (16), we obtain

$$\dot{V} \leq \nu^T N \nu,$$

where $\nu = \text{col}\{x, \dot{x}, \delta_0, \delta_1, \kappa\}$. Thus, the condition $N < 0$ implies exponential stability of the system (1), (7).

(ii) We show that $N < 0$ is feasible for small enough h . Since (1), (3) is stable, there exists $0 < P \in \mathbb{R}^{n \times n}$ such that $PD + D^T P < 0$. Then, there exist P_2, P_3 such that

$$\Lambda = \begin{bmatrix} P_2^T D + D^T P_2 & P - P_2^T + D^T P_3 \\ * & -P_3 - P_3^T \end{bmatrix} < 0.$$

To see this, one can take $P_2 = P$, $P_3 = \varepsilon I$ and apply the Schur complement to obtain

$$PD + D^T P + \frac{\varepsilon}{2} D^T D < 0,$$

which is true for small $\varepsilon > 0$. Taking $W_0 = W_1 = R_1 = h^{-1} I_m$ and applying the Schur complement, we obtain that $N < 0$ holds if

$$\Lambda + hF < 0,$$

where F is some matrix independent of h . Clearly, the latter holds for small enough h . \blacksquare

Remark 2. (Polytopic uncertainty). The results of Theorem 1 are applicable to polytopic-type uncertain A . Indeed, by applying the Schur complement to the square in A terms, we obtain that $N < 0$ is equivalent to

$$\begin{bmatrix} & & & & 0 \\ & & & & h [\bar{K}_1 C A]^T (W_1 + \frac{1}{4} R_1) \\ & \bar{N} & & & 0 \\ & & & & 0 \\ & & & & 0 \\ * & * & * & * & * \end{bmatrix} < 0, \quad (17)$$

where \bar{N} coincides with N except for the block

$$\bar{N}_{22} = -P_3 - P_3^T + h^2 [\bar{K}_0 C]^T W_0 [\bar{K}_0 C].$$

The LMI (17) is affine in A , therefore, if A resides in the uncertain polytope

$$A = \sum_{j=1}^M \mu_j A^{(j)}, \quad 0 \leq \mu_j \leq 1, \quad \sum_{j=1}^M \mu_j = 1,$$

one needs to solve² (17) simultaneously for the M vertices $A^{(j)}$ applying the same decision matrices P_2, P_3, W_1, R_1 .

Remark 3. In Selivanov and Fridman (2018), the system (1) was studied under the sampled-data feedback

$$u(t) = K_0 y(t_k) + K_1 y(t_{k-q}), \quad t \in [t_k, t_{k+1}), k \in \mathbb{N}_0, \quad (18)$$

where q is an integer delay. In the analysis, the errors due to sampling $y(t_k) - y(t)$ and $y(t_{k-q}) - y(t - qh)$ were multiplied by K_0 and K_1 that grow when $qh \rightarrow 0$. Consequently, one had to increase the discrete delay q while reducing the sampling period h to maintain K_0 and K_1 bounded. Here, the errors due to sampling are multiplied by \bar{K}_0 and \bar{K}_1 that do not depend on h (see δ_0 and δ_1 below (11)). This allows to use $q = 1$ (cf. (7) and (18)) and, therefore, smaller memory is required to implement (7) (see Example 1).

3. EXAMPLES

Example 1 (Liu and Fridman (2012)). Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = [1 \ 0] x(t) \quad (19)$$

with the uncertainty $g \in [-0.1, 0.1]$. This system is of relative degree two and cannot be stabilized by sampled-data controller $u(t) = K y(t_k)$, $t \in [t_k, t_{k+1})$. Consider the sampled-data controller (7) with

$$\bar{K}_0 = -0.25, \quad \bar{K}_1 = -0.0499.$$

² MATLAB codes for solving the LMIs are available at <https://github.com/AntonSelivanov/ROCOND18>

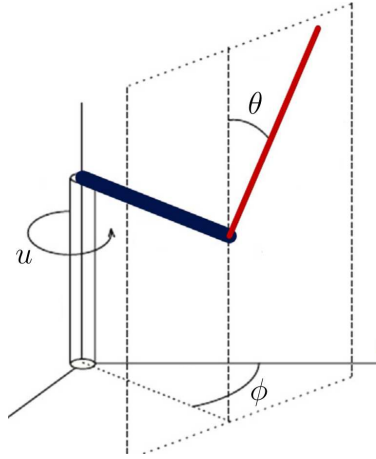


Fig. 1. Furuta pendulum³

These gains were obtained by taking $K_0 = -0.35$, $K_1 = 0.1$ as in Liu and Fridman (2012) and using (6) with $h = 0.499$ (the largest h obtained in Liu and Fridman (2012)). The LMIs of Remark 2 are feasible for $h \in (0, 0.258]$. Taking $h = 0.258$ in (6), we deduce that the sampled-data controller (7) with

$$K_0 = -0.4434, \quad K_1 = 0.1934, \quad t_k = 0.258 \cdot k$$

exponentially stabilizes (19). The system (19) under the sampled-data controller (18) with $q = 3$ has been studied in Liu and Fridman (2012); Seuret and Briat (2015); Selivanov and Fridman (2018). In our case $q = 1$, which leads to a smaller memory used in the implementation.

Example 2 (Ortega-Montiel et al. (2017)). Consider a linearized model of Furuta pendulum given by (1) with

$$\begin{bmatrix} A|B \\ \hline C|0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 37.377 & -0.515 & 0 & 0.142 & | & -35.42 \\ 0 & 0 & 0 & 1 & | & 0 \\ -8.228 & 0.113 & 0 & -0.173 & | & 43.28 \\ \hline 1 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \end{bmatrix} \quad (20)$$

and $x = \text{col}\{\theta, \dot{\theta}, \phi, \dot{\phi}\}$, where θ is the angular position of the pendulum and ϕ is the angle of the rotational arm (see Fig. 1). The control input u is proportional to the motor induced torque. Using the pole placement, we find that for

$$\bar{K}_0 = [1.2826 \ 0.0013], \quad \bar{K}_1 = [0.1209 \ 0.0086]$$

the eigenvalues of D defined below (12) are -1 , -1.1 , -1.2 , -1.3 . Therefore, the derivative-dependent controller (3) stabilizes the system (1), (20). The conditions of Theorem 1 are feasible for $h \in (0, 0.103]$. Taking $h = 0.103$ in (6), we deduce that the sampled-data controller (7) with

$$K_0 = [2.4566 \ 0.0845], \quad K_1 = [-1.1740 \ -0.0832],$$

and $t_k = 0.103 \cdot k$ exponentially stabilizes the Furuta pendulum (1), (20).

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³ The picture is taken from Ramírez-Neria et al. (2014)