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# Sampled-data $H_\infty$ filtering of a 2D heat equation under pointlike measurements

Anton Selivanov and Emilia Fridman

**Abstract**—The existing sampled-data observers for 2D heat equations use spatially averaged measurements, i.e., the state values averaged over subdomains covering the entire space domain. In this paper, we introduce an observer for a 2D heat equation that uses pointlike measurements, which are modeled as the state values averaged over small subsets that do not cover the space domain. The key result, allowing for an efficient analysis of such an observer, is a new inequality that bounds the  $L_2$ -norm of the difference between the state and its point value by the reciprocally convex combination of the  $L_2$ -norms of the first and second order space derivatives of the state. The convergence conditions are formulated in terms of linear matrix inequalities feasible for large enough observer gain and number of pointlike sensors. The results are extended to solve the  $H_\infty$  filtering problem under continuous and sampled in time pointlike measurements.

## I. INTRODUCTION

Partial differential equations model tremendous amount of processes: heat transfer, fluid dynamics, fusion reactions, wave propagation, etc. Such processes may require feedback control to remain stable, e.g., chemical reactors [1], oil drill strings [2], tokamaks [3], and rotating stall in axial compressors [4]. Here, we construct observers for 2D heat equations with continuous and sampled in time pointlike measurements. These observers can be combined with state-feedback controllers to stabilize 2D reaction-diffusion systems.

For 1D heat equations, *point* observers/controllers have been constructed and analyzed under continuous [5], [6], [7], [8], [9], [10] and sampled in time [7], [11] measurements.  $N$ -D diffusion equations with *averaged* measurements (i.e., the state values are averaged over subdomains covering the entire space domain) have been studied in [12], [13], [14]. Constructive analysis of 2D diffusion equations under pointlike measurements and actuators is a challenging problem, especially in the presence of time-delays and disturbances. In this paper, we develop a constructive method allowing to solve the  $H_\infty$  filtering problem in the case of a 2D reaction-diffusion system with continuous and sampled in time pointlike measurements.

Point measurements are often modeled using Dirac delta functions. This leads to certain difficulties in the well-posedness analysis and data sampling becomes hard to study due to the unboundedness of the corresponding input/output operators. We use a more convenient approach where pointlike measurements are presented as the average state value

over a small enough domain subset. This allows to avoid technical difficulties related to the well-posedness and allows to study data sampling. Similarly to [15], which studied 1D domains, we use the mean value theorem to present such measurements as the state point values. Inspired by [16], we derive an inequality that bounds the  $L_2$ -norm of the difference between the state and its point value by the reciprocally convex combination of the  $L_2$ -norms of the first and second order space derivatives of the state (Lemma 5). Combining this inequality with a Lyapunov functional, we derive the observer convergence conditions in terms of linear matrix inequalities that are feasible for a high enough observer gain and large enough number of sensors (Section II). The results are extended to solve the  $H_\infty$  filtering problem under continuous (Section III) and sampled in time (Section IV) pointlike measurements.

*Notations:*  $\|\cdot\|$  is the  $L^2$ -norm,  $\text{supp } f$  is the support of function  $f$ ,  $\text{conv}(G)$  is the convex hull.

The following lemmas will be used in the analysis.

*Lemma 1:* Let  $g_i \in L^2$ ,  $i = 1, \dots, n$ . Then

$$\left\| \sum_{i=1}^n g_i \right\|^2 \leq \sum_{i=1}^n \frac{1}{\alpha_i} \|g_i\|^2$$

for any  $\alpha_i > 0$  such that  $\sum_i \alpha_i = 1$ .

*Proof.* By the convexity of  $\|\cdot\|^2$ ,

$$\left\| \sum_{i=1}^n \alpha_i \frac{g_i}{\alpha_i} \right\|^2 \leq \sum_{i=1}^n \alpha_i \left\| \frac{g_i}{\alpha_i} \right\|^2 = \sum_{i=1}^n \frac{1}{\alpha_i} \|g_i\|^2. \quad \blacksquare$$

*Lemma 2 (Wirtinger's inequality):* For  $f \in H^1(a, b)$ ,

$$\|f\| \leq \frac{2(b-a)}{\pi} \|f'\| \quad \text{if } f(a) = 0 \text{ or } f(b) = 0,$$

$$\|f\| \leq \frac{(b-a)}{\pi} \|f'\| \quad \text{if } f(a) = 0 \text{ and } f(b) = 0.$$

*Proof.* See [17, Chapter 7.7].

*Lemma 3 (Exponential Wirtinger's inequality):* If  $\alpha \in \mathbb{R}$  and  $f \in H^1(a, b)$  is such that  $f(a) = 0$  or  $f(b) = 0$ , then

$$\int_a^b e^{2\alpha t} f^2(t) dt \leq e^{2|\alpha|(b-a)} \frac{4(b-a)^2}{\pi^2} \int_a^b e^{2\alpha t} \dot{f}^2(t) dt.$$

*Proof.* See [18, Lemma A.18].

*Lemma 4 (Jensen inequality):* If  $\rho: [a, b] \rightarrow [0, \infty)$  and  $f: [a, b] \rightarrow \mathbb{R}$  are such that the integration concerned is well-defined, then

$$\left[ \int_a^b \rho(s) f(s) ds \right]^2 \leq \int_a^b \rho(s) ds \int_a^b \rho(s) f^2(s) ds.$$

*Proof.* See [19, Lemma 1].

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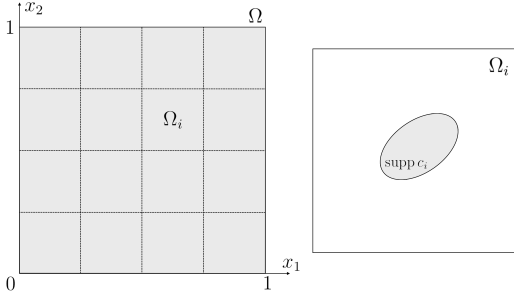


Fig. 1. Subdomains  $\Omega_i$  and the subset  $\text{supp } c_i \subset \bar{\Omega}_i$

## II. POINTLIKE OBSERVER FOR A 2D HEAT EQUATION

Consider the reaction-diffusion system

$$\begin{aligned} z_t(x, t) &= \Delta_D z(x, t) + az(x, t), \quad x \in \Omega, t > 0, \\ z|_{\partial\Omega} &= 0, \quad z|_{t=0} = z_0 \end{aligned} \quad (1)$$

defined on  $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$  with the state  $z: \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$ , reaction coefficient  $a$ , and diffusion term

$$\Delta_D z = \text{div}(D\nabla z), \quad 0 < D = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \in \mathbb{R}^{2 \times 2}. \quad (2)$$

*Remark 1:* Any open parallelogram  $\tilde{\Omega} \subset \mathbb{R}^2$  can be transformed to  $\Omega = (0, 1) \times (0, 1)$  using a nonsingular change of variable  $x = A\tilde{x} + b$ . In this case,  $D = A\tilde{D}A^T$ .

*Remark 2:* For simplicity, we consider a *linear* system. The results can be extended to Lipschitz and sector-bounded nonlinearities in a manner similar to [7], [13].

Let  $\Omega$  be divided into  $N$  square subdomains  $\Omega_i$  (Fig. 1) with a sensor placed in each  $\Omega_i$  providing the measurements

$$\begin{aligned} y_i(t) &= \int_{\Omega_i} c_i(\xi) z(\xi, t) d\xi, \\ 0 \leq c_i &\in L^2(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i = 1, \dots, N. \end{aligned} \quad (3)$$

For example,

$$c_i(\xi) = \begin{cases} \frac{1}{\varepsilon^2}, & |\xi - x_c^i|_\infty < \frac{\varepsilon}{2}, \\ 0, & |\xi - x_c^i|_\infty \geq \frac{\varepsilon}{2} \end{cases} \quad (4)$$

with a small  $\varepsilon \in (0, 1/\sqrt{N}]$  model point measurements at  $x_c^i \in \Omega_i$ . The case of  $\varepsilon = 1/\sqrt{N}$  was considered in [13].

We study the observer

$$\begin{aligned} \hat{z}_t(x, t) &= \Delta_D \hat{z}(x, t) + a\hat{z}(x, t) \\ &\quad + L \sum_{i=1}^N \chi_i(x) \left[ y_i(t) - \int_{\Omega_i} c_i(\xi) \hat{z}(\xi, t) d\xi \right], \\ \hat{z}|_{\partial\Omega} &= 0, \quad \hat{z}|_{t=0} = 0 \end{aligned} \quad (5)$$

with the injection gain  $L$  and characteristic functions

$$\chi_i(x) = \begin{cases} 1, & x \in \Omega_i, \\ 0, & x \notin \Omega_i, \end{cases} \quad i = 1, \dots, N. \quad (6)$$

The estimation error  $\bar{z}(x, t) = z(x, t) - \hat{z}(x, t)$  satisfies

$$\begin{aligned} \bar{z}_t &= \Delta_D \bar{z} + a\bar{z} - L \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) \bar{z}(\xi, t) d\xi, \\ \bar{z}|_{\partial\Omega} &= 0, \quad \bar{z}|_{t=0} = z_0. \end{aligned} \quad (7)$$

*Definition 1:* A (classical) solution of (7) is a function  $\bar{z} \in C^1([0, \infty); L^2(\Omega))$  such that  $\bar{z}(\cdot, t) \in H^2(\Omega) \cap H_0^1(\Omega)$  for  $t \geq 0$  and  $\bar{z}$  satisfies (7).

Since  $A: D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ ,  $Aw = \Delta_D w$ , with  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$  generates a  $C_0$ -semigroup [20, Theorem 7.2.7] and  $B: L^2(\Omega) \rightarrow L^2(\Omega)$ ,  $Bw = aw - L \sum_{i=1}^N \chi_i \int_{\Omega_i} c_i w$ , is bounded, the operator  $A + B$  generates a  $C_0$ -semigroup [21, Theorem 3.2.1]. By [21, Theorem 3.1.3], (7) has a unique classical solution for

$$z_0 \in D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

By the mean value theorem (this idea comes from [15]),

$$\int_{\Omega_i} c_i(\xi) \bar{z}(\xi, t) d\xi = \bar{z}(x^i(t), t),$$

where  $x^i(t) \in \text{conv}(\text{supp } c_i)$  for  $t \geq 0$  and  $i = 1, \dots, N$ . Denoting

$$\sigma(x, t) = L\bar{z}(x, t) - L \sum_{i=1}^N \chi_i(x) \bar{z}(x^i, t), \quad x \in \Omega, t \geq 0, \quad (8)$$

we present (7) as

$$\begin{aligned} \bar{z}_t &= \Delta_D \bar{z} + (a - L)\bar{z} + \sigma, \quad x \in \Omega, t > 0, \\ \bar{z}|_{\partial\Omega} &= 0, \quad \bar{z}|_{t=0} = z_0. \end{aligned} \quad (9)$$

If  $\sigma \equiv 0$ , then the system (9) is stable for a large enough injection gain  $L$ . If  $\Omega = (0, 1)$ , the error  $\sigma \neq 0$  can be bounded using Wirtinger's inequality as  $\|\sigma(\cdot, t)\| \leq 2/(N\pi) \|\bar{z}_x(\cdot, t)\|$ , which was used in [7] to prove the stability of (9) for large  $L$  and  $N$ . We prove the following lemma to bound the error  $\sigma$  in the case of  $\Omega = (0, 1)^2$ . This lemma refines [16, Lemma 4.1].

*Lemma 5:* Let  $f \in H^2((0, l)^2; \mathbb{R})$ ,  $f(0, 0) = 0$ . Then

$$\begin{aligned} \|f\|^2 &\leq \frac{1}{\alpha_1} \left(\frac{2l}{\pi}\right)^2 \left\| \frac{\partial f}{\partial x_1} \right\|^2 + \frac{1}{\alpha_2} \left(\frac{2l}{\pi}\right)^2 \left\| \frac{\partial f}{\partial x_2} \right\|^2 \\ &\quad + \frac{1}{\alpha_3} \left(\frac{2l}{\pi}\right)^4 \left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right\|^2 \end{aligned} \quad (10)$$

for any positive  $\alpha_1, \alpha_2, \alpha_3$  such that  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ .

*Proof.* Since  $\alpha_2 \in (0, 1)$ ,

$$\begin{aligned} \|f\|^2 &= \|f(\cdot, 0) + (f(\cdot, \cdot) - f(\cdot, 0))\|^2 \\ &\stackrel{\text{Lem.1}}{\leq} \frac{1}{1-\alpha_2} \|f(\cdot, 0)\|^2 + \frac{1}{\alpha_2} \|f(\cdot, \cdot) - f(\cdot, 0)\|^2 \\ &\stackrel{\text{Lem.2}}{\leq} \frac{1}{1-\alpha_2} \left(\frac{2l}{\pi}\right)^2 \|f_{x_1}(\cdot, 0)\|^2 + \frac{1}{\alpha_2} \left(\frac{2l}{\pi}\right)^2 \|f_{x_2}\|^2. \end{aligned}$$

Since  $\frac{\alpha_1}{1-\alpha_2} + \frac{\alpha_3}{1-\alpha_2} = 1$ ,

$$\begin{aligned} \|f_{x_1}(\cdot, 0)\|^2 &= \|f_{x_1}(\cdot, \cdot) + (f_{x_1}(\cdot, 0) - f_{x_1}(\cdot, \cdot))\|^2 \\ &\stackrel{\text{Lem.1}}{\leq} \frac{1-\alpha_2}{\alpha_1} \|f_{x_1}\|^2 + \frac{1-\alpha_2}{\alpha_3} \|f_{x_1}(\cdot, 0) - f_{x_1}(\cdot, \cdot)\|^2 \\ &\stackrel{\text{Lem.2}}{\leq} \frac{1-\alpha_2}{\alpha_1} \|f_{x_1}\|^2 + \frac{1-\alpha_2}{\alpha_3} \left(\frac{2l}{\pi}\right)^2 \|f_{x_1 x_2}\|^2. \end{aligned}$$

Combining these inequalities, we obtain (10).  $\blacksquare$

*Corollary 1:* Let  $f \in H^2((0, l)^2; \mathbb{R})$ ,  $f(0, 0) = 0$ ,  $\eta > 0$ . Then

$$\begin{aligned} \eta \|f\|^2 &\leq \lambda_1 \left(\frac{2l}{\pi}\right)^2 \left\| \frac{\partial f}{\partial x_1} \right\|^2 + \lambda_2 \left(\frac{2l}{\pi}\right)^2 \left\| \frac{\partial f}{\partial x_2} \right\|^2 \\ &\quad + \lambda_3 \left(\frac{2l}{\pi}\right)^4 \left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right\|^2 \end{aligned} \quad (11)$$

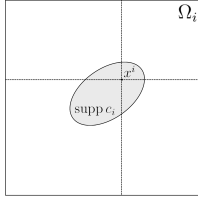


Fig. 2. Four rectangles cornered at  $x^i \in \text{supp } c_i$

for any  $\lambda_1, \lambda_2, \lambda_3$  satisfying

$$\text{diag}\{\lambda_1, \lambda_2, \lambda_3\} \geq \eta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \quad (12)$$

*Proof.* By the Schur complement, (12) is equivalent to

$$\begin{bmatrix} \eta^{-1} \text{diag}\{\lambda_1, \lambda_2, \lambda_3\} & \mathbf{1}_3 \\ \mathbf{1}_3^T & 1 \end{bmatrix} \geq 0,$$

where  $\mathbf{1}_3 = (1, 1, 1)^T$ , which is equivalent to

$$\begin{aligned} 0 &< \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}, \\ 0 &\leq 1 - \eta \mathbf{1}_3^T \text{diag}\{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\} \mathbf{1}_3 = 1 - \eta \sum_{i=1}^3 \lambda_i^{-1}. \end{aligned}$$

Thus, for

$$\alpha_1 = \frac{\eta}{\lambda_1}, \quad \alpha_2 = \frac{\eta}{\lambda_2}, \quad \alpha_3 = \frac{\eta}{\lambda_3},$$

we have  $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$ . Clearly, Lemma 5 remains true for  $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$  implying (11). ■

Each rectangle cornered at  $x^i \in \text{supp } c_i$  and lying in  $\Omega_i$  (see Fig. 2) has sides smaller than

$$l = \max_{i=1, \dots, N} \max_{\omega \in \partial \Omega_i} \max_{d \in \text{supp } c_i} |\omega - d|_\infty. \quad (13)$$

Applying Corollary 1 to  $\sigma$  defined in (8) on each of such rectangles and summing over them, we obtain

$$\begin{aligned} 0 &\leq -\eta \frac{\|\sigma\|^2}{L^2} + \lambda_1 \left(\frac{2l}{\pi}\right)^2 \|\bar{z}_{x_1}\|^2 \\ &\quad + \lambda_2 \left(\frac{2l}{\pi}\right)^2 \|\bar{z}_{x_2}\|^2 + \lambda_3 \left(\frac{2l}{\pi}\right)^4 \|\bar{z}_{x_1 x_2}\|^2 \end{aligned} \quad (14)$$

with  $\eta > 0$ ,  $\lambda_1, \lambda_2, \lambda_3$  satisfying (12). The positive terms in (14) can be made arbitrarily small by reducing  $l$ , i.e., by increasing the number of sensors  $N$ .

*Theorem 1:* Consider the system (1) with the measurements (3). For a given injection gain  $L$  and decay rate  $\alpha > 0$ , let there exist<sup>1</sup>

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0, \quad \eta > 0, \quad \lambda_i > 0, \quad i = 1, \dots, 6,$$

such that (12) is true,  $\Phi \leq 0$ , and  $\Phi_\nabla \leq 0$ , where

$$\begin{aligned} \Phi &= \begin{bmatrix} \Phi_{11} & 0 & 1 \\ * & \Phi_{22} & -\bar{p} \\ * & * & -\eta/L^2 \end{bmatrix}, \\ \Phi_{11} &= 2(a - L + \alpha) - (\lambda_5 + \lambda_6)\pi^2, \\ \Phi_{22} &= -\bar{p}\bar{d}^T - \bar{d}\bar{p}^T + \begin{bmatrix} 0 & \lambda_4 \\ \lambda_3(2l/\pi)^4 - 2\lambda_4 & 0 \\ \lambda_4 & 0 \end{bmatrix}, \\ \Phi_\nabla &= -2D + 2(a - L + \alpha)P + \frac{(2l)^2}{\pi^2} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \end{bmatrix} \end{aligned}$$

<sup>1</sup>MATLAB codes for solving the LMIs are available at <https://github.com/AntonSelivanov/CDC18>

with  $l$  defined in (13),  $\bar{p} = (p_1, 2p_2, p_3)^T$ , and  $\bar{d} = (d_1, 2d_2, d_3)^T$ . Then the state of the observer (5) exponentially converges to the state of the system (1) in the  $H_0^1$ -norm for any initial conditions  $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ :

$$\exists C > 0: \quad \|z(\cdot, t) - \hat{z}(\cdot, t)\|_{H_0^1} \leq C e^{-\alpha t} \|z_0\|_{H_0^1}.$$

*Proof.* Differentiating  $V_0 = \|\bar{z}\|^2$  along (9), we obtain

$$\dot{V}_0 = 2 \int_\Omega \bar{z} \bar{z}_t = 2 \int_\Omega \bar{z} [\Delta_D \bar{z} + (a - L)\bar{z} + \sigma].$$

Since  $\bar{z}|_{\partial\Omega} = 0$ , by the divergence theorem,

$$2 \int_\Omega \bar{z} \Delta_D \bar{z} = 2 \int_\Omega \bar{z} \text{div}(D\nabla \bar{z}) = -2 \int_\Omega (\nabla \bar{z})^T D \nabla \bar{z}.$$

Therefore,

$$\begin{aligned} \dot{V}_0 + 2\alpha V_0 &= -2 \int_\Omega (\nabla \bar{z})^T D \nabla \bar{z} \\ &\quad + 2(a - L + \alpha) \int_\Omega \bar{z}^2 + 2 \int_\Omega \bar{z} \sigma. \end{aligned} \quad (15)$$

If  $\sigma \equiv 0$ , (15) is nonpositive for a large  $L$  implying the exponential stability of (9) in  $L^2$ . To compensate  $\sigma \neq 0$ , we will use (14) that contains  $\|\bar{z}_{x_1 x_2}\|^2$ . For  $\bar{z} \in C_0^\infty$  integration by parts yields

$$0 = -2\lambda_4 \int_\Omega \bar{z}_{x_1 x_2}^2 + 2\lambda_4 \int_\Omega \bar{z}_{x_1 x_1} \bar{z}_{x_2 x_2}. \quad (16)$$

Since  $C_0^\infty$  is dense in  $H^2 \cap H_0^1$ , the latter holds for  $\bar{z} \in H^2 \cap H_0^1$ . To compensate  $\bar{z}_{x_1 x_1}$  and  $\bar{z}_{x_2 x_2}$ , we consider

$$V_1 = \int_\Omega (\nabla \bar{z}(x, t))^T P \nabla \bar{z}(x, t) dx. \quad (17)$$

Since  $\bar{z}|_{\partial\Omega} = 0$ , by the divergence theorem,

$$\dot{V}_1 = 2 \int_\Omega (\nabla \bar{z})^T P \nabla \bar{z}_t = -2 \int_\Omega \text{div}(P \nabla \bar{z}) \bar{z}_t.$$

Substituting  $\bar{z}_t$ , we obtain

$$\begin{aligned} \dot{V}_1 + 2\alpha V_1 &= -2 \int_\Omega \text{div}(P \nabla \bar{z}) \text{div}(D \nabla \bar{z}) \\ &\quad + 2(a - L + \alpha) \int_\Omega (\nabla \bar{z})^T P \nabla \bar{z} - 2 \int_\Omega \text{div}(P \nabla \bar{z}) \sigma, \end{aligned} \quad (18)$$

where we used the divergence theorem to obtain the second summand. Wirtinger's inequality (Lemma 2) implies

$$0 \leq -(\lambda_5 + \lambda_6)\pi^2 \int_\Omega \bar{z}^2 + \int_\Omega (\nabla \bar{z})^T \begin{bmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \end{bmatrix} \nabla \bar{z}. \quad (19)$$

Summing up (14)–(19), for  $V = V_0 + V_1$  we obtain

$$\dot{V} + 2\alpha V \leq \int_\Omega \varphi^T \Phi \varphi + \int_\Omega (\nabla \bar{z})^T \Phi_\nabla \nabla \bar{z} \leq 0,$$

where  $\varphi = (\bar{z}, \bar{z}_{x_1 x_1}, \bar{z}_{x_1 x_2}, \bar{z}_{x_2 x_2}, \sigma)^T$ . Thus,  $\dot{V} \leq -2\alpha V$  implying the exponential stability of (9) in the  $H_0^1$ -norm. ■

*Remark 3 (Feasibility of LMIs):* The LMIs of Theorem 1 are always feasible for a large enough injection gain  $L$  and small enough  $l$  defined in (13). Indeed,  $D > 0$  implies  $d_1 d_3 - d_2^2/\nu > 0$  for a large enough  $\nu < 1$ . Since

$$\begin{aligned} 2 \begin{bmatrix} 0 & -d_1 d_2 \\ -d_1 d_2 & 0 \end{bmatrix} &\leq 2 \text{diag}\{\nu d_1^2, d_2^2/\nu\}, \\ 2 \begin{bmatrix} 0 & -d_2 d_3 \\ -d_2 d_3 & 0 \end{bmatrix} &\leq 2 \text{diag}\{d_2^2/\nu, \nu d_3^2\}, \end{aligned}$$

for  $l = 0$ ,  $p_1 = d_3$ ,  $p_2 = 0$ ,  $p_3 = d_1$ , and  $\lambda_4 = d_1^2 + d_3^2$ ,

$$\Phi_{22} \leq \begin{bmatrix} -2(d_1 d_3 - \frac{d_2^2}{\nu}) & 0 & 0 \\ 0 & -2(1-\nu)\lambda_4 & 0 \\ 0 & 0 & -2(d_1 d_3 - \frac{d_2^2}{\nu}) \end{bmatrix} < 0.$$

Therefore,  $\Phi < 0$  for large enough  $L$  and  $\eta$ . Clearly,  $\Phi_{\nabla} < 0$  for a large enough  $L$  and (12) holds for large enough  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ . Thus, the LMIs of Theorem 1 are feasible for  $l = 0$ . By continuity, they remain so for a small enough  $l$ .

*Corollary 2:* The observer (5) provides exponentially converging state estimate of the system (1), (3) if the injection gain  $L$  is large enough and  $l$  defined in (13) is small enough (i.e., the number of sensors  $N$  is large enough).

*Remark 4 (Boundary conditions):* The results can be extended to (1) with the boundary conditions

$$z|_{\Gamma_D} = 0, \quad \frac{\partial z}{\partial \mathbf{n}}|_{\Gamma_N} = 0, \quad \partial\Omega = \Gamma_D \cup \Gamma_N, \quad \Gamma_D \cap \Gamma_N = \emptyset,$$

where  $\mathbf{n}$  denotes the normal to  $\Gamma_N$ . All the calculations of Theorem 1 remain valid except for (19), which according to the Wirtinger inequality (Lemma 2) should be replaced by

$$\begin{aligned} 0 &\leq -\lambda_5 c_1 \pi^2 \int_{\Omega} \bar{z}^2 + \lambda_5 \int_{\Omega} \bar{z}_{x_1}^2, \\ 0 &\leq -\lambda_6 c_2 \pi^2 \int_{\Omega} \bar{z}^2 + \lambda_6 \int_{\Omega} \bar{z}_{x_2}^2, \end{aligned}$$

where

$$c_1 = \begin{cases} 1, & \text{if } z(0, x_2) = z(1, x_2) = 0, \forall x_2 \in (0, 1) \\ \frac{1}{4}, & \text{if } z(0, x_2) = 0 \text{ or } z(1, x_2) = 0, \forall x_2 \in (0, 1) \\ 0, & \text{otherwise,} \end{cases}$$

$$c_2 = \begin{cases} 1, & \text{if } z(x_1, 0) = z(x_1, 1) = 0, \forall x_1 \in (0, 1) \\ \frac{1}{4}, & \text{if } z(x_1, 0) = 0 \text{ or } z(x_1, 1) = 0, \forall x_1 \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

*Remark 5 (3D domains):* If  $\Omega = (0, 1)^3$ , an upper bound for  $\sigma$  similar to (14) can be derived. This bound will involve the 3rd order space derivative, which we do not know how to compensate. Thus, it is not clear how to extend the proposed method to 3D domains.

### III. $H_{\infty}$ FILTERING OF A 2D HEAT EQUATION

Consider the reaction-diffusion system

$$\begin{aligned} z_t(x, t) &= \Delta_D z(x, t) + a z(x, t) + w(x, t), \quad t > 0, x \in \Omega \\ z|_{\partial\Omega} &= 0, \quad z|_{t=0} = z_0 \end{aligned} \quad (20)$$

defined on  $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$  with the state  $z: \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$ , diffusion term (2), reaction coefficient  $a$ , and disturbance  $w \in L^2(0, \infty; L^2(\Omega))$ .

Similarly to the previous section, the domain  $\Omega$  is divided into  $N$  square subdomains  $\Omega_i$  (Fig. 1) with a sensor placed in each  $\Omega_i$  providing the measurements

$$\begin{aligned} y_i(t) &= \int_{\Omega_i} c_i(\xi) z(\xi, t) d\xi + v_i(t), \\ 0 &\leq c_i \in L^2(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i = 1, \dots, N, \end{aligned} \quad (21)$$

where  $v_i \in L^2(0, \infty)$  is the measurement noise.

Consider the observer (5). The estimation error  $\bar{z}(x, t) = z(x, t) - \hat{z}(x, t)$  satisfies (cf. (7))

$$\begin{aligned} \bar{z}_t &= \Delta_D \bar{z} + a \bar{z} - L \sum_{i=1}^N \chi_i(x) \bar{z}(x^i, t) + w - v, \\ \bar{z}|_{\partial\Omega} &= 0, \quad \bar{z}|_{t=0} = z_0, \end{aligned} \quad (22)$$

where

$$v(x, t) = L \sum_{i=1}^N \chi_i(x) v_i(t).$$

We assume that  $w$  and  $v$  are such that (22) is well-posed. E.g., if  $w, v \in C^1([0, \infty), L^2)$ , the system (22) has a unique classical solution for any  $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$  [21, Theorem 3.1.3].

We say that (22) has an  $L^2$ -gain not greater than  $\gamma$ , if

$$\int_0^{\infty} \|\bar{z}(\cdot, t)\|^2 dt \leq \gamma^2 \int_0^{\infty} [\|w(\cdot, t)\|^2 + \|v(\cdot, t)\|^2] dt \quad (23)$$

for  $z_0 = 0$  and any  $w, v \in L^2(0, \infty; L^2(\Omega))$ .

*Theorem 2:* Consider the system (20) with the measurements (21). For a given injection gain  $L$  and decay rate  $\alpha > 0$ , let there exist<sup>2</sup>

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0, \quad \eta, \gamma_1, \gamma_2 > 0, \quad \lambda_i > 0, \quad i = 1, \dots, 6,$$

such that (12) is true,  $\Psi \leq 0$ , and  $\Phi_{\nabla} \leq 0$ , where

$$\Psi = \left[ \begin{array}{ccc|cc} & & & 1 & 1 \\ & \bar{\Phi} & & -\bar{p} & -\bar{p} \\ * & * & * & 0 & 0 \\ * & * & * & -\gamma_2 & 0 \\ * & * & * & * & -\gamma_2 \end{array} \right],$$

$\bar{\Phi}$  coincides with  $\Phi$  from Theorem 1 except for

$$\bar{\Phi}_{11} = 2(a - L + \alpha) - (\lambda_5 + \lambda_6)\pi^2 + \gamma_1,$$

$\bar{p} = (p_1, 2p_2, p_3)^T$ , and  $\Phi_{\nabla}$  is given in Theorem 1. Then (22) has an  $L^2$ -gain not greater than  $\gamma = \sqrt{\gamma_2/\gamma_1}$ .

*Proof.* Using (8), we present (22) as (cf. (9))

$$\begin{aligned} \bar{z}_t &= \Delta_D \bar{z} + (a - L)\bar{z} + \sigma + w - v, \quad x \in \Omega, t > 0, \\ \bar{z}|_{\partial\Omega} &= 0, \quad \bar{z}|_{t=0} = z_0. \end{aligned} \quad (24)$$

Differentiating  $V_0 = \|\bar{z}\|^2$  and  $V_1$  defined in (17) along (24) and using the divergence theorem, we obtain (cf. (15), (18))

$$\begin{aligned} \dot{V}_0 + 2\alpha V_0 &= -2 \int_{\Omega} (\nabla \bar{z})^T D \nabla \bar{z} + 2(a - L + \alpha) \int_{\Omega} \bar{z}^2 \\ &\quad + 2 \int_{\Omega} \bar{z} \sigma + 2 \int_{\Omega} \bar{z} [w - v], \end{aligned} \quad (25)$$

$$\begin{aligned} \dot{V}_1 + 2\alpha V_1 &= -2 \int_{\Omega} \operatorname{div}(P \nabla \bar{z}) \operatorname{div}(D \nabla \bar{z}) \\ &\quad + 2(a - L + \alpha) \int_{\Omega} (\nabla \bar{z})^T P \nabla \bar{z} - 2 \int_{\Omega} \operatorname{div}(P \nabla \bar{z}) \sigma \\ &\quad - 2 \int_{\Omega} \operatorname{div}(P \nabla \bar{z}) [w - v]. \end{aligned} \quad (26)$$

Summing up (14), (16), (19), (25), and (26), for  $V = V_0 + V_1$  we obtain

$$\begin{aligned} \dot{V} + 2\alpha V &+ \gamma_1 \|\bar{z}(\cdot, t)\|^2 - \gamma_2 [\|w(\cdot, t)\|^2 + \|v(\cdot, t)\|^2] \\ &\leq \int_{\Omega} \psi^T \Psi \psi + \int_{\Omega} (\nabla \bar{z})^T \Phi_{\nabla} \nabla \bar{z} \leq 0, \end{aligned} \quad (27)$$

where  $\psi = (\bar{z}, \bar{z}_{x_1}, \bar{z}_{x_2}, \sigma, w, -v)^T$ . Integrating (27) from 0 to  $\infty$  with  $\bar{z}(\cdot, 0) = 0$ , we obtain (23) with  $\gamma = \sqrt{\gamma_2/\gamma_1}$ . ■

<sup>2</sup>MATLAB codes for solving the LMIs are available at <https://github.com/AntonSelivanov/CDC18>

#### IV. SAMPLED-DATA $H_\infty$ FILTERING

Consider the reaction-diffusion system (20). Let the domain  $\Omega$  be divided into  $N$  square subdomains  $\Omega_i$  (Fig. 1) with a sensor placed in each  $\Omega_i$  providing the sampled in time measurements (cf. (21))

$$y_{i,k} = \int_{\Omega_i} c_i(\xi) z(\xi, t_k) d\xi + v_{i,k}, \quad (28)$$

$$0 \leq c_i \in L^\infty(\Omega_i), \quad \int_{\Omega_i} c_i = 1, \quad i = 1, \dots, N,$$

where  $v_{i,k}$  is the measurement noise and the sampling instants  $t_k$  with  $k \in \mathbb{N}$  satisfy

$$0 = t_1 < t_2 < \dots, \quad \lim t_k = \infty, \quad t_{k+1} - t_k \leq h.$$

We study the sampled-data observer (cf. (5))

$$\begin{aligned} \hat{z}_t(x, t) &= \Delta_D \hat{z}(x, t) + a \hat{z}(x, t) + L \sum_{i=1}^N \chi_i(x) \times \\ &\quad \left[ y_{i,k} - \int_{\Omega_i} c_i(\xi) \hat{z}(\xi, t_k) d\xi \right], \quad t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ \hat{z}|_{\partial\Omega} &= 0, \quad \hat{z}|_{t=0} = 0 \end{aligned} \quad (29)$$

with the injection gain  $L$  and characteristic functions  $\chi_i$  defined in (6). The estimation error  $\bar{z}(x, t) = z(x, t) - \hat{z}(x, t)$  satisfies (cf. (7), (22))

$$\begin{aligned} \bar{z}_t &= \Delta_D \bar{z} + a \bar{z} - L \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) \bar{z}(\xi, t_k) d\xi \\ &\quad + w - v, \quad t \in [t_k, t_{k+1}), \\ \bar{z}|_{\partial\Omega} &= 0, \quad \bar{z}|_{t=0} = z_0, \end{aligned} \quad (30)$$

where  $v(x, t) = L \sum_{i=1}^N \chi_i(x) v_{i,k}$  for  $t \in [t_k, t_{k+1})$ . The existence of a unique classical solution of (30) for  $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$  can be established using the step method.

**Theorem 3:** Consider the system (20) with the measurements (28). For a given injection gain  $L$ , decay rate  $\alpha > 0$ , and maximum sampling period  $h$ , let there exist<sup>3</sup>

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0, \quad \eta, \gamma_1, \gamma_2, \nu > 0, \quad \lambda_i > 0, \quad i = 1, \dots, 6,$$

such that (12) is true and  $\Upsilon \leq 0$ ,  $\Phi_\nabla \leq 0$ , where

$$\Upsilon = \begin{bmatrix} & & & & & & 1 & \nu h(a-L) \\ & & & & & & -\bar{p} & \nu h \bar{d} \\ & & & & & & 0 & \nu h \\ & & & & & & 0 & \nu h \\ & & & & & & 0 & \nu h \\ & & & & & & 0 & \nu h \\ \hline * & * & * & * & * & & -\nu & \nu h \\ * & * & * & * & * & & * & \Upsilon_{77} \end{bmatrix},$$

$$\Upsilon_{77} = -\frac{\pi^2 N \nu e^{-2\alpha h}}{4L^2 \max_i \|c_i\|_\infty},$$

$\Psi$  is given in Theorem 2,  $\bar{p} = (p_1, 2p_2, p_3)^T$ ,  $\bar{d} = (d_1, 2d_2, d_3)^T$ , and  $\Phi_\nabla$  is given in Theorem 1. Then (30) has an  $L^2$ -gain less or equal than  $\gamma = \sqrt{\gamma_2/\gamma_1}$ .

*Proof.* By the mean value theorem, for  $t \in [t_k, t_{k+1})$

$$\begin{aligned} L \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) \bar{z}(\xi, t_k) d\xi \\ &= L \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) \bar{z}(\xi, t) d\xi - \kappa(x, t) \\ &= L \sum_{i=1}^N \chi_i(x) \bar{z}(x^i(t), t) - \kappa(x, t) \\ &= L \bar{z}(x, t) - \sigma(x, t) - \kappa(x, t) \end{aligned}$$

<sup>3</sup>MATLAB codes for solving the LMIs are available at <https://github.com/AntonSelivanov/CDC18>

with  $x^i(t) \in \text{conv}(\text{supp } c_i)$ ,  $\sigma(x, t)$  defined in (8), and

$$\begin{aligned} \kappa(x, t) &= L \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) [\bar{z}(\xi, t) - \bar{z}(\xi, t_k)] d\xi, \\ &\quad x \in \Omega, \quad t \in [t_k, t_{k+1}). \end{aligned}$$

Then the error system (30) takes the form (cf. (24))

$$\begin{aligned} \bar{z}_t &= \Delta_D \bar{z} + (a - L) \bar{z} + \sigma + \kappa + w - v, \\ \bar{z}|_{\partial\Omega} &= 0, \quad \bar{z}|_{t=0} = z_0. \end{aligned} \quad (31)$$

Let  $V = V_0 + V_1 + V_\kappa$  with  $V_0 = \|\bar{z}\|^2$ ,  $V_1$  from (17), and

$$\begin{aligned} V_\kappa &= \frac{4\nu h^2}{\pi^2} e^{2\alpha h} \int_{t_k}^t e^{-2\alpha(t-s)} \int_{\Omega} \kappa_t^2(x, s) dx ds \\ &\quad - \nu \int_{t_k}^t e^{-2\alpha(t-s)} \int_{\Omega} \kappa^2(x, s) dx ds, \quad t \in [t_k, t_{k+1}). \end{aligned}$$

Due to Lemma 3,  $V_\kappa \geq 0$ . Moreover,  $V_\kappa$  does not grow at the jumps  $t_k$ , since  $V_\kappa(t_k) = 0$ . Differentiating  $V$  along (31), we have (cf. (25), (26))

$$\begin{aligned} \dot{V}_0 + 2\alpha V_0 &= -2 \int_{\Omega} (\nabla \bar{z})^T D \nabla \bar{z} + 2(a - L + \alpha) \int_{\Omega} \bar{z}^2 \\ &\quad + 2 \int_{\Omega} \bar{z} \sigma + 2 \int_{\Omega} \bar{z} \kappa + 2 \int_{\Omega} \bar{z} [w - v], \end{aligned} \quad (32)$$

$$\begin{aligned} \dot{V}_1 + 2\alpha V_1 &= -2 \int_{\Omega} \text{div}(P \nabla \bar{z}) \text{div}(D \nabla \bar{z}) \\ &\quad + 2(a - L + \alpha) \int_{\Omega} (\nabla \bar{z})^T P \nabla \bar{z} - 2 \int_{\Omega} \text{div}(P \nabla \bar{z}) \sigma \\ &\quad - 2 \int_{\Omega} \text{div}(P \nabla \bar{z}) \kappa - 2 \int_{\Omega} \text{div}(P \nabla \bar{z}) [w - v], \end{aligned} \quad (33)$$

$$\dot{V}_\kappa + 2\alpha V_\kappa = \frac{4\nu h^2}{\pi^2} e^{2\alpha h} \int_{\Omega} \kappa_t^2(x, t) dx - \nu \int_{\Omega} \kappa^2(x, t) dx. \quad (34)$$

The positive term in (34) can be bounded as

$$\begin{aligned} \int_{\Omega} \kappa_t^2(x, t) dx &= \int_{\Omega} \left( L \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) \bar{z}_t(\xi, t) d\xi \right)^2 dx \\ &= L^2 \int_{\Omega} \sum_{i=1}^N \chi_i(x) \left( \int_{\Omega_i} c_i(\xi) \bar{z}_t(\xi, t) d\xi \right)^2 dx \\ &\stackrel{\text{Lem.4}}{\leq} L^2 \int_{\Omega} \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} c_i(\xi) \bar{z}_t^2(\xi, t) d\xi dx \\ &\leq L^2 \max_i \|c_i\|_\infty \int_{\Omega} \sum_{i=1}^N \chi_i(x) \int_{\Omega_i} \bar{z}_t^2(\xi, t) d\xi dx \\ &= \max_i \|c_i\|_\infty \frac{L^2}{N} \int_{\Omega} \bar{z}_t^2(\xi, t) d\xi. \end{aligned}$$

Summing up (14), (16), (19), (32)–(34), we obtain

$$\begin{aligned} \dot{V} + 2\alpha V + \gamma_1 \|\bar{z}(\cdot, t)\|^2 - \gamma_2 [\|w(\cdot, t)\|^2 + \|v(\cdot, t)\|^2] \\ &\leq \int_{\Omega} v^T \Upsilon v + \int_{\Omega} (\nabla \bar{z})^T \Phi_\nabla \nabla \bar{z} \\ &\quad + \frac{4\nu h^2}{\pi^2} e^{2\alpha h} \max_i \|c_i\|_\infty \frac{L^2}{N} \int_{\Omega} \bar{z}_t^2(x, t) dx, \end{aligned}$$

where  $v = (\bar{z}, \bar{z}_{x_1 x_1}, \bar{z}_{x_1 x_2}, \bar{z}_{x_2 x_2}, \sigma, w, -v, \kappa)^T$  and  $\Upsilon$  is obtained from  $\Upsilon$  by eliminating the last block-column and block-row. Substituting (31) for  $\bar{z}_t$  and using the Schur complement, we obtain that  $\Upsilon < 0$  and  $\Phi_\nabla < 0$  guarantee

$$\dot{V} + 2\alpha V + \gamma_1 \|\bar{z}(\cdot, t)\|^2 - \gamma_2 [\|w(\cdot, t)\|^2 + \|v(\cdot, t)\|^2] \leq 0.$$

Integrating it from 0 to  $\infty$  with  $\bar{z}(\cdot, 0) = 0$ , we obtain (23) with  $\gamma = \sqrt{\gamma_2/\gamma_1}$ .  $\blacksquare$

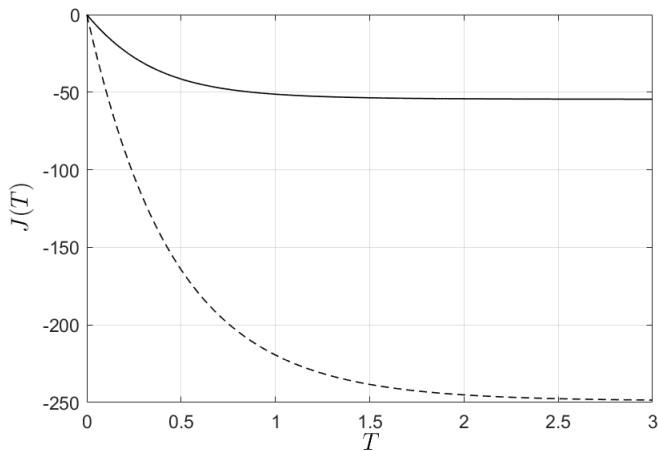


Fig. 3. Performance index  $J(T)$  on  $[0, 3]$  for continuous-time measurements (21) with  $\gamma = 2.4$  (solid line) and sampled-data measurements (28) with  $h = 10^{-3}$  and  $\gamma = 4.6$  (dashed line).

### V. EXAMPLE

Consider the system (20) with  $D = \text{diag}\{1, 0.8\}$  and  $a = 2\pi^2$ . Let the domain  $\Omega = (0, 1)^2$  be divided into  $N = 36$  squares of side length  $1/\sqrt{N} = 1/6$ . Let the measurements be given by (21) with  $c_i$  defined in (4), where  $x_c^i$  are the centers of  $\Omega_i$  and  $\varepsilon = 0.05$ . Then  $l = 1/(2\sqrt{N}) + \varepsilon/2 \approx 0.1$  according to (13). The LMIs of Theorem 2 are feasible for  $L = 5$ ,  $\gamma = 2.4$ ,  $\alpha = 0.01$ . Thus, the observer (5) provides  $H_\infty$  filtering of the system (20) with the  $L^2$ -gain not greater than  $\gamma = 2.4$ . Fig. 3 shows the evolution of

$$J(T) = \int_0^T [\|\bar{z}(\cdot, t)\|^2 - \gamma^2 \|w(\cdot, t)\|^2 - \gamma^2 \|v(\cdot, t)\|^2] dt$$

for  $z_0 \equiv 0$ ,  $w(x, t) = e^{-t} \sin(\pi x) \sin(\pi y)$ ,  $v_i(t) = e^{-t}$ . It remains negative implying that (23) is satisfied. For this choice of  $w$  and  $v_i$  the smallest  $L_2$ -gain obtained from the numerical simulations is  $\gamma = 1.2$ .

The LMIs of Theorem 3 are feasible for  $\gamma = 4.6$  and  $h = 10^{-3}$  (other parameters are the same). Therefore, the sampled-data observer (29) provides  $H_\infty$  filtering of the system (20) with the  $L^2$ -gain not greater than  $\gamma = 4.6$ .

### VI. CONCLUSIONS

Design of sampled-data observers for 2D parabolic systems with point measurements was an open problem [13]. This paper suggested a solution to this problem for linear 2D reaction-diffusion systems with the pointlike measurements modeled as the state values averaged over small subdomains. The solution is based on a novel bound on the  $L_2$ -norm of the difference between the state and its point value in terms of a reciprocally convex combination of the  $L_2$ -norms of the first and second order state derivatives. The results can be extended to semilinear systems as considered in [13]. Extension of the results to the controller design is a topic of the future research.

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