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An improved time-delay implementation of derivative-dependent feedback

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Abstract

We consider an LTI system of relative degree $r \geq 2$ that can be stabilized using $r-1$ output derivatives. The derivatives are approximated by finite differences leading to a time-delayed feedback. We present a new method of designing and analyzing such feedback under continuous-time and sampled measurements. This method admits essentially larger time-delay/sampling period compared to the existing results and, for the first time, allows to use consecutively sampled measurements in the sampled-data case. The main idea is to present the difference between the derivative and its approximation in a convenient integral form. The kernel of this integral is hard to express explicitly but we show that it satisfies certain properties. These properties are employed to construct the Lyapunov-Krasovskii functional that leads to LMI-based stability conditions. If the derivative-dependent control exponentially stabilizes the system, then its time-delayed approximation stabilizes the system with the same decay rate provided the time-delay (for continuous-time measurements) or the sampling period (for sampled measurements) are small enough.

Key words: LTI systems; delay-induced stabilization; derivative approximation; LMIs

1 Introduction

Control laws that depend on output derivatives are used to stabilize LTI systems with relative degrees greater than one. To estimate the derivatives, which can hardly be measured directly, one can use the finite differences, i.e., $\dot{y} \approx (y(t) - y(t-h))/h$. Such approximation leads to time-delayed feedback that preserves the stability if the delay $h > 0$ is small enough [1–3]. For a given h, the delay-induced stability can be checked using frequency-domain techniques [4–7] or complete Lyapunov-Krasovskii functionals [8–10], which give necessary and sufficient conditions.

The delay-induced stability can be also studied using linear matrix inequalities (LMIs) [11–13]. The advantage of LMIs is that, though being conservative, they allow for performance and robustness analysis, can cope with certain types of nonlinearities [14], and can deal with stochastic perturbations [15,16]. Simple and yet efficient LMIs for the delay-induced stability were obtained in [15,16]. The key idea was to use the Taylor's expansion of the delayed terms with the remainders in the integral form that are compensated by appropriate terms in the Lyapunov-Krasovskii functional. Compared to [11–13], the resulting LMIs have a lower order, contain less decision variables, and were proved to be feasible for small delays if the derivative-dependent feedback stabilizes the system.

Another advantage of LMI-based conditions is that they can be extended to sampled-data systems. This has been done using discretized Lyapunov functionals with a Wirtinger-based term in [17]. Another LMIs for sampled-data stabilization were derived in [18] by employing impulsive system representation and looped Lyapunov functionals. The high-order LMIs obtained in [17] and [18] contain many decision variables, which make them hard to solve numerically. Using the ideas of [15,16], simple LMIs for sampled-data delay-induced stabilization were derived in [19]. These conditions were proved to be feasible for a small enough sampling period if the continuous-time derivative-dependent feedback stabilizes the system.

In this paper, we essentially improve the results of [16] for continuous-time measurements (Section 2) and the results of [19] for sampled measurements (Section 3). Namely, we derive simple LMIs that are feasible for significantly larger values of time-delay (Remark 2) and sampling period (Remark 3). Such improvement is achieved using an original integral representation of the difference between the derivative and its approximation (Proposition 1). The kernel of this integral is hard to express explicitly but we show

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that it satisfies certain properties (Proposition 2). These properties are employed to construct Lyapunov-Krasovskii terms that bound the approximation errors and lead to LMI-based stability conditions. Compared to [16,19], such approach leads to a more natural design of the controller gains in the delayed feedback. Moreover, the considered sampled-data delayed controller uses *consecutive* measurements, while [19] used *distant* measurements (cf. (25) and (29)). All these improvements allow to use less memory and slower sampling when one uses time-delays to implement derivative-dependent feedback. Finally, we show that if the derivative-dependent controller exponentially stabilizes the system with a decay rate $\alpha' > 0$, then the LMIs are feasible for any decay rate $\alpha < \alpha'$ and small enough time-delay/sampling period.

The part of this paper corresponding to the sampled-data implementation of the first order derivative was presented in [20]. These results were used in [21] to study sampled-data implementation of PID control.

Notations: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbf{1}_r = [1, \ldots, 1]^T \in \mathbb{R}^r$, $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix, $\otimes$ stands for the Kronecker product, $\mathrm{diag}\{R_i\}_{i=1}^{r-1}$ is the block-diagonal matrix with R_i being on the diagonal, $0 < P \in \mathbb{R}^{n\times n}$ denotes that P is symmetric and positive-definite, C^i is a class of i times continuously differentiable functions.

Auxiliary lemmas

Lemma 1 (Exponential Wirtinger inequality [22]) *Let $f\colon [a,b] \to \mathbb{R}^n$ be an absolutely continuous function with a square integrable first order derivative such that $f(a) = 0$ or $f(b) = 0$. Then*

$$\int_a^b e^{2\alpha t} f^T(t) W f(t)\,dt \le e^{2|\alpha|(b-a)} \frac{4(b-a)^2}{\pi^2} \int_a^b e^{2\alpha t} \dot{f}^T(t) W \dot{f}(t)\,dt$$

for any $\alpha \in \mathbb{R}$ and $0 \le W \in \mathbb{R}^{n\times n}$.

Lemma 2 (Jensen's inequality [23]) *Let $\rho\colon [a,b] \to [0,\infty)$ and $f\colon [a,b] \to \mathbb{R}^n$ be such that the integration concerned is well-defined. Then for any $0 < Q \in \mathbb{R}^{n\times n}$,*

$$\left[\int_a^b \rho(s) f(s)\,ds\right]^T Q \left[\int_a^b \rho(s) f(s)\,ds\right] \le \int_a^b \rho(s)\,ds \int_a^b \rho(s) f^T(s) Q f(s)\,ds.$$

2 Continuous-time control

Consider the LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned} \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^l \tag{1}$$

with relative degree $r \ge 2$, i.e.,

$$CA^iB = 0, \quad i = 0, 1, \ldots, r-2, \qquad CA^{r-1}B \ne 0. \tag{2}$$

Relative degree is how many times the output $y(t)$ needs to be differentiated before the input $u(t)$ appears explicitly. In particular, (2) implies

$$y^{(i)} = CA^i x, \quad i = 0, 1, \ldots, r-1. \tag{3}$$

To prove (3), note that it is trivial for $i = 0$ and, if it has been proved for $i < r-1$, it holds for $i+1$:

$$y^{(i+1)} = \left(y^{(i)}\right)' \overset{(3)}{=} (CA^i x)' \overset{(1)}{=} CA^i[Ax + Bu] \overset{(2)}{=} CA^{i+1}x.$$

For LTI systems with relative degree r, it is common to look for a stabilizing controller of the form

$$u(t) = \bar{K}_0 y(t) + \bar{K}_1 y^{(1)}(t) + \ldots + \bar{K}_{r-1} y^{(r-1)}(t). \tag{4}$$

Remark 1 *The control law (4) essentially reduces the system's relative degree from $r \ge 2$ to $r = 1$. Indeed, due to (2), the transfer matrix of (1) has the form*

$$W(s) = \frac{\beta_r s^{n-r} + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

with $\beta_r = CA^{r-1}B \ne 0$. Taking $u(t) = \hat{K}_0 u_0(t) + \hat{K}_1 u_0^{(1)}(t) + \cdots + \hat{K}_{r-1} u_0^{(r-1)}(t)$, one has

$$\tilde{y}(s) = \frac{(\beta_r s^{n-r} + \cdots + \beta_n)(\hat{K}_{r-1} s^{r-1} + \cdots + \hat{K}_0)}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n} \tilde{u}_0(s),$$

where $\tilde{y}$ and $\tilde{u}_0$ are the Laplace transforms of y and u_0. If $\beta_r \hat{K}_{r-1} \ne 0$, the latter system has relative degree one. If it can be stabilized by $u_0 = Ky$ then (1) can be stabilized by (4) with $\bar{K}_i = \hat{K}_i K$.

The controller (4) depends on the output derivatives, which are hard to measure directly. Instead, the derivatives can be approximated by finite-differences $\tilde{y}_i(t) \approx y^{(i)}(t)$:

$$\begin{aligned}\tilde{y}_0(t) &= y(t),\\ \tilde{y}_i(t) &= \frac{\tilde{y}_{i-1}(t) - \tilde{y}_{i-1}(t-h)}{h}\\ &= \tfrac{1}{h^i} \textstyle\sum_{k=0}^{i} \binom{i}{k} (-1)^k y(t-kh), \quad i \in \mathbb{N}\end{aligned} \tag{5}$$

with a delay $h > 0$ and the binomial coefficients $\binom{i}{k} = \frac{i!}{k!(i-k)!}$. Replacing $y^{(i)}$ in (4) with their approximations $\tilde{y}_i$, we obtain the delay-dependent control

$$u(t) = \textstyle\sum_{i=0}^{r-1} \bar{K}_i \tilde{y}_i(t) \overset{(5)}{=} \sum_{i=0}^{r-1} K_i y(t-ih), \tag{6}$$

where we set [1] $y(t) = y(0)$ for $t < 0$ and

$$K_i = (-1)^i \sum_{j=i}^{r-1} \binom{j}{i} \frac{1}{h^j} \bar{K}_j, \quad i = 0, \ldots, r-1. \tag{7}$$

If (1) can be stabilized by the derivative-dependent control (4), then it can be stabilized by the delay-dependent control (6) with small enough delays [3]. In this section, we derive simple and yet efficient LMIs that allow to obtain appropriate value of the delay $h > 0$. The first step is to present the approximation error $y^{(i)}(t) - \tilde{y}_i(t)$ in a convenient form suitable for the analysis via Lyapunov-Krasovskii functionals.

[1] Then $y^{(i)}(0)$ with $i > 0$ are approximated by 0

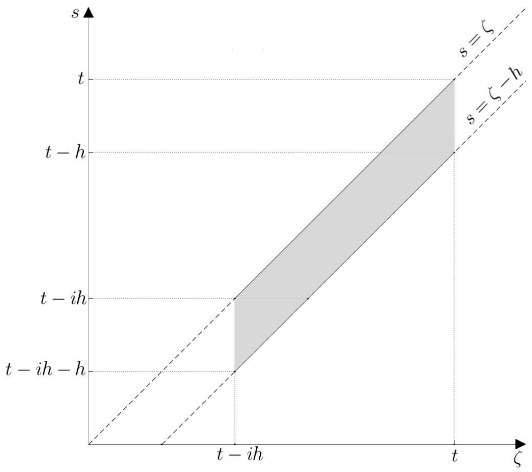

Fig. 1. Change of the integration order in $\mathcal{I}$

Proposition 1 *If $y \in C^i$ and $y^{(i)}$ is absolutely continuous with $i \in \mathbb{N}$, then $\tilde{y}_i$ defined in (5) satisfies*

$$\tilde{y}_i(t) = y^{(i)}(t) - \int_{t-ih}^{t} \varphi_i(t-s) y^{(i+1)}(s)\, ds, \tag{8}$$

where $\varphi_1(\xi) = \frac{h-\xi}{h}$ and for $i \in \mathbb{N}$,

$$\varphi_{i+1}(\xi) = \begin{cases} \int_0^{\xi} \frac{\varphi_i(\lambda)}{h}\, d\lambda + \frac{h-\xi}{h}, & \xi \in [0,h], \\ \int_{\xi-h}^{\xi} \frac{\varphi_i(\lambda)}{h}\, d\lambda, & \xi \in (h, ih), \\ \int_{\xi-h}^{ih} \frac{\varphi_i(\lambda)}{h}\, d\lambda, & \xi \in [ih, ih+h]. \end{cases} \tag{9}$$

Proof. For $i \in \mathbb{N}$, Taylor's expansion with the remainder in the integral form gives

$$y^{(i-1)}(t-h) = y^{(i-1)}(t) - y^{(i)}(t)h - \int_{t-h}^{t} (t-h-s) y^{(i+1)}(s)\, ds.$$

Reorganizing the terms, we obtain

$$\frac{y^{(i-1)}(t) - y^{(i-1)}(t-h)}{h} = y^{(i)}(t) - \int_{t-h}^{t} \frac{h-(t-s)}{h} y^{(i+1)}(s) ds. \tag{10}$$

Relations (5) and (10) imply (8) for $i = 1$. Let (8) be true for some $i \geq 1$. Then

$$\tilde{y}_{i+1}(t) \overset{(5)}{=} \frac{\tilde{y}_i(t) - \tilde{y}_i(t-h)}{h} \overset{(8)}{=} \frac{y^{(i)}(t) - y^{(i)}(t-h)}{h} - \mathcal{I}$$
$$\overset{(10)}{=} y^{(i+1)}(t) - \int_{t-h}^{t} \frac{h-(t-s)}{h} y^{(i+2)}(s)\, ds - \mathcal{I},$$

where

$$\begin{aligned} \mathcal{I} &= \int_{t-ih}^{t} \frac{\varphi_i(t-\zeta)}{h} y^{(i+1)}(\zeta)\, d\zeta \\ &\qquad - \int_{t-h-ih}^{t-h} \frac{\varphi_i(t-h-\zeta)}{h} y^{(i+1)}(\zeta)\, d\zeta \\ &= \int_{t-ih}^{t} \frac{\varphi_i(t-\zeta)}{h} \left[y^{(i+1)}(\zeta) - y^{(i+1)}(\zeta-h) \right] d\zeta \\ &= \int_{t-ih}^{t} \frac{\varphi_i(t-\zeta)}{h} \int_{\zeta-h}^{\zeta} y^{(i+2)}(s)\, ds\, d\zeta \\ &\overset{\text{Fig.1}}{=} \int_{t-h}^{t} \left[\int_s^t \frac{\varphi_i(t-\zeta)}{h}\, d\zeta \right] y^{(i+2)}(s)\, ds \\ &\quad + \int_{t-ih}^{t-h} \left[\int_s^{s+h} \frac{\varphi_i(t-\zeta)}{h}\, d\zeta \right] y^{(i+2)}(s)\, ds \\ &\quad + \int_{t-ih-h}^{t-ih} \left[\int_{t-ih}^{s+h} \frac{\varphi_i(t-\zeta)}{h}\, d\zeta \right] y^{(i+2)}(s)\, ds. \end{aligned}$$

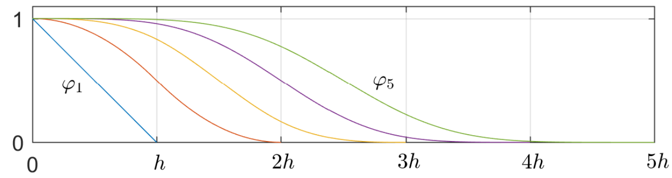

Fig. 2. Plots of φ_i for $i = 1, \ldots, 5$

Therefore, (8) holds for $i+1$ with

$$\varphi_{i+1}(t-s) = \begin{cases} \int_s^t \frac{\varphi_i(t-\zeta)}{h} d\zeta + \frac{h-(t-s)}{h}, & s \in [t-h, t], \\ \int_s^{s+h} \frac{\varphi_i(t-\zeta)}{h}\, d\zeta, & s \in (t-ih, t-h), \\ \int_{t-ih}^{s+h} \frac{\varphi_i(t-\zeta)}{h}\, d\zeta, & s \in [t-ih-h, t-ih]. \end{cases}$$

Taking $\lambda = t - \zeta$ and $\xi = t - s$, we obtain (9). ■

Using (3), the closed-loop system (1), (4) can be written as

$$\dot{x}(t) = Dx(t), \quad D = A + B \sum_{i=0}^{r-1} \bar{K}_i C A^i. \tag{11}$$

Using (3) and (8), the system (1), (6) can be written as

$$\dot{x}(t) = Dx(t) + B \sum_{i=1}^{r-1} \kappa_i(t) \tag{12}$$

with the same D and

$$\kappa_i(t) = -\bar{K}_i \int_{t-ih}^{t} \varphi_i(t-s) y^{(i+1)}(s)\, ds, \quad i = 1, \ldots, r-1. \tag{13}$$

If (4) stabilizes (1), then D is Hurwitz. In our analysis we derive the conditions ensuring that the errors κ_i do not ruin the stability of (12). For that sake we need several properties of the functions φ_i (see Fig. 2).

Proposition 2 *The functions φ_i defined in (9) satisfy*

1) $\varphi_i \in C^1[0, ih]$,
2) $\varphi_i' \leq 0$,
3) $0 \leq \varphi_i \leq 1$,
4) $\varphi_i(\xi) + \varphi_i(ih - \xi) = 1$,
5) $\int_0^{ih} \varphi_i(\xi)\, d\xi = \frac{ih}{2}$.

Proof. Most properties are proved using induction on i.

1) Clearly, $\varphi_1(\xi) = \frac{h-\xi}{h} \in C^1[0,h]$. If $\varphi_i \in C^1$, then

$$\varphi_{i+1}'(\xi) = \begin{cases} \frac{\varphi_i(\xi)}{h} - \frac{1}{h} & \xi \in [0,h], \\ \frac{\varphi_i(\xi) - \varphi_i(\xi-h)}{h} & \xi \in (h, ih), \\ -\frac{\varphi_i(\xi-h)}{h} & \xi \in [ih, ih+h] \end{cases} \tag{14}$$

is continuous on $[0, (i+1)h]$. (Continuity at h and ih follows from $\varphi_i(0) = 1$ and $\varphi_i(ih) = 0$, respectively.)

2) For $i = 1$ we have

$$\varphi_1'(\xi) = \left(\frac{h-\xi}{h} \right)' = -\frac{1}{h} < 0.$$

If 2) holds for some $i \geq 1$, then all expressions in (14) are negative since $\varphi_i(0) = 1$, $\varphi_i' \leq 0$, and $\varphi_i \geq 0$.

3) Relation $0 \leq \varphi_i$ easily proved using induction and (9). Relations $\varphi_i(0) = 1$ and $\varphi_i' \leq 0$ imply $\varphi_i \leq 1$.

4) Clearly, it is enough to prove 4) for $\xi \in [0, \frac{ih}{2}]$. For $i = 1$, $\varphi_1(\xi) + \varphi_1(h-\xi) = \frac{h-\xi}{h} + \frac{h-(h-\xi)}{h} = 1$.

Let 4) be true for some $i \geq 1$. If $\xi \in [0, h]$, the change of variable $\tilde{\lambda} = ih - \lambda$ in the second integral leads to

$$\begin{aligned}
&\varphi_{i+1}(\xi) + \varphi_{i+1}((i+1)h - \xi) \\
&\quad = \tfrac{1}{h}\int_0^\xi \varphi_i(\lambda)\,d\lambda + \tfrac{h-\xi}{h} + \tfrac{1}{h}\int_{ih-\xi}^{ih} \varphi_i(\lambda)\,d\lambda \\
&\quad = \tfrac{1}{h}\int_0^\xi \varphi_i(\lambda)\,d\lambda + \tfrac{h-\xi}{h} + \tfrac{1}{h}\int_0^\xi \varphi_i(ih-\tilde{\lambda})\,d\tilde{\lambda} \\
&\quad \overset{4)}{=} \tfrac{1}{h}\int_0^\xi 1\,d\lambda + \tfrac{h-\xi}{h} = 1.
\end{aligned}$$

If $\xi \in (h, \frac{ih}{2}]$, the change of variable $\tilde{\lambda} = ih - \lambda$ in the second integral leads to

$$\begin{aligned}
&\varphi_{i+1}(\xi) + \varphi_{i+1}((i+1)h - \xi) \\
&\quad = \tfrac{1}{h}\int_{\xi-h}^\xi \varphi_i(\lambda)\,d\lambda + \tfrac{1}{h}\int_{ih-\xi}^{ih-\xi+h} \varphi_i(\lambda)\,d\lambda \\
&\quad = \tfrac{1}{h}\int_{\xi-h}^\xi \varphi_i(\lambda)\,d\lambda + \tfrac{1}{h}\int_{\xi-h}^\xi \varphi_i(ih-\tilde{\lambda})\,d\tilde{\lambda} \\
&\quad \overset{4)}{=} \tfrac{1}{h}\int_{\xi-h}^\xi 1\,d\lambda = 1.
\end{aligned}$$

5) Using the change of variable $\tilde{\xi} = ih - \xi$, we obtain

$$\begin{aligned}
\int_0^{ih} \varphi_i(\xi)\,d\xi &= \int_0^{\frac{ih}{2}} \varphi_i(\xi)\,d\xi + \int_{\frac{ih}{2}}^{ih} \varphi_i(\xi)\,d\xi \\
&= \int_0^{\frac{ih}{2}} \varphi_i(\xi)\,d\xi + \int_0^{\frac{ih}{2}} \varphi_i(ih-\tilde{\xi})\,d\tilde{\xi} \\
&= \int_0^{\frac{ih}{2}} [\varphi_i(\xi) + \varphi_i(ih-\xi)]\,d\xi,
\end{aligned}$$

which implies 5) in view of 4). ■

Theorem 1 *Consider the LTI system* (1) *of relative degree* $r \geq 2$*, i.e., satisfying* (2).

(i) The delay-dependent feedback (6) *with a time-delay* $h > 0$ *and controller gains* (7) *exponentially stabilizes* (1) *with a decay rate* $\alpha > 0$ *if there exist*

$$0 < P \in \mathbb{R}^{n\times n}, \quad 0 < R_i \in \mathbb{R}^{m\times m}, \quad i = 1,\dots,r-1$$

such that[2] $M < 0$*, where* M *is the symmetric matrix composed from*

$$\begin{aligned}
M_{11} &= D^TP + PD + 2\alpha P \\
&\quad + \textstyle\sum_{i=1}^{r-2} \frac{(ih)^2}{4}\left[\bar{K}_iCA^{i+1}\right]^T R_i \left[\bar{K}_iCA^{i+1}\right], \\
M_{12} &= \mathbf{1}_{r-1}^T \otimes PB, \\
M_{13} &= \tfrac{(r-1)h}{2}\left[\bar{K}_{r-1}CA^{r-1}D\right]^T R_{r-1}, \\
M_{22} &= -\operatorname{diag}\{e^{-2\alpha ih}R_i\}_{i=1}^{r-1}, \\
M_{23} &= \tfrac{(r-1)h}{2}\mathbf{1}_{r-1} \otimes \left[\bar{K}_{r-1}CA^{r-1}B\right]^T R_{r-1}, \\
M_{33} &= -R_{r-1}
\end{aligned}$$

with $D = A + B\sum_{i=0}^{r-1}\bar{K}_iCA^i$.

(ii) If the derivative-dependent feedback (4) *with controller gains* $\bar{K}_i \in \mathbb{R}^{m\times l}$*,* $i = 0,\dots,r-1$*, stabilizes* (1) *with a decay rate* $\alpha' > 0$*, then for any* $\alpha \in (0,\alpha')$ *there exists a sufficiently small* $h > 0$ *such that the delay-dependent control* (6) *with the controller gains* (7) *stabilizes* (1) *with the decay rate* α.

Proof. (i) Consider $V = V_0 + \sum_{i=1}^{r-1} V_{\kappa i}$, where

$$\begin{aligned}
V_0 &= x^TPx, \\
V_{\kappa i} &= \tfrac{ih}{2}\int_{t-ih}^t e^{-2\alpha(t-s)}\psi_i(t-s)\times \\
&\qquad \left[\bar{K}_iy^{(i+1)}(s)\right]^T R_i \left[\bar{K}_iy^{(i+1)}(s)\right]\,ds
\end{aligned} \tag{15}$$

with

$$\psi_i(\xi) = \int_\xi^{ih} \varphi_i(\lambda)\,d\lambda, \quad i = 1,\dots,r-1. \tag{16}$$

Due to the properties of φ_i given in Proposition 2, we have

$\psi_i \in C^1[0, ih], \quad \psi_i(\xi) \geq 0, \quad \forall \xi \in [0, ih].$

Therefore, $V \geq 0$ is smooth for $t \geq (r-1)h$. We have

$\dot{V}_0 + 2\alpha V_0 \overset{(12)}{=} 2x^TPDx + 2x^TPB\sum_{i=1}^{r-1}\kappa_i + 2\alpha x^TPx$

with κ_i defined in (13). Proposition 2 implies

$\int_{t-ih}^t \varphi_i(t-s)\,ds = \psi_i(0) = \frac{ih}{2}, \quad i \in \mathbb{N}.$

Moreover, (16) implies

$\psi_i(ih) = 0, \quad \psi_i'(\xi) = -\varphi_i(\xi).$

Using these properties, we obtain

$$\begin{aligned}
\dot{V}_{\kappa i} + 2\alpha V_{\kappa i} &= \tfrac{(ih)^2}{4}\left[\bar{K}_iy^{(i+1)}(t)\right]^T R_i \left[\bar{K}_iy^{(i+1)}(t)\right] \\
&\quad - \tfrac{ih}{2}\int_{t-ih}^t e^{-2\alpha(t-s)}\varphi_i(t-s)\times \\
&\quad \left[\bar{K}_iy^{(i+1)}(s)\right]^T R_i \left[\bar{K}_iy^{(i+1)}(s)\right]\,ds \\
&\overset{\text{Lem.2}}{\leq} \tfrac{(ih)^2}{4}\left[\bar{K}_iy^{(i+1)}(t)\right]^T R_i \left[\bar{K}_iy^{(i+1)}(t)\right] \\
&\quad - e^{-2\alpha ih}\kappa_i^T(t)R_i\kappa_i(t).
\end{aligned} \tag{17}$$

Substituting $y^{(i+1)} \overset{(3)}{=} CA^{i+1}x$ for $i = 1,\dots,r-2$ and $y^{(r)} = CA^{r-1}\dot{x}$, we obtain

$$\begin{aligned}
\dot{V} + 2\alpha V &\leq \mu^T \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \mu \\
&\quad + \tfrac{(r-1)^2h^2}{4}\dot{x}^T\left[\bar{K}_{r-1}CA^{r-1}\right]^T R_{r-1}\left[\bar{K}_{r-1}CA^{r-1}\right]\dot{x},
\end{aligned}$$

where $\mu = \operatorname{col}\{x, \kappa_1, \dots, \kappa_{r-1}\}$. Substituting (12) for $\dot{x}$ and using the Schur complement, we deduce that $M < 0$ guarantees $\dot{V} \leq -2\alpha V$, which implies the exponential stability.

(ii) If (4) stabilizes (1) with a decay rate $\alpha' > 0$, for any $\alpha \in (0, \alpha')$ there exists $0 < P \in \mathbb{R}^{n\times n}$ such that

$$D^TP + PD + 2\alpha P < 0. \tag{18}$$

By the Schur complement, $M < 0$ is equivalent to

$$\begin{bmatrix} \widetilde{M}_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} + h^2F < 0, \quad -R_{r-1} < 0, \tag{19}$$

[2] MATLAB codes for solving the LMIs are available at `https://github.com/AntonSelivanov/Aut18a`

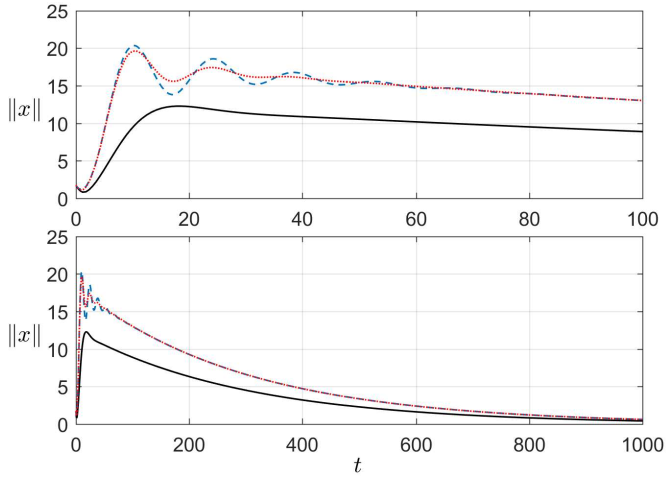

Fig. 3. *Example 1 (Chain of three integrators)*: dynamics of (1), (21) under the derivative-dependent feedback (4) (black solid line), time-delay feedback (6) with $h = 2.529$ (blue dashed line), and sampled-data feedback (25) with h= 1.436 (red dotted line).

where $\widetilde{M}_{11} = D^T P + PD + 2\alpha P$ and symmetric F does not depend on h. Due to (18), $\left[\begin{smallmatrix} \widetilde{M}_{11} & M_{12} \\ M_{12}^T & M_{22} \end{smallmatrix}\right] < 0$ for $R_i = cI_m$ with large enough $c \in \mathbb{R}$. Therefore, (19) holds for small enough h implying $M < 0$. By Theorem 1(i), (6) exponentially stabilizes (1) with the decay rate α. ■

Remark 2 *A different approach to the analysis of* (1), (6) *has been proposed in [16], where Taylor's expansion was used for each* $y(t - ih)$ *with* $i = 1, \dots, r-1$*:*

$$y(t-ih) = \sum_{j=0}^{r-1} \frac{y^{(j)}(t)}{j!}(-ih)^j + \int_{t-ih}^{t} \bar{\varphi}_i(t-s) y^{(r)}(s)\, ds. \tag{20}$$

Here

$$\bar{\varphi}_i(\xi) = -\frac{(\xi - ih)^{r-1}}{(r-1)!}, \quad i = 1, \dots, r-1.$$

The approximation errors were bounded using functionals similar to $V_{\kappa i}$ *from* (15)*. The values* $\int_0^{ih} \bar{\varphi}_i(\xi)\, d\xi$ *play a key role in such analysis: the smaller these values are, the smaller the effect of the errors is (see* (17)*). When* $h \to \infty$, $|\int_0^{ih} \bar{\varphi}_i(\xi)\, d\xi| = \frac{(ih)^r}{r!}$ *grow faster than* $\int_0^{ih} \varphi_i(\xi)\, d\xi = \frac{ih}{2}$ *used here. Thus, our results admit larger time-delay* h*.*

Moreover, in [16], the errors were multiplied by K_i *that grow when* $h \to 0$ *(similarly to* (7)*), while we multiply the errors by* $\bar{K}_i$ *independent of* h *(see* (13)*). This allows to obtain larger interval for the time-delay* h *(see Example 2).*

These benefits are achieved using an original representation (8)*, where the errors are related to the finite differences* $\tilde{y}_i$ *defined in* (5)*, while in [16] the errors were related to* $y(t - ih)$*. However, for* $r = 2$ *the results coincide, since* (8) *and* (20) *are equivalent.*

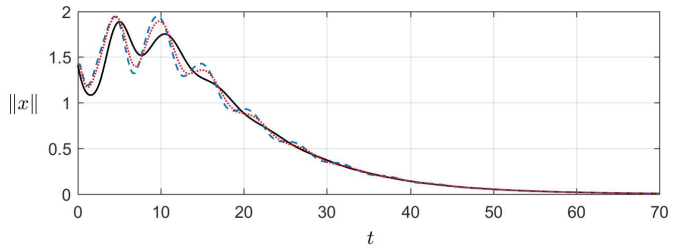

Fig. 4. *Example 2 (Chain of four integrators)*: dynamics of (1), (23) under the derivative-dependent feedback (4) (black solid line), time-delay feedback (6) with $h = 0.169$ (blue dashed line), and sampled-data feedback (25) with $h = 0.1$ (red dotted line).

Example 1 (Chain of three integrators). Consider (1) with

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right]. \tag{21}$$

These parameters satisfy (2) with the relative degree $r = 3$. The derivative-dependent control (4) with

$$\bar{K}_0 = -2 \times 10^{-4}, \quad \bar{K}_1 = -0.06, \quad \bar{K}_2 = -0.342 \tag{22}$$

stabilizes (1), (21). The LMIs of Theorem 1 are feasible for $h \in (0, 2.529]$, $\alpha = 0$. Therefore, the delay-dependent controller (6) also stabilizes the system (1), (21). The method developed in [16] leads to a smaller interval $h \in (0, 2.32]$. Fig. 3 shows $\|x\|$ for $x(0) = [1, -1, 1]^T$.

Example 2 (Chain of four integrators). Consider (1) with

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{23}$$

These parameters satisfy (2) with the relative degree $r = 4$. The derivative-dependent control (4) with

$$\bar{K}_0 = -0.0208, \bar{K}_1 = -0.32, \bar{K}_2 = -1.18, \bar{K}_3 = -0.7 \tag{24}$$

stabilizes (1), (23). These gains are taken from [16]. The LMIs of Theorem 1 are feasible for $h \in (0, 0.169]$, $\alpha = 0$. Therefore, the delay-dependent controller (6) also stabilizes the system (1), (23). The method developed in [16] leads to a smaller interval $h \in (0, 0.138]$. Fig. 4 shows $\|x\|$ for $x(0) = [1, 0, 0, -1]^T$.

3 Sampled-data control

In this section, we assume that only sampled in time measurement $y(t_k)$ are available to the controller, where $t_k = kh$ are the sampling instants with a sampling period $h > 0$ and $k \in \mathbb{N}_0$. The derivative-dependent controller (4) is approximated by the sampled-data controller

$$u(t) = \sum_{i=0}^{r-1} \bar{K}_i \tilde{y}_i(t_k) = \sum_{i=0}^{r-1} K_i y(t_{k-i}), \qquad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0 \tag{25}$$

with $\tilde{y}_i$ from (5) and K_i from (7). We set[3] $y(t_{k-i}) = y(t_0)$ for $k < i$. For $t \in [t_k, t_{k+1})$ with $k \geq r-1$, we present the sampled measurements as

$$\begin{aligned}\tilde{y}_0(t_k) &= y(t) - \int_{t_k}^t \dot{\tilde{y}}_0(s)\,ds = y(t) - \int_{t_k}^t \dot{y}(s)\,ds,\\ \tilde{y}_i(t_k) &= \tilde{y}_i(t) - \int_{t_k}^t \dot{\tilde{y}}_i(s)\,ds\\ &\overset{(8)}{=} y^{(i)}(t) - \int_{t-ih}^t \varphi_i(t-s)y^{(i+1)}(s)\,ds - \int_{t_k}^t \dot{\tilde{y}}_i(s)\,ds,\\ &\qquad\qquad i = 1,\ldots,r-1.\end{aligned}$$

Then the controller (25) can be written as

$u = \sum_{i=0}^{r-1} \bar{K}_i y^{(i)} + \delta_0 + \sum_{i=1}^{r-1}(\delta_i + \kappa_i)$,

where, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\delta_i(t) &= -\bar{K}_i \int_{t_k}^t \dot{\tilde{y}}_i(s)\,ds, && i = 0,\ldots,r-1,\\ \kappa_i(t) &= -\bar{K}_i \int_{t-ih}^t \varphi_i(t-s)y^{(i+1)}(s)\,ds, && i = 1,\ldots,r-1.\end{aligned} \tag{26}$$

The closed-loop system (1), (25) takes the form (cf. (12))

$$\dot{x} = Dx + B\delta_0 + B\sum_{i=1}^{r-1}(\delta_i + \kappa_i) \tag{27}$$

with D defined in (11). If (4) stabilizes (1), then D is Hurwitz. In our analysis we derive the conditions ensuring that the errors δ_i and κ_i do not ruin the stability of (27).

Theorem 2 *Consider the LTI system* (1) *of relative degree* $r \geq 2$*, i.e., satisfying* (2)*.*

(i) The sampled-data feedback (25) *with a sampling period* $h > 0$ *and controller gains* (7) *exponentially stabilizes* (1) *with a decay rate* $\alpha > 0$ *if there exist*

$$0 < P \in \mathbb{R}^{n\times n}, \quad 0 < W_0 \in \mathbb{R}^{m\times m},$$
$$0 < W_i \in \mathbb{R}^{m\times m}, \quad 0 < R_i \in \mathbb{R}^{m\times m}, \quad i = 1,\ldots,r-1$$

such that[4] $N < 0$*, where* N *is the symmetric matrix composed from*

$$\begin{aligned}N_{11} &= D^TP + PD + 2\alpha P\\ &\quad + \sum_{i=0}^{r-2} h^2 e^{2\alpha ih}\left[\bar{K}_iCA^{i+1}\right]^T W_i \left[\bar{K}_iCA^{i+1}\right]\\ &\quad + \sum_{i=1}^{r-2} \tfrac{(ih)^2}{4}\left[\bar{K}_iCA^{i+1}\right]^T R_i \left[\bar{K}_iCA^{i+1}\right],\\ N_{12} &= \mathbf{1}_r^T \otimes PB,\\ N_{13} &= \mathbf{1}_{r-1}^T \otimes PB,\\ N_{14} &= h\left[\bar{K}_{r-1}CA^{r-1}D\right]^T H,\\ N_{22} &= -\tfrac{\pi^2}{4}e^{-2\alpha h}\operatorname{diag}\{W_i\}_{i=0}^{r-1},\\ N_{24} &= h\mathbf{1}_r \otimes \left[\bar{K}_{r-1}CA^{r-1}B\right]^T H,\\ N_{33} &= -\operatorname{diag}\{e^{-2\alpha ih}R_i\}_{i=1}^{r-1},\\ N_{34} &= h\mathbf{1}_{r-1} \otimes \left[\bar{K}_{r-1}CA^{r-1}B\right]^T H,\\ N_{44} &= -H\end{aligned}$$

with

$$\begin{aligned}D &= A + B\sum_{i=0}^{r-1}\bar{K}_iCA^i,\\ H &= e^{2\alpha(r-1)h}W_{r-1} + \left(\tfrac{r-1}{2}\right)^2 R_{r-1}.\end{aligned} \tag{28}$$

(ii) If the derivative-dependent feedback (4) *with controller gains* $\bar{K}_i \in \mathbb{R}^{m\times l}$*,* $i = 0,\ldots,r-1$*, stabilizes* (1) *with a decay rate* $\alpha' > 0$*, then for any* $\alpha \in (0, \alpha')$ *there exists a sufficiently small sampling period* $h > 0$ *such that the sampled-data control* (25) *with the controller gains* (7) *stabilizes* (1) *with the decay rate* α*.*

Proof. (i) For $t \geq (r-1)h$ consider the functional

$V = V_0 + V_{\delta 0} + \sum_{i=1}^{r-1}(V_{\delta i} + V_{yi} + V_{\kappa i})$,

where V_0, $V_{\kappa i}$ are given in (15) and

$$\begin{aligned}V_{\delta i} &= h^2\int_{t_k}^t e^{-2\alpha(t-s)}\left[\bar{K}_i\dot{\tilde{y}}_i(s)\right]^T W_i\left[\bar{K}_i\dot{\tilde{y}}_i(s)\right]\,ds\\ &\quad -\tfrac{\pi^2}{4}e^{-2\alpha h}\int_{t_k}^t e^{-2\alpha(t-s)}\delta_i^T(s)W_i\delta_i(s)\,ds, t \in [t_k, t_{k+1})\\ V_{yi} &= h^2e^{2\alpha ih}\int_{t-ih}^t e^{-2\alpha(t-s)}\varphi_i(t-s)\times\\ &\qquad \left[\bar{K}_iy^{(i+1)}(s)\right]^T W_i\left[\bar{K}_iy^{(i+1)}(s)\right]\,ds.\end{aligned}$$

Since $\dot{\delta}_i(t) = -\bar{K}_i\dot{\tilde{y}}_i(t)$ and $\delta_i(t_k) = 0$, Lemma 1 implies $V_{\delta i} \geq 0$ for $i = 0,\ldots,r-1$. Since $\varphi_i \geq 0$ and $\psi_i \geq 0$, we have $V \geq 0$. Calculating the derivatives, we obtain

$$\begin{aligned}\dot{V}_0 + 2\alpha V_0 &\overset{(27)}{=} 2x^TPDx + 2x^TPB\delta_0\\ &\quad + 2x^TPB\sum_{i=1}^{r-1}(\delta_i+\kappa_i) + 2\alpha x^TPx,\\ \dot{V}_{\delta i} + 2\alpha V_{\delta i} &= h^2\left[\bar{K}_i\dot{\tilde{y}}_i\right]^T W_i\left[\bar{K}_i\dot{\tilde{y}}_i\right] - \tfrac{\pi^2}{4}e^{-2\alpha h}\delta_i^TW_i\delta_i.\end{aligned}$$

The functional V_{yi} is introduced to compensate the term $h^2[\bar{K}_i\dot{\tilde{y}}_i]^TW_i[\bar{K}_i\dot{\tilde{y}}_i]$ in the above expression. Since $\varphi_i(0) = 1$, $\varphi_i(ih) = 0$, and $\varphi_i' \leq 0$ (Proposition 2),

$$\begin{aligned}&\dot{V}_{yi} + 2\alpha V_{yi} = h^2e^{2\alpha ih}\left[\bar{K}_iy^{(i+1)}\right]^TW_i\left[\bar{K}_iy^{(i+1)}\right] + h^2e^{2\alpha ih}\times\\ &\int_{t-ih}^t e^{-2\alpha(t-s)}\varphi_i'(t-s)\left[\bar{K}_iy^{(i+1)}(s)\right]^TW_i\left[\bar{K}_iy^{(i+1)}(s)\right]ds\\ &\overset{\text{Lem.2}}{\leq} h^2e^{2\alpha ih}\left[\bar{K}_iy^{(i+1)}\right]^TW_i\left[\bar{K}_iy^{(i+1)}\right]\\ &\quad - h^2\left(\int_{t-ih}^t(-\varphi_i'(t-s))\,ds\right)^{-1}\int_{t-ih}^t\varphi_i'(t-s)\times\\ &\quad \left[\bar{K}_iy^{(i+1)}(s)\right]^T ds\ W_i\int_{t-ih}^t\varphi_i'(t-s)\left[\bar{K}_iy^{(i+1)}(s)\right]ds.\end{aligned}$$

Differentiating (8), we obtain

$$\dot{\tilde{y}}_i = -\int_{t-ih}^t \varphi_i'(t-s)y^{(i+1)}(s)\,ds, \quad i \in \mathbb{N}.$$

The latter and

$$\int_{t-ih}^t(-\varphi_i'(t-s))\,ds = \varphi_i(0) - \varphi_i(ih) = 1$$

lead to

$$\begin{aligned}\dot{V}_{yi} + 2\alpha V_{yi} &\leq h^2e^{2\alpha ih}\left[\bar{K}_iy^{(i+1)}\right]^TW_i\left[\bar{K}_iy^{(i+1)}\right]\\ &\quad - h^2\left[\bar{K}_i\dot{\tilde{y}}_i\right]^TW_i\left[\bar{K}_i\dot{\tilde{y}}_i\right].\end{aligned}$$

The term $-h^2[\bar{K}_i\dot{\tilde{y}}_i]^TW_i[\bar{K}_i\dot{\tilde{y}}_i]$ in the above expression will cancel the positive term of $\dot{V}_{\delta i} + 2\alpha V_{\delta i}$. The derivative of

[3] Then $y^{(i)}(0)$ with $i > 0$ are approximated by 0

[4] MATLAB codes for solving the LMIs are available at `https://github.com/AntonSelivanov/Aut18a`

$V_{\kappa i}$ is given in (17). Substituting $y^{(i+1)} \overset{(3)}{=} CA^{i+1}x$ for $i = 1, \dots, r-2$ and $y^{(r)} = CA^{r-1}\dot{x}$, we obtain

$$\dot{V} + 2\alpha V \leq \eta^T \bar{N}\eta + \dot{x}^T h^2 [\bar{K}_{r-1}CA^{r-1}]^T H [\bar{K}_{r-1}CA^{r-1}]\dot{x},$$

where $\eta = \mathrm{col}\{x, \delta_0, \dots, \delta_{r-1}, \kappa_1, \dots, \kappa_{r-1}\}$, H is defined in (28), and $\bar{N}$ is obtained from N by removing the last block-column and block-row. Substituting (27) for $\dot{x}$ and using the Schur complement, we deduce that $N < 0$ guarantees $\dot{V} \leq -2\alpha V$, which implies the exponential stability.

(ii) The proof is similar to the proof of Theorem 1(ii). ■

Remark 3 *In [19], the system* (1) *was studied under the sampled-data feedback*

$$u(t) = \textstyle\sum_{i=0}^{r-1} K_i y(t_k - q_i h), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0 \quad (29)$$

with integer delays $0 = q_0 < q_1 < \cdots < q_{r-1}$. *In [19], the errors due to sampling* $y(t_k - q_i h) - y(t - q_i h)$ *were multiplied by* K_i *that grow when* $q_i h \to 0$. *Consequently, one had to increase discrete delays* q_i *while reducing the sampling period* h *to maintain* K_i *bounded. Here, due to the representation* $u(t) = \sum_{i=0}^{r-1} \bar{K}_i \tilde{y}_i(t)$ *(see* (6)*), we can consider the errors due to sampling* $\tilde{y}_i(t_k) - \tilde{y}_i(t)$ *that are multiplied by* $\bar{K}_i$ *independent of* h *(see* δ_i *in* (26)*). This allows to use* $q_i = i$ *(cf.* (25) *and* (29)*) and, therefore, smaller memory is required to implement* (25) *(see Example 1).*

In addition, the results of [19] are based on [16], therefore, all the benefits of the current analysis mentioned in Remark 2 remain relevant for the sampled-data case if $r > 2$.

Example 1 (Chain of three integrators). Consider (1) with the parameters given in (21). The LMIs of Theorem 2 are feasible for the controller gains (22) with $h \in (0, 1.436]$, $\alpha = 10^{-3}$. Therefore, the sampled-data controller (25) exponentially stabilizes the system (1), (21). Fig. 3 shows $\|x\|$ for $x(0) = [1, -1, 1]^T$. The same example has been considered in [19], where a significantly smaller interval $h \in (0, 0.044]$ was obtained. Moreover, [19] used (29) with $q_1 = 30$, $q_2 = 60$, what required to keep 61 measurements $y(t_k), \dots, y(t_k - q_2 h)$ to implement the controller, while (25) uses only the last three: $y(t_k)$, $y(t_{k-1})$, $y(t_{k-2})$.

Example 2 (Chain of four integrators). Consider (1) with the parameters given in (23). The LMIs of Theorem 2 are feasible for the controller gains (24) with $h \in (0, 0.1]$, $\alpha = 0.01$. Therefore, the sampled-data controller (25) exponentially stabilizes the system (1), (23). Fig. 4 shows $\|x\|$ for $x(0) = [1, 0, 0, -1]^T$. The conditions of [19] are feasible for the controller (29) with $h \sim 10^{-6}$ and $q_i \sim 10^4$.

Example 3 (Furuta pendulum [24]). Consider the linearized model of the Furuta pendulum given by (1) with

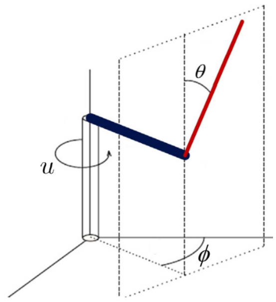

Fig. 5. Furuta pendulum [5]

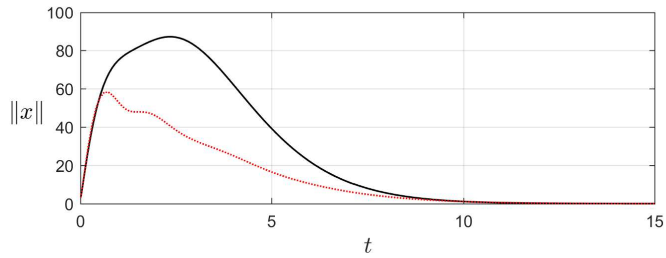

Fig. 6. *Example 3 (Furuta pendulum)*: dynamics of (1), (30) under the derivative-dependent feedback (4) (black solid line) and sampled-data feedback (25) with h=0.104 (red dotted line).

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 37.377 & -0.515 & 0 & 0.142 & -35.42 \\ 0 & 0 & 0 & 1 & 0 \\ -8.228 & 0.113 & 0 & -0.173 & 43.28 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad (30)$$

and $x = \mathrm{col}\{\theta, \dot{\theta}, \phi, \dot{\phi}\}$, where θ is the angular position of the pendulum and ϕ is the angle of the rotational arm (see Fig. 5). The control input u is proportional to the motor induced torque. Using the pole placement, we find that for

$$\bar{K}_0 = \begin{bmatrix} 1.2826 & 0.0013 \end{bmatrix}, \quad \bar{K}_1 = \begin{bmatrix} 0.1209 & 0.0086 \end{bmatrix}$$

the eigenvalues of D defined in (11) are -1, -1.1, -1.2, -1.3. Therefore, the derivative-dependent controller (4) stabilizes the system (1), (30). The conditions of Theorem 2 (with $\alpha = 0$) are feasible for $h \in (0, 0.104]$. Taking $h = 0.104$ in (7), we deduce that the sampled-data controller (25) with

$$K_0 = \begin{bmatrix} 2.4453 & 0.0837 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -1.1627 & -0.0824 \end{bmatrix},$$

and $t_k = 0.104 \cdot k$, $k \in \mathbb{N}_0$, exponentially stabilizes the Furuta pendulum (1), (30). Fig. 6 shows $\|x\|$ for $x(0) = [\pi, 0, 0, 0]^T$. The conditions of [19] are feasible for the controller (29) with $h \sim 10^{-4}$ and $q_1 \sim 10^3$.

[5] The picture is taken from [25]

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