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LIMIT THEOREMS AND FLUCTUATIONS FOR POINT VORTICES OF GENERALIZED EULER EQUATIONS

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ABSTRACT. We prove a mean field limit, a law of large numbers and a central limit theorem for a system of point vortices on the 2D torus at equilibrium with positive temperature. The point vortices are formal solutions of a class of equations generalising the Euler equations, and are also known in the literature as generalised inviscid SQG. The mean field limit is a steady solution of the equations, the CLT limit is a stationary distribution of the equations.

1. INTRODUCTION

The paper analyses the mean field limit and the corresponding fluctuations for the point vortex dynamics, at equilibrium with positive temperature, arising from a class of equations generalising the Euler equations. More precisely, we consider the family of models

$$\partial_t\theta + u\cdot\nabla\theta = 0,$$

on the two dimensional torus $\mathbb{T}_2$ with periodic boundary conditions and zero spatial average. Here $u = \nabla^\perp(-\Delta)^{-\frac{m}{2}}\theta$ is the velocity, and m is a parameter. When $m = 2$, the model corresponds to the Euler equations, when $m = 1$ this is the inviscid surface quasi-geostrophic (briefly, SQG).

As in the case of Euler equations, a family of *formal* solutions is given by point vortices, namely measure solutions described by

$$\sum_{j=1}^N \gamma_j\delta_{X_j(t)},$$

where $X_1, X_2, \dots, X_N$ are vortex positions and $\gamma_1, \gamma_2, \dots, \gamma_N$ are vortex intensities. Positions evolve according to (2.3), and intensities are constant by a generalized version of Kelvin's theorem. This evolution is Hamiltonian with Hamiltonian (2.4), and has a family of invariant distributions (2.5) indexed by a parameter β. Unfortunately when $m < 2$ the invariant distributions, written in terms of a density which is the exponential of the Hamiltonian, do not make sense since the Green function of the fractional Laplacian $(-\Delta)^{\frac{m}{2}}$ has a singularity which is too strong.

Nevertheless, the main aim of the paper is to realize the program developed in [CLMP92, CLMP95, Lio98, BG99] for point vortices for the Euler equations. To this aim we introduce a regularization of the Green function. At the level

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of the regularized problem all statements we are interested in (mean field limit, analysis of fluctuations) are not difficult, since the interaction among vortices is bounded. We recover the original problem in the limit of infinite vortices, since when the number of vortices N increases to ∞, we choose the regularization parameter ϵ so that it goes at the same time to 0. To ensure the validity of our result though, the speed of convergence of $\epsilon = \epsilon(N)$ must be at least logarithmically slow in terms of N.

Under the conditions $\beta > 0$ and $m < 2$, and when $\epsilon(N) \downarrow 0$, we prove propagation of chaos, namely vortices decorrelate and in the limit are independent. A law of large numbers holds and, in terms of θ, the limit is a stationary solution of the original equation. Likewise, a central limit theorem holds. The limit Gaussian distribution for the θ variable turns out to be a statistically stationary solution of the equations.

The paper is organized as follows. In Section 2 we introduce the model with full details, we give some preliminary results and we prepare the framework to state the main results. Section 3 contains the main results, as well as some consequences and additional remarks. Finally, Section 4 is devoted to the proof of the main results.

We conclude this introduction with a list of notations used throughout the paper. We denote by $\mathbb{T}_2$ the two dimensional torus, and by ℓ the normalized Lebesgue measure on $\mathbb{T}_2$. Given a metric space E, we shall denote by $C(E)$ the space of continuous functions on E, and by $\mathcal{P}(E)$ the set of probability measures on E. If $x \in E$, then δ_x is the Dirac measure on x. Given a measure μ on E, we will denote by $\mu(F) = \langle F, \mu \rangle = \int F(x)\,\mu(dx)$ the integral of a function F with respect to μ. Sometimes we will also use the notation $\mathbb{E}_\mu[F]$. We will use the operator $\otimes$ to denote the product between measures. We shall denote by $\lambda_1, \lambda_2, \dots$ the eigenvalues in non-decreasing order, and by $e_1, e_2, \dots$ the corresponding orthonormal basis of eigenvectors of $-\Delta$, where Δ is the Laplace operator on $\mathbb{T}_2$ with periodic boundary conditions and zero spatial average. With these positions, if $\phi = \sum_k \phi_k e_k$, then the fractional Laplacian is defined as

$$(-\Delta)^{\frac{\alpha}{2}}\phi = \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha}{2}} \phi_k e_k.$$

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2. The model

Consider the family of models,

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{2.1}$$

on the torus with periodic boundary conditions and zero spatial average, where the velocity $u = \nabla^{\perp}\psi$, and the stream function ψ is solution to the following problem,

$$(-\Delta)^{\frac{m}{2}}\psi = \theta,$$

with periodic boundary conditions and zero spatial average. Here m is a parameter. The case $m = 2$ corresponds to the Euler equation in vorticity formulation, $m = 1$ is the inviscid surface quasi-geostrophic equation (briefly, SQG), and for a general value is sometimes known in the literature as the inviscid *generalized surface quasi-geostrophic* equation. Here we will consider values $m < 2$ of the parameter.

2.1. **Generalities on the model.** We start by giving a short introduction to the main features of the model (2.1).

2.1.1. *Existence and uniqueness of solution.* The inviscid SQG has been derived in meteorology to model frontogenesis, namely the production of fronts due to tightening of temperature gradients and is an active subject of research. See [CMT94, HPL94, HPGS95] (see also [CFR04, Rod05]) for the first mathematical and geophysical studies about strong fronts. The generalized version of the equations bridges the cases of Euler and SQG and it is studied to understand the mathematical differences between the two cases.

As it regards existence, uniqueness and regularity of solutions, a local existence result is known, namely data with sufficient smoothness give local in time unique solutions with the same regularity of the initial condition, see for instance [CCC+12]. Unlike the Euler equation, it is not known if the inviscid SQG (as well as its generalized version) has a global solution. Actually, there is numerical evidence, see [CFMR05], of emergence of singularities in the generalized SQG, for $m \in [1, 2)$. On the other hand see [CGSI17] for classes of global solutions. Finally, [CCW11] presents a regularity criterion for classical solutions.

The state is different if one turns to weak solutions. Indeed existence of weak solutions is known since [Res95], see also [Mar08]. For existence of weak solution for the generalized SQG model one can see [CCC+12]. Global flows of weak solution with a (formal) invariant measure (corresponding to the measure in (2.2) with $\beta = 0$) as initial condition has been provided in [NPST17].

2.1.2. *Invariant quantities.* We turn to some simple (and in general formal, but that can be made rigorous on classical solutions) properties of the equation. As in the case of Euler equations, equation (2.1) can be solved by means of characteristics, in the sense that if θ is solution of (2.1) and $u = \nabla^\perp\theta$,

$$\begin{cases} \dot X = u(t, X_t), \\ X(0) = x, \end{cases}$$

then, at least formally,

$$\frac{d}{dt}\theta(t, X_t) = \partial_t\theta(t, X_t) + \dot X_t \cdot \nabla\theta(t, X_t) = (\partial_t\theta + u \cdot \nabla\theta)(t, X_t) = 0,$$

therefore $\theta(t, X_t) = \theta(0, x)$. This formally ensures conservation of the sign and of the magnitude (L^∞ norm) of θ.

It is not difficult to see that (2.1) admits an infinite number of conserved quantities, for instance of L^p norms of θ. We are especially interested in the quantity (that in the case $m = 2$ is the enstrophy),

$$\|\theta(t)\|_{L^2}^2 = \int_{\mathbb{T}_2} |\theta(t, x)|^2 \, d\ell$$

and in the quantity,

$$\int_{\mathbb{T}_2} \theta(t, x)\psi(t, x) \, d\ell = \|(-\Delta)^{-\frac{m}{4}}\theta\|_{L^2(\ell)}^2.$$

Here, unlike the case $m = 2$, this conserved quantity is not the kinetic energy. Formally, corresponding to these conserved quantities, in analogy with the invariant measures of the Euler equations [AC90], one can consider the invariant measures

$$\mu_{\beta,\alpha}(d\theta) = \frac{1}{Z_{\beta,\alpha}} e^{-\beta\|(-\Delta)^{-\frac{m}{4}}\theta\|^2 - \alpha\|\theta(t)\|_{L^2}^2} \, d\theta, \tag{2.2}$$

classically interpreted as Gaussian measures with suitable covariance (see Remark 3.5).

2.2. **The point vortex motion.** The point vortex motion is a powerful point of view to understand some of the phenomenological interesting properties of solutions of the Euler equations. Mathematical results about the general dynamics of point vortices [MP94] and about the connection with the equations [Sch96] are classical. The statistical mechanics approach to the description of point vortices, the central topic of this paper, dates back to some of the intuitions in the celebrated paper of Onsager [Ons49], and later developed in the physical literature [JM73, FR83, ES93, Kie93]. Results of mean field type are obtained in [CLMP92, CLMP95, Lio98]. Mean field limit results of point vortices with random intensities can be found in [Ner04, Ner05, KW12]. The analysis of fluctuations can be found in [BPP87, BG99] and in the recent [GR18].

The central topic of this paper is to give results about the mean field limit of a system of point vortices governed by (2.1). To be more detailed, if one considers a configuration of N point vortices located at $x_1, x_2, \dots, x_N$, with respective intensities $\gamma_1, \gamma_2, \dots, \gamma_N$, that is the measure

$$\theta(0) = \sum_{j=1}^{N} \gamma_j \delta_{x_j}$$

as the initial condition of (2.1), one can check that, formally, the solution evolves as a measure of the same kind, where the "intensities" γ_j remain constant (a generalized version of Kelvin's theorem about the conservation of circulation), and where the vortex positions evolve according to the system of equations

$$\begin{cases} \dot{X}_j = \sum_{k\neq j} \gamma_k \nabla^\perp G_m(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1, 2, \dots, N, \tag{2.3}$$

where G_m is the Green function of the operator $(-\Delta)^{\frac{m}{2}}$ on the torus with periodic boundary conditions and zero spatial average. The effective connection between the equations and the point vortex dynamics is not yet clear and will be discussed elsewhere. See also [FS18].

The motion of vortices is described by the Hamiltonian

$$H_N(\gamma^N, X^N) = \frac{1}{2} \sum_{j\neq k} \gamma_j \gamma_k G_m(X_j, X_k), \tag{2.4}$$

where $X^N = (X_1, X_2, \dots, X_N)$ and $\gamma^N = (\gamma_1, \gamma_2, \dots, \gamma_N)$.

A natural invariant distribution for the Hamiltonian dynamics (2.3) should be the measure

$$\mu_\beta^N(dX^N) = \frac{1}{Z_\beta^N} e^{-\beta H_N(X^N, \gamma^N)} \, d\ell^{\otimes N}, \tag{2.5}$$

where here and throughout the paper we denote by ℓ the normalized Lebesgue measure on $\mathbb{T}_2$. Due to the singularity of the Green function on the diagonal, which is of order $G_m(x, y) \sim |x - y|^{m-2}$, the density above is not integrable and thus the measure μ_β^N does not make sense.

2.3. **The regularized system.** To overcome this difficulty, we consider a regularization of the Green function. To define the regularization, notice that we can represent the Green function for the fractional Laplacian through the eigenvectors,

$$G_m(x, y) = \sum_{k=1}^{\infty} \lambda_k^{-\frac{m}{2}} e_k(x) e_k(y).$$

Given $\epsilon > 0$, consider the following regularization of the Green function,

$$G_{m,\epsilon}(x, y) = \sum_{k=1}^{\infty} \lambda_k^{-\frac{m}{2}} \, e^{-\epsilon \lambda_k} \, e_k(x) e_k(y). \tag{2.6}$$

Here, we have regularized the fractional Laplacian so that the new operator $D_{m,\epsilon}$ reads $D_{m,\epsilon} = (-\Delta)^{m/2} e^{-\epsilon\Delta}$ and the eigenvalues change from $\lambda^{m/2}$ to

$\lambda^{m/2} e^{\epsilon\lambda}$. We remark that, as long as $G_{m,\epsilon}$ is translation invariant and non-singular on the diagonal, the exact form of the regularization is not essential for our main results given in Section 3.

If we replace G_m by $G_{m,\epsilon}$ in (2.3), the motion is still Hamiltonian with Hamiltonian H_N^ϵ given by (2.4), with G_m replaced by $G_{m,\epsilon}$, namely

$$H_N^\epsilon(\gamma^N, X^N) = \frac{1}{2}\sum_{j\neq k} \gamma_j \gamma_k G_m^\epsilon(X_j, X_k).$$

In terms of invariant distributions, we want to consider a problem slightly more general and we shall randomize the intensities of vortices. The "quenched" case, namely the case with fixed intensities, will follow as a by-product, see Remark 3.6.

Let ν be a probability measure on the real line with support on a compact set $K_\nu \subset \mathbb{R}$[1]. The measure ν will be the *prior* distribution on vortex intensities. A natural invariant distribution for the regularized Hamiltonian dynamics with random intensities is

$$\mu^N_{\beta,\epsilon}(d\gamma^N, dX^N) = \frac{1}{Z^N_{\beta,\epsilon}} e^{-\frac{\beta}{N} H_N^\epsilon(\gamma^N, X^N)}\, d\ell^{\otimes N}\, d\nu^{\otimes N}, \tag{2.7}$$

where ℓ is the normalized Lebesgue measure on $\mathbb{T}_2$ and $Z^N_{\beta,\epsilon}$ is the normalization factor. In the above formula for the measure we have scaled the parameter β by N^{-1}, in analogy with the case $m = 2$. Indeed, for the Euler equation there is no nontrivial thermodynamic limit [FR83], and the interesting regime that provides interesting results is the mean field limit. See [MP94] for a physical motivation, and [CLMP92, CLMP95, Lio98] for the related mathematical results.

2.3.1. *Mean field limit of the regularized system.* The problem of finding the limit of measures $(\mu^N_{\beta,\epsilon})_{N\geqslant 1}$ is trivial, since the interaction among particles is bounded, and we only give an outline of the results.

The goal of this section is to show the existence of limit points for the measure (2.7) and to characterize them. We look at convergence of the distributions of a finite number of vortices. Therefore, we work with the so-called *correlation functions*, defined by

$$\rho^{N,k}_{\beta,\epsilon}(\gamma^k, X^k) := \int_{\mathbb{T}_2{}^{N-k}} \mu^N_{\beta,\epsilon}(d\gamma^{N-k}, dX^{N-k}),$$

namely the distribution of the first k vortices. This is not restrictive, by exchangeability of the measures (2.7).

In the following result we summarise the relevant estimates and therefore deduce weak convergence of the correlation functions.

[1] In other words we assume that intensities are bounded in size by a deterministic constant. Notice that in the case $m = 2$ this is in a way a requirement, see [CLMP92, CLMP95, Lio98].

Lemma 2.1. *There is a number* $C > 0$ *that depends (only) on* β *and* ϵ*, but not on* $k \geqslant 1$ *and* $N \geqslant 1$*, such that the following bounds hold,*

$$Z^N_{\beta,\epsilon} \leqslant C^N,$$
$$\rho^{N,k}_{\beta,\epsilon} \leqslant C^k e^{-\frac{\beta}{N} H^\epsilon_k(\gamma^k, X^k)},$$
$$\|\rho^{N,k}_{\beta,\epsilon}\|_{L^p} \leqslant C^k, \qquad p \in [1, \infty).$$

In particular, there is a sub-sequence $(N_j)_{j \geqslant 1}$ *such that*

$$\rho^{N_j,k}_{\beta,\epsilon} \rightharpoonup \rho^k_{\beta,\epsilon}$$

weakly in $L^p((K_\nu \times \mathbb{T}_2)^k)$*, for all* $k \geqslant 1$ *and* $p \in [1, \infty)$*.*

The proof of this lemma is a simpler version, due to the boundedness of the Green function, of corresponding results from [Ner04], and is therefore omitted.

To characterize the limit, consider the free energy functional on measures on $(K_\nu \otimes \mathbb{T}_2)^N$,

$$\mathcal{F}^\epsilon_N(\mu) = \mathcal{E}(\mu | \nu^{\otimes N} \otimes \ell^{\otimes N}) + \frac{\beta}{N} \int H^\epsilon_N(\gamma^N, X^N) \mu(d\gamma^N, dX^N)$$

where $\mathcal{E}$ is the relative entropy. It is not difficult to see that $\mu^N_{\beta,\epsilon}$ is the unique minimiser of the free energy. This can be carried to the limit. By convexity and subadditivity, we can define the entropy

$$\mathcal{E}_\star(\mu) = \lim_{N \to \infty} \frac{1}{N} \mathcal{E}(\rho^N | \nu^{\otimes N} \otimes \ell^{\otimes N}).$$

and thus the limit free energy as

$$\mathcal{F}^\epsilon_\star(\mu) = \mathcal{E}_\star(\mu) + \frac{1}{2}\beta \iint H^\epsilon_2(\gamma^2, X^2) \rho^2(d\gamma^2, dX^2), \tag{2.8}$$

where $(\rho^N)_{N \geqslant 1}$ are the correlation functions of μ. Here $\mathcal{E}_\star$ and $\mathcal{F}^\epsilon_\star$ are defined on exchangeable measures on $(K_\nu \times \mathbb{T}_2)^{\mathbb{N}}$ with absolutely continuous (with respect to powers of $\nu \otimes \ell$) correlation measures, with bounded densities.

As in [Ner04, Theorem 11], we have the following result.

Proposition 2.2 (Propagation of chaos)**.** *All limit points of* $(\mu^N_{\beta,\epsilon})_{N \geqslant 1}$ *are minima of the functional* $\mathcal{F}^\epsilon_\star$*.*

If $\mathcal{F}^\epsilon_\star$ *has a unique minimum* μ*, then* μ *is a product measure, namely there is a bounded function* ρ *such that all correlation functions* $(\rho^k_\mu)_{k \geqslant 1}$ *of* μ *have densities*

$$\rho^k_\mu(\gamma^k, x^k) = \rho(\gamma_1, x_1) \cdot \rho(\gamma_2, x_2) \cdot \dots \cdot \rho(\gamma_k, x_k).$$

In other words, propagation of chaos holds.

Since the measures $\mu^N_{\beta,\epsilon}$ are symmetric, each limit point $\mu^\epsilon_{\beta,\infty}$ will be exchangeable thus, by the De Finetti theorem, will be a superposition of product measures, namely

$$\mu^\infty_{\beta,\epsilon} = \int \mu^{\otimes \mathbb{N}} \pi(d\mu),$$

for a measure π on probability measures on $K_\nu \times \mathbb{T}_2$. As in [Ner04, Theorem 13], each measure π is concentrated on product measures $\mu^{\otimes\mathbb{N}}$ such that $\mu = \rho\,\nu\otimes\ell$ and the variational principle for $\mathcal{F}^\epsilon_\star$ can be read as a variational principle for ρ, in terms of the free energy

$$\mathcal{F}^\epsilon(\rho) = \int \rho\log\rho\,d\nu\,d\ell + \frac{1}{2}\beta\iint H^\epsilon_2(\gamma^2, x^2)\rho(\gamma_1, x_1)\rho(\gamma_2, x_2)\,d\nu^{\otimes 2}\,d\ell^{\otimes 2}. \tag{2.9}$$

The corresponding Euler-Lagrange equation, mean field equation in the language of [CLMP92, CLMP95, Lio98], is

$$\rho(\gamma, x) = \frac{1}{Z}e^{-\beta\gamma\psi_\rho(x)}, \tag{2.10}$$

where Z is the normalization constant, and ψ_ρ is the averaged stream function, that is $\psi_\rho(x) = \int\gamma G_{m,\epsilon}(x,y)\rho(\gamma,y)\,d\nu\,d\ell$. It is elementary to check that the function $\rho_0 = 1$ is a solution, with stream function $\psi_{\rho_0} = 0$. If $\mu_0 = (\rho_0\nu\otimes\ell)^{\mathbb{N}}$ is the product measure corresponding to ρ_0, then it is easy to check that $\mathcal{F}^\epsilon_\star(\mu_0) = 0$. If $\beta \geqslant 0$, the limit free energy $\mathcal{F}^\epsilon_\star$ is non–negative, and μ_0 is the unique minimum. As in [BG99], one can actually show that there is only one minimiser for small negative values of β, and thus propagation of chaos also holds for those values of β and limit measure μ_0.

2.3.2. *Back to the original problem.* The program outlined here for $(\mu^N_{\beta,\epsilon})_{N\geqslant 1}$ does not work at $\epsilon = 0$ from the very beginning, because, as already pointed out, the densities are too singular. On the other hand the limit free energy $\mathcal{F}^\epsilon_\star$ makes sense at $\epsilon = 0$, as well as the mean field equation (2.10). Moreover, as long as $\mathcal{F}^0_\star$ is convex and non–negative, the unique minimum is again μ_0. In Section 3 we prove that, by taking the limit of measures $(\mu^N_{\beta,\epsilon_N})_{N\geqslant 1}$, with a careful choice of the sequence $\epsilon_N \downarrow 0$, one can derive, at least when $\beta \geqslant 0$, propagation of chaos, a law of large numbers and a central limit theorem for the empirical density of the pair intensity-position of vortices.

Before that we wish to give some comments about the case when β is negative. First of all, we do not expect that $\nu\otimes\ell$ will be a minimiser for all negative values of β. This is true for all values of ϵ, in particular for the interesting case $\epsilon = 0$ and this is the reason we give a detailed computation below. The computation is similar to [Lio98, section 5.3].

Lemma 2.3. *Let $\epsilon \geqslant 0$ and $\beta < 0$. Then $\mu_0 = \nu\otimes\ell$ is not a minimiser of the free energy (2.9) for $\beta < \beta_0$, where*

$$\beta_0 := -\frac{\lambda_1^{\frac{m}{2}}e^{\epsilon\lambda_1}}{\nu(\gamma^2)},$$

and $\nu(\gamma^2) = \int\gamma^2\,d\nu$.

Proof. Let φ be bounded and with zero average with respect to $\nu\otimes\ell$, and set $\rho_t = 1 + t\varphi$, so that $\rho_t\nu\otimes\ell$ is a perturbation of μ_0 for t small. Clearly $\mathcal{F}^\epsilon(\rho_0) = 0$

and

$$\mathcal{F}^\epsilon(\rho_t) = \int \rho_t \log \rho_t \, d\nu \, d\ell + \frac{1}{2}\beta t^2 \|(-\Delta)^{-\frac{m}{4}} e^{\frac{1}{2}\epsilon\Delta} \bar{\varphi}\|^2_{L^2(\ell)}$$

where $\bar{\varphi}(x) = \int \gamma\varphi(\gamma, x)\, \nu(d\gamma)$. Expand the entropy around $t = 0$ and choose $\varphi = \gamma e_1$, to get

$$\mathcal{F}^\epsilon(\rho_t) = \mathcal{F}^\epsilon(\rho_0) + \frac{1}{2}t^2\big(1 + \beta\lambda_1^{-\frac{m}{2}} e^{-\epsilon\lambda_1} \nu(\gamma^2)\big) + o(t^2).$$

With the choice of β as in the statement, μ_0 cannot be a minimiser. □

The previous result can be read in terms of the equation for the averaged stream function. If ρ is a solution of the mean field equation (2.10), define the averaged scalar,

$$\theta_\rho(x) = \int \gamma\rho(\gamma, x)\nu(d\gamma) - \iint \gamma\rho(\gamma, x)\, d\nu\, d\ell,$$

and the averaged stream function $\psi_\rho = \int G_{m,\epsilon}(x, y)\theta_\rho(y)\, dy$, then ψ_ρ satisfies the following version of (2.10),

$$D_{m,\epsilon}\psi = \frac{\int \gamma\, e^{-\beta\gamma\psi_\rho}\, d\nu - \iint \gamma\, e^{-\beta\gamma\psi_\rho}\, d\nu\, d\ell}{\iint e^{-\beta\gamma\psi_\rho}\, d\nu\, d\ell}.$$

where $D_{m,\epsilon} = (-\Delta)^{\frac{m}{2}} e^{-\epsilon\Delta}$. For $\rho = 1$, $\theta_\rho = \psi_\rho = 0$. The linearisation of the above nonlinear equation around $\psi = 0$ yields the operator $D_{m,\epsilon} + \beta\nu(\gamma^2)I$, which is positive definite for $\epsilon \geqslant 0$ and β as in the previous lemma. At least when $\epsilon > 0$, due to uniform bounds on the minima that one can derive as in [BG99, Property 2.2], this shows that the previous lemma is optimal. In the case $\epsilon = 0$ unfortunately these bounds are not available and this is only an indication on what could happen.

We do not know if a law of large numbers and a central limit theorem hold for $-\beta_0 < \beta < 0$, or if Gaussian fluctuations hold up to the value β_0. This is the subject of a work in progress.

Remark 2.4. One can derive a large deviation principle, as in [BG99], for the regularized system at fixed $\epsilon > 0$. We have not been able to derive a large deviation principle in the limit $N \uparrow \infty$ and $\epsilon = \epsilon(N) \downarrow 0$, similarly to the results of the next section, due to a unsatisfactory control of the free energy, the expected rate function.

3. Main results

In this section we illustrate our main results, that is convergence of distributions of a finite number of vortices and propagation of chaos, and a law of large numbers and a central limit theorem for the point vortex system under the assumption of positive temperature $\beta > 0$. Our results are asymptotic both in the number of vortices *and* the regularization parameter ϵ, and thus they capture the behaviour of the original system (2.1). The results hold, though, only if the regularization parameter is allowed to go to zero with a speed, with respect to the number of vortices, which is at least *logarithmically slow*.

We know from Section 2.3.1 that, at finite ϵ, propagation of chaos holds and the limit distribution of a pair (position, intensity) is the measure $\nu \otimes \ell$. This is also the candidate limit when ϵ, N converge jointly to 0 and ∞. This is the first main result of this section.

Theorem 3.1 (Convergence of finite dimensional distributions)**.** *Assume* $m < 2$ *and* $\beta > 0$*, and fix a sequence* $\epsilon = \epsilon(N) \downarrow 0$ *so that*

$$\epsilon(N) \downarrow 0 \quad \text{as} \quad N \uparrow \infty, \qquad \epsilon(N) \geqslant C(\log N)^{-\frac{2}{2-m}} \tag{3.1}$$

with C *large enough (depending on* ν *and* β*). Then, as* $N \to \infty$*, the* k*-finite dimensional marginals of* $\mu^N_{\beta,\epsilon}$ *converge to* $(\nu \otimes \ell)^{\otimes k}$*. In particular, propagation of chaos holds.*

The proof of convergence of finite dimensional distribution will be given in Section 4.1.

We turn to the second limit theorem. Consider a system of N point vortices at equilibrium, with equilibrium measure (2.7), described by the N pairs $(\gamma^N_1, X^N_1), (\gamma^N_2, X^N_2), \dots, (\gamma^N_N, X^N_N)$ of intensity and position. Define the joint empirical distribution

$$\eta_N = \frac{1}{N}\sum_{j=1}^N \delta_{(\gamma^N_j, X^N_j)}$$

of intensity and position of point vortices.

Theorem 3.2 (Law of large numbers)**.** *Assume* $m < 2$ *and* $\beta > 0$*, and choose* $\epsilon = \epsilon(N)$ *as in the previous theorem. Then*

$$\eta_N \rightharpoonup \nu \otimes \ell, \qquad \text{in probability}$$

as $N \uparrow \infty$.

The proof of the law of large numbers is postponed to Section 4.2.

Remark 3.3. It is elementary to verify that convergence of η_N to $\ell \otimes \nu$ implies immediately convergence of the empirical pseudo-vorticity,

$$\theta_N = \frac{1}{N}\sum_{j=1}^N \gamma^N_j \delta_{X^N_j}$$

to $\nu(\gamma)\ell$, with $\nu(\gamma) = \int \gamma\, \nu(d\gamma)$. This yields a law of large numbers for the empirical pseudo-vorticity.

Finally, we can analyze fluctuations with respect to the limit stated in the previous theorem, namely the limit of the measures

$$\zeta_N = \sqrt{N}(\eta_N - \nu \otimes \ell)$$

to a Gaussian distribution. To this end define the operators $\mathscr{E}, \mathscr{G}$ as

$$\mathscr{G}\phi(x) := \int_{\mathbb{T}_2} G_m(x,y)\phi(y)\,\ell(dy),$$
$$\mathscr{E}\phi(\gamma,x) := \gamma\int_{K_\nu}\int_{\mathbb{T}_2} \gamma' G_m(x,y)\phi(\gamma',y)\,\nu(d\gamma')\ell(dy).$$

The operator $\mathscr{G}$ provides the solution to the problem $(-\Delta)^{\frac{m}{2}}\Phi = \phi$ with periodic boundary conditions and zero spatial average, and extends naturally to functions depending on both variables γ, x by acting on the spatial variable only. The proof of the following theorem will be the subject of Section 4.3.

Theorem 3.4 (Central limit theorem). *Assume $\beta > 0$ and choose $\epsilon = \epsilon(N)$ as in (3.1). Then $(\zeta_N)_{N\geqslant 1}$ converges, as $N \uparrow \infty$, to a Gaussian distribution with covariance $I - \beta(I + \beta\Gamma_\infty\mathscr{G})^{-1}\mathscr{E}$, in the sense that for every test function $\psi \in L^2(\nu\otimes\ell)$, $\langle\psi,\zeta_N\rangle$ converges in law to a real centred Gaussian random variable with variance*

$$\sigma_\infty(\psi)^2 := \langle I - \beta(I + \beta\Gamma_\infty\mathscr{G})^{-1}\mathscr{E}(\psi-\bar\psi), (\psi-\bar\psi)\rangle,$$

where $\bar\psi = (\nu\otimes\ell)(\psi)$ and $\Gamma_\infty = \nu(\gamma^2)$.

Remark 3.5. As in Remark 3.3, we can derive a central limit theorem for the empirical pseudo-vorticity θ_N Indeed, $\sqrt{N}(\theta_N - \nu(\gamma)\ell)$ converges to a Gaussian distribution with covariance $\Gamma_\infty(I + \beta\Gamma_\infty\mathscr{G})^{-1}$, in the sense that for every test function $\psi \in L^2(\ell)$, $\langle\sqrt{N}(\theta_N - \nu(\gamma)\ell), \psi\rangle$ converges in law to a real centred Gaussian random variable with variance

$$\tilde\sigma_\infty(\psi)^2 = \Gamma_\infty\langle(I + \beta\Gamma_\infty\mathscr{G})^{-1}(\psi-\bar\psi), (\psi-\bar\psi)\rangle.$$

The Gaussian measure obtained corresponds to the invariant measure (2.2) of the original system (2.1), when one takes $\alpha = 1/\Gamma_\infty$.

Remark 3.6 (Quenched results). The above results hold also in a "quenched" version, namely if intensities are non-random but given at every N. For instance, consider the result about convergence of finite dimensional distributions of vortices and propagation of chaos (Theorem 3.1). For every N, fix a family $\Gamma_N^q := (\gamma_j^N)_{j=1,2,\dots,N}$ and consider the quenched version of (2.7),

$$\mu_{\beta,\epsilon}^{\Gamma_N^q,N}(dx_1,\dots,dx_N) = \frac{1}{Z_{\beta,\epsilon}^{\Gamma_N^q,N}} e^{-\frac{\beta}{N}H_N^\epsilon(\gamma_1^N,\dots,\gamma_N^N,x_1,\dots,x_N)}\,d\ell^{\otimes N}.$$

If there is a measure $\nu_\star$ such that

$$\frac{1}{N}\sum_{j=1}^N \delta_{\gamma_j^N} \rightharpoonup \nu_\star, \qquad N\uparrow\infty, \tag{3.2}$$

and, due to our singular setting (in view of Lemma 4.4), if

$$\Big|\frac{1}{N}\sum_{j=1}^N(\gamma_j^N)^2 - \int\gamma^2\,\nu(d\gamma)\Big| G_{m,\epsilon_N}(0,0) \longrightarrow 0 \qquad N\uparrow\infty,$$

then the k-dimensional marginals of $\mu^{\Gamma^q_N,N}_{\beta,\epsilon}$ converge to $(\nu \otimes \ell)^{\otimes k}$, for all $k \geqslant 1$. Under the same assumptions, the law of large numbers also holds. To obtain the central limit theorem, one needs to assume some concentration condition on the convergence (3.2).

4. Proofs of the main results

Prior to the proof of our main results we state some preliminary results that will be useful in the rest of the section.

Lemma 4.1. *Let* $f \in L^3(\mathbb{T}_2)$ *with zero average on* $\mathbb{T}_2$, *then*

$$\Big| \int_{\mathbb{T}_2} e^{if(x)}\, d\ell - e^{-\frac{1}{2}\|f\|^2_{L^2}} \Big| \leqslant \|f\|^3_{L^3}.$$

Here the norms $\|\cdot\|_{L^2}$ *and* $\|\cdot\|_{L^3}$ *are computed with respect to the normalized Lebesgue measure* ℓ *on* $\mathbb{T}_2$.

Proof. Using the well-known inequalities

$$|\, e^{ix} - (1 + ix - \tfrac{1}{2}x^2)| \leqslant |x|^3,$$
$$|\, e^{-\frac{1}{2}x^2} - (1 - \tfrac{1}{2}x^2)| \leqslant |x|^3,$$

the proof is elementary. □

Lemma 4.2. *Let* $(\mu_N)_{N\geqslant 1}$, μ_∞ *be random probability measures on* $\mathbb{T}_2 \times K_\nu$. *Then* $(\mu_N)_{N\geqslant 1}$ *converges in law to* μ_∞ *if and only if for every* $\psi \in C(K_\nu \times \mathbb{T}_2)$,

$$\mathbb{E}\Big[e^{i\langle \psi, \mu_N\rangle}\Big] \longrightarrow \mathbb{E}\Big[e^{i\langle \psi, \mu_\infty\rangle}\Big]$$

Moreover, test functions can be taken in $C^1(K_\nu \times \mathbb{T}_2)$.

Proof. Set $E = \mathcal{P}(K_\nu \times \mathbb{T}_2)$ and recall that $K_\nu \times \mathbb{T}_2$ is a complete compact metric space, therefore E and $\mathcal{P}(E)$ are complete compact (thus separable) metric spaces for the topology of weak convergence.

For every $\psi \in C(K_\nu \times \mathbb{T}_2)$ define $\Phi_\psi \in C(E)$ as $\Phi_\psi(\mu) = \int \psi\, d\mu$. Consider the subset

$$\mathcal{M} = \{e^{i\Phi_\psi} : \psi \in C(K_\nu \times \mathbb{T}_2)\}$$

of $C(E)$. By Lemma 4.3 and Theorem 4.5 of [EK86], it is sufficient to prove that $\mathcal{M}$ is an algebra that separates the points of E. It is straightforward to check that $\mathcal{M}$ is an algebra. To prove that $\mathcal{M}$ separates points, consider $\mu, \nu \in E$ with $\Psi(\mu) = \Psi(\nu)$ for all $\Psi \in \mathcal{M}$. This reads

$$e^{i\langle \psi, \mu\rangle} = e^{i\langle \psi, \nu\rangle}$$

for all $\psi \in C(K_\nu \times \mathbb{T}_2)$. This readily implies that $\mu = \nu$. Likewise if $\psi \in C^1(K_\nu \times \mathbb{T}_2)$. □

In the proof of our limit theorems we will streamline and adapt to our setting an idea from [BPP87]. The key point is to give a representation of the equilibrium measure density in terms of a Gaussian random field. Here the condition $\beta > 0$ is crucial.

Lemma 4.3. *Let* $(x_1, x_2, \dots, x_N) \in \mathbb{T}_2{}^N$ *be* N *distinct points, and let* $\gamma_1, \gamma_2, \dots, \gamma_N \in K_\nu$*. Then*

$$e^{-\frac{\beta}{N} H_N^\epsilon(x^N,\gamma^N)} = \mathbb{E}_{U_{\beta,\epsilon}}\big[e^{\frac{i}{\sqrt{N}}\sum_{j=1}^N \gamma_j U_{\beta,\epsilon}(x_j)}\big]\, e^{\frac{1}{2N}\beta G_{m,\epsilon}(0,0)\sum_{j=1}^N \gamma_j^2},$$

where $U_{\beta,\epsilon}$ *is the periodic mean zero Gaussian random field on the torus with covariance* $\beta G_{m,\epsilon}$*, and* $\mathbb{E}_{U_{\beta,\epsilon}}$ *denotes expectation with respect to the probability framework on which* $U_{\beta,\epsilon}$ *is defined.*

Proof. The proof is elementary, since by definition of the random field $U_{\beta,\epsilon}$, the random vector $(U_{\beta,\epsilon}(x_1), U_{\beta,\epsilon}(x_2), \dots, U_{\beta,\epsilon}(x_N))$ is centred Gaussian with covariance matrix $(\beta G_{m,\epsilon}(x_j, x_k))_{j,k=1,2,\dots,N}$. Notice finally that by translation invariance, $G_{m,\epsilon}(x,x) = G_{m,\epsilon}(0,0)$. □

Lemma 4.4. *Assume there are a sequence of i. i. d. real random variables* $(X_k)_{k\geqslant 1}$ *such that there is* $M > 0$ *with* $0 \leqslant X_k \leqslant M$ *for all* k*, and a sequence of complex random variables* $(Y_k)_{k\geqslant 1}$ *such that* $\mathbb{E}Y_k \to L$*, a. s. and* $|Y_k| \leqslant M$ *for all* k*. Set* $S_n = \frac{1}{n}\sum_{k=1}^n X_k$*,* $S = \mathbb{E}[X_1]$*.*

If $F_n : [-S, M] \to \mathbb{R}$ *is a sequence of functions such that there is* $\alpha < \frac{1}{4}$ *with*

- $1 = F_n(0) \leqslant F_n(y) \leqslant e^{c_0 n^{2\alpha}}$ *for all* $y \in [-S, M]$*,*
- $\mathcal{B}_\delta := \sup_{|y|\leqslant\delta, n\geqslant 1} F_n(n^{-\alpha}y) \longrightarrow 1$ *as* $\delta \to 0$*,*

then

$$\mathbb{E}[F_n(S_n - S)Y_n] \longrightarrow L,$$

as $n \to \infty$*.*

Proof. Choose β such that $\alpha \leqslant \beta < \frac{1}{2}(1-2\alpha)$, fix $\delta > 0$ and set

$$A_n := \{n^\beta|S_n - S| \leqslant \delta\}.$$

By the Bernstein inequality there is $c_1 > 0$ such that

$$\mathbb{P}[A_n^c] \leqslant e^{-c_1 n^{1-2\beta}}. \tag{4.1}$$

In particular, $n^\beta(S_n - S) \to 0$ a. s.. Now,

$$\mathbb{E}[F_n(S_n - S)Y_n] = \mathbb{E}[F_n(S_n - S)Y_n \mathbb{1}_{A_n}] + \mathbb{E}[F_n(S_n - S)Y_n \mathbb{1}_{A_n^c}] =: \boxed{i} + \boxed{o}.$$

First, using the first assumption on F_n and (4.1),

$$\boxed{o} \leqslant M\, e^{c_0 n^{2\alpha}}\, \mathbb{P}[A_n^c] \leqslant M\, e^{c_0 n^{2\alpha} - c_1 n^{1-2\beta}} \longrightarrow 0,$$

by the choice of β. For the other term, let $\theta_\delta(y) = (y \wedge \delta) \vee (-\delta)$, then (recall that $\alpha \leqslant \beta$),

$$\begin{aligned}\boxed{i} &= \mathbb{E}[F_n(n^{-\alpha}\theta_\delta(n^\alpha(S_n - S)))Y_n \mathbb{1}_{A_n}] \\ &= \mathbb{E}\big[\big(F_n(n^{-\alpha}\theta_\delta(n^\alpha(S_n - S))) - 1\big)Y_n \mathbb{1}_{A_n}\big] + \mathbb{E}[Y_n \mathbb{1}_{A_n^c}].\end{aligned}$$

By (4.1), $\mathbb{E}[Y_n \mathbb{1}_{A_n^c}] \to L$, moreover,

$$\big|\mathbb{E}\big[\big(F_n(n^{-\alpha}\theta_\delta(n^\alpha(S_n - S))) - 1\big)Y_n \mathbb{1}_{A_n}\big]\big| \leqslant M(\mathcal{B}_\delta - 1)$$

and $\mathcal{B}_\delta \to 1$ as $\delta \to 0$ by the second assumption. The conclusion follows by first taking the limit in n, and then the limit in δ. □

4.1. **Proof of Theorem 3.1.** This section contains the proof of convergence of finite dimensional distributions of the equilibrium measure (2.7). To this end it is sufficient to prove convergence of the characteristic functions.

Fix $n \geqslant 1$, and assume $N \gg n$. By exchangeability of the measure $\mu^N_{\beta,\epsilon}$, it is sufficient to focus on the first n vortices $(\gamma_1, X_1), \dots, (\gamma_n, X_n)$. We have dropped here for simplicity the superscript ^{N}. Fix $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^k$ and $b = (b_1, b_2, \dots, b_n) \in (\mathbb{R}^2)^k$, we will write $a \cdot \gamma$ as a shorthand for $a_1\gamma_1 + \dots + a_n\gamma_n$, as well as $b \cdot X$ as a shorthand for $b_1 \cdot X_1 + \dots + b_n \cdot X_n$.

With these positions,

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}\left[e^{i(a\cdot\gamma+b\cdot X)}\right] = \frac{1}{Z^N_{\beta,\epsilon}} \int \dots \int e^{i(a\cdot\gamma+b\cdot x)}\, e^{-\frac{\beta}{N} H^\epsilon_N(\gamma,x)}\, d\ell^{\otimes N}\, d\nu^{\otimes N}.$$

By using Lemma 4.3, the formula above can be re-written as

$$\begin{aligned}\mathbb{E}_{\mu^N_{\beta,\epsilon}}&\left[e^{i(a\cdot\gamma+b\cdot X)}\right] = \\ &= \frac{1}{Z^N_{\beta,\epsilon}} \int \dots \int \mathbb{E}_{U^N_{\beta,\epsilon}}\left[e^{i(a\cdot\gamma+b\cdot x)+\frac{i}{\sqrt{N}}\sum_{j=1}^N \gamma_j U_{\beta,\epsilon}(x_j)}\right] e^{\frac{1}{2}\beta\Gamma_N G_{m,\epsilon}(0,0)}\, d\ell^{\otimes N}\, d\nu^{\otimes N},\end{aligned}$$

where

$$\Gamma_N := \frac{1}{N}\sum_{j=1}^N \gamma_j^2, \tag{4.2}$$

that by the law of large numbers converges in probability to $\mathbb{E}_\nu[\gamma^2] = \Gamma_\infty$.

We write now the space integral of the expectation in the integral above in a more compact way, to make our computations easier. Extend the vector b to 0, in the sense that $b_j = 0$ if $j \geqslant n+1$, and set

$$\begin{aligned} A^N_{\epsilon j}(b) &:= \int_{\mathbb{T}_2} e^{i\left(b_j\cdot x_j + \frac{\gamma_j}{\sqrt{N}} U_{\beta\epsilon}(x_j)\right)}\, d\ell, \\ B^N_{\epsilon j}(b) &:= e^{-\frac{\gamma_j^2}{2N}\|U_{\beta\epsilon}\|^2_{L^2(\ell)}} \int_{\mathbb{T}_2} e^{i b_j\cdot x_j}\, d\ell, \\ D^N_{\epsilon j}(b) &:= A^N_{\epsilon j}(b) - B^N_{\epsilon j}(b). \end{aligned}$$

We thus have

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}\left[e^{i(a\cdot\gamma+b\cdot X)}\right] = \frac{1}{Z^N_{\beta,\epsilon}} \int \dots \int e^{ia\cdot\gamma}\, \mathbb{E}_{U^N_{\beta,\epsilon}}\left[\prod_{j=1}^N A^N_{\epsilon j}(b)\right] e^{\frac{1}{2}\beta\Gamma_N G_{m,\epsilon}(0,0)}\, d\nu^{\otimes N},$$

Since we have the straightforward decomposition

$$\prod_{j=1}^N A^N_{\epsilon j}(b) = \prod_{j=1}^N B^N_{\epsilon j}(b) + \sum_{k=1}^N \Big(\prod_{j=1}^{k-1} A^N_{\epsilon j}(b)\Big)\cdot D^N_{\epsilon k}(b)\cdot\Big(\prod_{j=k+1}^N B^N_{\epsilon j}(b)\Big),$$

if we also set

$$\mathcal{L}(a,b) := \int \dots \int e^{\frac{1}{2}\beta(\Gamma_N-\Gamma_\infty)G_{m,\epsilon}(0,0)}\, e^{ia\cdot\gamma}\, \mathbb{E}_{U_{\beta\epsilon}}\left[\prod_{j=1}^N B^N_{\epsilon j}(b)\right] d\nu^{\otimes N},$$

and

$$\mathcal{E}(a,b) := \int\cdots\int e^{\frac{1}{2}\beta(\Gamma_N-\Gamma_\infty)G_{m,\epsilon}(0,0)}\,e^{ia\cdot\gamma}\,\cdot$$
$$\cdot\,\mathbb{E}_{U_{\beta\epsilon}}\Big[\sum_{k=1}^{N}\Big(\prod_{j=1}^{k-1}A_{\epsilon j}^N(b)\Big)D_{\epsilon k}^N(b)\Big(\prod_{j=k+1}^{N}B_{\epsilon j}^N(b)\Big)\Big]\,d\nu^{\otimes N},$$

we have that

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}\big[e^{i(a\cdot\gamma+b\cdot X)}\big] = \frac{1}{Z^N_{\beta\epsilon}}\,e^{\frac{1}{2}\beta\Gamma_\infty G_{m,\epsilon}(0,0)}\big(\mathcal{L}(a,b)+\mathcal{E}(a,b)\big)$$

If in particular we take $a = b = 0$, we obtain an analogous formula for the partition function,

$$Z^N_{\beta\epsilon} = e^{\frac{1}{2}\beta\Gamma_\infty G_{m,\epsilon}(0,0)}\big(\mathcal{L}(0,0)+\mathcal{E}(0,0)\big),$$

and in conclusion

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}\big[e^{i(a\cdot\gamma+b\cdot X)}\big] = \frac{\mathcal{L}(a,b)+\mathcal{E}(a,b)}{\mathcal{L}(0,0)+\mathcal{E}(0,0)} = \Bigg(\frac{\frac{\mathcal{L}(a,b)}{\mathcal{L}(0,0)}}{1+\frac{\mathcal{E}(a,b)}{\mathcal{L}(0,0)}} + \frac{\frac{\mathcal{E}(a,b)}{\mathcal{L}(0,0)}}{1+\frac{\mathcal{E}(a,b)}{\mathcal{L}(0,0)}}\Bigg).$$

It is sufficient now to prove that

$$\frac{\mathcal{L}(a,b)}{\mathcal{L}(0,0)} \longrightarrow \int\cdots\int e^{ia\cdot\gamma+ib\cdot x}\,d\ell^{\otimes n}\,d\nu^{\otimes n}, \qquad \frac{\mathcal{E}(a,b)}{\mathcal{L}(0,0)} \longrightarrow 0, \tag{4.3}$$

as $N\uparrow\infty$, $\epsilon=\epsilon(N)\downarrow 0$, for all a and b.

We first analyze $\mathcal{L}(a,b)/\mathcal{L}(0,0)$. If $(U_{\beta,\epsilon,k})_{k\geqslant1}$ are the components of $U_{\beta,\epsilon}$ with respect to the eigenvectors $e_1, e_2, \dots$, we notice that $(U_{\beta,\epsilon,k})_{k\geqslant1}$ are independent centred Gaussian random variables, and for each k, $U_{\beta,\epsilon,k}$ has variance βg_k^ϵ, where we have set for brevity $g_k^\epsilon := \lambda_k^{-m/2}\,e^{-\epsilon\lambda_k}$. Therefore, by Plancherel,

$$\frac{1}{2}\sum_{j=1}^{N}\|\tfrac{\gamma_j}{\sqrt{N}}U_{\beta,\epsilon}\|^2_{L^2(\ell)} = \frac{1}{2}\Gamma_N\sum_{k=1}^{\infty}U^2_{\beta,\epsilon,k},$$

and by independence,

$$\begin{aligned}\mathbb{E}_{U_{\beta\epsilon}}\Big[\prod_{j=1}^{N}B^N_{\epsilon j}(b)\Big] &= \Big(\prod_{j=1}^{n}\int_{\mathbb{T}_2}e^{ib_j\cdot x_j}\,d\ell\Big)\mathbb{E}_{U_{\beta\epsilon}}\Big[e^{-\frac{1}{2}\Gamma_N\sum_{k=1}^\infty U^2_{\beta,\epsilon,k}}\Big]\\ &= \Big(\prod_{j=1}^{n}\int_{\mathbb{T}_2}e^{ib_j\cdot x_j}\,d\ell\Big)\prod_{k=1}^{\infty}\mathbb{E}_{U_{\beta\epsilon}}\Big[e^{-\frac{1}{2}\Gamma_N U^2_{\beta,\epsilon,k}}\Big]\\ &= \Big(\prod_{j=1}^{n}\int_{\mathbb{T}_2}e^{ib_j\cdot x_j}\,d\ell\Big)\prod_{k=1}^{\infty}\frac{1}{(1+\beta g_k^\epsilon\Gamma_N)^{\frac{1}{2}}}\end{aligned}$$

using elementary Gaussian integration. Thus we have

$$\begin{aligned}\mathcal{L}(a,b) = \Big(\prod_{j=1}^{n}\int_{\mathbb{T}_2} e^{ib_j\cdot x_j}\,d\ell\Big)\Big(\prod_{k=1}^{\infty}\frac{1}{(1+\beta g_k^\epsilon\Gamma_\infty)^{\frac12}}\Big)\cdot\\ \cdot\int\cdots\int e^{ia\cdot\gamma}\,F_N(\Gamma_N-\Gamma_\infty)\,d\nu^{\otimes N},\end{aligned}\tag{4.4}$$

where

$$F_N(X) = e^{\frac12\beta X G_{m,\epsilon}(0,0)}\prod_{k=1}^{\infty}\Big(1+\frac{\beta g_k^\epsilon}{1+\beta g_k^\epsilon\Gamma_\infty}X\Big)^{-\frac12}.\tag{4.5}$$

If we prove that F_N meets the assumptions of Lemma 4.4, we immediately have the first claim of (4.3). Indeed, set

$$c_k = \frac{\beta g_k^\epsilon}{1+\beta g_k^\epsilon\Gamma_\infty},$$

then, by using the elementary inequality $\log(1+x)\geqslant x-\frac12 x^2$,

$$\begin{aligned}2\log F_N(x) &= \beta G_{m,\epsilon}(0,0)x - \sum_{k=1}^{\infty}\log(1+c_k x)\\ &\leqslant \Big(\beta G_{m,\epsilon}(0,0)-\sum_{k=1}^{\infty}c_k\Big)x+\frac12\Big(\sum_{k=1}^{\infty}c_k^2\Big)x^2\\ &\leqslant \Big(\beta G_{m,\epsilon}(0,0)-\sum_{k=1}^{\infty}c_k\Big)x+\frac12\Big(\sum_{k=1}^{\infty}c_k\Big)^2x^2.\end{aligned}$$

Since

$$0\leqslant\sum_k c_k\leqslant\beta\sum_k g_k^\epsilon = \beta G_{m,\epsilon}(0,0),$$

both assumptions of the lemma hold if there is $\alpha<\frac14$ such that $G_{m,\epsilon}(0,0)\lesssim N^\alpha$. On the other hand, it is elementary to see that

$$G_{m,\epsilon}(0,0) = \sum_{k=1}^{\infty}g_k^\epsilon = \sum_{k=1}\lambda_k^{-\frac{m}{2}}e^{-\epsilon\lambda_k}\approx\epsilon^{-\frac12(2-m)},\tag{4.6}$$

since $\lambda_k\sim k$, therefore our choice of $\epsilon=\epsilon(N)$ is sufficient to ensure the assumptions of Lemma 4.4 for F_N.

We turn to the analysis of $\mathcal{E}(a,b)/\mathcal{L}(0,0)$. By their definition, we have that $|A_{\epsilon j}^N(b)|\leqslant 1$ and $|B_{\epsilon j}^N(b)|\leqslant 1$, therefore,

$$\Big|\mathbb{E}_{\mathfrak{u}_{\beta\epsilon}}\Big[\sum_{k=1}^{N}\Big(\prod_{j=1}^{k-1}A_{\epsilon j}^N(b)\Big)D_{\epsilon k}^N(b)\Big(\prod_{j=k+1}^{N}B_{\epsilon j}^N(b)\Big)\Big]\Big|\leqslant\sum_{k=1}^{N}\mathbb{E}_{\mathfrak{u}_{\beta\epsilon}}[|D_{\epsilon k}^N(b)|],$$

and it remains to estimate the expectations of the terms $|D_{\epsilon j}^N(b)|$.

If $j \leqslant n$,

$$|D^N_{\epsilon j}(b)| \leqslant \int_{\mathbb{T}_2} \left| e^{\frac{\gamma_j}{\sqrt{N}} U_{\beta\epsilon}(x_j)} - e^{-\frac{\gamma_j^2}{2N}\|U_{\beta\epsilon}\|^2_{L^2(\ell)}} \right| d\ell$$
$$\leqslant \frac{|\gamma_j|}{\sqrt{N}} \int_{\mathbb{T}_2} |U_{\beta\epsilon}(x_j) + \|U_{\beta\epsilon}\|_{L^2(\ell)}| \, d\ell$$
$$\lesssim \frac{1}{\sqrt{N}} \|U_{\beta\epsilon}\|_{L^2(\ell)},$$

where we have used the elementary inequalities $|e^{ix} - 1| \leqslant |x|$ and $1 - e^{-\frac{1}{2}y^2} \leqslant |y|$. Thus for $j \leqslant n$,

$$\mathbb{E}_{U_{\beta\epsilon}}[|D^N_{\epsilon j}(b)|] \lesssim \frac{1}{\sqrt{N}} \mathbb{E}[\|U_{\beta\epsilon}\|^2_{L^2(\ell)}]^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{N}} \sqrt{G_{m,\epsilon}(0,0)},$$

since $U_{\beta\epsilon}$ is a Gaussian random field with covariance $\beta G_{m,\epsilon}$.

If on the other hand $j \geqslant n+1$, by Lemma 4.1,

$$\mathbb{E}_{U_{\beta\epsilon}}[|D^N_{\epsilon j}(b)|] \lesssim \frac{1}{N^{3/2}} \mathbb{E}_{U_{\beta\epsilon}}\left[\|\gamma_j U_{\beta\epsilon}\|^3_{L^3(\ell)}\right]$$
$$\leqslant \frac{1}{N^{3/2}} \left(\mathbb{E}_{U_{\beta\epsilon}}\left[\|\gamma_j U_{\beta\epsilon}\|^4_{L^4(\ell)}\right]\right)^{\frac{3}{4}}$$
$$\lesssim \frac{1}{N^{3/2}} G_{m,\epsilon}(0,0)^{\frac{3}{2}},$$

since

$$\mathbb{E}_{U_{\beta,\epsilon}}[\|U_{\beta,\epsilon}\|^4_{L^4(\ell)}] = \int_{\mathbb{T}_2} \mathbb{E}[U_{\beta,\epsilon}(x)^4] \, d\ell = = \int_{\mathbb{T}_2} 3\beta^2 G_{m,\epsilon}(x,x)^2 \, d\ell = 3\beta^2 G_{m,\epsilon}(0,0). \tag{4.7}$$

In conclusion

$$\mathcal{E}(a,b) \leqslant \int \cdots \int e^{\frac{1}{2}\beta(\Gamma_N - \Gamma_\infty) G_{m,\epsilon}(0,0)} \sum_{k=1}^N \mathbb{E}_{U_{\beta\epsilon}}[|D^N_{\epsilon k}(b)|] \, d\nu^{\otimes N}$$
$$\leqslant \left(\frac{n}{\sqrt{N}} \sqrt{G_{m,\epsilon}(0,0)} + \frac{N-n}{N^{\frac{3}{2}}} G_{m,\epsilon}(0,0)^{\frac{3}{2}} \right) \mathcal{E}_0$$
$$\lesssim \frac{1}{\sqrt{N}} (1 + G_{m,\epsilon}(0,0)^{\frac{3}{2}}) \mathcal{E}_0,$$

where we have set for brevity

$$\mathcal{E}_0 := \int \cdots \int e^{\frac{1}{2}\beta(\Gamma_N - \Gamma_\infty) G_{m,\epsilon}(0,0)} \, d\nu^{\otimes N}. \tag{4.8}$$

It is easy to see that by Lemma 4.4, $\mathcal{E}_0 \to 1$. Moreover, by our previous computations, see (4.4), we can write $\mathcal{L}(0,0)$ as

$$\mathcal{L}(0,0) = \left(\prod_{k=1}^\infty \frac{1}{1 + \beta \Gamma_\infty g^\epsilon_k} \right)^{\frac{1}{2}} \mathcal{L}_0,$$

with $\mathcal{L}_0 \to 1$. We have,

$$\prod_{k=1}^{\infty} \frac{1}{1+\beta\Gamma_\infty g_k^\epsilon} = e^{-\sum_k \log(1+\beta\Gamma_\infty g_k^\epsilon)} \geqslant e^{-\sum_k \beta\Gamma_\infty g_k^\epsilon} = e^{-\beta\Gamma_\infty G_{m,\epsilon}(0,0)}, \tag{4.9}$$

therefore

$$\frac{\mathcal{E}(a,b)}{\mathcal{L}(0,0)} \lesssim \frac{1}{\sqrt{N}}(1+G_{m,\epsilon}(0,0)^{\frac{3}{2}})\, e^{\beta\Gamma_\infty G_{m,\epsilon}(0,0)}\, \frac{\mathcal{E}_0}{\mathcal{L}_0}.$$

So it is sufficient to choose $\epsilon = \epsilon(N)$ so that

$$\frac{1}{\sqrt{N}}(1+G_{m,\epsilon}(0,0)^{\frac{3}{2}})\, e^{\beta\Gamma_\infty G_{m,\epsilon}(0,0)} \longrightarrow 0 \tag{4.10}$$

Using (4.6), we see immediately that it suffices to choose $\epsilon^{-\frac{1}{2}(2-m)} \leqslant c \log N$, with c small enough.

4.2. **Law of large numbers.** We turn to the proof of Theorem 3.2 on the weak law of large numbers for point vortices. We will broadly follow the same strategy of the previous section.

First of all, we notice that it is sufficient to prove convergence in law of η_N to $\ell\otimes\nu$. Moreover, in view of Lemma 4.2, it is sufficient to prove convergence of the characteristic functions over test functions $\psi \in C(K_\nu \times \mathbb{T}_2)$. Fix $\psi \in C(K_\nu \times \mathbb{T}_2)$, then by using Lemma 4.3,

$$\begin{aligned}
&\mathbb{E}_{\mu^N_{\beta,\epsilon}}[e^{i\langle\psi,\eta_N\rangle}] = \\
&\quad = \frac{1}{Z^N_{\beta,\epsilon}} \int \cdots \int \mathbb{E}_{U^N_{\beta,\epsilon}}\Big[e^{\frac{i}{\sqrt{N}}\sum_{j=1}^N \frac{1}{\sqrt{N}}\psi(\gamma_j,x_j)+\gamma_j U_{\beta,\epsilon}(x_j)}\Big]\, e^{\frac{1}{2}\beta\Gamma_N G_{m,\epsilon}(0,0)}\, d\ell^{\otimes N}\, d\nu^{\otimes N},
\end{aligned}$$

where Γ_N has been defined in (4.2). We set now some notations to shorten and simplify our formulas. Let $\ell(\psi)(\gamma) = \int \psi(\gamma,x)\,\ell(dx)$ and $\phi = \psi - \ell(\psi)$. For a function $a \in C(K_\nu)$, define

$$M_N(a) = \frac{1}{N}\sum_{j=1}^N a(\gamma_j). \tag{4.11}$$

Set moreover,

$$\begin{aligned}
A^N_{\epsilon j}(\phi) &:= \int_{\mathbb{T}_2} e^{\frac{i}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\phi(\gamma_j,x_j)+\gamma_j U_{\beta\epsilon}(x_j)\right)}\, d\ell,\\
B^N_{\epsilon j}(\phi) &:= e^{-\frac{1}{2N}\left\|\frac{1}{\sqrt{N}}\phi(\gamma_j,\cdot)+\gamma_j U_{\beta\epsilon}\right\|^2_{L^2(\ell)}},\\
D^N_{\epsilon j}(\phi) &:= A^N_{\epsilon j}(\phi) - B^N_{\epsilon j}(\phi).
\end{aligned}$$

We have the decomposition

$$\prod_{j=1}^N A^N_{\epsilon j}(\phi) = \prod_{j=1}^N B^N_{\epsilon j}(\phi) + \sum_{k=1}^N \Big(\prod_{j=1}^{k-1} A^N_{\epsilon j}(\phi)\Big)\cdot D^N_{\epsilon j}(\phi)\cdot\Big(\prod_{j=k+1}^N B^N_{\epsilon j}(\phi)\Big). \tag{4.12}$$

If we also set

$$\mathcal{L}(\psi) := \int\cdots\int e^{\frac{1}{2}\beta(\Gamma_N-\Gamma_\infty)G_{m,\epsilon}(0,0)}\, e^{iM_N(\ell(\psi))}\,\mathbb{E}_{U_{\beta\epsilon}}\Big[\prod_{j=1}^N B^N_{\epsilon j}(\phi)\Big]\,d\nu^{\otimes N},$$

and

$$\mathcal{E}(\psi) := \int\cdots\int e^{\frac{1}{2}\beta(\Gamma_N-\Gamma_\infty)G_{m,\epsilon}(0,0)}\, e^{iM_N(\ell(\psi))}\cdot$$
$$\cdot\,\mathbb{E}_{U_{\beta\epsilon}}\Big[\sum_{k=1}^N\Big(\prod_{j=1}^{k-1}A^N_{\epsilon j}(\phi)\Big)D^N_{\epsilon k}(\phi)\Big(\prod_{j=k+1}^N B^N_{\epsilon j}(\phi)\Big)\Big]\,d\nu^{\otimes N},$$

we have that

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}[e^{i\langle\psi,\eta_N\rangle}] = \frac{1}{Z^N_{\beta\epsilon}}\,e^{\frac{1}{2}\beta\Gamma_\infty G_{m,\epsilon}(0,0)}\big(\mathcal{L}(\psi)+\mathcal{E}(\psi)\big).$$

A similar formula can be obtained for $Z^N_{\beta\epsilon}$, therefore

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}[e^{i\langle\psi,\eta_N\rangle}] = \frac{\mathcal{L}(\psi)+\mathcal{E}(\psi)}{\mathcal{L}(0)+\mathcal{E}(0)}$$

It is sufficient now to prove that

$$\frac{\mathcal{L}(\psi)}{\mathcal{L}(0)}\longrightarrow e^{i\nu\otimes\ell(\psi)}\qquad\text{and}\qquad\frac{\mathcal{E}(\psi)}{\mathcal{L}(0)}\longrightarrow 0,$$

as $N\uparrow\infty$, $\epsilon=\epsilon(N)\downarrow 0$, for all ψ.

We first analyze $\mathcal{L}(\psi)/\mathcal{L}(0)$. Let $(U_{\beta,\epsilon,k})_{k\geqslant 1}$ and $(\phi_k)_{k\geqslant 1}$ be the component of $U_{\beta,\epsilon}$ and ϕ with respect to the eigenvectors $e_1, e_2, \dots$, and we set again $g^\epsilon_k := \lambda_k^{-m/2}\,e^{-\epsilon\lambda_k}$. By independence and elementary Gaussian integration,

$$\begin{aligned}\mathbb{E}_{U_{\beta\epsilon}}\Big[\prod_{j=1}^N B^N_{\epsilon j}(\phi)\Big] &= \mathbb{E}_{U_{\beta\epsilon}}\Big[e^{-\frac{1}{2N}\sum_{j=1}^N\|\frac{1}{\sqrt{N}}\phi(\gamma,\cdot)+\gamma_j U_{\beta,\epsilon}\|^2_{L^2(\ell)}}\Big]\\ &= e^{-\frac{1}{2N}M_N(\|\phi\|^2_{L^2(\ell)})}\prod_{k=1}^\infty\mathbb{E}_{U_{\beta,\epsilon}}\Big[e^{-\frac{1}{\sqrt{N}}M_N(\gamma\phi_k)U_{\beta,\epsilon,k}-\frac{1}{2}\Gamma_N U^2_{\beta,\epsilon,k}}\Big]\\ &= e^{-\frac{1}{2N}M_N(\|\phi\|^2_{L^2(\ell)})}\prod_{k=1}^\infty\Big(\frac{1}{(1+\beta\Gamma_N g^\epsilon_k)^{\frac{1}{2}}}\,e^{\frac{1}{2N}\frac{\beta M_N(\gamma\phi_k)^2 g^\epsilon_k}{1+\beta\Gamma_N g^\epsilon_k}}\Big).\end{aligned}$$

Thus we have

$$\mathcal{L}(\psi) = \Big(\prod_{k=1}^\infty\frac{1}{(1+\beta\Gamma_\infty g^\epsilon_k)^{\frac{1}{2}}}\Big)e^{i\nu\otimes\ell(\psi)}\,\mathcal{L}_0(\psi), \tag{4.13}$$

where

(4.14)
$$\mathcal{L}_0(\psi) := \int\cdots\int F_N(\Gamma_N-\Gamma_\infty)\,e^{i(M_N(\ell(\psi))-\nu\otimes\ell(\psi))}\cdot$$
$$\cdot\,e^{-\frac{1}{2N}M_N(\|\phi\|^2_{L^2(\ell)})}\,e^{\frac{1}{2N}\sum_{k=1}^\infty\frac{\beta M_N(\gamma\phi_k)^2 g^\epsilon_k}{1+\beta\Gamma_N g^\epsilon_k}}\,d\nu^{\otimes N},$$

and where F_N has been defined in (4.5). Since we look at the ratio $\mathcal{L}(\psi)/\mathcal{L}(0)$, it is sufficient to prove that $\mathcal{L}_0(\psi) \to 1$ as $N \uparrow \infty$ and $\epsilon = \epsilon(N) \downarrow 0$, for all ψ. In view of Lemma 4.4, we notice that

$$e^{i(M_N(\ell(\psi)) - \nu\otimes\ell(\psi))} \longrightarrow 1,$$

a. s. by the law of large numbers, and it is bounded. Likewise, the same holds for

$$e^{-\frac{1}{2N}M_N(\|\phi\|^2_{L^2(\ell)})} \longrightarrow 1.$$

Finally, since $M_N(\gamma\phi_k)^2 \leqslant \Gamma_N M_N(\phi_k^2)$,

$$\sum_{k=1}^\infty \frac{\beta M_N(\gamma\phi_k)^2 g_k^\epsilon}{1+\beta\Gamma_N g_k^\epsilon} \leqslant \sum_{k=1}^\infty M_N(\phi_k^2) = M_N(\|\phi\|^2_{L^2(\ell)}), \tag{4.15}$$

is bounded, we also have that

$$e^{\frac{1}{2N}\sum_{k=1}^\infty \frac{\beta M_N(\gamma\phi_k)^2 g_k^\epsilon}{1+\beta\Gamma_N g_k^\epsilon}} \longrightarrow 1,$$

and is bounded. Since we have proved in the previous section that F_N meets the assumptions of Lemma 4.4, we conclude that $\mathcal{L}_0(\psi) \to 1$.

We turn to the analysis of $\mathcal{E}(\psi)/\mathcal{L}(0)$. By Lemma 4.1 and formula (4.7),

$$\begin{aligned}
\mathbb{E}_{U_{\beta\epsilon}}[|D^N_{\epsilon j}(\phi)|] &\lesssim \frac{1}{N^{3/2}}\mathbb{E}_{U_{\beta\epsilon}}\Big[\big\|\tfrac{1}{\sqrt{N}}\phi(\gamma_j,\cdot) + \gamma_j U_{\beta\epsilon}\big\|^3_{L^3(\ell)}\Big] \\
&\leqslant \frac{1}{N^{3/2}}\Big(\mathbb{E}_{U_{\beta\epsilon}}\Big[\big\|\tfrac{1}{\sqrt{N}}\phi(\gamma_j,\cdot) + \gamma_j U_{\beta\epsilon}\big\|^4_{L^4(\ell)}\Big]\Big)^{\frac{3}{4}} \\
&\lesssim \frac{1}{N^{3/2}}(1 + G_{m,\epsilon}(0,0)^{\frac{3}{2}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{U_{\beta\epsilon}}\Big[\sum_{k=1}^N\Big(\prod_{j=1}^{k-1}A^N_{\epsilon j}(\phi)\Big)D^N_{\epsilon k}(\phi)\Big(\prod_{j=k+1}^N B^N_{\epsilon j}(\phi)\Big)\Big] &\leqslant \sum_{k=1}^N \mathbb{E}_{U_{\beta\epsilon}}[|D^N_{\epsilon k}(\phi)|] \\
&\lesssim \frac{1}{\sqrt{N}}(1 + G_{m,\epsilon}(0,0)^{\frac{3}{2}}),
\end{aligned}$$

so in conclusion

$$\mathcal{E}(\psi) \lesssim \frac{1}{\sqrt{N}}(1 + G_{m,\epsilon}(0,0)^{\frac{3}{2}})\mathcal{E}_0,$$

where $\mathcal{E}_0$ is defined as in (4.8), and $\mathcal{E}_0 \to 1$. Using the expression of $\mathcal{L}(0)$ given by (4.13), and formula (4.9), we have

$$\frac{\mathcal{E}(\psi)}{\mathcal{L}(0)} \lesssim \frac{1}{\sqrt{N}}(1 + G_{m,\epsilon}(0,0)^{\frac{3}{2}})\, e^{\beta\Gamma_\infty G_{m,\epsilon}(0,0)}\, \frac{\mathcal{E}_0}{\mathcal{L}_0(0)}.$$

As in formula (4.10), by our assumption on $\epsilon = \epsilon(N)$, the right-hand side converges to 0.

4.3. **Central limit theorem.** We finally turn to the proof of Theorem 3.4 on the fluctuations of point vortices. Again, the strategy is the same of the previous sections. By Lemma 4.2, it suffices to prove convergence of the characteristic functions over test functions ψ, with $\psi \in C^1(K_\nu \times \mathbb{T}_2)$, namely to prove that

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}[e^{i\langle\psi,\zeta_N\rangle}] \longrightarrow e^{-\frac{1}{2}\sigma_\infty(\psi)^2}$$

To this end fix $\psi \in C^1(K_\nu \times \mathbb{T}_2)$, and use the same notations of the previous section, namely $\phi = \psi - \ell(\psi)$ and the operator M_N defined in (4.11). Let $(\phi_k)_{k\geqslant 1}$ and $(G_{m,k})_{k\geqslant 1}$ be the Fourier coefficients of ϕ and G_m with respect to the basis of eigenvectors $e_1, e_2, \dots$. It is an elementary computation that

$$\sigma_\infty(\psi)^2 = \nu(\ell(\psi)^2) - \nu\otimes\ell(\psi) + \|\phi\|^2_{L^2(\nu\otimes\ell)} - \beta\sum_{k=1}^\infty \frac{G_{m,k}\nu(\gamma\phi_k)^2}{1+\beta\Gamma_\infty G_{m,k}}. \tag{4.16}$$

By using Lemma 4.3,

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}[e^{i\langle\psi,\zeta_N\rangle}] = \frac{1}{Z^N_{\beta,\epsilon}}\int\cdots\int e^{i\sqrt{N}(M_N(\ell(\psi))-\nu\otimes\ell(\psi))}\, e^{\frac{1}{2}\beta\Gamma_N G_{m,\epsilon}(0,0)}\,\cdot$$
$$\cdot\,\mathbb{E}_{u^N_{\beta,\epsilon}}\Big[e^{\frac{i}{\sqrt{N}}\sum_{j=1}^N \phi(\gamma_j,x_j)+\gamma_j u_{\beta,\epsilon}(x_j)}\Big]\, d\ell^{\otimes N}\, d\nu^{\otimes N},$$

where Γ_N is defined in (4.2). With the positions

$$A^N_{\epsilon j}(\phi) := \int_{\mathbb{T}_2} e^{\frac{i}{\sqrt{N}}\left(\phi(\gamma_j,x_j)+\gamma_j u_{\beta\epsilon}(x_j)\right)}\, d\ell,$$
$$B^N_{\epsilon j}(\phi) := e^{-\frac{1}{2N}\left\|\phi(\gamma_j,\cdot)+\gamma_j u_{\beta\epsilon}\right\|^2_{L^2(\ell)}},$$
$$D^N_{\epsilon j}(\phi) := A^N_{\epsilon j}(\phi) - B^N_{\epsilon j}(\phi).$$

the decomposition (4.12) still holds. Set also

$$E_N(\psi) = e^{i\sqrt{N}(M_N(\ell(\psi))-\nu\otimes\ell(\psi))},$$
$$\mathcal{L}(\psi) := \int\cdots\int e^{\frac{1}{2}\beta(\Gamma_N-\Gamma_\infty)G_{m,\epsilon}(0,0)}\, E_N(\psi)\mathbb{E}_{u_{\beta\epsilon}}\Big[\prod_{j=1}^N B^N_{\epsilon j}(\phi)\Big]\, d\nu^{\otimes N},$$

and

$$\mathcal{E}(\psi) := \int\cdots\int e^{\frac{1}{2}\beta(\Gamma_N-\Gamma_\infty)G_{m,\epsilon}(0,0)}\, E_N(\psi)\cdot$$
$$\cdot\,\mathbb{E}_{u_{\beta\epsilon}}\Big[\sum_{k=1}^N\Big(\prod_{j=1}^{k-1} A^N_{\epsilon j}(\phi)\Big) D^N_{\epsilon k}(\phi)\Big(\prod_{j=k+1}^N B^N_{\epsilon j}(\phi)\Big)\Big]\, d\nu^{\otimes N},$$

then, as in the previous sections,

$$\mathbb{E}_{\mu^N_{\beta,\epsilon}}[e^{i\langle\psi,\eta_N\rangle}] = \frac{\mathcal{L}(\psi)+\mathcal{E}(\psi)}{\mathcal{L}(0)+\mathcal{E}(0)},$$

and it is sufficient now to prove that

$$\frac{\mathcal{L}(\psi)}{\mathcal{L}(0)} \longrightarrow e^{-\frac{1}{2}\sigma_\infty(\psi)^2} \qquad \text{and} \qquad \frac{\mathcal{E}(\psi)}{\mathcal{L}(0)} \longrightarrow 0,$$

as $N \uparrow \infty$, $\epsilon = \epsilon(N) \downarrow 0$, for all ψ.

We first prove the convergence of the ratio $\mathcal{L}(\psi)/\mathcal{L}(0)$. Let $(U_{\beta,\epsilon,k})_{k\geqslant 1}$ and $(\phi_k)_{k\geqslant 1}$ be the components of $U_{\beta,\epsilon}$ and ϕ with respect to the eigenvectors $e_1, e_2, \dots$, and set again $g_k^\epsilon := \lambda_k^{-m/2} e^{-\epsilon\lambda_k}$. By Plancherel, independence, and elementary Gaussian integration,

$$\begin{aligned}\mathbb{E}_{U_{\beta\epsilon}}\Big[\prod_{j=1}^N B_{\epsilon j}^N(\phi)\Big] &= \mathbb{E}_{U_{\beta\epsilon}}\Big[e^{-\frac{1}{2N}\sum_{j=1}^N \|\phi(\gamma_j,\cdot)+\gamma_j U_{\beta,\epsilon}\|^2_{L^2(\ell)}}\Big]\\ &= e^{-\frac12 M_N(\|\phi\|^2_{L^2(\ell)})}\prod_{k=1}^\infty \mathbb{E}_{U_{\beta,\epsilon}}\Big[e^{-\frac12(\Gamma_N U^2_{\beta,\epsilon,k}+2M_N(\gamma\phi_k)U_{\beta,\epsilon,k})}\Big]\\ &= e^{-\frac12 M_N(\|\phi\|^2_{L^2(\ell)})}\prod_{k=1}^\infty\Big(\frac{1}{(1+\beta\Gamma_N g_k^\epsilon)^{\frac12}}\,e^{\frac{\beta g_k^\epsilon M_N(\gamma\phi_k)^2}{2(1+\beta\Gamma_N g_k^\epsilon)}}\Big).\end{aligned}$$

Thus we have

$$\mathcal{L}(\psi) = \Big(\prod_{k=1}^\infty \frac{1}{\sqrt{1+\beta\Gamma_\infty g_k^\epsilon}}\Big)\mathcal{L}_0(\psi),$$

with

$$\begin{aligned}\mathcal{L}_0(\psi) := \int\cdots\int F_N(\Gamma_N-\Gamma_\infty)E_N(\psi)\cdot\\ \cdot e^{-\frac12 M_N(\|\phi\|^2_{L^2(\ell)})}\,e^{\frac12\beta\sum_{k=1}^\infty \frac{g_k^\epsilon M_N(\gamma\phi_k)^2}{1+\beta\Gamma_N g_k^\epsilon}}\,d\nu^{\otimes N},\end{aligned}\tag{4.17}$$

and where F_N has been defined in (4.5). As in the previous proofs, it suffices to prove that $\mathcal{L}_0(\psi) \to e^{-\frac12\sigma_\infty(\psi)^2}$ as $N\uparrow\infty$ and $\epsilon = \epsilon(N)\downarrow 0$, for all ψ. Since we know already by the proof of Theorem 3.1 (see Section 4.1) that F_N verifies the assumptions of Lemma 4.4, it is sufficient to prove convergence in expectation of the other terms in $\mathcal{L}_0(\psi)$. First,

$$e^{-\frac12 M_N(\|\phi\|^2_{L^2(\ell)})} \longrightarrow e^{-\frac12\|\phi\|^2_{L^2(\nu\otimes\ell)}},$$

and

$$e^{\frac12\beta\sum_{k=1}^\infty\frac{g_k^\epsilon M_N(\gamma\phi_k)^2}{1+\beta\Gamma_N g_k^\epsilon}} \longrightarrow e^{\frac12\beta\sum_{k=1}^\infty\frac{G_{m,k}\nu(\gamma\phi_k)^2}{1+\beta\Gamma_\infty G_{m,k}}}$$

converge a. s. and in L^1 by the strong law of large numbers. The first term is obviously bounded, the second is bounded by the computations in (4.15). Here we can pass to the limit also in the sum using the smoothness of ϕ. Finally, by the central limit theorem for i. i. d. random variables,

$$E_N(\psi) \longrightarrow e^{-\frac12(\nu(\ell(\psi)^2)-\nu\otimes\ell(\psi)^2)}.\tag{4.18}$$

By recalling the explicit form of $\sigma_\infty(\psi)$ given in (4.16), we conclude that $\mathcal{L}_0(\psi)$ converges to $e^{-\frac12\sigma_\infty(\psi)^2}$.

We turn to the analysis of $\mathcal{E}(\psi)/\mathcal{L}(0)$. By Lemma 4.1 and formula (4.7),

$$\begin{aligned}\mathbb{E}_{\mu_{\beta\epsilon}}[|D^N_{\epsilon j}(\phi)|] &\lesssim \frac{1}{N^{3/2}}\mathbb{E}_{\mu_{\beta\epsilon}}\left[\left\|\phi(\gamma_j,\cdot)+\gamma_j U_{\beta\epsilon}\right\|^3_{L^3(\ell)}\right] \\ &\lesssim \frac{1}{N^{3/2}}(1+G_{m,\epsilon}(0,0)^{\frac{3}{2}}),\end{aligned}$$

therefore, as in the previous sections,

$$\mathcal{E}(\psi) \lesssim \frac{1}{\sqrt{N}}(1+G_{m,\epsilon}(0,0)^{\frac{3}{2}})\mathcal{E}_0,$$

with $\mathcal{E}_0 \to 1$. In conclusion,

$$\frac{\mathcal{E}(\psi)}{\mathcal{L}(0)} \lesssim \frac{1}{\sqrt{N}}(1+G_{m,\epsilon}(0,0)^{\frac{3}{2}})\, e^{\beta\Gamma_\infty G_{m,\epsilon}(0,0)}\, \frac{\mathcal{E}_0}{\mathcal{L}_0(0)}.$$

As in formula (4.10), by our assumption on $\epsilon = \epsilon(N)$, the right-hand side converges to 0.

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