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# Coarse homotopy groups

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## Abstract

In this note on coarse geometry we revisit coarse homotopy. We prove that coarse homotopy indeed is an equivalence relation, and this in the most general context of abstract coarse structures. We introduce (in a geometric way) coarse homotopy groups. The main result is that the coarse homotopy groups of a cone over a compact simplicial complex coincide with the usual homotopy groups of the underlying compact simplicial complex.

To prove this we develop geometric triangulation techniques for cones which we expect to be of relevance also in different contexts.

## KEYWORDS

coarse algebraic topology, coarse geometry, geometric triangulations, homotopy groups, simplicial geometry

## MSC (2010)

55Q05

## 1 | INTRODUCTION AND PRELIMINARIES

Our main results are the definition and computation of coarse homotopy and coarse homotopy groups, in the category of generalized coarse spaces, as introduced in particular by John Roe [8].

In this note, we discuss in detail the concept of coarse homotopy (and coarse homotopy equivalence). In particular, we check carefully that this is an equivalence relation, a result which seems not to be available in the literature. We use the “correct” notion of coarse homotopy, differing from the original one which has been shown to be inappropriate by being too flexible.

We then introduce a geometric version of coarse homotopy groups and show their basic properties (in particular that they form groups in the first place). The main computation is then the calculation of the coarse homotopy groups of cones on simplicial complexes: they are equal to the homotopy groups of the base of the cone. Preliminary results in this direction are contained in the Göttingen doctoral thesis of Behnam Norouzizadeh [6].

Along the way, we discuss that there is a canonical coarse structure on the (euclidean) cone of a simplicial complex. We also develop precise geometric triangulation techniques for cones of simplicial complexes which we expect to be of relevance in other contexts.

Before we get to these results, we start with preliminaries, introducing the coarse category and then deriving some gluing theorems for coarse maps, which are indispensable when working with geometric homotopy groups. We also work out some basics of triangulations and subdivisions which we will need.

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## 1.1 | The coarse category

Recall (compare e.g. [8]) that a (unital) *coarse structure* on a set  $X$  is a distinguished collection,  $\mathcal{E}$ , of subsets of the product  $X \times X$  called *entourages* such that:

- Any finite union of entourages is an entourage. Any subset of an entourage is an entourage.
- The union of all entourages is the entire space  $X \times X$ .
- The *inverse* of an entourage  $M$ :

$$M^{-1} = \{(y, x) \in X \times X \mid (x, y) \in M\}$$

is an entourage.

- The *composition* of entourages  $M_1$  and  $M_2$ :

$$M_1 M_2 = \{(x, z) \in X \times X \mid (x, y) \in M_1, (y, z) \in M_2 \text{ for some } y \in X\}$$

is an entourage.

- The diagonal,  $\Delta = \{(x, x) \mid x \in X\}$  is an entourage.

A space  $X$  equipped with a coarse structure is called a *coarse space*.

The above definition differs slightly from that of [4], but agrees with the definition of a *unital coarse structure* on a set in [5, 9]. We call an entourage *symmetric* if it is equal to its inverse and we write  $S(M) := M \cup M^{-1}$  for the symmetric entourage generated by the entourage  $M$ .

If  $X$  is a coarse space, and  $f, g : S \rightarrow X$  are maps into  $X$ , the maps  $f$  and  $g$  are termed *close* or *coarsely equivalent* if the set  $\{(f(s), g(s)) \mid s \in S\}$  is an entourage. We call a subset  $B \subseteq X$  *bounded* if the inclusion  $B \hookrightarrow X$  is close to a constant map.

The most important, and motivating, example of a coarse structure is the one of a proper metric space.

**Example 1.1.** Let  $X$  be a proper metric space (i.e. the closures of sets of finite diameter are compact). The *bounded coarse structure* on  $X$  is by definition the unital coarse structure formed by defining the entourages to be subsets of *R-neighbourhoods of the diagonal*:

$$D_R = \{(x, y) \in X \times X \mid d(x, y) < R\}; \quad R \in \mathbb{R}.$$

The bounded sets are simply those which are bounded with respect to the metric.

Let  $X$  and  $Y$  be coarse spaces. Then a map  $f : X \rightarrow Y$  is said to be *controlled* if for every entourage  $M \subseteq X \times X$ , the image

$$f[M] = \{(f(x), f(y)) \mid (x, y) \in M\}$$

is an entourage. A controlled map is called *coarse* if the inverse image of a bounded set is also bounded.

If  $X$  and  $Y$  are metric spaces equipped with their bounded coarse structures, a map  $f : X \rightarrow Y$  is controlled if and only if for all  $R > 0$  there exists  $S > 0$  such that if  $d(x, y) < R$  for  $x, y \in X$ , then  $d(f(x), f(y)) < S$  in the space  $Y$ .

We can form the category of all coarse spaces and coarse maps. We call this category the *coarse category*. We call a coarse map  $f : X \rightarrow Y$  a *coarse equivalence* if there is a coarse map  $g : Y \rightarrow X$  such that the composites  $g \circ f$  and  $f \circ g$  are close to the identities  $1_X$  and  $1_Y$  respectively.

Coarse spaces  $X$  and  $Y$  are said to be *coarsely equivalent* if there is a coarse equivalence between them. There is a similar notion of coarse equivalence between pairs of coarse spaces.

The following definition comes from [4].

**Definition 1.2.** Let  $X$  be a Hausdorff space. A coarse structure on  $X$  is said to be *compatible with the topology* if every entourage is contained in an open entourage, and the closure of any bounded set is compact.

Note that any coarse topological space is locally compact. In such a space, the bounded sets are precisely those which are precompact. It also follows from the definition that any precompact subset of the product of the space with itself is an entourage, and the closure of any entourage is an entourage.

**Example 1.3.** The bounded coarse structure on a proper metric space is compatible with the topology.

The purpose of this article is to develop some notions of homotopy theory in the coarse category. These homotopies have to end eventually, but the end time will be allowed to depend on the given point in the coarse space (and to go to infinity as one goes to infinity). This will be measured by coarse maps  $p : X \rightarrow \mathbb{R}_+$ , which we call “basepoint projection”, and which will be part of the structure for us.

**Example 1.4.** Let  $X$  be a proper metric space. Endow  $\mathbb{R}_+$  with the bounded coarse structure coming from the metric. Choose a point  $x_0 \in X$ . Then we have a basepoint projection  $p_{x_0} : X \rightarrow \mathbb{R}_+$  defined by the formula

$$p_{x_0}(x) = d(x, x_0).$$

Observe that for any two points  $x_0, y_0 \in X$  the maps  $p_{x_0}$  and  $p_{y_0}$  are close.

When proving results about coarse homotopies, the following lemma summarises many of the relevant properties of the coarse space  $\mathbb{R}_+$ . It is easy to check.

**Lemma 1.5.** *Let  $\mathbb{R}_+$  be the space  $[0, \infty)$  equipped with the bounded coarse structure arising from the usual metric. Then the following hold:*

- *Let  $M, N \subseteq \mathbb{R}_+ \times \mathbb{R}_+$  be entourages. Then the sets*

$$M + N = \{(u + x, v + y) \mid (u, v) \in M, (x, y) \in N\}$$

*and*

$$M - N = \{(u - x, v - y) \mid (u, v) \in M, (x, y) \in N, u \geq x, v \geq y\}$$

*are entourages.*

- *Let  $M \subseteq \mathbb{R}_+ \times \mathbb{R}_+$  be an entourage. Then the set*

$$Z(M) = \{(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x \leq u \leq y, x \leq v \leq y, (x, y) \in M\}$$

*is an entourage. Note that  $Z(Z(M)) = Z(M)$ .*

- *Let  $M \subseteq \mathbb{R}_+ \times \mathbb{R}_+$  be an entourage. Then the set*

$$\{(x + a, y + a) \mid a \in \mathbb{R}_+, (x, y) \in M\}$$

*is an entourage.*

**Proposition 1.6.** *Let  $X$  be a coarse space, and let  $f, g : X \rightarrow \mathbb{R}_+$  be coarse maps. Then the sum of  $f$  and  $g$  and the maximum of  $f$  and  $g$  are coarse maps.*

*Proof.* Let  $M \subseteq X \times X$  be an entourage. The images  $f[M]$  and  $g[N]$  are entourages. Observe that

$$(f + g)[M] = \{(f(x) + g(x), f(y) + g(y)) \mid (x, y) \in M\} \subseteq f[M] + g[N]$$

and

$$\begin{aligned} \max(f, g)[M] &= \{(\max\{f(x), g(x)\}, \max\{f(y), g(y)\}), (x, y) \in M\} \\ &\subseteq f[M] \cup g[M] \cup S(Z(S(f(M))) \cup Z(S(g(M))))). \end{aligned}$$

Hence by the above, the images  $(f + g)[M]$  and  $\max(f, g)[M]$  are entourages.

Now, let  $B \subseteq \mathbb{R}_+$  be bounded. Then we can choose  $a > 0$  such that  $B \subseteq [0, a]$ . Hence

$$(f + g)^{-1}[B] \subseteq \{x \in X \mid f(x) + g(x) \leq a\} \subseteq \{x \in X \mid f(x) \leq a\} = f^{-1}[0, a].$$

We see that the inverse image  $(f + g)^{-1}[B]$  is bounded. A similar argument tells us that the inverse image  $\max(f, g)^{-1}[B]$  is bounded.

So the maps  $f + g$  and  $\max(f, g)$  are both coarse, and we are done.  $\square$

**Definition 1.7.** Let  $X$  and  $Y$  be coarse spaces. Then the set-theoretic product  $X \times Y$  is equipped with the coarse structure defined by taking the entourages to be subsets of sets of the form  $M \times N$ , where  $M \subseteq X \times X$  and  $N \subseteq Y \times Y$  are entourages for the spaces  $X$  and  $Y$  respectively.

The product  $X \times Y$  is not a product in the category-theoretic sense. The problem is that the projections  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  are not in general coarse maps; the inverse images of bounded sets need not to be bounded.

## 1.2 | Pasting together maps

Many of the constructions of maps we are going to make are carried out in a piecewise manner, and we need criteria which make sure that a map which has good properties on the pieces does have such good properties globally.

For the following, recall that for metric spaces  $X$  and  $Y$  we call a map  $f : X \rightarrow Y$  *Lipschitz* if there is a constant  $C > 0$  such that  $d(f(x), f(y)) \leq Cd(x, y)$  for all  $x, y \in X$ . Certainly, any Lipschitz map is continuous. We call  $C$  the *Lipschitz constant* of  $f$ ; a Lipschitz map with Lipschitz constant  $C$  is called *C-Lipschitz*. A *bilipschitz homeomorphism* is an invertible Lipschitz map with Lipschitz inverse.

We will need the following properties of Lipschitz maps which are well known and easy to prove.

**Lemma 1.8.** *A continuous and piecewise smooth Riemannian map between smooth manifolds with a uniform bound on the norm of the differential is Lipschitz.*

*A composition of Lipschitz maps is Lipschitz.*

**Lemma 1.9.** *Let  $X$  be a geodesic metric space with a decomposition  $X = A \cup B$  for closed subsets  $A$  and  $B$ . Let  $Y$  be a metric space, and let  $f : X \rightarrow Y$  be a map such that the restrictions  $f|_A : A \rightarrow Y$  and  $f|_B : B \rightarrow Y$  are both  $C$ -Lipschitz. Then also  $f : X \rightarrow Y$  is  $C$ -Lipschitz.*

*More generally, if  $X = \bigcup_{i \in I} A_i$  is a union of closed subsets  $A_i$ , which is such that every compact subset is contained in a union of only finitely many of the  $A_i$ , and the restriction  $f|_{A_i} : A_i \rightarrow Y$  is  $C$ -Lipschitz for every  $i \in I$ , then the map  $f : X \rightarrow Y$  is also  $C$ -Lipschitz.*

*Proof.* Pick  $x, y \in X$ . If  $x, y \in A$  or  $x, y \in B$ , then  $d(f(x), f(y)) \leq Cd(x, y)$  by the Lipschitz condition on  $f|_A$ , and similar if  $x, y \in B$ . If  $x \in A$  and  $y \in B$  choose a geodesic  $\gamma : [0, d(x, y)] \rightarrow X$  from  $x$  to  $y$ . Then, there is a point  $z \in A \cap B$  on that geodesic. We obtain

$$d(f(x), f(y)) \leq d(f(x), f(z)) + d(f(z), f(y)) \leq Cd(x, z) + Cd(z, y) = Cd(x, y).$$

Here, the first inequality is the triangle inequality, the second the Lipschitz property of  $f|_A$  and  $f|_B$  and the third the geodesic property of  $\gamma$ .

The same proof gives the general statement for  $X = \bigcup_{i \in I} A_i$ , using the fact that any geodesic has compact image and therefore will involve only finitely many of the  $A_i$ .  $\square$

**Proposition 1.10.** *Let  $X$  be a proper metric space, considered as coarse space. Assume  $X = A \cup B$ . Assume this decomposition is coarsely excisive, i.e. for each  $R > 0$  there is  $S > 0$  such that  $U_R(A) \cap U_R(B) \subset U_S(A \cap B)$ , where  $U_R(Z) := \{x \in X \mid d(x, Z) \leq R\}$  for  $Z \subset X$  is the  $R$ -neighborhood of  $Z$ .*

*Assume  $f : X \rightarrow Y$  for a coarse space  $Y$  satisfies that  $f|_A : A \rightarrow Y$  and  $f|_B : B \rightarrow Y$  are coarse. Then also  $f : X \rightarrow Y$  is coarse.*

*Proof.* Firstly, if  $K \subset Y$  is bounded then  $f^{-1}(K) = (f|_A)^{-1}(K) \cup (f|_B)^{-1}(K)$  is the union of two bounded sets and therefore bounded.

Secondly, given  $R > 0$  choose  $S > 0$  such that  $U_R(A) \cap U_R(B) \subset U_S(A \cap B)$ . We have to show that the  $R$ -entourage  $\{(x, y) \in X \times X \mid d(x, y) \leq R\}$  in  $X \times X$  is mapped to an entourage of  $Y$ .

Let  $A_R = \{(x, y) \in A \times A \mid d(x, y) \leq R\}$  and  $B_R = \{(x, y) \in B \times B \mid d(x, y) \leq R\}$ . Then the  $R$ -entourage  $\{(x, y) \in X \times X \mid d(x, y) \leq R\}$  is the union of the sets  $A_R, B_R$ , and the set  $C := \{(x, y) \in A \times B \cup B \times A \mid d(x, y) \leq R\}$ . Hence, if  $(x, y) \in C$ , then  $x, y \in U_R(A) \cap U_R(B) \subset U_S(A \cap B)$ , i.e. the set  $C$  is contained in the  $S$ -entourage of  $A \cap B$  and therefore also in the  $S$ -entourage of  $A$ .

As the restrictions  $f|_A$  and  $f|_B$  are coarse maps, the above considerations imply that the images of  $A_R, B_R$ , and  $C$  are each entourages. Consequently, the map  $f$  is coarse.  $\square$

### 1.3 | Simplicial complexes

The following lemma summarizes metric properties of simplicial maps between geometric simplices, known by elementary geometry.

**Lemma 1.11.** *Let  $\sigma := \langle v_0, \dots, v_n \rangle \subset \mathbb{R}^N$  be a geometric  $n$ -simplex in  $\mathbb{R}^N$  spanned by  $(n + 1)$  vectors  $v_0, \dots, v_n$  in general position. Let  $w_0, \dots, w_n$  be vertices of a geometric  $k$ -simplex  $\tau \subset \mathbb{R}^M$ .*

*Then there is a unique affine linear map  $f : \sigma \rightarrow \tau$  sending  $v_i$  to  $w_i$ . The Lipschitz constant of  $f$  is bounded above by  $c(n, k, w) \max\{|w_i - w_j|\}$  where  $c(n, k, w)$  depends on the dimensions  $n$  and  $k$  of the simplices and in addition on a lower bound  $w$  on the width of  $\sigma$  defined to be the shortest distance from any vertex of  $\sigma$  to the opposite face.*

We need specific, geometric, triangulations of  $c(X)$  for a finite simplicial complex  $X$  embedded simplicially into  $\mathbb{R}^n$ . This can be achieved using standard subdivisions, as introduced by Whitney [11] and used by Dodziuk [2, Sect. 2].

**Definition 1.12.** A simplicial complex  $X$  is called *locally ordered* if there is a partial ordering on its vertices which restricts to a total ordering on the vertices of each simplex of  $X$ .

**Example 1.13.** A total order on the vertices of a simplicial complex of course also is a partial order. A barycentric subdivision has a canonical local order.

**Definition 1.14.** Let  $\sigma := \langle v_0, \dots, v_n \rangle \subset \mathbb{R}^N$  be a simplex realized as convex hull of the  $n + 1$  affinely independent vertices  $v_0, \dots, v_n \in \mathbb{R}^N$ . We define its standard subdivision  $S(\sigma)$  as the simplicial complex with vertices  $v_{ij} := (v_i + v_j)/2$  for  $0 \leq i < j \leq n$ .

On this set of vertices we define a partial order setting  $(i, j) \leq (k, l)$  if and only if  $k \leq i \leq j \leq l$ . By definition, the simplices of the standard subdivision are spanned by increasing sequences of vertices, making  $S(\sigma)$  locally ordered.

Given a locally ordered simplicial complex  $X$  define a standard subdivision  $S(X)$  by applying the standard decomposition to each simplex to obtain a simplicial decomposition of the whole simplicial complex. This is well defined due to the compatibility of the local orders of the vertices of the different simplicies. Note that the vertices of the standard subdivision inherit a partial order making it locally ordered which allows us to iterate the standard subdivision procedure.

**Definition 1.15.** Two geometric simplices  $\sigma, \tau \subset \mathbb{R}^N$  are *strongly similar* if one can be obtained from the other by translation and multiplication by a positive constant.

The great advantage of the standard subdivision is [2, Lem. 2.5]:

**Lemma 1.16.** *Let  $X$  be a finite simplicial complex embedded into  $\mathbb{R}^N$ , with a local order. There are only finitely many strong similarity types of the simplices of iterated standard subdivisions of  $X$ .*

We will also need several simplicial structures on  $X \times [0, 1]$  for a simplicial complex  $X$ .

**Definition 1.17.** Recall that, for a locally ordered simplicial complex  $X$  there is a canonical triangulation of  $X \times [0, 1]$  with the obvious simplicies in  $X \times \{0\}$  and  $X \times \{1\}$  coming from the triangulation of  $X$  and where in addition for any ordered simplex  $(v_0, \dots, v_k)$  of  $X$  and  $0 \leq j \leq k$  we get a new simplex spanned by  $(v_0, 0), \dots, (v_j, 0), (v_j, 1), \dots, (v_k, 1)$ .

We now define a “standard product subdivision” which restricts to the given triangulation on  $X \times \{0\}$  but to the standard subdivision  $S(X) \times \{1\}$  on the other end. The additional simplicies here are the following:

Whenever  $u_l \leq u_{l-1} \leq \dots \leq u_0 \leq v_0 < \dots < v_k \leq w_0 \leq \dots \leq w_l$  are vertices of a simplex of  $X$  such that  $(u_0, w_0) < (u_1, w_1) < \dots < (u_l, w_l)$  in the standard subdivision we get a simplex of the “standard product subdivision” of  $X \times [0, 1]$  spanned by  $(v_0, 0), \dots, (v_k, 0), ((u_0, w_0), 1), \dots, ((u_l, w_l), 1)$ . It is a little combinatorial exercise that these simplicies are indeed precisely and in unique way unions of the simplicies of the canonical triangulation of  $S(X) \times [0, 1]$  (which therefore further refines our standard product subdivision): the convex hull of  $\langle (u_0, w_0), \dots, (u_l, w_l) \rangle \times \{1\}$  and  $\langle v_0, \dots, v_k \rangle \times \{0\}$  as above is precisely the union of the convex hulls of  $\langle (u_0, w_0), \dots, (u_l, w_l) \rangle \times \{1\}$  and the simplicies in the standard subdivision of  $\langle v_0, \dots, v_k \rangle$  (times  $\{0\}$ ), and this way we obtain precisely the simplicies in the canonical triangulation of  $S(X) \times [0, 1]$ .

Therefore the described standard product subdivision indeed giving a triangulation of  $X \times [0, 1]$ .

## 2 | COARSE HOMOTOPY

To develop the notion of homotopy for coarse spaces we first consider cylinders. Our definition is inspired by [3, Sect. 3].

**Definition 2.1.** Let  $X$  be a coarse space, and let  $p : X \rightarrow \mathbb{R}_+$  be a coarse map. Then we define the  $p$ -cylinder

$$I_p X = \{(x, t) \in X \times \mathbb{R}_+ \mid t \leq p(x) + 1\}.$$

We have inclusions  $i_0 : X \rightarrow I_p X$  and  $i_1 : X \rightarrow I_p X$  defined by the formulas  $i_0(x) = (x, 0)$  and  $i_1(x) = (x, p(x) + 1)$ , respectively. The canonical projection  $q : I_p X \rightarrow X$  defined by the formula  $q(x, t) = x$  is a coarse map. The identities  $q \circ i_0 = 1_X$  and  $q \circ i_1 = 1_X$  clearly hold.

**Definition 2.2.** Let  $X$  and  $Y$  be coarse spaces. A *coarse homotopy* is a coarse map  $H : I_p X \rightarrow Y$  for some coarse map  $p : X \rightarrow \mathbb{R}_+$ .

We call coarse maps  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$  *coarsely homotopic* if there is a coarse homotopy  $H : I_p X \rightarrow Y$  such that  $f_0 = H \circ i_0$  and  $f_1 = H \circ i_1$ .

This map  $H$  is termed a *coarse homotopy* between the maps  $f_0$  and  $f_1$ .

Let  $f : X \rightarrow Y$  be a coarse map between coarse spaces. We call the map  $f$  a *coarse homotopy equivalence* if there is a coarse map  $g : Y \rightarrow X$  such that the composites  $g \circ f$  and  $f \circ g$  are coarsely homotopic to the identities  $1_X$  and  $1_Y$  respectively.

**Example 2.3.** Let  $X$  and  $Y$  be coarse spaces. Let  $p : X \rightarrow \mathbb{R}_+$  be any coarse map. Let  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$  be close coarse maps. Then we can define a coarse homotopy  $H : I_p X \rightarrow Y$  between the maps  $f_0$  and  $f_1$  by the formula

$$H(x, t) = \begin{cases} f_0(x), & t < 1, \\ f_1(x), & t \geq 1. \end{cases}$$

**Theorem 2.4.** *The notion of two coarse maps being coarsely homotopic is an equivalence relation.*

Before proving this theorem we need a technical lemma.

**Lemma 2.5.** Let  $q, p : X \rightarrow \mathbb{R}_+$  be coarse maps. Let us write  $I_{p+q}X = A \cup B$  where

$$A = \{(x, t) \in I_{p+q}X \mid t \leq p(x)\}; \quad B = \{(x, t) \in I_{p+q}X \mid t \geq p(x)\}.$$

Suppose that  $f : I_{p+q}X \rightarrow Y$  is a map such that the restrictions  $f|_A$  and  $f|_B$  are coarse maps. Then the map  $f$  is a coarse map.

*Proof.* It is clear that the inverse image under the map  $f$  of a bounded set is bounded, as the union of any two bounded sets is bounded. Let  $M \subseteq (X \times \mathbb{R}_+) \times (X \times \mathbb{R}_+)$  be an entourage. We need to show that the image  $f[M]$  is an entourage.

Since the restrictions  $f|_A$  and  $f|_B$  are coarse, we know that the sets  $f[M \cap (A \times A)]$  and  $f[M \cap (B \times B)]$  are entourages. We need to prove that the sets  $f[M \cap (A \times B)]$  and  $f[M \cap (B \times A)]$  are entourages. We will check only the first case; the second case is similar.

Without loss of generality, suppose that  $M = M_1 \times M_2 \cap I_{p+q}X$  where  $M_1 \subseteq X \times X$  and  $M_2 \subseteq \mathbb{R}_+ \times \mathbb{R}_+$  are symmetric entourages containing the diagonal, and with  $M_2 = Z(M_2)$ . We are here indulging in some mild abuse of notation involving the order of various factors in products. Consider points  $(x, s) \in A$  and  $(y, t) \in B$  such that  $((x, s), (y, t)) \in M$ .

The inequalities  $s \leq p(x)$  and  $p(y) \leq t$  hold. So either  $s \leq p(y) \leq t$  or  $p(y) \leq s \leq p(x)$ . The former yields that  $(p(y), t), (p(y), s) \in Z(M_2) = M_2$ ; the latter that  $(p(y), s) \in Z(p[M_1])$ . Since  $(s, t) \in M_2$ , in either case we have that  $(p(y), t), (p(y), s) \in Z(p[M_1])M_2$ . So if we let  $N$  be the entourage  $M_1 \times Z(p[M_1])M_2$  (which depends only on the entourage  $M$  and the coarse map  $p$ ), then  $((x, s), (y, p(y))) \in N \cap (A \times A)$  and  $((y, p(y)), (y, t)) \in N \cap (B \times B)$ .

Therefore

$$(f(x, s), f(y, t)) \in f[N \cap (A \times A)]f[N \cap (B \times B)].$$

Hence the image  $f[M \cap (A \times B)]$  is contained in the entourage  $f[N \cap (A \times A)]f[N \cap (B \times B)]$  and the map  $f$  is coarse.  $\square$

*Proof of Theorem 2.4.* The relation is reflexive by Example 2.3. Let  $p : X \rightarrow \mathbb{R}_+$  be a coarse map, and let  $H : I_pX \rightarrow Y$  be a coarse homotopy. Define a map  $\overline{H} : I_pX \rightarrow Y$  by the formula

$$\overline{H}(x, t) = H(x, p(x) + 1 - t).$$

We claim that the map  $\overline{H}$  is a coarse homotopy, thus proving that the relation of coarse homotopy is symmetric. To show this fact, it suffices to show that the *flip map*  $F : I_pX \rightarrow I_pX$  defined by the formula  $F(x, t) = (x, p(x) + 1 - t)$  is coarse.

Let  $M \subseteq X \times X$  and  $N \subseteq \mathbb{R}_+ \times \mathbb{R}_+$  be entourages. Observe that

$$F(M \times N) \subseteq M \times (p(M) + 1 - N)$$

which is an entourage by Lemma 1.5 as  $p$  is a coarse map.

Let  $A \subseteq X$  and  $B \subseteq \mathbb{R}_+$  be bounded sets. Then

$$F^{-1}[A \times B] \subseteq A \times (p(A) + 1 - B)$$

which is bounded since  $p$  is coarse, and so takes bounded sets to bounded sets. We conclude that the map  $F$  and hence the map  $\overline{H}$  are coarse.

We must now prove that the equivalence relation is transitive. Let  $p, p' : X \rightarrow \mathbb{R}_+$  be coarse maps. Then by Proposition 1.6, the sum  $p + p' + 1 : X \rightarrow \mathbb{R}_+$  is also coarse.

Consider coarse homotopies  $H : I_pX \rightarrow Y$  and  $H' : I_{p'}X \rightarrow Y$  such that  $H(x, p(x) + 1) = H'(x, 0)$  for all  $x \in X$ . Define a map  $H + H' : I_{p+p'+1}X \rightarrow Y$  by the formula

$$(H + H')(x, t) = \begin{cases} H(x, t), & 0 \leq t \leq p(x) + 1, \\ H'(x, t - (p(x) + 1)), & p(x) + 1 \leq t \leq p(x) + p'(x) + 2. \end{cases}$$



Then the map  $H + H'$  is a coarse map by Lemma 2.5. Transitivity now follows.  $\square$

The above notion of coarse homotopy is not quite the one used in older literature for the coarse category. However, as mentioned in [1], the conventional definition is not quite adequate for the purposes of coarse homology. Our definition is the appropriate remedy.

The definition of coarse homotopy contains the choice of the basepoint projection map  $p : X \rightarrow \mathbb{R}_+$ . It might seem that we lose too much control here. However, for most spaces we are interested in we can normalize this:

**Lemma 2.6.** *Let  $X$  be a path-metric space, considered as coarse space. For  $x_0 \in X$ , let  $p_0 : X \rightarrow \mathbb{R}_+$ ;  $x \mapsto d(x, x_0)$  be the standard basepoint projection. Let  $q : X \rightarrow \mathbb{R}_+$  be any coarse map. Then any coarse homotopy  $H : I_q X \rightarrow Y$  between  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  gives rise to a coarse homotopy  $\tilde{H} : I_{p_0} X \rightarrow Y$  between  $f$  and  $g$ .*

*The statement generalizes in the obvious way to  $X$  with finitely many path components.*

*Proof.* As  $X$  is a path metric space, it is well known that the coarse map  $q$  is *large scale Lipschitz*, i.e. there is  $L > 0$  such that  $|q(x) - q(y)| \leq Ld(x, y) + L$  for all  $x, y \in X$ . In particular,

$$|q(x)| \leq |q(x) - q(x_0)| + |q(x_0)| \leq Lp_0(x) + C$$

for  $C = L + |q(x_0)|$  and for all  $x \in X$ . Set  $q'(x) := C + Lp_0(x)$ . We just saw that  $q \leq q'$ . We can extend the homotopy  $H$  to  $H' : I_{q'} X \rightarrow Y$  by extending “constantly” for the additional time, i.e.  $H'(x, t) = g(x, t)$  if  $(x, t) \in I_{q'} X \setminus I_q X$ .

Finally, there is a canonical coarse equivalence  $\Psi : I_{p_0} X \rightarrow I_{q'} X$ , with

$$\Psi(x, t) := \begin{cases} (x, (C + 1)t), & 0 \leq t \leq 1, \\ (x, C + 1 + L(t - 1)), & 1 \leq t \leq p_0(x) + 1, \end{cases}$$

and we define  $\tilde{H} := H' \circ \Psi$  which has all the desired properties.  $\square$

The following example can be found in several places in the literature, for example following [7, Lem. 9.9]. We write out the argument again here in order to establish that everything is in order when we use our notion of coarse homotopy.

**Example 2.7.** Let  $M$  be a complete simply-connected Riemannian manifold of non-positive sectional curvature. The metric turns the manifold  $M$  into a coarse space. The exponential map  $\exp : \mathbb{R}^n \rightarrow M$  is a distance-increasing diffeomorphism. The inverse  $\log : M \rightarrow \mathbb{R}^n$  is therefore a coarse map.

We claim that the map  $\log$  is a coarse homotopy equivalence. The problem is that the inverse map  $\exp$  is not coarse; otherwise, the result would be trivial.

Let us call a map  $s : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a *radial shrinking* if it takes the form  $s(r, \theta) = (f(r), \theta)$  in polar coordinates, where the map  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a distance-decreasing differentiable map with positive derivative. Then it is clear that any radial shrinking is coarsely homotopic to the identity map. Moreover, it is not hard to see that also  $\exp \circ s \circ \log : M \rightarrow M$  is a coarse map coarsely homotopic to the identity.

Now, we can find a radial shrinking  $s$  such that the composite  $\exp \circ s$  is a coarse map. By the above remark, the composites  $\log \circ \exp \circ s$  and  $\exp \circ s \circ \log$  are coarsely homotopic to identity maps, and so the map  $\log$  is a coarse homotopy equivalence as claimed.

In particular, Euclidean space  $\mathbb{R}^n$  and hyperbolic space  $\mathbb{H}^n$  are coarsely homotopy equivalent.

### 3 | METRIC CONES

In this section we collect some basic properties of metric cones. In particular, we show that for a finite simplicial complex there is a canonical (euclidean) coarse structure (even metric structure up to bilipschitz equivalence) on the infinite cone.

Moreover, we prove a regularity result similar to the simplicial approximation theorem (and based on it): in our context every coarse map is coarsely equivalent to a Lipschitz map.

**Definition 3.1.** Let  $X$  be a subset of the unit sphere of some real Hilbert space  $H$ . Then we define the *metric cone with spherical base* (with the induced metric)

$$C(X) = \{tx \mid t \geq 0, x \in X\}.$$

If  $Y$  is a subset of some real Hilbert space  $H$  we define the *metric cone with flat base* (with the induced metric)

$$c(Y) := \{(hx, h) \mid h \geq 0, x \in Y\} \subset H \times \mathbb{R}.$$

For  $R \geq 0$  we set  $c_R(Y) := c(Y) \cap H \times [R, \infty)$ , that is to say  $c_R(Y)$  is the part of the cone of height at least  $R$ . If  $Y$  is compact then the inclusion  $c_R(Y) \hookrightarrow c(Y)$  is a coarse equivalence. Therefore, for us it usually is sufficient to consider only the part  $c_R(Y)$ , which is sometimes technically more convenient.

**Example 3.2.** Let  $Y = S^n$  be the whole unit sphere. Then  $C(S^n) = \mathbb{R}^{n+1}$ .

This definition is further reaching than it first appears. For example, every finite CW-complex is homeomorphic to a subset of the unit sphere of a Hilbert space, even of a finite dimensional one. However, it is not completely clear whether the resulting coarse space is uniquely defined, up to coarse equivalence, by the homeomorphism type of  $X$ . It is true, however, that a finite simplicial complex gives rise to a preferred coarse type of the metric cone (determined by the simplicial structure), what we discuss next.

**Lemma 3.3.** Let  $X$  be a connected finite simplicial complex. Let  $f : X \rightarrow \mathbb{R}^n$  and  $g : X \rightarrow S^m$  be PL-embeddings.

Form the cones  $c(f(X))$  and  $C(g(X))$ . We have a canonical homeomorphism

$$\Psi : c(f(X)) \rightarrow C(g(X)); (hf(x), h) \mapsto hg(x) \quad \text{for } x \in X, h \geq 0.$$

If we equip each cone with either the subspace metric obtained as restriction of the metric on  $\mathbb{R}^{n+1}$  or  $\mathbb{R}^{m+1}$ , or with the induced path metric, then the homeomorphism  $\Psi$  and the identity maps  $\text{id}_{c(f(X))}$  and  $\text{id}_{C(g(X))}$  applied when changing metrics are bilipschitz homeomorphisms.

In particular, the bilipschitz class does not depend on the chosen PL-embedding, on the question whether we use a spherical base as in  $C(g(X))$  or a euclidean base as in  $c(f(X))$ , nor on the question whether we use the induced metric from the embedding or the induced path metric.

The same result applies to  $c_R(f(X))$  for fixed  $R > 0$ .

*Proof.* It is well known that for the PL-embeddings  $f$  and  $g$  the subspace metric and the path metric on the image are bilipschitz equivalent. Moreover, because the maps are piecewise linear and  $X$  is compact, any two PL-embeddings either into  $\mathbb{R}^n$  or into  $S^m$  induce equivalent metrics on  $X$ .

Consider now the compact cones (the parts of the full cones with height between 0 and 1)  $c_f(1)$  and  $C_g(1)$ , where for  $R > 0$

$$c_f(R) := \{(hx, h) \in c(f(X)) \mid 0 \leq h \leq R\} \subset \mathbb{R}^n \times [0, R],$$

$$C_g(R) := \{tx \in C(g(X)) \mid 0 \leq t \leq R\} \subset B_R(0) \subset \mathbb{R}^{m+1}.$$

These are again PL-embedded simplicial complexes with the resulting induced metrics from the embeddings, so that the identity map and the restriction of  $\Psi$  are bilipschitz homeomorphisms for the restricted metrics and the induced path metrics.

Next, observe that for arbitrary  $R > 0$ , but fixed  $f, g$  the parts  $c_f(R)$  of the cones  $c_f(X)$  and  $C_g(R)$  of  $C_g(X)$  are just scaled versions of  $c_f(1)$  and  $C_g(1)$ . In particular, the identity maps (for the path metric versus the restricted metric) and the map

$\Psi$  (restricted to  $c_f(R)$ ) are just a scaling of the corresponding maps on  $c_f(1)$  and  $C_g(1)$ , respectively. This implies directly that these maps remain bilipschitz homeomorphisms with the same bilipschitz constant as the maps for  $R = 1$ .

This, in turn, implies that also the maps defined on the full cones are bilipschitz with the same bilipschitz constant, by the very definition of the Lipschitz property.  $\square$

Of course the spaces  $c(f(X))$  and  $C(g(X))$  are geodesic when equipped with path metrics.

Note that the space  $c_R(f(X))$  is not bilipschitz equivalent to the full cone  $c_f(X)$ .

**Definition 3.4.** Let  $X \subset \mathbb{R}^N$  be a finite simplicial complex simplicially embedded.

Write  $c(X) \subset \mathbb{R}^N \times [0, \infty)$  as the union of the convex hull of 0 and  $X \times \{1\}$ , the compact cone on  $X$  with the obvious simplicial structure and the infinitely many copies of  $X \times [0, 1]$  given as

$$ZX(n) := \{(hx, h) \mid x \in X, h \in [n, n+1]\} \quad \text{for } n = 1, 2, \dots$$

We now define a simplicial structure on  $c(X)$  as follows: we use the  $n$ -th standard subdivision of  $X$  on  $kX \times \{k\}$  for  $k \in \mathbb{N}$  with  $2^n \leq k < 2^{n+1}$  and the product simplicial structure of Definition 1.17 on  $ZX(k)$  compatible with the so given simplicial structure on the top and the bottom.

**Lemma 3.5.** *There are only finitely many strong similarity types in the simplicial structure of  $c(X)$  given in Definition 3.4. Moreover, the lengths of the edges are contained in a compact interval  $[a, b]$  with  $0 < a < b < \infty$ . In particular, there is a positive lower bound on the width of the simplices and an upper bound on the diameter.*

*Proof.* Scaling does not change the strong similarity type, therefore by Lemma 1.16 there are only finitely many strong similarity types among the simplices of the cross sections  $kX \times \{k\}$  for  $k \in \mathbb{N}$ . The remaining simplices are obtained from these by two procedures to obtain triangulations of  $X \times [0, 1]$  subdividing  $\sigma \times [0, 1]$  for a simplex  $\sigma$ , which results in finitely many new strong similarity types for each similarity type of  $\sigma$ , which are then also further scaled to obtain the simplices of  $c(X)$ . Furthermore, there are finitely many more simplices at the tip of the cone.

The lengths of the edges in our triangulation are bounded above because we perform a further standard subdivision of the cross-section (which halves each original edge) as soon as the complex is scaled by 2 in  $kX \times \{k\}$ . The standard subdivision procedure does only produce edges whose length is at least half the length of an edge of the original simplicial complex. Therefore, in the cross sections  $kX \times \{k\}$  the edges are never shorter than the shortest edge of the original triangulation of  $X$ . The statement about the lower and upper bound on the geometry of the simplices of the triangulation now follows immediately.  $\square$

The following proposition is needed for the technical heart of our construction to prove the main result, contained in Section 5. It says in a very precise way that concepts of coarse maps and coarse homotopies between cones of finite simplicial complexes can be reduced to proper Lipschitz maps and coarse Lipschitz homotopies.

**Proposition 3.6.** *Let  $X, Y \subset \mathbb{R}^N$  be finite geometric simplicial complexes with subcomplexes  $X_0 \subset X, Y_0 \subset Y$  and with cones  $c(X), c(Y)$  respectively. Then every coarse map of pairs  $\phi : (c(Y), c(Y_0)) \rightarrow (c(X), c(X_0))$  is close (i.e. coarsely equivalent) to a proper Lipschitz map of pairs  $f : (c_L(Y), c_L(Y_0)) \rightarrow (c(X), c(X_0))$  where we restrict the domain to the coarsely equivalent  $c_L(X)$  for a suitable  $L > 0$ . The map  $f$  can be chosen to be simplicial for triangulations of the cones as in Definition 3.4.*

*Moreover, if the map  $\phi$  is already Lipschitz when restricted to  $c(Y_1)$  for a further subcomplex  $Y_1$  of  $Y$ , then the maps  $\phi$  and the  $f$  constructed in the process and restricted to  $c_L(Y_1)$  are Lipschitz homotopic as maps of pairs  $(c_L(Y_1), c_L(Y_1 \cap Y_0)) \rightarrow (c(X), c(X_0))$ . Even better, the above map  $f$  can be replaced by a coarsely equivalent Lipschitz map  $\tilde{f}$ , which coincides with  $\phi$  on  $c(Y_1)$ .*

*Finally, suppose the maps  $\phi, \psi : (c(Y), c(Y_0)) \rightarrow (c(X), c(X_0))$  are equivalent by a coarse homotopy that is proper Lipschitz when restricted to  $c(Y_1)$ . Let  $f$  and  $g$  be proper Lipschitz maps constructed above, coarsely equivalent to  $\phi$  or  $\psi$ , respectively, with  $f|_{c(Y_1)} = \phi|_{c(Y_1)}$  and  $g|_{c(Y_1)} = \psi|_{c(Y_1)}$ . Then there is a proper Lipschitz homotopy of pairs between  $f$  and  $g$  which coincides with the original homotopy on  $c(Y_1)$ .*

*Proof.* The strategy is to replace our map by a simplicial map for suitable and regular enough triangulations. The Lipschitz property will then follow from Lemma 1.11.

We choose the triangulation of  $c(X)$  as in Definition 3.4.

To obtain the desired simplicial map we follow the method of proof of the classical simplicial approximation theorem [10, Sect. 3.4] and [12].

For this, choose  $R > 0$  such that  $\text{diam}(\phi(\text{Sk}(x))) \leq R$  for every vertex  $x$  in  $c(Y)$ , where  $\text{Sk}(x)$  is the closed star of the vertex  $x$ . This is possible due to Lemma 3.5 (which gives a uniform upper bound on the diameters of all such stars) and the fact that  $\phi$  is a coarse map between metric spaces.

Next, consider the triangulation of Definition 3.4 on  $c_1(X)$ . By Lemma 3.5, the simplices of this triangulation are obtained from finitely many congruence types, scaled by elements in  $[a, b]$  for a compact subset of  $(0, \infty)$ . This implies that there is  $r > 0$  such that every  $r$ -ball is contained in the open star of a vertex (the covering by open stars of simplices has Lebesgue number  $\geq r$ ). Dually, by just scaling we obtain: there is  $L' > 0$  such that for the  $C$ -scaled triangulation of Definition 3.4 on  $c_{L'}(X)$  every  $R$ -ball is contained in the open star of a simplex.

Use now the properness of the map  $\phi$  to choose a natural number  $L > 0$  such that  $\phi(c_L(Y)) \subset c_{L'}(X)$ . Now, the standard conditions for the proof of the simplicial approximation theorem of [10, Sect. 3.4] are satisfied: given any vertex  $v$  of our triangulation of  $c_L(Y)$ , the images of the collection of all vertices connected to  $v$  by an edge is contained in an open star of a vertex  $w_v$  of the chosen triangulation of  $c_{L'}(X)$ . Consequently, we can now define a simplicial map  $f : c_L(Y) \rightarrow c_{L'}(X)$  defined by sending each vertex  $v$  to an appropriate vertex  $w_v$ . Automatically, as in [10, Cor. 3.4.4] the subcomplex  $Y_0$  will be mapped to the subcomplex  $X_0$  by this construction. Moreover,  $f$  and  $\phi$  have distance at most  $D$ , where  $D$  is an upper bound on the diameters of the simplices of our scaled triangulation of  $c(X)$ . Upto scaling, there are only finitely many isometry types of simplices. By Lemma 1.11 and Lemma 1.9, the map  $f$  is globally Lipschitz.

The standard construction of simplicial approximation provides a well defined “straight line” homotopy of pairs  $H : c_L(Y) \times [0, 1] \rightarrow c_{L'}(X)$  between  $\phi$  and  $f$ , with  $H(x, t) = (1 - t)\phi(x) + tf(x)$  where our construction makes sure that this is indeed making sense and given by a path inside a simplex of  $c_{L'}(X)$ . In particular, throughout this homotopy  $X_0$  is mapped to  $Y_0$ . Of course, if  $\phi$  is not continuous also  $H$  is not. However, if  $\phi$  is Lipschitz then the triangle inequality implies that  $H$  is also Lipschitz, with Lipschitz constant determined by the maximum  $K$  of the Lipschitz constants of  $f$  and  $\phi$  and by  $D$ . Concretely, if  $x, y$  lie in the same simplex of  $Y$  (which suffices to consider) and  $0 \leq s, t \leq 1$  then

$$\begin{aligned} |H(x, t) - H(y, s)| &= |(1 - t)\phi(x) + tf(x) - (1 - s)\phi(y) - sf(y)| \\ &\leq (1 - t)|\phi(x) - \phi(y)| + t|f(x) - f(y)| + |t - s| \cdot |\phi(y) - f(y)| \\ &\leq Kd(x, y) + |t - s| \cdot D \end{aligned}$$

which implies the claim. The same argument applies when we restrict everything to a subcomplex  $c_L(Y_1)$  on which  $\phi$  is Lipschitz.

We now use this Lipschitz homotopy  $H$  to change  $f$  to coincide with  $\phi$  on the subcomplex  $c_L(Y_1)$ . For this, we use a geometric topological implementation of the fact that the inclusion of  $c_L(Y_1)$  into  $c_L(Y)$  is a cofibration. More specifically, consider the space  $Z := c_L(Y_1) \times [0, 1] \cup_{c_L(Y_1)} c_L(Y)$  where we use the embedding  $c_L(Y_1) \rightarrow c_L(Y_1) \times [0, 1]; y \mapsto (y, 1)$  to glue.

We now construct a map  $R : c_L(Y) \rightarrow Z$  which maps  $c_L(Y_1)$  to  $c_L(Y_1) \times \{0\}$  in the obvious way and which is the identity on all simplices of  $c_L(Y)$  not touching  $c_L(Y_1)$ .

Such a map is constructed by “stretching out” a simplex  $c\sigma$  of  $c_L(Y)$  with a face  $\tau := \sigma \cap c_L(Y_1)$  not equal to  $\sigma$  to  $\tau \times [0, 1] \cup_{\tau \times \{1\}} \sigma$ , i.e. by choosing (compatible with face restrictions) suitable maps

$$R_\sigma : \sigma \rightarrow \tau \times [0, 1] \cup_{\tau \times \{1\}} \sigma$$

sending the face  $\tau$  identically to  $\tau \times \{0\}$  and the complementary face  $\tau^\perp$  (spanned by all simplices of  $\sigma \setminus \tau$ ) identically to  $\tau^\perp$ . It is an elementary observation that this can be done, and that this can be done such that restricted to each simplex the map is Lipschitz (albeit not affine linear). But now, because up to scaling we have only finitely many configurations due to Lemma 3.5, it suffices to use finitely maps  $R_\sigma$  up to scaling to construct the map  $R$ . This implies that  $R$  is globally Lipschitz.

The map  $\tilde{f}$  is now defined as the composition of  $R$  with the union of  $f$  on  $c_L(Y) \subset Z$  and the homotopy  $H$  on  $c_L(Y_1) \times [0, 1] \subset Z$  which as a composition of unions of Lipschitz maps is still Lipschitz and also clearly coarsely equivalent to  $f$ .

The statement about homotopies follows from the general (relative) statement applied to the coarse homotopy which by Lemma 2.6 we can assume to be defined on  $I_p c(Y)$  for  $p : c(Y) \rightarrow \mathbb{R}_+$ ;  $(hy, h) \mapsto y$  the standard height projection. But, then  $I_p c(Y) = c(Y \times [0, 1])$ , so that indeed we are in the situation already discussed.  $\square$

**Definition 3.7.** For  $X, Y \subset \mathbb{R}^n$  we form the cones  $c(X) \subset X \times [0, \infty)$  and  $c(Y) \subset Y \times [0, \infty)$ . A map  $f : X \rightarrow Y$  induces a radial map  $c(f) : c(X) \rightarrow c(Y)$ ;  $(hx, h) \mapsto (hf(x), h)$ .

*Remark 3.8.* Similarly, for cones with spherical base one defines the radial map  $f_C$  induced by a map  $f$  between the bases of the cones. Unfortunately, the maps  $f_C$  and  $c(f)$  are not in general coarse. They are, however, if the initial map is Lipschitz.

**Proposition 3.9.** *Let  $X$  and  $Y$  be bounded subsets of Hilbert spaces (with diameter bounded by  $D$ ). Let  $f : X \rightarrow Y$  be a proper Lipschitz map. Then the induced map  $c(f)$  is a proper Lipschitz map. In particular, the map  $c(f)$  is coarse.*

*Proof.* If  $B \subseteq c(Y)$  is compact, then the inverse image  $c(f)^{-1}[B] \subseteq c(X)$  is also compact by the properness of  $f$ .

Let  $L$  be the Lipschitz constant of  $f$ . Let  $R > 0$ ,  $s, t \in \mathbb{R}_+$  and  $x, y \in X$ , and suppose that  $\|(sx, s) - (ty, t)\| < R$ . Then it follows that  $|s - t| < R$  and by the triangle inequality

$$s\|x - y\| \leq \|sx - ty\| + |s - t|\|y\| \leq R + RD.$$

Now

$$\begin{aligned} \|c(f)(sx, x) - c(f)(ty, t)\| &= \|(sf(x) - tf(y), s - t)\| \\ &\leq 2(s\|f(x) - f(y)\| + |s - t|\|f(y)\| + |s - t|) \\ &\leq 2(Ls\|x - y\| + (D + 1)|s - t|). \\ &\leq 2(L + 1)(D + 1) \cdot R \end{aligned}$$

so the map  $c(f)$  is Lipschitz with Lipschitz constant  $\leq 2(L + 1)(D + 1)$ .  $\square$

Furthermore, the condition that the map  $f : X \rightarrow Y$  is Lipschitz is not a severe one up to homotopy, as the next result shows.

**Lemma 3.10.** *Let  $(X, X_0)$  and  $(Y, Y_0)$  be pairs of finite simplicial complexes, equipped with simplicial metrics. Let  $f : (X, X_0) \rightarrow (Y, Y_0)$  be a continuous map. Then  $f$  is homotopic to a Lipschitz map.*

*Further, if  $f_0, f_1 : X \rightarrow Y$  are homotopic maps, and  $g_0, g_1 : X \rightarrow Y$  are Lipschitz maps homotopic to  $f_0$  and  $f_1$  respectively, then we have a Lipschitz map  $H : X \times [0, 1] \rightarrow Y$  such that  $H(-, 0) = g_0$  and  $H(-, 1) = g_1$ .*

*If the map  $f$  or the homotopy  $H$  is already simplicial (and hence Lipschitz) when restricted to a subcomplex  $A \subset X$  then we can choose the Lipschitz map and Lipschitz homotopy relative to  $A$  (i.e. restricted to  $A$  all maps and homotopies coincides with the given ones).*

*Proof.* By the relative simplicial approximation theorem [12], after a suitable subdivision of the simplicial structure of  $X$  the map  $f$  has a simplicial approximation  $g$ , which is homotopic to  $f$ , kept unchanged on the subcomplex  $A$  where it already was simplicial and still maps  $X_0$  to  $Y_0$ .

Restricted to each simplex with any chosen simplicial metric, the map  $g$  is Lipschitz, being affine linear between this simplex and a simplex of  $Y$ . The associated path metric is geodesic (by compactness of  $X$ ). Because there are only finitely many simplices involved, the map  $g$  is globally Lipschitz by Lemma 1.9.

Any two metrics we obtain by subdivision and the compatible choice of a simplicial metric on each simplex are bilipschitz equivalent.

The homotopy statement follows in the same way applying the relative simplicial approximation theorem to  $X \times [0, 1]$ .  $\square$

## 4 | COARSE HOMOTOPY GROUPS

In order to define coarse homotopy groups, we need a coarse analogue of a basepoint in topology.

**Definition 4.1.** Let  $X$  be a coarse space. An  $\mathbb{R}_+$ -basepoint for  $X$  is a coarse map  $i_0 : \mathbb{R}_+ \rightarrow X$ .

If  $Y$  is another coarse space with  $\mathbb{R}_+$ -basepoint  $j_0$ , then a coarse map  $f : X \rightarrow Y$  is termed  $\mathbb{R}_+$ -pointed if  $j_0 = f \circ i_0$ .

The above definition immediately suggests the following.

**Definition 4.2.** Let  $X$  be a coarse space. Then we define the 0-th coarse homotopy set,  $\pi_0^{\text{coarse}}(X)$ , to be the set of coarse homotopy classes of maps from  $\mathbb{R}_+$  to  $X$ .

For convenience, we write  $[i] \in \pi_0^{\text{coarse}}(X)$  to denote the coarse  $\mathbb{R}_+$ -homotopy class of a map  $i : \mathbb{R}_+ \rightarrow X$ .

**Example 4.3.** Let  $B$  be a bounded coarse space. Then there are no coarse maps  $\mathbb{R}_+ \rightarrow B$ , and so  $\pi_0^{\text{coarse}}(B) = \emptyset$ .

*Remark 4.4.* Computing this coarse homotopy set is more difficult than it might seem at first glance. The idea is of course that one counts the ‘‘components at infinity’’.

In particular, one would expect  $\pi_0^{\text{coarse}}(\mathbb{R}^n)$  to have two elements if  $n = 1$  and exactly one element if  $n \geq 2$ .

However, we can define many coarse maps  $\mathbb{R}_+ \rightarrow \mathbb{R}^2$ , for example an embedding as a ray (a radial map, and it is easy to see that these are all coarsely homotopic to each other), but also an embedding which slowly spirals around the origin and (to be a proper map) out to infinity. It is far from obvious how to homotop such a map to the radial inclusion.

It is a consequence of the main result, Theorem 5.6, of this paper that the above statements are true.

A coarse pair is a pair of coarse space  $(X, A)$  along with a coarse map  $k_A : A \rightarrow X$ .

**Definition 4.5.** Let  $(X, A)$  and  $(Y, B)$  be coarse pairs. A coarse map  $f : (X, A) \rightarrow (Y, B)$  is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow k_A & & \uparrow k_B \\ A & \xrightarrow{f} & B \end{array}$$

**Definition 4.6.** Let  $f, g : (X, A) \rightarrow (Y, B)$  be coarse maps such that  $f|_A = g|_A$ . A relative coarse homotopy between  $f$  and  $g$  is a coarse homotopy  $H : I_p X \rightarrow Y$  between the maps  $f, g : X \rightarrow Y$  such that  $H(a, t) = f(a)$  for all  $a \in A$  and  $t \leq p(a) + 1$ .

We call a coarse map of pairs  $f : (X, A) \rightarrow (Y, B)$  a relative coarse homotopy equivalence if there is a coarse map of pairs  $g : (Y, B) \rightarrow (X, A)$  such that the composites  $g \circ f$  and  $f \circ g$  are relatively coarsely homotopic to the identities  $1_X$  and  $1_Y$ , respectively.

The following definition is directly inspired by the classical definition of homotopy groups.

**Definition 4.7.** Let  $X$  be a coarse space with  $\mathbb{R}_+$ -basepoint  $i_0 : \mathbb{R}_+ \rightarrow X$ . For  $n \geq 1$  define the  $n$ -th coarse homotopy group  $\pi_n^{\text{coarse}}(X, i_0)$  to be the set of relative  $\mathbb{R}_+$ -pointed coarse homotopy classes of maps

$$F : (c([0, 1]^n), c(\partial[0, 1]^n)) \rightarrow (X, i_0[\mathbb{R}_+])$$

such that  $F|_{c(\partial[0, 1]^n)} = i_0 \circ p$ .

Here  $p : c([0, 1]^n) \rightarrow \mathbb{R}_+$ ;  $(x, h) \mapsto h$  just denotes the height variable of the cone. A homotopy is  $\mathbb{R}_+$ -pointed if it preserves the  $\mathbb{R}_+$ -basepoint throughout.

More generally, for a coarse pair  $k_A : A \rightarrow X$  with  $\mathbb{R}_+$ -basepoint  $i_0 : \mathbb{R}_+ \rightarrow A$  we define the relative  $n$ -th coarse homotopy “group”  $\pi_n^{\text{coarse}}(X, A, i_0)$  to be the set of relative  $\mathbb{R}_+$ -pointed coarse homotopy classes of maps

$$F : (c([0, 1]^n), c(\partial[0, 1]^n), c(\partial_+[0, 1]^n)) \rightarrow (X, A, i_0[\mathbb{R}_+])$$

such that  $F|_{c(\partial_+[0, 1]^n)} = i_0 \circ p$ .

Here  $\partial_+[0, 1]^n := \{(x_1, \dots, x_n) \in \partial[0, 1]^n \mid x_n > 0\}$ .

The following result is routine to check; the computations almost identically resemble those needed to check the corresponding in topology. For details, see for example [10, Sect. 7.2]. The main points to care about are the following:

- The piecewise defined coarse maps indeed are globally coarse maps, this follows immediately from Proposition 1.10.
- The usual homotopies can be used to define coarse homotopies on appropriate cylinders. This again works nicely and automatically, with cylinder  $I_p c([0, 1]^n)$  where  $p : c([0, 1]^n) \rightarrow \mathbb{R}_+$  is again the height projection.

**Proposition 4.8.** *Let  $n \geq 1$ . Let  $F, G : (c([0, 1]^n), c(\partial[0, 1]^n)) \rightarrow (X, i_0[\mathbb{R}_+])$  be such that  $F|_{c(\partial[0, 1]^n)} = G|_{c(\partial[0, 1]^n)} = i_0 \circ p$ . Define their product*

$$F * G : (c([0, 1]^n), c(\partial[0, 1]^n)) \rightarrow (X, i_0[\mathbb{R}_+])$$

by the formula

$$F * G(x_1, x_2, \dots, x_n, h) = \begin{cases} F(2x_1, x_2, \dots, x_n, h), & x_1 \leq h/2, \\ G(2x_1 - h, x_2, \dots, x_n, h), & h/2 \leq x_1 \leq h. \end{cases}$$

Then the operation  $[F] \cdot [G] = [F * G]$  turns the set  $\pi_n^{\text{coarse}}(X, i_0)$  into a group. Further,  $\pi_n^{\text{coarse}}(X, i_0)$  is abelian if  $n \geq 2$ . The unit is represented by the map  $i_0 \circ p : c([0, 1]^n) \rightarrow X$ .

For  $n \geq 2$ , the same formula makes sense for the relative homotopy groups and defines a group structure on them, abelian if  $n \geq 3$ .

We call the groups  $\pi_n^{\text{coarse}}(X, A, i_0)$  the *coarse homotopy groups* of  $(X, A)$ . The following result is also straightforward to prove, and resembles its classical analogue.

**Proposition 4.9.** *Let  $(X, A)$  and  $(Y, B)$  be  $\mathbb{R}_+$ -pointed coarse pairs and  $f : (X, A) \rightarrow (Y, B)$  be an  $\mathbb{R}_+$ -pointed coarse map. Then there is a functorially induced homomorphism*

$$f_* : \pi_n^{\text{coarse}}(X, A, i_0) \rightarrow \pi_n^{\text{coarse}}(Y, B, j_0)$$

defined by the formula  $f_*([F]) = [f \circ F]$ .

Further, if  $\mathbb{R}_+$ -pointed coarse maps  $f, g : (X, A, i_0) \rightarrow (Y, B, j_0)$  are  $\mathbb{R}_+$ -pointed relatively coarsely homotopic, then the homomorphisms  $f_*$  and  $g_*$  are equal.

**Proposition 4.10.** *If  $(X, A, i_0)$  is a  $\mathbb{R}_+$ -pointed coarse pair with map  $k : A \rightarrow X$ , the analogue of the usual construction in topology defines a long exact sequence of coarse homotopy groups or pointed sets*

$$\begin{aligned} \rightarrow \pi_2^{\text{coarse}}(A, i_0) \xrightarrow{k_*} \pi_2^{\text{coarse}}(X, i_0) \rightarrow \pi_2^{\text{coarse}}(X, A, i_0) \xrightarrow{\partial} \pi_1^{\text{coarse}}(A, i_0) \\ \xrightarrow{k_*} \pi_1^{\text{coarse}}(X, i_0) \rightarrow \pi_1^{\text{coarse}}(X, A, i_0) \rightarrow \pi_0^{\text{coarse}}(A, i_0) \rightarrow \pi_0^{\text{coarse}}(X, i_0) \end{aligned}$$

Here, the boundary map is (as usual) obtained by restricting to the subset of  $c[0, 1]^n$  with  $x_n = 0$ .

*Proof.* The proof just follows the standard pattern of the corresponding statement for ordinary homotopy groups, compare [10, Sect. 7.2]. There is one subtlety though: One has to convert certain homotopies  $H : I_p c[0, 1]^n \rightarrow X$  to maps  $\bar{H} : c[0, 1]^{n+1} \rightarrow X$ .

Usually, this is done by interpreting the homotopy parameter  $t$  of  $(hx, h, t)$  as the extra variable  $hx_{n+1}$ . This is permitted here, as well, as we can normalized the domain of the homotopies defined on  $c[0, 1]^n$  to be defined on  $I_p c([0, 1]^n)$  due to Lemma 2.6 where  $p(hx, h) = h$  is the standard height projection.

We leave the details to the reader.  $\square$

*Remark 4.11.* In classical topology, probably the most important application of the long exact sequence of homotopy groups of a pair is to a fibration  $F \rightarrow E \rightarrow B$ , where one proceeds to identify the (in general mysterious) relative homotopy groups of  $(E, F)$  with those of the base  $B$ .

A question for the future is whether there is a version of a coarse fibration which is as frequent as the fibrations in classical topology, and for which a corresponding statement holds for coarse homotopy groups.

## 5 | HOMOTOPY GROUPS OF CONES

For  $X \subset \mathbb{R}^n$  with basepoint  $x_0 \in X$  we have the corresponding  $\mathbb{R}_+$ -basepoint  $i_0 : \mathbb{R}_+ \rightarrow c(X)$ , the ray through  $x_0$ . The main result of this section is that for a wide class of spaces,  $X$ , the coarse homotopy group,  $\pi_n^{\text{coarse}}(c(X), i_0)$  is isomorphic to the ordinary homotopy group  $\pi_n(X, x_0)$ . In particular, we have isomorphisms

$$\pi_n^{\text{coarse}}(\mathbb{R}^{k+1}, i_0) \cong \pi_n^{\text{coarse}}(c(S^k), i_0) \cong \pi_n(S^k, x_0).$$

At first glance this result seems expected. At second glance, however, one realizes that this is not such a triviality, as already discussed in Remark 4.4 concerning  $\pi_0^{\text{coarse}}(\mathbb{R}^n)$  which of course persists to higher degrees.

If  $X$  is a finite simplicial complex there is a canonical bilipschitz class of metric cones  $c(X)$  coming from a PL-embedding of  $X$  into Euclidean space, as discussed in Section 3. If  $x_0 \in X$  is a basepoint, then the cone  $c(X)$  has an induced  $\mathbb{R}_+$ -basepoint  $i_0 : \mathbb{R}_+ \rightarrow c(X)$  defined by the formula  $i_0(t) = (tx_0, t)$ .

**Definition 5.1.** We define the homomorphism

$$\Psi : \pi_n(X, x_0) \rightarrow \pi_n^{\text{coarse}}(c(X), i_0)$$

by setting  $\Psi([f]) = [c(f)]$  where  $f : ([0, 1]^n, \partial[0, 1]^n) \rightarrow (X, x_0)$  is a Lipschitz map. The equivalence class on the left is that of relative Lipschitz homotopy, and that on the right is relative coarse  $\mathbb{R}_+$ -homotopy.

Note that it follows from Lemma 3.10 that the set of continuous homotopy classes of continuous maps here is the same as the set of Lipschitz homotopy classes of Lipschitz maps, so that the map  $\Psi$  is well defined. By the construction of the group structures, it is a group homomorphism.

The main result in this article is that the map  $\Psi$  is an isomorphism. We prove this by constructing an inverse.

The following result is the technical heart of our construction. It says that we can homotop to radial Lipschitz maps. In the statement of the result and the proof, we will write points in  $c_L(X)$  as pairs  $(hx, h)$  with  $x \in X$ ,  $h \geq L$ .

**Proposition 5.2.** *Let  $X$  and  $Y$  be finite simplicial complexes PL-embedded into  $\mathbb{R}^n$  with subcomplexes  $X_0 \subset X$  and  $Y_0 \subset Y$ , respectively. Let  $f : (c(X), c(X_0)) \rightarrow (c(Y), c(Y_0))$  be a coarse map. Then, if we restrict the map  $f$  to  $c_L(X)$  for some suitable  $L > 0$ , it is coarsely homotopic, as a map of pairs, to a radial proper Lipschitz map  $g$ .*

*Suppose  $X_1 \subset X$  is a subcomplex such that the restriction of  $f$  to  $c(X_1)$  is already a radial Lipschitz map. Then we can chose  $g$  such that  $g|_{c(X_1)} = f|_{c(X_1)}$ , and the coarse homotopy between  $f$  and  $g$  can be chosen such that its restriction to  $c(X_1)$  is the concatenation of a homotopy of the form  $(hx, h, t) \mapsto \rho(h, t)f(x, L)$  with its inverse, where  $0 \leq t \leq h$ .*

*Proof.* By Proposition 3.6, we can assume that  $f$  is a proper Lipschitz map on  $c_L(X)$  with values in  $c_{L'}(Y)$  for some  $L, L' \geq 1$ . To simplify notation, we assume  $L = L' = 1$ , the general case is just a technical modification.

We construct our homotopy in several steps.



First, define

$$g(hx, h) := f(\sqrt{hx}, \sqrt{h}).$$

We define a proper Lipschitz homotopy  $F$  between  $g$  and  $f$  by the formula

$$F(hx, h, t) := f((h-t)x, h-t); \quad 0 \leq t \leq h - \sqrt{h}.$$

Secondly, define

$$u(hx, h) := \sqrt{h}f(x, 1).$$

We define a proper Lipschitz homotopy between  $u$  and  $g$  by

$$G(hx, h, t) := \left( \frac{\sqrt{h}}{\frac{t}{\sqrt{h}} + 1} \right) f\left( \left( \frac{t}{\sqrt{h}} + 1 \right) x, \frac{t}{\sqrt{h}} + 1 \right); \quad 0 \leq t \leq h - \sqrt{h}.$$

Finally, let

$$v(hx, h) := hf(x, 1).$$

Then  $v$  is a radial proper Lipschitz map on  $c_1(X)$ . We define the proper Lipschitz homotopy  $H$  between  $u$  and  $v$  by

$$H(hx, h, t) := (t + \sqrt{h})f(x, 1); \quad 0 \leq t \leq h - \sqrt{h}.$$

We have given explicit formulas for the maps and the homotopies. Substituting  $t = 0$  or  $t = h - \sqrt{h}$  into the homotopies, it is immediate to see that they are homotopies between the maps as claimed. We have to justify the following facts:

1. The maps are proper.
2. The maps are globally Lipschitz.
3. The homotopies are indeed coarse homotopies, i.e. the domains are permitted.
4. All maps send  $c(X_0)$  to  $c(Y_0)$ .
5. The restriction of the maps and homotopies to  $c(X_1)$ , when the original map  $f$  is radial, have the required form.

The domain of the homotopies is contained in  $I_{p', c_1}(X)$  with  $p' : c_1(X) \rightarrow \mathbb{R}_+ : (hx, h) \mapsto h - \sqrt{h}$  which is a proper Lipschitz map and therefore a coarse map.

By construction, all maps constructed send  $c(X_0)$  to  $c(Y_0)$ . If  $f$  is radial, i.e.  $f(hx, h) = hf(x, 1)$  then the first homotopy  $F$  reduces to  $F(hx, ht, t) = (h-t)f(x, 1)$  (for  $0 \leq t \leq h - \sqrt{h}$ ), the second homotopy  $G$  is constant, and the third homotopy  $H$  becomes  $H(hx, h, t) = (t + \sqrt{h})f(x, 1)$  for  $0 \leq t \leq h - \sqrt{h}$  which indeed is precisely the inverse of  $F$ .

It remains to check that all maps defined are proper and Lipschitz, using that  $f$  itself is proper and Lipschitz.

The homotopy  $F$  is the composition of  $f$  and a map  $\alpha : I_{p', c_1}(X) \rightarrow c_1(X)$  for which it is elementary to check that it is proper and Lipschitz.

To check that  $G$  and  $H$  are globally Lipschitz is slightly more tedious, but again an elementary exercise, using Lemma 5.3 and Lemma 5.4. Their properness follows from the fact that the norm of the values tends to infinity as  $h \rightarrow \infty$ .

Let us give some of the details of the proof of the Lipschitz property of  $G$ , the most tedious to write down. Consider  $h(x, 1) \in c_1(X)$  and  $t < s \leq h - \sqrt{h}$ . Then

$$\begin{aligned}
|G(hx, h, t) - G(hx, h, s)| &\leq \left| \frac{\sqrt{h}}{\frac{t}{\sqrt{h}} + 1} - \frac{\sqrt{h}}{\frac{s}{\sqrt{h}} + 1} \right| \left| f\left(\left(\frac{t}{\sqrt{h}} + 1\right)(x, 1)\right) \right. \\
&\quad \left. + \frac{\sqrt{h}}{\frac{s}{\sqrt{h}} + 1} \left| f\left(\left(\frac{t}{\sqrt{h}} + 1\right)(x, 1)\right) - f\left(\left(\frac{s}{\sqrt{h}} + 1\right)(x, 1)\right) \right| \right| \\
&\leq h \left| \frac{1}{t + \sqrt{h}} - \frac{1}{s + \sqrt{h}} \right| \left( L \left( \frac{t}{\sqrt{h}} + 1 \right) L \right) + \frac{h}{s + \sqrt{h}} L |t - s| \frac{L}{\sqrt{h}} \\
&\leq h \frac{1}{(t + \sqrt{h})^2} |t - s| \left( \frac{L}{\sqrt{h}} (t + \sqrt{h}) L \right) + \frac{\sqrt{h}}{s + \sqrt{h}} L^2 |t - s| \\
&\leq 2L^2 |t - s| = 2L^2 |(hx, h, t) - (hx, h, s)|.
\end{aligned}$$

The first inequality is just the triangle inequality. For the second, we use the Lipschitz property of  $f$  (with Lipschitz constant  $L$ ) which implies in particular also that  $|f(x)| \leq L|x|$  for all  $|x| \geq 1$  by comparing to  $f(0)$  and making  $L$  bigger depending on  $f(0)$ , if necessary. We also use that by the compactness of  $X$  we can choose  $L$  such that  $|(x, 1)| \leq L$  for all  $x \in X$ .

For the third inequality we use that the derivative  $y \mapsto -(y + \sqrt{h})^{-2}$  of  $y \mapsto (y + \sqrt{h})^{-1}$  is monotonically increasing in absolute value and use the mean value theorem.

For the last inequality, we just use that  $s, t \geq 0$ .

By Lemma 5.3, the Lipschitz property of  $G$  follows now if we establish a similar uniform inequality for  $G(hx, h, t) - G(ry, r, t)$  for  $x, y \in X$ ,  $1 \leq h < r$ , and  $t \leq h - \sqrt{h}$  which can be obtain by similar elementary computations, using also Lemma 5.4. Details are left to the reader.  $\square$

We used the following Lipschitz criterion for coarse homotopies.

**Lemma 5.3.** *Let  $X, Y$  be metric space, let  $p_0 : X \rightarrow [0, \infty); x \mapsto d(x, x_0)$  be a basepoint projection for  $x_0 \in X$ . Let  $H : I_{p_0}X \rightarrow Y$  be a map.*

*If there is  $C > 0$  such that for each  $t_0 \in [0, \infty)$  and each  $z \in X$  the restrictions to the  $t_0$ -time slice  $S_{t_0} := X \times \{t_0\} \cap I_{p_0}X$  and the  $z$ -slice*

$$H_{S_{t_0}} : S_{t_0} \rightarrow Y; \quad H_{\{z\} \times [0, \infty) \cap I_{p_0}X} : \{z\} \times [0, \infty) \cap I_{p_0}X \rightarrow Y$$

*are  $C$ -Lipschitz, then  $H$  is globally  $2C$ -Lipschitz.*

*Proof.* This uses the fact that there are enough points in  $I_{p_0}X$  to interpolate; specifically, let  $(x, t)$  and  $(z, s) \in I_{p_0}X$  with  $p_0(x) \geq p_0(z)$ . By definition of  $I_{p_0}X$  then  $s \leq p_0(z) \leq p_0(x)$  and therefore also  $(x, s) \in I_{p_0}X$ . Consequently, by the triangle inequality,

$$\begin{aligned}
d(f(x, t), f(y, s)) &\leq d(f(x, t), f(x, s)) + d(f(x, s), f(y, s)) \\
&\leq Cd((x, t), (x, s)) + Cd((x, s), (y, s)) = C|t - s| + Cd(x, y) \\
&\leq 2Cd((x, t), (y, s)).
\end{aligned}$$

$\square$

**Lemma 5.4.** *Assume that  $X \subset \mathbb{R}^N$  is bounded, i.e. there is  $C > 0$  such that  $|x| \leq C$  for all  $x \in X$ . For  $(hx, h), (ry, r) \in c(X) \subset \mathbb{R}^N \times [0, \infty)$  and  $t \leq \min\{h, r\}$  we then have*

$$d((tx, t), (ty, t)) \leq (1 + (C + 1))d((hx, y), (ry, r)).$$

*Proof.* Observe:

$$\begin{aligned} d((tx, t), (ty, t)) &= td(x, y) \leq d((hx, h), (hy, h)) \\ &\leq d((hx, h), (ry, r)) + d((ry, r), (hy, h)) \\ &= d((hx, h), (ry, r)) + |r - h| \cdot |y, t| \\ &\leq (1 + (C + 1))d((hx, y), (ry, r)). \end{aligned}$$

□

Proposition 5.2 is not quite good enough for our purposes because the homotopy constructed there would, for example, not preserve an  $\mathbb{R}_+$ -basepoint. However, we have enough control such that we can perform a “padding” construction in our specific situation, where the domain is  $c([0, 1]^n)$  and where the map is radial on one of the faces.

**Lemma 5.5.** *Let  $f : (c([0, 1]^n), c(\partial[0, 1]^n)) \rightarrow (c(Y), c(Y_0))$  be a coarse map of coarse pairs. Set  $D := [0, 1]^{n-1} \times \{1\}$  and assume that  $f|_{c(D)} = c(u)$  is radial for a PL-map  $u$ .*

*Then we can find a coarse homotopy of pairs from  $f$  to a radial map such that the restriction to  $c(D)$  is equal to  $f|_{c(D)}$  throughout the coarse homotopy.*

*Proof.* Let

$$\begin{aligned} i &: [0, 1]^{n-1} \times [0, 1/2] \rightarrow [0, 1]^n; (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{n-1}, 2x_n) \quad \text{and} \\ p &: [0, 1]^{n-1} \times [1/2, 1] \rightarrow D; (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, 1). \end{aligned}$$

Define a map  $\bar{f} : c([0, 1]^n) \rightarrow Y$  by  $\bar{f}|_{c([0, 1]^{n-1} \times [0, 1/2])} := f \circ c(i)$  and  $\bar{f}|_{c([0, 1]^{n-1} \times [1/2, 1])} := f \circ c(p)$ , i.e. we squeeze  $f$  into the lower half of  $c([0, 1]^n)$  and then extend constantly in the  $x_n$ -coordinate.

There is an obvious coarse homotopy between  $f$  and  $\bar{f}$  whose restriction to  $c(D)$  is  $f|_{c(D)}$  throughout the homotopy.

Now we construct the required coarse homotopy from  $\bar{f}$  to a radial map whose restriction to  $c(D)$  remains constant. For this, we use the homotopy  $H$  provided by Proposition 5.2 on  $I_p c([0, 1]^{n-1} \times [0, 1/2])$ . On the top part of the domain of this homotopy, where the initial map was radial, i.e. of the form  $c(u)$  for a map  $u : [0, 1]^{n-1} \times \{1/2\} \rightarrow Y$ , we know that  $H(h(x, t)) = \rho(h, t) \cdot (u(x), 1)$  with a real valued function  $\rho$  with  $\rho(h, t) = \rho(h, 1 - t)$ . We then simply extend the homotopy to  $I_p c([0, 1]^{n-1} \times [1/2, 1])$  by setting

$$H(h(x_1, \dots, x_n), 1, t) := \begin{cases} \rho(h, t(1 - 2x_n))(u(x_1, \dots, x_{n-1}, 1/2), 1), & 0 \leq t \leq 1/2, \\ \rho(h, (1 - t)(1 - 2x_n))(u(x_1, \dots, x_{n-1}, 1/2), 1), & 1/2 \leq t \leq 1. \end{cases}$$

It is clear that this procedure does the job. □

We now formulate and prove the main result of this paper.

**Theorem 5.6.** *Let  $X$  be a finite simplicial complex with subcomplex  $X_0$  and base vertex  $x_0 \in X_0$ . Choose a PL-embedding into  $\mathbb{R}^n$  and identify  $X$  with its image and let  $i_0 : [0, \infty) \rightarrow c(X); t \mapsto (tx_0, t)$  be associated to  $x_0$ . Then the homomorphism  $\Psi : \pi_n(X, X_0, x_0) \rightarrow \pi_n^{\text{coarse}}(c(X), c(X_0), i_0)$  of Definition 5.1 is an isomorphism.*

*Proof.* We want to construct an inverse  $\Phi$  to  $\Psi$ . For this, let  $f : c([0, 1]^n) \rightarrow c(X)$  be a coarse map representing an element  $[f] \in \pi_n^{\text{coarse}}(c(X), c(X_0), i_0)$ .

Observe that there is a PL-homeomorphism  $[0, 1]^n \rightarrow [0, 1]^n$  mapping  $\partial_+[0, 1]^n$  to the set  $D$  of Lemma 5.5. We can therefore apply Lemma 5.5 and get a coarse homotopy which is constant on  $c(\partial_+([0, 1]^n))$  (meaning it is an  $\mathbb{R}_+$ -pointed coarse homotopy) to a radial map  $c(u)$  for a PL-map  $u : ([0, 1]^n, \partial[0, 1]^n, \partial_+[0, 1]^n) \rightarrow (X, X_0, x_0)$ .

Of course, we want to set  $\Phi([f]) := [u]$ . It is then obvious that  $\Phi \circ \Psi = \text{id}$  and  $\Psi \circ \Phi = \text{id}$ .

But we have to show that the map  $\Phi$  is really well defined.

For this, we could replace  $f$  by  $g$ , coarsely homotopies through a coarse homotopy  $H_1$ . Moreover, we have to chose a coarse homotopy  $H_0$  from a radial map  $c(u)$  to  $f$  and  $H_2$  from  $g$  to a radial map  $c(v)$ . All homotopies are  $\mathbb{R}_+$ -pointed and map the boundary of  $[0, 1]^n$  to  $c(X_0)$ . We can concatenate  $H_0, H_1, H_2$  and reinterpret the domain of the homotopy as  $c([0, 1]^{n+1})$ . Being  $\mathbb{R}_+$ -pointed, this concatenation is radial when restricted to  $c(\partial_+[0, 1]^n \times [0, 1])$ .

Proposition 5.2 almost allows us to replace  $f$  by a coarsely equivalent radial map based on some  $u : [0, 1]^n \rightarrow X$  such that  $[f] = [c(u)] \in \pi_n^{\text{coarse}}(c(X), c(X_0), i_0)$  such that  $[u] \in \pi_n(X, X_0, x_0)$  would be a candidate for  $\Phi([f])$ .

The problem is that the construction of Proposition 5.2 does not preserve the coarse basepoint  $i_0$ . Fortunately, Proposition 5.2 provides enough control on the part of the domain where the map is already radial, in particular in our case on  $c(\partial_+[0, 1]^n)$ . Moreover, it is radial on  $c([0, 1]^n \times \{0, 1\})$  because the beginning and end of the concatenated coarse homotopies are radial.

Now, by Lemma 5.5 there is a coarse homotopy to a radial map, and that new map coincides with the old one where it is already radial. We can reinterpret that map as (the cone of) a homotopy between  $u$  and  $v$  which is pointed. This shows that indeed the map  $\Phi$  is well defined.  $\square$

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