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# EXTERIOR POWERS OF THE ADJOINT REPRESENTATION AND THE WEYL RING OF $E_8$

ANDREA BRINI

ABSTRACT. I derive explicitly all polynomial relations in the character ring of  $E_8$  of the form  $\chi_{\wedge^k \mathfrak{e}_8} - \mathfrak{p}_k(\chi_1, \dots, \chi_8) = 0$ , where  $\wedge^k \mathfrak{e}_8$  is an arbitrary exterior power of the adjoint representation and  $\chi_i$  is the  $i^{\text{th}}$  fundamental character. This has simultaneous implications for the theory of relativistic integrable systems, Seiberg–Witten theory, quantum topology, orbifold Gromov–Witten theory, and the arithmetic of elliptic curves. The solution is obtained by reducing the problem to a (large, but finite) dimensional linear problem, which is amenable to an efficient solution via distributed computation.

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## 1. INTRODUCTION

**1.1. The problem.** Let  $\mathcal{G}$  be a complex, simple, simply-connected Lie group of rank  $r$  and  $V \in \text{Rep}_{\mathbb{C}}(\mathcal{G})$  an element of its representation ring. We may view the latter, upon taking characters  $V \rightarrow \chi_V$ , as the Weyl ring of Ad-invariant regular functions on  $\mathcal{G}$ , or equivalently, as the ring of  $\mathcal{W}(\mathcal{G})$ -invariant regular functions on the Cartan torus, where  $\mathcal{W}(\mathcal{G})$  is the Weyl group of  $\mathcal{G}$ . It is a basic fact in Lie theory that this is a polynomial ring over the integers,  $\text{Rep}_{\mathbb{C}}(\mathcal{G}) \simeq \mathbb{Z}[\chi_1, \dots, \chi_r]$ , where  $\chi_j$ ,  $j = 1, \dots, r$  denotes the character of the  $j^{\text{th}}$  fundamental representation of  $\mathcal{G}$ .

In this paper we will be concerned with a special instance of the following

**Problem 1.1.** *Given a finite-dimensional representation  $V$  of  $\mathcal{G}$ , find polynomials  $\mathfrak{p}_k^V \in \mathbb{Z}[x_1, \dots, x_r]$ ,  $k = 1, \dots, \dim_{\mathbb{C}} V$ , such that*

$$\det_V(g - \mu \mathbf{1}) = \sum_k \mathfrak{p}_k^V(\chi_1(g), \dots, \chi_r(g))(-\mu)^{\dim V - k} \in \mathbb{Z}[\chi_1(g), \dots, \chi_r(g)][\mu]. \quad (1.1)$$

for all  $g \in \mathcal{G}$  and  $k = 1, \dots, \dim_{\mathbb{C}} V$ . Equivalently, given an arbitrary exterior power of  $V$ , determine the corresponding polynomial relations in  $\text{Rep}(\mathcal{G})$  of the form

$$0 = \chi_{\wedge^k V} - \mathfrak{p}_k^V(\chi_1, \dots, \chi_r). \quad (1.2)$$

In other words, Problem 1.1 asks to find explicit expressions for characteristic polynomials in a given representation  $V$  (alternatively, of antisymmetric characters of  $V$ ) in terms of polynomials in the fundamental characters. For example, if  $\mathcal{G} = \text{SL}_N(\mathbb{C})$  and  $V = V_{\omega_1}$  is the defining representation of  $\text{SL}_N(\mathbb{C})$ , we have simply

$$\mathfrak{p}_k^V(\chi_1, \dots, \chi_r) = \chi_k \quad (1.3)$$

since  $V_{\omega_k} = \wedge^k V_{\omega_1}$ .

Let  $\mathfrak{d}_0(V)$  denote the dimension of the zero-weight space of  $V$ . A case of particular importance for applications is when the characteristic polynomial  $\det_V(g - \mu \mathbf{1})$  (respectively:  $\det_V(g - \mu \mathbf{1})(1 - \mu)^{-\mathfrak{d}_0(V)}$ ) in (1.1) is irreducible over  $\text{Rep}(\mathcal{G})$ : this amounts to  $V$  being a minuscule (respectively: quasi-minuscule) irreducible representation. In the quasi-minuscule setting, Problem 1.1 is computationally easy for most Dynkin types and quasi-minuscule representations (see [3, 4]), with one single, egregious exception: this is  $\mathcal{G} = E_8$  and  $V = \mathfrak{e}_8$ , which is of formidable complexity. The purpose of the present paper is to present a solution of this exceptional case, which had previously been announced in [4, Appendix A].

**1.2. Motivation: six places of appearance of Problem 1.1.** As it stands, Problem 1.1 is of purely representation theoretic character. At the same time, my motivation for looking at it is mostly extrinsic in nature, and it is eminently geometrical: there are indeed six different classes of questions in Geometry and Mathematical Physics that are simultaneously answered by giving an explicit solution to Problem 1.1, and in particular to the case when  $(\mathcal{G}, V) = (E_8, \mathfrak{e}_8)$ , as follows.

1.2.1. *Integrable systems, spectral curves, and the relativistic Toda chain.* Let  $\mathcal{T}$  denote the maximal torus of  $\mathcal{G}$ . A central object in the theory of algebraically complete integrable systems is the datum of a rational map

$$\mathcal{L} : C_{\mathfrak{g}} \times \mathcal{P} \rightarrow \mathcal{G} \quad (1.4)$$

from the product of a  $2r$ -complex algebraic symplectic variety  $\mathcal{P}$  and a fixed smooth curve  $C_{\mathfrak{g}}$  with  $h^{1,0}(C_{\mathfrak{g}}) = \mathfrak{g}$  to  $\mathcal{G}$ , called the *Lax map*. Associated to  $\mathcal{L}$  is the family of *spectral curves*  $\mathcal{C}[\mathcal{L}]$

$$\begin{array}{ccc} \Sigma_u \hookrightarrow & \mathcal{C}[\mathcal{L}] & \\ \downarrow & \downarrow & \\ u \hookrightarrow & \mathcal{B} & \end{array} \quad \text{pt} \quad (1.5)$$

where

- $\mathcal{B} = \chi_{\bullet}(\mathcal{T})$  is the image of the torus under the fundamental regular characters  $\chi_i$ ,  $i = 1, \dots, r$ ; the natural co-ordinates  $u$  on  $\mathcal{B}$ ,  $u_i \triangleq \chi_i(g)$  for  $g \in \mathcal{G}$ , give a co-ordinate chart on  $\mathcal{T}$ ;
- for fixed  $u = (u_i = \chi_i(\mathcal{L}))_i$ ,  $\Sigma_u$  is the compact Riemann surface

$$\Sigma_u = \overline{\mathbb{V} \left( \det_V(\mathcal{L} - \mu \mathbf{1}) \right)} \quad (1.6)$$

given by the smooth completion (normalisation of the projective closure) of the algebraic curve in  $\mathbb{A}^1 \times C_{\mathfrak{g}}$  given by the vanishing locus of the characteristic polynomial of  $\mathcal{L}$  at fixed  $u$ .

More explicitly,

$$\det_V(\mathcal{L} - \mu \mathbf{1}) = \sum_{k=0}^{\dim V} (-\mu)^{\dim V - k} \chi_{\wedge^k V} \mathcal{L} \in \mathcal{M}_{C_{\mathfrak{g}}}[\mu; u_1, \dots, u_r] \quad (1.7)$$

where  $\mathcal{M}_{C_{\mathfrak{g}}}$  is the ring of meromorphic functions on  $C_{\mathfrak{g}}$ . A Hamiltonian dynamical system can then be defined on  $\mathcal{P}$  in terms of isospectral flows on  $\mathcal{L}$ ; in particular, any Ad-invariant function of  $\mathcal{L}$  is an integral of motion, and a maximal involutive set of these is given by the fundamental characters  $u_i(\mathcal{L})$ . For fixed  $u$ , the resulting flow is linear on the Picard group of the irreducible components  $\Sigma_u$ , and the dynamics is independent of  $V$  so long as  $V \neq \mathbf{1}$  [8, 19, 24, 25]: so for simplicity, we may assume  $V$  to be minuscule (resp. quasi-minuscule for  $\mathcal{G} = E_8$ ), which entails that  $\Sigma_u$  (resp. the reduced component of  $\Sigma_u$ ) is generically irreducible.

A central example is given by the periodic relativistic Toda chain of type  $\mathcal{G}$ : in this case  $C_{\mathfrak{g}} \simeq \mathbb{P}^1$ , and in an affine co-ordinate  $\lambda$  on the base  $\mathbb{P}^1$  the Lax map satisfies [3, 4]

$$\partial_{\lambda} u_i(\mathcal{L}) = \delta_{i, i_{\text{top}}} \left( 1 - \frac{1}{\lambda^2} \right), \quad (1.8)$$

where  $V_{i_{\text{top}}}$  is the top dimensional fundamental representation. This means that (1.7) becomes

$$\det_V(\mathcal{L} - \mu \mathbf{1}) = \sum_{k=0}^{\dim V} (-\mu)^{\dim V - k} \chi_{\wedge^k V} \mathcal{L} \in \mathbb{Z}[\mu, \lambda^{\pm}; u_1, \dots, u_r] \quad (1.9)$$

where the last step requires expanding  $\chi_{\wedge^k V}$  as a polynomial in  $\chi_i$  from the solution of Problem 1.1, and using (1.8). A complete presentation for the family of spectral curves (1.5), and the complete solution for the dynamics of the underlying integrable model, follows thus from solving Problem 1.1 for the given pair  $(\mathcal{G}, V)$ .

1.2.2. *Gauge theory I : Seiberg–Witten theory.* The Toda systems of the previous section have a central place in the study of supersymmetric quantum field theories [15, 23, 26]. In particular, constructing explicitly the family of spectral curves (1.5) encodes the solution of the low energy effective dynamics for  $\mathcal{N} = 1$  supersymmetric gauge theories with no hypermultiplets on  $\mathbb{R}^{1,3} \times S^1$ : in this setting, the fundamental characters  $u_i$  are the semiclassical, Weyl-invariant gauge parameters and the full effective action up to two derivatives in the supercurvature of the gauge field, including all-order instanton corrections, is recovered via period integrals of  $\log \mu d \log \lambda$  on  $\Sigma_u$ . I refer the reader to [3, 4, 26] for a fuller discussion of the link between the relativistic Toda chain and this class of five-dimensional quantum field theories.

The case  $\mathcal{G} = E_8$  has been outstanding since the solution of the Wilsonian dynamics of the theory was proposed shortly after the celebrated work of Seiberg–Witten [23, 26, 29]; the same considerations for the relativistic Toda chain recast this problem into solving Problem 1.1 for  $(E_8, \mathfrak{e}_8)$ .

1.2.3. *Gauge theory II: Chern–Simons theory.* Let  $\mathfrak{g}$  be a simple complex Lie algebra  $\mathfrak{g}$  of type  $A_n$ ,  $D_n$  or  $E_n$ . Let  $V_{\mathfrak{g}}$  be the defining representation for  $\mathfrak{g} = \mathfrak{sl}(n+1)$ , the vector representation for  $\mathfrak{g} = \mathfrak{so}(2n)$ , and the  $\mathbf{27}_{E_6}$ ,  $\mathbf{56}_{E_7}$  and  $\mathbf{248}_{E_8} = \mathfrak{e}_8$  for  $\mathfrak{g} = \mathfrak{e}_{6,7,8}$  respectively, and denote by  $\Gamma_{\mathfrak{g}} < SU(2)$  the finite order subgroup of  $SU(2)$  of the same Dynkin type of  $\mathfrak{g}$  associated to  $\mathfrak{g}$  by the classical McKay correspondence. Considerations about large  $N$  duality in gauge theory led [3, 4] to propose that the family of curves (1.5) for  $(\mathcal{G}, V) = (\exp(\mathfrak{g}), V_{\mathfrak{g}})$  determines the asymptotics of the Witten–Reshetikhin–Turaev invariant of spherical 3-space forms  $M_{\mathfrak{g}} = S^3/\Gamma_{\mathfrak{g}}$  via the Chekhov–Eynard–Orantin topological recursion [5, 11, 12]. In particular, the case  $\Gamma_{\mathfrak{e}_8} = \mathbb{I}_{120}$  being the binary icosahedral group gives the distinguished case of the Poincaré integral homology sphere  $M_{\mathfrak{e}_8} = \Sigma(2, 3, 5)$ . The all-order asymptotic expansion of the quantum invariants of the Poincaré sphere is fully determined by periods of  $\log \mu d \log \lambda$  and the topological recursion on the spectral curves (1.5) for the  $\mathcal{G}$ -relativistic Toda chain – and therefore, ultimately, by solving Problem 1.1 for  $(\mathcal{G}, V) = (E_8, \mathfrak{e}_8)$ .

1.2.4. *Algebraic enumerative geometry.* Let  $\mathcal{O}_{\mathbb{P}^1}(-1)$  denote the tautological bundle on the complex projective line, and let  $X_{\mathfrak{g}} \triangleq [\mathcal{O}(-1)_{\mathbb{P}^1}^{\oplus 2}/\Gamma_{\mathfrak{g}}]$  be the fibrewise quotient stack of  $\text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \rightarrow \mathbb{P}^1$  by the action of the finite group  $\Gamma_{\mathfrak{g}}$  as in Section 1.2.3. The *Gromov–Witten potential* of  $X_{\mathfrak{g}}$  is a formal generating series of virtual counts of twisted stable curves to the orbifold  $X_{\mathfrak{g}}$ , to all genera, degrees, and insertions of twisted cohomology classes [1]. It was proposed and non-trivially checked in [3, 4] that the full Gromov–Witten potential of  $X_{\mathfrak{g}}$  coincides with the Chern–Simons partition function of  $M_{\mathfrak{g}}$ ; therefore the full curve counting information on  $X_{\mathfrak{g}}$  is encoded into (1.5), and solved by Problem 1.1, as an instance of 1-dimensional mirror symmetry for Calabi–Yau manifolds.

1.2.5. *Orbits of affine Weyl groups and Frobenius manifolds.* A suitable degeneration<sup>1</sup> of the family (1.5) was proved in [4] to provide a new 1-dimensional mirror for the orbifold quantum co-homology of the orbifold projective line  $\mathbb{P}_{\mathfrak{g}}^1 = [\mathbb{C}^* \backslash \mathbb{C}^2 // \Gamma_{\mathfrak{g}}]$ , or, equivalently [28], to the Frobenius manifold structure on the orbits of the affine Weyl group of the same ADE type of  $\Gamma_{\mathfrak{g}}$  [9]. This allowed [4] to solve the long-standing problem [9] of determining flat co-ordinates for the Saito metric for all ADE types, and gave a higher genus reconstruction theorem by the Chekhov–Eynard–Orantin topological recursion as a bonus. Once more, the central tool in the theorem is the closed-form presentation of the family of spectral curves (1.5), and therefore, the solution of Problem 1.1 in all ADE types; particularly,  $(\mathcal{G}, V) = (E_8, \mathfrak{e}_8)$ .

1.2.6.  *$\mathbb{Q}$ -extensions with Galois group  $\mathcal{W}(E_8)$ .* A subject of particular interest in arithmetic geometry is the construction of extensions of the rationals with Galois group equal to the full Weyl group of  $E_8$  [30, 32]. By a theorem of Shioda [30, Thm .7.2], the Galois action on the extension arises from the action of the full Weyl group of  $E_8$  on the Mordell–Weil lattice of a non-isotrivial elliptic curve  $E$  over  $\mathbb{Q}(t)$ , and in turn, from the datum of a degree 240, monic integer polynomial  $\Psi \in \mathbb{Z}[x]$  whose splitting field has  $\mathcal{W}(E_8)$  as Galois group: a few explicit instances of these polynomials were constructed in [18, 21, 31]. Solving Problem 1.1 for  $(\mathcal{G}, V) = (E_8, \mathfrak{e}_8)$  gives manifestly a  $\mathbb{Z}^8$ -worth of candidate such integer polynomials<sup>2</sup> upon specialising  $\chi_i \in \mathbb{Z}$ : and for generic integral values [21], the resulting polynomial has the full Weyl group of  $E_8$  as Galois group<sup>3</sup>. The solution of Problem 1.1 generates in this way an infinite wealth of examples of multiplicatively excellent families of elliptic surfaces (in the sense of [21]) of type  $E_8$ .

1.3. **Sketch of the solution.** I briefly describe here the strategy employed to solve Problem 1.1 for  $(\mathcal{G}, V) = (E_8, \mathfrak{e}_8)$ , which was announced in my previous work [4, Appendix C]. In its crudest terms, what we would like to achieve amounts to enforcing

$$\chi_{\wedge^k \mathfrak{e}_8} = \mathfrak{p}_k(\chi_1, \dots, \chi_8) \in \mathbb{Z}[\chi_1, \dots, \chi_8] \quad (1.10)$$

for an unknown polynomial  $\mathfrak{p}_k$  as an identity of regular functions on the Cartan torus – that is, as an identity between integral Laurent polynomials. However, a direct calculation of  $\mathfrak{p}_k$  in this vein is unviable because of the exceptional complexity of  $E_8$ , even for small values of  $k$ . We find a workaround that breaks up the task of determining  $\mathfrak{p}_k$  in (1.10) into three main steps:

**Casimir bound and finite-dimensional reduction:** An *a priori* bound on the number of monomials appearing in  $\mathfrak{p}_k$  in (1.10) holds. The set of allowed monomials has cardinality  $\approx 10^6$ , and (1.10) reduces then to a finite-dimensional linear problem of the same rank upon taking sufficiently many numerical “sampling points”  $g \in \mathcal{T}$  and evaluating  $\chi_{\wedge^k \mathfrak{e}_8}(g)$  and  $\chi_i(g)$  at  $g$ .

<sup>1</sup>This is essentially given by a suitably defined  $\lambda \rightarrow \infty$  limit of (1.8).

<sup>2</sup>The path followed in [18] to find one such example hinges on constructing such a polynomial precisely as an adjoint characteristic polynomial for  $E_8^{\mathbb{C}}$ , upon factoring out the contribution of zero roots.

<sup>3</sup>For sufficient conditions for this to happen, see e.g. [18, Lemma 3.2].

**Partition of the monomial set:** For generic sampling sets, the resulting linear problem is size-wise three orders of magnitude beyond the reach of practical calculations. There exist however special choices of the sampling set such that the original linear problem is equivalent to  $\approx 4 \times 10^3$  linear sub-systems of size varying from one to  $\approx 3 \times 10^3$ . This can be realised by constructing the sampling set numerically using Newton–Raphson inversion, evaluating derivatives of characters in exponentiated linear co-ordinates on the torus up to sufficiently high order, and establishing rigorous analytic bounds to perform an exact integer rounding.

**Partition of the sampling set and distributed computing:** The problem can then be solved effectively on a computer: most of the runtime comes from the construction of the linear subsystems, by carrying out the generation of the sampling set and the evaluation of derivatives of  $\chi_i$  and  $\chi_{\wedge^k \mathfrak{e}_8}$ , and it is in the order of a few years. On the other hand the calculation can be effectively parallelised by a suitable segmentation of the sampling set, and subsequent distribution among different processor cores: this allowed to reduce in our computer implementation the total absolute clock-time taken by the entire computation to about six weeks on a small departmental cluster. The result is available as a computer package at

<http://tiny.cc/E8Char> .

and a description of the individual files is given in Appendix A.

The last two points indicate quite clearly that this project had a very substantial computational component to it. I discuss rather diffusely the details of its concrete implementation in Sections 2.3 and 3.

**1.4. Organisation of the paper.** The paper is organised as follows. In Section 2 we review the problem and explain the Casimir bound that reduces Problem 1.1 to a finite-dimensional, generically dense linear problem of rank  $\approx 10^6$ ; the Section ends with the central statement (Theorem 2.4) that suitable choices of sampling sets lead to a block-reduction of this linear problem to  $\approx 4 \times 10^3$  linear sub-systems over the rationals by suitable partitions of the set of admissible monomials; both the reduction and the solution of these linear problems can be performed effectively by parallel computation. This statement is justified in Section 3, which occupies the main body of the paper: after reviewing general computational strategies in Section 2.3.1, I describe the partition of the original problem in Section 3.1, and the exact computation of the reduced linear problems using semi-numerical methods and analytic bounds for exact integer roundings in Sections 3.2–3.4; additional details of the computer implementation are given in Section 3.5. Finally, Section 4 contains some applications to the construction of explicit integral polynomials with Galois group  $\text{Weyl}(E_8)$ ; the reader is referred to [4] for further applications of the results obtained here to integrable systems ( $E_8$  relativistic Toda lattices),  $E_8$  Seiberg–Witten theory, quantum invariants of the Poincaré sphere, and Frobenius manifolds on orbits of the affine Weyl group of type  $E_8$ . Use and abuse of notation throughout the text will be in accordance with Table 1.

$E_8$	The Dynkin type specified by the diagram of Figure 1 The complex Lie group of the same Dynkin type
$\mathfrak{e}_8$	The Lie algebra of type $E_8$ The adjoint representation $\text{Ad}_{E_8}$ of $E_8$
$\mathfrak{h}_{\mathfrak{e}_8}$	The Cartan subalgebra of $\mathfrak{e}_8$
$\mathcal{T}_{E_8}$	The Cartan torus $\mathcal{T}_{E_8} = \exp(\mathfrak{h}_{\mathfrak{e}_8})$ of $E_8$
$g$	A regular element of $E_8$
$\mathcal{W}(E_8)$	The Weyl group $N_{\mathcal{T}_{E_8}}/\mathcal{T}_{E_8}$
$h$ (resp. $\exp(h)$ )	A regular element of $\mathfrak{h}_{\mathfrak{e}_8}$ (resp. an element of a Weyl orbit in $\mathcal{T}_{E_8}$ conjugate to a regular $g \in E_8$ )
$\Pi = \{\alpha_1, \dots, \alpha_8\}$	The set of simple roots of $\mathfrak{e}_8$
$\Omega = \{\omega_1, \dots, \omega_8\}$	The set of fundamental weights of $\mathfrak{e}_8$
$\Delta$ (resp. $(\Delta^\pm)$ )	The set of all (resp. positive/negative) roots of $\mathfrak{e}_8$
$V_\omega$	The irreducible representation with highest weight $\omega$
$V_i$	The $i^{\text{th}}$ fundamental representation, $i = 1, \dots, 8$
$\Gamma(V)$ (resp. $\Gamma_{\text{red}}(V)$ )	The weight system of the representation $V$ (resp. the weight system without multiplicities)
$\chi_\omega$	The formal character of $V_\omega$
$\chi_i$	The formal character of $V_i$
$\rho$	The Weyl vector of $\mathfrak{e}_8$
$(t_1, \dots, t_8)$	Linear co-ordinates on the Cartan subalgebra of $\mathfrak{e}_8$ w.r.t. the co-root basis $\Pi^*$
$(Q_1, \dots, Q_8)$	$(\exp(t_1), \dots, \exp(t_8))$
$D^c$	$\frac{\partial^{ \mathfrak{c} }}{\partial Q_1^{c_1} \dots \partial Q_8^{c_8}}$
$\mathfrak{D}^c$	$\frac{\partial^{ \mathfrak{c} }}{\partial \chi_1^{c_1} \dots \partial \chi_8^{c_8}}$
$ v \in \mathbb{Z}^8 $	$\sum_{i=1}^8  v_i $
$C_{E_8}$	The Cartan matrix of $E_8$

TABLE 1. Notation employed throughout the text.

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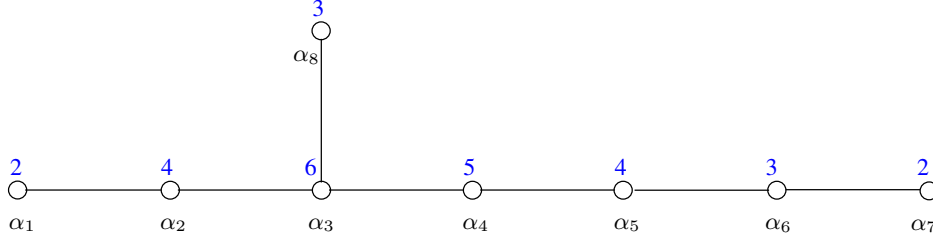


FIGURE 1. The Dynkin diagram of type  $E_8$ . The numbers in blue are the components of the highest root in the  $\alpha$ -basis.

FLINT C library [17] for multi-precision arithmetics over the integers and the rationals; C source code for all calculations is available upon request. This research was partially supported by the ERC Grant no. 682603 (PI: T. Coates) and the EPSRC Fellowship grant EP/S003657/1.

## 2. EXTERIOR CHARACTER RELATIONS

**2.1. Introduction.** We start by recasting Problem 1.1 in the following form. The cofactor expansion of the characteristic polynomial reduces the problem to finding polynomials  $\mathbf{p}_k \in \mathbb{Z}[\chi_1, \dots, \chi_8]$  such that

$$\chi_{\wedge^k \epsilon_8} = \mathbf{p}_k(\chi_1, \dots, \chi_8), \quad (2.1)$$

as in (1.10).

**Definition 2.1.** Let  $V \in \text{Rep}(\mathcal{G})$ . We call the presentation of  $\chi_V$  as a polynomial in the generators  $\chi_k$  of the representation ring the polynomial character decomposition for  $\chi_V$ .

Computing the polynomial character decomposition of  $\chi_{\wedge^k \epsilon_8}$  can be simplified somewhat for large  $k$ . Since the adjoint representation is real, we have  $\chi_{\wedge^k \epsilon_8} = \chi_{\wedge^{248-k} \epsilon_8}$ , meaning we can restrict ourselves to  $k \leq 124$ . Moreover, the characteristic polynomial in any representation factors as a product over irreducible factors corresponding to Weyl orbits: in the case of the adjoint, we have

$$\det_{\epsilon_8}(g - \mu \mathbf{1}) = (\mu - 1)^8 \mathfrak{Q}_{240}(\mu; g) \quad (2.2)$$

for a degree 240 polynomial  $\mathfrak{Q}_{240}(\mu; g)$ , corresponding to the contribution of non-zero roots. Then (2.2) imposes polynomial (in fact linear) relations in  $\text{Rep}(E_8)$ ,

$$\chi_{\wedge^j \epsilon_8} \in \mathbb{Q}[\chi_{\epsilon_8, \wedge^2 \epsilon_8, \dots, \wedge^{120} \epsilon_8}], \quad j = 121, \dots, 124, \quad (2.3)$$

meaning that in (2.1) we only<sup>4</sup> need to worry about  $k \leq 120$ . We can then equivalently phrase Problem 1.1 for  $(E_8, \epsilon_8)$  as follows.

<sup>4</sup>It is natural to ask whether we could strengthen (2.3) for  $k < 120$  (and possibly all the way down to  $k = 9$ ), i.e. whether

$$\chi_{\wedge^j \epsilon_8} \in \mathbb{Q}[\chi_{\epsilon_8, \wedge^2 \epsilon_8, \dots, \wedge^8 \epsilon_8}], \quad j = 9, \dots, 124, \quad (2.4)$$

I am unable to conclude from the structure of the root system of  $E_8$ , nor do I in fact expect, that such a statement holds.

**Problem 2.1.** For all  $1 \leq k \leq 120$  and  $\iota \in \mathbb{N}_0^8$ , determine  $N_\iota^{(k)} \in \mathbb{Z}$  such that

$$\chi_{\wedge^k \mathbf{e}_8} = \sum_{\iota} N_\iota^{(k)} \prod_{l=1}^8 \chi_l^{\iota_l}. \quad (2.5)$$

There is in principle a systematic recursive method to determine  $N_\iota^{(k)}$ , as follows. Note that  $\wedge^k \mathbf{e}_8$  is highly reducible as soon as  $k > 1$ ; let

$$\wedge^k \mathbf{e}_8 = \bigoplus_{i=1}^{f_k} \mathfrak{m}_i^{(k)} V_{\sum_l j_{i,l}^{(k)} \omega_l} \quad (2.6)$$

be its decomposition into irreducible summands, with  $\mathfrak{m}_i^{(k)}, f_k, j_{i,l}^{(k)} \in \mathbb{N}$ . Then Problem 2.1 can be solved in two steps: by determining first the multiplicities  $\mathfrak{m}_i^{(k)}$ , and then solving Problem 2.1 for each irreducible summand in (2.6). This last step can be achieved recursively by computing, for a dominant weight  $\varpi = \sum_k \varpi_k \omega_k$ , the product

$$V_{\varpi - \omega_{\bar{k}}} \otimes V_{\omega_{\bar{k}}} = V_{\varpi} \oplus \bigoplus_{b=1}^{f_{\varpi}} \mathfrak{m}_i^{\varpi} V_{\sum_l j_{i,l}^{\varpi} \omega_l} \quad (2.7)$$

where  $\mathfrak{m}_i^{(\varpi)}, f_{\varpi}, j_{i,l}^{\varpi} \in \mathbb{N}$  with notation adapted from (2.6), and  $\bar{k}$  is an index labelling the non-vanishing component of  $\varpi$  in the direction of a fundamental weight with weight system of minimal cardinality:

$$\bar{k} \in I(\varpi) := \{k = 1, \dots, 8 \mid \varpi_k \neq 0\} \quad \text{s.t.} \quad \dim V_{\omega_{\bar{k}}} = \min_{j \in I(\varpi)} \dim V_{\omega_j}. \quad (2.8)$$

This is easily shown to be a recursion with induction data given by the fundamental weights and the zero weight.

**Example 2.1.** Take  $k = 3$ . We first decompose  $\wedge^3 \mathbf{e}_8$  into irreducibles to get

$$\wedge^3 \mathbf{e}_8 = V_0 + V_{\omega_5} + V_{\omega_6} + V_{2\omega_7} + V_{\omega_1} \quad (2.9)$$

Upon taking characters, we get

$$\chi_{\wedge^3 \mathbf{e}_8} = 1 + \chi_5 + \chi_6 + \chi_{2\omega_7} + \chi_1 \quad (2.10)$$

so the only bit we need to compute is the polynomial decomposition of  $\chi_{2\omega_7}$  in the tensor algebra. In this case we have only one non-vanishing component of  $\varpi := 2\omega_7$ , and we can compute explicitly from (2.7) that

$$V_{\varpi - \omega_7} \otimes V_{\omega_7} = V_{\omega_7} \otimes V_{\omega_7} = V_{2\omega_7} \oplus V_0 \oplus V_{\omega_7} \oplus V_{\omega_1} \oplus V_{\omega_6}. \quad (2.11)$$

Combining (2.10) and (2.11) and taking characters gives

$$\chi_{\wedge^3 \mathbf{e}_8} = \chi_5 - \chi_7 + \chi_7^2. \quad (2.12)$$

While this example is relatively simple, carrying out the same procedure for  $k$  all the way up to  $k = 120$  is unfeasible, both for the growth in complexity of the tensor product decompositions in (2.7) and the recursive procedure outlined above as  $k$  grows, and especially because of the difficulty in computing explicitly the multiplicities of plethysms such as (2.10) for all  $k$ , which is a hard problem in its own right (and one that is unsolved to date, see [2, 20, 22, 27] for an account).

So we need to circumvent both problems, and possibly, we need to do so in a way that works uniformly with respect to the exterior exponent  $k$ . I will describe in the next section four alternative methods to accomplish this, each having its upsides and shortcomings.

**2.2. Character decompositions and an a priori bound.** I will start by establishing sufficient vanishing conditions for the numbers  $N_\iota^{(k)}$  in Problem 2.1. Let

$$e_{jk} := \sum_j (C_{E_8}^{-1})_{jk} \quad (2.13)$$

be the  $k^{\text{th}}$  component of  $\omega_j$  in the root basis. Notice that this is always a positive integer, since  $E_8$  has trivial centre.

**Definition 2.2.** *We call the finite set*

$$\mathfrak{J} := \left\{ \iota \in (\mathbb{Z}_+)^8 \mid \sum_j \iota_j e_{jk} \leq 2 \sum_j e_{jk} \right\} \quad (2.14)$$

the set of admissible exponents of the exterior algebra  $\wedge \mathfrak{e}_8$ .

**Lemma 2.1.** *We have that*

$$\iota \notin \mathfrak{J} \Rightarrow N_\iota^{(k)} = 0 \quad \forall k. \quad (2.15)$$

*Proof.* By definition of the exterior power, weights in  $\Gamma(\wedge^k \mathfrak{e}_8)$  have the form  $\mu_{i_1 \dots i_k} = \sum_{j=1}^k \alpha_{i_j}$  with  $i_j \neq i_l$  for  $j \neq l$ . When  $k = 120$ , there is a unique element in the weight module with  $i_j > 0 \forall j = 1, \dots, 120$ , which is precisely  $2\rho = \sum_{\alpha \in \Delta^+} \alpha$ . Therefore, for any  $\mu \in \Gamma(\wedge^{120} \mathfrak{e}_8)$ , we have that  $2\rho - \mu = \sum_{i=1}^8 n_i \alpha_i$  with all  $n_i \in \mathbb{Z}_{\geq 0}$ : equivalently, for any  $\mu$  in  $\Gamma(\wedge^k \mathfrak{e}_8)$ , we must have  $\mu \preceq 2\rho$ , where  $\lambda \preceq \mu$  is the usual partial order on the weights indicating that  $\mu - \lambda$  is a non-negative integral linear combination of the simple roots. Consider now the representation space version of (2.5),

$$\wedge^k \mathfrak{e}_8 = \bigoplus_{\iota} N_\iota^{(k)} \bigotimes_{l=1}^8 V_{\omega_l}^{\iota_l}, \quad (2.16)$$

and consider the subset of indices  $\iota_j \in \mathbb{N}_0$  such that  $\sum_j \iota_j \omega_j \not\preceq 2\rho$ . Since the corresponding one-dimensional weight space must appear with zero coefficient on the l.h.s. of (2.16), the weighted occurrences of this weight space must sum up with total coefficient equal to zero on its r.h.s.. Consider now the  $\mathbb{Z}$ -linear map

$$\begin{aligned} \mathfrak{f} : \mathbb{N}^8 &\longrightarrow \mathbb{N}^{10} \\ \{\iota_j\}_{j=1}^8 &\longrightarrow \{k_l\}_{l=1}^{10} \end{aligned} \quad (2.17)$$

induced by the factorisation

$$\prod_j \dim V_{\omega_j}^{\iota_j} = 2^{k_1} 3^{k_2} 5^{k_3} 7^{k_4} 11^{k_5} 13^{k_6} 17^{k_7} 19^{k_8} 23^{k_9} 31^{k_{10}}. \quad (2.18)$$

It is immediate to verify that  $\mathfrak{f}$  maps the positive 8-orthant injectively into the 10-orthant, as the reader can verify by writing down explicitly the expression of  $\mathfrak{f}_l(\iota) = \sum_{l_j} \mathfrak{m}_{l_j}^{\mathfrak{f}} \iota_j$  and verifying that the integral  $10 \times 8$  matrix  $\mathfrak{m}^{\mathfrak{f}}$  has maximal rank. This implies that each summand  $\bigotimes_l V_{\omega_l}^{\iota_l}$  in (2.16) is uniquely determined by the exponents  $\iota_l$  of the tensor powers. Let then  $\iota_j^{\max}$  be the indices labelling the summand of maximal dimension on the r.h.s. of (2.16). This summand contains the dominant weight  $\sum_j \iota_j^{\max} \omega_j \not\leq 2\rho$  in its weight module, and by construction it is the unique direct summand in (2.16) containing it, so  $N_{\iota}^{(k)} = 0$ . Induction on the maximal dimension gives  $N_{\iota}^{(k)} = 0$  for all  $\sum_j \iota_j \omega_j \not\leq 2\rho$ . The lemma then follows from writing down the condition  $\sum_j \iota_j \omega_j \leq 2\rho$  in the root basis.  $\square$

**Remark 2.2.** It is obvious that (2.15) could be further refined to provide stronger bounds for any fixed  $k < 120$ . It will suffice in our discussion to consider just the uniform bound above, and treat all values of  $k$  simultaneously in Problem 2.1, as will be clear momentarily.

**Remark 2.3.** The condition of being admissible gives an *a priori* bound on the range of  $\iota_j$  in the sum on the right hand side of (2.5),

$$\chi_{\wedge^k \mathfrak{e}_8} = \sum_{\iota \in \mathfrak{J}} N_{\iota}^{(k)} \prod_{l=1}^8 \chi_l^{\iota_l} \quad (2.19)$$

with possibly at most  $|\mathfrak{J}| = 950077$  monomials appearing with non-zero coefficient in (2.19) for any given value of  $k$ . Crude as it might appear, the sufficient vanishing condition of Lemma 2.1 is remarkably close to be necessary as well for sufficiently big  $k$ ; we will be able to check *a posteriori* that

$$\text{card}\left\{\iota \in \mathfrak{J} \mid N_{\iota}^{(k)} \neq 0 \text{ for some } k\right\} = 949468, \quad (2.20)$$

$$\sup_k \text{card}\left\{\iota \in \mathfrak{J} \mid N_{\iota}^{(k)} \neq 0\right\} = 949256 \quad (k = 118), \quad (2.21)$$

meaning that most ( $\approx 99.94\%$ ) admissible monomials will appear with non-vanishing coefficient in the polynomial character decomposition of  $\chi_{\wedge^k \mathfrak{e}_8}$  for some  $k$ , and most (up to  $\approx 99.91\%$ ) will appear for a fixed  $k$  big enough.

**2.3. Computing the solution of Problem 2.1.** Lemma 2.1 reduces Problem 2.1 to a finite dimensional linear problem which can in principle be fed to a computer solver; in practice, however, this will require a substantial degree of additional sophistication. The main idea of our solution is that the large linear problem Problem 2.1 is non-trivially equivalent to a big number of small linear problems over  $\mathbb{Q}$ , that are explicitly solvable in parallel (and within our lifetimes; typically in the order of a few weeks when implemented on a small compute cluster).

2.3.1. *Four paths to polynomial character decompositions.* Lemma 2.1 reduces Problem 2.1 to the following finite dimensional linear problem: impose (2.19) as an identity of regular functions on  $\mathcal{T}_{E_8}$ ,

$$\chi_V = \sum_{w \in \Gamma(V)} \prod_i Q_i^{w_i} \in \mathbb{Z}[Q_1^\pm, \dots, Q_8^\pm], \quad (2.22)$$

and then read off  $N_l^{(k)}$  by plugging  $V = \wedge^k \mathfrak{e}_8$  and  $V = V_j$  onto either side of (2.19), and equating the coefficients. One immediate drawback of such a brute-force approach is the sheer size of these Laurent polynomials – from Lemma 2.1, there are  $\prod_k (1 + 2 \sum_j e_{jk}) \approx 3.2 \times 10^{17}$  weight spaces appearing in (2.19), thus rendering this approach unwieldy. I describe below four computational methods that are both more effective and are also complementary in what they allow to compute.

**Method 1 (Numerical sampling):** The simplest thing to do is to consider a *sampling set* of  $|\mathcal{J}|$  numerical values for the conjugacy class  $[g] = [\exp(h)] = [(Q_1^{(i)}, \dots, Q_8^{(i)})] \in \mathcal{T}_{E_8}/\mathcal{W}_{E_8}$  for  $i \in \mathcal{J}$ . Upon evaluating the regular characters  $\chi_{\wedge^k \mathfrak{e}_8}(\exp h)$ ,  $\chi_j(\exp h)$  on either side of (2.19), and for generic sampling sets, we are left with a rank- $|\mathcal{J}|$  linear system with  $N_l^{(k)}$  as unknowns to solve for.

This is a straightforward method which furthermore works uniformly in  $k$ . However, for a generic choice of sampling values, the linear system is dense and extremely poorly-conditioned – the corresponding matrix is a generalised Vandermonde matrix minor – which implies that exact arithmetics is required for its solution. Memory-wise this places a bound for the size at around

$$r_{\max} := 3.5 \times 10^3. \quad (2.23)$$

Since  $|\mathcal{J}| \approx 10^6$ , this means that we're off by three orders of magnitude here.

**Method 2 (Q-expansions):** As an alternative, we could consider to Taylor-expand  $\chi_{\wedge^k \mathfrak{e}_8}(g)$  in  $\chi_i$  at a value  $g = \exp h$  such that  $\chi_i(g) = 0$  for all  $i$ . It is immediate to verify, for example, that

$$t(h_0) = -\frac{2i\pi}{31}(6, 3, 15, 1, 12, -4, 5, 0) \quad (2.24)$$

in linear co-ordinates for  $\mathcal{T}_{E_8}$  results in  $\chi_i(\exp h_0)$  being zero for all  $i$ : a direct calculation of  $\chi_{\wedge^k \mathfrak{e}_8}(\exp h_0)$  in (2.19) returns the constant term  $N_{(0, \dots, 0)}^{(k)}$  for all  $k$ . In the same vein,  $N_l^{(k)}$  can be computed by taking higher order  $\chi_i$ -derivatives of  $\chi_{\wedge^k \mathfrak{e}_8}$  at  $h = h_0$  in the following three steps:

- (1) we first compute  $Q_j$ -derivatives of  $\chi_{\wedge^k \mathfrak{e}_8}$  at  $h = h_0$  from (2.22), using the explicit structure of  $\Gamma(\mathfrak{e}_8) = \Delta$ ;
- (2) the Jacobian  $\partial_{Q_j} \chi_i$  may be computed combining basic relations in  $\text{Rep}(E_8)$  with Newton identities;

(3) finally, higher order  $\chi_j$  derivatives of  $\chi_{\wedge^k \mathfrak{e}_8}$  can be obtained by the multi-variate Faà di Bruno formula<sup>5</sup> [6], converting  $\partial_{\chi_1^{\iota_1} \dots \chi_1^{\iota_8}}^{|\iota|}$  into its expression as a  $|\iota|^{\text{th}}$ -order differential operator in  $(Q_j)_j$ .

The main drawback here is the factorial growth in  $|\iota|$  of the number of terms in the Faà di Bruno formula, which practically limits its use to

$$|\iota| \lesssim 5 =: d_{\max}. \quad (2.25)$$

This is a very long way from the value

$$|\iota^{\max}| := \sup_{\iota \in \mathfrak{J}} \sum_j \iota_j = 31 \quad (2.26)$$

for the highest sum of admissible exponents in (2.14).

**Method 3 (Numerical interpolation):** A further, numerical way to determine  $N_\iota^{(k)}$  is as follows. Suppose that, for every  $\iota$ , we could find a solution  $h_\iota$  of the system of algebraic equations in  $Q$

$$\chi_j(\exp h_\iota) = \sum_{w \in \Gamma(V_{\omega_j})} \prod_l Q_l^{w_l} = u_j^{(\iota)} \quad (2.27)$$

for given  $\{(u_1, \dots, u_8)^{(\iota)} \in \mathbb{C}^8\}_{\iota \in \mathfrak{J}}$ . Then  $\mathfrak{p}_k(\chi_1, \dots, \chi_8)$  can be recovered by polynomial interpolation of  $\chi_{\wedge^k \mathfrak{e}_8}(Q(u^{(\iota)}))$  in  $\chi_1, \dots, \chi_8$  through the given values of  $\chi_j(\exp h) = u_j^{(\iota)}$ : even if the solution  $Q(u_i^{(\iota)})$  cannot in general be found exactly, one might still hope to compute an approximate, generic root of the algebraic system (2.27) numerically; and if this is done with sufficiently high precision, the sought-for interpolating coefficients  $N_\iota^{(k)}$  can be reliably computed by rounding to the nearest integer.

This semi-numerical approach is effective in computing e.g.  $N_\iota^{(k)}$  when all but one  $\iota_j \neq 0$ : in this case interpolation is done in only one variable and the number of sampling points necessary is in the order of a few dozens. The main problem with the general case is that it requires  $|\mathfrak{J}|$  interpolating points, hence we would be faced with a large interpolating matrix to invert and the same problem of **Method 1** above. Alternatively, we could resort to interpolation on the lattice hypercube  $\times_j [0, \iota_j^{\max}] \cap \mathbb{Z}^8$  with  $\iota_j^{\max} = \sup_{\iota \in \mathfrak{J}} \iota_j$ , with a trivially invertible Vandermonde matrix but  $\prod \iota_j^{\max} \approx 6.5 \times 10^9$  high-precision numerical inversions to perform. On the type of computer systems we employed<sup>6</sup>, a single numerical root is typically found in  $\approx 10$  sec to the necessary precision, so the total runtime would take  $\approx 2 \times 10^3$  years on a single CPU.

**Method 4 ( $p$ -adic expansions):** We could also consider a version of **Method 1** where we sample at integers and reduce mod  $p$  for sufficiently large  $p$ . This works efficiently for low  $k$ , but it is a challenge to determine the behaviour for higher exterior powers and, furthermore, it is hard to place an *a priori* bound on the largest  $p$  required.

<sup>5</sup>We discuss a version of this formula in Proposition 3.2.

<sup>6</sup>These were all 64-bit systems with quad-core CPUs and up to 48 Gb of RAM.

Each of these methods individually fails to address Problem 2.1; we have in particular bounds

$$r_{\max} \approx 3.5 \times 10^3, \quad d_{\max} \approx 5 \quad (2.28)$$

for the size of the linear systems we can solve and the order of the derivatives respectively for **Method 1** and **Method 2**: the bound on the size comes from memory constraints when using exact linear solving algorithms, whilst the bound on the order is a runtime bound for the use of the Faà di Bruno formula in many variables.

Nonetheless, it is still possible to combine **Methods 1-4** in such a way that a complete answer of the problem can be obtained efficiently by distributed computation. Suppose that  $(Q_1^{(\kappa)}, \dots, Q_8^{(\kappa)}) \in \mathcal{T}_{E_8}$  is a point on the Cartan torus such that the value of the  $j^{\text{th}}$  fundamental character,  $u_j^{(\kappa)} := \chi_j(Q^{(\kappa)})$  in (2.27), is zero for all but a few values of  $j$ . Then, upon evaluating  $\chi_j$  at such  $Q^{(\kappa)}$ , all terms with  $\iota_j \neq 0$  drop out of the r.h.s. of (2.19), leaving a smaller number of unknown coefficients  $N_\iota^{(k)}$  to solve for; case in point, (2.24) is an example where only one term survives.

This means that if we could choose judiciously the elements  $(Q_1^{(\kappa)}, \dots, Q_8^{(\kappa)})$  in **Method 1** such that any given subset of the fundamental characters are vanishing at  $Q^{(\kappa)}$ , the original linear problem (2.19) can be reduced to  $2^8$  linear sub-systems of smaller size: any  $\phi \in \mathbb{N}^8$  with  $\phi_j \in \{0, 1\}$  determines a subset  $\mathfrak{J}_\phi \subset \mathfrak{J}$  with  $\phi_j$  equal to zero or one for  $\iota_j = 0$  or  $\iota_j > 0$  respectively, and this is a partition of the set of admissible exponents. We can further reduce the size of these systems by using **Method 2** and considering derivatives  $\frac{\partial^{|\mathbf{c}|} \chi_{\wedge^k \epsilon_8}}{\partial \chi_1^{c_1} \dots \partial \chi_8^{c_8}}(Q_1^{(\kappa)}, \dots, Q_8^{(\kappa)})$ : indeed, computing  $\chi_j$ -derivatives of given order  $c_j$  of the l.h.s. of (2.19) and evaluating at points with  $u_j^{(\kappa)} = \chi_j(Q^{(\kappa)}) = 0$  entails that only summands with  $\iota_j = c_j$  will appear on the r.h.s.. I claim that the procedure above can be performed effectively and in a way compatible with (2.28) in the following theorem, the proof of which will occupy the largest part of Section 3.

As above, let  $\phi \in \{0, 1\}^8$  denote an 8-tuple of integers equal to either 0 or 1, and let  $i_1(\phi)$  (resp.  $i_2(\phi)$ ) be the index of the non-vanishing component of  $\phi$  such that  $\dim V_{\omega_{i_1(\phi)}}$  is maximal (resp.  $\dim V_{\omega_{i_2(\phi)}}$  is next-to-maximal) in  $\{\dim V_{\omega_i}\}_{i|\phi_i \neq 0}$ ; e.g.

$$\phi = (0, 1, 0, 0, 1, 0, 1, 1) \Rightarrow i_1(\phi) = 2; i_2(\phi) = 5. \quad (2.29)$$

Let  $m_1, m_2 \in \{1, \dots, d_{\max}\}$ . We write

$$\begin{aligned} \mathfrak{J}_\phi^{(m_1, m_2)} &= \{\iota \in \mathfrak{J} | \iota_j = 0 \Leftrightarrow \phi_j = 0, \iota_{i_1(\phi)} = m_1, \iota_{i_2(\phi)} = m_2\}, \\ \mathfrak{J}_\phi^{(m_1)} &= \{\iota \in \mathfrak{J} | \iota_j = 0 \Leftrightarrow \phi_j = 0, \iota_{i_1(\phi)} = m_1, \iota_{i_2(\phi)} > d_{\max} - m_1\}, \\ \mathfrak{J}_\phi^{(> d_{\max})} &= \{\iota \in \mathfrak{J} | \iota_j = 0 \Leftrightarrow \phi_j = 0, \iota_{i_1(\phi)} > d_{\max}\}, \end{aligned} \quad (2.30)$$

and we write  $\aleph$  to indicate any of the superscripts  $(m_1, m_2)$ ,  $(m_1)$ , or  $(> d_{\max})$  where  $1 < m_1 + m_2 \leq d_{\max}$  and  $1 \leq m_1 \leq d_{\max}$ . Clearly, we have

$$\mathfrak{J} = \bigsqcup_{\phi, \aleph} \mathfrak{J}_\phi^\aleph. \quad (2.31)$$

**Theorem 2.4.** *There exists a choice of  $u_j^\iota \in \mathbb{Q} + i\mathbb{Q}$ ,  $\iota \in \mathfrak{J}_\phi^\aleph$ , such that*

i. Problem 2.1 is equivalent to solving, for every  $(\phi, \aleph)$ , the linear system over the rationals

$$\sum_{\kappa \in \mathfrak{J}_\phi^\aleph} (\mathcal{A}_\phi^\aleph)_{\iota\kappa} N_\kappa^{(k)} = (\mathcal{B}_\phi^\aleph)_\iota^{(k)}, \quad (2.32)$$

where

(a)

$$(\mathcal{A}_\phi^\aleph)_{\iota\kappa} = \Re \left( \prod_j (u_j^\iota)^{\kappa_j} \right), \quad (2.33)$$

as in **Method 1**;

(b)  $(\mathcal{B}_\phi^\aleph)_\iota^{(k)}$  is computed from  $Q$ -derivatives of order at most  $d_{\max}$  of  $\chi_{\wedge^k \epsilon_8}$  at points  $Q^\iota \in \mathcal{T}_{E_8}$  such that  $\chi_j(Q^\iota) = u_j^\iota$ , as in **Method 2**;

ii. the pre-images  $Q^\iota = Q(u^\iota)$  can be obtained numerically via **Method 3** to precision  $2^{-M_0}$ ,  $M_0 \in \mathbb{N}$ , and there exist a priori analytical bounds on the resulting numerical error on  $(\mathcal{B}_\phi^\aleph)_\iota^{(k)} \in \mathbb{Q}$  at the points  $Q^\iota = Q(u^\iota)$  such that for sufficiently large  $M_0$ , the numerical calculation of  $(\mathcal{B}_\phi^\aleph)_\iota^{(k)}$  via **Method 3** can be rounded rigorously to its exact rational value;

iii. the system (2.32) is non-degenerate, of rank lower than  $r_{\max}$  for all  $(\phi, \aleph)$ , and can be solved effectively using  $p$ -adic expansions as in **Method 4**.

The next Section gives a detailed account of both the proof of Theorem 2.4 and its concrete computer implementation, leading to the calculation of all  $N_\iota^{(k)}$  in (2.19). The reader who is more interested in the applications of the solution of Problem 2.1 via Theorem 2.4 may wish to skip the next section and move on directly to Section 4.

### 3. PROOF OF THEOREM 2.4

**3.1. Partition of the monomial set.** The key point we explained at the end of the previous section is that while *generic* choices of sampling sets in **Method 1** give rise to large, dense linear systems, there are *special* choices of the numerical values of  $Q^{(\iota)} \in \mathcal{T}_{E_8}$  that reduce the calculation of  $N_\iota^{(k)}$  to the solution of a large number of linear problems of much smaller size. We explore this idea in detail in this section.

Let

$$\begin{aligned} \varphi &: \mathfrak{J} \rightarrow \{0, 1\}^8 \\ \iota &\rightarrow (\theta(\iota_1), \dots, \theta(\iota_8)) \end{aligned} \quad (3.1)$$

where for  $n \in \mathbb{N}$

$$\theta(n) = \begin{cases} 1 & n \neq 0 \\ 0 & n = 0 \end{cases}, \quad (3.2)$$

and let  $\mathfrak{J}_\phi = \varphi^{-1}(\phi)$  for  $\phi \in \{0, 1\}^8$ . The map  $\varphi$  maps an 8-tuple of admissible exponents  $(\iota_1, \dots, \iota_8)$  into an 8-tuple  $(\varphi_1(\iota), \dots, \varphi_8(\iota))$  with  $\varphi_j(\iota) = \theta(\iota_j)$  equal to one if  $\iota_j$  is non-zero, and zero otherwise.

Its fibres  $\mathfrak{J}_\phi$  partition the monomial set into  $2^8$  subsets of cardinality varying from 1 to  $\approx 3 \times 10^4$ ,

$$\bigsqcup_{\phi} \mathfrak{J}_\phi = \mathfrak{J}. \quad (3.3)$$

We furthermore define a strict weak ordering on the set of admissible exponents as

$$\iota < \kappa \iff |\varphi(\iota)| < |\varphi(\kappa)|, \quad (3.4)$$

and for subsets  $\mathfrak{J}, \mathfrak{K}$  we will write  $\mathfrak{J} < \mathfrak{K}$  if  $\iota < \kappa, \forall \iota \in \mathfrak{J}, \kappa \in \mathfrak{K}$ .

The crucial point is now as follows: consider a set of points on the torus  $(Q_l^{(\iota)})_l \in \mathcal{T}_{E_8}, \iota \in \mathfrak{J}$ , such that

$$\chi_j(Q^{(\iota)}) \in \varphi_j(\iota) \mathbb{C}. \quad (3.5)$$

Evaluating (2.19) at  $Q^{(\iota)}$  we have that only the coefficients  $N_\kappa^{(k)}$  with  $\varphi_k(\kappa) \leq \varphi_k(\iota)$  will appear on the r.h.s. of (2.19). Then the linear problems

$$\mathfrak{L}_\phi \triangleq \left\{ \chi_{\wedge k_{E_8}}(Q^{(\iota)}) - \sum_{\substack{\kappa < \iota \\ \varphi_l(\kappa) \leq \varphi_l(\iota)}} N_\kappa^{(k)} \chi_1^{\kappa_1} \cdots \chi_8^{\kappa_8} = \sum_{\substack{\kappa \in \mathfrak{J}_{\varphi(\iota)} \\ \varphi(\iota) = \phi}} N_\kappa^{(k)} \chi_1^{\kappa_1} \cdots \chi_8^{\kappa_8} \right\} \quad (3.6)$$

are fully determined by the solution of  $\mathfrak{L}_{\phi'}$  for  $|\phi'| < |\phi|$ , are each of size  $|\mathfrak{J}_\phi|$ , and their solution for all  $\phi$  solves Problem 2.1: the end result is to break up our original linear system into  $2^8 = 256$  linear subsystems of size varying from 1 to  $\approx 3 \times 10^4$ .

**3.1.1. Refining the partition.** The original linear problem for generic sampling values as described in **Method 1** is theoretically solved in  $\mathcal{O}(|\mathfrak{J}|^3)$  time; the special choice (3.5) breaks it up to  $\approx 10^2$  sub-systems of size at most  $\approx 10^{-2}|\mathfrak{J}|$ , giving an improvement of a factor more than  $10^4$  in the expected runtime using floating-point precision. However this isn't satisfactory yet, since we are still off by one order of magnitude in the size of the individual systems we can practically solve,

$$\sup_{\phi} \text{rank} \mathfrak{L}_\phi \gg r_{\max}. \quad (3.7)$$

In order to reduce  $\mathfrak{L}_\phi$  to sub-systems of smaller size we refine the partition of  $\mathfrak{J}$  by considering proper subsets of  $\mathfrak{J}_\phi$  having two fixed exponents equal to, or greater than, an integer less than  $d_{\max}$ . To this aim we introduce maps  $\vartheta_\phi^{(1)}$  and  $\vartheta_\phi^{(2)}$ , with

$$\begin{aligned} \vartheta_{\phi, (l)} &: \mathfrak{J}_\phi \rightarrow \mathbb{N}_0, & l = 1, 2, \\ & \iota &\rightarrow \iota_{\sigma(\phi, l)} \end{aligned} \quad (3.8)$$

where  $\sigma(\phi, l)$  is defined by

$$\begin{aligned} \dim V_{\omega_{\sigma(\phi, 1)}} &= \max\{\dim V_{\omega_j}\}_{\phi_j \neq 0}, \\ \dim V_{\omega_{\sigma(\phi, 2)}} &= \max\{\dim V_{\omega_j}\}_{\phi_j \neq 0, j \neq \sigma(\phi, 1)}. \end{aligned} \quad (3.9)$$

They can be used to refine the strict weak ordering (3.4) on  $\mathfrak{J}$  as follows

$$\iota < \kappa \iff \begin{cases} |\varphi(\iota)| < |\varphi(\kappa)|, & \text{or} \\ \varphi(\iota) = \varphi(\kappa) \text{ and } \iota_{\sigma(\phi,1)} \leq d_{\max}, \kappa_{\sigma(\phi,1)} > d_{\max}, & \text{or} \\ \varphi(\iota) = \varphi(\kappa) \text{ and } \iota_{\sigma(\phi,1)}, \iota_{\sigma(\phi,2)}, \kappa_{\sigma(\phi,1)} \leq d_{\max}, \kappa_{\sigma(\phi,2)} > d_{\max}. \end{cases} \quad (3.10)$$

Let  $m_l \in \mathbb{N}$ ,  $l = 1, 2$  with  $m_1 + m_2 \leq d_{\max} = 5$ , and introduce for fixed  $\phi$  the following 16 subsets of  $\mathfrak{J}_\phi$ :

$$\mathfrak{J}_\phi^{(m_1, m_2)} \triangleq \vartheta_{\phi,1}^{-1}(m_1) \cap \vartheta_{\phi,2}^{-1}(m_2), \quad (3.11)$$

$$\mathfrak{J}_\phi^{(m_1)} \triangleq \vartheta_{\phi,1}^{-1}(m_1) \cap \vartheta_{\phi,2}^{-1}(\{m_2 > d_{\max} - m_2\}), \quad (3.12)$$

$$\mathfrak{J}_\phi^{(>d_{\max})} \triangleq \vartheta_{\phi,1}^{-1}(\{m_1 > d_{\max}\}). \quad (3.13)$$

Clearly,

$$\mathfrak{J} = \bigsqcup_{\phi} \bigsqcup_{m_1+m_2 \leq d_{\max}} \mathfrak{J}_\phi^{(m_1, m_2)} \bigsqcup_{1 \leq m_1 \leq d_{\max}} \mathfrak{J}_\phi^{(m_1)} \sqcup \mathfrak{J}_\phi^{(>d_{\max})} \quad (3.14)$$

and

$$\iota < \kappa < \lambda \quad \forall \iota \in \mathfrak{J}_\phi^{(m_1, m_2)}, \kappa \in \mathfrak{J}_\phi^{(m_1)}, \lambda \in \mathfrak{J}_\phi^{(>d_{\max})}. \quad (3.15)$$

We now have

$$\sup \left\{ |\mathfrak{J}_\phi^{(m_1, m_2)}|, |\mathfrak{J}_\phi^{(m_1)}|, |\mathfrak{J}_\phi^{(>d_{\max})}| \right\}_{m_1, m_2, \phi} = \left| \mathfrak{J}_{(1,1,0,0,1,1,1,1)}^{(>d_{\max})} \right| = 3027 \ll r_{\max}, \quad (3.16)$$

so if the linear problem  $\mathfrak{L}_\phi$  in (3.6) could be reduced down to linear sub-systems of rank  $|\mathfrak{J}_\phi^{(m_1, m_2)}|$ ,  $|\mathfrak{J}_\phi^{(m_1)}|$  and  $|\mathfrak{J}_\phi^{(>d_{\max})}|$ , these could now be individually solved explicitly. This can be done by mimicking (3.6): for  $\iota \in \mathfrak{J}_\phi$ , consider  $(Q_l^{(\iota)})_l \in \mathcal{T}_{E_8}$  such that

$$\chi_j(Q^{(\iota)}) = u_j^{(\iota)} \in \varphi_j(\iota) \left\{ \begin{array}{ll} \delta_{j, \sigma(\phi,1)} \delta_{j, \sigma(\phi,2)}, & \iota \in \mathfrak{J}_\phi^{(m_1, m_2)}, \\ \delta_{j, \sigma(\phi,1)}, & \iota \in \mathfrak{J}_\phi^{(m_1)}, \\ 1, & \iota \in \mathfrak{J}_\phi^{(>d_{\max})}, \end{array} \right\} \mathbb{C}. \quad (3.17)$$

and define

$$\begin{aligned} \mathfrak{L}_\phi^{(m_1, m_2)} &\triangleq \left\{ \frac{1}{(m_1)!(m_2)!} \frac{\partial^{m_1+m_2}}{\partial \chi_{\sigma(\phi,1)}^{m_1} \partial \chi_{\sigma(\phi,2)}^{m_2}} \chi_{\wedge^k \mathfrak{e}_8}(Q^{(\iota)}) - \sum_{\kappa \in \mathfrak{P}_\phi^{(m_1, m_2)}} N_\kappa^{(k)} \prod_{\substack{j \neq \sigma(\phi, l) \\ l=1,2}} \chi_j^{\kappa_j}(Q^{(\iota)}) \right\}_{\iota \in \mathfrak{J}_\phi^{(m_1, m_2)}}, \\ &= \sum_{\kappa \in \mathfrak{J}_\phi^{(m_1, m_2)}} N_\kappa^{(k)} \prod_{\substack{j \neq \sigma(\phi, l) \\ l=1,2}} \chi_j^{\kappa_j}(Q^{(\iota)}) \Big|_{\iota \in \mathfrak{J}_\phi^{(m_1, m_2)}}, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \mathfrak{L}_\phi^{(m_1)} &\triangleq \left\{ \frac{1}{(m_1)!} \frac{\partial^{m_1} \chi_{\wedge^k \mathfrak{e}_8}}{\partial \chi_{\sigma(\phi,1)}^{m_1}}(Q^{(\iota)}) - \sum_{\kappa \in \mathfrak{P}_\phi^{(m_1)}} N_\kappa^{(k)} \prod_{j \neq \sigma(\phi,1)} \chi_j^{\kappa_j}(Q^{(\iota)}) \right\}_{\iota \in \mathfrak{J}_\phi^{(m_1)}}, \\ &= \sum_{\kappa \in \mathfrak{J}_\phi^{(m_1)}} N_\kappa^{(k)} \prod_{j \neq \sigma(\phi,1)} \chi_j^{\kappa_j}(Q^{(\iota)}) \Big|_{\iota \in \mathfrak{J}_\phi^{(m_1)}}, \end{aligned} \quad (3.19)$$

$$\mathfrak{L}_\phi^{(>d_{\max})} \triangleq \left\{ \chi_{\wedge^k \mathfrak{e}_8}(Q^{(\iota)}) - \sum_{\kappa \in \mathfrak{P}_\phi^{(>d_{\max})}} N_\kappa^{(k)} \prod_j \chi_j^{\kappa_j}(Q^{(\iota)}) = \sum_{\kappa \in \mathfrak{J}_\phi^{(>d_{\max})}} N_\kappa^{(k)} \prod_j \chi_j^{\kappa_j}(Q^{(\iota)}) \right\}_{\iota \in \mathfrak{J}_\phi^{(>d_{\max})}} \quad (3.20)$$

where

$$\begin{aligned} \mathfrak{P}_\phi^{(m_1, m_2)} &:= \left\{ \kappa \in \mathfrak{J} \mid \kappa < \iota, \kappa_{\sigma(\phi, l)} = m_l \ \forall l = 1, 2, \iota \in \mathfrak{J}_\phi^{(m_1, m_2)} \right\}, \\ \mathfrak{P}_\phi^{(m_1)} &:= \left\{ \kappa \in \mathfrak{J} \mid \kappa < \iota, \kappa_{\sigma(\phi, 1)} = m_1 \ \forall \iota \in \mathfrak{J}_\phi^{(m_1)} \right\} \bigcup_{m_2=1}^{d_{\max}-m_1} \mathfrak{J}_\phi^{(m_1, m_2)}, \\ \mathfrak{P}_\phi^{(>d_{\max})} &:= \left\{ \kappa \in \mathfrak{J} \mid \kappa < \iota, \iota \in \mathfrak{J}_\phi^{(>d_{\max})} \right\} \bigcup_{m_1+m_2 \leq d_{\max}} \mathfrak{J}_\phi^{(m_1, m_2)} \bigcup_{m_1=1}^{d_{\max}} \mathfrak{J}_\phi^{(m_1)}. \end{aligned} \quad (3.21)$$

Since  $\mathfrak{P}_\phi^{(m_1, m_2)} \subsetneq \mathfrak{P}_\phi^{(m_1)} \subsetneq \mathfrak{P}_\phi^{(>d_{\max})}$ , we have that  $\{\mathfrak{L}_\phi^{(m_1, m_2)}\}_{m_1, m_2}$ ,  $\{\mathfrak{L}_\phi^{(m_1)}\}_{m_1}$  and  $\mathfrak{L}_\phi^{(>d_{\max})}$  can be solved in this order to give inhomogeneous linear systems of rank  $\{|\mathfrak{J}_\phi^{(m_1, m_2)}|\}_{m_1, m_2}$ ,  $\{|\mathfrak{J}_\phi^{(m_1)}|\}_{m_1}$  and  $|\mathfrak{J}_\phi^{(>d_{\max})}|$  respectively.

Let us summarise what we have done so far in this section: the original Problem 2.1 is equivalent to the  $16 \times 256 = 4096$  numerical inhomogeneous systems of linear equations

$$\begin{aligned} \mathfrak{L}_\phi^{(m_1, m_2)} &: \sum_{\kappa \in \mathfrak{J}_\phi^{(m_1, m_2)}} \left( \mathcal{A}_\phi^{(m_1, m_2)} \right)_{\iota \kappa} N_\kappa^{(k)} = \left( \mathcal{B}_\phi^{(m_1, m_2)} \right)_\iota^{(k)}, \\ \mathfrak{L}_\phi^{(m_1)} &: \sum_{\kappa \in \mathfrak{J}_\phi^{(m_1)}} \left( \mathcal{A}_\phi^{(m_1)} \right)_{\iota \kappa} N_\kappa^{(k)} = \left( \mathcal{B}_\phi^{(m_1)} \right)_\iota^{(k)}, \\ \mathfrak{L}_\phi^{(>d_{\max})} &: \sum_{\kappa \in \mathfrak{J}_\phi^{(>d_{\max})}} \left( \mathcal{A}_\phi^{(>d_{\max})} \right)_{\iota \kappa} N_\kappa^{(k)} = \left( \mathcal{B}_\phi^{(>d_{\max})} \right)_\iota^{(k)}, \end{aligned} \quad (3.22)$$

where, writing  $\aleph$  to indicate any of the symbols  $(m_1, m_2)$ ,  $(m_1)$ , or  $(> d_{\max})$ , the matrices  $\mathcal{A}_\phi^\aleph$  (resp.  $\mathcal{B}_\phi^\aleph$ ) are computed from the r.h.s. (resp. l.h.s.) of (3.18)–(3.20) with  $\chi_j$  (resp.  $\chi_{\wedge^k \mathfrak{e}_8}$  and  $\chi_j$ ) evaluated at points  $Q^{(\iota)} \in \mathcal{T}_{E_8}$  satisfying (3.17). The inhomogeneous piece  $(\mathcal{B}_\phi^\aleph)_\iota^{(k)}$  depends, by (3.18)–(3.20), on all  $N_{\iota'}^{(k)}$  with  $\iota' < \iota$ : therefore, if  $\mathfrak{J} \subset \mathfrak{K}$ ,  $\mathfrak{K} \subset \mathfrak{J}$  are any two elements in the partition (3.14) of  $\mathfrak{J}$ , the corresponding linear problems in (3.22) must be solved sequentially if  $\mathfrak{J} < \mathfrak{K}$ , but they can be solved in parallel if  $\mathfrak{J}$  is incomparable with  $\mathfrak{K}$  under the weak order  $<$ . Furthermore, the determination and solution of (3.22) is compatible with the computability bounds (2.28): writing out  $\mathfrak{L}_\phi^\aleph$  only requires the calculation of  $Q$ -derivatives of  $\chi_{\wedge^k \mathfrak{e}_8}$  up to order  $d_{\max}$ , and solving it involves inverting linear operators  $\mathcal{A}_\phi^\aleph$  of rank  $\ll r_{\max}$ . Moreover, since the latter does not depend on  $k$ , the inversion solves (3.22) for all  $k$  in a single go.

**3.2. The computation of  $\mathfrak{L}_\phi^\aleph$ .** Let us show how the  $\mathfrak{L}_\phi^\aleph$  can be computed in practice. Let  $u^{(\iota)} \in \mathbb{C}^8$  be as in (3.17) and let  $Q^{(\iota)} = Q(u^{(\iota)})$  be any pre-image of  $u^{(\iota)}$ . Given  $u^{(\iota)}$ , the matrices  $(\mathcal{A}_\phi^\aleph)_{\iota \kappa}$  are immediately computed as

$$(\mathcal{A}_\phi^\aleph)_{\iota \kappa} = \prod_j \left( u_j^{(\iota)} \right)^{\kappa_j}. \quad (3.23)$$

The calculation of  $(\mathcal{B}_\phi^\aleph)_l^{(k)}$  is however significantly more involved, as it requires to compute  $\chi_j$ -derivatives of  $\chi_{\wedge^k \mathfrak{e}_8}(Q)$  up to order  $d_{\max}$  at  $Q = Q^{(\iota)}$  for all  $k$ . We describe how this is done in the rest of this section.

3.2.1. *Step 1:  $Q$ -derivatives from  $\Gamma(\omega_j)$ .* The first step is to record the value of all  $Q$ -derivatives of  $\chi_{\wedge^k \mathfrak{e}_8}$  at  $Q = Q(u^{(\iota)})$  for  $\iota \in \mathfrak{J}_\phi^\aleph$  up to a sufficiently high order. To this aim, write

$$\gamma_\aleph = \begin{cases} m_1 + m_2, & \aleph = (m_1, m_2), \\ m_1, & \aleph = m_1, \\ 0, & \aleph > d_{\max}. \end{cases} \quad (3.24)$$

Let

$$\mathcal{C}_k = \left\{ c \in (\mathbb{Z}_0^+)^8 \mid \sum_i c_i = k \right\} \quad (3.25)$$

be the set of compositions of  $k \in \mathbb{Z}^+$  of length eight. Consider the total order on  $\mathcal{C}_k$  obtained by sorting ascendingly with respect to the values of the components  $c_1, \dots, c_8$  taken in this order, and denote  $\mathcal{C}_k^i$ ,  $i = 1, \dots, |\mathcal{C}_k| = \binom{k+7}{7}$  for the  $i^{\text{th}}$  element of  $\mathcal{C}_k$  as a totally ordered set. Then what we want to compute, for any  $c \in \mathcal{C}_{\gamma_\iota}$ , is

$$\mathcal{D}(c, \iota, V) \triangleq D^c \chi_V(Q(u^{(\iota)})) = \frac{\partial^{|c|} \chi_V}{\partial Q_1^{c_1} \dots \partial Q_8^{c_8}}(Q(u^{(\iota)})) = \sum_{w \in \Gamma(V)} \prod_i (w_i - c_i + 1)_{c_i} Q_i^{w_i - c_i} \Big|_{Q=Q(u^{(\iota)})}, \quad (3.26)$$

for  $V = \wedge^k \mathfrak{e}_8$  and  $V = V(\omega_j)$ , where we wrote  $(a)_n$  for the Pochhammer symbol  $\Gamma(a+n)/\Gamma(a)$ .

The sums in (3.26) are unwieldy in that form, since we have up to  $|\Gamma_{\text{red}}(V_{\omega_3})| = 186481$  summands (weight spaces not counted with multiplicities) when computing (3.26) for the fundamental representations  $V = V(\omega_j)$ , and up to  $|\Gamma_{\text{red}}(V_{\wedge^{120} \mathfrak{e}_8})| \approx 10^{17}$  for the exterior powers of the adjoint,  $V = \wedge^k \mathfrak{e}_8$ . It is however possible to reduce the calculation of (3.26) to sums having at most  $|\Gamma_{\text{red}}(\omega_1)| = 2401$ ,  $|\Gamma_{\text{red}}(\omega_7)| = |\Delta^+ \cup \Delta^- \cup \mathbf{0}| = 241$ , or  $|\Gamma_{\text{red}}(\omega_8)| = 26401$  terms, as follows. To compute (3.26) for  $V = \wedge^k \mathfrak{e}_8$ , we first evaluate the virtual power sum characters

$$\tilde{\mathcal{D}}_{k,j}(c, \iota) \triangleq \frac{\partial^{\gamma_\iota} \chi_{\mathfrak{e}_8}}{\partial Q_1^{c_1} \dots \partial Q_8^{c_8}}(Q^k(u^{(\iota)})) = \sum_{w \in \Gamma(\omega_j)} \prod_i (w_i - c_i + 1)_{c_i} Q_i^{kw_i - c_i} \Big|_{Q=Q(u^{(\iota)})}, \quad (3.27)$$

and then recursively compute (3.26) using Newton identities:

$$\mathcal{D}(c, \iota, \wedge^k V(\omega_j)) = \sum_{m=1}^k (-1)^{m+1} \sum_{c'+c''=c} \left( \prod_l \binom{c_l}{c'_l} \right) \tilde{\mathcal{D}}_{m,j}(c', \iota) \mathcal{D}(c'', \iota, \wedge^{k-m} V(\omega_j)). \quad (3.28)$$

for  $j = 7$ . To evaluate the fundamental characters  $\mathcal{D}(c, \iota, V(\omega_i))$ , we compute directly from (3.26) for the three smallest-dimensional fundamental representations corresponding to  $i = 1, 7, 8$ ; for the remaining five, we make use of the following identities in  $\text{Rep}(E_8)$ , which can be easily proved in

the same vein of Example 2.1:

$$V_{\omega_6} = \wedge^2 \mathbf{e}_8 \ominus \mathbf{e}_8, \quad (3.29)$$

$$V_{\omega_5} = \wedge^3 \mathbf{e}_8 \oplus \mathbf{e}_8 \ominus \mathbf{e}_8 \otimes \mathbf{e}_8, \quad (3.30)$$

$$V_{\omega_4} = \wedge^4 \mathbf{e}_8 \oplus \wedge^3 \mathbf{e}_8 \oplus \wedge^2 \mathbf{e}_8 \ominus \wedge^2 \mathbf{e}_8 \otimes \mathbf{e}_8 \ominus V_{\omega_5}, \quad (3.31)$$

$$V_{\omega_3} = \wedge^5 \mathbf{e}_8 \oplus 2 \wedge^4 \mathbf{e}_8 \oplus 2 \wedge^2 \mathbf{e}_8 \ominus \mathbf{e}_8 \ominus \wedge^3 \mathbf{e}_8 \otimes \mathbf{e}_8 \ominus V_{\omega_4}, \quad (3.32)$$

$$V_{\omega_2} = \wedge^2 V(\omega_1) \oplus V_{\omega_1} - V_{\omega_1} \otimes \mathbf{e}_8 + V_{\omega_8}. \quad (3.33)$$

Let now

$$m_j^{\max} = \begin{cases} 2 & j = 1, \\ 5 & j = 7, \\ 1 & j = 8. \end{cases} \quad (3.34)$$

$\mathcal{D}(c, \iota, V(\omega_j))$ ,  $j = 2, \dots, 6$  can then be calculated by (3.29)–(3.33) knowing  $\mathcal{D}(c, \iota, \wedge^{m_j^{\max}} V(\omega_j))$  with  $j = 1, 7, 8$ ; and the latter is computed by (3.26)–(3.28) as an explicit polynomial in  $\tilde{\mathcal{D}}_{k,j}(c, \iota)$ ,  $k = 1, \dots, m_j^{\max}$ , for the same values of  $j$ . All in all, we have proved that there exist explicit polynomials over the rationals such that

$$\begin{aligned} \mathcal{D}(c, \iota, V(\omega_j)) &\in \mathbb{Q} \left[ \left\{ \tilde{\mathcal{D}}_{1,1}(c, \iota), \tilde{\mathcal{D}}_{2,1}(c, \iota), \{\tilde{\mathcal{D}}_{l,7}(c, \iota)\}_{l=1}^5, \tilde{\mathcal{D}}_{1,8}(c, \iota) \right\}_{c \in \mathcal{C}_{\delta_i}} \right], \\ \mathcal{D}(c, \iota, \wedge \mathbf{e}_8) &\in \mathbb{Q} \left[ \left\{ \tilde{\mathcal{D}}_{l,7}(c, \iota) \right\}_{\substack{l=1, \dots, 120 \\ c \in \mathcal{C}_{\delta_i}}} \right]. \end{aligned} \quad (3.35)$$

**3.2.2. Step 2:  $\chi_\omega$ -derivatives, graph expansions and the Faà di Bruno formula.** The second step consists of computing  $\chi_j$ -derivatives of  $\chi_{\wedge \mathbf{e}_8}(g)$  at  $\chi_j(g) = u^{(\iota)}$  from knowledge of its  $Q$ -derivatives at  $Q = Q(u^{(\iota)})$ . To start with, let

$$\mathcal{J}_{ij} \triangleq \left( \frac{\partial \chi_i}{\partial Q_j} \right) \in \text{Mat}(8, \mathbb{C}[Q^\pm]) \quad (3.36)$$

denote the Jacobian matrix of the fundamental characters  $\chi_i$  with respect to the exponentiated linear co-ordinates  $Q_j$ , which we assume here to be non-singular at  $Q(u^{(\iota)})$  for all  $\iota$ . Let again  $\iota \in \mathfrak{I}$ , suppose  $\gamma_\iota > 0$  and let  $c \in \mathcal{C}_{\gamma_\iota - 1}$ . What we want to compute is the numerical matrix

$$\mathcal{J}_{c, \iota}^{\text{inv}} \triangleq \frac{\partial^{|\mathcal{C}|} \mathcal{J}^{-1}}{\partial Q_1^{c_1} \dots \partial Q_8^{c_8}} \Big|_{Q=Q(u^{(\iota)})}. \quad (3.37)$$

Iterated differentiation of

$$\frac{\partial \mathcal{J}^{-1}}{\partial Q_j} = -\mathcal{J}^{-1} \frac{\partial \mathcal{J}}{\partial Q_j} \mathcal{J}^{-1}. \quad (3.38)$$

leads to the following easy

**Proposition 3.1.** *We have*

$$\mathcal{J}_{c, \iota}^{\text{inv}} = \sum_{r=1}^{|\mathcal{C}|} \sum_{d^{(1)} + \dots + d^{(r)} = c} (-1)^r \mathcal{J}_{0, \iota}^{\text{inv}} \prod_{l=1}^r \mathcal{J}_{d^{(l)}, \iota} \mathcal{J}_{0, \iota}^{\text{inv}}, \quad (3.39)$$

where we defined for  $d \leq c \in \mathcal{C}_{\gamma_\iota-1}$

$$\mathcal{J}_{d,\iota} \triangleq \frac{\partial^{|d|} \mathcal{J}}{\partial Q_1^{d_1} \dots \partial Q_8^{d_8}} \Big|_{Q=Q(u^{(\iota)})}. \quad (3.40)$$

On the r.h.s. of (3.39),  $\mathcal{J}_{0,\iota}^{\text{inv}}$  is computed from  $\mathcal{D}(d, \iota, V(\omega_j))$  with  $|d| = 1$ ; the latter gives the numerical expression for the Jacobian matrix  $\mathcal{J}$  at  $Q = Q(u^{(\iota)})$ , and  $\mathcal{J}_{0,\iota}^{\text{inv}}$  is by definition its inverse. And as for  $\mathcal{J}_{d(r),\iota}$ , its entries are just given by values  $\mathcal{D}(d, \iota, V(\omega_j))$  for  $|d| = 1, \dots, \gamma_\iota$ ; therefore  $\mathcal{J}_{c,\iota}^{\text{inv}}$  is computed by (3.39) as an explicit polynomial expression in  $\mathcal{D}(d, \iota, V(\omega_j))$  and  $\mathcal{J}_{0,\iota}^{\text{inv}}$ , which in turn are determined by  $\tilde{\mathcal{D}}_{k,j}(c, \iota)$ ,  $k = 1, \dots, m_j^{\text{max}}$ ,  $j = 1, 7, 8$ , all of which were computed in Step 1 above.

What we will do next, armed with  $\mathcal{J}_{c,\iota}^{\text{inv}}$  computed as in (3.39), is to convert<sup>7</sup> the differential operator  $\mathfrak{D}^c \triangleq \partial_{\chi_1^{c_1} \dots \chi_8^{c_8}}^{|c|}$  into a differential operator in  $(Q_1, \dots, Q_8)$  at  $Q = Q(u^{(\iota)})$ ; at the end of the day this will provide a presentation of the Faà di Bruno formula for the partial derivatives of composite functions in several variables, which appears to be new in this form. We start by the following

**Definition 3.1.** For  $k_L, k_R, n \in \mathbb{N}$ , an  $n$ -decorated Faà di Bruno graph  $\mathbf{G}$  of order  $(k_L, k_R)$  is a decorated, ordered, oriented graph  $(V = V_L \cup V_R, E)$ , satisfying

- (1)  $|V_L| = k_L$ ,  $|V_R| = k_R \leq k_L$ ;
- (2)  $V_L = \{v_1^{(L)}, \dots, v_k^{(L)}\}$  with  $v_i^{(L/R)} \geq v_j^{(L/R)} \Leftrightarrow i \leq j$ ,  $v_i^{(L)} \leq v_j^{(R)} \forall i, j$ ;
- (3) every vertex in  $V_L$  ( $V_R$ ) has exactly one outgoing (incoming) oriented edge attached to it;
- (4) any  $v \in V_R$  is a leaf;
- (5) if  $v_i^{(L)}$  is adjacent to  $v_j^{(R)}$ ,  $v_s^{(L)}$  is adjacent to  $v_t^{(R)}$ , and  $i < s$ , then  $j < t$ ;
- (6) if  $v_i^{(L)}$  is adjacent to  $v_j^{(L)}$ , and  $i > j$ , then the attaching edge is oriented from  $j$  to  $i$ ;
- (7)  $i : V_L \rightarrow \{1, \dots, n\}$ .

A helpful way to visualise the definition above is the following (see Figure 2): the datum of  $\mathbf{G}$  gives two sets of vertices,  $V_L$  and  $V_R$ , which we can arrange in two vertical columns in the plane; the number of left vertices is greater than that of right vertices (Property 1,2). Each left vertex has exactly one outgoing arrow emanating from it (Property 3), which can be of either of two types: a vertical arrow ending on another left vertex higher up in the column (Property 6), or a horizontal arrow ending on a right vertex; horizontal arrows are not allowed to cross (Property 5). Left vertices are decorated with a positive integer less than or equal to  $n$ .

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<sup>7</sup>Stated more invariantly, we want to look at the co-ordinate expression, in the chart induced by the exponentiated linear co-ordinates  $Q$  on  $\mathcal{T}_{E_8}$ , of  $\mathfrak{D}^c$  as a map  $J_{u^{(\iota)}}^{|c|} \mathcal{T}_{E_8} \rightarrow \mathbb{C}$  from the fibre at  $Q = Q(u^{(\iota)})$  of the  $|c|^{\text{th}}$  jet bundle on  $\mathcal{T}_{E_8}$  to  $\mathbb{C}$ .

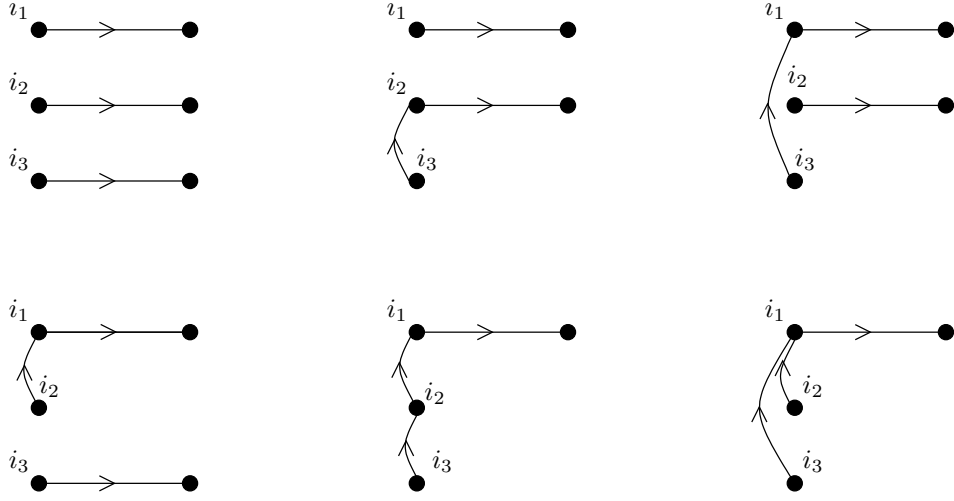


FIGURE 2. The Faà di Bruno graphs for  $k_L = 3$ .

We denote  $\text{FG}_{k_L, k_R, n}$  the set of  $n$ -decorated Faà di Bruno graphs of order  $(k_L, k_R)$ . The graphs for  $k_L = 3$  are depicted in Figure 2; in general, there are  $k_L!$  un-decorated graphs with fixed value of  $k_L$ .

**Proposition 3.2** (Multi-variate Faà di Bruno formula). *Let  $\delta > 0$ ,  $x_0 \in \mathbb{R}^n$ ,  $k_L \in \mathbb{N}$ , and  $(f_l)_l \in C^{k_L}(B_\delta(x_0))$  ( $l = 1, \dots, n$ ), and suppose  $\det \mathcal{J} \neq 0$  with  $\mathcal{J}_{ij} = \partial_{x_i} f_j(x_0)$  in local co-ordinates  $x_i$  around  $x_0$ . Then, for  $\epsilon := \sup_{x \in (B_\delta(x_0))} \|f(x) - f(x_0)\|$  and  $F \in C^{k_L}(B_\epsilon(f(x_0)))$ , we have*

$$\frac{\partial^{k_L} F}{\partial f_{i_1} \dots \partial f_{i_{k_L}}} = \sum_{\substack{k_R \leq k_L \\ \mathbf{G} \in \text{FG}_{k_L, k_R, n}}} \sum_{j_1, \dots, j_{k_L} = 1}^n \prod_{r=1}^{k_L} \left( \prod_{v_l^{(L)} \in \text{In}(v_r^{(L)})} \partial_{x_{j_l}} \right) \mathcal{J}_{i_r, j_r}^{-1} \prod_{s=1}^{k_R} \left( \prod_{v_m^{(R)} \in \text{In}(v_s^{(R)})} \partial_{x_{j_m}} (F \circ f) \right) \quad (3.41)$$

where for  $v \in V(\mathbf{G})$ ,  $\mathbf{G} \in \text{FG}_{k_L, k_R, n}$ , we indicate by  $\text{In}(v) \subset V_L$  as the set of tails of arrows having  $v$  as their head.

*Proof.* The claim follows by keeping track of the combinatorics of the Leibniz formula for the composition of linear differential operators of the form  $\partial_{f_i} = \sum_j (\mathcal{J})_{ij}^{-1} \partial_{x_j}$ : this can be encoded into graphical rules that boil down to the properties of Faà di Bruno graphs in Definition 3.1. Consider the order  $k_L$  differential operator

$$\frac{\partial^{k_L}}{\partial f_{i_1} \dots \partial f_{i_{k_L}}} \Big|_{f(x_0)} = \prod_{l=1}^{k_L} \frac{\partial}{\partial f_{i_l}} \Big|_{f(x_0)} = \sum_{j_1, \dots, j_{k_L}} \tilde{\mathcal{J}}_{i_1, j_1, \dots, i_{k_L}, j_{k_L}}, \quad (3.42)$$

where

$$\tilde{\mathcal{J}}_{i_1, j_1, \dots, i_{k_L}, j_{k_L}} \triangleq \prod_{l=1}^{\overrightarrow{k_L}} \left( \mathcal{J}_{i_l j_l}^{-1} \frac{\partial}{\partial x_{j_l}} \right) \Big|_{x_0} \quad (3.43)$$

and the order of the product in (3.43) has been chosen from left to right (the sum over  $j_l$  in (3.42) yet making any particular choice of ordering immaterial, as the l.h.s. is obviously symmetric in

$i_1, \dots, i_{k_L}$ ). To each factor in the ordered product (3.43) we associate a vertex  $v_l^{(L)}$ ,  $l = 1, \dots, k_L$  in the plane, and we align the set of vertices vertically and order them from bottom to top corresponding to the rightwards order in (3.43). Then for each factor in (3.43) is a linear differential operator  $\mathcal{J}_{i_l j_l}^{-1} \frac{\partial}{\partial x_{j_l}}$  which can act on the factors to its right in either of the two following ways:

- it can hit  $\mathcal{J}_{i_m j_m}^{-1}$ ,  $m > l$ : in that case we draw an oriented edge from  $v_l^{(L)}$  to  $v_m^{(L)}$ ;
- it can move across all Jacobian factors to its right; in that case we draw a leaf  $v_{\sigma_l}^{(R)}$  next to  $v_l^{(L)}$  to its right, with an oriented edge from the latter to the former. The labels  $\sigma_l$  for the right vertices  $v_{\sigma_l}^{(R)}$  are chosen to respect the order of the left vertices and to take consecutive values from 1 to  $k_R$  for some positive integer  $k_R \neq k_L$ .

Each of these possibilities defines a Faà di Bruno graph with a decoration of the left vertices by  $i_l$ , the row index of  $\mathcal{J}_{i_l j_l}^{-1}$ . The Leibniz rule states that (3.43) reduces to a sum over all these possibilities, each giving an order  $k_R$  differential operator whose coefficients are products of derivatives of  $\mathcal{J}^{-1}$ , with the orders of differentiation of the factors summing up to  $k_L - k_R$ . Plugging the resulting sum over  $k_R$  and  $\text{FG}_{k_L, k_R, n}$  into (3.42) and acting on  $F \in C^{k_L}(B_\epsilon(f(x_0)))$  gives (3.41).

□

**Corollary 3.3.** *We have*

$$(\mathcal{B}_\phi^{\mathbb{N}})^{(k)} \in \mathbb{Q} \left[ \left\{ \tilde{\mathcal{D}}_{1,1}(c, \iota), \tilde{\mathcal{D}}_{2,1}(c, \iota), \{\tilde{\mathcal{D}}_{l,7}(c, \iota)\}_{l=1}^{\max(k,5)}, \tilde{\mathcal{D}}_{1,8}(c, \iota) \right\}_{c \in \mathcal{C}_{\delta_i}}, \mathcal{J}_{0,\iota}^{\text{inv}} \right] \quad (3.44)$$

This follows immediately from Propositions 3.1 and 3.2 and the description of Step 1 in Section 3.2.1, which give explicit expressions for the  $\mathbb{Q}$ -polynomials on the r.h.s.. It should be noted that  $\mathcal{J}_{0,\iota}^{\text{inv}}$  is also determined, albeit not polynomially, by the basic building blocks  $\{\tilde{\mathcal{D}}_{l,m_l^{\max}}(c, \iota)\}_{j=1,7,8}$ .

**3.3. Computing  $(\mathcal{B}_\phi^{\mathbb{N}})^{(k)}$  exactly.** There are still quite a few more hurdles yet to overcome in writing down (3.22), as follows.

- (1) The main problem is that the whole method of the previous section hinges on finding an (arbitrary, but exact) pre-image  $Q(u^\iota)$  of  $\chi_j(Q) = u_j^\iota$  in (3.17). This requires solving a non-linear system of eight polynomial equations of high degree in eight variables, which cannot be achieved analytically but for very special values of  $u_j^\iota$ .
- (2) Of course, as in **Method 3**, one could try to solve (3.17) for  $Q^{(\iota)} = Q(u^{(\iota)})$  numerically. The resulting linear problems (3.22) will then have numerically approximated entries, and one can then hope to determine  $N_i^{(k)} \in \mathbb{Z}$  by rounding up the numerical solution of (3.22) to the nearest integer – and with appropriate analytic bounds on the propagation of the numerical error, this rounding will be exact. But even then, the density and ill-conditioning of  $\mathcal{L}_\phi^{\mathbb{N}}$  rule out the use of floating-point arithmetics in its solution: the absolute value of the minimal principal value of  $\mathcal{A}_\phi^{\mathbb{N}}$  can be of order as little as  $\approx 10^{-5000}$ , meaning that tiny

rounding errors in the numerical evaluation of  $\chi_{\wedge^k \epsilon_8}$  lead to massive errors  $\gg \mathcal{O}(1)$  after solving for  $N_\iota^{(k)}$  in  $\mathfrak{L}_\phi^{\mathbb{N}}$ , making an exact rounding to the nearest integer unfeasible.

- (3) Finally, one should be aware that the choice of points  $Q^{(\iota)} \in \mathcal{T}_{E_8}$  in (3.17) could potentially lead to a singular  $\mathcal{A}_\phi^{\mathbb{N}}$  in (3.22). This might appear to be an issue of lesser importance, as generic choices in (3.17) lead to non-degenerate linear problems; however, as we will see momentarily, to deal with (1)-(2) above we will have to consider a highly constrained choice of sampling values  $Q^{(\iota)}$ , so this point will become relevant as well.

The main problem to deal with is that we need to be able to a) write and b) solve the linear problems (3.22) exactly. It turns out that **Methods 2-4** can be used to achieve both of these objectives.

**3.3.1. Exactly rounded floating point arithmetics.** For the latter point, our strategy is to consider sampling values  $(Q_\iota^{(\iota)})_\iota \in \mathcal{T}_{E_8}$  such that

$$\chi_j(Q^{(\iota)}) = u_j^{(\iota)} \in \varphi_j(\iota) \left\{ \begin{array}{ll} \delta_{j,\sigma(\phi,1)} \delta_{j,\sigma(\phi,2)}, & \iota \in \mathfrak{J}_\phi^{(m_1, m_2)}, \\ \delta_{j,\sigma(\phi,1)}, & \iota \in \mathfrak{J}_\phi^{(m_1)}, \\ 1, & \iota \in \mathfrak{J}_\phi^{(>d_{\max})}, \end{array} \right\} \mathfrak{S} \quad (3.45)$$

where  $\mathfrak{S} \subset (\mathbb{Q} + i\mathbb{Q})$ . Then, since  $N_\iota^{(k)} \in \mathbb{Z}$ , the real part of  $\mathfrak{L}_\phi^{\mathbb{N}}$  becomes a linear system over  $\mathbb{Q}$  (or  $\mathbb{Z}$ ), which can be solved using exact arithmetics, provided we can find an exact pre-image of  $u_j^{(\iota)}$  in (3.45) – which as explained above we cannot do analytically in general. What we will do instead will be to find an approximate solution  $(\hat{Q}^{(\iota)}, \delta_{\hat{Q}^{(\iota)}})$  with  $Q^{(\iota)}$  contained in a ball of radius  $\delta_{\hat{Q}^{(\iota)}}$  from  $\hat{Q}^{(\iota)}$ . We will then compute from that a pair  $((\hat{\mathcal{B}}_\phi^{\mathbb{N}})^{(k)}, \delta_{\hat{\mathcal{B}}_\phi^{\mathbb{N}}(k)})$  out of which the exact value of  $(\mathcal{B}_\phi^{\mathbb{N}})_\iota^{(k)} \in \mathbb{Q} + i\mathbb{Q}$  can be uniquely and exactly determined.

We get started with the following

**Definition 3.2.** Let  $M \in \mathbb{N}$ . We denote

$$\text{FPC}_M = \left\{ x \in \mathbb{C} \mid \text{Mant}_2(\Re(x)) = \sum_{k=1}^M a_k 2^{-k}, \text{Mant}_2(\Im(x)) = \sum_{k=1}^M b_k 2^{-k} \right\}, \quad (3.46)$$

the set of complex floating-point numbers with  $M$ -bit precision, where  $a_k, b_k \in \{0, 1\}$ , and in (3.46) we denoted

$$\text{Mant}_2(x) = 2^{-\text{Exp}_2(x)} x, \quad \text{Exp}_2(x) = \lceil \log_2 x \rceil. \quad (3.47)$$

We shall also write

$$\varepsilon_x := 2^{\text{Exp}_2(x) - M}. \quad (3.48)$$

**Assumption 3.1.** Throughout this section and in all expressions in Step 1 and 2 leading up to the calculation of  $(\mathcal{B}_\phi^{\mathbb{N}})_\iota^{(k)}$  in Corollary 3.3, we suppose that

- $(Q^{(\iota)})_\iota \in \text{FPC}_M$ ;

- any expression of the form  $y = f(Q^{(\iota)})$  for a complex-valued function  $f : \mathbb{C} \rightarrow \mathbb{C}$  will be shorthand for the composition of  $\text{rnd}_M \circ f$  where  $\text{rnd}_M : \mathbb{C} \rightarrow \text{FPC}_M$  is the rounding to the nearest element in  $\text{FPC}_M$  in absolute value.

In other words, we assume to be able to compute a floating-point expression for  $(\mathcal{B}_\phi^{\times})_\iota^{(k)} \in \text{FPC}_M$  in Corollary 3.3 starting with  $(Q^{(\iota)})_l \in \text{FPC}_M$  with *exact roundings* throughout. On a computer implementation of Step 1 and 2, this can be carried out by working if necessary with internal precision greater than  $M$  in the calculation of functions  $f(x)$  for  $x \in \text{FPC}_M$ , and then rounding-off-to nearest when storing the value of  $y = f(x)$  as an  $M$ -bit complex floating point. We have used the GNU MPC library [10] (building on the GNU GMP/MPFR multi-precision libraries [13, 16]) to achieve this.

**3.3.2. Newton–Raphson inversion.** Let now  $u^{(\iota)}$  be as in (3.45) and suppose we want to find an approximate pre-image  $Q^{(\iota)} \in \text{FPC}_M$  of  $u^{(\iota)} = \chi_j(Q)$  with  $M$ -bit precision. We can do this via the following adaptation of the Newton–Raphson method.

**Algorithm 3.1.** Suppose let  $Q_0 = \exp(t(h_0))$  with  $h_0$  given in (2.24), and consider the double sequence of linear systems

$$\sum_j \mathcal{J}_{ij}(Q^{(\iota, k-1, l/n)})(Q^{(\iota, k, l/n)} - Q^{(\iota, k-1, l/n)}) = \chi_i(Q^{(\iota, k-1, l/n)}) - \frac{lu_i^{(\iota)}}{n}, \quad k, n, l \geq 1. \quad (3.49)$$

For all  $\iota$  set  $Q^{(\iota, 0, 1)} = Q_0$ ,  $n = 1$ ,  $l = 1$ , and iterate the following:

- assuming  $\det \mathcal{J}_{ij}(Q^{(\iota, k-1, l/n)}) \neq 0$ , solve (3.49) for  $Q^{(\iota, k, n)}$  recursively in  $k$ ;
- if there exists  $k_{l/n} \in \mathbb{N}$  such that  $|Q^{(\iota, k_{l/n}, l/n)} - Q^{(\iota, k_{l/n}-1, l/n)}| < \varepsilon_{Q^{(\iota, k_{l/n}, l/n)}}$ , we let  $Q^{(\iota, 0, (l+1)/n)} := Q^{(\iota, k_{l/n}, l/n)}$  and repeat the previous step. If there is no such  $k_{l/n}$ , let  $n \rightarrow n + 1$ ,  $l \rightarrow \lfloor (n + 1)l/n \rfloor$  and repeat the previous step.

The adaptive re-scaling of the value of  $u^{(\iota)}$  on the r.h.s. of (3.49) ensures that the process converges to  $Q^{(\iota)} := Q^{(\iota, k_1, 1)} \in \text{FPC}_M$ , such that the  $\varepsilon_{Q^{(\iota)}}$ -ball  $B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})$  contains at least one exact pre-image of  $u_i^{(\iota)} = \chi_i(Q)$ .

**3.3.3. Bounds and exact roundings.** In the following, for any  $x \in \text{FPC}_M$  and any symbol  $\delta_x$ , write  $\hat{\delta}_x := \delta_x + \varepsilon_x$ . The first question we would like to ask is what kind of bounds can be put on the error in the computation of  $\hat{D}_{i,j}(c, \iota)$  under Assumption 3.1 given the rounding error  $\varepsilon_{Q^{(\iota)}}$  in the

calculation of  $Q^{(\iota)}$ . For  $c = 0$ , we could bound this a priori by  $\delta_{\tilde{\mathcal{D}}_{i,j}(0,\iota)}(\varepsilon_{Q^{(\iota)}})$  defined as

$$\begin{aligned} \sup_{Q', Q'' \in B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})} \left| \chi_i(Q'^j) - \chi_i(Q''^j) \right| &\leq \sup_{Q \in B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})} \left| \chi_i(Q^j) \right| - \inf_{Q \in B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})} \left| \chi_i(Q^j) \right| \\ &\leq \sum_{w \in \Gamma(\omega_i)} \left[ \prod_i \left( |Q_i^{(\iota)}|^{jw_i} + (-1)^{\text{sign}w_i} \epsilon^{jw_i} \right) - \prod_i \left( |Q_i^{(\iota)}|^{jw_i} - (-1)^{\text{sign}w_i} \epsilon^{jw_i} \right) \right] \\ &=: \delta_{\tilde{\mathcal{D}}_{i,j}(0,\iota)}(\varepsilon_{Q^{(\iota)}}) - \varepsilon_{\tilde{\mathcal{D}}_{i,j}(0,\iota)}(\varepsilon_{Q^{(\iota)}}) \end{aligned} \quad (3.50)$$

where the second line is obtained by repeated use of the triangle inequality, and the last line defines  $\delta_{\tilde{\mathcal{D}}_{i,j}(0,\iota)}$  as the sum of the theoretical error bound on the l.h.s. and a rounding error term due to the floating-point round-off. With  $c \neq 0$ , we can use the fact that the real and imaginary parts of  $\chi_i(Q)$  are harmonic in  $Q$  to obtain a uniform bound for the the order- $|c|$  derivatives. Recall that if  $f \in C^\infty(\Omega \subset \mathbb{R}^n)$  is harmonic in  $\Omega$ , and  $\Omega' \subsetneq \Omega$  is compact, we have (see for example [14, Thm 2.10])

$$\sup_{\Omega'} |D^c f| \leq \left( \frac{n|c|}{\text{dist}(\Omega', \partial\Omega)} \right)^{|c|} \sup_{\Omega} |f|. \quad (3.51)$$

Using (3.51) with  $\Omega' = B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})$ ,  $\Omega = B_{2\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})$ , we get

$$\begin{aligned} \sup_{Q', Q'' \in B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})} \left| D^c \chi_i(Q'^j) - D^c \chi_i(Q''^j) \right| &\leq (64|c|)^{|c|} \delta_{\tilde{\mathcal{D}}_{i,j}(0,\iota)}(2\varepsilon_{Q^{(\iota)}}) \\ &=: \delta_{\tilde{\mathcal{D}}_{i,j}(c,\iota)}(\varepsilon_{Q^{(\iota)}}) - \varepsilon_{\tilde{\mathcal{D}}_{i,j}(c,\iota)}(\varepsilon_{Q^{(\iota)}}) \end{aligned} \quad (3.52)$$

where once again we included in the definition of the error bound  $\delta_{\tilde{\mathcal{D}}_{i,j}(c,\iota)}$  a contribution accounting for the rounding in FPC<sub>M</sub>. From this we recursively get a bound on the error terms in an exactly-rounded FPC evaluation of the antisymmetric characters of  $V(\omega_j)$ ,

$$\begin{aligned} \delta_{\mathcal{D}(c,\iota,\wedge^{k-m}V(\omega_j))} - \varepsilon_{\mathcal{D}(c,\iota,\wedge^{k-m}V(\omega_j))} &:= \sum_{m=1}^k \sum_{c'+c''=c} \left( \prod_l \binom{c_l}{c'_l} \right) \left( |\tilde{\mathcal{D}}_{m,j}(c',\iota)| \delta_{\mathcal{D}_m(c'',\iota,\wedge^{k-m}V(\omega_j))} \right. \\ &\quad \left. + |\mathcal{D}(c'',\iota,\wedge^{k-m}V(\omega_j))| \delta_{\tilde{\mathcal{D}}_{m,j}(c',\iota)} + 2^{-M} \right) \\ &\geq \sup_{Q', Q'' \in B_{\varepsilon_{Q^{(\iota)}}}(Q^{(\iota)})} \left| D^c \chi_{\wedge^k \varepsilon_8}(Q') - D^c \chi_{\wedge^k \varepsilon_8}(Q'') \right| \end{aligned} \quad (3.53)$$

where we assumed (and can verify a posteriori) that  $\delta_{\tilde{\mathcal{D}}_{m,j}(c',\iota)} \delta_{\mathcal{D}(c'',\iota,\wedge^{k-m}V(\omega_j))} \leq 2^{-M}$ .

Now, by Propositions 3.1 and 3.2 and (3.35), we have that

$$\mathfrak{D}^c \chi_{\wedge^k \varepsilon_8} = \mathcal{L}_{c,\iota}((D^d \chi_{\wedge^k \varepsilon_8})_{|d| \leq |c|}) \quad (3.54)$$

for a linear functional  $\mathcal{L}_{c,\iota}$  independent on  $k$ : this linear operator encodes the change-of-variable when writing the differential operator  $\mathfrak{D}_c = \partial_{\chi_1^{c_1} \dots \chi_8^{c_8}}^{|c|}$  in exponentiated linear co-ordinates  $Q$  on  $\mathcal{T}_{E_8}$  (Proposition 3.2), which gives a linear combination of  $D^d$  at  $Q^{(\iota)}$  with  $|d| \leq |c|$ . So to figure out how the  $\varepsilon$ -error in  $Q^{(\iota)}$  propagates to the l.h.s. of (3.54), we need to estimate a bound for the

2-norm of  $\mathcal{L}_{c,\ell}$ ,

$$\begin{aligned} \delta_{\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}} - \varepsilon_{\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}} &:= \|\mathcal{L}_{c,\ell}\|_2 \|(\delta_{D^d \chi_{\wedge^k \epsilon_8}})_{|d| \leq |c|}\|_2 \\ &\geq \sup_{Q', Q'' \in B_{\varepsilon_{Q^{(\ell)}}}(Q^{(\ell)})} \left| \mathcal{L}_{c,\ell} \left( D^d \chi_{\wedge^k \epsilon_8}(Q') - D^d \chi_{\wedge^k \epsilon_8}(Q'') \right) \right|. \end{aligned} \quad (3.55)$$

This can be done as follows: from Proposition 3.1, we have

$$\|\mathcal{J}_{c,\ell}^{\text{inv}}\|_2 \leq \sum_{r=1}^{|c|} \sum_{d^{(1)} + \dots + d^{(r)} = c} \|\mathcal{J}_{0,\ell}^{\text{inv}}\|_2 \prod_{l=1}^r \|\mathcal{J}_{d^{(l)},\ell}\|_2 \|\mathcal{J}_{0,\ell}^{\text{inv}}\|_2 \quad (3.56)$$

and we can place upper limits for the summands by bounding each operator 2-norm in  $\text{Mat}(8, \mathbb{C})$  with the Frobenius norm,  $\|\cdot\|_2 \rightarrow \|\cdot\|_F$ , on which upper bounds can be straightforwardly found in terms of  $u_i^{(\ell)}$  by using (3.51):

$$\|\mathcal{J}_{c,\ell}^{\text{inv}}\|_2 \leq \sum_{r=1}^{|c|} \sum_{d^{(1)} + \dots + d^{(r)} = c} \|\mathcal{J}_{0,\ell}^{\text{inv}}\|_F \prod_{l=1}^r \|\mathcal{J}_{d^{(l)},\ell}\|_F \|\mathcal{J}_{0,\ell}^{\text{inv}}\|_F, \quad (3.57)$$

$$\|\mathcal{J}_{d,\ell}\|_F^2 \leq 816|d+1|^{|d|+1} \left[ \sum_j \left( |u_j^{(\ell)}| + \delta_{\mathcal{D}(0,\ell,V(\omega_j))}(2\varepsilon_{Q^{(\ell)}}) \right)^2 \right] \quad (3.58)$$

$$\|\mathcal{J}_{0,\ell}^{\text{inv}}\|_F^2 \leq 128 \left[ \sum_j \left( |Q_j^{(\ell)}| + 2\varepsilon_{Q^{(\ell)}} \right)^2 \right]. \quad (3.59)$$

Suppose that  $\|\mathcal{J}_{0,\ell}^{\text{inv}}\|_F > 1$ . Then by Proposition 3.2 and (3.57)–(3.59), the largest contribution to  $\|\mathcal{L}_{c,\ell}\|_2$  comes from the Faà di Bruno graph with a single horizontal arrow emanating from the top-left vertex, and all vertical arrows ending on it. Therefore,

$$\begin{aligned} \|\mathcal{L}_{c,\ell}\|_2 &\leq |c|! \sum_{r=1}^{|c|} 2^{\frac{17}{2}r} \left[ \sum_j \left( |Q_j^{(\ell)}| + 2\varepsilon_{Q^{(\ell)}} \right)^2 \right]^{r/2} \left[ \sum_j \left( |u_j^{(\ell)}| + \delta_{\mathcal{D}(0,\ell,V(\omega_j))}(2\varepsilon_{Q^{(\ell)}}) \right)^2 \right]^{r/2} \\ &\times \sum_{d^{(1)} + \dots + d^{(r)} = c} \prod_{l=1}^r 16 |d^{(l)} + 1|^{|d^{(l)}|+1}/2}, \end{aligned} \quad (3.60)$$

placing accordingly a bound on  $\delta_{\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}}$  in (3.55).

The main question now is whether it is possible to choose judiciously the sampling set  $\mathfrak{S} \subset \mathbb{Q} + i\mathbb{Q}$  in (3.45) for  $\{u^{(\ell)}\}_{\ell \in \mathcal{J}}$  such that we can employ the bounds (3.55)–(3.60) to control the size of the error  $\delta_{\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}}$ , and hence rigorously round  $\chi_{\wedge^k \epsilon_8}(Q^{(\ell)})$  to its exact value as a complex number with rational real and imaginary parts. One naïve possibility is to choose  $\mathfrak{S} \subset \mathbb{Z}$ , which would lead to  $\chi_{\wedge^k \epsilon_8}(Q^{(\ell)}) \in \mathbb{Z}$ ; however one immediately faces the problem that that even for small integer values of  $u_i^{(\ell)}$ , the values of the power sums  $\tilde{D}_{k,j}(c,\ell)$  may be as large as  $\approx 10^{40}$  for large  $k$ . This implies that the loss of significant digits coming from the round-off errors  $\varepsilon_{\tilde{D}_{k,j}(c,\ell)}$  will be correspondingly large, and lead eventually to a very poorly controlled bound on  $\delta_{\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}}$ . The above indicates that to compute  $(\mathcal{B}_{\phi}^{\mathfrak{N}})$  exactly we need to

- pick values in  $\mathfrak{S}$  that are sufficiently close to zero, so that the growth of the values of  $\mathcal{D}_{k,j}(c, \iota)$  and  $\mathcal{D}(c, \iota, \wedge^k \omega_j)$  is under control;
- have sufficiently many sample points, with  $|\mathfrak{S}|$  large enough to ensure that  $\mathcal{A}_\phi^\mathfrak{N}$  is always non-singular<sup>8</sup>;
- find an a priori reliable prescription to round the final values for  $\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}$  to a number in  $\mathbb{Q} + i\mathbb{Q}$ .

A particularly convenient choice is to take

$$\mathfrak{S} := \left\{ \frac{l}{2} + i\frac{m}{2}, l \in \{-3, \dots, 3\}; m \in \{-2, \dots, 2\} \right\}, \quad (3.61)$$

so that  $|\mathfrak{S}| = 40$ . Then for  $u_j^{(\iota)} \in \mathfrak{S}$ , we would expect to have

$$\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}(Q^{(\iota)}) \in 2^{-|\iota^{\max}|}(\mathbb{Z} + i\mathbb{Z}) \quad (3.62)$$

from (2.5) and (2.26). The bounds computed in the previous section can be combined together to prove the following

**Theorem 3.4.** *For any  $(\phi, \mathfrak{N})$ , there exists a choice of values  $\{u_j^{(\iota)}\}_{\iota \in \mathfrak{J}}$  in (3.45), with*

$$\mathfrak{S} = \begin{cases} \{-e_k, \dots, e_k - 1, e_k\}, & \phi_k = |\phi| = 1, \\ \left\{ \frac{l}{2} + i\frac{m}{2} \right\}_{\substack{l=-3, \dots, 3 \\ m=-2, \dots, 2}}, & |\phi| > 1 \end{cases} \quad (3.63)$$

and of approximate pre-images  $\{Q_j^{(\iota)} \in \text{FPC}_M\}_{\iota \in \mathfrak{J}}$  satisfying the following:

- in (3.23),  $\det \Re \mathcal{A}_\phi^\mathfrak{N} \neq 0$  for all  $(\phi, \mathfrak{N})$ ;
- $0 \neq |\det \mathcal{J}(Q^\iota)| \in \text{FPC}_M$ ,  $\|\mathcal{J}^{\text{inv}}\|_F$  in (3.36);
- for  $M \geq M_0 := 494$ , we have that

$$2^{|\iota^{\max}|} \delta_{\mathfrak{D}^c \chi_{\wedge^k \epsilon_8}} < \frac{1}{2}. \quad (3.64)$$

*Proof.* We can prove the first two points by exhibiting a choice of values  $\{(u, Q)^{(\iota)}\}_{\iota \in \mathfrak{J}}$ , where  $Q^{(\iota)}$  is found with Algorithm 3.1; we claim that one such can be found, and its explicit form for all  $\iota$  is available upon request. The third point follows from evaluating the analytic bounds (3.50)–(3.60) for the set of pre-images  $Q^{(\iota)}$  thus found; we refer the reader to the ancillary files described in Appendix A for the calculation of all the relevant quantities.  $\square$

Let us pause to rephrase the content of Theorem 3.4 once more. The theorem states that there exists a choice of  $u^{(\iota)}$  compatible with (3.45) and (3.61) such that  $\Re \mathcal{A}_\phi^\mathfrak{N}$  is a non-singular linear system over  $\mathbb{Z}$  for all  $(\phi, \mathfrak{N})$ ; and furthermore, there exists a sufficiently large integer  $M_0$  such that, denoting  $Q^{(\iota)} \in \mathbb{C}$  and  $\widehat{Q}^{(\iota)} \in \text{FPC}_{M_0}$ , respectively, a pre-image of  $u^{(\iota)}$  in (3.45) and an  $M$ -bit precision

<sup>8</sup>An immediate lower bound here is  $|\iota^{\max}| = 31$ , corresponding to  $\phi_j = \delta_{j7}$ : this is the system computing coefficients of monomials of the form  $\chi_7^k$ ,  $k = 1, \dots, |\iota^{\max}| = 31$ .

approximation thereof, the value of  $\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}(\widehat{Q}^{(\iota)}) \in \text{FPC}_{M_0}$  computed under Assumption 3.1 with  $M \geq M_0$  must satisfy

$$2^{|\iota|^{\max}} \mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}(Q^{(\iota)}) \in B_{\frac{1}{2}}(\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}(\widehat{Q}^{(\iota)})), \quad (3.65)$$

by the analytic bounds (3.50)–(3.60). Now, the l.h.s. is a point in the two-dimensional lattice  $\mathbb{Z}^2 \simeq \mathbb{Z} + i\mathbb{Z}$ ; and the Theorem constrains it to lie in a disk of radius  $< 1/2$  centred at  $\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}(\widehat{Q}^{(\iota)})$ : so it's the *unique* (if at all existent) integer lying in that disk. Therefore, the exact value of  $\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}(Q^{(\iota)})$  is determined by  $\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}(\widehat{Q}^{(\iota)})$  computed from Step 1 and 2 under Assumption 3.1. This concludes the calculation of  $\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}|_{\chi_j = u_j^{(\iota)}}$ , and thus  $(\mathcal{B}_\phi^\aleph)_{\iota}^{(k)}$ , from (3.45).

**3.4. Solving exactly for  $N_\iota^{(k)}$ .** Theorem 3.4 allows us to write the exact form of  $\mathfrak{L}_\phi^\aleph$  subordinate to a choice of sampling set (3.45) and (3.61); all is left to do to provide the solution of Problem 2.1 is to solve explicitly for  $N_\iota^{(k)}$  the linear problem  $\mathfrak{L}_\phi^\aleph$  for all  $\phi$  and  $\aleph$ . By Theorem 3.4, and since  $N_\iota^{(k)} \in \mathbb{Z}$ , we just need to consider the real part of  $\mathfrak{L}_\phi^\aleph$ , which is a linear system over  $\mathbb{Q}$ . This can be solved exactly using **Method 4** by Dixon's  $p$ -adic lifting [7]: we fix a prime  $p$ , invert  $\Re \mathcal{A}_\phi^\aleph \bmod p$  by LU decomposition, and solve for  $N_\iota^{(k)} \bmod p^Q$  (for  $Q$  large enough). This allows to reconstruct  $N_\iota^{(k)}$  from its  $p$ -adic expansion in  $\mathcal{O}(r^3 \log^2 r)$  time, where  $r = |\mathfrak{J}_\phi^\aleph|$ . For this computation, the `fmpz_mat` module of the FLINT (Fast Library for Number Theory) C library has been systematically employed; see [17] and references therein for details.

**3.5. Implementation: runtime estimates and distributed computation.** An estimate of the runtime required can be illustrated by the following two tables. For the first six rows, the operations in Table 2 are in  $M + 1 = 495$ -bit precision with correct rounding-to-nearest, whereas in the last row multi-precision rational arithmetic is used. The first five rows are for a *fixed*  $\iota \in \mathfrak{J}_\phi^\aleph$  and *all*  $c \in \mathcal{C}_{\gamma_\aleph}$ ; the last two indicate estimates for all  $\iota \in \mathfrak{J}_\phi^\aleph$  for given  $(\phi, \aleph)$ .

Operation	Walltime estimate
$Q = Q(u^{(\iota)})$ (Algorithm 3.1)	$\approx 10$ sec
$\mathcal{D}(c, \iota, \wedge^k \mathfrak{e}_8)$ , (3.27)–(3.28)	$\approx \binom{ c +8}{8}$ 5 sec
$\mathcal{D}(c, \iota, V(\omega_i))$ , (3.27)–(3.33)	$\approx \binom{ c +8}{8}$ 1 sec
$\mathcal{J}_{c, \iota}^{\text{inv}}$ (3.39)	$\approx \binom{ c +8}{8}$ .1 sec
$\mathfrak{D}_c$ (Definition 3.1, (3.41))	$\approx \binom{ c +8}{8}$ .1 sec
$\{\mathfrak{D}_c \chi_{\wedge^k \mathfrak{e}_8}(u^{(\iota)})\}_{\iota \in \mathfrak{J}_\phi^\aleph}$	$\approx \binom{ c +8}{8}  \mathfrak{J}_\phi^\aleph  \times 5 \times 10^{-2}$ sec
$N_\iota^{(k)}$	$\approx  \mathfrak{J}_\phi^\aleph ^3 \log^2  \mathfrak{J}_\phi^\aleph  \times 10^{-5}$ sec

TABLE 2. Wall-clock estimate of the sequence of operations in Sections 3.2 and 3.3.

Because  $|\mathfrak{J}| \approx 10^6$ , and  $\binom{d_{\max}+8}{8} \approx 10^3$ , the total runtime would be in the order of a few dozen years, but there are a number of ways to reduce it considerably. Define

$$(\zeta_{\aleph, \phi})_i = \begin{cases} \delta_{i\sigma(\phi, 1)} + \delta_{i\sigma(\phi, 2)}, & \aleph = (m_1, m_2), \\ \delta_{i\sigma(\phi, 1)}, & \aleph = m_1, \\ 0, & \aleph > d_{\max}. \end{cases} \quad (3.66)$$

and consider monomial subsets  $\mathfrak{J}_\phi^{\aleph}, \mathfrak{J}_{\phi'}^{\aleph'}$  such that

$$\phi_i + (\zeta_{\aleph, \phi})_i = \phi'_i + (\zeta_{\aleph', \phi'})_i \quad \forall i. \quad (3.67)$$

Then the condition (3.45) is the same for both subsets: in particular, if  $|\mathfrak{J}_\phi^{\aleph}| \leq |\mathfrak{J}_{\phi'}^{\aleph'}|$ , admissible sampling values  $u^{(\iota)}$  for  $\iota \in \mathfrak{J}_\phi^{\aleph}$  are also admissible as sampling values for  $\mathfrak{J}_{\phi'}^{\aleph'}$ , since (3.45) is the same: the calculation of the first five rows can then be performed only once. The repetition allows to drastically reduce the number of values  $Q = Q(u^{(\iota)})$  on which the first five operations of Table 2 have to be performed by a factor of about five, down to  $\approx 2 \times 10^5$  points, and a total runtime for these operation of  $\approx 7.5$  yrs.

Furthermore, computations at different sampling values are entirely independent of each other, which means that they can be effectively parallelised purely by segmentation of the sampling set. Since these processes do not interact with each other, distributing the first batch of five calculations in Table 2 to  $N$  CPU cores gives a theoretical net factor of  $N$  in the reduction of the absolute walltime for the computation: with  $N \approx 100$ , corresponding to two 4-CPU machines with 12 processor cores each, this is reduced down to  $\approx 4$  weeks. Similar considerations apply for the last two sets of computations in Table 2:  $\mathfrak{L}_\phi^{\aleph}$  and  $\mathfrak{L}_{\phi'}^{\aleph'}$  can be computed and solved in parallel whenever  $\mathfrak{J}_\phi^{\aleph}$  and  $\mathfrak{J}_{\phi'}^{\aleph'}$  are incomparable under the weak order  $<$  of (3.10), for an extra  $\approx 1.5$  weeks worth of elapsed real time for the computation of  $N_\iota^{(k)}$ .

**3.6. Results and ancillary files.** The calculations described in the previous section were carried out at the Maths Compute Cluster at Imperial College London, and to a smaller extent, at the **Omega** and **Muse** compute clusters of the Université de Montpellier. The results of the computation, including the coefficients  $N_\iota^{(k)}$  as well as the auxiliary quantities  $Q^{(\iota)}$ ,  $u^{(\iota)}$  and  $\delta_\iota^{\max} := \sup_{k,c} 2^{|\iota|^{\max}} \delta_{\mathfrak{D}^c \chi_{\wedge^k \mathfrak{e}_8}}$  relevant for Theorem 3.4 are stored in binary files available at

<http://tiny.cc/E8Char>,

which also comprises of a *Mathematica* notebook to access them. Their structure is described in Appendix A. the polynomial character decomposition  $\mathfrak{p}_k$  of  $\chi_{\wedge^k \mathfrak{e}_8}$  is reproduced in Appendix B up to  $k = 11$ .

## 4. APPLICATIONS

A big chunk of the “Applications” section of this paper is really [4]; the polynomial character decompositions found here were used there to construct the spectral curves of the  $\widehat{E}_8$ -relativistic Toda chain (Section 1.2.1) and the Seiberg–Witten curves of  $E_8$  minimally supersymmetric Yang–Mills theory on  $\mathbb{R}^{1,3} \times S^1$  (Section 1.2.2), as well as to prove an all-genus version of the Gopakumar–Ooguri–Vafa correspondence (Section 1.2.3) for the Poincaré sphere, and to provide a mirror theorem for the Fano orbicurve of type  $E_8$  (Section 1.2.5). We provide here some examples and non-examples of characteristic polynomials having Galois group the Weyl group of  $E_8$  (Section 1.2.6).

**4.1. Some cyclotomic cases.** Let us first start with some non-examples, where the splitting field of the characteristic polynomial (1.1) is given by a finite sequence of cyclotomic extensions of the rationals.

**Example 4.1.** The obvious one – a sanity check really – is given by specialising

$$\chi_i(g) = \dim V(\omega_i) \quad (4.1)$$

corresponding to  $g = 1$ . Then obviously we must have

$$\chi_{\wedge^k \epsilon_8}(g) = \binom{248}{k} \quad (4.2)$$

which is (non-trivially) verified by substituting (4.1) into (2.5), leading to to

$$\Omega_{240}(\mu) = (\mu - 1)^{240}. \quad (4.3)$$

**Example 4.2.** A more interesting case is

$$\chi_i(g) = 0. \quad (4.4)$$

Our solution of Problem 2.1 then leads to the following result for the antisymmetric adjoint traces,

$$\chi_{\wedge^k \epsilon_8}(g) = \begin{cases} 0 & 31 \nmid k, \\ \binom{8}{k/31} & \text{else.} \end{cases} \quad (4.5)$$

Therefore,

$$\Omega_{240}(\mu) = (\Phi_{31}(\mu))^8 = \left( \sum_{i=0}^{30} \mu^i \right)^8, \quad (4.6)$$

where  $\Phi_n$  denotes the  $n^{\text{th}}$  cyclotomic polynomial,

$$\Phi_n(\mu) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} \left( x - \exp \frac{2\pi i k}{n} \right), \quad (4.7)$$

so in this case the splitting field of  $\Omega_{240}$  is  $\mathbb{Q}(\exp(2\pi i/31))$ , and the Galois group is  $\mathbb{Z}/(31\mathbb{Z})$ .

**Example 4.3.** A third notable example, already considered in [4], is

$$(\chi_i)_i(g) = (1, 3, 0, 3, -3, 3, -2, -2). \quad (4.8)$$

Geometrically, this corresponds to the super-singular limit of the Toda spectral curves computed in [4]; equivalently, the superconformal point of the geometrically engineered  $E_8$  Yang–Mills theory on  $\mathbb{R}^{1,4}$ ; the zero 't Hooft coupling point of Chern–Simons theory on the Poincaré sphere; and the conifold point for the orbifold Gromov–Witten theory of  $[\mathcal{O}(-1)_{\mathbb{P}^1}^{\oplus 2}/\mathbb{I}_{120}]$ . We obtain in this case:

$$\begin{aligned} \Omega_{240} &= \Phi_2^2(\mu) \Phi_6^3(\mu) \Phi_3^5(\mu) \Phi_5^5(\mu) \Phi_{18}^4(\mu) \Phi_9^3(\mu) \Phi_{15}^2(\mu) \Phi_{30}^3(\mu) \Phi_{45}^2(\mu) \Phi_{90}^3(\mu) \\ &= (\mu + 1)^2 (\mu^2 - \mu + 1)^3 (\mu^2 + \mu + 1)^5 (\mu^4 + \mu^3 + \mu^2 + \mu + 1)^5 (\mu^6 - \mu^3 + 1)^4 \\ &\quad (\mu^6 + \mu^3 + 1)^3 (\mu^8 - \mu^7 + \mu^5 - \mu^4 + \mu^3 - \mu + 1)^2 (\mu^8 + \mu^7 - \mu^5 - \mu^4 - \mu^3 + \mu + 1)^3 \\ &\quad (\mu^{24} - \mu^{21} + \mu^{15} - \mu^{12} + \mu^9 - \mu^3 + 1)^2 (\mu^{24} + \mu^{21} - \mu^{15} - \mu^{12} - \mu^9 + \mu^3 + 1)^3. \end{aligned} \quad (4.9)$$

**Example 4.4.** As a final example, consider

$$(\chi_i)_i(g) = (0, -1, 0, 0, 1, 0, 0, 0). \quad (4.10)$$

This example was found heuristically while constructing a set of sampling values  $Q^{(i)}$  satisfying the three conditions of Theorem 3.4. This was done by generating sampling values with a uniform random measure and testing *a posteriori* that they were admissible under the conditions of Theorem 3.4; only in one case did we find a set of values in  $\mathfrak{S}$  for which  $\det \mathcal{J}(Q^{(i)}) = 0$ , which is precisely (4.10). The singularity of the Jacobian is a sufficient condition for the Galois group of the splitting field of  $\mathfrak{Q}_{240}$  to fail to be the full Weyl group – this indicates that the corresponding group element is on the boundary of a Weyl chamber, and hence the map from the Weyl group to the Galois group has a kernel containing the corresponding reflection. This is however not necessary – Example 4.2 is a counter-example to this.

Switching back to calculations, the solution of Problem 2.1 gives

$$\chi_{\wedge^k \mathfrak{e}_8}(g) = \begin{cases} (-1)^{k+j} \tau_{\lfloor \frac{j}{4} \rfloor} & k = \lfloor \frac{7j}{2} \rfloor, j \in \mathbb{N} \\ 0 & \text{else,} \end{cases} \quad (4.11)$$

where  $\{\tau_j\}_{j=0}^9 = \{1, 13, 82, 334, 985, 2233, 4030, 5914, 7144\}$ . This gives

$$\begin{aligned} \mathfrak{Q}_{240}(\mu) &= \Phi_1^2(\mu) \Phi_2^{10}(\mu) \Phi_3(\mu) \Phi_7^9(\mu) \Phi_{12}^4(\mu) \Phi_{14}^{10}(\mu) \Phi_{84}^4(\mu) \\ &= (\mu - 1)^2 (\mu + 1)^{10} (\mu^2 + \mu + 1) (\mu^4 - \mu^2 + 1)^4 (\mu^6 - \mu^5 + \mu^4 - \mu^3 + \mu^2 - \mu + 1)^{10} \\ &\quad (\mu^6 + \mu^5 + \mu^4 + \mu^3 + \mu^2 + \mu + 1)^9 (\mu^{24} + \mu^{22} - \mu^{18} - \mu^{16} + \mu^{12} - \mu^8 - \mu^6 + \mu^2 + 1)^4. \end{aligned} \quad (4.12)$$

**4.2. The Jouve–Kowalski–Zywina polynomial.** Let us move on to consider the irreducible examples for which the splitting field has Galois group the full Weyl group  $\mathcal{W}(E_8)$ . In [18], Jouve, Kowalski and Zywina give an example of such an explicit integral polynomial in their Appendix, precisely by considering the characteristic polynomial of a specific group element  $g \in E_8$ : hence this should be reproducible from the general knowledge of  $N_i^{(k)}$  for a specific choice of  $\chi_i$ . I will just content myself to reproduce their result from the solution of Problem 2.1 without proof: set

$$(\chi_i)_i(g) = (38412, 699221720, 8046927290936, 42593483592, 175914484, 531468, 1044, 4538992). \quad (4.13)$$

I claim that these correspond to the values for the regular fundamental characters of the special group element  $g$  chosen in [18]; this can be checked directly from the definition of  $g$  in [18, Section 3]. Using the solution found for  $N_i^{(k)}$ , we get

$$\begin{aligned} (\chi_{\wedge^k \mathfrak{e}_8}(g))_{k=1}^{120} &= (1044, 532512, 177003376, 43147804716, 8230609109252, 1280164588118952, \\ &\quad 167041280674255148, 18671692028344452040, 1816777039210236799436, \dots \\ &\quad \dots, 1.0653573147903152502124530638226214941071412605209 \times 10^{108}) \end{aligned} \quad (4.14)$$

from which we get

$$\Omega_{240}(\mu) = \mu^{120} \sum_{k=0}^{120} q_k (\mu + \mu^{-1})^k \quad (4.15)$$

with

$$\begin{aligned} q_k = & (-1036, 524076, -172657460, 41688975082, -7871527038772, 1211012431626440, \\ & -156184748605164508, 17242140511966984109, -1655565532193307303324, \\ & \dots, 3.6558789498396792285456042110665879483526916202580 \dots \times 10^{86}) \end{aligned} \quad (4.16)$$

where  $q_k$  match all coefficients in the final table of [18, Appendix B].

**4.3. The group theory lift of the Shioda polynomial.** Another example of unreduced Galois group is provided by Shioda in [30, Prop. 2.2] by exhibiting an explicit even integral polynomial  $\Psi_{240}(v)$  (see *ibidem*, Eq. (2.3)) satisfying the sufficient conditions of [18, Lemma 3.2] for the Galois group to be the full Weyl group of  $E_8$ . It turns out that his example is a Lie-algebraic limit of the characteristic polynomials constructed here: it can be verified directly using Newton–Girard identities and the explicit form of the root module for  $\mathfrak{e}_8$  that there exists a regular element  $h \in \mathfrak{e}_8$  such that

$$\Psi(v) = v^{-8} \det_{\mathfrak{e}_8}(v - h) = \prod_{\alpha \in \Delta} (v - \alpha \cdot h) \quad (4.17)$$

from which we can read off directly the value of the regular fundamental characters  $\chi_i(e^h)$ . We obtain from [31, Eq. (2.3)] that

$$\chi_i(e^h) = (3811, 6967872, 6529441632, 128894934, 1829184, 23404, 184, 129266)_i. \quad (4.18)$$

We can use the solution of Problem 2.1 to construct the characteristic polynomial for the group theory lift of (4.17), which by the same token has Galois group the full Weyl group. We get

$$\tilde{\Omega}_{240}(\mu) = \mu^{120} \sum_{k=0}^{120} \tilde{q}_k (\mu^k + \mu^{-k}) \quad (4.19)$$

with

$$\begin{aligned} \tilde{q}_k = & (-1, -176, -22152, -1680656, -119102436, -5862463536, -264263737336, \\ & -13802737369104, -447652929177642, -3448304727308240, 469573988622035048, \\ & 22268833325609862288, 474582729791826806540, 1387986435785748203824, \\ & \dots, 8.68482933306707881880061436471 \dots \times 10^{87}). \end{aligned} \quad (4.20)$$

## APPENDIX A. AUXILIARY FILES

The **E8Char** package available at <http://tiny.cc/E8Char> consists of five binary files containing numerical data for the set of admissible exponents  $\iota$ , the coefficients  $N_\iota^{(k)}$  of the solution of Problem 2.1, sampling values, pre-images and error bounds  $u^\iota$ ,  $Q^\iota$  and  $\delta_\iota^{\max}$ , plus a *Mathematica*

notebook and further weight data in ASCII format for  $\Gamma(\omega_i)$ ,  $i = 1, 7, 8$ . The structure of the binary files is described below.

Filename	Description
DictFile.bin	This stores the admissible exponents $\iota$ as an 8-bit integer array $\{\{\iota_j\}_{j=1}^8\}_{\iota \in \mathcal{J}}$ .
SolFileE8.bin	This stores the solution coefficients $N_\iota^{(k)}$ as a 64-bit integer array $\{\{N_\iota^{(k)}\}_{k=1}^{120}\}_{\iota \in \mathcal{J}}$ .
USamples.bin	This stores the sampling values $u^\iota =: (p^\iota/q^\iota + ir^\iota/s^\iota)$ as an 8-bit integer array $\{\{p_j^\iota, q_j^\iota, r_j^\iota, s_j^\iota\}_{j=1}^8\}_{\iota \in \mathcal{J}}$ .
AllRoots.bin	This stores the 496-bit precision pre-images $Q^\iota$ of the sampling values $u^\iota$ as a 512-bit floating point array $\{\{\Re Q_j^\iota, \Im Q_j^\iota\}_{j \in \mathcal{J}}\}_{\iota \in \mathcal{J}}$ . Each 512-bit block is structured as follows: the first 16-bit limb represents the exponent as a signed 16-bit integer; the next bit represents the sign; the remaining 495 bits store the mantissa.
Delta.bin	This stores the maximum values on the upper bounds $\delta_\iota^{\max}$ as a 64-bit floating point array $\{\delta_\iota\}_{\iota \in \mathcal{J}}$ .

The C source code for the calculations in Sections 3.2, 3.3 and 3.5 leading up to the results of the above files is available upon request.

## APPENDIX B. TABLE OF EXTERIOR CHARACTER RELATIONS

The table below gives the polynomial character decompositions of  $\chi_{\wedge^k \mathfrak{e}_8}$  for  $k$  up to 11.

$k$	$\chi_{\wedge^k \mathfrak{e}_8} = \mathfrak{p}_k(\chi_1, \dots, \chi_8)$
1	$\chi_7$
2	$\chi_6 + \chi_7$
3	$\chi_5 - \chi_7 + \chi_7^2$
4	$\chi_4 - \chi_6 + \chi_6 \chi_7$
5	$\chi_3 - \chi_7 + \chi_5 \chi_7 - 2 \chi_6 \chi_7 - \chi_7^2 + \chi_7^3$
6	$-\chi_1^2 - \chi_1^3 + 2 \chi_1 \chi_2 - \chi_3 - \chi_1 \chi_4 + \chi_1 \chi_5 - \chi_1 \chi_6 - \chi_1^2 \chi_6 + \chi_2 \chi_6 + \chi_5 \chi_6 - \chi_6^2 - 2 \chi_1 \chi_7 + \chi_1^2 \chi_7 + \chi_2 \chi_7 + \chi_4 \chi_7 - 2 \chi_6 \chi_7 + \chi_1 \chi_6 \chi_7 - \chi_7^2 + 2 \chi_1 \chi_7^2 + \chi_6 \chi_7^2 - \chi_1 \chi_8 + \chi_2 \chi_8 - \chi_6 \chi_8 - \chi_7 \chi_8$
7	$\chi_1 + 2 \chi_1^2 + \chi_1^3 - \chi_1 \chi_2 + \chi_2^2 - \chi_3 - 2 \chi_1 \chi_3 + \chi_4 + \chi_1 \chi_4 - \chi_5 - 3 \chi_1 \chi_5 - 2 \chi_1^2 \chi_5 + 2 \chi_2 \chi_5 + \chi_5^2 + \chi_6 + 3 \chi_1 \chi_6 + 2 \chi_1^2 \chi_6 - \chi_2 \chi_6 + \chi_4 \chi_6 - 2 \chi_5 \chi_6 + \chi_6^2 + \chi_1 \chi_6^2 + 2 \chi_7 + 4 \chi_1 \chi_7 - \chi_1^2 \chi_7 - 2 \chi_1^3 \chi_7 - 2 \chi_2 \chi_7 + 4 \chi_1 \chi_2 \chi_7 - \chi_3 \chi_7 + \chi_4 \chi_7 - 4 \chi_5 \chi_7 + 2 \chi_1 \chi_5 \chi_7 + 4 \chi_6 \chi_7 + \chi_1 \chi_6 \chi_7 + \chi_7^2 - 6 \chi_1 \chi_7^2 + \chi_1^2 \chi_7^2 + 2 \chi_2 \chi_7^2 + 2 \chi_5 \chi_7^2 - 2 \chi_6 \chi_7^2 - 4 \chi_7^3 + 2 \chi_1 \chi_7^3 + \chi_7^4 - \chi_1 \chi_8 - \chi_1^2 \chi_8 + \chi_2 \chi_8 - \chi_4 \chi_8 - \chi_1 \chi_6 \chi_8 - \chi_7 \chi_8 - \chi_1 \chi_7 \chi_8 - \chi_6 \chi_7 \chi_8 + \chi_1 \chi_8^2$

8	$ \begin{aligned} & -\chi_1^4 + \chi_2 + 3\chi_1^2\chi_2 - \chi_2^2 + \chi_3 - 3\chi_1\chi_3 - \chi_4 + 2\chi_1\chi_4 - \chi_1^2\chi_4 + \chi_2\chi_4 + 2\chi_5 - \chi_1\chi_5 \\ & + \chi_1^2\chi_5 - 2\chi_2\chi_5 + 2\chi_4\chi_5 - 2\chi_5^2 + 2\chi_1\chi_6 + \chi_1^2\chi_6 - \chi_1^3\chi_6 + 2\chi_1\chi_2\chi_6 - 2\chi_3\chi_6 + \\ & \chi_4\chi_6 - \chi_5\chi_6 + \chi_1\chi_5\chi_6 + \chi_6^2 + \chi_1\chi_6^2 - 2\chi_7 + 5\chi_1\chi_7 - \chi_1^2\chi_7 + \chi_2\chi_7 + 2\chi_1\chi_2\chi_7 - \\ & 3\chi_3\chi_7 + \chi_1\chi_4\chi_7 + \chi_5\chi_7 - 3\chi_1\chi_5\chi_7 + \chi_6\chi_7 + 2\chi_1\chi_6\chi_7 + 2\chi_2\chi_6\chi_7 + \chi_5\chi_6\chi_7 + \chi_6^2 \\ & \chi_7 + 3\chi_7^2 - \chi_1\chi_7^2 + 2\chi_1^2\chi_7^2 - \chi_2\chi_7^2 + 2\chi_4\chi_7^2 - 4\chi_5\chi_7^2 + 3\chi_1\chi_6\chi_7^2 + \chi_7^3 - 2\chi_1\chi_7^3 + \\ & \chi_6\chi_7^3 - 2\chi_7^4 + \chi_8 + \chi_1\chi_8 - \chi_2\chi_8 + \chi_1\chi_2\chi_8 - 3\chi_3\chi_8 + 2\chi_4\chi_8 - 3\chi_5\chi_8 - 2\chi_1\chi_5\chi_8 \\ & + 2\chi_6\chi_8 + 4\chi_7\chi_8 - 3\chi_1\chi_7\chi_8 - 2\chi_1^2\chi_7\chi_8 + 3\chi_2\chi_7\chi_8 + 2\chi_6\chi_7\chi_8 - 4\chi_7^2\chi_8 + \chi_1\chi_7^2 \\ & \chi_8 - \chi_8^2 + \chi_8^3 \end{aligned} $
9	$ \begin{aligned} & -\chi_1^2 - \chi_1^3 + \chi_2 + 2\chi_1\chi_2 + \chi_1\chi_3 + \chi_1^2\chi_3 - 2\chi_2\chi_3 - \chi_4 - \chi_1\chi_4 + \chi_1^2\chi_4 - 2\chi_2\chi_4 \\ & + 2\chi_4^2 + 2\chi_1\chi_5 + \chi_1^2\chi_5 - 2\chi_2\chi_5 - 3\chi_3\chi_5 - \chi_5^2 + \chi_6 + \chi_1\chi_6 - \chi_1^2\chi_6 + 4\chi_2\chi_6 + \\ & \chi_1\chi_2\chi_6 - 2\chi_3\chi_6 - \chi_4\chi_6 + \chi_1\chi_4\chi_6 + \chi_5\chi_6 - 2\chi_1\chi_5\chi_6 + 3\chi_6^2 + 2\chi_1\chi_6^2 + \chi_2\chi_6^2 + 2\chi_6^3 \\ & - \chi_1\chi_7 + \chi_1^2\chi_7 + 2\chi_1^3\chi_7 + 5\chi_2\chi_7 - 3\chi_1\chi_2\chi_7 + \chi_2^2\chi_7 + 2\chi_3\chi_7 - 4\chi_1\chi_3\chi_7 - \chi_4\chi_7 \\ & + 2\chi_1\chi_4\chi_7 + 5\chi_5\chi_7 - 6\chi_1\chi_5\chi_7 - 2\chi_1^2\chi_5\chi_7 + 2\chi_2\chi_5\chi_7 + 4\chi_6\chi_7 + 5\chi_1\chi_6\chi_7 + \chi_1^2 \\ & \chi_6\chi_7 + 2\chi_2\chi_6\chi_7 + 3\chi_4\chi_6\chi_7 - 4\chi_5\chi_6\chi_7 + 6\chi_6^2\chi_7 + 2\chi_1\chi_6^2\chi_7 - 2\chi_7^2 + 10\chi_1\chi_7^2 - \\ & \chi_1^2\chi_7^2 - 2\chi_1^3\chi_7^2 - 4\chi_2\chi_7^2 + 4\chi_1\chi_2\chi_7^2 - 2\chi_3\chi_7^2 + \chi_4\chi_7^2 - 5\chi_5\chi_7^2 + 2\chi_1\chi_5\chi_7^2 + 2\chi_6\chi_7^2 \\ & + 2\chi_1\chi_6\chi_7^2 + \chi_6^2\chi_7^2 + 4\chi_7^3 - 7\chi_1\chi_7^3 + 2\chi_1^2\chi_7^3 - 3\chi_6\chi_7^3 - 2\chi_7^4 - \chi_1^3\chi_8 + 2\chi_1\chi_2\chi_8 - \\ & \chi_3\chi_8 + \chi_4\chi_8 - 2\chi_1\chi_4\chi_8 + \chi_1\chi_5\chi_8 - \chi_1^2\chi_6\chi_8 + \chi_2\chi_6\chi_8 - \chi_5\chi_6\chi_8 + \chi_7\chi_8 - 2\chi_1\chi_7\chi_8 \\ & + \chi_4\chi_7\chi_8 - 2\chi_5\chi_7\chi_8 - \chi_6\chi_7\chi_8 - 2\chi_1\chi_6\chi_7\chi_8 - \chi_7^2\chi_8 + \chi_6\chi_7^2\chi_8 - \chi_1\chi_8^2 + \chi_2\chi_8^2 + \\ & \chi_5\chi_8^2 - 2\chi_7\chi_8^2 + \chi_1\chi_7\chi_8^2 + \chi_7^2\chi_8^2 \end{aligned} $
10	$ \begin{aligned} & -\chi_1 - \chi_1^2 + \chi_1^3 + \chi_1^4 - \chi_1\chi_2 - 2\chi_1^2\chi_2 + \chi_3 - \chi_1^2\chi_3 + \chi_2\chi_3 - \chi_4 - \chi_1\chi_4 + 2\chi_1^2\chi_4 \\ & + \chi_1^3\chi_4 - \chi_2\chi_4 - 3\chi_1\chi_2\chi_4 + \chi_3\chi_4 - 2\chi_4^2 - 2\chi_1\chi_5 - 2\chi_1^2\chi_5 + 3\chi_2\chi_5 + 2\chi_3\chi_5 + \\ & \chi_1\chi_4\chi_5 + 2\chi_5^2 + \chi_1\chi_5^2 - \chi_6 - \chi_1\chi_6 + 2\chi_1^3\chi_6 - 3\chi_1\chi_2\chi_6 + 2\chi_3\chi_6 - 2\chi_1\chi_3\chi_6 - 3 \\ & \chi_4\chi_6 - 4\chi_1\chi_5\chi_6 + \chi_2\chi_5\chi_6 - \chi_5^2\chi_6 + 2\chi_1\chi_6^2 + \chi_1^2\chi_6^2 - \chi_2\chi_6^2 - \chi_5\chi_6^2 + \chi_6^3 + \chi_1\chi_6^3 - \\ & 2\chi_7 + 2\chi_1^2\chi_7 - \chi_1^3\chi_7 - \chi_1^4\chi_7 - \chi_2\chi_7 - \chi_1\chi_2\chi_7 + 3\chi_1^2\chi_2\chi_7 - \chi_2^2\chi_7 + 3\chi_3\chi_7 - 2 \\ & \chi_1\chi_3\chi_7 - 9\chi_4\chi_7 - \chi_1^2\chi_4\chi_7 - 2\chi_2\chi_4\chi_7 + 2\chi_1\chi_5\chi_7 - \chi_2\chi_5\chi_7 + 3\chi_4\chi_5\chi_7 - \chi_5^2\chi_7 - 3 \\ & \chi_6\chi_7 + 8\chi_1\chi_6\chi_7 + \chi_1^2\chi_6\chi_7 - \chi_1^3\chi_6\chi_7 - \chi_2\chi_6\chi_7 + 2\chi_1\chi_2\chi_6\chi_7 - 3\chi_3\chi_6\chi_7 - 6\chi_4\chi_6\chi_7 \\ & - \chi_5\chi_6\chi_7 - 2\chi_1\chi_5\chi_6\chi_7 + 3\chi_6^2\chi_7 + 5\chi_1\chi_6^2\chi_7 + 2\chi_6^3\chi_7 - 5\chi_7^2 + 4\chi_1\chi_7^2 - 2\chi_1^2\chi_7^2 \\ & + \chi_1^3\chi_7^2 + 3\chi_2\chi_7^2 - 5\chi_4\chi_7^2 + 2\chi_1\chi_4\chi_7^2 + 9\chi_5\chi_7^2 - 4\chi_1\chi_5\chi_7^2 + \chi_1^2\chi_6\chi_7^2 + \chi_2\chi_6\chi_7^2 - 2 \\ & \chi_5\chi_6\chi_7^2 + 2\chi_6^2\chi_7^2 - \chi_7^3 + 2\chi_1\chi_7^3 - \chi_2\chi_7^3 + 4\chi_4\chi_7^3 - \chi_5\chi_7^3 - 4\chi_6\chi_7^3 + 4\chi_7^4 - 3\chi_1\chi_7^4 \\ & - \chi_6\chi_7^4 - \chi_8 - 3\chi_1\chi_8 - \chi_1^2\chi_8 + \chi_2\chi_8 - 2\chi_1\chi_2\chi_8 + 3\chi_3\chi_8 + \chi_1\chi_3\chi_8 - 3\chi_4\chi_8 + \\ & \chi_1\chi_4\chi_8 + 2\chi_5\chi_8 + \chi_1\chi_5\chi_8 - \chi_1^2\chi_5\chi_8 + 2\chi_2\chi_5\chi_8 + \chi_5^2\chi_8 - 2\chi_6\chi_8 - 4\chi_1\chi_6\chi_8 - \chi_1^2 \\ & \chi_6\chi_8 - \chi_4\chi_6\chi_8 - \chi_5\chi_6\chi_8 - 2\chi_1\chi_6^2\chi_8 - 5\chi_7\chi_8 + 2\chi_1\chi_7\chi_8 - \chi_2\chi_7\chi_8 + \chi_1\chi_2\chi_7\chi_8 - \\ & \chi_3\chi_7\chi_8 - \chi_4\chi_7\chi_8 + 2\chi_5\chi_7\chi_8 + \chi_1\chi_5\chi_7\chi_8 - 3\chi_6\chi_7\chi_8 - \chi_1\chi_6\chi_7\chi_8 + 2\chi_7^2\chi_8 - 2\chi_1^2 \\ & \chi_7^2\chi_8 + 2\chi_2\chi_7^2\chi_8 + 3\chi_5\chi_7^2\chi_8 - 4\chi_6\chi_7^2\chi_8 - 2\chi_7^3\chi_8 + 2\chi_1\chi_7^3\chi_8 + 2\chi_7^4\chi_8 - \chi_8^2 + \\ & \chi_1\chi_8^2 - \chi_4\chi_8^2 - \chi_6\chi_8^2 + \chi_1\chi_6\chi_8^2 - \chi_7\chi_8^2 + 2\chi_1\chi_7\chi_8^2 - \chi_7^2\chi_8^2 - \chi_8^3 \end{aligned} $

11	$ \begin{aligned} & 3\chi_7^6 + 3\chi_7^5 + 5\chi_5\chi_7^4 - 15\chi_6\chi_7^4 + 2\chi_8\chi_7^4 - 8\chi_7^4 - 2\chi_1^2\chi_7^3 - 5\chi_6^2\chi_7^3 - \chi_8^2\chi_7^3 + \chi_1\chi_7^3 + 2\chi_2\chi_7^3 + \\ & 4\chi_3\chi_7^3 - 5\chi_4\chi_7^3 + 2\chi_1\chi_5\chi_7^3 + 14\chi_5\chi_7^3 - 6\chi_1\chi_6\chi_7^3 - 10\chi_6\chi_7^3 + 2\chi_1\chi_8\chi_7^3 + \chi_6\chi_8\chi_7^3 + 4\chi_8\chi_7^3 \\ & - 10\chi_7^3 + 2\chi_5^2\chi_7^2 + 2\chi_1\chi_6^2\chi_7^2 + 12\chi_6^2\chi_7^2 - \chi_1\chi_8^2\chi_7^2 - 9\chi_1\chi_7^2 + 2\chi_2\chi_7^2 + 5\chi_3\chi_7^2 - \chi_1\chi_4\chi_7^2 \\ & - 3\chi_4\chi_7^2 - \chi_1^2\chi_5\chi_7^2 + 7\chi_1\chi_5\chi_7^2 + 2\chi_2\chi_5\chi_7^2 - 5\chi_5\chi_7^2 + 3\chi_4\chi_6\chi_7^2 - 13\chi_5\chi_6\chi_7^2 + 17\chi_6\chi_7^2 + \\ & 3\chi_1^2\chi_8\chi_7^2 + \chi_1\chi_8\chi_7^2 - 2\chi_2\chi_8\chi_7^2 + \chi_4\chi_8\chi_7^2 + 2\chi_5\chi_8\chi_7^2 + 3\chi_1\chi_6\chi_8\chi_7^2 - 11\chi_6\chi_8\chi_7^2 - 7\chi_8\chi_7^2 \\ & + 9\chi_7^2 - 3\chi_1^3\chi_7 + 13\chi_6^3\chi_7 + 2\chi_1^2\chi_7 + 4\chi_1\chi_5^2\chi_7 + 8\chi_5^2\chi_7 - \chi_1^2\chi_6^2\chi_7 + 9\chi_1\chi_6^2\chi_7 + \chi_2\chi_6^2\chi_7 - \\ & 5\chi_5\chi_6^2\chi_7 + 24\chi_6^2\chi_7 + \chi_1^2\chi_8^2\chi_7 - 2\chi_1\chi_8^2\chi_7 - \chi_2\chi_8^2\chi_7 - \chi_5\chi_8^2\chi_7 + \chi_6\chi_8^2\chi_7 + 3\chi_8^2\chi_7 + 9\chi_1\chi_7 \\ & + 3\chi_1\chi_2\chi_7 - 7\chi_2\chi_7 + 2\chi_1\chi_3\chi_7 - 8\chi_3\chi_7 + 5\chi_1^2\chi_4\chi_7 - 2\chi_1\chi_4\chi_7 - 8\chi_2\chi_4\chi_7 + 3\chi_4\chi_7 - \\ & 8\chi_1\chi_5\chi_7 + \chi_2\chi_5\chi_7 + 5\chi_3\chi_5\chi_7 - 8\chi_4\chi_5\chi_7 - 19\chi_5\chi_7 + 2\chi_1^2\chi_6\chi_7 + 13\chi_1\chi_6\chi_7 - 3\chi_2\chi_6\chi_7 \\ & - 10\chi_3\chi_6\chi_7 - 15\chi_1\chi_5\chi_6\chi_7 - 21\chi_5\chi_6\chi_7 + 14\chi_6\chi_7 - \chi_1^3\chi_8\chi_7 - \chi_1^2\chi_8\chi_7 - 4\chi_6^2\chi_8\chi_7 - \\ & 8\chi_1\chi_8\chi_7 + \chi_1\chi_2\chi_8\chi_7 - 2\chi_2\chi_8\chi_7 + 4\chi_3\chi_8\chi_7 - 3\chi_1\chi_4\chi_8\chi_7 - 3\chi_4\chi_8\chi_7 + 6\chi_1\chi_5\chi_8\chi_7 + \\ & 2\chi_5\chi_8\chi_7 - 2\chi_1^2\chi_6\chi_8\chi_7 - 10\chi_1\chi_6\chi_8\chi_7 + 2\chi_2\chi_6\chi_8\chi_7 + 2\chi_5\chi_6\chi_8\chi_7 - 5\chi_6\chi_8\chi_7 - 2\chi_8\chi_7 + \\ & 2\chi_7 + \chi_1^4 + 3\chi_6^4 + 3\chi_1^3 + 5\chi_1\chi_6^3 + 7\chi_6^3 - \chi_1\chi_8^3 + \chi_6\chi_8^3 + \chi_8^3 + 3\chi_1^2 + 2\chi_2^2 + \chi_3^2 - \chi_1\chi_4^2 \\ & + 5\chi_1\chi_5^2 + 4\chi_2\chi_5^2 + 9\chi_5^2 + 3\chi_1^2\chi_6^2 + 10\chi_1\chi_6^2 + \chi_1\chi_2\chi_6^2 - 6\chi_3\chi_6^2 - 4\chi_1\chi_5\chi_6^2 - 4\chi_5\chi_6^2 + \\ & 6\chi_6^2 + 2\chi_1^2\chi_8^2 + 3\chi_1\chi_8^2 - 3\chi_2\chi_8^2 + 2\chi_4\chi_8^2 - \chi_1\chi_5\chi_8^2 - 4\chi_5\chi_8^2 + \chi_1\chi_6\chi_8^2 + \chi_6\chi_8^2 + \chi_8^2 + \chi_1 \\ & - 2\chi_1^2\chi_2 - 3\chi_1\chi_2 - 3\chi_1^2\chi_3 - 5\chi_1\chi_3 + 3\chi_2\chi_3 - \chi_3 + \chi_1^2\chi_4 + 2\chi_1\chi_4 - 2\chi_1\chi_2\chi_4 - \chi_2\chi_4 \\ & - \chi_3\chi_4 + \chi_1^4\chi_5 - 2\chi_1^3\chi_5 - 10\chi_1^2\chi_5 + 2\chi_2^2\chi_5 - 7\chi_1\chi_5 - 4\chi_1^2\chi_2\chi_5 + 2\chi_1\chi_2\chi_5 + 8\chi_2\chi_5 + \\ & 4\chi_1\chi_3\chi_5 + 7\chi_3\chi_5 - 4\chi_4\chi_5 - \chi_5 + 2\chi_1^3\chi_6 + 7\chi_1^2\chi_6 + \chi_2^2\chi_6 + \chi_5^2\chi_6 + 7\chi_1\chi_6 - \chi_1\chi_2\chi_6 - \\ & 3\chi_2\chi_6 - 5\chi_1\chi_3\chi_6 - 6\chi_3\chi_6 + \chi_1^2\chi_4\chi_6 - \chi_1\chi_4\chi_6 - 4\chi_2\chi_4\chi_6 + \chi_4\chi_6 - 4\chi_1^2\chi_5\chi_6 - 9\chi_1\chi_5\chi_6 \\ & + \chi_2\chi_5\chi_6 + \chi_4\chi_5\chi_6 - 8\chi_5\chi_6 + 2\chi_6 - \chi_1^3\chi_8 - \chi_1^2\chi_8 - 2\chi_5^2\chi_8 - \chi_1\chi_6^2\chi_8 + \chi_1\chi_8 + \chi_1\chi_2\chi_8 \\ & + 2\chi_1\chi_3\chi_8 + \chi_1^2\chi_4\chi_8 - 2\chi_2\chi_4\chi_8 + 2\chi_1^2\chi_5\chi_8 + 3\chi_1\chi_5\chi_8 + 2\chi_5\chi_8 - \chi_1^3\chi_6\chi_8 - \chi_1\chi_6\chi_8 + \\ & 3\chi_1\chi_2\chi_6\chi_8 - \chi_2\chi_6\chi_8 - 3\chi_3\chi_6\chi_8 + \chi_4\chi_6\chi_8 \end{aligned} $
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