Article:

Berger, T. orcid.org/0000-0002-5207-6221 and Betina, A. (Submitted: 2019) On Siegel eigenvarieties at Saito-Kurokawa points. arXiv. (Submitted)

https://doi.org/10.48550/arXiv.1902.05885

ON SIEGEL EIGENVARIETIES AT SAITO-KUROKAWA POINTS

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Abstract. We study the geometry of the Siegel eigenvariety $\mathcal{E}_\Delta$ of paramodular tame level Δ associated to a squarefree $N \in \mathbb{N}_+$ at certain points having a critical slope. For $k \geq 2$ let f be a cuspidal eigenform of $\mathrm{S}_{2k-2}(\Gamma_0(N))$ ordinary at a prime $p \nmid N$ with sign $\epsilon_f = -1$ and write α for the unit root of the Hecke polynomial of f at p. Let $\mathrm{SK}(f)_\alpha$ be the semi-ordinary p-stabilization of the Saito-Kurokawa lift of the cusp form f to $\mathrm{GSp}(4)$ of weight (k,k) of tame level Δ. Under the assumption that the dimension of the Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ attached to f is at most one and some mild assumptions on the mod p representation $\bar{\rho}_f$ associated to f, we show that the rigid analytic space $\mathcal{E}_\Delta$ is smooth at the point x corresponding to $\mathrm{SK}(f)_\alpha$. This means that there exists a unique irreducible component of $\mathcal{E}_\Delta$ specializing to x, and we also show that this irreducible component is not globally endoscopic. Finally we give an application to the Bloch-Kato conjecture, by proving under some mild assumptions on $\bar{\rho}_f$ that the smoothness failure of $\mathcal{E}_\Delta$ at x yields that $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) \geq 2$.

1. Introduction

Let p be a prime number. Eigenvarieties are p-adic rigid analytic spaces interpolating the Hecke eigenvalues of automorphic representations of a particular reductive group G of finite slope eigenvalues for Hecke operators at p, fixed tame level away from p and varying weights. Following the seminal works of Hida [Hid86] and Coleman-Mazur [CM98] their geometry has been studied by many people, e.g. Bellaïche and Chenevier [BC06], Majumdar [Maj15] and Bellaïche and Dimitrov [BD16] for $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q})$, and by Bellaïche and Chenevier [Bel08], [BC09] for unitary groups.

Andreatta, Iovita and Pilloni constructed in [AIP15] an eigenvariety parametrizing locally analytic overconvergent cuspidal Siegel eigenforms of genus two, principal level N and finite slope, and they proved that the Siegel eigenvariety of tame level 1 is étale over the weight space at certain classical non-critical points of regular cohomological weights with Iwahoric level at p. The proof uses the classicality criteria for overconvergent Siegel cusp forms of Hida [Hid02, Prop.3.6], Tilouine and Urban [TU99, Thm.3.2], Pilloni [Pil11, Thm.2] and the multiplicity one theorem of Arthur's classification for GSp_4 [Art04].

Both authors acknowledge support from the EPSRC Grant EP/R006563/1.

We investigate in this work the geometry of the Siegel eigenvariety $\mathcal{E}_\Delta$ of paramodular level N at the points corresponding to Saito-Kurokawa lifts of ordinary cusp forms for $\mathrm{GL}_2(\mathbb{Q})$ (which have a critical slope), including the case of the non-cohomological weight $(2,2)$.

In order to state our results, we recall some facts and fix some notations: Let N be a squarefree integer prime to p. For a prime ℓ the paramodular subgroup of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ is defined as $\Delta_\ell = \gamma \mathrm{M}_4(\mathbb{Z}_\ell)\gamma^{-1} \cap \mathrm{GSp}_4(\mathbb{Q}_\ell)$ for $\gamma = \mathrm{diag}[1,1,\ell,1]$. We write $\Delta := \prod_{\ell|N} \Delta_\ell \cap \mathrm{GSp}_4(\mathbb{Q})$ for the paramodular congruence subgroup of level N. If $N=1$ we put $\Delta = \mathrm{GSp}_4(\mathbb{Z})$.

Let $f \in \mathrm{S}_{2k-2}(\Gamma_0(N), K_f)$ be a weight $2k-2$ cuspidal N-new eigenform for $\mathrm{GL}_2(\mathbb{Q})$ with coefficient field K_f. Assume that f has an ordinary p-stablization and denote it by f_α, where $U_p(f_\alpha) = \alpha.f_\alpha$.

The L-function $L(f,s)$ attached to f satisfies the following functional equation:

$$L(f,s) = \epsilon_f L(f, 2k-2-s).$$

We have that $\epsilon_f = (-1)^{\mathrm{ord}_{s=k-1} L(f,s)}$. Assume until the end of this paper that $\epsilon_f = -1$[1], which means that there exists a lift $\mathrm{SK}(f)$ to a weight (k,k) cuspform of level Δ called the Saito-Kurokawa lift of f. It satisfies

$$L^N(\mathrm{SK}(f), \mathrm{spin}, s) = \zeta^N(s-k+1)\zeta^N(s-k+2)L^N(s,f).$$

When $N=1$ this lift was constructed by Maass, Andrianov and Zagier; Gritsenko generalized it to any level N. A representation theoretic approach building on results of Piatetski-Shapiro and Waldspurger is discussed in [Sch07].

In order to p-adically deform $\mathrm{SK}(f)$, one must first choose a semi-ordinary[2] *p-stabilization* of $\mathrm{SK}(f)$, that is an eigenform of tame level the paramodular group Δ and sharing the same eigenvalues as $\mathrm{SK}(f)$ away from p and of finite slope. Denote by π_α the p-stablization of $\mathrm{SK}(f)$ such that $U_0(\pi_\alpha) = \alpha.\pi_\alpha$, and $U_1(\pi_\alpha) = p.\alpha.\pi_\alpha$ where U_0, U_1 are the Hecke operators attached to $\mathrm{diag}[1,1,p,p]$ (U_0 is often denoted by U_p), $\mathrm{diag}[1,p,p^2,p]$, and U_1 has been renormalized to have a good p-adic interpolation (see for example [SU06, Thm.2.4.14]).

Let $\mathcal{E}_\Delta$ be Siegel eigenvariety of tame paramodular level Δ (see appendix §B.4). It is reduced and equidimensional of dimension 2, and endowed with a morphism

$$\kappa : \mathcal{E}_\Delta \to \mathcal{W}$$

called the weight map (which is locally finite and torsion-free), where the weight space $\mathcal{W}$ is the rigid analytic space over $\mathbb{Q}_p$ such that $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}((\mathbb{Z}_p^\times)^2, \mathbb{C}_p^\times)$.

[1] When $N=1$, one has $\epsilon_f = (-1)^{k-1}$.

[2] Semi-ordinary means that the eigenvalue for the Hecke operator U_0 is a p-adic unit. Following Tilouine-Urban this is also called Siegel ordinary.

The cuspidal eigenform π_α defines a point x of $\mathcal{E}_\Delta$. Write L for the residue field of x, a finite extension of $\mathbb{Q}_p$. Note that the slopes of U_0 and U_1 are locally constant on $\mathcal{E}_\Delta$, and equal to 0 for U_0 and 1 for U_1 locally at x. This means that the cuspform π_α has a critical slope since it does not satisfy the small slope condition of [AIP15, Thm. 7.3.1].

One can show that there exists a pseudo-character $\mathrm{Ps} = \mathrm{Ps}_{\mathcal{E}_\Delta} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta)$ of dimension 4 such that the specialization $\mathrm{Ps}(y)$ of Ps at a classical point $y \in \mathcal{E}_\Delta(\bar{\mathbb{Q}}_p)$ is the trace of the semi-simple p-adic Galois representation $\rho_y : G_{\mathbb{Q}} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_p)$ of dimension 4 attached to a cuspidal Siegel eigenform g_y corresponding to y (i.e. $L(g_y, \mathrm{spin}, s) = L(\rho_y, s)$). For $y = x = \pi_\alpha$ we have

$$\mathrm{Ps}(\pi_\alpha) = \epsilon_p^{1-k} + \epsilon_p^{2-k} + \mathrm{Tr}\,\rho_f,$$

where ρ_f is the p-adic Galois representation attached to f (i.e. $L(f,s) = L(\rho_f, s)$) and ϵ_p is the p-adic cyclotomic character.

Let $\mathcal{T}$ be the local ring of $\mathcal{E}_\Delta$ at x for the rigid topology, $\mathfrak{m}$ the maximal ideal of $\mathcal{T}$ and Λ the local ring of $\mathcal{W}$ for the rigid topology at the weight $\kappa(x)$ of x (they are both Henselian rings). Note that $\mathcal{T}$ is an equidimensional ring of dimension 2.

Definition 1.1. We say that an *irreducible* affinoid $\mathcal{Z} \subset \mathcal{E}_\Delta$ of dimension 2 is *stable* if and only if the reducibility locus of the pseudo-character $\mathrm{Ps}_{\mathcal{Z}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{Z})$ given by the composition of $\mathrm{Ps}_{\mathcal{E}_\Delta}$ with the natural morphism $\mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{O}(\mathcal{Z})$ is strictly contained in $\mathcal{Z}$ (i.e. of dimension less or equal to 1). Otherwise, we say that $\mathcal{Z}$ is an endoscopic irreducible affinoid of $\mathcal{E}_\Delta$ of dimension 2.

Let $\bar{\rho}_f : G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_2(k(L))$ be the residual representation (i.e. mod p) attached to ρ_f, where $k(L)$ is the residue field of L, and let $\pi_f = \bigotimes_{\ell} \pi_{f,\ell}$ be the automorphic representation attached to f.

We will recall the assumptions used in the Taylor-Wiles isomorphism (i.e. R=T) [TW95] and [Wil88]:

- ($\mathbf{AI}_{\mathbb{Q}}$) The restriction of $\bar{\rho}$ to $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ is absolutely irreducible.
- (**Reg**) $\bar{\rho}_f$ is p-distinguished and $\alpha \neq 1$ when $k = 2$.
- (**Min**) For any prime $\ell \mid N$, $\bar{\rho}_f|_{I_\ell}$ is unipotent and non-trivial and $a_\ell = -\ell^{k-2}$ (i.e $\pi_{f,\ell} \simeq \mathrm{St} \otimes \xi$, where ξ is the unramified character with $\xi(\ell) = -1$).

Under the assumptions ($\mathbf{AI}_{\mathbb{Q}}$), (**Reg**) and (**Min**), the local Noetherian ring R^{ord} representing the p-ordinary minimally ramified deformation of $\bar{\rho}_f$ is isomorphic to the local component of the semi-local p-ordinary Hecke algebra $\mathfrak{h}^{\mathrm{ord}}$ of level Np^∞ whose maximal ideal corresponds to the modular form $f_\alpha \mod p$ (see [Hid86] for its construction).

Andreatta-Iovita-Pilloni pose the following question in [AIP15, §.8]:

Open problem. Let $x(g)$ be a classical point of the Siegel eigenvariety $\mathcal{E}_N$ of tame level the principal congruence subgroup of level N. Is the map $\kappa : \mathcal{E}_N \to \mathcal{W}$ unramified at $x(g)$?

Let $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ, the completed local ring of $\mathcal{W}$ at $\kappa(x)$, $\mathcal{T}' = \mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T}$ be the local ring of the fiber $\kappa^{-1}(\kappa(x)) \subset \mathcal{E}_\Delta$ at x (since κ is locally finite, $\mathcal{T}'$ is an Artinian algebra), and let $\mathfrak{t}_{\pi_\alpha}$ (resp. $\mathfrak{t}^0_{\pi_\alpha}$) be the Zariski tangent space of $\mathcal{T}$ (resp. $\mathcal{T}'$, i.e the relative tangent space of $\kappa^\# : \Lambda \to \mathcal{T}$).

Let $\omega_p : G_\mathbb{Q} \to \mathbb{Z}_p^\times$ be the Teichmüller character and $L_p(f_\alpha, \omega_p^{-1}, .) \in \Lambda := \bar{\mathbb{Z}}_p[\![T]\!]$ be the Manin-Vishik p-adic L-function attached to $f_\alpha \otimes \omega_p^{-1}$ (see e.g. [Kat04, Thm.16.2]), and let

$$\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = \ker(\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1)) \to \mathrm{H}^1(\mathbb{Q}_p, \rho_f(k-1) \otimes B_{\mathrm{crys}}) \oplus_{\ell \nmid p} \mathrm{H}^1(I_\ell, \rho_f(k-1)))$$

be the Selmer group attached to f.

Our main result is the following theorem describing the local geometry of the rigid analytic space $\mathcal{E}_\Delta$ (equidimensional of dimension 2) at π_α:

Theorem A (see §.3 and §.8.2)**.**
Put $s = \dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$.

(i) Assume that $k \geq 2$, $\pi_{f,\ell}$ *is special (Steinberg or twisted Steinberg) at every prime* $\ell \mid N$ *and* (**Reg**)*. Then all the irreducible affinoids of* $\mathcal{E}_\Delta$ *of dimension* 2 *specializing to* π_α *are stable.*

(ii) Assume that $k \geq 3$, (**Min**), (**AI**$_\mathbb{Q}$) *and* (**Reg**)*, then*

$$2 \leq \dim \mathfrak{t}_{\pi_\alpha} \leq 1 + s^2 \ \textit{and} \ \dim \mathfrak{t}^0_{\pi_\alpha} \leq s^2.$$

Moreover, if $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$*, then* $\mathcal{E}_\Delta$ *is smooth at* π_α*, and the reducibility locus of the pseudo-character* $\mathrm{Ps}_\mathcal{T} : G_\mathbb{Q} \to \mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{T}$ *is the closed irreducible smooth subscheme of* $\mathrm{Spec}\,\mathcal{T}$ *of dimension* 1 *associated to the Saito-Kurokawa lift of the ordinary Hida family* $\mathcal{F}$ *passing through* f_α*, and it is even a principal Weil divisor of* $\mathrm{Spec}\,\mathcal{T}$*.*

(iii) Assume that $k = 2$, (**Min**), (**AI**$_\mathbb{Q}$), (**Reg**) *and* $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$*, then*

$$2 \leq \dim \mathfrak{t}_{\pi_\alpha} \leq 1 + s^2 \ \textit{and} \ \dim \mathfrak{t}^0_{\pi_\alpha} \leq s^2.$$

Moreover, if $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(1)) = 1$*, then* $\mathcal{E}_\Delta$ *is smooth at* π_α*, and the reducibility ideal of the pseudo-character* $\mathrm{Ps}_\mathcal{T} : G_\mathbb{Q} \to \mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{T}$ *is principal.*

A key step in the proof is the determination of the schematic reducibility locus of the pseudo-character $\mathrm{Ps}_\mathcal{T} : G_\mathbb{Q} \to \mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{T}$ carried by $\mathcal{E}_\Delta$ at x, and our approach uses pseudo-representations of p-adic families of cuspidal Siegel eigenforms and p-adic Hodge theory. We provide a more detailed sketch of the proof in section 1.1.

A direct consequence of (ii) and (iii) of the above theorem is that under these assumptions there exists a unique irreducible component of $\mathcal{E}_\Delta$ specializing to π_α when the Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ is 1-dimensional.

The smoothness of the eigencurve at critical points is a crucial ingredient for the construction of a family of p-adic L functions on an open neighborhood of these points, see e.g. [Bel12]. Our result on the smoothness of $\mathcal{E}_\Delta$ opens up the possibility of constructing a family of p-adic L-functions in a neighbourhood of π_α, a challenging question in Iwasawa theory.

Using results about Λ-adic Selmer groups we exhibit many examples where the Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ is 1-dimensional (see Appendix §.C). We also have an example of an elliptic curve satisfying all the assumptions of (iii) of the above theorem (see §.C).

Corollary 1.2.

(i) Assume that $k \geq 3$, (**Min**), ($\mathbf{AI}_{\mathbb{Q}}$), (**Reg**). *If the rigid analytic space $\mathcal{E}_\Delta$ is singular at π_α then* $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) \geq 2$.

(ii) Assume that $k = 2$, (**Min**), ($\mathbf{AI}_{\mathbb{Q}}$), (**Reg**) *and $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$. If the rigid analytic space $\mathcal{E}_\Delta$ is singular at π_α, then* $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) \geq 2$.

Hence we have a geometric criterion to detect if $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) \geq 2$. Thus, the question of finding a lower bound of the dimension of the Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ can be reduced to certain computations of spaces of semi-ordinary p-adic modular cuspforms for GSp_4.

It turns out that the geometry of $\mathcal{E}_N$ at x depends on the tame level. When we change the tame level to the principal Siegel congruence subgroup $\Gamma(N)$ it is in general non-smooth. In particular, the answer to the question in [AIP15] is negative if N is not prime.

Theorem B (see Corollary 9.4)**.** *Assume that $\ell_1.\ell_2 \mid N$, where $\{\ell_i\}_{\{1,2\}}$ are prime numbers and assume that f is Steinberg at both these primes. Then the eigenvariety $\mathcal{E}_N$ is singular at π_α and has at least two irreducible endoscopic components specializing to π_α.*

1.1. **Sketch of the proof of Theorem A.** Using [SU06, Thm.3.2.9] we show that any endoscopic irreducible affinoid $\mathcal{Z} \subset \mathcal{E}_\Delta$ of dimension 2 specializing to π_α is the Yoshida lift of the Hida family $\mathcal{F}$ passing through f_α and a Coleman family $\mathcal{F}'$ passing through an overconvergent form sharing the same system of Hecke eigenvalues for $\{T_\ell\}_{\ell \nmid Np}$ and U_p as the critical Eisenstein series $E_2^{\mathrm{crit}_p}$ of weight 2. We prove that $\mathcal{F}'$ is necessarily special at some $\ell_0 \mid N$ and $\mathcal{F}$ is special at every $\ell \mid N$. Hence the classical specializations in sufficiently high weights of $\mathcal{Z}$ are Yoshida lifts of two cuspidal eigenforms such that the local automorphic representation at ℓ_0 of both are special. In fact, it follows from the classification of Roberts and Schmidt that no such Yoshida lift exists for tame level Δ. This establishes that all irreducible components of $\mathcal{E}_\Delta$ containing x are stable.

Hence by localizing the pseudo-character $\mathrm{Ps}_{\mathcal{E}_\Delta} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta)$ of dimension 4 at the local Henselian ring $\mathcal{T}$, we get a pseudo-character $\mathrm{Ps}_{\mathcal{T}} : G_{\mathbb{Q}} \to \mathcal{T}$ deforming $\mathrm{Ps}(x)$ which is generically irreducible on each irreducible component containing x. Following the results of [BC09], we obtain a GMA matrix $S = \mathcal{T}[G_{\mathbb{Q}}]/\ker(\mathrm{Ps}_{\mathcal{T}})$ with orthogonal idempotents lifting the natural idempotents of the semi-simple representation $\varrho = \epsilon_p^{2-k} \oplus \rho_f \oplus \epsilon_p^{1-k}$.

The total reducibility ideal $\mathcal{I}^{\mathrm{tot}}$ of $\mathrm{Ps}_{\mathcal{T}}$ is defined to be the smallest ideal I of $\mathcal{T}$ such that

$$\mathrm{Ps}_{\mathcal{T}} \mod I = T_1 + T_2 + T_3$$

for pseudocharacters T_i with $T_i \mod \mathfrak{m} = Tr(\rho_i)$ for $\rho_1 = \epsilon_p^{2-k}$, $\rho_2 = \rho_f$, $\rho_3 = \epsilon_p^{1-k}$. By results of [BC09] it is controlled by the entries of the GMA S (see Proposition 4.4). These in turn give rise to S-extensions of ρ_i by ρ_j for $i \neq j$. We prove in Theorem 7.7 when $s := \dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ that $\mathcal{I}^{\mathrm{tot}}$ is principal (or more generally we bound the number of its generators by s^2) by proving that these extension satisfy the required local properties to lie in the corresponding Selmer groups $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-2)) = 0$ (a deep result of Kato [Kat04]), $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p) \overset{\mathrm{Kummer}}{\simeq} \mathbb{Z}^\times \otimes L = 0$ and $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$, which we assume to be at most 1-dimensional.

This local analysis forms the technical heart of the paper. At p we use that any representation ρ_z attached to a classical point $z \in \mathcal{Z}$ of $\mathcal{E}_\Delta$ containing x is semi-ordinary (i.e.$\dim \rho_z^{I_p} \geq 1$). Using this we prove in §4 and §6 that any S-extension W (resp. W') occuring in the cohomology group $\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1))$ (resp. $\mathrm{H}^1(\mathbb{Q}, \rho_f(k-2))$) is in fact ordinary at p, in the sense that $W^{I_p} \neq 0$, $(W')^{I_p} \neq 0$ and Frob_p acts on them by α. Therefore, W (resp. W' when $k \geq 3$) is ordinary in the sense of Fontaine-Perrin-Riou (so de Rham), and hence crystalline since $\mathrm{H}^1_g(\mathbb{Q}_p, \rho_f(k-i)) = \mathrm{H}^1_f(\mathbb{Q}_p, \rho_f(k-i))$ for $i \in \{1, 2\}$.

To prove the crystallinity of the S-extensions in $\mathrm{Ext}^1_{G_{\mathbb{Q}}}(\epsilon_p^{1-k}, \epsilon_p^{2-k})$ we apply in §5 the results of [BC09] §4 on the analytic continuation of crystalline periods for the smallest Hodge-Tate weight in families of p-adic Galois representations occurring in a torsion free coherent module. To this end we establish in section §B that classical points which are old at p are very Zariski dense in $\mathcal{E}_\Delta$. To be able to study the period we are interested in we need to consider the quotient by the line fixed by inertia due to semi-ordinarity. At a classical point $z \in \mathcal{Z}$ of cohomological weight (l_1, l_2) the smallest Hodge-Tate weight of the 3-dimensional $G_{\mathbb{Q}_p}$-representation $\rho_z/\rho_z^{I_p}$ is $l_2 - 2$ and $\dim \mathcal{D}_{\mathrm{crys}}(\rho_z/\rho_z^{I_p})^{U_1/U_0(z)p^{l_2-2}} = 1$ when ρ_z is crystalline.

This allows us to prove that the S-extensions occuring in $\mathrm{Ext}^1_{G_{\mathbb{Q}}}(\epsilon_p^{1-k}, \epsilon_p^{2-k})$ have a crystalline period equal to

$$\lim_{z_n\in\mathcal{E}, z_n\to x} U_1/U_0(z)p^{l_2(z)-2} = U_1/U_0(x)p^{k-2} = p^{k-1}.$$

This means that for any S-extension $V \in \mathrm{Ext}^1_{L[G_{\mathbb{Q}}^{Np}]}(\epsilon_p^{1-k}, \epsilon_p^{2-k})$, we have $\mathcal{D}_{\mathrm{crys}}^{\Phi=p^{k-1}}(V) \neq 0$ and $\mathcal{D}_{\mathrm{crys}}^{\Phi=p^{k-2}}(V) \neq 0$, so that $\dim \mathcal{D}_{\mathrm{crys}}(V) = 2$, i.e. that V is crystalline at p.

For $\ell \mid N$ we apply local Euler's characteristic formula and Tate's duality to show that $\mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-i))$ are trivial[3] for $i = 1, 2$. Thus, the S-extensions occuring in the cohomology group $\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1))$ (resp. $\mathrm{H}^1(\mathbb{Q}, \rho_f(k-2))$) are unramified outside p. For proving that the S-extensions occuring in $\mathrm{H}^1(\mathbb{Q}, \epsilon_p)$ are unramified at $\ell \mid N$ we use the semi-continuity of the rank of the monodromy operator attached to the Weil-Deligne representation at ℓ of p-adic families and that the rank is generically one for families of paramodular tame level.

Having bounded the number of generators of $\mathcal{I}^{\mathrm{tot}}$ by s^2 we determine in §8 the local ring $A := \mathcal{T}/\mathcal{I}^{\mathrm{tot}}$ by proving that the completion $\widehat{A}$ of A with respect to its maximal ideal is isomorphic to the universal ring representing the p-ordinary minimally ramified deformations of ρ_f, and which is isomorphic also to the completed local ring of the eigencurve $\mathcal{C}_N$ of tame level N at f_α (thanks to the $R = T$ isomorphism of Taylor-Wiles). The latter is known to be regular thanks to Hida's control theorem[4] [Hid86]. Since $\mathcal{T}$ is equidimensional of dimension 2, $\mathcal{T}/\mathcal{I}^{\mathrm{tot}} = A$ is regular of dimension one (implied by $\widehat{A}$ being regular) and $\mathcal{I}^{\mathrm{tot}}$ is principal when $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ (or more generally generated by at most s^2 elements), it follows that the generator of $\mathcal{I}^{\mathrm{tot}}$ is a regular local parameter of $\mathcal{T}$ when $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ (or more generally, we obtain the desired bound of the Zariski tangent space of $\mathcal{T}$).

This means that the tangent space of $\mathcal{T}$ is of dimension 2 when $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ and $\mathcal{T}$ is regular of dimension 2. Thus the rigid analytic space $\mathcal{E}_\Delta$ is smooth at x, and as a consequence, $\mathcal{E}_\Delta$ has a unique irreducible component (of dimension 2) specializing to x.

However, for the case when $k = 2$ (i.e Thm.A(iii)), we need to prove in addition that the S-extensions occuring $\mathrm{H}^1(\mathbb{Q}, \rho_f)$ are crystalline at p. This seems difficult to establish (see Remark 6.2). But we know that these extensions are ordinary in the sense that they have an unramified line on which Frob_p acts by α, and so they belong to a Greenberg's type Selmer group $\mathrm{Sel}_{\mathbb{Q},f_\alpha}$ attached to $\rho_f^\vee(-1)$ (see §.6.1). Moreover, we know from the Iwasawa main conjecture for GL_2 that the Pontryagin dual of the Λ-adic Greenberg's Selmer group of f_α is a torsion Λ-module, and its characteristic ideal contains the p-adic L function $L_p(f_\alpha, \omega_p^{-1}, .)$

[3] This is where the assumption that $a_\ell = -\ell^{k-2}$ at every prime $\ell \mid N$ is crucial.

[4] Hida's control theorem (or more generally Coleman classicality criterion) yields that $\mathcal{C}_N$ is étale over the weight space at f_α.

(see [SU14, Thm.3.25]). Hence, the condition that $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$ is sufficient for the vanishing of $\mathrm{Sel}_{\mathbb{Q}, f_\alpha}$.

1.2. **Relationship to other results in the literature.** Bellaïche-Chenevier studied in [BC09] the geometry of some eigenvarieties X attached to unitary Shimura varieties at points with reducible Galois representation and gave applications to the Bloch-Kato conjecture. They focus on points $z \in X$ with Galois representation given by $\mathbb{1} \oplus \epsilon_p \oplus \rho_z$, where ρ_z is an irreducible n-dimensional representation anti-ordinary at p. They proved that at $z \in X$, the local Galois deformation at p is irreducible on every Artinian thickening of z (the reducibility locus at z of the pseudo-character carried by X is the maximal ideal of $\mathcal{O}_{X,z}$). It should be pointed out that our setting is quite different since the reducibility locus at π_α of the pseudo-character $\mathrm{Ps}_{\mathcal{E}_\Delta}$ is given by a principal Weil divisor of the 2-dimensional affine scheme $\operatorname{Spec} \mathcal{T}$ and corresponds on the modular side to the Saito-Kurokawa lift of the Hida family passing through f_α. A further difference between these settings lies in the position of the Hodge-Tate weights and their distribution between the different pieces of the reducible Galois representations $\mathbb{1} \oplus \epsilon_p \oplus \rho_z$ and $\rho_{\pi_\alpha} := \epsilon_p^{1-k} \oplus \epsilon_p^{2-k} \oplus \rho_f$. More precisely, while the smallest Hodge-Tate of ρ_{π_α} is zero and occurs in the 2-dimensional representation ρ_f, the smallest Hodge-Tate weight of $\mathbb{1} \oplus \epsilon_p \oplus \rho_z$ is -1 and occurs in the one dimensional sub-representation ϵ_p, and ρ_z has no Hodge-Tate weights equal to $\{0, -1\}$, and this difference makes the proof of the crystalinity of the $S := \mathcal{T}[G_\mathbb{Q}]/\ker(\mathrm{Ps}_\mathcal{T})$-extensions occuring in $\mathrm{Ext}^1_{G_\mathbb{Q}}(\epsilon^{1-k}, \epsilon_p^{2-k})$ (in our setting) more subtle than [BC09, Prop.8.2.14] (see §.5). In addition, we investigate also in this paper the geometry of $\mathcal{E}_\Delta$ at Saito-Kurokawa points π_α of non-cohomological weights (i.e when $k = 2$) and in that case ρ_{π_α} has only two Hodge-Tate weights $\{0, 1\}$ (with multiplicity two).

Skinner-Urban constructed in [SU06, Thm.2.4.10] a semi-ordinary eigenvariety as an admissible open of $\mathcal{E}_N$. Using a deep automorphic argument they established the existence of a stable semi-ordinary p-adic cuspidal eigenfamily $\mathcal{Y}$ of dimension 2 specializing to π_α (see [SU06, Thm.4.2.7]), with fewer assumptions on the level and the local representation ρ_f at $\ell \mid Np$ than us (they assumed only that f is ordinary at p). They then applied the lattice construction of [Urb01] (generalizing Ribet's Lemma to higher dimensions) to obtain a non-trivial extension in $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$.

In [BK17] short crystalline, minimal, essentially self-dual deformations of non-semisimple mod p Galois representations $\overline{\rho}_{SK(f)}$ with $\overline{\rho}^{\mathrm{ss}}_{SK(f)} = \overline{\epsilon}_p^{2-k} \oplus \overline{\rho}_f \oplus \overline{\epsilon}_p^{1-k}$ are studied. In this analysis the principality of the total reducibility ideal of the universal pseudodeformation of $\mathrm{Tr}(\overline{\rho})$ to $\mathcal{O}_L$-algebras also played a crucial role.

Hernandez constructed in [Her17] a three dimensional p-adic eigenvariety for the group $\mathrm{U}(2,1)(E)$, where E is a quadratic imaginary field in which p is inert (the Picard modular surface has an empty ordinary locus in that case), and gave an application by reproving particular cases of the Bloch-Kato conjecture for Galois characters of E.

Acknowledgement. We would like to thank Riccardo Brasca, Kris Klosin, Vincent Pilloni, Jacques Tilouine, Chris Skinner and Eric Urban for helpful communications related to the topics of this article.

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Notation and some remarks.

(i) Let $\mathbb{Q}_p(1)$ denote the $G_\mathbb{Q}$ representation of dimension 1 on which $G_\mathbb{Q}$ acts by the p-adic cyclotomic character $\epsilon_p : G_\mathbb{Q} \twoheadrightarrow \mathbb{Z}_p^\times \hookrightarrow \mathbb{Q}_p^\times$.

(ii) The Hodge-Tate-Sen weight of $\mathbb{Q}_p(1)$ is -1 and its Sen polynomial is $X+1$ (we are following the geometric convention).

(iii) Let B_{crys} denote the crystalline period ring endowed with the semi-linear Frobenius Φ and the natural $G_{\mathbb{Q}_p}$-action.

(iv) Let $t \in B_{\mathrm{crys}}$ be the element on which $G_{\mathbb{Q}_p}$-acts by ϵ_p and $\Phi(t) = p.t$. Note that t generates the maximal ideal of the integral de Rham periods ring B^+_{dR}; i.e $B^+_{\mathrm{dR}}/t.B^+_{\mathrm{dR}} \simeq \mathbb{C}_p$ as $G_{\mathbb{Q}_p}$-modules.

(v) Let $B^+_{\mathrm{crys}} \subset B_{\mathrm{crys}}$ denote the ring of period defined in [PP94, Exposé II, §.2.3].

(vi) Let V be a $G_{\mathbb{Q}_p}$-representation of finite dimension over a p-adic field L. Let $\mathcal{D}_{\mathrm{crys}}(V)$ denote the L-vector space $(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ of dimension at most $\dim_L V$. And we denote again by Φ for the semi-linear action given by $\Phi \otimes \mathrm{Id}_V$ on $\mathcal{D}_{\mathrm{crys}}(V)$. Denote also by $\mathcal{D}^+_{\mathrm{crys}}(V)$ for $(B^+_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$.

(vii) Let $x \in \mathcal{E}_N$ be a classical point such that the Galois representation ρ_x attached to x is crystalline. Then the (Φ, Γ)-module attached to V is trianguline in the sense of Colmez. However, the triangulation can be given by non étale (Φ, Γ)-submodules, and hence $V_{|G_{\mathbb{Q}_p}}$ is not necessarily ordinary at p.

(viii) Remark that $\mathcal{D}^+_{\mathrm{crys}}(\epsilon_p) = 0$, $\mathcal{D}_{\mathrm{crys}}(\epsilon_p) = \mathbb{Q}_p.t^{-1}$ (t^{-1} is not in B^+_{crys}), and $\mathcal{D}^+_{\mathrm{crys}}(\epsilon_p^{-1}) = \mathbb{Q}_p.t$.

(ix) Let $\mathbb{1}$ be the trivial representation of dimension 1.

(x) We shall always write Frob_ℓ for the geometric Frobenius at the prime ℓ.

(xi) Let $\alpha \in \mathbb{Q}$, we shall denote $\mathcal{E}_N^\alpha$ for the admissible open locus of $\mathcal{E}_N$ defined by $| U_0U_1 |_p = \alpha$.

(xii) We write $G_\mathbb{Q}^{Np}$ for the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside of Np and ∞. For any $G_\mathbb{Q}$-geometric representation V we define the Bloch-Kato

Selmer groups

$$\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, V) = \ker(\mathrm{H}^1(\mathbb{Q}, V) \to \mathrm{H}^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}}) \oplus_{\ell \nmid p} \mathrm{H}^1(I_\ell, V))$$

and

$$\mathrm{H}^1_f(\mathbb{Q}, V) = \ker(\mathrm{H}^1(\mathbb{Q}, V) \to \mathrm{H}^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}})).$$

(xiii) Let A be a ring and M be a finite length A-module. We shall always denote by $l(M)$ for the length of M as A-module.

2. Some properties of automorphic p-adic representations

In this section we recall some facts about the Galois representations associated to classical and Siegel modular forms.

2.1. **Elliptic modular forms.** Let ρ_f be the Galois representation attached to a Hecke eigencusp form $f \in \mathrm{S}_{2k-2}(\Gamma_0(N))$ in the sense that $L(\rho_f, s) = L(f, s)$. We note that $\rho_f^\vee \simeq \rho_f(2k-3)$ by the duality of 2-dimensional representations. It is known that ρ_f is de Rham and that its Hodge-Tate-Sen weights are $(2k-3, 0)$. Moreover, ρ_f is crystalline at p since $p \nmid N$.

Since f_α is ordinary at p, $(\rho_f)_{|G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \psi & * \\ 0 & \psi^{-1}\epsilon_p^{3-2k} \end{pmatrix}$, where $\psi : G_{\mathbb{Q}_p} \to \bar{\mathbb{Q}}_p^\times$ is the unramified character such that $\psi(\mathrm{Frob}_p) = \alpha = U_p(f_\alpha)$ and $\det \rho_f = \epsilon_p^{3-2k}$. Note that the characteristic polynomial of the semi linear Frobenius Φ acting of $\mathcal{D}_{\mathrm{crys}}(\rho_f)$ is equal to the p-th Hecke polynomial of f.

Proposition 2.1. *Let $\ell \mid N$ be a prime number.*

(i) Assume that $\pi_{f,\ell} \simeq \mathrm{St} \otimes \xi$ (i.e $a_\ell(f) = -\ell^{k-2}$), then

$$\dim \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\rho_f, \epsilon_p^{2-k}) = \dim \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\epsilon_p^{1-k}, \rho_f) = \dim \mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-2)) = 0.$$

(ii) Assume that $\pi_{f,\ell}$ is special at ℓ, then

$$\dim \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\rho_f, \epsilon_p^{1-k}) = \dim \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\epsilon_p^{2-k}, \rho_f) = \dim \mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-1)) = 0.$$

Remark 2.2. When $k = 2$, the assumption that $a_\ell = -1$ when $\ell \mid N$ is a prime holds if and only if the abelian variety A_f attached to the weight 2 cuspidal eigenform f has non-split multiplicative reduction at ℓ.

Proof. We know, in fact, that $(\rho_f)|_{G_{\mathbb{Q}_\ell}} = \begin{pmatrix} \psi_\ell^{-1} & * \\ 0 & \psi_\ell^{-1}\epsilon_p^{-1} \end{pmatrix}$ with infinite image of inertia, where ψ_ℓ is an unramified character such that $\psi_\ell(\mathrm{Frob}_\ell) = a_\ell(f)$. Note that by [Miy89, Theorem 4.6.17(2)] $a_\ell^2(f) = \ell^{2k-4}$. Our assumption on a_ℓ implies that $\mathrm{H}^0(\mathbb{Q}_\ell, \rho_f(k-1)) = \mathrm{H}^0(\mathbb{Q}_\ell, \rho_f(k-2)) = 0$.

By applying the Euler characteristic formula and Tate duality, we obtain:

$$\dim \mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-1)) = \dim \mathrm{H}^0(\mathbb{Q}_\ell, \rho_f(k-1)) + \dim \mathrm{H}^0(\mathbb{Q}_\ell, (\rho_f(k-1))^\vee(1)).$$

Since $\rho_f^\vee = \rho_f(2k-3)$ (the duality for 2-dimensional representations), the above equality yields that

$$\dim \mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-1)) = 0. \tag{1}$$

The other cases are proved similarly.

□

2.2. **Siegel modular forms.** We define the abstract Hecke algebra $\mathcal{H}_N$ as the $\mathbb{Z}$-algebra generated by the Hecke operators $T_{\ell,1}, T_{\ell,2}, \mathrm{S}_\ell$ for $\ell \nmid Np$ and the Hecke operators U_0, U_1 at p, where $T_{\ell,1}$ (resp. $T_{\ell,2}$, S_ℓ) is the Hecke operator attached to $\mathrm{diag}[1,1,\ell,\ell]$ (resp. $\mathrm{diag}[1,\ell,\ell^2,\ell]$, $\mathrm{diag}[\ell,\ell,\ell,\ell]$), and U_0, U_1.

We recall the p-adic properties of Galois representation arising from Siegel modular eigenforms. The following theorem has been proved by Laumon and Weissauer (see [Wei05] and [Lau05]).

Theorem 2.3. *Let π be a Siegel modular eigenform of central character ω_π of level $\Gamma(N)$ and of cohomological weight $k = (l_1, l_2)$ with corresponding Hecke character $\lambda_\pi : \mathcal{H}_N \to \overline{\mathbb{Q}}_p^*$. Then there exist a p-adic field L_π finite over $\mathbb{Q}_p$ and a continuous representation $\rho_\pi : G_\mathbb{Q} \to \mathrm{GL}_4(L_\pi)$ unramified outside Np and such that for all $\ell \nmid Np$,*

$$\det(X.\mathrm{Id} - \rho_\pi(\mathrm{Frob}_\ell)) = P_{\pi,\ell}(X),$$

where $P_{\pi,\ell}(X)$ is the Hecke-Andrianov polynomial at ℓ attached to π. Moreover, we have the symplectic relation :

$$\overset{\vee}{\rho_\pi} \simeq \rho_\pi \otimes \chi_\pi^{-1}, \tag{2}$$

and $\det \rho_\pi = \chi_\pi^2$. Moreover, we have also the following relation between the similitude character χ_π and the central character:

$$\omega_\pi \epsilon_p^{3-l_1-l_2} = \chi_\pi.$$

We have also the following properties at p of ρ_π following from the works of Chai-Faltings, Laumon, Taylor, Urban and Weissauer (see [Lau05], [Urb05], [Tay93] [Wei05] and [FC90]).

Theorem 2.4. *Under the notations of the above theorem we have :*

(i) *The Galois representation ρ_π is of Hodge-Tate (even de Rham*[5]*) and their Hodge-Tate weights are* $\{0, l_2-2, l_1-1, l_1+l_2-3\}$.

(ii) *If π is old at p, then the p-adic representation ρ_π is crystalline at p, and the characteristic polynomial of Φ acting on $\mathcal{D}_{\mathrm{crys}}(\rho_\pi)$ is the Hecke polynomial at p. The eigenvalues of the semi-linear Frobenius Φ acting on $\mathcal{D}_{\mathrm{crys}}(\rho_\pi)$ are*

$$\{\lambda_\pi(U_0), \lambda_\pi(U_1.U_0^{-1})p^{l_2-2}, \lambda_\pi(U_0.U_1^{-1})^{-1}p^{l_1-1}, \lambda_\pi(U_0)^{-1}p^{l_1+l_2-3}\}.$$

(iii) *Assume that π is semi-ordinary at p (i.e of finite slope for $\mathbb{U} = U_0U_1$ and U_0 acts by a p-adic unit), then*

$$(\rho_\pi)_{|G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \phi_\pi & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & \phi_\pi^{-1}\epsilon_p^{-l_1-l_2+3} \end{pmatrix},$$

where $\phi_\pi : G_{\mathbb{Q}_p} \to \bar{\mathbb{Q}}_p^\times$ is the unramified character having $\lambda_\pi(U_0)$ as value at Frob_p.

One has the following remark for the distinctness of the Hodge-Tate weights of ρ_π.

Remark 2.5. It follows from Arthur's classification that π is weakly equivalent to a generic representation. Hence [Wei05, Thm.III] implies that the Hodge-Tate weights of ρ_π are distinct.

Corollary 2.6. *Assume that π is old at p, non endoscopic and cohomological. Let Z_π be the $G_{\mathbb{Q}_p}$-stable line of $(\rho_\pi)_{|G_{\mathbb{Q}_p}}$ on which $G_{\mathbb{Q}_p}$ acts by ϕ_π, then the subspace $G_{\mathbb{Q}_p}$-stable W_π of dimension 2 of the quotient of $(\rho_\pi)_{|G_{\mathbb{Q}_p}}$ by Z_π is crystalline with Hodge-Tate weight $(l_1 - 1, l_2 - 2)$. Moreover, the eigenvalues of the semi-linear Frobenius Φ acting on $\mathcal{D}_{\mathrm{crys}}(W_\pi)$ are $\lambda_\pi(U_1U_0^{-1})p^{l_2-2}$ and $\lambda_\pi(U_0U_1^{-1})p^{l_1-1}$.*

Remark 2.7. Note that the p-adic Galois representation attached to a cuspidal Siegel eigenform is not necessarily irreducible. Schmidt makes the consequences of Arthur's classification for GSp_4 explicit in [Sch18]. All cuspidal automorphic representations are either of type (G), (Y), (B), (Q), or (P). The latter three are CAP representations, with type (P) for the Siegel parabolic being the Saito-Kurokawa type representations. Type (Y) representations are endoscopic representations ("of Yoshida type"). Type (G) representations are "stable" in the sense that their transfer to GL_4 stays cuspidal, and therefore their Galois representations are expected to be irreducible.

[5]Chai and Faltings constructed a smooth toroidal compactifcation of the Siegel modular scheme and they obtained ρ_π from the etale cohomology of the toroidal compactification with coefficients in a local system given by algebraic representations.

2.3. **Properties at $\ell \neq p$ of a p-adic representation arising form a Siegel cusp form.** We have the following result on the local properties of ρ_π at the primes $\ell \mid N$ (compare [SU06, Conj.3.1.7]) proved by [Mok14, Theorem 3.5] (local-global compatibility up to Frobenius semi-simplification) and [Sor10, Corollary 1] (monodromy rank 1). Mok [Mok14] used Arthur's classification for GSp_4, whose proof was completed by Gee-Taibi in [GT18].

Theorem 2.8. *Under the notations of Theorem 2.3, and assuming that π is non-CAP and non-endoscopic and $\pi^\Delta \neq 0$, the rank of the monodromy operator of the Weil-Deligne representation attached to the Galois representation $(\rho_\pi)_{|G_{\mathbb{Q}_\ell}}$ is at most one when $\ell \mid N$.*

3. Non existence of endoscopic components of $\mathcal{E}_\Delta$ specializing to π_α

Let $\mathcal{C}_N$ be the p-adic eigencurve of tame level N constructed using the Hecke operators U_p and $T_\ell, <\ell>$ for $\ell \nmid Np$. Recall that $\mathcal{C}_N$ is reduced and there exists a flat and locally finite morphism $w : \mathcal{C}_N \to \mathcal{V}$, called the weight map, where $\mathcal{V}$ is the rigid space over $\mathbb{Q}_p$ representing homomorphisms $\mathbb{Z}_p^\times \to \mathbb{G}_m$ (it is a disjoint union of open unit disks $\mathrm{Spm}\, \mathbb{Z}_p[\![T]\!][1/p]$). The eigencurve $\mathcal{C}_N$ was introduced by Coleman-Mazur in the case where the tame level is one (see [CM98]), and by Buzzard and Chenevier for any tame level (see [Buz07] and [Che04] for more details).

Let $\epsilon_p^{\kappa_1} : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{W})^\times$ (resp. $\epsilon_p^{\kappa_2} : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{W})^\times$) be the universal character specializing to $\epsilon_p^{k_1}$ (resp. $\epsilon_p^{k_2}$) at $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2 \subset \mathcal{W}$. Note that the derivative of $\epsilon_p^{\kappa_1}$ (resp. $\epsilon_p^{\kappa_2}$) at 1 is the analytic function $\kappa_1 \in \mathcal{O}(\mathcal{W})$ (resp. $\kappa_2 \in \mathcal{O}(\mathcal{W})$) and the evaluation of (κ_1, κ_2) at any point $\underline{k} \in \mathcal{W}$ is (k_1, k_2).

Coleman, Gouvea and Jochnowitz proved in [CGJ95] that the p-adic modular form

$$G_2(q) = \frac{\zeta(-1)}{2} + \sum_{n=1}^{\infty} \sigma(n) q^n, \text{ where } \sigma(n) = \sum_{d|n} d$$

is not overconvergent, however the p-ordinary p-stabilization $E_2^{ord_p}(q) = G_2(q) - p.G_2(q^p)$ of $G_2(q)$ is classical, hence the critical p-stabilization $E_2^{\mathrm{crit_p}} = G_2(q) - G_2(q^p)$ of $G_2(q)$ is not overconvergent. On the other hand, any ordinary ℓ-stabilization $E_2^{\mathrm{crit_p}, ord_\ell}$ of $E_2^{\mathrm{crit_p}}$ is an overconvergent modular form of weight two and level $\Gamma_0(\ell p)$. Note that $a_{\ell'}(E_2^{\mathrm{crit_p}, ord_\ell}) = 1 + \ell'$ where $\ell' \nmid \ell.p$, and $a_\ell(E_2^{\mathrm{crit_p}, ord_\ell}) = 1$, $a_p(E_2^{\mathrm{crit_p}, ord_\ell}) = p$.

$E_2^{\mathrm{crit_p}, ord_\ell}$ is a cuspidal overconvergent form of tame level $\Gamma_0(\ell)$ since each constant term of its q-expansion is trivial at each cusp of the multiplicative ordinary locus of the the rigid curve attached to the semi-stable modular curve $X_1(\Gamma_1(4\ell) \cap \Gamma_0(p))/\mathbb{Z}_p$ (these cusps are in the $\Gamma_0(p)$-orbit of the standard cusp ∞).

The following proposition is a consequence of [SU06, Thm.3.3.10] and [SU06, 3.2.9].

Proposition 3.1. *Assume* (**Reg**), $k \geq 2$ *and let* $\mathcal{Z}$ *be an irreducible affinoid of* $\mathcal{E}_\Delta$ *of dimension* 2 *specializing to* x *such that the pseudo-character* $\mathrm{Ps}_{\mathcal{Z}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{Z})$ *is reducible, then* $\mathcal{Z}$ *is globally endoscopic. More precisely, there exist an integer* M *(a power of* N*), an affinoid subdomain* $\mathcal{X} = \operatorname{Spm} R$ *of* $\mathcal{Z}$ *containing* x*, an affinoid* $\mathcal{U} \subset \mathcal{C}_M$ *specializing to* f_α*, and an affinoid* $\mathcal{U}^1 \subset \mathcal{C}_M$ *specializing to the system of Hecke eigenvalues of* $E_2^{\mathrm{crit_p}}$ *away from* M*, and a morphism* $j : \mathcal{X} \subset \mathcal{E}_\Delta \to \mathcal{U} \times_{\mathbb{Q}_p} \mathcal{U}^1$ *such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "j=(j_1,j_2)"] \arrow[d, "\kappa"] & \mathcal{U} \times_{\mathbb{Q}_p} \mathcal{U}^1 \arrow[d, "(w\times w)"] \\ \mathcal{W} \arrow[r, "(w_1,w_2)"] & \mathcal{V} \times \mathcal{V} \end{tikzcd}$$

where $(w_1, w_2)(k_1, k_2) = (k_1.k_2[-2], k_1.k_2^{-1}.[2])$ *and* $[n]$ *means the character* ϵ_p^n*. For any* $\lambda_x : \operatorname{Spm} \mathbb{C}_p \to \mathcal{X}$*, we have*

$$\lambda_x(P_\ell(X)) = (X^2 - a(\ell)(j_1(x))X + \ell^{-1} w_1(x)(\ell)\chi(\ell)) \times (X^2 - \epsilon_p^{\kappa_2(x)}(\ell)\ell^{-2} a(\ell)(j_2(x))X + \ell^{-5}\epsilon^{2\kappa_2(x)} w_2(x)(\ell)\chi(\ell)),$$

where $P_\ell(X) \in \mathcal{O}(\mathcal{X})[X]$ *is the Hecke-Andrianov polynomial at* $\ell \nmid Np$ *and* χ *is the Dirichlet character attached to the central character of the family* $\mathcal{X}$*. Moreover, we have also* $U_0(x) = a(p)(j_1(x))$ *and* $U_1(x) = a(p)(j_2(x)).a(p)(j_1(x))$*.*

Proof. Since $\mathcal{Z}$ specializes to x and ρ_f is absolutely irreducible, a subconstituent of the pseudo-character $\mathrm{Ps}_{\mathcal{Z}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{Z}) \to \mathcal{O}_{\mathcal{Z},x}$ is a pseudo-character of dimension 2 whose reducibility locus is of dimension at most one. (One can rule out the existence of a 3-dimensional irreducible constituent by specializing at sufficiently regular classical weights and applying the argument from the proof of Case A(iii) in [SU06, Theorem 3.2.1].) Hence one can find a sufficiently small affinoid neighborhood $\mathcal{X} = \operatorname{Spm} R$ of x with an odd representation $\varrho_1 : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ specializing to the 2-dimensional odd representation ρ_f and such that any classical specialization of ϱ_1 is irreducible, and a representation $\varrho_2 : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ specializing to $\epsilon^{1-k} \oplus \epsilon^{2-k}$ with $\operatorname{Tr}\varrho_1 + \operatorname{Tr}\varrho_2 = \mathrm{Ps}_{\mathcal{X}}$. Moreover, the p-regularity assumption on x (when $k = 2$) and [SU06, Prop.3.3.6] yield (after shrinking again $\mathcal{X}$ to a smaller affinoid which we denote again by $\mathcal{X}$) that ϱ_1 is ordinary at p (in the sense that $\varrho_1^{I_p}$ is a direct summand in ϱ_1 of rank 1). Hence, Theorem [SU06, 3.2.9] implies that any specialization of $\mathcal{X}$ at a classical point $z \in \mathcal{X}$ of a cohomological weight is CAP or endoscopic. Since the Krull dimension of $\mathcal{X}$ is 2, then $\mathcal{X}$ contains a Zariski dense set Σ of classical points of non parallel very regular weights (see Cor.B.4), and then the specialization of $\mathcal{X}$ at these points can not be a CAP form (see [Urb01, Prop.3.3]) and hence necessarily endoscopic by Theorem [SU06, 3.2.9]. Thus $\mathcal{X}$ has a Zariski dense set of classical endoscopic points and hence it is globally endoscopic.

Note that $\mathcal{X}$ has a point with an algebraic weight all of whose Hodge-Tate weights are of multiplicity one (by remark 2.5) and classical points which are old at p are very Zariski dense in $\mathcal{X}$ (see Cor.B.4). One may choose a dense set of classical points of $\mathcal{X}$ old at p, sharing the same Dirichlet character associated to their central characters and endoscopic. Finally, we can now apply [SU06, Thm.3.3.10] to get the desired assertion.

□

One has the following proposition which will be crucial to classify further the Galois representations attached to irreducible components of $\mathcal{E}_\Delta$ specializing to x.

Proposition 3.2.

(i) Let $\mathcal{Y}$ be an irreducible component of the p-adic Eigencurve $\mathcal{C}_N$ of tame level N specializing to the system of Hecke eigenvalues of $E_2^{\mathrm{crit_p}}$ away from N and $\rho_{\mathcal{U}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{\mathcal{U}})$ the Galois representation attached to $\mathcal{U}$, where $K_{\mathcal{U}}$ is the field of fractions of some connected affinoid subdomain $\mathcal{U}$ of $\mathcal{Y}$ containing $E_2^{\mathrm{crit_p}}$, then $\rho_{\mathcal{U}}$ is Steinberg at least one prime $\ell \mid N$ (hence $N \neq 1$).

(ii) Assume that $N \geq 2$ and that f is special at every $\ell \mid N$. Let $\mathcal{F}$ be the Hida family specializing to f_α, then the $\mathbb{I}$-adic Galois representation $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(Q(\mathbb{I}))$ attached to $\mathcal{F}$ is Special at every $\ell \mid N$.

Proof. 1) Let $A := \mathcal{O}_{\mathcal{Y},y}$ be the local ring of $\mathcal{Y} \subset \mathcal{C}_N$ at the point y corresponding to the system of Hecke eigenvalues of $E_2^{\mathrm{crit_p}}$ away from N. One has a pseudo-character

$$G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{C}_N) \tag{3}$$

sending Frob_r to the Hecke operator T_r, where $r \nmid Np$ is a prime number. The localization of the pseudo-character (3) at A gives rise to a pseudo-character

$$\mathrm{Ps}_A : G_{\mathbb{Q}} \to A$$

of dimension 2 and specializing to $\epsilon_p^{-1} \oplus \mathbb{1}$ modulo the maximal ideal of $\mathcal{O}_{\mathcal{Y},y}$. Moreover, Ps_A is the trace of a 2-dimensional irreducible Galois representation $\rho_A : G_{\mathbb{Q}} \to \mathrm{GL}_2(Q(A))$ (since $\mathcal{Y}$ corresponds to a cuspidal Coleman family). Hence, we obtain from ρ_A a non-trivial cohomology class c_y in $\mathrm{H}^1(\mathbb{Q}, \epsilon_p)$ (see [BC09, §.1.5]). The cohomology class c_y corresponds to an extension $V = \mathbb{Q}_p^2$ of ϵ_p^{-1} by $\mathbb{1}$ unramified outisde Np. It is knows that for any classical point y' in $\mathcal{C}_N$, the semi-simple p-adic Galois representation $\rho_{y'} : G_{\mathbb{Q}} \to \mathrm{GL}(V_{y'})$ of dimension 2 attached to the modular form corresponding to y' (i.e.$\mathrm{Tr}\,\rho_{y'}$ is the specialization of (3) at y') has a crystalline periods equal to $U_p(y')$ (see [Kis03]) and it corresponds to its smaller Hodge-Tate weight which is zero (i.e.$\mathcal{D}_{\mathrm{crys}}(V_{y'})^{\Phi=U_p(y')} \neq 0$), hence by using the analytic continuation of the crystalline periods U_p on the Eigencurve $\mathcal{C}_N$ (see [BC09, Thm.4.3.6]), one has $\mathcal{D}_{\mathrm{crys}}(V)^{\Phi=U_p(y)} = \mathcal{D}_{\mathrm{crys}}(V)^{\Phi=p} \neq 0$ (note that $U_p(y) = U_p(E_2^{\mathrm{crit_p}}) = p$). Thus, c_y is crystalline extension of of ϵ_p^{-1} by $\mathbb{1}$, and it belongs to

$$\mathrm{H}^1_f(G_{\mathbb{Q}}^{Np}, \epsilon_p) = \ker(\mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \epsilon_p) \to \mathrm{H}^1(\mathbb{Q}_p, \epsilon_p \otimes \mathrm{B}_{\mathrm{crys}})).$$

Let us proceed now by contradiction. Assume that $\rho_{\mathcal{U}}$ is not Steinberg at any $\ell \mid N$ (i.e the rank of the monodromy operator of the Weil-Deligne representation attached to $\rho_{\mathcal{U}}$ by [BC09, Lemma 7.8.14] at any ℓ is zero), hence $\rho_{\mathcal{U}}$ is principal series or supercuspidal, which implies that for any $\ell \mid N$, the image of the inertia group I_ℓ by $\rho_{\mathcal{U}}$ is finite (we also have a natural inclusion $K_{\mathcal{U}} \subset Q(\mathcal{O}_{\mathcal{Y},y})$, and then semi-simple and reducible. Moreover, $\epsilon_p^{-1} \oplus \mathbb{1}$ is trival on I_ℓ when $\ell \nmid p$, hence $\rho_{\mathcal{U}}$ is unramified outside p.

Thus, the extension c_y is not Steinberg at any $\ell \mid N$ (hence unramified outside p) and it belongs necessarily to $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p)$ which is trivial (the Kummer map provides an isomorphism $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p) \simeq \mathbb{Z}^\times \otimes \mathbb{Q}_p$). Thus, the cohomology class c_y is trivial, contradicting the fact that $\rho_{\mathcal{Y}}$ is absolutely irreducible.

ii) It follows from the semi-continuity of the rank of the monodromy operator of the Weil-Deligne representation at any $\ell \mid N$ of $\rho_{\mathcal{F}}$ (see [BC09, Prop.7.18]) and the fact that ρ_f is special at any ℓ.

□

Using results of Roberts and Schmidt we can show, in fact, that no endoscopic irreducible components $\mathcal{Z} \subset \mathcal{E}_\Delta$ as in Proposition 3.1 exist:

Theorem 3.3.

Assume (**Reg**) *and $k \geq 2$, then any irreducible affinoid $\mathcal{Z}$ of $\mathcal{E}_\Delta$ of dimension two containing x is stable.*

Proof. First consider the case that $N = 1$. Assume $\mathcal{Z} \subset \mathcal{E}_\Delta = \mathcal{E}$ is not stable. Then the Pseudo-character $\mathrm{Ps}_{\mathcal{Z}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{Z})$ is reducible.

Hence, after shrinking $\mathcal{Z}$ to a smaller affinoid subdomain $\Omega = \mathrm{Spm}\, R$ containing x, Propositions 3.1 and 3.2 yield that the pseudo-character $\mathrm{Ps}_R : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{Z}) \to R$ is reducible and Ω must be globally endoscopic and it is the Yoshida lift of the irreducible components $\mathcal{U} \subset \mathcal{C}_M$ passing through f_α and $\mathcal{U}^1 \subset \mathcal{Y}$ of $\mathcal{C}_M$ specializing to the system of Hecke eigenvalues of $E_2^{\mathrm{crit_p}}$. It follows from Proposition 3.2 that the p-adic representation attached to the p-adic family $\mathcal{Y}$ should be ramified at some prime $\ell_0 \neq p$, and yielding a contradiction since $\mathcal{Z}$ is of tame level 1.

For $N > 1$ we argue as follows: Again assume that $\mathcal{Z} \subset \mathcal{E}_\Delta$ is not stable. Then there exists an affinoid subdomain $\Omega = \mathrm{Spm}\, R$ of $\mathcal{Z}$ containing x such that the Pseudo-character $\mathrm{Ps}_R : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{Z}) \to R$ is reducible. Hence, Propositions 3.1 and 3.2 yield that Ω must be globally endoscopic and contains a point with non-parallel classical weight (l_1, l_2) specializing to a Yoshida lift of classical eigencuspforms of tame level $\Gamma_0(N)$ and weight $l_1 + l_2 - 2$ and $l_1 - l_2 + 2 > 2$, respectively, and such that both are Steinberg at ℓ_0. In fact, no such Yoshida lift (of tame level the paramodular group Δ) exists, as we can see by considering the local representations of the corresponding automorphic representations: By Proposition 3.2(i) there exists $\ell_0 \mid N$ such that both the corresponding local representations of $\mathrm{GL}_2(\mathbb{Q}_{\ell_0})$ are Steinberg or twisted Steinberg by a non-trivial unramified quadratic character, depending on their Atkin-Lehner eigenvalue at ℓ_0. By the following result of Roberts and Schmidt their Yoshida lift corresponds to a local representation of $\mathrm{GSp}_4(\mathbb{Q})$ which has no paramodular fixed vector under the paramodular subgroup Δ_ℓ.

□

Remark 3.4. When $N = 1$, Skinner-Urban used in [SU02] a simpler argument to obtain a contradiction and their argument is based on the fact that $E_2^{\mathrm{crit}_p}$ is a p-adic modular form but not overconvergent, and so $\mathcal{Y}$ can not specialize to it (since the specializations of $\mathcal{Y}$ are overconvergent).

Proposition 3.5 (Roberts-Schmidt). *Let τ_1, τ_2 be either a Steinberg representation* St *of* $\mathrm{GL}_2(\mathbb{Q}_\ell)$ *(or Steinberg representation twisted by unramified quadratic character). Via the endoscopic embedding these define a local packet for* $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ *with two elements, neither of which has fixed vectors under the paramodular subgroup Δ_ℓ.*

Proof. By table (16) in [SS13] the local packets are either $\{\mathrm{Va}, \mathrm{Va}^*\}$ or $\{\mathrm{VIa}, \mathrm{VIb}\}$. By [RS07] Theorem 3.4.3 and Table A.15 none of these have fixed vectors under Δ_ℓ. □

4. The GMA S and ordinarity of S-extensions occurring in $\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1))$

Recall that Theorem 3.3 implies that all irreducible components of $\mathcal{E}_\Delta$ passing through x are stable, and that $\mathcal{T}$, the local ring of $\mathcal{E}_\Delta$ at x, is reduced and equidimensional of dimension 2 since $\mathcal{E}_\Delta$ is reduced and equidimensional of dimension 2. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{T}$ and L be the residue field of $\mathcal{T}$.

Let A be a reduced Noetherian ring. Recall that the total fraction ring of A is the fraction ring $Q(A) := \mathcal{S}^{-1}A$ where $\mathcal{S} \subset A$ is the multiplicative subset of nonzerodivisors of A. We check at once that the natural map $A \to \mathcal{S}^{-1}A$ is injective and flat, and that the non-zerodivisors of A are invertible in $\mathcal{S}^{-1}A$. Moreover, since A is Noetherian the zero divisors of A are the elements of the union of the (finitely many) minimal primes ideal of A, so $\mathcal{S}^{-1}A = \prod_{\mathcal{P}_i} A_{\mathcal{P}_i}$, where $\mathcal{P}_i$ runs over the minimal prime ideals of A. Moreover, each $A_{\mathcal{P}_i}$ is a field, since it is reduced, local and of Krull dimension equal to zero. Let $K = \prod K_i$ be the total field of fractions of the reduced equidimensional ring $\mathcal{T}$, where K_i is the localisation of $\mathcal{T}$ at a minimal prime ideal.

Definition 4.1 (Definition/Proposition)**.** The pseudo-character[6]

$$\mathrm{Ps}_{\mathcal{T}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{T}$$

is residually multiplicity free and the corresponding Cayley-Hamilton faithful algebra

$$S := \mathcal{T}[G_{\mathbb{Q}}]/\ker \mathrm{Ps}_{\mathcal{T}}$$

can by [BC09, Thm.1.4.4(i)] be equipped with the structure of a GMA (in the sense of [BC09, Defn. 1.3.1]). It is of finite type and torsion-free as $\mathcal{T}$-module. Since $\mathcal{T}$ is reduced we further have an associated Galois representation $\rho_K : G_{\mathbb{Q}} \to \mathrm{GL}_4(K)$ by [BC09, Thm.1.4.4(ii)]. Note that $\rho_K : G_{\mathbb{Q}} \to \mathrm{GL}_4(K)$ is absolutely irreducible, since all the minimal prime ideals of $\mathcal{T}$ correspond to stable irreducible components of $\mathcal{E}_\Delta$ passing through x (so each Galois representation $\rho_{K_i} : G_{\mathbb{Q}} \to \mathrm{GL}_4(K_i)$ is irreducible).

Assume until the end of this paper that $\alpha \neq 1$ when $k = 2$ (which we will refer to as "p-adic regularity"). Recall that $\varrho = \begin{pmatrix} \epsilon_p^{2-k} & 0 & 0 \\ 0 & \rho_f & 0 \\ 0 & 0 & \epsilon_p^{1-k} \end{pmatrix}$ is the Galois representation attached to π_α in a basis such that $\varrho(\tau) \sim \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$, where the eigenvalues of $\tau \in G_{\mathbb{Q}_p}$ are all distinct (since $\alpha \neq 1$ when $k = 2$) (necessarily in this basis $\varrho(G_{\mathbb{Q}_p}) \sim \begin{pmatrix} \epsilon_p^{2-k} & 0 & 0 & 0 \\ 0 & \psi & * & 0 \\ 0 & 0 & \psi^{-1}\epsilon_p^{3-2k} & 0 \\ 0 & 0 & 0 & \epsilon_p^{1-k} \end{pmatrix}$).

Remark 4.2. Note that the character ϕ_{π_α} of Theorem 2.4(iii) equals ψ since $U_p(f_\alpha) = \alpha.f_\alpha$ and $U_0(\pi_\alpha) = \alpha.\pi_\alpha$.

Since all reducible components of ϱ have multiplicity one, [BC09, Thm.1.4.4] implies that there exist orthogonal idempotents $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of $S = \mathrm{Im}(\mathcal{T}[G_{\mathbb{Q}}] \to M_4(K))$ lifting the idempotents e_1, e_2, e_3 of ϱ, and corresponding respectively to $\epsilon_p^{2-k}, \rho_f, \epsilon_p^{1-k}$. Moreover, we can see S as

$$S = \begin{pmatrix} \mathcal{T} & M_{1,2}(\mathcal{T}_{1,2}) & \mathcal{T}_{1,3} \\ M_{2,1}(\mathcal{T}_{2,1}) & M_2(\mathcal{T}) & M_{2,1}(\mathcal{T}_{2,3}) \\ \mathcal{T}_{3,1} & M_{1,2}(\mathcal{T}_{3,2}) & \mathcal{T} \end{pmatrix},$$

[6] The pseudo-character $\mathrm{Ps}_{\mathcal{T}}$ is obtained by composing $\mathrm{Ps}_{\mathcal{E}_\Delta}$ with the localization map $\mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{T}$.

where $\mathcal{T}_{i,j}$ are fractional ideals of K ($\mathcal{T}_{i,j}$ are finite type $\mathcal{T}$-modules).

Put $\rho_1 = \epsilon_p^{2-k}$, $\rho_2 = \rho_f$ and $\rho_3 = \epsilon_p^{1-k}$. We recall Bellaïche and Chenevier's definition of reducibility ideals:

Definition 4.3 ([BC09] Definition 1.5.2, Proposition 1.5.1)**.** Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_s)$ be a partition of the set $\mathcal{I} = \{1, 2, 3\}$. The ideal of reducibility $I^{\mathcal{P}}$ (associated to $\mathrm{Ps}_{\mathcal{T}}$ and the partition $\mathcal{P}$) is the smallest ideal I of $\mathcal{T}$ with the the property that there exist pseudocharacters $T_1, \ldots, T_s : \mathcal{T}/I[G_{\mathbb{Q}}] \to \mathcal{T}/I$ such that

(i) $\mathrm{Ps}_{\mathcal{T}} \otimes \mathcal{T}/I = \sum_{l=1}^{s} T_l$,
(ii) for each $l \in \{1, \ldots, s\}$, $T_l \otimes L = \sum_{i \in \mathcal{P}_l} \mathrm{trace}\rho_i$.

Proposition 4.4 ([BC09] Proposition 1.5.1, [BK17] Corollary 6.5)**.** *One has*

$$I^{\mathcal{P}} = \sum_{\substack{(i,j) \\ i,j \text{ not in the same } \mathcal{P}_l}} \mathcal{T}_{i,j}\mathcal{T}_{j,i}.$$

For $\mathcal{P} = \{\{1\}, \{2\}, \{3\}\}$ we write

$$\mathcal{I}^{\mathrm{tot}} := \mathcal{I}^{\mathcal{P}} = \mathcal{T}_{3,1}\mathcal{T}_{1,3} + \mathcal{T}_{2,3}\mathcal{T}_{3,2} + \mathcal{T}_{1,2}\mathcal{T}_{2,1}.$$

Let $\mathcal{T}'_{i,j} = \mathcal{T}_{i,k}\mathcal{T}_{k,j}$ for i, j, k distinct. Since the maximal ideal $\mathfrak{m}$ of $\mathcal{T}$ contains the total reducibility ideal $\mathcal{I}^{\mathrm{tot}}$ [BC09, Theorem 1.5.5] implies that for $i \neq j \in \{1, 2, 3\}$ there exists an injective homomorphism of L-modules

$$\mathrm{Hom}(\mathcal{T}_{i,j}/\mathcal{T}'_{i,j}, L) \hookrightarrow \mathrm{H}^1(\mathbb{Q}_{Np}, \rho_i \otimes \rho_j^{\vee} \otimes L). \tag{4}$$

Theorem 4.5. *Assume that $\alpha \neq 1$ when $k = 2$. For $(i, j) = (1, 2)$ the injective homomorphism of L-modules of (4) gives rise to*

$$\mathrm{Hom}(\mathcal{T}_{1,2}/\mathcal{T}'_{1,2}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) \tag{5}$$

Proof. The proof of [BC09, Theorem 1.5.5] tells us that the homomorphism (4) is given by

$$\begin{aligned} \mathrm{Hom}(\mathcal{T}_{1,2}/\mathcal{T}'_{1,2}, L) &\hookrightarrow \mathrm{H}^1(\mathbb{Q}, \rho_f(k-1)) \\ h &\mapsto (g \to h(\bar{b}_1(g), \bar{b}_2(g))\rho_f^{-1}(g)), \end{aligned} \tag{6}$$

where $(\bar{b}_1(g), \bar{b}_2(g))$ is the class of $t_{1,2}(g) = (b_1(g), b_2(g)) \in M_{1,2}(\mathcal{T}_{1,2})$ in $M_{1,2}(\mathcal{T}_{1,2}/\mathcal{T}'_{1,2})$. The classical points old at p and of regular weights form a very Zariski dense set Σ in every irreducible component of $\mathcal{E}_\Delta$ specializing to x (see Lemma B.2 and [SU06, Prop.3.3.6]). By Theorem 2.4, the set of Hodge-Tate-Sen weights of the semi-simple representation ρ_y attached to any point $y \in \mathcal{E}_\Delta$ corresponding to a classical cuspidal Siegel eigenform old at p of weight (l_1, l_2) is $\{0, l_2 - 2, l_1 - 1, l_1 + l_2 - 3\}$ and ρ_y is crystalline at p.

On the other hand, for any $y \in \Sigma$, let us denote by $\rho_y : G_{\mathbb{Q}} \to \mathrm{GL}_4(L_y)$ the semi-simple p-adic Galois representation attached to the Siegel eigenform corresponding to y (i.e.$\mathrm{Tr}\,\rho_y$ is the specialization of the universal pseudo-character $\mathrm{Ps}_{\mathcal{E}_\Delta} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta)$ at y). Theorem 2.4 implies that $\dim \mathcal{D}^+_{\mathrm{crys}}(\rho_y)^{\Phi=U_0(y)} = 1$, and then $(\mathrm{Ps}_{\mathcal{E}_\Delta}, \Sigma, U_0, \{\kappa_i\})$ is a weakly refined family (in the sense of [BC09, def.4.2.7]) since $U_0 \in \mathcal{O}(\mathcal{E}_\Delta)^\times$. Note also that condition (*) of [BC09, Def.4.2.7] is satisfied since we have a torsion free morphism $\kappa : \mathcal{E}_\Delta \to \mathcal{W}$; condition (v) of [BC09, Def.4.2.7] is satisfied by Lemma A.6, Lemma B.2 and Corollary B.4 (so the classical points of $\mathcal{E}_\Delta$ which are old at p accumulate to x).

Moreover, $\dim \mathcal{D}^+_{\mathrm{crys}}(\varrho)^{\Phi=\alpha=U_0(\pi_\alpha)} = 1$ by regularity assumption on ϱ at p. Hence, [BC09, Thm.4.3.6] implies that any $G_{\mathbb{Q}}$-representation V corresponding to a cohomology class in the image of the morphism (6) satisfies

$$\dim \mathcal{D}^+_{\mathrm{crys}}(V)^{\Phi=\alpha} = 1.$$

We use this to first prove that V is crystalline at p. One can see V as the following $G_{\mathbb{Q}}^{Np}$-extension:

$$0 \to \epsilon_p^{2-k} \to V \to \rho_f \to 0.$$

Let $\tilde{\rho} = \begin{pmatrix} \epsilon_p^{2-k} & * \\ 0 & \rho_f \end{pmatrix}$ be the realization of V by a matrix. The restriction to $G_{\mathbb{Q}_p}$ of $\tilde{\rho}$ has the form $\begin{pmatrix} \epsilon_p^{2-k} & b & c \\ 0 & \psi & * \\ 0 & 0 & \psi^{-1}\epsilon_p^{3-2k} \end{pmatrix}$. Hence, we have an extension of $G_{\mathbb{Q}_p}$-modules

$$0 \to \begin{pmatrix} \epsilon_p^{2-k} & b \\ 0 & \psi \end{pmatrix} \to \tilde{\rho}_{|G_{\mathbb{Q}_p}} \to \psi^{-1}\epsilon_p^{3-2k} \to 0.$$

Let $V^0 \subset V$ be the L-vector space of dimension 2 on which $G_{\mathbb{Q}_p}$ acts by $\begin{pmatrix} \epsilon_p^{2-k} & b \\ 0 & \psi \end{pmatrix}$. By applying the left exact functor $\mathcal{D}^+_{\mathrm{crys}}(.)^{\Phi=\alpha}$ to the above exact sequence, we obtain

$$\mathcal{D}^+_{\mathrm{crys}}(V^0)^{\Phi=\alpha} \simeq \mathcal{D}^+_{\mathrm{crys}}(\tilde{\rho}_{|G_{\mathbb{Q}_p}})^{\Phi=\alpha}.$$

Since $\dim \mathcal{D}_{\mathrm{crys}}(V)^{\Phi=\alpha} = 1$, we get $\dim \mathcal{D}^+_{\mathrm{crys}}(V^0)^{\Phi=\alpha} = 1$. Hence, $V_0 = \begin{pmatrix} \epsilon_p^{2-k} & b \\ 0 & \psi \end{pmatrix}$ is crystalline at p which implies that the cohomology class of b in $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\psi, \epsilon_p^{2-k})$ is trivial (i.e.$\tilde{\rho}_{|G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \epsilon_p^{2-k} & 0 & c \\ 0 & \psi & * \\ 0 & 0 & \psi^{-1}\epsilon_p^{3-2k} \end{pmatrix}$). Thus, $\tilde{\rho}$ is ordinary in the sense of Fontaine and Perrin-Riou [PR94] and then semi-stable (hence de Rham) at p. Therefore the extension V gives a cohomology class in

$$\mathrm{H}^1_g(G_{\mathbb{Q}}^{Np}, \rho_f(k-1)) = \ker(\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1)) \to \mathrm{H}^1(\mathbb{Q}_p, \rho_f(k-1) \otimes B_{\mathrm{dR}})).$$

Since $\mathrm{H}^1_g(G_{\mathbb{Q}}^{Np}, \rho_f(k-1)) \simeq \mathrm{H}^1_f(G_{\mathbb{Q}}^{Np}, \rho_f(k-1))$ (see e.g. [SU06, Lemme 4.1.3]) we deduce that V is crystalline at p.

Finally, the restriction of the map

$$\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1)) \to \mathrm{H}^1(I_\ell, \rho_f(k-1)),$$

when $\ell \mid N$ is trivial, since it factors through the restriction

$$\mathrm{H}^1(\mathbb{Q}, \rho_f(k-1)) \to \mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-1)),$$

and the cohomology group $\mathrm{H}^1(\mathbb{Q}_\ell, \rho_f(k-1))$ is trivial thanks to Proposition 2.1.

□

4.1. **Symplectic relation and the the anti-involution τ on S.** Recall that

$$\mathrm{Ps}_{\mathcal{E}_\Delta} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta) \tag{7}$$

are pseudo characters of dimension 4 and since the classical points of $\mathcal{E}_\Delta$ are Zariski dense, the relation (2) implies that the pseudo-character $\mathrm{Ps}_{\mathcal{T}}$ is invariant under the anti-involution

$$\tau : \mathcal{T}[G_{\mathbb{Q}}] \to \mathcal{T}[G_{\mathbb{Q}}] \text{ sending } g \to \chi_x.g^{-1},$$

where χ_x is a character $G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{U})^\times$ interpolating the similitude character of the $G_{\mathbb{Q}}$-semi-simple representations whose trace correspond to the classical specializations of the pseudo-character $\mathrm{Ps}_{\mathcal{U}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{O}(\mathcal{U})$ for an enough small affinoid neighborhood $\mathcal{U}$ of x. More precisely, χ_x is equal to $\omega_{\mathcal{U}}.\epsilon_p^{-\kappa_1-\kappa_2+3}$, where $\omega_{\mathcal{U}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{U})^\times$ is the character interpolating the central character of the classical specializations of $\mathcal{U}$.

Hence τ yields an anti automorphism on S given by $\rho_K \circ \tau$ and it follows from [BC09, Lemma.1.8.3] that we can choose our idempotent $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of S lifting the idempotents e_1, e_2, e_3 attached respectively to $\epsilon_p^{2-k}, \rho_f, \epsilon_p^{1-k}$, and such that $\tilde{e}_{\tau(1)} = \tilde{e}_3$ (τ preserves the idempotent corresponding to ρ_f, and switches the idempotents corresponding to ϵ_p^{1-k}, ϵ_p^{2-k}).

By (4) there exists an injection

$$\mathrm{Hom}(\mathcal{T}_{2,3}/\mathcal{T}'_{2,3}, L) \hookrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}}^{Np}}(\epsilon_p^{1-k}, \rho_f) \simeq \mathrm{H}^1(\mathbb{Q}, \rho_f(k-1)). \tag{8}$$

Proposition [BC09, Prop.1.8.6] yields immediately the following corollary.

Corollary 4.6. *The image of* (8) *lands in* $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ *and has dimension equal to the dimension of the image of the morphism* (5).

5. Crystallinity of the S-extensions occuring in $\mathrm{H}^1(\mathbb{Q}, \epsilon_p)$

In this section we show using the analytic continuation of the crystalline periods in a family of p-adic $G_{\mathbb{Q}_p}$-representations of generic rank 3 interpolating $\{\rho_z/\rho_z^{I_p}, z \in \mathcal{E}_\Delta^{|U_0|_p=1}\}$ the crystallinity of the S-extensions occuring in $\mathrm{H}^1(\mathbb{Q}, \epsilon_p)$. Assume in this section (**Reg**) and that $k \geq 2$.

By (4) we have a natural injection

$$\begin{aligned} \mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}'_{1,3}, L) &\hookrightarrow \mathrm{Ext}^1_{G^{Np}_{\mathbb{Q}}}(\epsilon_p^{1-k}, \epsilon_p^{2-k}) \simeq \mathrm{H}^1(G^{Np}_{\mathbb{Q}}, \epsilon_p) \\ h &\mapsto (g \to \frac{h(\bar{t}_{1,3}(g))}{\epsilon_p^{1-k}(g)}), \end{aligned} \tag{9}$$

where $\bar{t}_{1,3}(g)$ is the class of $t_{1,3}(g) \in \mathcal{T}_{1,3}$ in $\mathcal{T}_{1,3}/\mathcal{T}'_{1,3}$.

Now we have to determine the exact image of the injective morphism (9). In [BC09, §1.5.4], Bellaïche-Chenevier introduce a left ideal $M_3 = S.E_3$ of $S \subset M_4(K)$ which is the third column of the GMA matrix S and hence it is a projective left S-module (see [BC09, 1.3.3] for the definition of E_3), and they proved in [BC09, Thm.1.5.6] and [BC09, Lemma.4.3.9] the following results:

(i) There exists an exact sequence of S-left modules

$$0 \to E \to M_3/\mathfrak{m}M_3 \to \epsilon_p^{1-k} \to 0 \tag{10}$$

(ii) Any simple S-subquotient of E occurs in the set $\{\rho_f, \epsilon_p^{2-k}\}$ (in particular it is not isomorphic to ϵ_p^{1-k}).
(iii) The image of the morphism (9) consists of extensions occurring as quotient of the $S/\mathfrak{m}S$-module $M_3/\mathfrak{m}M_3 \oplus \epsilon_p^{2-k}$ by an S-submodule $\mathcal{Q}$ such that the S-simple subquotient of $\mathcal{Q}$ occurs in $\{\rho_f, \epsilon_p^{2-k}\}$ (in particular it is not isomorphic to ϵ_p^{1-k}).

We will need the following additional property:

Lemma 5.1. *Let* S_p *be the subring generated by the image of* $G_{\mathbb{Q}_p}$ *in* S*. Then the* S_p*-simple subquotients of* $\mathcal{Q}$ *occur in* $\{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{3-2k}\}$.

Proof. Let Ps_p be the restriction of $\mathrm{Ps}_\mathcal{T}$ to S_p. By [BC09, Lemma 1.2.7] we have $S/\mathrm{rad}(S) \cong \overline{S}/\ker\overline{\mathrm{Ps}}$ and $\mathrm{S}_p/\mathrm{rad}(\mathrm{S}_p) \cong \overline{S}_p/\ker\overline{\mathrm{Ps}}_p$, hence $\mathrm{rad}(S) \cap \mathrm{S}_p \subset \mathrm{rad}(\mathrm{S}_p)$, and we obtain a morphism $\mathrm{S}_p/\mathrm{rad}(S)\cap\mathrm{S}_p \twoheadrightarrow \mathrm{S}_p/\mathrm{rad}(\mathrm{S}_p) = \overline{S}_p/\ker\overline{\mathrm{Ps}}_p \cong \prod_{i=1}^4 \mathrm{End}_L(\rho_i)$, where $\rho_i \in \{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{2k-3}, \epsilon_p^{1-k}\}$. In particular, one can see that all $\{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{2k-3}, \epsilon_p^{1-k}\}$ are simple S_p-modules. Now, we claim that any simple S_p-representation occurs in $\{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{2k-3}, \epsilon_p^{1-k}\}$, and it follows immediately from the injection $\mathrm{S}_p/\mathrm{rad}(S) \cap \mathrm{S}_p \hookrightarrow S/\mathrm{rad}(S) \simeq \overline{S}/\ker\overline{\mathrm{Ps}} \cong \prod_{i=1}^3 \mathrm{End}_L(\rho_i)$ whose image is $\prod_{i=1}^3 \mathrm{End}_L((\rho_i)_{|G_{\mathbb{Q}_p}})$ (so $\mathrm{S}_p/\mathrm{rad}(\mathrm{S}_p)$ is a semi-simple quotient of $\prod_{i=1}^3 \mathrm{End}_L((\rho_i)_{|G_{\mathbb{Q}_p}})$).

The rest of the lemma follows from the fact (see (10)(iii)) that the S-module $\mathcal{Q}$ has a Jordan-Holder sequence, all subquotients of which are isomorphic to either ρ_f or ϵ_p^{2-k}, and it has a refinement as S_p-module for which the S_p-simple subquotients occur in $\{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{2k-3}\}$. □

We recall that a torsion-free A-module is a module over a ring A such that 0 is the only element annihilated by a regular element (i.e non-zero-divisor of A) of the ring. A coherent sheaf $\mathcal{F}$ over a rigid analytic space X is a sheaf of $\mathcal{O}_X^{\mathrm{rig}}$-modules such that there exists an admissible covering of X by affinoid subdomains $\{U_i = \mathrm{Spm}\, R_i\}$ of X for which the restriction $\mathcal{F}_{|U_i}$ is associated to $\tilde{M}_i$ and M_i is a finite type R_i-module.

The sheaf $\mathcal{F}$ is said to be torsion-free if all those modules M_i are torsion-free over their respective rings. Alternatively, $\mathcal{F}$ is torsion-free if and only if it has no local torsion sections.

Lemma 5.2.

(i) One has $M_3 \subset K^4$ and $M_3.K = K^4$. Moreover, M_3 is a $\mathcal{T}$-torsion-free lattice of the representation $\rho_K \to \mathrm{GL}_4(K)$.

(ii) The natural morphism $M_3 \to M_3 \otimes_{\mathcal{T}} K$ is injective and the natural morphism

$$M_3 \otimes_{\mathcal{T}} K \to M_3.K$$

is an isomorphism.

Proof. i) Note that the finite type $\mathcal{T}$-module M_3 corresponds to the third column of the GMA matrix $S \subset M_4(K)$, hence $M_3 \subset K^4$. Since $\rho_K : G_{\mathbb{Q}} \to S^\times \subset \mathrm{GL}_4(K)$ is irreducible, then $M_3.K$ is necessarily of rank 4 over K.

ii) Recall that $M_3 \otimes_{\mathcal{T}} K = M_3 \otimes_{\mathcal{T}} \mathcal{S}^{-1}\mathcal{T}$, where $\mathcal{S}$ is the set of non-zero divisors of $\mathcal{T}$. Hence, $M_3 \otimes_{\mathcal{T}} K = \mathcal{S}^{-1}M_3$ and the injection follows from the fact that M_3 is torsion-free. Moreover, to see that the natural surjection $M_3 \otimes_{\mathcal{T}} K \twoheadrightarrow M_3.K = K^4$ is an isomorphism, comparing the ranks is sufficient, and it is enough to see that $M_3 \otimes_{\mathcal{T}} K$ contains $M_3.K$ (which is obvious from the inclusion $M_3 \to M_3 \otimes_{\mathcal{T}} K$). □

Theorem 5.3. *The image of the injective morphism of L-modules*

$$\mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}'_{1,3}, L) \hookrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}}^{Np}}(\epsilon_p^{1-k}, \epsilon_p^{2-k}) \simeq \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \epsilon_p)$$

lands in $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p)$.

Proof. To simplify notation, let M denote the finite type S-module M_3. We recall that M is a torsion-free finite type $\mathcal{T}$-module, because S is of finite type over $\mathcal{T}$ and $M \subset S \subset M_4(K)$. According to [BC09, Lemma.4.3.7], there exists an open affinoid neighborhood $\mathcal{U} = \operatorname{Spm} A$ of x inside $\mathcal{E}_\Delta$ such that we can extend M to an analytic torsion-free coherent sheaf $\tilde{\mathcal{M}}$ over $\mathcal{U}$ ($\mathcal{M}$ is the A-module associated to $\tilde{\mathcal{M}}$) and such that:

(i) $Q(A) \otimes \mathcal{M} = Q(A)^4$ (i.e the generic rank of $\mathcal{M}$ is 4 [7])

(ii) $\mathcal{M} \otimes_A \mathcal{T} = M$ (i.e the stalk of $\tilde{\mathcal{M}}$ at x is M).

(iii) The A-module $\mathcal{M}$ carries a continuous action of $G_{\mathbb{Q}}$ compatible with the action of $G_{\mathbb{Q}}$ on its localization M at x, and the generic representation $G_{\mathbb{Q}} \to \mathrm{GL}_4(Q(A))$ is semi-simple and its trace is just the trace given by $\mathrm{Ps}_A : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta) \to A$.

On the other hand, by semi-ordinarity at p, the action of I_p on $Q(A)^4$ stabilizes a line $(Q(A)^4)^{I_p}$ on which Frob_p acts by U_0. Let $\tilde{\mathcal{L}}$ be the subsheaf of $\mathcal{M}$ given by $(Q(A)^4)^{I_p} \cap \mathcal{M}$ (i.e the sections of $\tilde{\mathcal{L}}$ are the sections of $\tilde{\mathcal{M}}$ on which I_p acts trivially and Frob_p acts by U_0). Moreover, $\tilde{\mathcal{L}}$ is the coherent sheaf associated to the A-submodule $\mathcal{L}$ of $\mathcal{M}$ given by the elements which are invariant under the actions of the inertia I_p and on which Frob_p acts by U_0.

Let $\tilde{\mathcal{M}}'_+$ be the quotient presheaf $\tilde{\mathcal{M}}/\tilde{\mathcal{L}}$ and $\tilde{\mathcal{M}}'$ be the sheaf associated to the presheaf $\tilde{\mathcal{M}}'_+$, and it is $\widetilde{\mathcal{M}/\mathcal{L}}$ since $\mathcal{U}$ is an affinoid, and is endowed naturally with an action of $G_{\mathbb{Q}_p}$.

Let $M' := \mathcal{M}' \otimes_A \mathcal{T}$. Since $M_K := M \otimes_{\mathcal{T}} K = M.K = K^4$, it is obvious that M' is a $\mathcal{T}$-submodule of $K^4/(K^4)^{I_p}$, where $(K^4)^{I_p}$ means the I_p-invariant subspace on which Frob_p acts by U_0. Hence, M' is a finite type torsion-free $\mathcal{T}$-module of generic rank 3 over K, and the regularity assumption when $k = 2$ yields that the $G_{\mathbb{Q}_p}$-semi-simplification $M' \otimes_{\mathcal{T}} L$ doesn't contain ψ.

Similarly, since $Q(A) \subset K$ and $(K^4)^{I_p} \cap \mathcal{M} = \mathcal{M}^{I_p}$, we obtain that $\mathcal{M}'$ injects into $K^4/(K^4)^{I_p}$ and $\mathcal{M}'$ is torsion-free over A and with generic rank equal to 3. Moreover, the regularity assumption yields that the $G_{\mathbb{Q}_p}$-semi-simplification of its specialization at $\pi_\alpha = x$ does not contain the character $\psi_{|G_{\mathbb{Q}_p}}$.

In fact, Corollary 2.6 implies the characteristic polynomial of the semi-linear Frobenius Φ acting on the crystalline module of almost of the classical specializations y of $\mathcal{M}'$ has no root equal to $U_0(y)$.

Let $Z = V(\mathcal{I}) \subset \mathcal{U}$ be the Zariski closed set defined by the ideal $\mathcal{I}$ generated by the 4-th Fitting ideal Fitt_4 of the A-module $\mathcal{M}$ and by the 3-rd Fitting ideal Fitt_3 of the A-module $\mathcal{M}'$, then any point y lies in $Z = V(\mathrm{Fitt}_4)$ (resp. $V(\mathrm{Fitt}_3)$) if and only if $\dim_{k(y)}(\mathcal{M}(y)) \geq 5$ (resp. $\dim_{k(y)}(\mathcal{M}'(y)) \geq 4$), where $\mathcal{M}(y)$ (resp. $\mathcal{M}'(y)$) is the fiber of $\mathcal{M}$(resp. $\mathcal{M}'$) at y and $k(y)$ is the residue field at y.

[7] We have to choose $\operatorname{Spm} A$ small enough in the aim that it is connected and it contains no more irreducible components than $\operatorname{Spec} \mathcal{T}$, to have a natural inclusion $Q(A) \subset K$.

Thus $\mathcal{U} - V(\mathcal{I})$ is the biggest admissible open subset of $\mathcal{U}$ on which $\mathcal{M}$ (resp. $\mathcal{M}'$) can be locally generated (on stalks) by 4 elements (resp. 3 elements). Moreover, since the coherent $\mathcal{M}$ (resp. $\mathcal{M}'$) is generically of rank 4 (resp. 3) and torsion-free then one can deduce that the coherent sheaf $\mathcal{M}$ (resp. $\mathcal{M}'$) is locally free of rank 4 (resp. 3) on the admissible open $\mathcal{U} - Z = \mathcal{U}'$ ($\mathcal{U}'$ does not necessarily contain x). Thus, the direct summand $\mathcal{M}'$ of $\mathcal{M}$ is also locally free of rank 3 on $\mathcal{U}'$. Hence one can deduce that the Hodge-Tate weights of the specialization of $\mathcal{M}'$ at classical points of $\mathcal{U}'$ of weight $l_1 > l_2 + 1$ and having crystalline representation (they form a very Zariski dense set) are $l_2 - 2, l_1 - 1, l_1 + l_2 - 3$; and then $l_2 - 2$ is the smallest Hodge-Tate weight (see Corollary 2.6).

In addition, if $\mathcal{M}'(y)$ (resp. $\mathcal{M}(y)$) denotes the specialization of $\mathcal{M}'$ (resp. $\mathcal{M}$) at a very classical point $y \in \mathcal{U}'$. We can enlarging Z if it is necessary to have that for any $y \in \mathcal{U}'$, $\mathcal{M}(y)^{ss} = \mathcal{M}(y)$. Now, if $y \in \mathcal{U}'$ is a classical point of weight (l_1, l_2) and ρ_y is a crystalline representation at p, then the eigenvalues of the semi-linear Frobenius Φ acting on $\mathcal{D}_{\mathrm{crys}}(\mathcal{M}'(y))$ are $\lambda_y(U_1U_0^{-1})p^{l_2-2}$, $\lambda_y(U_0U_1^{-1})p^{l_1-1}$ and $\lambda_y(U_0^{-1})p^{l_2+l_1-3}$, where $\lambda_y : \mathrm{Spm}\, L_y \to \mathcal{E}_\Delta$ is the morphism corresponding to y. When $y = x$, we have $\lambda_x(U_1U_0^{-1}) = p$ ($\underline{k} = (k, k)$ is the weight of π_α).

The exact sequence (10) (i.e.ϵ_p^{1-k} occurs with multiplicity one in $\mathcal{M}'/\mathfrak{m}\mathcal{M}'$), the regularity assumption (i.e.$\alpha \neq 1$) of ϱ at p when $k = 2$, and the fact that $\mathcal{M}' \otimes_A \mathcal{T} = M'$ (since $\mathcal{M}^{I_p} \otimes_A \mathcal{T} = M^{I_p}$), yield that

$$\dim \mathcal{D}^+_{\mathrm{crys}}(\mathcal{M}'(x)^{ss})^{\Phi = p^{k-1}} = \dim \mathcal{D}^{\Phi = p^{k-1}}_{\mathrm{crys}}(\epsilon_p^{1-k}) = 1. \tag{11}$$

Hence, one has (after a twist by ϵ^{k-2})

$$\dim \mathcal{D}^+_{\mathrm{crys}}(\mathcal{M}'(x)^{ss}(k-2))^{\Phi = p} = 1. \tag{12}$$

Since the set Σ of classical points of $\mathcal{E}_\Delta$ of cohomological weights and old at p (i.e having a crystalline representation) of $\mathcal{E}_\Delta$ are very Zariski dense (see Cor.B.4), it follows from Lemma A.7 that $\Sigma \cap \mathcal{U} - (\Sigma \cap Z)$ is Zariski dense in $\mathcal{U}$, and hence we obtain a refined family

$$(G_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathcal{U}}(\mathcal{M}'), \Sigma \cap \mathcal{U} - (\Sigma \cap Z), \{\kappa_i\}, U_0/U_1 \in \mathcal{O}(\mathcal{E}_\Delta)^\times)$$

of generic rank equal to 3 over K. Note also that condition (*) of [BC09, Def.4.2.7] is satisfied since we have a torsion free morphism $\kappa : \mathcal{E}_\Delta \to \mathcal{W}$; the condition (v) of [BC09, Def.4.2.7] is satisfied by Lemma A.6, Lemma B.2 and Corollary B.4 (so $\Sigma \cap \mathcal{U} - (\Sigma \cap Z)$ accumulate to x).

Since $\mathcal{M}' \otimes_A \mathcal{T} = M'$, it follows from [BC09, Thm.3.4.1] that

$$\dim \mathcal{D}^+_{\mathrm{crys}}(M'/\mathfrak{m}M'(k-2))^{\Phi = p} = 1.$$

Then

$$\dim \mathcal{D}^{+}_{\mathrm{crys}}(M'/\mathfrak{m}M')^{\Phi=p^{k-1}} = 1.$$

Finally, by [BC09, Thm.1.5.6] any $S/\mathfrak{m}S$-extension V of ϵ_p^{1-k} by ϵ_p^{2-k} (i.e occurring in the image of the morphism (9)) is a quotient of $M/\mathfrak{m}M \oplus \epsilon_p^{2-k}$ by an S-submodule $\mathcal{Q}$ (see (iii) of (13)).

However, by the regularity assumption at p the non-trivial unramified character $\psi_{|G_{\mathbb{Q}_p}}$ does not occur in $V_{|G_{\mathbb{Q}_p}} \in \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\epsilon_p^{1-k}, \epsilon_p^{2-k})$, which implies that $V_{|G_{\mathbb{Q}_p}}$ is a quotient of $M'/\mathfrak{m}M' \oplus \epsilon_p^{2-k}$. Thus we obtain a surjection of $G_{\mathbb{Q}_p}$-modules

$$M'/\mathfrak{m}M' \oplus \epsilon_p^{2-k} \overset{\pi'}{\twoheadrightarrow} V_{|G_{\mathbb{Q}_p}}, \tag{13}$$

with kernel isomorphic to a quotient of the $G_{\mathbb{Q}_p}$-module $\mathcal{Q}$.

Since the semi simplification of $M_3/\mathfrak{m}M_3$ is isomorphic to the representation

$$\rho_f^{n_1} \oplus (\epsilon_p^{2-k})^{n_2} \oplus \epsilon_p^{1-k}$$

by (10), the regularity assumption at p on ϱ when $k=2$ (i.e.$\alpha \neq 1$), and the fact that the S_p-simple subquotients of $\mathcal{Q}$ do not equal ϵ_p^{1-k} by Lemma 5.1 (they occur in $\{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{3-2k}\}$), one has

$$\mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(\ker(\pi')) = 0.$$

Thus the surjective morphism (13) yields the following injection

$$\mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(M'/\mathfrak{m}M') \hookrightarrow \mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(V),$$

and implies that $\mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(V) \neq 0$ (since $\dim \mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(M'/\mathfrak{m}M') = 1$).

On the other hand, by applying the left exact functor $\mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(.)$ to the exact sequence

$$0 \to \epsilon_p^{2-k} \to V \to \epsilon_p^{1-k} \to 0,$$

and using the fact that $\mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(\epsilon_p^{2-k}) = 0$ and $\dim \mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(\epsilon_p^{1-k}) = 1$, we obtain that $\dim \mathcal{D}^{\Phi=p^{k-1}}_{\mathrm{crys}}(V) = 1$ (since it is non-zero by the above discussion). Hence the characteristic polynomial of Φ has two roots $\{p^{k-2}, p^{k-1}\}$ yielding that $\dim \mathcal{D}_{\mathrm{crys}}(V) = 2$ and that V is crystalline, so $V \otimes \epsilon_p^{k-2}$ is also crystalline at p.

It remains to proof that the image of the map

$$\mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}'_{1,3}, L) \hookrightarrow \mathrm{H}^1_f(G^{Np}_{\mathbb{Q}}, \epsilon_p)$$

consists of extensions which are unramified outside p. Let ℓ denote a prime number dividing N (so prime to p), note that any $G_{\mathbb{Q}_\ell}$-extension of ϵ_p^{-1} by $\mathbb{1}$ is trivial or its restriction to the inertia has the following form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and hence the monodromy operator of the Weil-Deligne representation attached to its 2-dimensional $G_{\mathbb{Q}_\ell}$-representation is of rank 1 (i.e a Steinberg type).

We know that the rank over L of the monodromy operator attached to the Weil-Deligne representation corresponding to $(\rho_f)_{|G_{\mathbb{Q}_\ell}}$ is one (since we assumed that ρ_f is a twisted Steinberg at every prime $\ell \mid N$).

Recall that the $G_{\mathbb{Q}}$-coherent sheaf $\mathcal{M}$ is locally free of rank 4 on the admissible open $\mathcal{U} - Z = \mathcal{U}'$ and it admits a Weil-Deligne representation $(r_{\mathcal{U}}, N_{\mathcal{U}})$ by [BC09, Lemma.7.8.14] at ℓ (for which $N_{\mathcal{U}} \in \mathrm{End}_A(\mathcal{M})$. Since the rank of the monodromy operator of the Weil-Deligne representation attached to the specializations of $(r_{\mathcal{U}}, N_{\mathcal{U}})$ at classical points of non-endoscopic, non-CAP points $\mathcal{U}'$ is at most 1 by Theorem 2.8, [BC09, Prop.7.8.19(ii)] implies that the generic rank over K of the monodromy operator of the Weil-Deligne representation attached $(r_{\mathcal{U}}, N_{\mathcal{U}})$ is also 1 (since it is non-trivial at x). Therefore, the generic rank of the monodromy $N_K = N_{\mathcal{U}} \otimes K$ operator of the Weil-Deligne representation attached to $(\rho_K)_{|G_{\mathbb{Q}_\ell}}$ is one.

Let S_ℓ be the image of $\mathcal{T}[G_{\mathbb{Q}_\ell}]$ inside S. Thanks to Proposition 2.1, one can apply [BC09, Lemma.8.2.11] [8] to $\mathcal{P} = \{\epsilon_p^{1-k}, \epsilon_p^{2-k}\}$), and we obtain that there exists idempotents $(\tilde{e_1}, \tilde{e_2}, \tilde{e_3})$ of S lifting the idempotents attached respectively to $\epsilon_p^{2-k}, \epsilon_p^{1-k}, \rho_f$ and such that $\tilde{e} = \tilde{e_1} + \tilde{e_2}$ is in the center of S_ℓ (see [BC09, Lemma.8.2.12]), and hence S_ℓ is block diagonal of type $(2,2)$ in S. Thus,

$$\mathrm{S}_\ell/\mathfrak{m}\mathrm{S}_\ell = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \rho_f \end{pmatrix}, \text{ and } \mathrm{S}_\ell = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & M_{2,2}(\mathcal{T}) \end{pmatrix}.$$

By [BC09, Lemma.7.8.14] one can see N_K as element of S_ℓ. By (10)(iii) it is enough to prove that $\tilde{e}N_K \in \tilde{e}\mathrm{S}_\ell$ is trivial for showing that the image of $\mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}'_{1,3}, L) \hookrightarrow \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \epsilon_p)$ gives rise to classes unramified at ℓ.

As an element of S_ℓ we know that $N_K = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$ and it is of rank 1 as discussed before. By [BC09, Prop.7.8.8] applied to $(1-\tilde{e})\mathrm{S}_\ell(1-\tilde{e})$ we further know that the rank of $(1-\tilde{e})N_K$ is one, using that ρ_f is a twisted Steinberg at ℓ (the rank of the monodromy operator of $WD_\ell(\rho_f)$ is one) and the surjection $(1-\tilde{e}).S/\mathfrak{m}S.(1-\tilde{e}) \twoheadrightarrow \rho_f$. Hence, $\tilde{e_i}N_K = 0$ for $i \in \{1,2\}$, which yields that $\tilde{e}N_K = 0$.

□

The proof of Theorem 5.3 yields the following corollary.

[8] The assumption that $\pi_{f,\ell} \simeq \mathrm{St} \otimes \xi$ is crucial to prove the vanishing of $\mathrm{H}^1(G_{\mathbb{Q}_\ell}, \rho_f(k-2))$.

Corollary 5.4. *There exists a representation $\rho_{\mathcal{M}'} : G_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathcal{U}}(\mathcal{M}')$, where $\mathcal{M}'$ is a torsion-free coherent sheaf on an admissible open affinoid $\mathcal{U} = \mathrm{Spm}\, A \subset \mathcal{E}_\Delta$ containing x and $\rho_{\mathcal{M}'}$ is of generic rank 3 over the total ring of fractions K of $\mathcal{U}$ such that:*

(i) There exists a very Zariski dense set $\Sigma' \subset \mathcal{U}$ such that the specialization of the representation $\rho_{\mathcal{M}'}$ at any point z of Σ' gives rise to a crystalline $G_{\mathbb{Q}_p}$-representation ρ'_z of dimension 3, with Hodge-Tate-Sen weights given by $(\kappa_2 - 2, \kappa_1 - 1, \kappa_1 + \kappa_2 - 3)$.

(ii) The smallest Hodge-Tate weight of ρ'_z is $\kappa_2(z) - 2$ and $U_1/U_0 \in \mathcal{O}(\mathcal{E}_\Delta)^\times$ interpolates the crystalline period of the smallest Hodge-Tate weight. In other words, one has

$$\dim \mathcal{D}_{\mathrm{crys}}(\rho'_z)^{\Phi = U_1/U_0(z) p^{\kappa_2(z)-2}} = 1.$$

(iii) Let $M' := \mathcal{M}' \otimes_A \mathcal{T}$, then for any cofinite ideal $\mathcal{J}$ of $\mathcal{T}$ one has that

$$l(\mathcal{D}^+_{\mathrm{crys}}(M'/\mathcal{J}M' \otimes (\epsilon_p^{\kappa_2 - 2}))^{\Phi = U_1/U_0}) = l(\mathcal{T}/\mathcal{J}).$$

(iv) The Sen operator of $\mathcal{D}_{\mathrm{sen}}(M'/\mathcal{J}M')$ is annihilated by the Polynomial

$$(T - (\kappa_2 - 2))(T - (\kappa_1 - 1))(T - (\kappa_1 + \kappa_2 - 3)).$$

Proof. i) and ii) follows directly from the proof of Theorem 5.3 and [BC09, Thm.1.5.6]. Thus, it remains to show iii) and iv), which follows immediately from similar arguments to those already used to prove of Thm.5.3, [BC09, Thm.3.4.1] and [BC09, Lemma.4.3.3](i). □

6. Crystallinity of the S-extensions occurring in $\mathrm{H}^1(\mathbb{Q}, \rho_f(k-2))$

By (4) we have a natural injection

$$\mathrm{Hom}(\mathcal{T}_{3,2}/\mathcal{T}'_{3,2}, L) \hookrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}}^{Np}}(\rho_f, \epsilon_p^{1-k}) \simeq \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \rho_f(k-2)). \tag{14}$$

Now we have to determine the exact image of the injective morphism (14). As in section 5 we apply the results of [BC09, Thm.1.5.6] and [BC09, Lemma.4.3.9] for the left ideal $M_2 = S.E_2$ of S given by the second column of the GMA matrix S:

(i) There exists an exact sequence of S-left modules

$$0 \to E' \to M_2/\mathfrak{m}M_2 \to \rho_f \to 0 \tag{15}$$

(ii) Any simple S-subquotients of E' is not isomorphic to ρ_f and they occur in the set $\{\epsilon_p^{1-k}, \epsilon_p^{2-k}\}$.

(iii) The image of the morphism (14) consists of extensions occurring as quotient of the $S/\mathfrak{m}S$-module $M_2/\mathfrak{m}M_2 \oplus \epsilon_p^{1-k}$ by an S-submodule $\mathcal{Q}'$ whose S-simple subquotients occur in the set $\{\epsilon_p^{1-k}, \epsilon_p^{2-k}\}$.

Since ρ_K is absolutely irreducible and M_2 is a finite type torsion free $\mathcal{T}$-module we again have $M_2.K = K^4$.

Theorem 6.1. *Assume that $\pi_{f,\ell} = \mathrm{St} \otimes \xi$ for any $\ell \mid N$, and that $\alpha \neq 1$ when $k = 2$. Let $\mathcal{T}'_{3,2}$ be the $\mathcal{T}$-module $\mathcal{T}_{3,1}\mathcal{T}_{1,2} \subset \mathcal{T}_{3,2}$, then:*

(i) There exists an injective homomorphism of L-modules

$$\mathrm{Hom}(\mathcal{T}_{3,2}/\mathcal{T}'_{3,2}, L) \hookrightarrow \ker(\mathrm{H}^1(\mathbb{Q}, \rho_f(k-2)) \to \mathrm{H}^1(\mathbb{Q}_p, \rho_f/\rho_f^{I_p}(k-2)) \oplus_{\ell \nmid p} \mathrm{H}^1(I_\ell, \rho_f(k-2))). \tag{16}$$

(ii) Assume that $k \geq 3$, then

$$\mathrm{Hom}(\mathcal{T}_{3,2}/\mathcal{T}'_{3,2}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-2)). \tag{17}$$

Proof.

i) By (15) we have a surjective morphism of S-modules $\pi : M_2/\mathfrak{m}M_2 \twoheadrightarrow \rho_f$ whose kernel does not contain ρ_f and whose semi-simplification contains only $G_{\mathbb{Q}}$-representations lying in the set $\{\epsilon_p^{1-k}, \epsilon_p^{2-k}\}$. Moreover, our assumptions yield that the irreducible constituents of the semi-simplification of $\varrho_{|G_{\mathbb{Q}_p}}$ are without multiplicity, hence $M_2^{I_p} := \{x \in M_2, \forall g \in I_p, g.x = x \text{ and } \mathrm{Frob}_p .x = U_0.x\}$ is not contained in $\mathfrak{m}M_2$. Let $V \in \mathrm{Ext}^1_{G_{\mathbb{Q}}}(\rho_f, \epsilon_p^{1-k}) = \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \rho_f(k-2))$ be in the image of (17). By (15) (iii) we have an exact sequence of left S-modules

$$0 \to \mathcal{Q}' \to M_2/\mathfrak{m}M_2 \oplus \epsilon_p^{1-k} \to V \to 0.$$

Similar to Lemma 5.1 we can show that $\mathcal{Q}'$ has no $L[G_{\mathbb{Q}_p}]$-simple subquotients equal to ψ or $\psi^{-1}\epsilon_p^{2k-3}$. This shows that the image of $M_2^{I_p}$ in V is non-zero. It follows that

$$V^{I_p} \neq 0.$$

Moreover, since Frob_p acts on $M_2^{I_p}$ by U_0, the action of Frob_p on V^{I_p} is given by ψ. If the realization of V is given by $\tilde{\rho} = \begin{pmatrix} \epsilon_p^{1-k} & * \\ 0 & \rho_f \end{pmatrix}$ then the restriction of $\tilde{\rho}$ to $G_{\mathbb{Q}_p}$ is given by $\begin{pmatrix} \epsilon_p^{1-k} & 0 & c \\ 0 & \psi & * \\ 0 & 0 & \psi^{-1}\epsilon_p^{3-2k} \end{pmatrix}$ since $V^{I_p} \neq 0$. Finally, it remains to show that the extensions V are unramified at every prime $\ell \mid N$, and this fact follows immediately from Proposition 2.1.

ii) The fact that $k \geq 3$ implies that $3 - 2k < 1 - k$ and hence $\tilde{\rho}$ is ordinary in the sense of Fontaine and Perrin-Riou and hence de Rham at p. Therefore the extension V gives a cohomology class in $\mathrm{H}^1_g(G_{\mathbb{Q}}^{Np}, \rho_f(k-2))$ which is isomorphic to $\mathrm{H}^1_f(G_{\mathbb{Q}}^{Np}, \rho_f(k-2))$ by [SU06, Lemme 4.1.3].

$\square$

Remark 6.2. For $k = 2$ ordinarity/crystallinity of the extension would require us to prove additionally that $\tilde{\rho}/\tilde{\rho}^{I_p} \cong \begin{pmatrix} \epsilon^{-1} & c \\ 0 & \psi^{-1}\epsilon^{-1} \end{pmatrix}$ is a trivial extension. This would follow, e.g. if one could prove that the generator of $\mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \rho_f)$ (which is conjecture to be 1-dimensional by Jannsen, see e.g. has no line fixed by inertia at p). See section 6.1 below for an alternative approach in this case.

Similarly to Corollary 4.6, [BC09, Prop.1.8.6] yields immediately the following corollary.

Corollary 6.3. *The image of the natural injective morphism of L-modules*

$$\mathrm{Hom}(\mathcal{T}_{2,1}/\mathcal{T}_{2,3}\mathcal{T}_{3,1}, L) \hookrightarrow \mathrm{H}^1(\mathbb{Q}, \rho_f(k-2))$$

is isomorphic to the image of (14) *(which is described in Thm.6.1).*

6.1. **On the vanishing of the Greenberg's Selmer group attached to f_α.** Assume in this subsection that $k = 2$ and let

$$\mathrm{Sel}_{\mathbb{Q},f_\alpha} = \ker(\mathrm{H}^1(\mathbb{Q}_{Np}, \rho_f) \to \mathrm{H}^1(\mathbb{Q}_p, \rho_f/\rho_f^{I_p}) \bigoplus_{\ell \nmid p} \mathrm{H}^1(I_\ell, \rho_f))$$

be the Greenberg-type Selmer group we used in Theorem 6.1(i) attached to the ordinary elliptic cuspform f_α. In the literature, Greenberg's Selmer group is often defined using the representation $\rho_f^\vee(-1)$ (arithmetic Frobenius convention). The p-adic representation $\rho_f^\vee$ corresponds to the Tate module $T_p(A_f)$ of the abelian variety A_f, and ρ_f is the Galois representation obtained from the p-adic étale cohomology of A_f. We remark also that for $k = 2$ the condition at p is weaker than the "usual" condition for the ordinary representation ρ_f (which would require the class to be split at p). Our condition of having an I_p-fixed quotient for the extension $\begin{pmatrix} \rho_f & * \\ 0 & 1 \end{pmatrix}$ (or dually an I_p-fixed line for $\begin{pmatrix} \epsilon^{-1} & * \\ 0 & \rho_f \end{pmatrix}$) is the one that would normally be required for $\rho_f(1) \cong \rho_f^\vee$.

Note that ρ_f is not critical in the sense of Deligne. We use Iwasawa theory for the cyclotomic $\mathbb{Z}_p$-extension to bound $\mathrm{Sel}_{\mathbb{Q},f_\alpha}$: It follows from Kato [Kat04] that the the Pontryagin dual of the Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty,f_\alpha}$ is a torsion Λ-module with characteristic ideal $g(T) \in \Lambda$. Furthermore, according to the Iwasawa main conjecture (Kato's bound, see e.g. [SU14, Thm.3.25]), $g(T) \mid L_p(f, \omega^{-1}, .)$. Hence $\dim \mathrm{Sel}_{\mathbb{Q},f_\alpha} = 0$ when $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$ (see [BK17, Prop.2.10] and [BK17, Thm.2.11] for more details). Moreover, it follows from the control theorem for the Λ-adic Greenberg's Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty,f_\alpha}$ (see [Och01]) that $g(T = p) \neq 0$ is a necessary condition for the vanishing of $\mathrm{Sel}_{\mathbb{Q},f_\alpha}$.

7. Schematic reducibility locus of the pseudo-character $\mathrm{Ps}_{\mathcal{T}}$ on $\operatorname{Spec}\mathcal{T}$ and applications to the Bloch-Kato conjecture

Recall that we view S as the generalized matrix attached to the pseudo-character

$$\mathrm{Ps}_{\mathcal{T}} : G_{\mathbb{Q}} \to \mathcal{T}$$

with respect to a set of idempotents compatible with the anti-involution τ and have

$$S = \begin{pmatrix} \mathcal{T} & M_{1,2}(\mathcal{T}_{1,2}) & \mathcal{T}_{1,3} \\ M_{2,1}(\mathcal{T}_{2,1}) & M_2(\mathcal{T}) & M_{2,1}(\mathcal{T}_{2,3}) \\ \mathcal{T}_{3,1} & M_{1,2}(\mathcal{T}_{3,2}) & \mathcal{T} \end{pmatrix},$$

where $\mathcal{T}_{i,j}$ are fractional ideals of K that satisfy $\mathcal{T}_{i,j}\mathcal{T}_{j,k} \subset \mathcal{T}_{i,k}$ and $\mathcal{T}_{i,j}\mathcal{T}_{j,i} \subset \mathfrak{m}$.

In this section we will compute the total reducibility ideal $\mathcal{I}^{\mathrm{tot}} \subset \mathcal{T}$ (see Definition 4.3). By Proposition 4.4 it is given by

$$\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{3,1}\mathcal{T}_{1,3} + \mathcal{T}_{2,3}\mathcal{T}_{3,2} + \mathcal{T}_{1,2}\mathcal{T}_{2,1}. \tag{18}$$

The following lemma follows directly from the anti involution $\tau : S \to S$ and the fact that $\mathrm{Ps}_{\mathcal{T}}$ is invariant under the action of τ.

Lemma 7.1. *One always has:*
$\mathcal{T}_{2,3}\mathcal{T}_{3,2} = \mathcal{T}_{1,2}\mathcal{T}_{2,1}$.

Proof. This is proved exactly as in Lemma [BC09, 8.2.16] using the anti involution τ. □

Hence, the above lemma implies that

$$\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{3,1}\mathcal{T}_{1,3} + \mathcal{T}_{1,2}\mathcal{T}_{2,1}. \tag{19}$$

Lemma 7.2. *We have, in fact, that*

$$\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{1,2}\mathcal{T}_{2,1} = \mathcal{T}_{2,3}\mathcal{T}_{3,2}.$$

Proof. We first show that $\mathcal{T}_{1,3} = \mathcal{T}'_{1,3} = \mathcal{T}_{1,2}\mathcal{T}_{2,3}$. By Theorem 5.3 we have an injective map

$$\mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}'_{1,3}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p).$$

Note that the Kummer map provides an isomorphism

$$\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p) \simeq \mathbb{Z}^{\times} \otimes L.$$

Hence $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p)$ is trivial, and then $\mathcal{T}_{1,3}/\mathcal{T}'_{1,3} = 0$ by Nakayama's lemma ($\mathcal{T}_{1,3}$ is of finite type over $\mathcal{T}$ since S is). Thus, we have

$$\mathcal{T}_{1,3} = \mathcal{T}_{1,2}\mathcal{T}_{2,3}. \tag{20}$$

It is easy to see that

$$\begin{aligned}\mathcal{I}^{\mathrm{tot}} &= \mathcal{T}_{3,1}\mathcal{T}_{1,3} + \mathcal{T}_{1,2}\mathcal{T}_{2,1}\\ &= \mathcal{T}_{1,2}\mathcal{T}_{2,3}\mathcal{T}_{3,1} + \mathcal{T}_{1,2}\mathcal{T}_{2,1}\\ &= \mathcal{T}_{1,2}\mathcal{T}_{2,1}, \text{ since } \mathcal{T}_{2,3}\mathcal{T}_{3,1} \in \mathcal{T}_{2,1}.\end{aligned} \tag{21}$$

□

Corollary 7.3. *One has*

$$\mathcal{T}'_{1,2} = \mathcal{I}^{\mathrm{tot}}.\mathcal{T}_{1,2}.$$

Proof. Since $\mathcal{T}_{1,3} = \mathcal{T}_{1,2}\mathcal{T}_{2,3}$ by relation (20) we get $\mathcal{T}'_{1,2} = \mathcal{T}_{1,3}\mathcal{T}_{3,2} = \mathcal{T}_{1,2}\mathcal{T}_{2,3}\mathcal{T}_{3,2}$. On the other hand, we have by Lemma 7.1 that $\mathcal{T}_{2,3}\mathcal{T}_{3,2} = \mathcal{T}_{1,2}\mathcal{T}_{2,1}$, and we have also by Lemma 7.2 $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{1,2}\mathcal{T}_{2,1} = \mathcal{T}_{2,3}\mathcal{T}_{3,2}$. Thus $\mathcal{T}'_{1,2} = \mathcal{T}_{1,3}\mathcal{T}_{3,2} = \mathcal{T}_{1,2}\mathcal{T}_{2,3}\mathcal{T}_{3,2} = \mathcal{I}^{\mathrm{tot}}\mathcal{T}_{1,2}$.

□

7.1. **Application to Bloch-Kato conjecture.** Since we have assumed that the sign ϵ_f of $L(f,s)$ is -1, the functional equation

$$L(f,s) = -L(f,1-s)$$

yields that $L(f,s)$ vanishes at the central value $k-1$. The Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q},\rho_f(k-1))$ classifies the extensions with everywhere good reduction and one can think of the Bloch-Kato conjecture as a generalization of the Birch and Swinnerton-Dyer conjecture for the motive M_f corresponding to f of weight $2k-2 \geq 2$. On has the following application related to the Bloch-Kato conjecture:

Corollary 7.4. *Assume that $k \geq 2$, $\pi_{f,\ell} \simeq \mathrm{St} \otimes \xi$ (i.e $a_\ell = -\ell^{k-2}$) for any $\ell \mid N$ and* (**Reg**)*, then there exists an injection*

$$\mathrm{Hom}(\mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q},\rho_f(k-1)), \tag{22}$$

and $\dim \mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2} \geq 1$.

Proof. The following injection follow from Theorem 4.5 and Corollary 7.3:

$$\mathrm{Hom}(\mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2}, L) \simeq \mathrm{Hom}(\mathcal{T}_{1,2}/\mathcal{I}^{\mathrm{tot}}.\mathcal{T}_{1,2}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q},\rho_f(k-1)) \tag{23}$$

Moreover, $\mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2} \neq \{0\}$ since $\rho_K : G_{\mathbb{Q}} \to \mathrm{GL}_4(K)$ is absolutelly irreducible (so $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{1,2}\mathcal{T}_{2,1} \neq (0)$).

□

Proposition 7.5. $\dim \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \epsilon_p^{-1}) = 1$.

Proof. It follows from [Maj15, Prop.2.2]. □

Assume now that $\bar{\rho}$ is absolutely irreducible. Let $\mathbb{I}$ be the finite flat integral extension of the Iwasawa algebra $\mathbb{Z}_p[\![T]\!]$ generated by the coefficients of a Hida family $\mathcal{F}$ specializing to f_α ($\mathcal{F}$ is unique up to Galois conjugacy) and let $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I})$ be the p-adic Galois representation attached to $\mathcal{F}$. Let $\chi_{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathbb{Z}_p[\![T]\!]^\times$ be the universal character given by the composition of the p-adic cyclotomic character $\epsilon_p : G_{\mathbb{Q}} \to 1 + p^\nu \mathbb{Z}_p$ with the tautological character $1 + p^\nu\mathbb{Z}_p \to \mathbb{Z}_p[\![1 + p^\nu\mathbb{Z}_p]\!]^\times \simeq \mathbb{Z}_p[\![T]\!]^\times$, where $\nu = 1$ if $p \geq 3$ and $\nu = 2$ if $p = 2$. It follows from the work of Nekovar [Nek06, Prop.4.2.3] that the $\mathbb{I}$-adic Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}} \otimes \chi_{\mathrm{univ}}^{-1/2})$ is of finite type over the Iwasawa algebra $\mathbb{Z}_p[\![T]\!]$, and so over $\mathbb{I}$ since $\mathbb{I}$ is finite flat over $\mathbb{Z}_p[\![T]\!]$.

Corollary 7.6. *Assume that $\bar{\rho}$ is absolutely irreducible,* (**Min**) *and* (**Reg**)*, then the generic rank of the $\mathbb{I}$-adic Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}} \otimes \chi_{\mathrm{univ}}^{-1/2})$ is at least one (i.e. $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}} \otimes \chi_{\mathrm{univ}}^{-1/2})$ has a non torsion class over $\mathbb{I}$).*

Proof. It follows from Corollary 7.4 that the $\mathbb{I}$-adic Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}} \otimes \chi_{\mathrm{univ}}^{-1/2})$ specializes at infinitely many classical points of $\mathrm{Spm}\,\mathbb{I}[1/p]$ to a non-trivial Selmer group. Hence, the generic rank of $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}} \otimes \chi_{\mathrm{univ}}^{-1/2})$ over $\mathbb{I}$ is non zero. □

7.2. Bounding the number of generators of $\mathcal{I}^{\mathrm{tot}}$.

Theorem 7.7. *Assume* (**Reg**) *and that* $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$.

(i) There exists idempotents $\{e'_1, e'_2, e'_3\}$ of S lifting the idempotents of ϱ attached to $\{\epsilon_p^{2-k}, \rho_f, \epsilon_p^{1-k}\}$ such that S has the following form

$$S = \begin{pmatrix} \mathcal{T} & M_{1,2}(\mathcal{T}) & \mathcal{T} \\ M_{2,1}(\mathcal{I}^{\mathrm{tot}}) & M_2(\mathcal{T}) & M_{2,1}(\mathcal{T}) \\ \mathcal{T}_{3,1} & M_{1,2}(\mathcal{I}^{\mathrm{tot}}) & \mathcal{T} \end{pmatrix},$$

where $\mathcal{T}_{3,1} = \mathcal{J} \subset \mathcal{I}^{\mathrm{tot}}$ is an ideal.

(ii) Assume $k \geq 3$. Then $\mathcal{I}^{\mathrm{tot}} = \mathcal{J} = \mathcal{T}_{3,1}$ and $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}.g + (\mathcal{I}^{\mathrm{tot}})^2$ for an element g in $\mathcal{I}^{\mathrm{tot}}$, and yielding that the reducibility ideal $\mathcal{I}^{\mathrm{tot}}$ is principal and generated by g.

(iii) Assume that $k = 2$ and $\dim \mathrm{Sel}_{\mathbb{Q},f_\alpha} = 0$, then $\mathcal{I}^{\mathrm{tot}} = \mathcal{J} = \mathcal{T}_{3,1}$ and $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}.g + (\mathcal{I}^{\mathrm{tot}})^2$ for an element g in $\mathcal{I}^{\mathrm{tot}}$, and the reducibility ideal $\mathcal{I}^{\mathrm{tot}}$ is principal and generated by g.

Remark 7.8. Using results about Λ-adic Selmer groups we exhibit many examples where the Selmer group $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ is 1-dimensional (see Appendix §.C).

Proof. i) By Theorem 4.5 and Corollary 7.3, we have the following:

$$\mathrm{Hom}(\mathcal{T}_{1,2}/\mathcal{I}^{\mathrm{tot}}.\mathcal{T}_{1,2}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) \tag{24}$$

Moreover, since $\mathcal{I}^{\mathrm{tot}} \subset \mathfrak{m}$, we have an injection

$$\mathrm{Hom}(\mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2}, L) \hookrightarrow \mathrm{Hom}(\mathcal{T}_{1,2}/\mathcal{I}^{\mathrm{tot}}.\mathcal{T}_{1,2}, L).$$

By the assumption on the dimension of $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$ we get $\dim \mathrm{Hom}(\mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2}, L) \leq 1$. On the other hand, the fact that $\rho_K : G_{\mathbb{Q}} \to \mathrm{GL}_4(K)$ is irreducible implies that $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{1,2}\mathcal{T}_{2,1} \neq 0$ and hence $\dim \mathrm{Hom}(\mathcal{T}_{1,2}/\mathfrak{m}.\mathcal{T}_{1,2}, L) = 1$.

Thus, Nakayama's lemma implies that the $\mathcal{T}$-modules $\mathcal{T}_{1,2}$ is a monogenic $\mathcal{T}$-module.

Since $\mathcal{T}_{1,2}$ is a fractional ideal of K and each component ρ_{K_i} of ρ_K is absolutely irreducible, the annihilator of the generator of $\mathcal{T}_{1,2}$ over $\mathcal{T}$ is trivial. Hence, $\mathcal{T}_{1,2}$ is a free rank one $\mathcal{T}$-module. Moreover, the symmetry under the anti-involution implies that $\mathcal{T}_{1,2} \simeq \mathcal{T}_{2,3}$ and hence $\mathcal{T}_{2,3}$ is also a free $\mathcal{T}$-module of rank one.

Let $\alpha \in K$ (resp. $\beta \in K$) be a generator of $\mathcal{T}_{1,2}$ (resp. of $\mathcal{T}_{2,3}$) as $\mathcal{T}$-module. A direct computation shows that one can choose $e'_1 = \alpha.e_1, e'_2 = e_2, e'_3 = \beta^{-1}.e_3$ as a suitable basis of idempotents.

Moreover, we recall that we have an injection by Theorem 5.3

$$\mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}_{1,2}\mathcal{T}_{2,3}, L) = \mathrm{Hom}(\mathcal{T}_{1,3}/\mathcal{T}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \epsilon_p) = \{0\}.$$

Hence, Nakayama's lemma implies that $\mathcal{T}_{1,3} = \mathcal{T}$. Now, we can conclude from the fact that $\mathcal{T}_{1,2}\mathcal{T}_{2,1} = \mathcal{T}_{2,3}\mathcal{T}_{3,2} = \mathcal{I}^{\mathrm{tot}}$ that $\mathcal{T}_{2,1} = \mathcal{T}_{3,2} = \mathcal{I}^{\mathrm{tot}}$.

ii) By (4) applied with $(i,j) = (3,2)$ and $(3,1)$, respectively, applying Theorem 6.1 and Cor.6.3 for $(i,j) = (3,2)$ and using that $\mathcal{T}_{3,2} = \mathcal{I}^{\mathrm{tot}}$, $\mathcal{T}'_{3,2} = \mathcal{T}_{3,1}\mathcal{T}_{1,2} = \mathcal{J}$, $\mathcal{T}_{3,1} = \mathcal{J}$, and $\mathcal{T}'_{3,1} = \mathcal{T}_{3,2}\mathcal{T}_{2,1} = (\mathcal{I}^{\mathrm{tot}})^2$ we get injective morphisms

$$\begin{gathered}\mathrm{Hom}(\mathcal{I}^{\mathrm{tot}}/\mathcal{J}, L) \hookrightarrow \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-2)) \\ \mathrm{Hom}(\mathcal{J}/(\mathcal{I}^{\mathrm{tot}})^2, L) \hookrightarrow \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \epsilon_p^{-1})\end{gathered} \tag{25}$$

One has $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-2)) = 0$ (by a deep result of Kato [Kat04]), hence Nakayama's lemma applied to $\mathcal{I}^{\mathrm{tot}}/\mathcal{J}$ yields that $\mathcal{I}^{\mathrm{tot}} = \mathcal{J}$. Moreover, the ideal $\mathcal{I}^{\mathrm{tot}}$ is non-zero since ρ_K is irreducible. Thus, the fact that $\dim \mathrm{H}^1(G_{\mathbb{Q}}^{Np}, \epsilon_p^{-1}) \leq 1$ (by Proposition 7.5) yields that $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}.g + (\mathcal{I}^{\mathrm{tot}})^2$ and g is a generator of the ideal $\mathcal{I}^{\mathrm{tot}}$.

iii) The assertion follows from similar arguments to those already used to prove i), ii) and the fact that $\mathrm{Hom}(\mathcal{I}^{\mathrm{tot}}/\mathcal{J}, L) \hookrightarrow \mathrm{Sel}_{\mathbb{Q},f_\alpha}$ by Thm.6.1 and Cor.6.3. ∎

One has the following general bound of the number of generators of $\mathcal{I}^{\mathrm{tot}}$:

Corollary 7.9. *Let $s := \dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1))$. Assume* (**Reg**) *and assume also that* $\dim \mathrm{Sel}_{\mathbb{Q},f_\alpha} = 0$ *if $k=2$. Then $\mathcal{I}^{\mathrm{tot}}$ is generated by at most s^2 elements.*

Proof. It follows from (4.5) and Corollary 7.3 that $\mathcal{T}_{1,2}$ (resp. $\mathcal{T}_{2,3}$) is generated by at most s elements. Moreover, it follows from Theorem 6.1 and Cor.6.3 that $\mathcal{T}_{2,1} = \mathcal{T}_{2,3}\mathcal{T}_{3,1}$ and $\mathcal{T}_{3,1} = \mathcal{T}_{3,2}\mathcal{T}_{2,1} + g.\mathcal{T}$. Thus, $\mathcal{T}_{2,1} = (\mathcal{T}_{3,2}\mathcal{T}_{2,1} + g.\mathcal{T})\mathcal{T}_{2,3} = \mathcal{I}^{\mathrm{tot}}\mathcal{T}_{2,1} + g.\mathcal{T}_{2,3}$. Hence, $\mathcal{T}_{2,1}$ is generated also by at most s elements and then $\mathcal{I}^{\mathrm{tot}} = \mathcal{T}_{1,2}\mathcal{T}_{2,1}$ is generated by at most s^2 elements.

□

8. Smoothness of $\mathcal{E}_\Delta$ at π_α

The goal of this section is to prove that $A := \mathcal{T}/\mathcal{I}^{\mathrm{tot}}$ is a regular ring of dimension one (so it is a DVR) and that $\mathcal{T}$ is a regular ring of dimension two.

8.1. **Modularity and $\mathcal{R}^{\mathrm{ord}} = \widehat{\mathcal{O}}_{\mathcal{C}_N,f_\alpha}$.** Recall that $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ is the irreducible odd p-adic representation corresponding to f_α.

We consider the following deformation problem attached to ρ_f: for B any local L-Artinian algebra with maximal ideal $\mathfrak{m}_B$ and residue field $B/\mathfrak{m}_B = L$, we define $\mathcal{D}_{\rho_f}(B)$ as the set of strict equivalence classes of representations $\rho_B : G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_2(B)$ lifting ρ_f (that is ρ_B mod $\mathfrak{m}_B \simeq \rho_f$) and which are ordinary at p in the sense that:

$$\rho_{B|G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \psi_{1,B} & * \\ 0 & \psi_{2,B} \end{pmatrix}, \tag{26}$$

where $\psi_{1,B} : G_{\mathbb{Q}_p} \to B^\times$ is an unramified character, and such that they are minimally ramified at every $\ell \mid N$. Let $\mathcal{D}'_{\rho_f}$ be the subfunctor of $\mathcal{D}_{\rho_f}$ of deformation with constant determinant (so equal to $\det \rho_f = \epsilon_p^{3-2k}$).

It follows from Schlessinger's criterion that $\mathcal{D}_{\rho_f}$ is represented by a Noetherian local ring $\mathcal{R}^{\mathrm{ord}}$. Let $\rho^{\mathrm{ord}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{R}^{\mathrm{ord}})$ be the universal p-ordinary deformation of ρ_f. Recall that we have a locally finite flat map $w : \mathcal{C}_N \to \mathcal{V}$ (see §B.5). The determinant of ρ^{ord} is a deformation of $\det \rho_f$, and yields that $\mathcal{R}^{\mathrm{ord}}$ is an $\widehat{\mathcal{O}}_{\mathcal{V},w(f_\alpha)}$-algebra, since the complete local ring $\widehat{\mathcal{O}}_{\mathcal{V},w(f_\alpha)}$ of $\mathcal{V}$ at $w(f_\alpha)$ represents the deformation of $\det \rho_f$ to GL_1 (see [BD16, §.6]). Note that $L[\![T]\!] = \widehat{\mathcal{O}}_{\mathcal{V},w(f_\alpha)}$ (since the weight space is smooth and of dimension one), and that $\mathcal{R}' = \mathcal{R}^{\mathrm{ord}}/T.\mathcal{R}^{\mathrm{ord}}$ represents the functor $\mathcal{D}'_{\rho_f}$. Denote by Λ_1 the local ring $L[\![T]\!]$.

Let $\mathcal{O}$ be the ring of integers of L and $\mathfrak{C}_{\mathcal{O}}$ the category of $\mathcal{O}$-local complete Noetherian algebras with residue field isomorphic to $\mathbb{F}$ the residue field of $\mathcal{O}$, and whose morphisms are local $\mathcal{O}$-homomorphisms inducing the identity on residue fields. We assume in the rest of this paper that the following conditions are satisfied by the residual representation $\bar{\rho}_f$ attached to ρ_f:

- (**AI**$_{\mathbb{Q}}$) The restriction of $\bar\rho$ to $G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ is absolutely irreducible.
- (**Reg**) $\bar\rho_f$ is p-distinguished and $\alpha \neq 1$ when $k=2$.
- (**Min**) For any prime $\ell \mid N$, one has $\bar\rho_f|_{I_\ell}$ is unipotent and non-trivial and $a_\ell = -\ell^{k-2}$.

Schlesinger's criterion implies that the functor $\mathcal{D}^{\mathrm{ord}}_{\bar\rho_f}$ of p-ordinary minimally ramified deformations of $\bar\rho_f$ unramified outside of Np to objects of $\mathfrak{C}_{\mathcal{O}}$ is representable by $(\mathcal{R}^{\mathrm{ord}}_{\bar\rho_f}, \widetilde{\rho}^{\mathrm{ord}})$. The deformation $\det\widetilde{\rho}^{\mathrm{ord}}$ of $\det\bar\rho$ endows $\mathcal{R}_{\bar\rho_f}$ naturally with a structure of a $\mathbb{Z}_p[\![T]\!]$-algebra.

Note that the representation ρ_f is a minimal ordinary deformation of $\bar\rho_f$ corresponding to a morphism $\theta_f : \mathcal{R}^{\mathrm{ord}}_{\bar\rho_f} \to \mathcal{O}$ inducing ρ_f. Thus $\mathcal{R}^{\mathrm{ord}}$ is isomorphic to the completion of $\mathcal{R}^{\mathrm{ord}}_{\bar\rho_f}$ with respect to the kernel of θ_f (a height one prime ideal).

Let $h^{\mathrm{ord}}(Np^\infty)$ be the universal semi-local p-ordinary Hecke constructed by Hida in [Hid86] and $\mathfrak{h}$ the local component of $h^{\mathrm{ord}}(Np^\infty)$ corresponding to $f_\alpha \mod p$. Recall that the formal affine scheme Spf $h^{\mathrm{ord}}(Np^\infty)$ is a Raynaud model of the ordinary locus $\mathcal{C}_N^{\mathrm{full},ord}$ of the full eigencurve $\mathcal{C}_N^{\mathrm{full}}$ (see §.B.5).

Let $\mathbb{Z}_p[\![T]\!]$ be the Iwasawa algebra in one variable. We have a natural finite flat morphism[9]

$$w^* : \mathbb{Z}_p[\![T]\!] \to \mathfrak{h}$$

which is étale at classical weight ≥ 2 by Hida's control theorem (see [Hid86]).

Since f_α is ordinary at p, it defines a point f_α of the connected component of $\mathcal{C}_N^{\mathrm{full},ord}$ of the ordinary locus of the eigencurve $\mathcal{C}_N^{\mathrm{full}}$ with Raynaud model Spf$\mathfrak{h}$. Moreover, there exists a unique morphism $\varphi_{f_\alpha} : \mathfrak{h} \to L$ sending T_ℓ to $\mathrm{Tr}\,\rho_f(\mathrm{Frob}_\ell)$ for all primes $\ell \nmid Np$ and U_p to α.

Let $\mathcal{P}_{f_\alpha}$ be the height one prime ideal $\ker\varphi_{f_\alpha}$ and $\mathbb{T}$ the completed local ring of $\mathrm{Spec}\,\mathfrak{h}$ at $\mathcal{P}_{f_\alpha}$. It follows from the work of Taylor-Wiles ([Wil88] and [TW95]) that there exists an isomorphism $\mathcal{R}^{\mathrm{ord}}_{\bar\rho_f} \simeq \mathfrak{h}$ of $\mathbb{Z}_p[\![T]\!]$-algebras.

By the previous discussion this implies an isomorphism [10]

$$\mathcal{R}^{\mathrm{ord}} \simeq \mathbb{T} = \widehat{\mathcal{O}}_{\mathcal{C}_N^{\mathrm{full}}, f_\alpha} \simeq \Lambda_1, \tag{27}$$

where $\widehat{\mathcal{O}}_{\mathcal{C}_N^{full}, f_\alpha}$ is the completed local ring of the eigencurve $\mathcal{C}_N^{\mathrm{full}}$ at f_α. Moreover, by Nyssen and Rouquier's results ([Nys96] and [Rou96]), $\mathcal{R}^{\mathrm{ord}}$ is generated by the trace of its universal representation and hence we have

$$\mathcal{R}^{\mathrm{ord}} = \mathbb{T} = \widehat{\mathcal{O}}_{\mathcal{C}_N, f_\alpha}, \tag{28}$$

where $\widehat{\mathcal{O}}_{\mathcal{C}_N, f_\alpha}$ is the completed local ring of the eigencurve $\mathcal{C}_N$ at f_α.

[9] At the level of the generic fibers, the morphism $w^* : \mathbb{Z}_p[\![T]\!] \to \mathfrak{h}$ induced by the weight map $w : \mathcal{C}_N^{\mathrm{full,ord}} \to \mathcal{V}$.

[10] The isomorphism $\mathbb{T} \simeq \Lambda_1$ follows from the fact that the local morphism $w^* : \mathbb{Z}_p[\![T]\!] \to \mathfrak{h}$ is étale at the height one prime ideal corresponding to f_α.

8.2. **Regularity of $\mathcal{T}/\mathcal{I}^{\rm tot}$ when $k \geq 3$.** Recall that $(\kappa_1, \kappa_2) \subset (\mathcal{O}(\mathcal{W}))^2$ are the universal weights interpolating k_1, k_2 (they are the derivative at 1 of $\epsilon_p^{\kappa_1}, \epsilon_p^{\kappa_2}$). Hence one can see κ_i as global section in $\mathcal{O}(\mathcal{E}_\Delta)$ via the weight map $\kappa : \mathcal{E}_\Delta \to \mathcal{W}$. Recall also that $\epsilon_p^{-\kappa_1}$ and $\epsilon_p^{-\kappa_2}$ specialize at $\underline{k} = (k_1, k_2)$ to the characters $\epsilon_p^{-k_1}, \epsilon_p^{-k_2}$, respectively.

We will need the following results about the reducibility ideal:

Proposition 8.1. *Assume that $k \geq 3$. Then $\kappa_1 - \kappa_2 \in \mathcal{I}^{\rm tot}$.*

Proof. Since the Hodge-Tate-Sen weight $k-1$ occurs with multiplicity one in $\mathcal{M}'(x)$, Proposition 8.3 below and [BC09, Thm.4.3.4] (i.e the "constant weight lemma") applied to the family of p-adic representations $\rho_{\mathcal{M}'} : G_{\mathbb{Q}_p} \to \mathrm{GL}(\mathcal{M}')$ defined in Corollary 5.4 yields that

$$(\kappa_1 - 1) - (\kappa_2 - 2) - 1 = \kappa_1 - \kappa_2 \in \mathcal{I}^{\rm tot}.$$

□

Proposition 8.2. *Assume that $k = 2$. Then $\kappa_1 - \kappa_2 \in \mathcal{I}^{\rm tot}$.*

Proof. We have previously constructed (see Corollary 5.4) a representation

$$\rho_{\mathcal{M}'} : G_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathcal{U}}(\mathcal{M}')$$

of generic rank 3, where $\mathcal{M}'$ is a torsion-free coherent sheaf on an open affinoid $\mathcal{U} = \mathrm{Spm} A \subset \mathcal{E}_\Delta$ containing x. It follows from Thm.2.4(iii) that the representation $\rho_{\mathcal{M}'} : G_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathcal{U}}(\mathcal{M}')$ has a sub-representation $\rho_{\mathcal{M}''} : G_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathcal{U}}(\mathcal{M}'')$ generically of dimension 2 ($\mathcal{M}''$ is torsion-free $\mathcal{O}_{\mathcal{U}}$-module), and it specialization at the very Zariski dense set Σ' of $\mathcal{U}$ gives a crystalline representation ρ''_z of dimension 2 and with Hodge-Tate-Sen weights given by $(\kappa_1 - 1, \kappa_2 - 2)$ (the smallest Hodge-Tate weight of any z of Σ is $\kappa_2(z) - 2$) and $U_1/U_0 \in \mathcal{O}(\mathcal{E}_\Delta)^\times$ interpolating the crystalline period of the smallest Hodge-Tate weight (i.e.$\dim \mathcal{D}_{\rm crys}(\rho''_z)^{\Phi = U_1/U_0(z) p^{p^{\kappa_2(z)-2}}} = 1$).

Let $M'' := \mathcal{M}'' \otimes_A \mathcal{T} = \mathcal{M}''_x$ be the stalk of $\mathcal{M}''$ at x. Similar arguments to those already used to prove [BC09, Lemma.4.3.3](i) yield that the Sen operator of $\mathcal{D}_{\rm sen}(M''/\mathcal{J}M'')$ is annihilated by the Polynomial

$$(T - (\kappa_2 - 2))(T - (\kappa_1 - 1)).$$

Moreover, the specialization of the pseudo-character $\mathrm{Tr}\, \rho_{\mathcal{M}''}$ at x is equal to $\epsilon_p^{2-k} \oplus \epsilon_p^{1-k}$. Thanks to Proposition 8.3 below, one can apply [BC09, Thm.4.3.4] (the "constant weight lemma") to the family of p-adic representations $\rho_{\mathcal{M}''} : G_{\mathbb{Q}_p} \to \mathrm{GL}(\mathcal{M}'')$ to claim that

$$(\kappa_1 - 1) - (\kappa_2 - 2) - 1 = \kappa_1 - \kappa_2 \in \mathcal{I}^{\rm tot}.$$

□

Let A be the local quotient ring $\mathcal{T}/\mathcal{I}^{\mathrm{tot}}$ of dimension ≤ 2. Note that A is Henselian, since $\mathcal{T}$ is Henselian (the local ring of a rigid analytic space for the rigid topology is always Henselian).

Let $\mathrm{Ps}_A : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta) \to A$ be the natural pseudo-character of dimension 4. Moreover, $\mathrm{Ps}_A = \Psi_1 + \Psi_2 + \mathrm{Tr}_A$ such that $\mathrm{Tr}_A : G_{\mathbb{Q}} \to A$ is a pseudo-character lifting the pseudo-character $\mathrm{Tr}(\rho_f)$ and $\{\Psi_i\}_{i=1,2} : G_{\mathbb{Q}} \to A^\times$ are characters lifting respectively ϵ_p^{2-k} and ϵ_p^{1-k}. Moreover, since ρ_f is absolutely irreducible, $\mathrm{Tr}_A : G_{\mathbb{Q}} \to A$ is the trace of a deformation $\rho_A : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ of ρ_f. The deformation $\det \rho_A$ of $\det \rho_f$ yields a natural local morphism of $\bar{\mathbb{Q}}_p$-algebras $\Lambda_1 \to A$ (see [BD16, §.6]).

Proposition 8.3. *For any cofinite ideal $\mathcal{J} \subset \mathcal{T}$ containing $\mathcal{I}^{\mathrm{tot}}$. We have:*

(i) $\mathcal{D}^+_{\mathrm{crys}}(M'/\mathcal{J}M' \otimes (\epsilon_p^{\kappa_2-2}))^{\Phi=U_1/U_0}$ *is a free rank one $\mathcal{T}/\mathcal{J}$-module.*

(ii) $\mathcal{D}^+_{\mathrm{crys}}(M''/\mathcal{J}M'' \otimes (\epsilon_p^{\kappa_2-2}))^{\Phi=U_1/U_0}$ *is a free rank one $\mathcal{T}/\mathcal{J}$-module, where M'' be the stalk of $\mathcal{M}''$ at x.*

Proof. i) Recall that in the proof of Theorem 5.3 and Corollary 5.4, we have constructed a family of p-adic representations $\rho_{\mathcal{M}'} : G_{\mathbb{Q}_p} \to \mathrm{GL}_{\mathcal{U}}(\mathcal{M}')$ over an affinoid $\mathcal{U} := \mathrm{Spm}\, B \subset \mathcal{E}_\Delta$ containing x, and such that $\mathcal{M}'$ is a torsion-free quotient of $\mathcal{M}$ of generic rank 3 and $\mathcal{M}/\mathcal{M}^{I_p} = \mathcal{M}'$ (the generic rank of $\mathcal{M}$ over $\mathcal{U}$ is 4). By [BC09, Thm.1.5.6] we have surjections

$$M = \mathcal{M} \otimes_B \mathcal{T} \twoheadrightarrow M/\mathcal{J}M \twoheadrightarrow \Psi_2 \mod \mathcal{J},$$

such that any semi-simple S-subquotient of the S-module $\ker(M/\mathcal{J}M \to \Psi_2 \mod \mathcal{J})$ occurs in $\{\rho_f, \epsilon_p^{2-k}\}$ (any S-simple module is necessarily an $S/\mathfrak{m}S$-module).

On the other hand, since $\mathcal{M}/\mathcal{M}^{I_p} = \mathcal{M}'$, the surjection $M/\mathcal{J}M \twoheadrightarrow \Psi_2 \mod \mathcal{J}$ must factor through

$$M'/\mathcal{J}M' \twoheadrightarrow \Psi_2 \mod \mathcal{J} \tag{29}$$

for $M' = \mathcal{M}' \otimes_B \mathcal{T}$.

We recall that

$$l(\mathcal{D}^+_{\mathrm{crys}}(M'/\mathcal{J}M' \otimes (\epsilon_p^{\kappa_2-2}))^{\Phi=U_1/U_0}) = l(\mathcal{T}/\mathcal{J}).$$

On the other hand, it follows from the fact that the semi-simple subquotients of

$$\ker(M'/\mathcal{J}M' \to \Psi_2 \mod \mathcal{J})$$

occur in $\{\epsilon_p^{2-k}, \psi, \psi^{-1}\epsilon_p^{3-2k}\}$ that

$$\mathcal{D}_{\mathrm{crys}}(\ker(M'/\mathcal{J}M' \to \Psi_2 \mod \mathcal{J}) \otimes \epsilon_p^{\kappa_2-2})^{\Phi=U_1/U_0} = \{0\}.$$

Therefore, $l(\mathcal{D}^+_{\mathrm{crys}}(\Psi_2 \otimes (\epsilon_p^{\kappa_2-2}) \mod \mathcal{J})^{\Phi=U_1/U_0}) = l(\mathcal{T}/\mathcal{J})$. Thus, [BC09, Lemma.3.3.9] yields that

$$\mathcal{D}^+_{\mathrm{crys}}(\Psi_2 \otimes (\epsilon_p^{\kappa_2-2}) \mod \mathcal{J})^{\Phi=U_1/U_0} \text{ is a free rank one } \mathcal{T}/\mathcal{J}\text{-module,} \tag{30}$$

and then

$$\mathcal{D}^+_{\mathrm{crys}}(M'/\mathcal{J}M' \otimes (\epsilon_p^{\kappa_2-2}))^{\Phi=U_1/U_0}) \text{ is a free rank one } \mathcal{T}/\mathcal{J}\text{-module.}$$

ii) The assertion follows from i), the fact that composition $M''/\mathcal{J}M'' \to M'/\mathcal{J}M' \twoheadrightarrow \Psi_2 \mod \mathcal{J}$ is surjective and that $\mathcal{D}^+_{\mathrm{crys}}(M'/M'' \otimes \mathcal{T}/\mathcal{J} \otimes (\epsilon_p^{\kappa_2-2}))^{\Phi=U_1/U_0}) = 0$.

□

Proposition 8.4. *Assume that $k \geq 2$, then the local ring A is topologically generated by the image of* $\mathrm{Tr}(\rho_A)$ *over Λ_1.*

Proof. Let A' be the subring of A topologically generated by the image of the trace $\mathrm{Tr}(\rho_A)$ over Λ_1. Note that the polarisation of Ps_A described in section 4.1 implies that $\det \rho_A$ is given by the character $\epsilon_p^{-\kappa_1-\kappa_2+3}$. By Propositions 8.1 and 8.2, one has $\kappa_1 - \kappa_2 \in \mathcal{I}^{\mathrm{tot}}$. Hence, the image of the character $\epsilon_p^{\kappa_1} = \epsilon_p^{\kappa_2} \mod \mathcal{I}^{\mathrm{tot}}$ in A lies in A'.

We will show in the following that $\Psi_2 = \epsilon_p^{1-\kappa_2}$, which by the polarisation of Ps_A shows $\Psi_1 = \epsilon_p^{2-\kappa_1}$. As Ps_A is surjective onto A by construction of $\mathcal{E}_\Delta$ this will establish the proposition.

First, (30) yields that for any ideal $\mathcal{J} \subset \mathcal{T}$ of $\mathcal{T}$ of cofinite length and such that $\mathcal{I}^{\mathrm{tot}} \subset \mathcal{J}$, the character $\Psi_2 \otimes \epsilon_p^{\kappa_2-2} \mod \mathcal{T}/\mathcal{J}$ is a crystalline $L[G_{\mathbb{Q}_p}]$-representation, because the $\mathcal{T}/\mathcal{J}$-module $\mathcal{D}^+_{\mathrm{crys}}(\Psi_2 \otimes \epsilon_p^{\kappa_2-2} \mod \mathcal{J})^{\Phi=U_1/U_0}$ is free of rank one over $\mathcal{T}/\mathcal{J}$.

Hence, the constant weight lemma (see [BC09, Prop.2.5.4]) implies that $\Psi_2 \otimes (\epsilon_p^{\kappa_2-2}) \otimes \epsilon_p \mod \mathcal{J}$ is of Hodge-Tate weight 0 and crystalline, therefore unramified. Thus, by class field theory we deduce that $\Psi_2 \otimes (\epsilon_p^{\kappa_2-2}) \otimes \epsilon_p \mod \mathcal{J}$ is the trivial character (since $\mathbb{Q}$ has a unique $\mathbb{Z}_p$-extension).

Thus, $\Psi_2 \mod \mathcal{J} = \epsilon_p^{1-\kappa_2} \mod \mathcal{J}$. Then the Krull intersection theorem implies that $\Psi_2 \mod \mathcal{I}^{\mathrm{tot}} = \epsilon^{1-\kappa_2}$ (since $\mathcal{I}^{\mathrm{tot}}$ is the intersection of the cofinite ideals of $\mathcal{T}$ containing $\mathcal{I}^{\mathrm{tot}}$).

□

Proposition 8.5. *The representation ρ_A is p-ordinary and minimal.*

Proof. According to [BC09, Thm.1.5.6] and [BC09, Lemma.4.3.9], there exists a $\mathcal{T}$-module $M \subset K^4$ of generic rank 4 endowed with a $G_{\mathbb{Q}}$-continuous action which is generically given by the semi-simple representation

$$\rho_K : G_{\mathbb{Q}} \to S^\times \subset \mathrm{GL}_4(K),$$

and equipped with a surjection $\pi : M/\mathcal{I}^{\mathrm{tot}}M \twoheadrightarrow \rho_A$ such that the S-simple subquotients of its kernel are either ϵ_p^{1-k} or ϵ_p^{2-k}.

Since $\mathcal{T}$ is reduced and ρ_K is semi-ordinary ($\rho_K^{I_p}$ is of dimension one and Frob_p acts on it by U_0) and $\alpha \neq 1$ when $k = 2$, we again have (as in §6) that M^{I_p} is not contained in $\mathfrak{m}M$. Since the S-simple subquotients of $\ker \pi$ do not contain ρ_f and contain only the representations in the set $\{\epsilon_p^{1-k}, \epsilon_p^{2-k}\}$, the regularity assumption further implies that the image of M^{I_p} under the surjection $\pi' : M/\mathfrak{m}M \twoheadrightarrow \rho_f$ is non-zero and hence the image of M^{I_p} under the surjection $\pi : M/\mathcal{I}^{\mathrm{tot}}M \twoheadrightarrow \rho_A$ is non-zero and it is not contained in $\mathfrak{m}A^2$.

Thus, we have an exact sequence of $A[G_{\mathbb{Q}_p}]$-modules:

$$0 \to \rho_A^{I_p} \to \rho_A \to \rho_A/\rho_A^{I_p} \to 0. \tag{31}$$

Since $\rho_A/\rho_A^{I_p} \otimes_A L$ is of rank one Nakayama's lemma implies that $\rho_A/\rho_A^{I_p}$ and $\rho_A^{I_p}$ are monogenic A-modules and generated respectively by y_1, y_2. Therefore y_1, y_2 generate A^2 and they must even form a basis of A^2. Hence the exact sequence (31) splits as A-modules and yields that ρ_A is p-ordinary.

We shall now prove that ρ_A is minimally ramified at every $\ell \mid N$. Let ℓ be a prime number dividing N. From the proof of Theorem 5.3 we know that there exist idempotents $(\tilde{e_1}, \tilde{e_2}, \tilde{e_3})$ of S lifting the idempotents attached respectively to $\epsilon_p^{2-k}, \epsilon_p^{1-k}, \rho_f$ such that $\tilde{e} = \tilde{e_1} + \tilde{e_2}$ is in the center of $\mathrm{S}_\ell = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & M_{2,2}(\mathcal{T}) \end{pmatrix}$, the image of $\mathcal{T}[G_{\mathbb{Q}_\ell}]$ inside S. We also recall that N_K, the monodromy operator corresponding to the Weil-Deligne representation attached to $G_{\mathbb{Q}_\ell} \to \mathrm{S}_\ell^\times$, can be viewed as an element of S_ℓ, has rank 1 by Proposition 2.8 and satisfies $\tilde{e_3} N_K \tilde{e_3} \neq 0$. For $N := \tilde{e_3} N_K \tilde{e_3} \in M_2(\mathcal{T})$ we know that N is non-trivial modulo $\mathfrak{m}_\mathcal{T}$ (since the rank of the monodromy operator of $WD_\ell(\rho_f)$ is one) and so the morphism $\rho_A|_{G_{\mathbb{Q}_\ell}} : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\mathcal{T}) \to \mathrm{GL}_2(A)$ is also minimally ramified.

□

Theorem 8.6. *Assume that $k \geq 2$, $(\mathbf{Min})$, $(\mathbf{AI}_\mathbb{Q})$ and $(\mathbf{Reg})$. Then one has:*

(i) The local ring A is regular of dimension one and there is an isomorphism $\mathcal{R}^{\mathrm{ord}} \simeq \widehat{A}$.
(ii) The local ring $\widehat{A}$ is étale over Λ_1 and $A \simeq \mathcal{O}_{SK(\mathcal{F}),x}$.
(iii) Assume that $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ and $k \geq 3$, then the local ring $\mathcal{T}$ is regular of dimension 2, i.e. $\mathcal{E}_\Delta$ is smooth at x.
(iv) Assume that $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ and $k \geq 3$, then the reducibility ideal of the pseudo-character $\mathrm{Ps}_\mathcal{T} : G_\mathbb{Q} \to \mathcal{T}$ corresponds to the principal Weil divisor (closed subset of dimension one) of $\mathrm{Spec}\,\mathcal{T}$ corresponding to the Saito-Kurokawa family $SK(\mathcal{F})$ specializing to x.

(v) *Assume that* $k = 2$, $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$ *and* $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$, *then the rigid analytic space* $\mathcal{E}_\Delta$ *is smooth at* π_α, *and the reducibility locus of the pseudo-character* $\mathrm{Ps}_{\mathcal{T}} : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{E}_\Delta) \to \mathcal{T}$ *is a principal Weil divisor of* $\mathrm{Spec}\,\mathcal{T}$, *and corresponds to the Saito-Kurokawa lift of the ordinary Hida family* $\mathcal{F}$ *passing through* f_α.

Proof. (i) We recall that Hida's control theorem (see [Hid86]) yields that $\mathbb{T}$ is a discrete valuation ring, and hence $\mathcal{R}^{\mathrm{ord}}$ is also a discrete valuation ring (since $\mathcal{R}^{\mathrm{ord}} \simeq \mathbb{T}$). Propositions 8.4 and 8.5 provide us with a surjective morphism

$$\mathcal{R}^{\mathrm{ord}} \twoheadrightarrow \widehat{A}.$$

It remains to show that the Krull dimension of A is at least one. This is a consequence of the fact that A surjects onto the local ring $\mathcal{O}_{SK(\mathcal{F}),x}$ at x of the Saito-Kurokowa family $SK(\mathcal{F})$ specializing to π_α (see [SU06, Prop.4.2.5], §.B.5) and $\mathcal{O}_{SK(\mathcal{F}),x}$ is of dimension one (since $SK(\mathcal{F}) \subset \mathcal{E}_\Delta$ is an irreducible closed analytic set of dimension one).

Thus the surjective morphism $\mathcal{R}^{\mathrm{ord}} \twoheadrightarrow \widehat{A}$ is necessarily an isomorphism of discrete valuation rings.

(ii) The étaleness follows from (28) and (i). The isomorphism $A \simeq \mathcal{O}_{SK(\mathcal{F}),x}$ follows from the fact (noted already in (i) of the proof) that the discrete valuation ring A surjects onto the 1-dimensional local ring $\mathcal{O}_{SK(\mathcal{F}),x}$. Hence, they are necessarily isomorphic.

(iii) We have to show that the tangent space of $\mathcal{T}$ is of dimension 2. Since the Krull dimension is always less or equal to the dimension of the tangent space, we have to show that the maximal ideal $\mathfrak{m}$ of $\mathcal{T}$ has at most two generators. Note that $\mathcal{I}^{\mathrm{tot}} = (g)$ (see Thm.7.7) and $A = \mathcal{T}/(g)$ is regular of dimension 1. Hence $\mathfrak{m}$ has at most two generators. Thus $\mathcal{T}$ is regular.

(iv) This follows from the fact that $\mathcal{I}^{\mathrm{tot}} = (g)$ and $\mathcal{O}_{SK(\mathcal{F}),x} = A = \mathcal{T}/(g)$.

(v) The assumption that $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$ yields that the $\dim \mathrm{Sel}_{\mathbb{Q},f_\alpha} = 0$ (see §.6.1), and hence Thm.7.7 implies that $\mathcal{I}^{\mathrm{tot}} = (g)$. Moreover, it follows from i) that $A = \mathcal{T}/(g)$ is regular of dimension 1, and hence $\mathcal{T}$ is a regular local ring of dimension 2.

□

One has the following general bound of the Zariski tangent space of $\pi_\alpha \in \mathcal{E}_\Delta$.

Corollary 8.7. *Assume* (**Min**), ($\mathbf{AI}_{\mathbb{Q}}$), (**Reg**) *and assume also that* $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$ *if* $k = 2$, *then:*

$2 \leq \dim \mathfrak{t}_{\pi_\alpha} \leq 1 + (\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)))^2$ *and* $\dim \mathfrak{t}^0_{\pi_\alpha} \leq (\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)))^2$.

Proof. The assertion follows immediately from Corollary 7.9 (i.e $\mathcal{I}^{\mathrm{tot}}$ is generated by at most s^2 elements) and from Theorem 8.6 (i.e $A = \mathcal{T}/\mathcal{I}^{\mathrm{tot}}$ is étale over $\Lambda_1 \simeq \Lambda/(\kappa_1 - \kappa_2)$).

□

9. Smoothness failure of $\mathcal{E}_N$ at π_α when N is square free and not prime

We prove in this subsection that our main results fail when we change the tame level to $\Gamma(N)$. In this subsection we can remove the assumption on the global root number ϵ_f being -1 as there exists a Saito-Kurokawa lift of level $\Gamma(N)$ for either sign (see [Sch07]).

Let $\mathcal{Y}_\ell$ be a cuspidal Coleman family of slope 1 of level ℓ and specializing to $E_2^{\mathrm{crit_p},ord_\ell}$. Then the Galois representation $\rho_{\mathcal{Y}_\ell}$ attached to $\mathcal{Y}_\ell$ is necessarily Steinberg at ℓ, otherwise, as in Proposition 3.2 we will obtain a non-trivial cohomology class of $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q},\epsilon_p)$, and it is know that $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q},\epsilon_p)$ is trivial.

Remark 9.1.

(i) The Atkin-Lehner eigenvalue of the classical specializations of $\mathcal{Y}_\ell$ at ℓ is constant and equal to -1.

(ii) The Hida family $\mathcal{F}$ specializing to f_α is special at every $q \mid N$ and the Atkin-Lehner eigenvalue of the classical specializations of $\mathcal{F}$ at every $q \mid N$ is constant.

(iii) According to [Maj15], the weight map $w : \mathcal{C}_\ell \to \mathcal{V}$ is étale at $E_2^{\mathrm{crit_p},\mathrm{ord}_\ell}$, and since w is locally finite, one can shrink any affinoid neighborhood of $E_2^{\mathrm{crit_p},\mathrm{ord}_\ell}$ to ensure that it will be étale over $\mathcal{V}$ (see Proposition A.5).

We can therefore apply the following result:

Proposition 9.2 ([SS13] Prop.3.1)**.** *Let $f_1 \in \mathrm{S}_{k_1}(N_1)$, $f_2 \in \mathrm{S}_{k_2}(N_2)$ be newforms of squarefree level with even integers $k_1 \geq k_2 \geq 2$ and $M := \gcd(N_1, N_2) > 1$. Assume that the Atkin-Lehner eigenvalues of f_1 and f_2 for $\ell \mid M$ coincide. Put $N = \mathrm{lcm}(N_1, N_2)$. Then there exists a non-zero holomorphic Yoshida lift of level $\Gamma(N)$ and weight $((k_1+k_2)/2, (k_1-k_2+4)/2)$ with corresponding Galois representation $\rho_{f_1} \oplus \rho_{f_2}(\frac{k_1-k_2}{2})$. For $p \nmid N$ there exists a p-stabilisation of this lift (of Iwahori level at p) with U_0-eigenvalue α_1 and U_1-eigenvalue[11] $\alpha_1\alpha_2 p^{\frac{k_1-k_2-2}{2}}$, where α_i are roots of the Hecke polynomial of f_i at p for $i = 1, 2$.*

Proof. For the existence of the lift of level $\Gamma(N)$ see [SS13] Prop.3.1. For the p-stabilisation of the principal unramified series see [MY14] §7.1.1, but we use the normalization of [SU06] §2.4.16. □

Theorem 9.3. *Let $\ell \mid N$ be a prime number for which f is Steinberg, $\mathcal{U}^1$ be an affinoid subdomain of the p-adic eigencurve $w_2 : \mathcal{C}_\ell \to \mathcal{V}$ of tame level ℓ containing $E_2^{\mathrm{crit_p},\mathrm{ord}_\ell}$, corresponding to a Coleman family $G = \sum_{n=1}^{\infty} a(n, G)q^n$, and such that it is étale over the weight space $\mathcal{V}$.*

[11] For the different normalisation (34) of the U_1 operator on the eigenvariety this corresponds to the constant eigenvalue $\alpha_1\alpha_2$.

Let $\mathcal{U}^0$ be an affinoid subdomain of the ordinary locus $\mathcal{C}_N^{\mathrm{ord}}$ of the p-adic eigencurve $\mathcal{C}_N$ of tame level N containing f_α and corresponding to the Hida family $\mathcal{F} = \sum_{n=1}^{\infty} a(n,\mathcal{F})q^n$, and such that it is étale [12] *over the weight space $\mathcal{V}$.*

There exists a Zariski closed immersion $\lambda_{\mathrm{Yo}} : \mathcal{U}^0 \times_{\mathbb{Q}_p} \mathcal{U}^1 \hookrightarrow \mathcal{E}_N$ with image denoted by $\mathrm{Yo}(\mathcal{F},\mathcal{U}^1)$ *and such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{U}^0 \times_{\mathbb{Q}_p} \mathcal{U}^1 \arrow[r, "\lambda_{\mathrm{Yo}}"] \arrow[d, "w_1 \times w_2"] & \mathcal{E}_N \arrow[d, "\kappa"] \\ \mathcal{V} \times \mathcal{V} \arrow[r, "\lambda_\kappa"] & \mathcal{W} \end{tikzcd}$$

where $\lambda_\kappa(2k_1, 2k_2) = (k_1+k_2, k_1-k_2+2)$ and the morphism λ_{Yo} corresponds to the morphism

$$\lambda_{\mathrm{Yo}}^* : \mathcal{O}(\mathcal{E}_N) \to \mathcal{O}(\mathcal{U}^0)\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(\mathcal{U}^1)$$

defined by

$\lambda_{\mathrm{Yo}}^*(P_\ell(X)) = (X^2 - a(\ell,\mathcal{F})X + \ell^{-3}\kappa_1\kappa_2(\ell))(X^2 - \kappa_2(\ell)\ell^{-2}a(\ell,G)X + \ell^{-3}\kappa_2(\ell).\kappa_1(\ell))$, *for any $\ell \nmid Np$,*

where $P_\ell(X) \in \mathcal{O}(\mathcal{E}_N)[X]$ is the Hecke-Andrianov polynomial at $\ell \nmid Np$, and $\lambda_{\mathrm{Yo}}^*(U_0) = a(p,\mathcal{F})$, and $\lambda_{\mathrm{Yo}}^*(U_1) = a(p,\mathcal{F}) \times a(p,G)$.

Proof. One can choose the affinoids $\mathcal{U}^0 \subset \mathcal{C}_N$ and $\mathcal{U}^1 \subset \mathcal{C}_\ell$ étale over the weight space and small enough such that there exist $\epsilon, v \in \mathbb{R}$ and the Banach sheaf ω_ϵ^κ on $\bar{X}(v) \times W$ of locally analytic v-overconvergent p-adic familes (see §.B.3), where $W = \mathrm{Spm}\, R$ is an affinoid of the weight space $\mathcal{W}$ given by $w_1(\mathcal{U}^0) \times_{\mathbb{Q}_p} w_2(\mathcal{U}^1)$. Let $\mathcal{T}_{W,1}$ be the affinoid $\mathbb{Q}_p$-algebra generated over R by the image of the abstract Hecke algebra $\mathcal{H}_N$ in the space of endomorphisms of the sections of $\varinjlim_{v\to 0} \mathrm{H}^0(\bar{X}(v) \times W, \omega_\epsilon^\kappa)$ with slope ≤ 1. By construction of $\mathcal{E}_N$ (see §.B.3), $\mathcal{E}_{N,W}^1 = \mathrm{Spm}\, \mathcal{T}_{W,1}$ is an affinoid subdomain of $\mathcal{E}_N$. Let $\theta : \mathcal{H}_N \twoheadrightarrow \mathcal{T}_{W,1}$ be the natural surjection and J be the kernel of θ generated by $g_1, ...g_n$.

On the other hand, let λ be the morphism

$$\lambda : \mathcal{H}_N \to \mathcal{O}(\mathcal{U}^0)\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(\mathcal{U}_1)$$

defined by

$\lambda_{\mathrm{Yo}}^*(P_\ell(X)) = (X^2 - a(\ell,\mathcal{F})X + \ell^{-3}\kappa_1\kappa_2(\ell))(X^2 - \kappa_2(\ell)\ell^{-2}a(\ell,G)X + \ell^{-3}\kappa_2(\ell).\kappa_1(\ell))$, for any $\ell \nmid Np$,

where $P_\ell(X) \in \mathcal{H}_N[X]$ is the Hecke-Andrianov polynomial at $\ell \nmid Np$, and $\lambda_{\mathrm{Yo}}^*(U_0) = a(p,\mathcal{F})$, and $\lambda_{\mathrm{Yo}}^*(U^1) = a(p,\mathcal{F}) \times a(p,G)$.

[12] According to Hida's control theorem, the weight map $w_1 : \mathcal{C}_N \to \mathcal{V}$ is étale at f_α, and since w is locally finite, one can shrink any affinoid neighborhood of f_α to ensure that it will be étale over $\mathcal{V}$.

It is enough to prove that for every $1 \leq i \leq n$, $\lambda(g_n) = 0$. Note that the classical points old at p of $\mathcal{U}^0$,$\mathcal{U}^1$ form a dense set Σ of $\mathcal{U}^0 \times_{\mathbb{Q}_p} \mathcal{U}^1$. It follows from Proposition 9.2 that the points Σ lift to a set $\tilde{\Sigma}$[13] of points of $\mathcal{E}^1_{N,W}$. Hence, for any $1 \leq i \leq n$, the specialization of $\lambda(g_i)$ at the points of the dense subset Σ of $\mathcal{U}^0 \times_{\mathbb{Q}_p} \mathcal{U}^1$ is trivial, yielding that

$$\lambda(g_i) = 0 \text{ for any } 1 \leq i \leq n. \tag{32}$$

Hence, we obtain a surjective homomorphism

$$\mathcal{O}(\mathcal{E}^1_{N,W}) \twoheadrightarrow \mathcal{O}(\mathcal{U}^0)\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(\mathcal{U}^1),$$

yielding a morphism $\mathcal{U}^0 \times_{\mathbb{Q}_p} \mathcal{U}^1 \to \mathcal{E}^1_{N,W}$, and its image is an irreducible component of $\mathcal{E}^1_{N,W}$. □

Corollary 9.4. *Assume $N > 1$ is squarefree and not prime. Assume that f is Steinberg for at least two primes $\ell_i \mid N, i = 1, 2$. Then the Siegel eigenvarieties $\mathcal{E}_N$ of tame level N is singular at x and there exists at least two p-adic families specializing to π_α.*

Proof. If f is Steinberg at ℓ_1 and ℓ_2, then by the previous theorem we get two irreducible components of $\mathcal{E}_N$ (they are endoscopic) specializing to π_α by taking $\mathcal{U}^1$ arising from $\mathcal{Y}_{\ell_i}$. □

A direct consequence of the above corollary is that $\kappa : \mathcal{E}_N \to \mathcal{W}$ is ramified at π_α. Let $\mathrm{S}_k(N)^{|\mathbb{U}|_p=1}[\![\pi_\alpha]\!]$ be the generalized eigenspace attached to π_α inside the L-vector space of locally analytic overconvergent Siegel cusp forms $\mathrm{S}_k(N)^{|\mathbb{U}|_p=1}$ of tame level $\Gamma(N)$ and slope 1 for $\mathbb{U}$.

Corollary 9.5. *One has* $\dim_L \mathrm{S}_k(N)^{|\mathbb{U}|_p=1}[\![\pi_\alpha]\!] \geq 2$.

Proof. Since $\mathcal{W}$ is smooth at $\kappa(x)$ and $\mathcal{E}_N$ is singular at x, the local ring $\mathcal{T}_0 = \mathcal{O}_{\mathcal{G},x}/\mathfrak{m}_{\mathcal{O}_{W,\kappa(x)}}\mathcal{O}_{\mathcal{G},x}$ of the fiber of $\kappa^{-1}(\kappa(x))$ at x is Artinian with a non-trivial tangent space (since κ is necessarily ramified at x in this case). On the other hand, it follows from the construction of eigenvarieties that the local ring $\mathcal{T}_0$ at x of the fiber $\kappa^{-1}\kappa(x)$ acts faithfully on $\mathrm{S}_k(N)^{|\mathbb{U}|_p=1}[\![\pi_\alpha]\!]$. Hence, $\dim \mathrm{S}_k(N)^{|\mathbb{U}|_p=1}[\![\pi_\alpha]\!] \geq 2$ (since $\dim_L \mathcal{T}_0 \geq 2$). □

[13]Any point of Σ corresponds to a 2-tuple of old forms (f_1, g_1) at p. Hence, f_1 (resp. g_1) is the p-ordinary (resp. p-critical) p-stabilization of a classical form of level $\Gamma_0(N)$ (resp. $\Gamma_0(\ell)$) denoted by f_1^{old} (resp. g_1^{old}). So we can consider the Yoshida lift of (f_1^{old}, g_1^{old}) and take its semi-ordinary p-stabilization which gives a point of $\tilde{\Sigma} \subset \mathcal{E}^1_N$.

Appendix A. Some basic facts about rigid analytic geometry

We shall recall in this section the notions of "very Zariski dense" subset of a rigid analytic space and discuss accumulation points of a Zariski dense set and irreducible components of rigid analytic spaces. Moreover, we will recall some basic properties of finite and torsion-free morphisms of affinoid spaces.

The following proposition is an analogue to [Ber93, Prop.2.1.6] for $\mathbb{Q}_p$-rigid analytic spaces.

Proposition A.1. *Let $g : X \to Y$ be a finite morphism between two $\mathbb{Q}_p$-affinoid spaces, $y \in f(X) \subset Y$, and $g^{-1}(y) = \{x_1, x_2, ..., x_n\}$, then there exists a small affinoid neighborhood $\mathcal{U}_{i_0}$ of y in Y such that $g^{-1}(\mathcal{U}_{i_0}) = \bigcup_{1\leq k\leq n} V_k^{i_0}$, and $V_k^{i_0} \cap V_j^{i_0} = \{0\}$, when $k \neq j$. Moreover, for any $1 \leq k \leq n$, the domains $\{V_k^i, i \in I$ and $i \leq i_0\}$ form a basis of neighborhood of x_k when $\mathcal{U}_i$ varies in a family $\{\mathcal{U}_i, i \in I$ and $i \leq i_0\}$ of basis of affinoids containing y.*

Proof. Let B (resp. A) be the affinoid $\mathbb{Q}_p$-algebra corresponding to X (resp. Y), and $\varphi : A \to B$ be the finite morphism corresponding to g. Let B_y be the finite $\mathcal{O}_{Y,y}$-algebra $B \otimes_A \mathcal{O}_{Y,y}$; thanks to [Ber93, Thm.2.1.5] the local ring $\mathcal{O}_{Y,y}$ is Henselian, hence

$$B_y =^{14} \prod_{x_i \in g^{-1}(y)} \mathcal{O}_{X,x_i}.$$

On the other hand, one has

$$B_y = B \otimes_A \ \mathcal{O}_{Y,x} = B \otimes_A \ \varinjlim_{\mathcal{U}_i} \mathcal{O}_Y(\mathcal{U}_i) = \varinjlim_{\mathcal{U}_i} \ B \otimes_A \ \mathcal{O}_Y(\mathcal{U}_i),$$

where $\{\mathcal{U}_i, i \in I\}$ runs over the affinoid neighorhood of y.

Hence, we have

$$\varinjlim_{\mathcal{U}_i, i\in I} B \otimes_A \ \mathcal{O}_Y(\mathcal{U}_i) =^{15} \varinjlim_{\mathcal{U}_i, i\in I} B \widehat{\otimes}_A \ \mathcal{O}_Y(\mathcal{U}_i) = \varinjlim_{\mathcal{U}_i, i\in I} \ \mathcal{O}_X(g^{-1}(\mathcal{U}_i)) = \prod_{x_i \in g^{-1}(y)} \mathcal{O}_{X,x_i}.$$

Thus, each local component $\mathcal{O}_{X,x_j}$ of $\prod_{x_i\in g^{-1}(y)} \mathcal{O}_{X,x_i}$ corresponds to an idempotent e_j of B_y. So there exist an $i_0 \in I$ and orthogonal idempotents $\{\tilde{e}_j, 1 \leq j \leq n\}$ of $\mathcal{O}_X(g^{-1}(\mathcal{U}_{i_0}))$ whose image in B_y is $\{e_j, 1 \leq j \leq n\}$ and corresponding respectively to $\{x_1, ...x_n\}$. Thus, $\mathcal{O}_X(g^{-1}(\mathcal{U}_{i_0})) = \prod_{\tilde{e}_j, 1\leq j\leq n} \tilde{e}_j.\mathcal{O}_X(g^{-1}(\mathcal{U}_{i_0}))$, and hence each affinoid subdomain $\operatorname{Spm} \tilde{e}_k.\mathcal{O}_X(g^{-1}(\mathcal{U}_{i_0}))$ of X corresponds to a connected component $V_k^{i_0}$ of $g^{-1}(\mathcal{U}_{i_0})$ containing x_k. Hence, $g^{-1}(\mathcal{U}_{i_0}) = \bigcup_{1\leq k\leq n} V_i^{i_0}$, and $V_l^{i_0} \cap V_k^{i_0} = \{0\}$, when $l \neq k$.

[14] Since B_y is finite over the Henselian ring $\mathcal{O}_{Y,y}$, it is necessarily a product of local Henselian rings.

[15] Since B is finite over A, $B \widehat{\otimes}_A \ \mathcal{O}_Y(\mathcal{U}_i) = B \otimes_A \ \mathcal{O}_Y(\mathcal{U}_i)$.

Finally, the rest of the assertion follows from the fact that

$$\varinjlim_{i \leq i_0} \mathcal{O}_X(V_k^i)) = \mathcal{O}_{X,x_k},$$

and the inductive limit is taken on the connected component V_k^i of $g^{-1}(\mathcal{U}_i)$ containing x_k, when $\mathcal{U}_i$ varies over the affinoid neighborhoods of x_k inside $\mathcal{U}_{i_0}$.

□

We recall that F is an irreducible component of a $\mathbb{Q}_p$-separated reduced rigid analytic space X, if F is the image of a connected component of the normalization X^{nor} of X via the normalization morphism $X^{\mathrm{nor}} \to X$ (see [Con99]). Moreover, when X is a reduced affinoid $\mathrm{Spm}\, A$, then the irreducible components of X correspond to $\mathrm{Spm}\, A/\mathcal{P}$, where $\mathcal{P}$ is a minimal prime ideal of A.

We recall also that a subset Z of a reduced $\mathbb{Q}_p$-rigid analytic space X is said to be Zariski-dense if the only analytic subset of X containing Z is X itself.

Example A.2. The set $S = \{(1/p^n, 1/p^m),\ \text{where } n \in \mathbb{Z}, m \in \mathbb{N}\}$ of the rigid affine plane $\mathbb{A}_2^{rig}$ of dimension 2 is Zariski dense but for any open affinoid subdomain $\mathcal{U} \subset \mathbb{A}_2^{\mathrm{rig}}$, the set $\mathcal{U} \cap S$ is not Zariski dense in $\mathcal{U}$ (it follows from the maximum modulus principle).

This example motivates the notion of a very Zariski dense set of a rigid analytic space (see also [Bel10, def.II.5.1]):

Definition A.3.

(i) Let X be a $\mathbb{Q}_p$-separated reduced rigid analytic space over $\mathbb{Q}_p$, and $\Sigma \subset X$ be a Zariski dense subset. We say that Σ is *very* Zariski-dense in X if for every $x \in \Sigma$ there is a basis of open affinoid neighborhoods $\mathcal{U}$ of x in X such that $\Sigma \cap \mathcal{U}$ is Zariski-dense in $\mathcal{U}$.

(ii) We say that a subset Z of a $\mathbb{Q}_p$-separated rigid analytic space Y accumulates at $y \in Y$ if there is a basis of affinoid neighborhoods $U \subset Y$ of y such that $U \cap Z$ is Zariski-dense in U.

Remark A.4. Let X be a separated $\mathbb{Q}_p$-rigid space, $\{F_i\}$ be the irreducible components of X and $\mathcal{U}$ be an admissible open of X. Then it follows from [Con99, cor.2.2.9] that each irreducible component of $\mathcal{U}$ is contained in a unique F_i and for any i, $\mathcal{U} \cap F_i$ is empty or the union of irreducible components of $\mathcal{U}$.

Proposition A.5.

(i) Let $g : X \to Y$ be a finite flat morphism between two $\mathbb{Q}_p$-affinoid spaces such that X is equidimensional and Y is irreducible. Assume that g is étale at a Zariski dense set

Σ of points of X, then after shrinking X to a smaller admissible open X' of X, the restriction $g : X' \to g(X')$ is étale and $g(X')$ is an admissible open of Y.

(ii) Let $g : X \to Y$ be a finite morphism between rigid analytic spaces, then for any irreducible component F of X, $g(X)$ is a closed irreducible component of Y.

Proof. i) It is known that g is étale outside of the support of the relative differential sheaf $\Omega_{X/Y}$. Moreover, since g is étale at a Zariski dense set of points of X, the support Z of $\Omega_{X/Y}$ is a Zariski closed set of X of dimension $< \dim X$ (since Σ is Zariski dense in all irreducible components of X by [Con99, Prop.2.2.8]). Hence, $g_{|X-Z} : X - Z \to Y$ is étale, and the image of the Zariski *open* [16] $X - Z$ under g is a Zariski open of Y (a flat morphism is Zariski open).

ii) The assertion follows from the fact that a finite morphism is Zariski closed and [Con99, Proposition.2.2.3].

□

The following proposition was proved by Chenevier in [Che04] using base change arguments. We give in the following a more direct proof:

Proposition A.6. *Let $g : X \to Y$ be a finite torsion-free morphism between two reduced $\mathbb{Q}_p$-affinoid spaces and such that Y is irreducible. Then :*

(i) X is equidimensional of dimension equal to $\dim Y$ and the image of each irreducible component of X under g is Y.

(ii) Let Σ be a Zariski dense set of Y, then $g^{-1}(\Sigma)$ is Zariski dense in X.

Proof.

i) Let B (resp. A) be the affinoid algebra corresponding to X (resp. Y) and $g^* : A \to B$ be the finite torsion-free morphism corresponding to g. Since Y is irreducible and reduced, A is a domain. Let $\mathcal{P}$ be a minimal prime ideal of B corresponding to an irreducible component F of X, it follows from the fact that B is a torsion-free finite A-algebra that the morphism $A \to B/\mathcal{P}$ is injective (since the zero divisors of a reduced Noetherian ring are the union of its minimal prime ideals). Moreover, the image of the natural composition $F \to X \to Y$ is dense, because $A \to B/\mathcal{P}$ is injective (so the image of the morphism $\operatorname{Spec} B/\mathcal{P} \to \operatorname{Spec} A$ is Zariski dense) and $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are Jacobson schemes (so $\operatorname{Spm} A$ is Zariski dense in $\operatorname{Spec} A$, and the same for B).

However, g is also finite, and then Zariski closed. Hence, the irreducible component F of X surjects onto Y, and since the morphism $A \to B/\mathcal{P}$ is injective and finite, then $\dim F = \dim Y$

[16] Note that a Zariski open U of rigid analytic space X is not necessarily an affinoid subdomain of X. Take $X = \operatorname{Spm} \mathbb{Q}_p < T >$ and $U = D(T)$ the locus where T is invertible; it is clear that U doesn't satisfy the maximal modulus principle for the function $1/T$, and hence U is not an affinoid. However, any Zariski open is an admissible open for the rigid topology.

(it follows from the Going-up theorem), and hence X is equidimensional of dimension equal to $\dim Y$.

ii) A subset $\Sigma' \subset X$ is a Zariski dense set of a reduced affinoid X if and only if for any irreducible component F of X (see [Con99, Prop.2.2.8], $\Sigma' \cap F$ is a Zariski dense set of F. Thus, it is enough to prove the assertion when X is reduced an irreducible. Assume that X is irreducible and that Σ is a Zariski dense set of Y. Let $\Sigma' \subset X$ denote the subset $g^{-1}(\Sigma)$ of X. Since g is finite and torsion-free, then g is closed for the Zariski topology and surjective, and then the Zariski closure of Σ' is necessarily an analytic subspace $Z \subset X$ of dimension equal to $\dim Y = \dim X$, because $g(Z)$ is a Zariski closed set of Y containing Σ (so $g(Z)$ contains Y the closure of Σ). Hence, Z is finite and surjects on Y and it follows that $Z = X$, since they have the same dimension and X is irreducible.

□

Lemma A.7. *Let $\mathcal{U} = \operatorname{Spm} A$ be an equidimensional affinoid of dimension 2, F be a Zariski closed subset of $\mathcal{U}$ of dimension ≤ 1, $\mathcal{U}'$ be the admissible open given by $\mathcal{U} - F$. Let Σ be Zariski dense set of $\mathcal{U}$, then $\Sigma' = \Sigma \cap \mathcal{U}'$ is Zariski dense in $\mathcal{U}$ and in $\mathcal{U}'$.*

Proof. Note that $\Sigma = \Sigma' \cup (\Sigma \cap F)$. Hence, the Zariski closure $\bar{\Sigma}$ of Σ is equal to the union of the Zariski closure $\bar{\Sigma}'$ of Σ' with the closure $\overline{\Sigma \cap F}$ of $\Sigma \cap F$. On the other hand, $\bar{\Sigma} = \mathcal{U}$ and it is equidimensional of dimension 2, and $\overline{\Sigma \cap F} \subset F$ is of dimension at most one. Hence, $\bar{\Sigma}' = \mathcal{U}$, yielding that Σ' is dense in $\mathcal{U}$ and so in $\mathcal{U}'$.

□

Appendix B. On the very Zariski density of classical points in the Eigenvariety $\mathcal{E}_\Delta$

The goal of this section is to recall quickly the construction of the Siegel eigenvarieties and to prove that classical points which are old at p and of cohomological weights are very Zariski dense in them.

B.1. **The Weight space $\mathcal{W}$.** Recall that the connected components of $\mathcal{W}$ are naturally indexed by $\mathcal{W}^{a,b}$, where $(a,b) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$. The classical weights $(k_1, k_2) \in \mathcal{W}^{a,b}$ are congruent to $(a,b) \mod p-1$, in other words, the discrete part of the restriction of any character of $\mathcal{W}^{a,b}(\mathbb{C}_p)$ to $\mathbb{Z}/p\mathbb{Z}^\times$ is (ω_p^a, ω_p^b), where ω_p is the Teichmüller character. In addition, the formal scheme Spf $\mathbb{Z}_p[\![T_1, T_1]\!]$ is a Raynaud's formal model of any connected component[17] $\mathcal{W}^{a,b}$of the weight space $\mathcal{W}$.

[17]Note that Spm $\mathbb{Z}_p[\![T_1, T_1]\!][1/p] = \mathcal{W}^{a,b}$, and Spm $\mathbb{Z}_p[\![T_1, T_1]\!][1/p]$ is the open disk of dimension 2 and radius 1.

Now, let $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2$, any morphism $\underline{k} : (\mathbb{Z}_p^\times)^2 \to \mathbb{Q}_p^\times$ sending $(z_1, z_2) \to z_1^{k_1}.z_2^{k_2}$ give a point of $\mathcal{W}(\mathbb{Q}_p)$ and which denote again by $\underline{k}$, and we call it "an algebraic weight". More generally, any character of $\mathcal{W}(\mathbb{C}_p)$ which is a product of a character $\underline{k} \in \mathbb{Z}^2 \subset \mathcal{W}(\mathbb{Q}_p)$ with a finite character $\chi : (\mathbb{Z}_p^\times)^2 \to \bar{\mathbb{Q}}_p^\times$ is called "an arithmetic character" and denoted by $(\underline{k}, \chi)$.

Lemma B.1. *The classical weights $\mathbb{Z}^2$ of $\mathcal{W}(\mathbb{Q}_p)$ are very Zariski dense in the weight space $\mathcal{W}$.*

Proof. It follows from the Weierstrass preparation theorem that the set $\mathbb{Z}^2$ of integral weights is Zariski-dense in $\mathcal{W}$. Moreover, the p-adic topology on the union of open discs $\mathcal{W}$ induces by restriction the topology on $\mathbb{Z}^2$ for which we have a natural basis of neighborhood of $\underline{k} \in \mathbb{Z}^2$ given by the congruence classes modulo $p^n(p-1)$ for all n. Hence $\mathbb{Z}^2$ is very Zariski dense. $\square$

B.2. **Geometric Siegel cuspforms.** Let G denote the algebraic group GSp_4 and $\Gamma(N)$ be the open compact subgroup of $G(\widehat{\mathbb{Z}})$ of level N given by $\{\gamma \in G(\widehat{\mathbb{Z}}) \mid \gamma = \mathbb{1}_4 \mod N\}$.

Assume now that $N \geq 5$, and let $X/\mathbb{Z}_p$[18] be the Siegel scheme of level $\Gamma(N) \cap I_1$, where I_r is the standard Iwahoric at p of G given by $\{\gamma \in \mathrm{GSp}_4(\mathbb{Z}_p) \mid \gamma \mod p^r \in B(\mathbb{Z}/p^r\mathbb{Z})\}$ and B is the Borel of GSp_4. There exists a universal abelian scheme A/X with identity section e and we let $\omega := e^*(\Omega_{A/X})$ be the conormal sheaf. Note that ω is a locally free sheaf of rank 2 over X. Let $\bar{X}$ denote a toroidal compactification of X (it is not unique and depends on a combinatorial choice, see [FC90]), $\bar{A}$ be the semi-abelian scheme extending A to $\bar{X}$ and $D = \bar{X}/X$ be the normal crossing divisor at infinity. The sheaf ω extends to a locally free sheaf of rank 2 over $\bar{X}$, which we again denote by ω.

The classical cuspidal Siegel forms of level $\Gamma(N) \cap I_1$ and weight $k = (k_1, k_2)$ and coefficients in a p-adic field L (we have $k_1 \geq k_2$) are the elements of $\mathrm{H}^0(\bar{X}_L, \omega_L^k(-D))$[19], where ω^k is the locally free sheaf $\mathrm{Sym}^{k_1-k_2}\omega \otimes \det \omega^{k_2}$, and ω_L^k is the base change of ω^k to $\bar{X}_L$. Let $\bar{X}^{\mathrm{rig}}/\mathbb{Q}_p$ be the rigid analytic space given by taking the *generic fiber* of the formal scheme given by the completion of $\bar{X}$ along its special fiber, and writing again ω for the analytification of ω, and let $\bar{X}^{\mathrm{ord}}$ be the multiplicative ordinary locus of $\bar{X}^{\mathrm{rig}}$ (it is not an affinoid), then the p-adic (resp. v-overconvergent) cuspidal Siegel modular forms of tame level $\Gamma(N)$ weight $k = (k_1, k_2)$ and coefficients in L are $\mathrm{H}^0(\bar{X}_L^{\mathrm{ord}}, \omega_L^k(-D))$ (resp. $\mathrm{H}^0(\bar{X}_L^{\mathrm{ord}}(v), \omega_L^k(-D)))$, where $\bar{X}(v)$ is the v-overconvergent neighborhood of the multiplicative ordinary locus $\bar{X}^{\mathrm{ord}}$ (note that $D \subset \bar{X}^{\mathrm{ord}}$, since $\bar{A}$ is toric over D).

[18] The generic fiber $X/\mathbb{Q}_p$ is smooth, and the special fiber $X/\mathbb{F}_p$ is singular, and it even has vertical components with respect to the Kottwitz-Rapoport stratification.

[19] It follows from Koecher principle that $\mathrm{H}^0(\bar{X}_L, \omega_L^k)$ does not depend on the choice of the toroidal compactification $\bar{X}$ of X.

Andreatta, Iovita and Pilloni constructed for any weight $k \in \mathcal{W}(\mathbb{C}_p)$ and certain parameters $v, w \in \mathbb{R}_+^\times$ a Banach sheaf ω_w^k over $\bar{X}(v)$, and a natural sheaf monomorphism $\omega^k \hookrightarrow \omega_w^k$ when $k = (k_1, k_2)$ is classical (see [AIP15]), and they describe precisely the cokernel of that monomorphism. The sheaf ω_w^k is isomorphic locally for the étale topology to the w-analytic induction of the Borel $B(\mathbb{Z}_p)$ to the Iwahoric of GL_2 with respect to the character k. Note that any character $k \in \mathcal{W}(\mathbb{C}_p)$ is locally analytic by [AIP15, §2.2] and hence ω_w^k is a non-zero Banach sheaf (The sections of ω_w^k are congruent to the image of the Hodge-Tate map by [AIP15, Prop.4.3.1]).

The p-adic modular forms obtained by this interpolation are locally analytic overconvergent (not necessarily overconvergent), however those satisfying the slope condition of [AIP15, Thm. 7.1.1] are overconvergent (see also [AIP15, Prop.2.5.1.] and [AIP15, Prop.7.2.1]). Note that this construction is independent of the choice of the toroidal compactification of $\bar{X}$ (see [L7̈4, Thm.1.6.1] and [AIP15, Prop.5.5.2]) and we denote the corresponding eigenvariety by $\mathcal{E}_N$.

B.3. **Local charts of the variety $\mathcal{E}_N$ and density of classical points of $\mathcal{E}_N$.**

Let $\chi : (\mathbb{Z}/p\mathbb{Z}^\times)^2 \to \mathbb{Q}_p^\times$ be a character, $\mathcal{W}^\chi$ the connected component of $\mathcal{W}$ corresponding to χ, and $\mathcal{E}_N^\chi$ the union of connected components of $\mathcal{E}_N$ given by the restriction of $\mathcal{E}_N$ to $\mathcal{W}^\chi$.

For $w, v \in \mathbb{R}$ let $W = \operatorname{Spm} R$ be a small enough affinoid subdomain of $\mathcal{W}^\chi$ to ensure the existence of the Banach sheaf $\omega_w^\kappa(-D)$ of $\bar{X}(v) \times \operatorname{Spm} R$ interpolating the Banach sheaf $\omega_w^k(-D)$ of w-analytic v-overconvergent Siegel cusp form when k varies in $\operatorname{Spm} R$ (κ denotes here the tautological character $(\mathbb{Z}_p^\times)^2 \to R^\times$).

On the other hand, let $S_\kappa^\dagger$ be the Frechet R-module of ϵ-overconvergent cuspidal Siegel families over the affinoid R and given by

$$\varinjlim_{v\to 0, w\to\infty} \mathrm{H}^0(\bar{X}(v) \times \operatorname{Spm} R, \omega_w^\kappa(-D)).$$

The action of the Hecke operator $\mathbb{U} = U_0.U_1$ is completely continuous on the Frechet R-module $S_\kappa^\dagger$. Let $\mathcal{T}_{W,r}$ be the image of the Hecke algebra generated over R by the image of $\mathcal{H}_N$ in $S_\kappa^{\dagger, v\leq r}$, where $S_\kappa^{\dagger,\leq r}$ is the R-finite submodule of $S_\kappa^\dagger$ of slope at most r for $\mathbb{U} = U_0.U_1$.[20] It follows from the results of [Bel10, §.II] that

$$\mathcal{E}_{N,W}^r := \operatorname{Spm} \mathcal{T}_{W,r}, \tag{33}$$

is an affinoid subdomain of $\mathcal{E}_N$ and by construction $\mathcal{E}_{N,W}^r$ is finite and torsion-free over W and the $\{\mathcal{E}_{N,W}^r\}$ form an admissble covering of $\mathcal{E}$.

[20]Note that the action of $\mathbb{U}$ is completely continuous on $S_\kappa^\dagger$, so we have a slope decomposition.

Since the ordinary locus of any toroidal compactification of the Siegel modular scheme is not an affinoid, we cannot prove that the specialization

$$\mathrm{H}^0(\bar{X}(v) \times_{\bar{\mathbb{Q}}_p} \operatorname{Spm} R, \omega_w^\kappa) \to \mathrm{H}^0(\bar{X}(v), \omega_w^k)$$

is surjective and that $\mathrm{H}^0(\bar{X}(v) \times \operatorname{Spm} R, \omega_w^\kappa)$ is a projective R-Banach module. However, Andreatta-Iovita-Pilloni proved in [AIP15, Prop.8.2.3.3] a control theorem for cuspidal families and that $\mathrm{H}^0(\bar{X}(v) \times \operatorname{Spm} R, \omega_w^\kappa(-D))$ is a projective R-Banach module, by projecting the sheaf $\omega_w^\kappa(-D)$ to the minimal compactification of the Siegel modular scheme, and using the fact that small v-overconvergent neighborhoods of the multiplicative ordinary locus of the minimal compactification of the Siegel modular scheme are affinoid spaces, and the deep descent result [AIP15, Prop.8.2.2.4].

Skinner-Urban constructed in [SU06, §2] a semi-ordinary eigenvariety $\mathcal{E}_N^{|U_0|_p=1} \subset \mathcal{E}_N$ for overconvergent Siegel cusp forms of tame level $\Gamma(N)$ and genus 2 by interpolating the locally free sheaf ω^k inside a Banach sheaf ω_w^κ over the weight space $\mathcal{W}$ using the Igusa tower. That construction is a special case of the construction given by Andreatta-Iovita and Pilloni in [AIP15] of the eigenvariety $\mathcal{E}_N$, since the linearization of the Hodge-Tate map

$$\mathrm{HT}_{H_n^D} : H_n^D \to \omega_{H_n}$$

is surjective on the multiplicative ordinary locus ($H_n \subset \bar{A}$ is the level n canonical subgroup and H_n^D is its Cartier dual), and the fact that any semi-ordinary (i.e of slope 0 for U_0) p-adic Siegel cuspforms of finite slope for U_1 overconverges to a strict neighborhood of the ordinary locus. For the latter note that under the iteration of the Hecke correspondances at p, an overconvergent neighborhood of X^{ord} accumulates around the multiplicative ordinary ordinary locus X^{ord}. The correspondence U_0 improves the radius of overconvergence. Hence, the functional equation $U_0.g = U_0(g).g$ allows us to extend g to a bigger neighborhood of the multiplicative ordinary locus when $U_0(f) \neq 0$ (the function degree of [Pil11, Thm.3.1.] increases under the iteration of U_0). Meanwhile, one can use a similar functional equation for U_1 to get classicality at the level of the sheaves when the slope satisfies the condition of [AIP15, Prop.7.3.1].

By construction of $\mathcal{E}_N$ we have an algebra homomorphism $\mathcal{H}_N \to \mathcal{O}_{\mathcal{E}_N}^{\mathrm{rig}}(\mathcal{E}_N)$, and the image lands in the subring $\mathcal{O}_{\mathcal{E}_N}^{\mathrm{rig}}(\mathcal{E}_N)^+$ given by the global section bounded by 1 on $\mathcal{E}_N$. Therefore, the canonical application "system of eigenvalues" induces a correspondence between the systems of eigenvalues for Hecke operators occuring in $\mathcal{H}_N$ of locally analytic overconvergent cuspidal Siegel eigenforms of tame level $\Gamma(N)$ and weight $k \in \mathcal{W}(\mathbb{C}_p)$ having nonzero $\mathbb{U}$-eigenvalue, and the set of $\mathbb{C}_p$-valued points of weight $k = (k_1, k_2)$ on the Siegel eigenvariety $\mathcal{E}_N$. Note that for any overconvergent form g corresponding to a point of $\mathcal{E}_N$ of weights (l_1, l_2),

$$g \mid U_1 = p^{l_2-3} U_1(g).g; \tag{34}$$

we renormalize U_1 in the aim to have a good p-adic interpolation (see for example [SU06, Thm.2.4.14]).

One has the following Lemmas proving the very Zariski density of the classical points having a crystalline representation at p in $\mathcal{E}_N$, which is important for applying further the results of [BC09, §4] (see the hypothesis (HT) of [BC09, §.3.3.2]).

Lemma B.2. *Let $z \in \mathcal{E}_N$ be a classical point, then there exists an affinoid neighborhood Ω of z in $\mathcal{E}_N$ of constant slopes for U_0, U_1 and such that the old at p classical points of regular weights of Ω are very Zariski dense in it, $\kappa(\Omega)$ is an open affinoid subdomain of $\mathcal{W}$, and each irreducible component of Ω surjects to $\kappa(\Omega)$.*

Proof. Note that $\mathcal{E}_N$ is admissibly covered by $\{\mathcal{E}^r_{N,W}\}$. Hence, there exists an affinoid subdomain $\mathcal{E}^r_{N,W}$ of $\mathcal{E}_N$ containing z and surjecting on the affinoid subdomain $W \subset \mathcal{W}$. By construction of $\mathcal{E}_N$, the slopes of U_0, U_1 are locally constant. Then Prop.A.1 and Prop.A.6 yields that we can shrink $\mathcal{E}^r_{N,W}$ to a smaller open affinoid subdomain Ω of $\mathcal{E}_N$ containing z and with constant slope S_1 (resp. S_2) for the Hecke operator U_0 (resp. U_1) and such that $\kappa(\Omega)$ is an open affinoid subdomain of $\mathcal{W}$, and $\kappa : \Omega \to \kappa(\Omega)$ is finite and torsion-free (so the restriction of κ to any irreducible component of Ω is surjective by Prop.A.6).

Since Ω contains the classical point z, then the points of Ω with weights satisfying the small slope conditions of [AIP15, Thm.7.1.1] form a Zariski dense set in Ω, because the *algebraic* points (l_1, l_2) of $\kappa(\Omega)$ satisfying the inequality of the small slope conditions of [AIP15, Thm.7.1.1] form a Zariski dense set of $\kappa(\Omega)$ (so their preimage is dense in Ω by Prop.A.6). Moreover, it follows from the criterion of classicality of overconvergent forms that the points satisfying the small slope conditions of [AIP15, Thm.7.1.1] are necessarily classical. Actually, Prop.A.1, Prop.A.6 and Lemma B.1 show that classical points of Ω are very Zariski-dense in it. Finally, the assertion follows from the fact that the classical points of Ω with sufficiently *regular* weights satisfy the assumptions of [SU06, Thm.2.4.17], and hence they are old at p. □

Corollary B.3. *Let $\mathcal{E}^{\mathrm{ord},1}_N$ be the admissible open of $\mathcal{E}_N$ defined by*

$$\mathcal{E}^{\mathrm{ord},1}_N := \{x \in \mathcal{E}_N, \mid U_0(x) \mid_p = 1, \mid U_1(x) \mid_p = p^{-1}\},$$

$C \in \mathbb{N}_{>1}$, and Σ_C be the set of points of $\mathcal{E}^{\mathrm{ord},1}_N$ of "algebraic weights" (k_1, k_2) satisfying $k_1 > k_2 + C \geq \mathrm{Max}(9, C)$. Then:

(i) The overconvergent cuspforms of Σ_C are classical and old at p.

(ii) The set Σ_C is very Zariski dense in $\mathcal{E}_N^{\mathrm{ord},1}$.

(iii) The point x of $\mathcal{E}_N^{\mathrm{ord},1}$ corresponding to π_α is an accumulation point of Σ_C.

Proof. The points of Σ_C have slope equal to 1, Iwahoric level at p and satisfy the slope condition $1 < k_1 - k_2 + 1, k_2 >> 0$ of the classicality criterion for overconvergent Siegel cuspforms. Hence they are necessarily classical. A direct computation shows that the points of Σ_C satisfy the assumptions of [SU06, Thm.2.4.17], and hence they are necessarily old at p.

Since the algebraic weights (k_1, k_2) with $k_1 > k_2 + C \geq \mathrm{Max}(9, C)$ are very Zariski dense in $\mathcal{W}$ (see Lemma B.1), the assertion of (ii) and (iii) follows directly from the argument already used to proof Lemma B.2.

□

B.4. **Siegel eigenvariety of paramodular level N.** Let $\mathcal{E}_\Delta$ be the Siegel eigenvariety of tame level the paramodular group Δ. Since the classical Siegel cuspforms of level $\Delta \cap I_1$ are necessarily of level $\Gamma(N) \cap I_1$, the results of [Bel10, II.5.] yields that there exists a natural closed immersion $\iota : \mathcal{E}_\Delta \hookrightarrow \mathcal{E}_N$ compatible with the system of Hecke eigenvalues and the weights:

$$\begin{tikzcd} & \mathcal{E}_\Delta \arrow[dl, "\iota"'] \arrow[dr, "\kappa"] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathcal{E}_N \arrow[rr, "\kappa"'] & & \mathcal{W} \end{tikzcd}$$

Since the restricted Hecke algebra $\mathcal{H}_{Np}$ generated over $\mathbb{Z}$ by the Hecke operators $T_{\ell,1}, T_{\ell,2}, \mathrm{S}_\ell$ for $\ell \nmid Np$ acts semi-simply on classical cuspidal Siegel paramodular eigenforms of cohomological weights, [Bel10, Lemma.I.9.1] implies that $\mathcal{E}_\Delta$ is reduced. Note also that $\mathcal{E}_\Delta$ is equidimensional of dimension 2.

Corollary B.4. *Let $\mathcal{E}_\Delta^{\mathrm{ord},1}$ be the admissible open of $\mathcal{E}_\Delta$ defined by*

$$\mathcal{E}_\Delta^{\mathrm{ord},1} := \{x \in \mathcal{E}_\Delta, \mid U_0(x) \mid_p = 1, \mid U_1(x) \mid_p = p^{-1}\},$$

$C \in \mathbb{N}_{>1}$, and Σ_C be the set of points of $\mathcal{E}_\Delta^{\mathrm{ord},1}$ of "algebraic weights" (k_1, k_2) satisfying $k_1 > k_2 + C \geq \mathrm{Max}(9, C)$. Then:

(i) The overconvergent cuspforms of Σ_C are classical and old at p.

(ii) The set Σ_C is very Zariski dense in $\mathcal{E}_\Delta^{\mathrm{ord},1}$.

(iii) The point x of $\mathcal{E}_\Delta^{\mathrm{ord},1}$ corresponding to π_α is an accumulation point of Σ_C.

Proof. It follows immediately from Corollary B.3 and the fact that a subset of an affinoid space is a Zariski dense if and only if its intersection with any irreducible component is Zariski dense in that irreducible component (see [Con99, Prop.2.2.8]).

□

B.5. **The Coleman-Mazur eigencurve.** It follows from the construction of the eigencurve $\mathcal{C}_N$ that there exists a morphism $\mathbb{Z}[T_l, U_p]_{\ell\nmid Np} \to \mathcal{O}(\mathcal{C}_N)$ such that the application defined by taking the system of Hecke eigenvalues $\mathcal{C}_N(\mathbb{C}_p) \to \mathrm{Hom}(\mathbb{Z}[T_l, U_p]_{\ell\nmid Np}, \mathbb{C}_p)$ induces a correspondence between the systems of Hecke eigenvalues for $\{T_l, U_p\}_{\ell\nmid Np}$ of normalised overconvergent modular eigenforms with Fourier coefficients in $\mathbb{C}_p$, of tame level N and of weight $w \in \mathcal{V}(\mathbb{C}_p)$, finite slope and the set of $\mathbb{C}_p$-valued points of weight w on the eigencurve $\mathcal{C}_N$.

Let $\mathcal{C}_N^{\mathrm{full}}$ be the full eigencurve of tame level N constructed using the Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid Np$.

There exists a natural locally finite surjective morphism $\mathcal{C}_N^{\mathrm{full}} \to \mathcal{C}_N$ (it is not injective when $N \geq 4$). There is a natural bijection between $\mathcal{C}_N^{\mathrm{full}}(\mathbb{C}_p)$ and the set of overconvergent eigenforms with finite slope, tame level N and weight in $\mathcal{V}(\mathbb{C}_p)$, which sends g to the system of eigenvalues $\{(T_\ell(g))_{\ell\nmid Np}, (U_\ell(g))_{\ell\mid Np}\}$.

By construction of the full eigencurve, the ordinary locus of $\mathcal{C}_N^{\mathrm{full}}$ (the open-closed locus where $\mid U_p \mid_p = 1$) has a formal model Spf $h^{\mathrm{ord}}(Np^\infty)$. Moreover, the irreducible components of the ordinary locus of $\mathcal{C}_N^{\mathrm{full}}$ correspond to the irreducible components of $\mathrm{Spec}\, h^{ord}(Np^\infty)$, and hence to Galois orbit of Hida families of tame level N.

It follows from Hida [Hid86] (the "control theorem") that the eigencurve $\mathcal{C}_N$ is étale over the weight space at all classical ordinary points of cohomological weight. This result has been generalized to all non-critical *p-regular* [21] classical points of cohomological weight by Coleman and Mazur [CM98, 7.6.2]. Their argument is based on showing that the generalized eigenspace of a such form consists only of classical forms (using the classicality criterion of [Col96]) and that the multiplicity of the operator U_p is exactly one by p-regularity. However, the étaleness of the weight map can fail in weight one (see [CV03] and [BD16]). In particular, the eigencurve is not Gorenstein (so singular) at p-irregular weight one Eisenstein series (see [BDP18]).

Thus, $\mathcal{C}_N$ is smooth at f_α (since it is étale over $\mathcal{V}$ at f_α), then there is a unique component of $\mathcal{C}_N$ specializing to f_α. Let $\mathcal{F} = \sum_{n=1}^{\infty} a(n, \mathcal{F})q^n$ denote the unique, up to Galois conjugacy, Hida family specializing to f_α. Recall that $\mathbb{I}$ is the finite integral extension of $\mathbb{Z}_p[\![T]\!]$ generated by the Fourier coefficients of $\mathcal{F}$, and let $\mathfrak{X}_{\mathbb{I}}$ denote the irreducible component of $\mathcal{C}_N$ corresponding to $\mathcal{F}$ ($\mathcal{X}(\mathbb{C}_p) = \mathrm{Hom}_{alg}(\mathbb{I}, \mathbb{C}_p)$). One can see that the classical specialization of the family $\mathcal{F}$ of weight $2k-2$ have a constant sign of the functional equation of their L-function, and if their weight $2k-2$ is congruent to a constant $a \mod p-1$, then they belong to the same connected component $\mathcal{V}^a$ of $\mathcal{V}$ ($\mathcal{V}^a(\mathbb{C}_p) = \mathrm{Hom}(1 + p^\nu \mathbb{Z}_p, \mathbb{C}_p^\times) = \mathrm{Hom}_{alg}(\mathbb{Z}_p[\![T]\!], \mathbb{C}_p)$), where $\nu = 2$ when $p \geq 3$ and $\nu = 4$ when $p = 2$.

[21]Conjecturally, any classical eigenform of cohomological weight is p-regular (i.e its Hecke polynomial at p has distinct roots).

Skinner-Urban constructed in [SU06, Prop.4.2.5] a Siegel cuspidal eigenfamily $SK(\mathcal{F})$ of parallel weight and tame level Δ and such that it is the Saito-Kurokawa lift to GSp(4) of the Hida family $\mathcal{F}$.

Proposition B.5 ([SU06] Prop.4.2.5)**.** *There exists a Zariski closed immersion $\lambda_{\mathcal{F}} : \mathfrak{X}_{\mathbb{I}} \hookrightarrow \mathcal{E}^1_{\Delta}$ with image denoted by $\mathfrak{V}$ and such that the following diagram commutes*

$$\begin{tikzcd} \mathfrak{X}_{\mathbb{I}} \arrow[r, "\lambda_{\mathcal{F}}"] \arrow[d, "w"] & \mathcal{E}^1_{\Delta} \arrow[d, "\kappa"] \\ \mathcal{V}^{2a} \arrow[r, "\lambda_w"] & \mathcal{W}^{a+1,a+1} \end{tikzcd}$$

where $\lambda_w(2k-2) = (k,k)$ and the morphism $\lambda_{\mathcal{F}}$ corresponds to the morphism

$$\lambda^*_{\mathcal{F}} : \mathcal{O}(\mathcal{E}^1_{\Delta}) \to \mathbb{I}[1/p] = \mathcal{O}(\mathfrak{X}_{\mathbb{I}})$$

defined by

$$\lambda^*_{\mathcal{F}}(P_\ell(X)) = (X- <\ell>^{1/2})(X- <\ell>^{1/2}\ \ell^{-1})(X^2 - a_{\ell,\mathcal{F}}X + \ell <\ell> \omega_p^a(\ell)),\ \text{for any } \ell \nmid Np,$$

where $<\ell>$ is the image of $\ell \nmid Np$ via the composition $1+p^\nu\mathbb{Z}_p \to \mathbb{Z}_p[\![1+p^\nu\mathbb{Z}_p]\!]^\times \to \mathcal{O}(\mathcal{V})^\times$, $P_\ell(X) \in \mathcal{O}(\mathcal{E}^1_\Delta)[X]$ is the Hecke-Andrianov polynomial at $\ell \nmid Np$ and $\lambda^*_{\mathcal{F}}(U_0) = a(p,\mathcal{F}), \lambda^*_{\mathcal{F}}(U_1) = p.a(p,\mathcal{F})$.

Appendix C. Some examples where $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1) = 1$

Using Nekovar's result [Nek06, Prop.4.2.3] about $\mathbb{I}$-adic Selmer groups mentioned before Corollary 7.6 we can exhibit infinitely many examples of modular forms f of weight $k \geq 3$ such that they satisfy the condition $\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_f(k-1)) = 1$ in Theorem 7.7. This requires finding suitable elliptic curves with ordinary reduction at p and considering their corresponding Hida family $\mathcal{F}$. One such example is discussed in section 9.1 of [BK17], where for $p = 5$ and $N = 731$ the residual Selmer group

$$\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \overline{\rho}_{E,p}(1)) = \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{E,p}(1) \otimes \mathbb{Q}_p/\mathbb{Z}_p)[p] = \mathrm{Sel}_p(E)[p]$$

of the rank 1 elliptic curve E (Cremona label 731a1) is calculated to have order 5 (since the order of vanishing of $L(f,s)$ at $s = 1$ is one we know that the BSD conjecture holds). This elliptic curve has non-split reduction at both primes dividing N and good ordinary reduction at 5, with $a_5(E) = -1$ and therefore $\alpha \neq 1$. In addition this example satisfied the condition $L_p(f_\alpha, \omega_p^{-1}, T = p) \neq 0$.

In the following assume that f is the p-ordinary stabilization of the weight two cuspform attached to a rank 1 elliptic curve $E/\mathbb{Q}$. Recall that $\mathbb{I}$ is the finite flat extension of $\mathbb{Z}_p[\![T]\!]$ generated by the Fourier coefficients of the Hida family $\mathcal{F}$ specializing to f ($\mathbb{I}$ is an integral domain).

Note that the cohomology groups $\mathrm{H}^i_{f,\mathrm{unr}}(G_{\mathbb{Q}}^{Np}, \rho_{\mathcal{F}} \otimes \chi_{\mathrm{univ}}^{-1/2})$ of the Selmer complex are of finite type over $\mathbb{I}$ when $i \in \{1, 2\}$ (see [Nek06, Prop.4.2.3]).

Let $\mathcal{P}_f \subset \mathbb{I}$ be the height one prime ideal corresponding to the system of Hecke eigenvalues of f. We have the following control theorem proved by Nekovar [Nek06, (0.15.1.1)]

(35)

$$0 \to \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}) \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{P}_f}) \otimes_{\mathbb{I}_{\mathcal{P}_f}} \mathbb{I}_{\mathcal{P}_f}/\mathcal{P}_f \to \mathrm{H}^1_{f,unr}(\mathbb{Q}, \rho_f(1)) \to \mathrm{H}^2_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}) \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{P}_f})[\mathcal{P}_f].$$

where $\mathrm{H}^2_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}) \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{P}_f})[\mathcal{P}_f]$ means the submodule annihilated by the prime ideal $\mathcal{P}_f$.

Since $\dim \mathrm{H}^1_{f,unr}(\mathbb{Q}, \rho_f(1)) = 1$ Nakayama's lemma applied to (35) yields that the $\mathbb{I}_{\mathcal{P}_f}$-module $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}) \otimes \mathbb{I}_{\mathcal{P}_f})$ is a monogenic. Moreover, it follows from Corollary 7.6 that $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}) \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{P}_f})$ is a torsion-free $\mathbb{I}_{\mathcal{P}_f}$-module, so

$$\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}) \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{P}_f}) = \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2})) \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{P}_f}$$

is a free rank one $\mathbb{I}_{\mathcal{P}_f}$-module. Thus there exists a principal Zariski open $D(s)$ of $\operatorname{Spec} \mathbb{I}$ (where $s \in \mathbb{I}$) such that the localization of $\mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}))$ at the non-vanishing locus $D(s)$ is a free rank one $\mathbb{I}[1/s]$-module. On the other hand, let $\mathcal{U} \subset D(s)$ be the Zariski open defined as the complementary of the support [22] of the $\mathbb{I}$-torsion part of $\mathrm{H}^2_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}}(\chi_{\mathrm{univ}}^{-1/2}))$. Note that the classical points of $\mathcal{U}$ are Zariski dense, hence (35) yields that all the classical specialization $\mathcal{F}_z$ of the Hida family $\mathcal{F}$ at a point $z \in \mathcal{U}$ of weight k_z satisfy

$$\dim \mathrm{H}^1_{f,\mathrm{unr}}(\mathbb{Q}, \rho_{\mathcal{F}_z}(k-1)) = 1.$$

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