Higher lattices, discrete two-dimensional holonomy and topological phases in (3+1)D with higher gauge symmetry

Alex Bullivant*[1,2], Marcos Calçada†[2,3], Zoltán Kádár‡[2], João Faria Martins§[2], and Paul Martin¶[2]

[1]School of Physics and Astronomy, University of Leeds, Leeds, LS2 9JT, United Kingdom
[2]Department of Pure Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom
[3]Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa, Ponta Grossa, PR, Brazil

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Abstract

Higher gauge theory is a higher order version of gauge theory that makes possible the definition of 2-dimensional holonomy along surfaces embedded in a manifold where a gauge 2-connection is present. In this paper we study Hamiltonian models for discrete higher gauge theory on a lattice decomposition of a manifold. We show that a construction for higher lattice gauge theory is well-defined, including in particular a Hamiltonian for topological phases of matter in 3+1 dimensions. Our construction builds upon the Kitaev quantum double model, replacing the finite gauge connection with a finite gauge 2-group 2-connection. Our Hamiltonian higher lattice gauge theory model is defined on spatial manifolds of arbitrary dimension presented by slightly *combinatorialised* CW-decompositions (2-lattice decompositions), whose 1-cells and 2-cells carry discrete 1-dimensional and 2-dimensional holonomy data. We prove that the ground-state degeneracy of Hamiltonian higher lattice gauge theory is a topological invariant of manifolds, coinciding with the number of homotopy classes of maps from the manifold to the classifying space of the underlying gauge 2-group.

The operators of our Hamiltonian model are closely related to discrete 2-dimensional holonomy operators for discretised 2-connections on manifolds with a 2-lattice decomposition. We therefore address the definition of discrete 2-dimensional holonomy for surfaces embedded in 2-lattices. Several results concerning the well-definedness of discrete 2-dimensional holonomy, and its construction in a combinatorial and algebraic topological setting are presented.

Keywords: Kitaev Model; topological phases in 3+1D; topological quantum computing; topological quantum field theory; higher gauge theory; surface holonomy; crossed module; lattice gauge theory.

1 Introduction

In the absence of external symmetries, a topological phase of matter is characterised by a local, gapped, quantum many-body Hamiltonian whose effective (infra-red) field theory is described by a topological quantum field theory (TQFT) [36, 56, 73, 70]. A topological phase is therefore diffeomorphism invariant, and thus insensitive to local perturbations in the sense that the amplitudes of physical processes are global topological invariants. It is this latter property that makes topological phases of matter candidates for the implementation of fault tolerant quantum computing [70, 58, 47, 56].

*E-mail address: A.L.Bullivant@leeds.ac.uk
†E-mail address: mcalcada@uepg.br
‡E-mail address: zokadar@gmail.com
§E-mail address: j.fariamartins@leeds.ac.uk
¶E-mail address: p.p.martin@leeds.ac.uk

Due to a lack of local observables, experimentally distinguishing different topological phases can be a difficult task from a microscopic point of view. Instead the characterising properties of topological phases are most efficiently described by their emergent behaviours. Signatures for the presence of topological order include ground state degeneracies which depend on the spatial topology of the material in question [71, 45], universal negative corrections to the entanglement entropy [46, 39, 50] and the presence of stable topological excitations which provide non-trivial representations of their respective motion groups [49, 43]. This means the braid group for point particles (anyons) in 2+1D [56] and the loop braid group for loop excitations in 3+1D [68].

In 2+1D there exist several constructions for TQFTs (see for instance [66]). Path-integral models arise from Chern-Simons-Witten theory [75] and from BF theory [1], while the discrete realisation of BF-theory coincides with the Turaev-Viro [66]/Barrett-Westbury [6] state-sum (see [2]). In contrast, in 3+1D a framework general enough to capture all features of 4D topology is still lacking [73]. Nevertheless, we have the Crane-Yetter TQFT [25, 66] and its generalisations [73, 27]; and the Yetter homotopy 2-type TQFT [76, 61, 35], derived from a strict finite 2-group [4]. All of these 3+1D TQFTs give rise to topological invariants which at most depend on the homotopy 2-type, signature and spin-structure of space-time [66, 35], or are conjectured to do so.

One successful approach to understanding candidate models for $(d+1)$D topological phases has been to define *Hamiltonian realisations* of $(d{+}1)$D TQFTs [62, 73]. This means that a finite dimensional Hilbert space $V(M, L)$, and an exactly solvable (the sum of mutually commuting projectors) Hamiltonian $H_L\colon V(M, L) \to V(M, L)$ is assigned to each d-manifold M, with a given *lattice decomposition* L (e.g. L can be a triangulation or a CW-decomposition of M). The constructions of both $V(M, L)$ and H_L should be local on L. To say that such a *Hamitonian schema* [62] is a Hamiltonian realisation of the TQFT $\mathcal{Z}$ roughly means that given a d-manifold M each ground state vector space $GS(M, L)$ of H_L is canonically isomorphic to $\mathcal{Z}(M)$. (In particular this implies that the ground state degeneracy $\dim(GS(M, L))$ does not depend on L and it is a topological invariant of M.) In 2+1D, this Hamiltonian realisation approach has been successfully achieved in the case of Dijkgraaf-Witten topological gauge theories [28, 47, 42, 69] and the Turaev-Viro TQFT, giving rise to the so-called string-net models [51]. The Kitaev quantum-double model [47] can be seen as a Hamiltonian realisation for the Dijkgraaf-Witten TQFT [28] with trivial cocycle and thus also for finite-group BF-theory. Similar ideas were applied to the 3+1D Crane-Yetter TQFT [68, 67], giving rise to the Walker-Wang model.

A Hamiltonian realisation of Yetter's homotopy 2-type TQFT was constructed in [73, 23]. This is a higher gauge theory version of Kitaev quantum-double model [47]. In this paper we continue to develop these Hamiltonian higher gauge models for topological phases. We note that topological phases protected by higher gauge symmetry are also proposed in [44].

Higher gauge theory [3, 5] is a generalisation of ordinary gauge theory with further levels of structure and symmetry. A key feature of higher gauge theory is parallel transport along surfaces embedded in a manifold where a gauge 2-connection is present [3, 5, 34, 65]. In higher gauge theory, instead of local gauge symmetry groups we have *local gauge symmetry 2-groups* [4, 3, 5], which in this paper will always be *strict.* Strict 2-groups are higher order (categorified) notion of groups, thereby possessing one additional layer of structure in comparison to groups. Strict 2-groups, recall, are equivalent to *crossed modules of groups* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ [22, 18, 7, 4]. Here $\partial\colon E \to G$ is a map of groups and $\triangleright$ is a left action of G on E by automorphisms, satisfying some compatibility relations: the 1st and 2nd Peiffer relations. As in [34], throughout the body of this paper, we will mainly use the language of crossed modules. This will facilitate using algebraic topological techniques as developed e.g. in [11, 13] – we will review these techniques here.

In this paper, completing the programme initiated in [23], we define an exactly solvable Hamiltonian model for higher lattice gauge theory on manifolds M of any dimension, here called the "higher Kitaev model". This lifts Kitaev's quantum-double model [47] for topological phases from finite group topological gauge theory to finite 2-group topological higher gauge theory [38, 33]. Typically M will be a 3-dimensional manifold, and the higher Kitaev model is proposed to be a model for (3+1)-dimensional topological phases [68, 73, 69, 23, 67, 49, 24, 44, 52]. We prove that the ground state degeneracy of the higher Kitaev model is a topological (in fact homotopy) invariant of manifolds. Specifically, we show that the ground state degeneracy is given by the number of homotopy classes of maps from M to the classifying space of the underlying gauge symmetry 2-group [18, 35]; hence the ground state degeneracy is closely related [35] to Yetter homotopy 2-type TQFT. (The precise relation appears in [23], where a proof that the higher Kitaev model is a Hamiltonian realisation of Yetter homotopy 2-type TQFT is given.)

1.1 Higher lattices and higher lattice gauge theory

Our model utilises ideas from higher lattice gauge theory [59, 44]. Similarly to [47], we take lattice gauge theory as the starting point (with its good connection to physical observation [74, 48]) and lift the structure through the process of categorification. We thereby replace the gauge group with a gauge 2-group and a gauge connection discretised on a lattice with a discretised higher gauge connection, here called a *fake-flat 2-gauge configuration*. Therefore we enrich the local variables of lattice gauge theory (holonomies along edges) to include non-abelian 2-dimensional holonomies along the faces of the lattice; recall again that 2-dimensional holonomies feature prominently in higher gauge theory; see [5, 34, 3, 65].

The model constructed in this paper extends and formalises the proposal of [23] for a Hamiltonian model for the Yetter homotopy 2-type TQFT [76], from triangulated manifolds to manifolds with a slightly combinatorialised version of CW-decompositions (here called *2-lattice decompositions* — see Definition 22). Hence a 2-lattice decomposition L represents a manifold M as a disjoint union of i-cells, where i is an arbitrary non-negative integer, where each i-cell homeomorphic to the interior of the i-disk $[0,1]^i$. As customary, 0-cells are called vertices, 1-cells are called edges, 2-cells are called faces (or plaquettes), and 3-cells are called blobs. These 2-lattice decomposition are considerably less rigid than triangulations. Therefore using 2-lattice decompositions of manifolds, as opposed to triangulations, to decompose a manifold into smaller pieces has the advantage that fewer cells are needed to decompose a manifold, leading to microscopic Hilbert spaces of much smaller rank. We illustrate this fact by describing two small models for discrete higher gauge theory in the 3-sphere; see §5.1.5.

Many constructions in this paper would still work if we use CW-complex decompositions of manifolds rather than 2-lattice decomposition; however a lot of the combinatorial flavour presented in the final construction of the Hamiltonian model would be lost. By using 2-lattice decompositions instead of triangulations some combinatorics is taken away; therefore, despite the fact that our model is fully combinatorial, some algebraic topology will be required in proving that the model is well-defined.

By passing from triangulations to 2-lattices, we hence demonstrate the internal consistency of the model in [23], which tacitly assumed that discrete 2-dimensional holonomy of a discrete higher gauge field is well-defined, for instance when proving in *loc cit* that the ground state degeneracy is a topological invariant derived from Yetter TQFT, and as such that our model is a Hamiltonian realisation of Yetter TQFT.

Prominent in this paper is the concept of a fake-flat 2-gauge configuration in a 2-lattice, to be a discretised model of a higher gauge field; as well as the construction of discrete 2-dimensional holonomy operators for surfaces cellularly embedded in a 2-lattice. (Fake-flat 2-gauge configurations are in line with the framework for higher lattice gauge theory of [59, 35] and also appear in formal homotopy quantum field theory constructions; see [60]). We carefully construct these discrete 2-dimensional holonomy operators, in an algebraic topological (§3.4) and in a combinatorial manner (§3.5), and, using algebraic topology, prove that this discrete 2-dimensional holonomy is gauge invariant and independent of the way we combine the faces of a particular CW-decomposition of the 2-sphere and of the 2-disk. These are results, of intrinsic interest. They provide a combinatorial construction of the 2-dimensional holonomy of a higher order bundle, completing its differential geometrical construction discussed for example in [5, 63, 34, 3].

In sections 2, 3 and 4 we lift the construction of ordinary lattice gauge theory to a higher setting, as outlined in [23, 59]. Let us summarise the general procedure.

A gauge configuration of ordinary lattice gauge theory with gauge group G is given by a map from the set of (by definition oriented) edges of the lattice into G. The well-definedness of lattice gauge theory can be expressed by saying that there is a lattice groupoid supporting well-defined groupoid maps (here called *discrete parallel transport functors* §3.1) to a gauge group any time a gauge configuration is given. The 'lattice groupoid' is a groupoid version of the free category over a graph (see for example [54, 41]) for a suitable graph derived from the lattice. It is the freeness that makes discrete parallel transport functors well-defined. A 'suitable graph' is (it is claimed) the 1-skeleton of a suitable CW-complex decomposition of physical space. If we aim for *topological* field theory then in principle any sufficiently regular CW-complex will do. Normally there is a notion of local structure — chunks of space that are independent of each other, which collectively encode extended structure. In this sense, the 'big story' of lattice gauge theory is that the free groupoid over a suitable lattice is an adequate model of physical space.

Our first task here is to construct a well-defined lift of these notions to the higher setting. The main tool is the concept of a lattice 2-groupoid (to be a model of space in lattice higher gauge theory), which in this paper is constructed in an algebraic topological language as the fundamental crossed module $\Pi_2(M^2, M^1, M^0)$ (see [11] and [18, Chapter 6]) of a certain filtered space associated to a 2-lattice decomposition L of the manifold M; see §3.3. We will make a very strong use of a freeness result for the lattice 2-groupoid, which essentially is a classical freeness theorem of Whitehead [72, 10], transported to the groupoid setting by Brown and Higgins; see [18, 6.8] and [13, 14, 15]. Whitehead's theorem provides also an equivalent combinatorial definition of the lattice 2-groupoid of (M, L), i.e. of a pair consisting of a manifold with a 2-lattice decomposition.

Let M be a manifold with a 2-lattice decomposition L. Given a crossed module $\mathcal{G} = (\partial\colon E \to G, \triangleright)$, representing the underlying gauge symmetry 2-group, a 2-gauge configuration is defined as a map assigning an element of the group E to each (pointed and oriented) face of L and an element of G to each (oriented) edge of L. Physically relevant configurations furthermore satisfy a certain compatibility condition — called *fake-flatness*. This is a discretised version of the well-established fake-flatness condition for differential geometrical 2-connections; see [3, 5, 63, 34]. The term fake-flatness was seemingly first used in [9].

In analogy to lattice gauge theory, we prove that any fake-flat 2-gauge configuration $\mathcal{F}$ extends uniquely to a crossed module map (called a *discrete parallel transport 2-functor*) from the lattice 2-groupoid of (M, L) into the underlying gauge 2-group $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ (in the form of a crossed module); see §3.3. These discrete parallel transport 2-functors are a discrete version of the differential-geometrical parallel transport 2-functors of [5, 63, 34]. Given an oriented 2-disk or 2-sphere Σ embedded in M, as a subcomplex, and a vertex (to be a base-point) v of Σ, we can then combine the 1-dimensional and 2-dimensional holonomies of the constituting pieces of Σ, and obtain an E-valued 2-dimensional holonomy $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)$ of the fake-flat 2-gauge configuration $\mathcal{F}$ along Σ. These are the 2-dimensional holonomy operators previously referred to. By using some basic algebraic topology, and the fact that the oriented mapping class groups of the 2-sphere and of the 2-disk both are trivial, we can then provide algebraic-topological and combinatorial descriptions of $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)$, and also show that the discrete 2-dimensional holonomy $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)$ of $\mathcal{F}$ along Σ depends only on the base-point v (in a way controlled by the action of G on E) and on the surface orientation, and not on any other data such as the order of multiplication of constituent 2-cells. This latter result does not apply (in this form) to other surfaces since the mapping class group is then more complicated: in general an isotopy class of embeddings is needed to define the 2-dimensional holonomy of a 2-gauge connection along an embedded surface. For discussion see [34, 65].

Playing a prominent role in the construction of our model, we introduce gauge transformations between fake-flat 2-gauge configurations. Gauge transformations initially come in two different types: vertex and edge types. These correspond to the thin and fat gauge transformation of [33]. Vertex and edge gauge transformations obey a semi-direct product structure, and can be assembled into a group of gauge operators, which acts on the set of fake-flat 2-gauge configurations. This action is explicitly constructed using a double category derived from the crossed module $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ [15, 34], and ultimately originates from a groupoid of fake-flat 2-gauge configurations and 'full gauge transformations between them', which we will carefully construct in §4.3. We prove in §4.3.2 that gauge transformations preserve the 2-dimensional holonomy of fake-flat 2-gauge configurations along cellularly embedded 2-spheres in M, up to acting by an element of G.

As mentioned in the previous paragraph, a major underpinning construction is that of a groupoid of fake-flat 2-gauge configurations and full gauge transformation between them §4.3. The latter groupoid can be seen as a combinatorial description of a certain groupoid of crossed complex (a generalisation of crossed modules) maps and their homotopies, which appeared in the work of Brown and Higgins on tensor products and homotopies of crossed complexes; see [16, 11]. This point of view will be essential when we discuss the ground state degeneracy of the higher Kitaev model in §5.2.

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a crossed module, representing the underlying gauge symmetry 2-group. Let L be a 2-lattice decomposition of M. A fake-flat 2-gauge configuration $\mathcal{F}$ in (M, L) is said to be 2-flat along a cellularly embedded 2-sphere Σ if the 2-dimensional holonomy $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)$ of $\mathcal{F}$ along Σ is the identity element of E. This 2-flatness of $\mathcal{F}$ along a 2-sphere $\Sigma \subset M$ is preserved by gauge transformations. A fake-flat 2-gauge configuration $\mathcal{F}$ in (M, L) is said to be *2-flat* if it is 2-flat along the boundaries of all 3-cells of L.

A crucial fact that we will use in this paper is the following one, a consequence of the work of Brown and Higgins [14, 15, 16, 17]; a more recent reference by Brown, Higgins and Siviera is [18, Chapters 9 and 11]. A 2-flat 2-gauge configuration $\mathcal{F}$ naturally yields a map $f_{\mathcal{F}}\colon M \to B_{\mathcal{G}}$, defined up to homotopy, from

M into the classifying space $B_{\mathcal{G}}$ of the crossed module $\mathcal{G}$; classifying spaces of crossed modules are defined in [18, 17, 11] and also [35, 30]. Moreover, by [17, THEOREM A] and [18, Theorem 11.4.19], it follows that given two 2-flat 2-gauge configurations $\mathcal{F}$ and $\mathcal{F}'$, then $f_{\mathcal{F}}, f_{\mathcal{F}'} : M \to B_{\mathcal{G}}$ are homotopic if, and only if, the 2-flat 2-gauge configurations $\mathcal{F}$ and $\mathcal{F}'$ are connected by a full gauge transformation. These facts will play a primary role in the proof that the ground state degeneracy of our model is a topological invariant of manifolds M, counting the number of homotopy classes of maps from M into $B_{\mathcal{G}}$; see §5.2.

Overview of the paper

In Section 2, we recap and fix conventions for: crossed modules, fundamental crossed modules, CW-complexes and 2-lattices, defined in §2.4. In section 3, we firstly define and discuss fake-flat 2-gauge configurations (called "cellular formal $\mathcal{C}$-maps" in [60]); see §3.2. In §3.3, we define the lattice 2-groupoid for a pair (M, L), consisting of a manifold M with a 2-lattice decomposition L, and show how fake-flat 2-gauge configurations give rise to 2-dimensional discrete parallel transport 2-functors, from the lattice 2-groupoid of (M, L) into the gauge crossed module $\mathcal{G}$. In §3.4 we give an algebraic topological definition of the 2-dimensional holonomy of a fake-flat 2-gauge configuration along a 2-sphere and along a 2-disk. In §3.5, we give a combinatorial definition of 2-dimensional holonomy along 2-disks and 2-spheres, and prove that the two definitions of 2-dimensional holonomy coincide.

In Section 4 we discuss gauge transformations between fake-flat 2-gauge configurations defined on a 2-lattice. In particular we define a group of gauge operators and prove that it acts on the set of fake-flat 2-gauge configurations in a way such that the 2-dimensional holonomy along cellularly embedded 2-spheres is preserved. The underpinning groupoid of fake-flat 2-gauge configurations and full gauge transformations between them is constructed in §4.3.

In Section 5, we address a Hamiltonian model for higher lattice gauge theory on a pair (M, L), consisting of a manifold M with 2-lattice decomposition L. This will be our proposal for a higher gauge theory version of Kitaev quantum-double model for topological phases: the *higher Kitaev model*. The underlying Hilbert space of our model is the free vector space on the set of all fake-flat 2-gauge configurations, and hence coincides with the Hilbert space in [23] for triangulated manifolds. In §5.1 we explicitly construct the higher Kitaev model, and give detailed description of all operators involved. In §5.1.3 we define the local operator algebra. The Hamiltonian §5.1.4 for the higher Kitaev model is a sum of three mutually commuting terms. We have two sums over 1-cells and 2-cells, respectively, constructed by using the action of the group of gauge operators, which impose higher gauge invariance along gauge transformations of vertex and edge types; and one sum over 3-cells, imposing 2-flatness along their boundary 2-sphere. A comparison with the Kitaev model is done in §5.1.6.

In §5.2 we show that the dimension of the ground state of the higher Kitaev model is given by the number of homotopy classes of maps from the space manifold M to the classifying space of the gauge crossed module $\mathcal{G}$ and therefore ground state degeneracy is a topological (in fact homotopy) invariant of M. (At this point we needed again to appeal to some basic algebraic topology for crossed modules and crossed complexes as given in [18, 30, 35].) This ground state degeneracy can be proven to coincide with Yetter's invariant on $M \times S^1$ (the level D invariant of the TQFT); see [23] for a proof of this fact.

Contents

2 Preliminaries on crossed modules, CW-complexes and 2-lattices

As mentioned in the Introduction, in higher gauge theory, instead of gauge symmetry groups we have gauge symmetry 2-groups [3, 5], which in this paper will always be strict. It is well-known that the category of strict 2-groups is equivalent to the category of crossed modules of groups; see e.g. [18, §2.5] and [4, 7, 22]. Throughout the body of this paper, only crossed modules will be mentioned. Using the language of crossed modules facilitates the use of homotopy theoretical techniques, as developed in [11, 13, 16, 18, 17], of which we will take advantage here. Higher gauge theory formalised by using crossed modules appears in [34].

In §3.2 we give the key definition of a fake-flat 2-gauge configuration on a 2-lattice decomposition of a manifold. This makes extensive use of crossed modules; we assemble some fundamental definitions in §2.1 and §2.2. Then in §2.3 we recall some facts about CW-complexes which we will need in §2.4 for defining 2-lattice decompositions of manifolds.

Remark 1. In this paper we will use $\mathrm{bd}(M)$ to denote the boundary of a manifold M. (We avoid the common notation ∂M, in order not to overuse the symbol ∂, which appears in several other contexts.)

2.1 Crossed modules (of groups and of groupoids)

Crossed modules of groups are discussed in [4, 11, 30]. Crossed modules of groupoids, discussed extensively in this paper, appear in [18, §6.2] and [11, 35, 19]. Crossed modules of groups and groupoids can be used for formalising 2-dimensional (2D) notions of holonomy (surface holonomy), in the same way that groups appear in the formulation of the holonomy of a connection in a principal bundle.

Definition 2 (Crossed modules of groups; Peiffer relations)**.** Let E and G be groups. A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ of groups is given by a group map $\partial\colon E \to G$, together with a left action $\triangleright$ of G on E by automorphisms, such that the relations below, called *Peiffer relations*, hold for each $g \in G$ and $e, e' \in E$:

$$\text{1st Peiffer relation} \qquad \partial(g \triangleright e) = g\partial(e)g^{-1}, \tag{1}$$

$$\text{2nd Peiffer relation} \qquad \partial(e) \triangleright e' = ee'e^{-1}. \tag{2}$$

Example 3. The crossed module $\mathcal{G} = \mathcal{G}_{32} := (\partial : \mathbb{Z}_3^+ \to \mathbb{Z}_2^\times, \triangleright)$, where $\mathbb{Z}_3^+ = \{0, 1, 2\}$ is the additive group of integers modulo 3 and $\mathbb{Z}_2^\times = \{\pm 1\}$ acts on $\mathbb{Z}_3$ as $z \triangleright e = ze$. The boundary map ∂ sends everything to $+1$.

Example 4 (From groups to crossed modules I)**.** Given a group G, let $\mathrm{Aut}(G)$ be the group of automorphisms of G. Clearly $\mathrm{Aut}(G)$ acts in G by automorphisms as $f \triangleright g = f(g)$, for each $f \in \mathrm{Aut}(G)$ and each $g \in G$. Let $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$ be the morphism that sends $g \in G$ to the inner automorphism $\mathrm{Ad}_g\colon x \in G \mapsto gxg^{-1} \in G$, obtained by conjugating by g. Then $\mathcal{AUT}(G) = (\mathrm{Ad}\colon G \to \mathrm{Aut}(G), \triangleright)$ is a crossed module.

Example 5 (From groups to crossed modules II)**.** If G is a group, then $(\{1\} \to G)$ and $(\mathrm{id}\colon G \to G, \mathrm{Ad})$, where Ad is the adjoint action, are crossed modules. If G is abelian then $(G \to \{1\})$ is also a crossed module.

Let us now discuss crossed modules of groupoids. Let $G = (\sigma, \tau\colon G_1 \to G_0)$ denote a groupoid [41, 11] with set of objects G_0; set of morphisms G_1; and source and target maps $\sigma, \tau\colon G_1 \to G_0$. We represent the morphisms $\gamma \in G_1$ as $a \xrightarrow{\gamma} b$. Thus $\sigma(\gamma) = a$ and $b = \tau(\gamma)$. Given $a, b \in G_0$, the set of morphisms $a \to b$ is $\hom(a, b) = \{\gamma \in G_1 \colon \sigma(\gamma) = a \text{ and } \tau(\gamma) = b\}$. The composition map in G yields for each triple (a, b, c) of objects a map $\circ\colon \hom(a, b) \times \hom(b, c) \to \hom(a, c)$, which we represent as (notice composition order):

$$\big(a \xrightarrow{\gamma} b\big) \circ \big(b \xrightarrow{\gamma'} c\big) = \big(a \xrightarrow{\gamma\gamma'} c\big).$$

A *totally intransitive groupoid* E is a groupoid of the form $E = (\beta, \beta\colon E_1 \to E_0)$. (Thus source and target maps coincide.) Given $x \in E_0$, we let $\mathrm{Aut}(x) = \{e \in E_1 : \beta(e) = x\}$, which is a group. And then E is isomorphic to the totally intransitive groupoid given by $\sqcup_{x \in E_0} \mathrm{Aut}(x)$, with the obvious composition and map $\beta\colon \sqcup_{x \in E_0} \mathrm{Aut}(x) \to E_0$. Hence a totally intransitive groupoid can be seen as a disjoint union of groups.

A *left groupoid action* $\triangleright$ [11, 19], by automorphisms, of the groupoid $G = (\sigma, \tau\colon G_1 \to C)$ on $E = (\beta, \beta\colon E_1 \to C)$, a totally intransitive groupoid with the same set of objects as G, is given by a set map:

$$(\gamma, e) \in \{(\gamma', e') \in G_1 \times E_1 : \tau(\gamma') = \beta(e')\} \longmapsto \gamma \triangleright e \in E_1,$$

such that whenever compositions and actions are well-defined:

$$\beta(\gamma \triangleright e) = \sigma(\gamma), \qquad (\gamma\gamma') \triangleright e = \gamma \triangleright (\gamma' \triangleright e) \qquad \text{and} \qquad \gamma \triangleright (ee') = (\gamma \triangleright e)(\gamma \triangleright e').$$

Definition 6 (Crossed module of groupoids)**.** Let E and G be groupoids with the same object set, with E totally intransitive. A *crossed module of groupoids* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by a groupoid map $\partial\colon E \to G$, which is the identity on objects, together with a left action of G on E, by automorphisms, such that the Peiffer relations (1,2) are satisfied, whenever actions and compositions make sense. (Full equations are in [19].)

Given $\mathcal{G} = (\partial\colon E \to G, \triangleright)$, we call E the *top groupoid* of $\mathcal{G}$ and G the *underlying groupoid* of $\mathcal{G}$.

Definition 7 (Crossed module map)**.** A map $(\psi, \phi)\colon (\partial\colon E \to G, \triangleright) \to (\partial\colon E' \to G', \triangleright)$ of crossed modules of groupoids is given by two groupoid maps $\psi\colon E \to E'$ and $\phi\colon G \to G'$, which are compatible with actions and boundary maps in the obvious way. (Full equations are in [35, §1.1.1].)

Since groups can be considered to be groupoids with a single object, we will see group crossed modules as particular cases of crossed modules of groupoids.

2.2 Example: the fundamental crossed module of a triple of spaces

The main example of a crossed module of groupoids is a topological one crucial to our construction. Our main references are [18, §2.1, §2.2 and §6] and [11]. We will need to review some algebraic topology definitions.

Definition 8. (See e.g. [26, p.17] and [18, 11].) Let Y be a locally path-connected space, and $C \subset Y$ any subset (in this paper C will always be finite). The *fundamental groupoid of Y, with object set C*, denoted $\pi_1(Y, C)$, is as follows. The set of objects of $\pi_1(Y, C)$ is C. Given $c, d \in C$, the set of morphism $\hom(c, d)$ is the set of equivalence classes of paths $\gamma\colon [0, 1] \to Y$, such that $\gamma(0) = c$ and $\gamma(1) = d$, where two paths $c \to d$ are equivalent if they are homotopic in Y, relative to the end-points (i.e. end-points remain stable during the homotopy). The composition in $\pi_1(Y, C)$ is given by concatenation (and rescaling) of representative paths.

If γ is a path in Y, the equivalence class to which it belongs in $\pi_1(Y, C)$ is denoted by $[\gamma]$. A morphism in $\pi_1(Y, C)$ from c to d is denoted as $c \xrightarrow{[\gamma]} d$ or simply by $c \xrightarrow{\gamma} d$ if no ambiguity arises.

Remark 9. Let $c \in C$. The group of morphisms $c \to c$ in the groupoid $\pi_1(Y, C)$ is exactly the fundamental group $\pi_1(Y, c)$. Let $S^1 = \mathrm{bd}([0, 1]^2)$, with a base point $*$ at $(0, 0)$; recall Rem. 1. Morphisms $c \to c$ hence can equivalently be seen as pointed homotopy classes of maps $(S^1, *) \to (Y, c)$.

Relative homotopy groups, including $\pi_2(X, Y, c)$, of pointed pairs of spaces (c being the base-point) are classical in homotopy theory and are defined e.g. in [40, p.343]. In this paper, we will use relative homotopy groupoids $\pi_2(X, Y, C)$, with a set $C \subset Y$ of base-points; see [18, §1.6, §6.2 and §6.3]. These are totally intransitive groupoids built as $\pi_2(X, Y, C) = \sqcup_{c \in C} \pi_2(X, Y, c)$. Let us give a quick review.

Definition 10 (The totally intransitive groupoid $\pi_2(X, Y, C)$)**.** Let X be a locally path-connected space. Let $Y \subset X$ be a locally path-connected subspace of X. Choose a subset C of Y. In this paper, C will always intersect non-trivially each path-component of X and of Y. For each $c \in C$, consider the *relative homotopy group* $\pi_2(X, Y, c)$. This group is made out of homotopy classes of maps $\Gamma\colon [0, 1]^2 \to X$ such that:

1. $\Gamma\big(([0, 1] \times \{0\}) \cup (\{0, 1\} \times [0, 1])\big) = \{c\}$,
2. $\Gamma\left([0, 1] \times \{1\}\right) \subset Y$.

Specifically, two such maps $\Gamma, \Gamma'\colon [0, 1]^2 \to X$ are said to be homotopic if there exists a homotopy $J\colon [0, 1]^3 \to X$, connecting Γ and Γ', such that for all $u \in [0, 1]$ the slice of J at u, namely $(t, s) \mapsto J_u(t, s) = J(t, s, u)$, satisfies the properties 1 and 2. The multiplication in $\pi_2(X, Y, c)$ is through horizontal juxtaposition of maps $[0, 1]^2 \to X$, followed by rescaling in the horizontal direction.

We can thus define a totally intransitive groupoid $\pi_2(X, Y, C) \doteq \bigsqcup_{c \in C} \pi_2(X, Y, c)$, with set of objects C.

Let (X, Y, C) be as in Def. 10. The elements $[\Gamma] \in \pi_2(X, Y, c)$, or simply $\Gamma \in \pi_2(X, Y, c)$, if no confusion arises, are visualised as:

$$\Gamma = \begin{array}{ccc} c & \xrightarrow{\partial(\Gamma)} & c \\ c \uparrow & \Gamma & \uparrow c \\ c & \xrightarrow{\;c\;} & c \end{array} .$$

Let $c \in C$. As indicated by the diagram above, if we restrict a $\Gamma \in \pi_2(X, Y, c)$ to the top of the square $[0, 1]^2$, this gives rise to an element $\partial(\Gamma) \in \pi_1(Y, c)$. This yields a group map $\partial\colon \pi_2(X, Y, c) \to \pi_1(Y, c)$. Putting all of these group maps together, yields a groupoid map $\partial\colon \pi_2(X, Y, C) \to \pi_1(Y, C)$, which is the identity on objects. We also have an action of the groupoid $\pi_1(Y, C)$ on the totally intransitive groupoid $\pi_2(X, Y, C)$, as indicated in figure 1. Details are in [18, §2.2 and §6.1] and (in the pointed case) [40, pp 355].

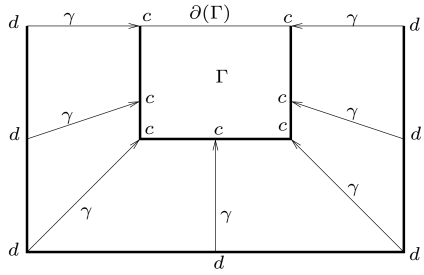


Figure 1: The action of an element $\gamma \in \pi_1(Y, C)$, with $\gamma(0) = d$ and $\gamma(1) = c$ on a $\Gamma \in \pi_2(X, Y, c)$.

Theorem 11. *(JHC Whitehead, [18, §2.2, §6] and [11, 13]) Let (X, Y, C) be a triple of spaces, as in Def. 10. Considering the natural action $\triangleright$ of the groupoid $\pi_1(Y, C)$ on the totally intransitive groupoid $\pi_2(X, Y, C)$, and the boundary map $\partial\colon \pi_2(X, Y, C) \to \pi_1(Y, C)$, we have a crossed module of groupoids, called the* fundamental crossed module of (X, Y, C). *The fundamental crossed module of (X, Y, C) is denoted as:*

$$\Pi_2(X, Y, C) \;=\; (\partial\colon \pi_2(X, Y, C) \to \pi_1(Y, C), \triangleright).$$

Remark 12. Let (X, Y) be a pair of spaces and $c \in Y$. Recall that the underlying set of the group $\pi_2(X, Y, c)$ can also be defined as the set of all maps $f\colon [0,1]^2 \to X$, such that $f(*) = c$, where $* = (0, 0)$, and $f(\mathrm{bd}[0,1]^2) \subset Y$, up to a homotopy $H\colon (x, t) \in [0,1]^2 \times [0,1] \mapsto f_t(x) \in X$, such that, for each t, $f_t(*) = c$ and $f_t(\mathrm{bd}[0,1]^2) \subset Y$. The boundary map $\partial\colon \pi_2(X, Y, x) \to \pi_1(Y, c)$ is obtained by restricting f to $\mathrm{bd}([0,1]^2)$; see Rem. 9.

Analogously [40, Chapter IV], the underlying set of the relative homotopy group $\pi_3(X, Y, c)$ can be defined as the set of all maps $f\colon [0,1]^3 \to X$ such that $f(*) = c$, where $* = (0, 0, 0)$, and $f(\mathrm{bd}[0,1]^3) \subset Y$, up to a homotopy $H\colon (x, t) \in [0,1]^3 \times [0,1] \mapsto f_t(x) \in X$ such that, for each t, $f_t(*) = c$ and $f_t(\mathrm{bd}[0,1]^3) \subset Y$. We also have a boundary map $\partial\colon \pi_3(X, Y, c) \to \pi_2(Y, c)$ obtained by restricting f to $\mathrm{bd}([0,1]^3)$.

Example 13 (The fundamental crossed module of the disk)**.** Let $D^2 = [0,1]^2$ and $S^1 = \mathrm{bd}(D^2)$. Let $v \in S^1$ be any point. Then $\Pi_2(D^2, S^1, v) \cong (\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}, \triangleright)$, where $a \triangleright b = b$, for each $a, b \in \mathbb{Z}$. To see this, look at the end of the homotopy long exact sequence of (D^2, S^1, v); see e.g. [40, Chapter IV]. This yields $\{0\} \cong \pi_2(D^2, v) \to \pi_2(D^2, S^1, v) \xrightarrow{\partial} \pi_1(S^1, v) \cong \mathbb{Z} \to \pi_1(D^2, v) \cong \{1\}$. Details are in e.g. [30].

2.3 CW-complexes

Let D^n denote the closed n-disk in the form $D^n = [0,1]^n$. The open n-disk is $\mathrm{int}(D^n) = (0,1)^n$. Also put:

$$\mathrm{bd}(D^n) = S^{n-1} = D^n \setminus \mathrm{int}(D^n)$$

— the boundary of the n-disk. Let $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let us briefly review the definition of CW-complexes [40, Appendix], [37] and [53]. We will use the definition given in [40, Prop A2] and [53, Chapter II].

Definition 14 (CW-complex)**.** A CW-complex $(X, \{\phi^n_a\}_{a\in L^n, n\in\mathbb{N}})$ is a Hausdorff topological space X, a collection of sets L^0, L^1, L^2, ..., and, for each $n \in \mathbb{N}$, a family of continuous maps $\{\phi^n_a\colon D^n \to X\}_{a\in L^n}$ (the 'characteristic maps of the closed n-cells') satisfying conditions 1,2,3 and 4, below.

Let the set $c^n_a \doteq \phi^n_a(\mathrm{int}(D^n)) \subset X$. It is called an *open cell of dimension* n, and is given the induced topology. Put $\overline{c^n_a} \doteq \phi^n_a(D^n) \subset X$. It is called a *closed cell of dimension* n, and is given the induced topology. Put $\mathrm{bd}(\overline{c^n_a}) \doteq \phi^n_a(\mathrm{bd}(D^n)) \subset X$. It is called the boundary of c^n_a. (Note that $\overline{c^n_a}$ need not be a ∂-manifold, hence $\mathrm{bd}(\overline{c^n_a})$ might not be a manifold boundary, though this will be imposed when we define 2-lattices.) Then:

1. Each characteristic map $\phi^n_a\colon D^n \to X$ restricts to a homeomorphism $\mathrm{int}(D^n) \to \phi^n_a(\mathrm{int}(D^n)) \subset X$.

2. The open cells c^n_a where $n \in \mathbb{N}$ and $a \in L^n$, form a partition of X. (I.e. they are pairwise disjoint and their union is X.)

3. Each $\mathrm{bd}(\overline{c^n_a})$ is contained in the union of a finite number of open cells of dimension $< n$.

4. A set $F \subset X$ is closed if, and only if, $(\phi_a^n)^{-1}(F)$ is closed in D^n, for each $n \in \mathbb{N}$ and each $a \in L^n$.

A CW-complex is called finite if L^n is finite for each $n \in \mathbb{N}$ and $L^n = \emptyset$ for all but a finite subset of $n \in \mathbb{N}$. We write X for $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$. The data $\{\phi_a^n \colon D^n \to X\}_{a \in L^n, n \in \mathbb{N}}$ is called a CW-decomposition of X.

Definition 15 (Subcomplex)**.** (See [40, p. 520]) A *subcomplex* of a CW-complex $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ is a subspace $A \subset X$ which is the union of open cells of X, such that the closure in X of each of these open cells is contained in A.

A subcomplex A can be made into a CW-complex $\big(A, \{\phi_b^n\}_{b \in L_A^n, n \in \mathbb{N}}\big)$, where for each $n \in \mathbb{N}$, we put $L_A^n = \{c \in L^n : \overline{c_a^n} \subset A\}$. (For a proof see e.g. [40, p. 520].)

Definition 16 (CW-pair. CW-decomposition of pairs)**.** A *CW-pair* [40, p. 7] is a pair of spaces (X, Y) where X has a CW-decomposition and Y is the underlying space of a subcomplex of X. Such CW-decompositions of X will be called *CW-decompositions of* (X, Y).

Definition 17. The *n-skeleton* X^n of a CW-complex X is the subspace given by the union of all the open cells of dimensions $\leq n$, with the induced topology. Note that X^n is a subcomplex of X, hence a CW-complex.

Remark 18 (CW-complexes: properties and nomenclature)**.** For proofs see e.g. [40, Appendix] and [53, 37].

- Condition 4. of the definition of a CW-complex is redundant if X has only a finite number of cells; see [40, p. 521]. (Essentially this follows since a finite union of closed sets is always closed). In this paper we will only deal with finite CW-complexes, so condition 4. of Def. 14 will not be mentioned again.
- Cf. [53, p. 6], as the notation suggests, the closed cell $\overline{c_a^n} \subset X$ is the closure in X of the open cell c_a^n.
- The *attaching map* of each closed n-cell $\overline{c_a^n}$ is the restriction of $\phi_a^n \colon D^n \to X$ to $\mathrm{bd}(D^n)$, namely:

$$\psi_a^n \colon \mathrm{bd}(D^n) \to \mathrm{bd}(\overline{c_a^n}) \subset X^{n-1} \subset X.$$

  The underlying topological space of the n-skeleton X^n of X is homeomorphic to the space obtained from X^{n-1} by attaching $\sqcup_{a \in L^n} D^n$ to it, along the attaching maps of the closed n-cells.

Definition 19 (Cellular map)**.** Given CW-complexes X and Y, a map $f \colon X \to Y$ is called cellular if $f(X^n) \subset Y^n$, for all $n \in \mathbb{N}$.

Definition 20 (Abstract cells)**.** If $(X, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ is a CW-complex, we call L^n the set of abstract n-cells.

Abstract n-cells are in one-to-one correspondence with open n-cells and with closed n-cells. If a is an abstract n-cell, the closed and open n-cells it corresponds to are (respectively) $\overline{c_a^n} = \phi_a^n(D^n)$ and $c_a^n = \phi_a^n(\mathrm{int}(D^n))$.

Remark 21 ((Geometric) vertices, edges, plaquettes (or faces), and blobs)**.** Abstract 0, 1, 2 and 3-cells of a CW-complex will sometimes be called called vertices, edges, plaquettes (or faces), and blobs, respectively. Closed 0, 1, 2 and 3-cells will sometimes be called geometric vertices, geometric edges, geometric plaquettes (or faces), and geometric blobs.

2.4 2-lattices

Simplicial complexes give rise to CW-complexes; but simplicial complexes are very rigid, therefore a large number of simplices is usually required to triangulate a manifold. CW-complexes allow for the decomposition of a manifold into fewer cells; however they are too general for our purposes, since the attaching maps of the closed cells can be highly singular, making it harder to use CW-complexes in combinatorial frameworks. In order to simplify our discussion later, we will consider CW-complexes which are 2-lattices, defined below.

If $S^n = \mathrm{bd}(D^{n+1})$ is the n-sphere, the base-point $*$ of it is defined to be $* = (0, \ldots, 0)$.

Definition 22 (2-lattices. Base point of a cell)**.** Let M be a topological manifold, with a CW-complex decomposition $\Delta_M = (M, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$. This Δ_M is called a 2-lattice for M if, for each $n \in \mathbb{N}$ and each n-cell $a \in L^n$:
(1) A CW-decomposition Z_a of $S^{n-1} = \mathrm{bd}(D^n)$ is given for which the base-point $* = (0, \dots, 0)$ is a 0-cell, and such that the attaching map $\psi_a^n \colon S^{n-1} \to M^{n-1}$ of the corresponding closed n-cell $\overline{c_a^n}$ is cellular. (Note that in particular (1) implies that $\psi_a^n(*) = x_a$ is a closed 0-cell of M, for each $a \in L^n$ and each $n \in \mathbb{N}$. The image $\psi_a^n(*) = x_a$ is called the *base-point* of the closed n-cell $\overline{c_a^n}$.)
(2) One of the following two conditions holds:

- The attaching map $\psi_a^n \colon S^{n-1} \to M^{n-1}$ of the corresponding n-cell $\overline{c_a^n}$ is constant.
- The attaching map $\psi_a^n \colon S^{n-1} \to M^{n-1}$ of the corresponding n-cell $\overline{c_a^n}$ is an embedding (i.e. it is a homeomorphism onto its image). Moreover, for each closed i-cell c of Z_a, it holds that $\psi_a^n(c)$ is a closed i-cell c_L of M, and the restriction of $\psi_a^n \colon S^{n-1} \to M^{n-1}$ to c is a homeomorphism $c \to c_L$.

(3) If $b \in L^3$, we impose that the attaching map $\psi_b^3 \colon S^2 \to M^2$ of the closed 3-cell $\overline{c_b^3}$ is an embedding and furthermore that the boundary $\psi_b^3(S^2) = \mathrm{bd}(\overline{c_b^3})$ of the 3-cell $\overline{c_b^3}$ is a subcomplex of M^2.

The space M is then said to have a 2-lattice decomposition.
A 2-lattice $(M, \{\phi_a^n\}_{a \in L^n, n \in \mathbb{N}})$ will usually be denoted as (M, L), or $\big(M, L = (L^0, L^1, \dots)\big)$.

Remark 23. In practice, when defining a particular 2-lattice decomposition of M, normally only the closed n-cells will be made explicit, as it will always be clear that, for each n-cell a, an attaching map $\psi_a^n \colon S^{n-1} \to M^{n-1}$ can be found which is cellular by using a suitable CW-decomposition of the $(n-1)$-sphere. This does not fully determine a CW-decomposition, as some ambiguity rests on the actual characteristic maps of the n-cells. However the topological space M, all closed cells, all i-skeletons M^i, and hence the crossed modules $\Pi_2(M, M^1, M^0)$ and $\Pi_2(M^2, M^1, M^0)$ will be defined with no ambiguity. This is all we need for this paper.

Remark 24 (Lax 2-lattice)**.** A CW-complex satisfying only (1) of the definition of 2-lattices is called a lax-2-lattice. All combinatorial constructions in this paper are still true for lax-2-lattices, with the obvious modifications. In particular, the combinatorial construction of the 2-dimensional holonomy operators in §3.5 remains almost unaltered. The only issue is that the description of edge and vertex gauge spikes in §5.1.1 then requires a lot more cases, especially when it comes to edge operators.

Remark 25. Let Y be a subcomplex of a CW-complex X. If X is a 2-lattice then clearly so is Y.

Example 26. Evidently, 1-dimensional CW-complexes are always 2-lattices. The circle $\{z \in \mathbb{C} \colon |z| = 1\}$ can be given a 2-lattice decomposition with two vertices (i.e. 0-cells) at $z = \pm 1$ and closed 1-cells at $\{z \in \mathbb{C} \colon |z| = 1 \wedge \Im(z) \geq 0\}$ and $\{z \in \mathbb{C} \colon |z| = 1 \wedge \Im(z) \leq 0\}$. Here $\Im(z)$ denotes the imaginary part of z.

Example 27. An example of a CW-complex which cannot be made into a 2-lattice is given by attaching $D^2 = [0,1]^2$ to $\{z \in \mathbb{C} \colon |z| = 1\}$ along $(x, y) \in \mathrm{bd}(D^2) \mapsto \exp(x 2\pi i \sin(2\pi/x))$, prolonged by continuity to $\mathrm{bd}(D^2) \cap \{x = 0\}$. This is the type of singular attaching maps we want to avoid by restricting to 2-lattices.

Example 28. Consider the 2-sphere $S^2 = \mathrm{bd}(D^3)$, with the CW-decomposition arising from the polyhedral structure of $D^3 = [0,1]^3$. Let Y be the space obtained from S^2 by attaching D^3 along $\psi \colon S^2 \to S^2$ defined as:

$$(x, y, z) \in \mathrm{bd}(D^3) \stackrel{\psi}{\longmapsto} \begin{cases} (x, y, z) \text{ if } z \geq 1/2 \\ (x, y, 1 - z) \text{ if } z \leq 1/2 \end{cases}$$

This CW-decomposition of Y is not a 2-lattice since the attaching map of its unique 3-cell is not an embedding.

Example 29 (Two 2-lattice decompositions of the 3-sphere S^3)**.** Let us in this example model the 2- and 3-spheres as being $S^2 = \{x \in \mathbb{R}^3 \colon |x| = 1\}$ and $S^3 = \{x \in \mathbb{R}^4 \colon |x| = 1\}$. The following are two 2-lattice decompositions of the 3-sphere.
$(S^3, L_{\mathfrak{g}})$**:** We consider the 3-sphere S^3 with the globe decomposition $L_{\mathfrak{g}} = (\{v\}, \{t\}, \{P, P'\}, \{b, b'\})$ as follows. We firstly consider a CW-decomposition L of S^2 with a unique closed 0-cell v at the point of zero

latitude and longitude, and a unique closed 1-cell t making the equator, oriented eastwards. We have two closed 2-cells, P, P', one for each hemisphere, attaching along the equator (oriented eastwards). See Fig. 2.

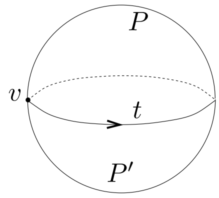


Figure 2: A 2-lattice decomposition L of the 2-sphere S^2.

To get from S^2 to S^3 we now need to add two additional 3-cells b, b' attaching on each side of the 2-sphere.

(S^3, L_0): We can choose an even simpler 2-lattice decomposition L_0 of S^3, having unique 0- and 2-cells (resulting in S^2), and two 3-cells b and b', as above, attaching along each side of the 2-sphere.

Definition 30 (Notation: $\partial_L(P)$ and $\partial_L(b)$)**.** Let (M, L) be a 2-lattice. Let $P \in L^2$ be a geometric 2-cell (i.e. plaquette; Def. 21). Let $\psi_P^2\colon \mathrm{bd}(D^2) \to M^1$ be the attaching map of the corresponding closed 2-cell $\overline{c_P^2}$ (i.e. geometric plaquette). By definition, $* = (0,0) \in S^1$ and $\psi_P^2(*)$ is a closed 0-cell x_P of M. Hence ψ_P^2 is a pointed map $(S^1, *) \to (M^1, x_P)$. Passing to the pointed homotopy class of $\psi_P^2\colon (S^1, *) \to (M^1, x_P)$ yields an element $\partial_L(P) \in \pi_1(M^1, x_P) \subset \pi_1(M^1, M^0)$; cf. Rem. 9. Here $\subset$ means inclusion of a groupoid into another.

Analogously, if $b \in L^3$ is a blob (i.e. an abstract 3-cell), then the attaching map $\psi_b^3\colon S^2 = \mathrm{bd}([0,1]^3) \to M^2$ of the corresponding closed 3-cell $\overline{c_b^3}$ (geometric blob) sends the base-point $* = (0,0,0)$ of S^2 to a 0-cell y_b (the base-point of $\overline{c_b^3}$; see Def. 22). Hence the attaching map ψ_b^3 is a pointed map $\psi_b^3\colon (S^2, *) \to (M^2, y_b)$. Passing to the pointed homotopy class of $\psi_b^3\colon (S^2, *) \to (M^2, y_b)$ gives rise to an element $\partial_L(b) \in \pi_2(M^2, y_b)$.

Definition 31 (Notation: $\iota_L(P)$ and $\iota_L(b)$)**.** Cf. Rem. 9 and 12. Continuing Def. 30, let $P \in L^2$ be a plaquette. Then the characteristic map $\phi_P^2\colon D^2 \to \overline{c_P^2} \subset M^2$ of the corresponding closed 2-cell induces a pointed map $(D^2, S^1, *) \to (M^2, M^1, x_P)$. Passing to the relative homotopy class of ϕ_P^2 yields an element $\iota_L(P) \in \pi_2(M^2, M^1, x_P)$. Analogously if $b \in L^3$ is an abstract 3-cell, then the characteristic map $\phi_b^3\colon D^3 \to \overline{c_b^3} \subset M^3$ yields an element $\iota_L(b) \in \pi_3(M^3, M^2, y_b)$.

Note that given $P \in L^2$ and $b \in L^3$ it holds that:

$$\partial\big(\iota_L(b)\big) = \partial_L(b) \qquad \text{and} \qquad \partial\big(\iota_L(P)\big) = \partial_L(P). \tag{3}$$

Definition 32 (2-lattice decomposition of pairs and triples of spaces)**.** Given a pair (M, N) of topological manifolds (i.e. N is a submanifold of M), we say that a 2-lattice decomposition of M is a 2-lattice decomposition of (M, N) if N is a subcomplex of M. (Note that the CW-decomposition of N rendered from the fact that N is a subcomplex of M is always a 2-lattice decomposition; see Rem. 25.) If L is a 2-lattice decomposition of M that yields a 2-lattice decomposition of (M, N), we let L_N be the induced 2-lattice decomposition of N. CW-decompositions of a triple (X, Y, Z) of manifolds are defined analogously.

If L is a 2-lattice decomposition of (M, Σ), we say that Σ is *cellularly embedded* in (M, L).

2.5 Paths on the lattice: the lattice groupoid of the 2-lattice (M, L)

Free groupoids on graphs are discussed in [12, 41, 18]. Ref. [11] in addition addresses groupoid presentations.

Definition 33 (Directed graph; totally intransitive graph)**.** A directed graph $(V, E) = (\sigma, \tau\colon E \to V)$ is a pair of sets V and E, the sets of vertices and of edges of (V, E), together with a pair of maps $\sigma\colon E \to V$ and $\tau\colon E \to V$, called the source and target maps.

The maps identify, given an edge e, its source $\sigma(e)$ and target $\tau(e)$ (also called initial and end-points). Edges of (V, E) may be represented as $x \xrightarrow{e} y$, where $x = \sigma(e)$ and $y = \tau(e)$.

A graph map $(V, E) \to (V', E')$ is given by a pair of set maps $V \to V'$ and $E \to E'$ compatible with initial and end-point of edges.

A totally intransitive graph is a graph for which source and target maps coincide.

Definition 34. The functor U sends a groupoid $G = (\sigma, \tau \colon G_1 \to G_0)$ to its underlying graph UG — simply forget the composition in G. Its left adjoint takes a graph to the free groupoid on the graph; see [41].

A directed graph (V, E) gives rise to another graph $(V, E \sqcup E^{-1})$ obtained by adding formal reverses to the edges of (V, E). Here E^{-1} is the set of symbols $\{e^{-1} \colon e \in E\}$, and we put $\sigma(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \sigma(e)$.

Definition 35 (Free groupoid on a graph. Granular path on a graph.)**.** Let (V, E) be a directed graph. A granular path $v \xrightarrow{\gamma} v'$ from vertex v to v' on (V, E) is a path on $(V, E \sqcup E^{-1})$, i.e. a sequence $\gamma = t_1^{\theta_1} \dots t_n^{\theta_n}$ where $t_i \in E$ and $\theta_i \in \{\pm 1\}$, such that the initial point of $t_i^{\theta_i}$ coincides with the final point of $t_{i-1}^{\theta_{i-1}}$, and also $v = \sigma(t_1^{\theta_1})$ and $v' = \tau(t_n^{\theta_n})$. Granular paths $v \to v$ include the empty path $\emptyset_v$ at v.

We define an equivalence relation on granular paths as follows. Firstly granular paths $\gamma, \gamma' \colon v \to v'$ are related if we can modify γ into γ' by deleting a subpath of the form $t^{\pm 1} t^{\mp 1}$. (Initial and end points of granular paths remain stable under this relation.) Now take the symmetric-reflexive-transitive closure of this relation. If γ is a granular path the equivalent class to which it belongs is denoted $[\gamma]$.

The *(free) groupoid* $\mathsf{FG}\langle V, E\rangle$ is the groupoid with object set V; arrows given by the set of equivalence classes of granular paths; and arrows $v \xrightarrow{[\gamma]} w$ and $w \xrightarrow{[\gamma']} u$ are composed by $\big(v \xrightarrow{[\gamma][\gamma']} u\big) = \big(v \xrightarrow{[\gamma\gamma']} u\big)$. (Note that this composition is well-defined.)

The notion of *freeness* will be important in our construction. We will use freeness in the context of crossed module of groupoids. The latter is a non-trivial construction, so to prepare for this we now take the opportunity to recall what it means to say that the groupoid is 'free' in Def. 35 above.

Definition 36. [41] A groupoid C is free over a graph G if there is a graph map $P : G \to UC$ satisfying the following property. For every groupoid B and each graph map $D : G \to UB$ there is a unique groupoid map $D' : C \to B$ so that the diagram below commutes:

$$\begin{array}{ccc} G & \xrightarrow{P} & UC \\ & \searrow^{D} & \downarrow{\scriptstyle UD'} \\ & & UB \end{array} \tag{4}$$

Straightfoward computations, analogous to the free-group construction prove that:

Lemma 37. [12, 8.2.1],[11],[41] *The groupoid* $\mathsf{FG}\langle V, E\rangle$ *is free over* (V, E). □

Let (M, L) be a 2-lattice (more generally a CW-complex). Recall that M^i is the i-skeleton of (M, L). Note that the characteristic maps $\phi_t^1 \colon [0, 1] \to M^1$ of the 1-cells give (L^0, L^1) the structure of a directed graph. Given $t \in L^1$ put $\sigma(t) = \phi_t^1(0)$ and $\tau(t) = \phi_t^1(1)$, where we identified M^0 and L^0.

Definition 38 (Granular path in a 2-lattice)**.** A granular path on a 2-lattice (M, L) is a granular path on the graph (L^0, L^1); Def. 35. Hence granular paths γ in (M, L) are obtained by formally chaining together closed 1-cells of M and their reverses: $\gamma = t_1^{\theta_1} t_2^{\theta_2} \dots t_n^{\theta_n}$, where $t_1, \dots, t_n$ are closed 1-cells, such that the initial point of $t_i^{\theta_i}$ is the end-point of $t_{i-1}^{\theta_{i-1}}$

The fundamental groupoid $\pi_1(M^1, M^0)$ is defined in Def. 8. Its set of objects is M^0. Let $u \xrightarrow{t} v$ be an edge in L^1. Let $\overline{c_u^0}, \overline{c_v^0} \in M^0$ be the closed 0-cells corresponding to the abstract 0-cells $u, v \in L^0$. The characteristic map $\phi_t^1 \colon [0, 1] \to M^1$ of t is such that $\phi_t^1(0) = \overline{c_u^0}$ and $\phi_t^1(1) = \overline{c_v^0}$. Passing to the homotopy class of ϕ_t^1, relative to the boundary $\{0, 1\}$ of $[0, 1]$, yields a morphism $\iota_L(t)$ in the homotopy groupoid $\pi_1(M^1, M^0)$; cf. Rem. 31. Since $\mathsf{FG}\langle L^0, L^1\rangle$ is free, ι_L extends to a groupoid map

$$\iota \colon \mathsf{FG}\langle L^0, L^1\rangle \to \pi_1(M^1, M^0).$$

The following can be seen as a generalisation of the van Kampen theorem [40], for spaces with a set of base points. A proof is in [12, 9.1.5]. This holds more generally for CW-complexes.

Theorem 39. *Let* (M, L) *be a 2-lattice. The groupoid map* $\iota \colon \mathsf{FG}\langle L^0, L^1\rangle \to \pi_1(M^1, M^0)$ *is an isomorphism. Hence* $\pi_1(M^1, M^0)$ *is isomorphic to the free groupoid* $\mathsf{FG}\langle L^0, L^1\rangle$*, with set of objects being* $M^0 = L^0$ *and with a free generator* $u \xrightarrow{t} v$ *for each edge* $t \in L^1$*. (Here* u *and* v *are the source and target of* t*.)* □

In particular, for any group G and for any map $f\colon L^1 \to G$ there exists a unique groupoid map $f'\colon \pi_1(M^1, M^0) \to G$ whose value on each $\iota_L(t)$, $t \in L^1$ an edge, is $f(t)$. (The same holds if G is a groupoid, except that we must pay attention to sources and targets.) As we will see in §3.2, this observation lies at the heart of the realisation of gauge theory that we will lift to the higher case.

We will hence see $\pi_1(M^1, M^0) \cong \mathsf{FG}\langle L^0, L^1\rangle$ as being the lattice groupoid of (M, L).

3 Higher order gauge configurations and discrete 2D holonomy for surfaces embedded in 2-lattices

In order to establish a template for the 'higher' construction, we start with a suitable characterisation of *ordinary* gauge configurations, and of their holonomy along cellularly embedded circles S^1.

3.1 Gauge configurations, discrete 1D parallel transport and holonomy along circles

Definition 40 (Gauge configuration)**.** Let G be a group. A *gauge configuration* on a 2-lattice (M, L) is a map

$$\mathcal{F}^1\colon L^1 \to G$$

We write $\mathcal{F}^1(t) = g_t$, for each edge $t \in L^1$.

By the freeness of $\pi_1(M^1, M^0) \cong \mathsf{FG}\langle L^0, L^1\rangle$ (Lem. 37 and Thm. 39) a gauge configurations $\mathcal{F}^1$ extends to a uniquely defined groupoid morphism

$$\Phi_{\mathcal{F}^1}\colon \pi_1(M^1, M^0) \to G.$$

Here G is regarded as a groupoid with one object. All groupoid maps $\mathsf{FG}\langle L^0, L^1\rangle \to G$ arise this way. Hence:

Theorem 41 (The discrete parallel transport of a gauge configuration)**.** *Let (M, L) be a 2-lattice. Let G be a group. The correspondence $\mathcal{F}^1 \mapsto \Phi_{\mathcal{F}^1}$ yields a one-to-one correspondence between gauge configurations $\mathcal{F}^1$ and groupoid maps $\Phi_{\mathcal{F}^1}\colon \pi_1(M^1, M^0) \to G$.* □

Those groupoid maps $\Phi_{\mathcal{F}^1}\colon \pi_1(M, M^1) \to G$ associated to a gauge configuration $\mathcal{F}^1$ will sometimes be called *discrete parallel transport functors*, in analogy with the differential-geometrical construction in [64, 34].

Definition 42. Let $\gamma = t_1^{\theta_1} t_2^{\theta_2} \dots t_n^{\theta_n}$ be a granular path in (M, L). Let $\mathcal{F}^1$ be a gauge configuration. Put:

$$g_\gamma = g_{t_1}^{\theta_1} g_{t_2}^{\theta_2} \dots g_{t_n}^{\theta_n}.$$

Let $[\gamma] \in \pi_1(M^1, M^0)$ be the equivalence class of γ. Given Thm. 39, it is clear that: $\Phi_{\mathcal{F}^1}([\gamma]) = g_\gamma$.

Definition 43 (Holonomy along a circle: combinatorial definition)**.** Let $\mathcal{F}^1\colon L^1 \to G$ be a gauge configuration on a 2-lattice decomposition (M, L). Let C be an oriented circle S^1 embedded in M. Suppose that L is a 2-lattice decomposition of (M, C); Def. 32. Let $v \in C \cap M^0 = C^0$. Starting at the vertex v, the path around the circle C in the positive direction therefore traces a granular path $\gamma = t_1^{\theta_1} t_2^{\theta_2} \dots t_n^{\theta_n}$, connecting v to v. The holonomy $\mathrm{Hol}^1_v(\mathcal{F}^1, C, L)$ of $\mathcal{F}^1$, along C, with initial point v, is defined as:

$$\mathrm{Hol}^1_v(\mathcal{F}^1, C, L) = g_\gamma = \Phi_{\mathcal{F}^1}([\gamma]) \in G.$$

Note that the holonomy $\mathrm{Hol}^1_v(\mathcal{F}^1, C, L)$ of $\mathcal{F}^1$ along C depends on the chosen starting point $v \in C \cap M^0$ only by conjugation by an element of G.

Remark 44 (Holonomy along a circle: algebraic topological definition)**.** Recall $S^1 = \partial D^2$. Choose a homeomorphism $f\colon \partial D^2 \to C$, preserving the orientation, sending the base-point $* = (0, 0)$ of ∂D^2 to v. By elementary algebraic topology (as $\pi_1(S^1) = \mathbb{Z}$), any two such homeomorphisms are homotopic, relative to $*$. Let $i_v(C) = f_*(1)$, where $f_*\colon \pi_1(S^1, *) \cong \mathbb{Z} \to \pi_1(C, v) \subset \pi_1(C, C^0)$ is the induced map on homotopy groups. Clearly $i_v(C) = t_1^{\theta_1} t_2^{\theta_2} \dots t_n^{\theta_n}$, as in Def. 43. Let $\mathcal{F}^1_C$ be the restriction of $\mathcal{F}^1$ to the induced lattice decomposition L_C of C; see Def. 32. It hence clearly holds that:

$$\mathrm{Hol}^1_v(\mathcal{F}^1, C, L) = \Phi_{\mathcal{F}^1_C}(i_v(C)).$$

Here $\Phi_{\mathcal{F}^1_C}\colon \pi_1(C, C^0) \to G$ is the discrete parallel transport of $\mathcal{F}^1_C$.

Remark 45. Although gauge configurations can formally be defined separately from Hamiltonians, as above, they have no physical meaning without an associated Hamiltonian. In particular parts of the structure of space-time are encoded *in a model* not in the gauge configuration but in the Hamiltonian. We are not ready to give the 'higher' Hamiltonian §5.1.4 (the higher Kitaev model) that will be the central focus of this paper, but we can already give an illustrative 'standard' example, which also serves as a template for the Kitaev quantum double model [47]; see §5.1.6. Given a lattice (M, L) and a group G, and hence the set $(\mathsf{FG}\langle L^0, L^1\rangle, G)$ of functors between $\mathsf{FG}\langle L^0, L^1\rangle$ and G, we may define for each character $\chi : G \to \mathbb{C}$ a Wilson action $H_\chi : (\mathsf{FG}\langle L^0, L^1\rangle, G) \to \mathbb{R}$ by (cf. Def. 30):

$$H_\chi(F) = \sum_{P \in L^2} Re(\chi(F(\partial_L(P))))$$

where $Re : \mathbb{C} \to \mathbb{R}$ is the real part (see e.g. Wilson [74], [48, §8], [55, §10.2] or [57, §1.2]). Note that this depends strongly on the cell-decomposition of M, as well as M. The *main* thing to note at this point is that the sum is over plaquettes, thus the Hamiltonian is sensitive to the 2-dimensional structure in the lattice (whereas the gauge configuration 'sees' only the underlying graph L^1). We will return to this point later.

3.2 Higher order gauge configurations

In this paper, we consider fake-flat 2-gauge configurations on a 2-lattice (M, L), as discretised models for higher gauge fields [5, 3, 34]. Instead of a gauge group we have a crossed module of groups; Def. 2. The main aim is to extend Thm. 41, Def. 43 and Rem. 44 to the case of fake-flat 2-gauge configurations. This yields 2-dimensional (2D) notions of parallel transport which restrict to notions of 2D holonomy along surfaces, cellularly embedded in M. We will address the 2-sphere and 2-disk case, which play an important role in higher Kitaev models.

3.2.1 Fake-flat 2-gauge configurations

Continuing the work of Yetter and Porter [76, 61], fake-flat 2-gauge configurations on CW-complexes were defined in [30, 35, 29]. Their algebraic topology interpretation was developed therein, following the work of Brown and Higgins on fundamental crossed modules of pairs of spaces and 2-dimensional van Kampen theorem [13, 14, 15, 18]. Homotopy quantum field theory applications of fake-flat 2-gauge configurations appear in [60] (and were there called "formal $\mathcal{C}$ maps"). The inherent (and independently addressed) differential-geometric higher gauge theory for 2-bundles with a 2-connection appears in [5, 64, 63, 34, 65]. The term "fake-flatness" appeared originally in the context of gerbes with connection; see [9].

Definition 46 (2-gauge configuration)**.** Let $\mathcal{G} = (\partial_\mathcal{G} : E \to G, \triangleright)$ be a crossed module of groups. Given $\mathcal{G}$, a 2-gauge configuration $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2)$, based on a 2-lattice $(M, L) = \big(M, L = (L^0, L^1, L^2, \dots)\big)$, is given by:

- A map $\mathcal{F}^1 : L^1 \to G$, denoted: $t \in L^1 \mapsto g_t \in G$, or $t \in L^1 \mapsto \mathcal{F}^1(t) \in G$.
- A map $\mathcal{F}^2 : L^2 \to E$, denoted: $P \in L^2 \mapsto e_P \in E$, or $P \in L^2 \mapsto \mathcal{F}^2(P) \in E$.

A 2-gauge configuration gives rise to a groupoid map $\Phi_\mathcal{F} = \Phi_{\mathcal{F}^1} : \pi_1(M^1, M^0) \to G$; see Thm. 41.

We mainly consider *fake-flat* 2-gauge configurations. Let us explain what this means.

Definition 47 (Fake-flat 2-gauge configuration)**.** A 2-gauge configuration $\mathcal{F} = (\mathcal{F}^2, \mathcal{F}^1)$, based on a 2-lattice (M, L), is said to be fake-flat if for each plaquette $P \in L^2$ it holds that (recall the notation of Rem. 30):

$$\partial_\mathcal{G}(e_P) = \Phi_{\mathcal{F}^1}(\partial_L(P)).$$

Given a crossed module $\mathcal{G}$, we denote the set of fake-flat 2-gauge configurations in (M, L) as $\Theta(M, L, \mathcal{G})$.

Let us give more explanation on the definition of fake-flatness. This is one of the points where the fact that we are restricting to 2-lattices Def. 22 makes our discussion a lot simpler. One more definition is needed.

Definition 48 (Granular boundary $\partial_L^Q(P)$ of a plaquette)**.** Let $P \in L^2$ be a plaquette of a 2-lattice (M, L). Let $\psi_P^2 \colon S^1 \to M^1$ be the attaching map of the correspondent closed 2-cell $\overline{c_P^2}$. We are given a CW-decomposition Z_P of S^1, which contains $* \in S^1$ as a 0-cell, such that $\psi_P^2 \colon S^1 \to M^1$ is cellular and satisfies the conditions of Def. 22. We will in addition suppose that all characteristic maps $\phi_{\gamma_i} \colon [0,1] \to S^1 = \mathrm{bd}([0,1]^2)$ of the closed 1-cells $\gamma_1, \ldots, \gamma_n$ of Z_P are oriented counterclockwise; see Fig. 3. We also assume that the 1-cells $\gamma_1, \ldots, \gamma_n$ of Z_P appear in that order, as we "travel" counterclockwise from $*$ to $*$ around S^1.

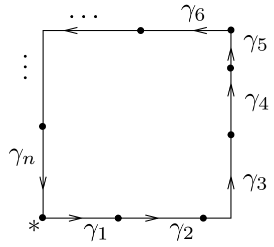


Figure 3: The CW-decomposition Z_P of the 1-sphere S^1.

We let $x_P = \psi_P^2(*) \in M^1$ be the base-point of the closed 2-cell $\overline{c_P^2}$ corresponding to P. Suppose that ψ_P^2 is not constant. Then for each $i \in \{1, \ldots, n\}$, $\psi_P^2(\gamma_i)$ is a closed 1-cell t_i of M and ψ_P^2 restricts to a homeomorphism $\gamma_i \to t_i$. The closed 1-cell $t_i \subset M$ is oriented by its characteristic map. We put $\theta_i = 1$ if the restriction $\gamma_i \to t_i$ of ψ_P^2 preserves orientation and $\theta_i = -1$, otherwise. The granular boundary of P is defined to be the following granular path in (M, L) (Def. 38), connecting x_P to x_P: $\partial_L^Q(P) = t_1^{\theta_1} t_2^{\theta_2} \ldots t_n^{\theta_n}$. Otherwise, if $\psi_P^2 \colon S^1 \to M^1$ satisfies $\psi_P^2(S^1) = x_P$, we define the granular boundary of P as $\partial_L^Q(P) = \emptyset_{x_P}$.

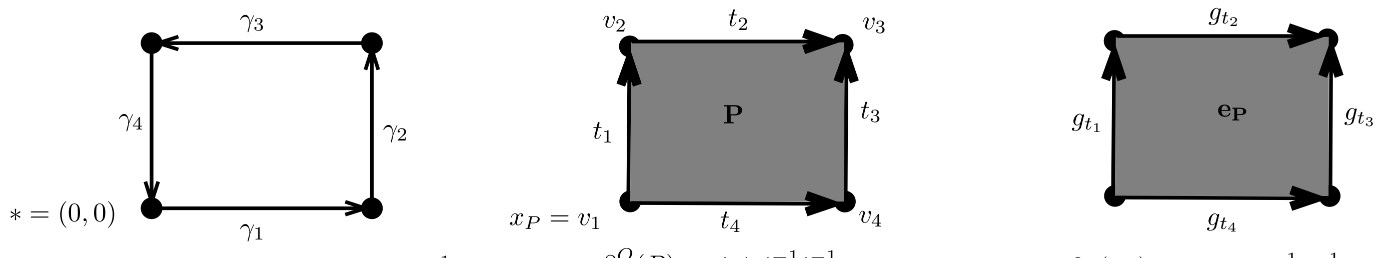
An example appears in Ex. 50.

Let $P \in L^2$. By passing to the equivalence class of the granular path $\partial_L^Q(P)$ (cf. the construction of $FG\langle L^0, L^1 \rangle \cong \pi_1(M^1, M^0)$ in Def. 35 and Prop. 37) yields $\partial_L(P) \in \pi_1(M^1, M^0)$ in Def. 30. Hence:

Proposition 49. *Let (M, L) be a 2-lattice. Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a crossed module of groups. A 2-gauge configuration $\mathcal{F} = (\mathcal{F}^2, \mathcal{F}^1)$ is fake-flat if, and only if:*

- *For each plaquette P for which ψ_P^2 is not constant, putting $\partial_L^Q(P) = t_1^{\theta_1} t_2^{\theta_2} \ldots t_n^{\theta_n}$, it holds that:*
$$\partial_{\mathcal{G}}(e_P) = g_{t_1}^{\theta_1} \ldots g_{t_n}^{\theta_n} = \Psi_{\mathcal{F}^1}([\partial L^Q(P)]). \tag{5}$$
- *If P is a plaquette for which $\psi_P^2(S^1) = x_P$ it should hold that $e_P \in \ker(\partial_{\mathcal{G}} \colon E \to G) \subset E$.*

Example 50. Consider the square $D^2 = [0,1]^2$, with the 2-lattice decomposition indicated in the middle of Fig. 4, namely $L = (L^0, L^1, L^2) = (\{v_1, v_2, v_3, v_4\}, \{t_1, t_2, t_3, t_4\}, \{P\})$. (Abstract cells and the corresponding closed cells are denoted in the same way.) The geometric 2-cell $\overline{e_P^2} = P$ attaches along the identity map $\psi_P^2 \colon S^1 \to S^1$, hence the attaching map $\psi_P^2 \colon S^1 \to S^1$ is positively oriented. The CW-decomposition Z_P of $S^1 = \mathrm{bd}([0,1]^2)$ (Def. 22) has a vertex for each corner and a positively oriented edge for each side of $[0,1]^2$.



Z_P, the CW-decomposition of S^1 $\quad$ $\partial_L^Q(P) = t_4 t_3 t_2^{-1} t_1^{-1}$ $\quad$ $\partial_{\mathcal{G}}(e_P) = g_{t_4} g_{t_3} g_{t_2}^{-1} g_{t_1}^{-1}$

Figure 4: A 2-lattice decomposition L of D^2, where Z_P is the corresponding CW-decomposition of S^1: cf. Def, 22. The base-point x_P of P is v_1. We also show a fake-flat 2-gauge configuration in (D^2, L).

For this example, the granular boundary of the plaquette P is $\partial_L^Q(P) \; = t_4 t_3 t_2^{-1} t_1^{-1}$. Hence a 2-gauge configuration of $([0,1]^2, L)$ is given by four elements $g_{t_1}, g_{t_2}, g_{t_3}, g_{t_4}$ of G, the colours of the edges, t_1, t_2, t_3, t_4, and an element $e_P \in E$, colouring P. The fake flateness conditions says: $\partial_{\mathcal{G}}(e_P) = g_{t_4} g_{t_3} g_{t_2}^{-1} g_{t_1}^{-1}$.

Let $\Theta(M, L, \mathcal{G})$ denote the set of fake-flat 2-gauge configurations. Note that $\Theta(M, L, \mathcal{G})$ is non-empty. In particular the 'naive vacuum' Ω_1 given by $e_P = 1_E$ for all plaquettes P of (M, L) and $g_t = 1_G$ for all edges t is fake-flat. Here 1_G and 1_E denote the identities of G and E.

3.3 On Whitehead theorem, 2-gauge configurations and the lattice 2-groupoid

Let (M, L) be a 2-lattice. Passing to the 0, 1 and 2-skeletons of the corresponding CW-decomposition of M, yields a triple (M^2, M^1, M^0) of locally path-connected spaces, where M^0 intersects non-trivially any path-connected component of M^1 and M^2. Utilising Def. 10, we can form the fundamental crossed module $\Pi_2(M^2, M^1, M^0)$; Thm. 11. This crossed module plays the role of lattice 2-groupoid of (M, L).

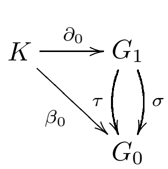
Observe that to make use of our fake-flat 2-gauge configurations we need corresponding lifts of Lem. 37 and Thm. 39. Analogously to Thm. 39, the crossed module $\Pi_2(M^2, M^1, M^0)$ is free on the attaching maps of the geometric plaquettes (i.e. closed 2-cells) of (M, L). This result (which holds in the general case of CW-complexes) is an old result due to JHC Whitehead [72]. Modern treatments can be found in [18, 11, 10, 13, 8].

Consider groupoids $H = (\sigma, \tau \colon H_1 \to H_0)$ and $H' = (\sigma, \tau \colon H_1' \to H_0')$. Throughout this subsection, we use the following notation. If $f \colon H \to H'$ is a groupoid map, put $f_{\mathrm{MOR}} \colon H_1 \to H_1'$ to be the restriction of f to morphisms and $f_{\mathrm{OBJ}} \colon H_0 \to H_0'$ to be the restriction of f to objects. If $(\partial \colon E \to G)$ is a crossed module of groupoids, thus E and G have the same set C of objects, it holds that $\partial_{\mathrm{OBJ}} \colon C \to C$ is the identity map.

In order not to excessively load our formulae, we use the same notation for the groupoid $\pi_2(M^2, M^1, M^0)$ and for its set of morphisms, and the same for $\pi_1(M^1, M^0)$. Which one is meant is clear from the context. The coinciding source and target maps in the groupoid $\pi_2(M^2, M^1, M^0)$ are given by the obvious map:

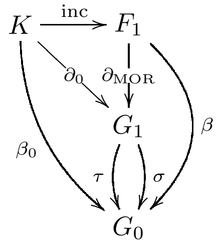
$$\beta \colon \pi_2(M^2, M^1, M^0) \doteq \bigsqcup_{v \in M^0} \pi_2(M^2, M^1, v) \to M^0.$$

First we specify what "crossed module freeness" is. Let $G = (\sigma, \tau \colon G_1 \to G_0)$ be a groupoid. Let also K be a set mapping to G_0, through a map β_0. Suppose also that we have a map $\partial_0 \colon K \to G_1$ that makes the diagram below commute (therefore $\beta_0 = \tau \circ \partial_0$ and $\beta_0 = \sigma \circ \partial_0$):



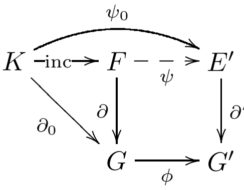
(6)

(This in particular means that ∂_0 is a map from K into the set of arrows (morphisms) in G that have the same source and target.) Let $F = (\beta, \beta \colon F_1 \to G_0)$ be a totally intransitive groupoid with the same set of objects as G. We say that a crossed module of groupoids $(\partial \colon F \to G, \triangleright)$ is free on $\partial_0 \colon K \to G_1$ (or more precisely on $\partial_0 \colon K \to G_1$ and $\beta_0 \colon K \to G_0$ as in (6)) if there exists an inclusion (set) map inc: $K \to F_1$ making the diagram below commute:

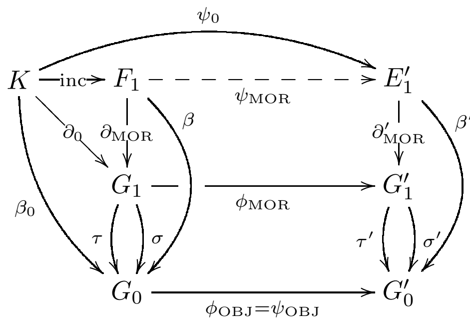


(7)

such that the following universal property is satisfied: *Given any crossed module $(\partial' \colon E' \to G', \triangleright)$ of groupoids, where $G' = (\sigma', \tau' \colon G_1' \to G_0')$ and $E' = (\beta', \beta' \colon E_1' \to G_0')$, and any groupoid map $\phi \colon G \to G'$,* *and any set map $\psi_0\colon K \to E'_1$, such that $\partial'_{\mathrm{MOR}} \circ \psi_0 = \phi_{\mathrm{MOR}} \circ \partial_0$, there exists a unique groupoid map $\psi\colon F \to E'$, with $\psi_{\mathrm{OBJ}} = \phi_{\mathrm{OBJ}}$, making the diagram below commutative:*



or leaving in all information:



(8)

and so that the pair (ψ, ϕ) of groupoid maps is a crossed module map $(\partial\colon F \to G) \to (\partial'\colon E' \to G')$.

Lemma 51. *Given $\partial_0\colon K \to G_1$ as in (6), the totally intransitive groupoid $F = (\beta, \beta\colon F_1 \to G_0)$, i.e. the top groupoid of the free crossed module on $\partial_0\colon K \to G_1$, is uniquely specified up to isomorphism by the universal property above. A model for F is the following. First of all note that we have a totally intransitive graph K' having G_0 as set of vertices and the set of edges being the set of all pairs (g, k), where $g \in G_1$ and $k \in K$ are such that $\tau(g) = \beta_0(k)$. The coinciding source and target maps of K' are given by $(g, k) \mapsto \sigma(g)$. Edges of K' therefore take the form:*

$$\sigma(g) \xrightarrow{(g,k)} \sigma(g), \text{ where } g \in G_1 \text{ and } k \in K \text{ are such that } \tau(g) = \beta_0(k).$$

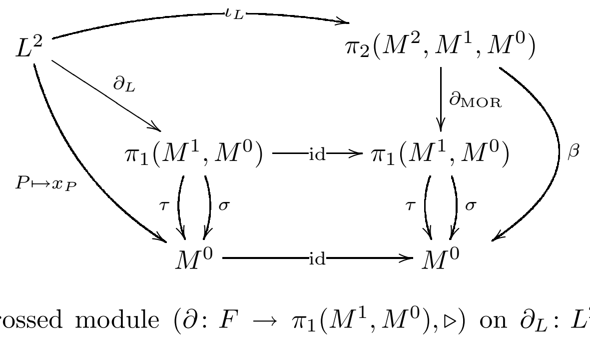
We then form the free groupoid $FG(K')$, which is a totally intransitive groupoid having G_0 as set of objects. We have a groupoid map $\partial\colon FG(K') \to G$ which is the identity on objects and on generating morphisms is:

$$\partial_{\mathrm{MOR}}\big(\sigma(g) \xrightarrow{(g,k)} \sigma(g)\big) = \sigma(g) \xrightarrow{g\partial_0(k)g^{-1}} \sigma(g).$$

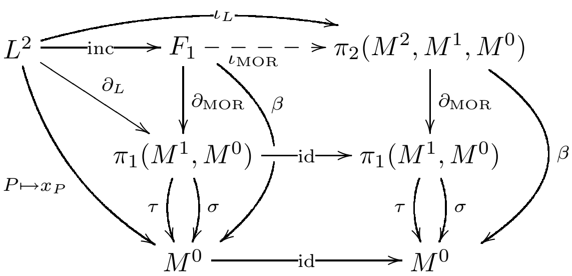
The groupoid $FG(K')$ has a natural left action by automorphisms of the groupoid G. On generators of $FG(K')$, the action takes the following form: If $g, h \in G_1$ are such that $\tau(h) = \sigma(g)$, and $k \in K$ is such that $\beta_0(k) = \tau(g)$, put $h.(g, k) = (hg, k)$. Then together with the map $\partial\colon FG(K') \to G$, nearly all conditions that crossed modules of groupoids must satisfy (Def. 2 and 6) hold, except for the 2nd Peiffer relation. The groupoid F is obtained from $FG(K')$ by dividing out the 2nd Peiffer relations, in the obvious way.

Proof. Routine. Details are in [18, §7.3(ii)] and [11]. □

Let (M, L) be a 2-lattice. Recall Rem. 30 and 31, and Def. 22. Going back to $\Pi_2(M^2, M^1, M^0)$, to each plaquette $P \in L^2$ we can associate elements $\partial_L(P) \in \pi_1(M^1, x_P) \subset \pi_1(M^1, M^0)$ and $\iota_L(P) \in \pi_2(M^2, M^1, x_P)$, where $x_P = \psi^2_P(*)$ is the base-point of the closed 2-cell $\overline{c^2_P}$ corresponding to P. Also $\partial_{\mathrm{MOR}}(\iota_L(P)) = \partial_L(P)$. In particular we have a commutative diagram as in (6,7,8):



Hence we can form the free crossed module $(\partial\colon F \to \pi_1(M^1, M^0), \triangleright)$ on $\partial_L\colon L^2 \to \pi_1(M^1, M^0)$, where $F = (\beta, \beta\colon F_1 \to M^0)$. And we have a unique groupoid morphism $\iota\colon F \to \pi_2(M^2, M^1, M^0)$, which is the identity on objects, that makes the diagram below commutative, and so that $(\iota, \mathrm{id})\colon (F \to \pi_1(M^1, M^0), \triangleright) \to \Pi_2(M^2, M^1, M^0)$ is a crossed module map:



(9)

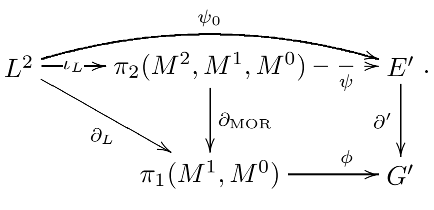
Theorem 52 (Whitehead Theorem)**.** *Let (M, L) be a 2-lattice, or indeed any CW-complex. Then the crossed module $\Pi_2(M^2, M^1, M^0) = (\partial\colon \pi_2(M^2, M^1, M^0) \to \pi_1(M^1, M^0), \triangleright)$ is free on $\partial_L\colon L^2 \to \pi_1(M^1, M^0)$. Specifically, the map $\iota\colon F \to \pi_2(M^2, M^1, M^0)$ defined from (9) is an isomorphism of groupoids, and, moreover, the pair $(\iota, \mathrm{id})\colon (\partial\colon F \to \pi_1(M^1, M^0), \triangleright) \to \Pi_2(M^2, M^1, M^0)$ is an isomorphism of crossed modules.*

Proof. This result is a particular case of Corollary 7.11 of [11]. See [18, §5.4] and [13, 14, 15], where Whitehead's theorem is deduced from the more general 2-dimensional van Kampen theorem, and also [8, 10]. (We note that Whitehead's original proof was done for crossed modules of groups rather than of groupoids, and only considered spaces with a single base-point.) □

Remark 53. Note that Whitehead's theorem together with the construction of free crossed modules (Lem. 51) implies that the totally intransitive groupoid $\pi_2(M^2, M^1, M^0)$ is generated by $\sigma(g) \xrightarrow{g \triangleright \iota_L(P)} \sigma(g)$, where $g \in \pi_1(M^1, M^0)$ and $P \in L^2$ is such that: $x_P = \tau(g)$; recall that $\sigma(g)$ and $\tau(g)$ are the initial and end-points of g and x_P is the base-point of P, and recall Def. 31 for notation. This will have a primary role in the construction of the 2-dimensional holonomy of a fake-flat 2-gauge configuration along a cellularly embebbed surface in §3.4.

We will only use the universal property (8) in the case when $\mathcal{G} = (\partial'\colon E' \to G', \triangleright)$ is a crossed module of groups. In this case there is not much to worry about maps on object sets of groupoids, as E' and G' are groupoids with a single object. We hence will not display the related part of the commutative diagrams. Given groupoid maps $f\colon \pi_1(M^1, M^0) \to G'$ and $f'\colon \pi_2(M^2, M^1, M^0) \to E'$ we put $f_{\mathrm{MOR}} = f$ and $f'_{\mathrm{MOR}} = f'$. In $\mathcal{G}$, we put $\partial'_{\mathrm{MOR}} = \partial'$. Recall that we use the same notation for the groupoid $\pi_2(M^2, M^1, M^0)$ and for its set of morphisms, and the same for $\pi_1(M^1, M^0)$.

Whitehead's theorem implies the following. Consider the inclusion map $\iota_L\colon L^2 \to \pi_2(M^2, M^1, M^0)$, with $\partial_{\mathrm{MOR}} \circ \iota_L = \partial_L$, cf. Rem. 30 and 31. If G' is a group, $(\partial'\colon E' \to G', \triangleright)$ is a crossed module of groups, and $\phi\colon \pi_1(M^1, M^0) \to G'$ is a groupoid map, then given any set map $\psi_0\colon L^2 \to E'$ such that $\phi \circ \partial_L = \partial' \circ \psi_0$, there exists a unique groupoid map $\psi\colon \pi_2(M^2, M^1, M^0) \to E'$ making the diagram below commutative and also making the pair (ψ, ϕ) a crossed module map $\Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ (thus (ψ, ϕ) is compatible with boundaries and groupoid actions):



(10)

3.3.1 The discrete 2-dimensional (2D) parallel transport of a fake-flat 2-gauge configuration

As promised in §3.1, we now state and prove the analogue of Thm. 41 for fake-flat 2-gauge configurations.

Let (M, L) be a 2-lattice and $\mathcal{G} = (\partial_{\mathcal{G}}\colon E \to G, \triangleright)$ be a group crossed module.

Theorem 54 (The discrete 2D parallel transport of a fake-flat 2-gauge configuration)**.** *There exists a one-to-one correspondence between fake-flat 2-gauge configurations $\mathcal{F} = (\mathcal{F}^2, \mathcal{F}^1)$ in (M, L) and crossed module maps $(\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}})\colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$. (Note $\Psi_{\mathcal{F}}\colon \pi_2(M^2, M^1, M^0) \to E$ and $\Phi_{\mathcal{F}}\colon \pi_1(M^1, M^0) \to G$ therefore are groupoid maps, compatible with boundary maps and groupoid actions in the obvious way.)*

In analogy with the differential-geometric construction of 2-dimensional parallel transport 2-functors attached to 2-connections on 2-bundles [63, 34], the crossed module map $(\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}})\colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ associated to a fake-flat 2-gauge configurations $\mathcal{F}$ will be called the *discrete 2D parallel transport 2-functor of $\mathcal{F}$*.

Proof. Recall the notation introduced after Thm 52. A fake-flat 2-gauge configuration $\mathcal{F} = (\mathcal{F}^2\colon L^2 \to E\ ,\ \mathcal{F}^1\colon L^1 \to G)$ is an assignment $\gamma \mapsto g_\gamma$ of an element of G to each 1-cell γ of L, and an assigment $P \mapsto e_P$ of an element of E to each 2-cell P, satisfying the fake-flatness condition of Def. 47. Whitehead's theorem (Thm 52) states that the fundamental crossed module $\Pi_2(M, M^1, M^0)$, with set M^0 of base points – a crossed module of groupoids, is isomorphic to the free crossed module on the map $\partial_L\colon L^2 \to \pi_1(M^1, M^0)$.

Since the groupoid $\pi_1(M^1, M^0)$ is free on the 1-cells, the assignment $\mathcal{F}^1\colon \gamma \in L^1 \mapsto g_\gamma \in G$ uniquely extends to a groupoid map $\Phi_{\mathcal{F}} = \Phi_{\mathcal{F}^1}\colon \pi_1(M^1, M^0) \to G$. The fake-flatness condition means that the outer part of the diagram below commutes:

$$\begin{array}{ccccc} & & \overset{\mathcal{F}^2}{\frown\!\frown\!\frown\!\frown\!\longrightarrow} & & \\ L^2 & \xrightarrow{\ \iota_L\ } & \pi_2(M^2, M^1, M^0) & \overset{\Psi_{\mathcal{F}}}{\dashrightarrow} & E\ . \\ & \searrow^{\partial_L} & \big\downarrow{\scriptstyle \partial_{\mathrm{MOR}}} & & \big\downarrow{\scriptstyle \partial_{\mathcal{G}}} \\ & & \pi_1(M^1, M^0) & \xrightarrow{\ \Phi_{\mathcal{F}}\ } & G \end{array}$$

And by applying the universal property defining free crossed modules of groupoids, in the particular form of (10), we can see that a gauge configuration $\mathcal{F} = (\mathcal{F}^2, \mathcal{F}^1)$ can be extended (uniquely) to a crossed module map $(\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}})\colon \Pi_2(M, M^1, M^0) \to \mathcal{G}$, and all crossed module maps $\Pi_2(M, M^1, M^0) \to \mathcal{G}$ arise this way. $\square$

Cf. §3.1. We have now explained the crossed module analogue of discrete parallel transport functors (Thm 41), in terms of discrete 2D parallel transport 2-functors. In the next two subsections §3.4 and §3.5, we address how the 2D parallel transport of a fake-flat 2-gauge configuration can be used to define notions of discrete 2D holonomy along surfaces Σ cellularly embedded in a 2-lattice (M, L). We will only deal with the case when Σ is the 2-disk D^2 or the 2-sphere S^2. In these cases, which are the ones needed to define higher Kitaev models, a 2D holonomy can be associated to cellularly embedded surfaces $\Sigma \subset M$. 2D holonomy along 2-disks and 2-spheres is particularly simple to formulate, given that the corresponding oriented mapping class groups are trivial.

For surfaces Σ not homeomorphic to S^2 or to D^2, additional information is needed to define a meaningful 2D holonomy for a cellular embedding of Σ into (M, L). Namely (assuming orientability) we must choose an isotopy class of homeomorphisms $\Sigma' \to \Sigma$, where Σ' is the boundary of an unknotted handlebody in $\mathbb{R}^3$; see [34, 65]. We will address this more general 2D holonomy in a forthcoming publication.

3.4 Algebraic topological definition of 2D holonomy along 2-disks and 2-spheres

Let us fix a crossed module of groups $\mathcal{G} = (\partial_{\mathcal{G}}\colon E \to G, \triangleright)$. In this subsection we use elementary algebraic topology to define precisely and concisely the 2-dimensional (2D) holonomy of a fake-flat 2-gauge configuration along cellularly embedded 2-disks and 2-spheres; as such we present the 2D analogue of Rem. 44. A combinatorial definition of this 2D holonomy (therefore the analogue of Def. 43) will be dealt with in §3.5.

3.4.1 The 2-disk case

Let (M, L) be a 2-lattice. Let Σ be a surface embedded in M. Suppose that Σ is homeomorphic to the 2-disk D^2. Suppose in addition that Σ is oriented. Furthermore (cf. Def. 32) suppose that L is a 2-lattice decomposition of the triple $(M, \Sigma, \mathrm{bd}(\Sigma))$, where $(\Sigma, \mathrm{bd}(\Sigma))$ is a pair homeomorphic to (D^2, S^1). We have an induced CW-decomposition of $(\Sigma, \mathrm{bd}(\Sigma))$. Note $\Sigma = \Sigma^2$ (the 2-skeleton of Σ) and $\mathrm{bd}(\Sigma) \subset \Sigma^1$.

Choose a base point $v \in \mathrm{bd}(\Sigma) \cap M^0$. Since $\pi_2(\Sigma^1, \mathrm{bd}(\Sigma), v)$ is trivial, the homotopy exact sequences of the triple $(\Sigma, \Sigma^1, \mathrm{bd}(\Sigma))$ and of the pair $(\Sigma^1, \mathrm{bd}(\Sigma))$ imply that the inclusion $(\Sigma, \mathrm{bd}(\Sigma)) \to (\Sigma, \Sigma^1)$ yields injections $\pi_2(\Sigma, \mathrm{bd}(\Sigma), v) \to \pi_2(\Sigma, \Sigma^1, v)$ and $\pi_1(\mathrm{bd}(\Sigma), v) \to \pi_1(\Sigma^1, v)$. We can thus see the crossed module $\Pi_2(\Sigma, \mathrm{bd}(\Sigma), v) \cong (\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z})$ (Ex. 13) as canonically included in $\Pi_2(\Sigma, \Sigma^1, v)$.

Let $* = (0,0)$ be the common base point of D^2 and $S^1 = \mathrm{bd}(D^2)$. By elementary algebraic topology – since $\pi_2(D^2, S^1, *) = \mathbb{Z}$ – any two pointed homeomorphisms $(D^2, S^1) \to (\Sigma, \mathrm{bd}(\Sigma))$ preserving orientation are homotopic as maps of pointed pairs $(D^2, S^1) \to (\Sigma, \mathrm{bd}(\Sigma))$. This is used in the definition below.

Definition 55 (Notation: $\partial_v(\Sigma, L)$ and $\iota_v(\Sigma, L)$)**.** Cf Rem. 30 and 31. Let Σ be an oriented surface homeomorphic to D^2. Let $v \in \mathrm{bd}(\Sigma)$, the boundary of Σ. We will be mainly interested in the case when $v \in \mathrm{bd}(\Sigma) \cap M^0$. Consider a pointed orientation preserving homeomorphism $j\colon (D^2, S^1, *) \to (\Sigma, \mathrm{bd}(\Sigma), v)$. We let $j_*\colon \Pi_2(D^2, S^1, *) \to \Pi_2(\Sigma, \mathrm{bd}(\Sigma), v)$ be given by the induced map on crossed modules. Let $\partial_v(\Sigma, v) \in \pi_1(\mathrm{bd}(\Sigma), v) \subset \pi_1(\Sigma^1, v)$ be $\partial_v(\Sigma, v) = j_*(1)$, where 1 is the positive (counterclockwise) generator of $\pi_1(S^1, v) \cong \mathbb{Z}$. Hence $\partial_v(\Sigma, v)$ is a positively oriented loop along the boundary $\mathrm{bd}(\Sigma)$ of the 2-disk Σ, starting and ending at v. Analogously put $\iota_v(\Sigma, L) = j_*(1) \in \pi_2(\Sigma, \mathrm{bd}(\Sigma), v) \subset \pi_2(\Sigma, \Sigma^1, v)$, where 1 is now the positive generator of $\pi_2(D^2, S^1, v) \cong \mathbb{Z}$.

Note that by construction (cf. Ex 13):

$$\begin{aligned} \partial\big(\iota_v(\Sigma, L)\big) &= \partial_v(\Sigma, L), \\ \partial_v(\Sigma, L) \triangleright \iota_v(\Sigma, L) &= \iota_v(\Sigma, L). \end{aligned} \tag{11}$$

Lemma 56 (Dependence of $\partial_v(\Sigma, L)$ and $\iota_v(\Sigma, L)$ on $v \in \mathrm{bd}(\Sigma)$)**.** *Suppose that $v \in M^0 \cap \mathrm{bd}(\Sigma)$. Choose another $v' \in M^0 \cap \mathrm{bd}(\Sigma)$. Consider a path γ in $\mathrm{bd}(\Sigma)$, connecting v' to v. (There are two different possible homotopy classes $[\gamma]$ for γ.) Then passing to the corresponding element $[\gamma] \in \pi_1(\Sigma^1, \Sigma^0)$, it holds that:*

$$\begin{aligned} [\gamma] \triangleright \iota_v(\Sigma, L) &= \iota_{v'}(\Sigma, L), \text{ in } \pi_2(\Sigma, \Sigma^1, \Sigma^0); \\ [\gamma]\, \partial_v(\Sigma, L)\, [\gamma]^{-1} &= \partial_{v'}(\Sigma, L), \text{ in } \pi_1(\Sigma^1, \Sigma^0). \end{aligned} \tag{12}$$

Proof. Follows from geometric considerations and the fact that $\pi_1(S^1, *)$ acts trivially on $\pi_2(D^2, S^1, *)$. □

Let now $\mathcal{G} = (\partial_{\mathcal{G}}\colon E \to G, \triangleright)$ be a crossed module of groups.

Definition 57 (2D holonomy $\mathrm{Hol}_v(\mathcal{F}, \Sigma, L)$ of a fake-flat 2-gauge configuration $\mathcal{F} = (\mathcal{F}^2, \mathcal{F}^1)$ along $\Sigma \cong D^2$, with initial point v)**.** Let M be a topological manifold. Let Σ be an oriented disk embedded in M. Let L be a 2-lattice decomposition of $(M, \Sigma, \mathrm{bd}(\Sigma))$; see Def. 32. Let $v \in \mathrm{bd}(\Sigma) \cap M^0$. Let $\mathcal{F}$ be a fake-flat 2-gauge configuration in (M, L), and $\mathcal{F}_\Sigma$ be its restriction to the induced 2-lattice decomposition of Σ. Let $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma})\colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$ be the 2D parallel transport 2-functor of $\mathcal{F}_\Sigma$; Thm. 54. We define the 2D holonomy of $\mathcal{F}$ along Σ, with initial point v, as:

$$\mathrm{Hol}_v(\mathcal{F}, \Sigma, L) = \big(\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L), \mathrm{Hol}^1_v(\mathcal{F}, \Sigma, L)\big) = \Big(\Psi_{\mathcal{F}_\Sigma}(\iota_v(\Sigma, L)), \Phi_{\mathcal{F}_\Sigma}(\partial_v(\Sigma, L))\Big) \in E \times G.$$

In the conditions of Def. 57, note that:

$$\partial_{\mathcal{G}}\left(\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)\right) = \mathrm{Hol}^1_v(\mathcal{F}, \Sigma, L). \tag{13}$$

This is because, by (11) and the fact that $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma})\colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$ is a crossed module map:

$$\begin{aligned} \partial_{\mathcal{G}}\left(\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)\right) &= \partial_{\mathcal{G}}(\Psi_{\mathcal{F}_\Sigma}(\iota_v(\Sigma, L)) = \Phi_{\mathcal{F}_\Sigma}\big(\partial(\iota_v(\Sigma, L))\big) \\ &= \Phi_{\mathcal{F}_\Sigma}(\partial_v(\Sigma, L)) = \mathrm{Hol}^1_v(\mathcal{F}, \Sigma, L). \end{aligned}$$

Remark 58 (Dependence of 2D holonomy on base points)**.** The 2D holonomy $\mathrm{Hol}_v(\mathcal{F}, \Sigma, L)$ of a fake-flat 2-gauge configuration along a cellularly embedded 2-disk depends on the choice of a base point $v \in \mathrm{bd}(\Sigma) \cap M^0$. However, the dependence is mild. Cf. Rem. 56. Choose any granular path $v' \xrightarrow{\gamma} v$, in the boundary of the disk Σ, from the new base point v' to the initial base point v. Let $[\gamma]$ be the corresponding element of $\pi_1(\Sigma^1, \Sigma^0) \cong \mathsf{FG}\langle L^0, L^1\rangle$. Then, since $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma})\colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$ is a crossed module map:

$$\begin{aligned} \mathrm{Hol}^2_{v'}(\mathcal{F}, \Sigma, L) &= \Psi_{\mathcal{F}_\Sigma}\big(\iota_{v'}(\Sigma, L)\big) \\ &= \Psi_{\mathcal{F}_\Sigma}\big([\gamma] \triangleright \iota_v(\Sigma, L)\big) \\ &= \Phi_{\mathcal{F}_\Sigma}([\gamma]) \triangleright \Psi_{\mathcal{F}_\Sigma}\big(\iota_v(\Sigma, L)\big). \end{aligned}$$

Hence:

$$\mathrm{Hol}^2_{v'}(\mathcal{F}, \Sigma, L) = \Phi_{\mathcal{F}_\Sigma}([\gamma]) \triangleright \mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L). \tag{14}$$

3.4.2 The 2-sphere case

We resume the notation and ideas of §3.4.1. Let $* = (0,0,0)$ be the base-point of $S^2 = \mathrm{bd}(D^3)$. Let M be a topological manifold. Cf. Def. 32, let L be a 2-lattice decomposition of (M, Σ), where $\Sigma \subset M$ is oriented and homeomorphic to the 2-sphere S^2. Choose a base point $v \in \Sigma \cap M^0$. By elementary algebraic topology, any two orientation-preserving homeomorphisms $f\colon S^2 \to \Sigma$ preserving base-points are pointed homotopic.

Since $\pi_2(\Sigma^1, v) \cong \{0\}$, the final bits of the homotopy exact sequence of the pointed pair (Σ, Σ^1), namely $\{0\} \cong \pi_2(\Sigma^1, v) \to \pi_2(\Sigma, v) \xrightarrow{i} \pi_2(\Sigma, \Sigma^1, v) \xrightarrow{\partial} \pi_1(\Sigma^1, v)$, yield a monomorphism $i\colon \pi_2(\Sigma, v) \to \pi_2(\Sigma, \Sigma^1, v)$, thus an isomorphism $i\colon \pi_2(\Sigma, v) \to \ker(\partial)$. Hence $\pi_2(\Sigma, v)$ can be seen as included in the set of morphisms of the groupoid $\pi_2(\Sigma, \Sigma^1, \Sigma^0)$.

Definition 59 (Notation: $\overline{\iota_v}(\Sigma)$)**.** Let S^2 carry the orientation arising from its embedding into $\mathbb{R}^3$. Let Σ be an oriented manifold homeomorphic to S^2. Choose a base point $v \in \Sigma$. Choose an orientation preserving homeomorphism $f\colon (S^2, *) \to (\Sigma, v)$. Let 1 be the positive generator of $\pi_2(S^2, *) \cong \mathbb{Z}$. We define $\overline{\iota_v}(\Sigma) \in \pi_2(\Sigma, v) \subset \pi_2(\Sigma, \Sigma^1, \Sigma^0)$, to be $\overline{\iota_v}(\Sigma) = f_*(1)$, where $f_*\colon \pi_2(S^2, *) \to \pi_2(\Sigma, v)$ is the induced map at the level of second homotopy groups. Note that (cf. Def. 35), it hence follows that $\partial\big(\overline{\iota_v}(\Sigma)\big) = \emptyset_v$, where ∂ is the boundary map in the crossed module of groupoids $\Pi_2(\Sigma, \Sigma^1, \Sigma^0) = (\partial\colon \pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \pi_1(\Sigma^1, \Sigma^0))$.

In what follows, we will frequently not distinguish $\overline{\iota_v}(\Sigma) \in \pi_2(\Sigma, v)$ from $i(\overline{\iota_v}(\Sigma)) \in \pi_2(\Sigma, \Sigma^1, v)$.

Remark 60. Let v and v' be different points in the 0-skeleton $\Sigma^0 = \Sigma \cap M^0$ of Σ. Let $[\gamma]$ be any path in Σ^1 connecting v' to v, considered up to homotopy relative to the end-points. Then $\overline{\iota_{v'}}(\Sigma) = [\gamma] \triangleright \overline{\iota_v}(\Sigma)$.

Definition 61 (2D holonomy $\mathrm{Hol}_v(\mathcal{F}, \Sigma, L)$ of a fake-flat 2-gauge configuration $\mathcal{F}$ along $\Sigma \cong S^2$, with initial point v)**.** Let Σ be an oriented manifold homeomorphic to the 2-sphere. Let L be a 2-lattice decomposition of (M, Σ). Let $v \in \Sigma \cap M^0$. Let $\mathcal{F}$ be a fake-flat 2-gauge configuration in (M, L). Let $\mathcal{F}_\Sigma$ be the restriction of $\mathcal{F}$ to the induced 2-lattice decomposition of Σ. Let $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma})\colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$ be the 2D parallel transport 2-functor of $\mathcal{F}_\Sigma$; see Thm. 54. We define the 2D holonomy of $\mathcal{F}$ along Σ, with initial point v, as:

$$\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) = \Psi_{\mathcal{F}_\Sigma}\big(i(\overline{\iota_v}(\Sigma, L))\big) \in E.$$

Remark 62. Continuing Def. 61, note that since $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma})\colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$ is a crossed module map:

$$\begin{aligned} \partial_{\mathcal{G}}\big(\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)\big) &= \partial_{\mathcal{G}} \circ \Psi_{\mathcal{F}_\Sigma}\big(i(\overline{\iota_v}(\Sigma))\big) \\ &= (\Phi_{\mathcal{F}_\Sigma} \circ \partial)\big(i(\overline{\iota_v}(\Sigma))\big) \\ &= \Phi_{\mathcal{F}_\Sigma}(\emptyset_v) = 1_G. \end{aligned}$$

So it always holds that $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) \in \ker(\partial_{\mathcal{G}}) \subset E$, if $\Sigma \cong S^2$. This is not the case for the 2-disk; cf. (13).

Lemma 63 (Dependence of 2D holonomy along 2-spheres on base points and orientations)**.** *We resume the notation of Def. 61. Let $v, v' \in \mathrm{bd}(\Sigma) \cap M^0$ be two base points. Let $\gamma = t_1^{\theta_1} \dots t_n^{\theta_n}$ be a granular path in Σ^1, from v' to v; see Def. 38. Recall $g_\gamma = g_{t_1}^{\theta_1} \dots g_{t_n}^{\theta_n} = \Phi_{\mathcal{F}^1}([\gamma])$; see Def. 42. We then have:*

$$\mathrm{Hol}^2_{v'}(\mathcal{F}, \Sigma, L) = g_\gamma \triangleright \mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L).$$

Furthermore, if Σ^* is Σ with the opposite orientation, then:

$$\mathrm{Hol}^2_v(\mathcal{F}, \Sigma^*, L) = \big(\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)\big)^{-1}.$$

Proof. Let $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma})\colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G} = (\partial_{\mathcal{G}}\colon E \to G, \triangleright)$ be (Thm. 54) the crossed module map (i.e. the discrete parallel transport 2-functor) yielded by the restriction $\mathcal{F}_\Sigma$ of $\mathcal{F}$ to Σ. Then:

$$\begin{aligned} \mathrm{Hol}^2_{v'}(\mathcal{F}, \Sigma, L) &= \Psi_{\mathcal{F}_\Sigma}\big(i\big(\overline{\iota_{v'}}(\Sigma)\big)\big) \\ &= \Psi_{\mathcal{F}_\Sigma}\big([\gamma] \triangleright i\big(\overline{\iota_v}(\Sigma)\big)\big), \text{ by Rem. 60} \\ &= \Phi_{\mathcal{F}_\Sigma}([\gamma]) \triangleright \Psi_{\mathcal{F}_\Sigma}\big(i\big(\overline{\iota_v}(\Sigma)\big)\big) \\ &= g_\gamma \triangleright \mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L). \end{aligned}$$

Let Σ^* be Σ with the opposite orientation. Then $i\big(\overline{\iota_v}(\Sigma^*)\big) = \big(i\big(\overline{\iota_v}(\Sigma)\big)\big)^{-1}$. Hence:

$$\mathrm{Hol}^2_v(\mathcal{F}, \Sigma^*, L) = \Psi_{\mathcal{F}_\Sigma}\big(i\big(\overline{\iota_v}(\Sigma^*)\big)\big) = \Psi_{\mathcal{F}_\Sigma}\big(i\big(\overline{\iota_v}(\Sigma)^{-1}\big)\big) = \big(\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)\big)^{-1}.$$

□

3.5 Combinatorial definition of 2D holonomy along 2-disks and 2-spheres

We now prepare a combinatorial description of the 2D holonomy of a fake-flat 2-gauge configuration along cellularly embedded 2-disks and 2-spheres. Some algebraic topology preliminaries are yet still needed.

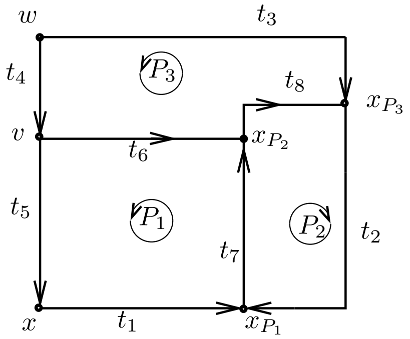
3.5.1 Algebraic topology preliminaries for the 2-disk case

Let Σ be an oriented manifold homeomorphic to the 2-disk $D^2 = [0,1]^2$. Hence we have an orientation of $\mathrm{bd}(\Sigma) \cong S^1$ as well. Let $L = (L^0, L^1, L^2)$ be a 2-lattice decomposition (Def. 32) of $(\Sigma, \mathrm{bd}(\Sigma)) \cong (D^2, S^1)$. Choose $v \in \mathrm{bd}(\Sigma)$, to be a 0-cell of L. It will look more or less like the pattern in Fig. 5. (Here and in other diagrams later, we put oriented circles inside the plaquettes in order to indicate the orientation of their attaching maps; this is redundant as orientations can be inferred from the form of their granular boundary.)

$\gamma_1 = t_5 t_1$

$\gamma_2 = t_6$

$\gamma_3 = t_6 t_8$



$\partial_L^Q(P_3) = t_3^{-1} t_4 t_6 t_8$

$\partial_L^Q(P_1) = t_7 t_6^{-1} t_5 t_1$

$\partial_L^Q(P_2) = t_8 t_2 t_7$

$\partial_v^Q(\Sigma, L) = t_5 t_1 t_2^{-1} t_3^{-1} t_4$

$$\partial_v^Q(\Sigma, L) \cong \big(\gamma_1\, \partial_L^Q(P_1)\, \gamma_1^{-1}\big)\ \big(\gamma_2\, \partial_L^Q(P_2)^{-1}\, \gamma_2^{-1}\big)\ \big(\gamma_3\, \partial_L^Q(P_3)\, \gamma_3^{-1}\big)$$
$$\iota_v(\Sigma, L) = \big([\gamma_1] \triangleright \iota_L(P_1)\big)\, \big([\gamma_2] \triangleright \iota_L(P_2)^{-1}\big)\, \big([\gamma_3] \triangleright \iota_L(P_3)\big)$$

Figure 5: A 2-lattice decomposition of $(\Sigma, \mathrm{bd}(\Sigma)) \cong (D^2, S^1)$. As shown, the attaching maps of the plaquettes P_1 and P_3 are oriented counterclockwise, whereas P_2 attaches clockwise. The base point of P_i is x_{P_i}. The granular boundaries $\partial_L^Q(P_i)$ of the plaquettes $P_i, i = 1,2,3$ are also shown; see Def. 48. The remaining information in the figure will be explained in Def. 65 and Ex. 68.

Recall that the definition of $\partial_v(\Sigma, L)$ and $\iota_v(\Sigma, L)$, which are given in Def. 55.

Remark 64. The homotopy exact sequence of the pointed pair (Σ, Σ^1, v) gives an exact sequence:

$$\{0\} \cong \pi_2(\Sigma, v) \to \pi_2(\Sigma, \Sigma^1, v) \xrightarrow{\partial} \pi_1(\Sigma^1, v) \to \pi_1(\Sigma, v) \cong \{1\}. \tag{15}$$

(see [40, pp 344]). Therefore $\partial\colon \pi_2(\Sigma, \Sigma^1, v) \to \pi_1(\Sigma^1, v)$ is an isomorphism. Hence, if $e \in \pi_2(\Sigma, \Sigma^1, v)$:

$$\partial(e) = \partial_v(\Sigma, L) \iff e = \iota_v(\Sigma, L). \tag{16}$$

Definition 65 (Granular boundary $\partial_v^Q(\Sigma, L)$ of a 2-disk Σ)**.** Choose a base-point $v \in \mathrm{bd}(\Sigma)$, to be a 0-cell. Now go around $\mathrm{bd}(\Sigma)$, following its orientation, starting at v until you go back to v. Along the way we pass by the geometric 1-cells $t_1, t_2, \dots, t_n$ of $\mathrm{bd}(\Sigma)$, in that order. Put $\theta_i = 1$ if the characteristic map $\phi^1_{t_i}\colon [0,1] \to \mathrm{bd}(\Sigma)$ of t_i is oriented positively, and $\theta_i = -1$ otherwise. Cf. Fig 5, the granular boundary $\partial_v^Q(\Sigma, L)$ of Σ is the following granular path (cf. Defs. 35, 38) in $\mathrm{bd}(\Sigma)$, from v to v:

$$\partial_v^Q(D^2, L) = t_1^{\theta_1} \dots t_n^{\theta_n}.$$

By passing to morphisms in $\pi_1(\Sigma^1, \Sigma^0) \cong \mathsf{FG}\langle \Sigma^0, \Sigma^1 \rangle$, we hence have $[\partial_v^Q(\Sigma, L)] = \partial_v(\Sigma, L)$; Def. 35, 55.

If we allow the cancellation of consecutive pairs of a 1-cell and its inverse, thus considering granular paths up to equivalence (Def. 35,) we can express – however not uniquely – $[\partial_v^Q(\Sigma, L)]$ as a product of granular boundaries (cf. Def. 48 and Thm. 39) of plaquettes (or their inverses), each of which is in addition conjugated by a (possibly trivial) granular path in Σ^1, connecting the base-point v of $\mathrm{bd}(\Sigma)$ with the base-point of each plaquette. More precisely:

Lemma 66. *Let L be a 2-lattice decomposition of $(\Sigma, \mathrm{bd}(\Sigma))$. Choose a base point $v \in \mathrm{bd}(\Sigma)$, the boundary of Σ, to be a 0-cell. There exists a positive integer N, plaquettes $P_i, i = 1, \ldots, N$ in L^2 (plaquettes can be repeated), integers $\theta_i \in \{\pm 1\}$ (where $i = 1, \ldots, N$), as well as granular paths $\gamma_1, \ldots, \gamma_N$ in Σ^1, connecting v to the base point of each plaquette P_i, such that the following equivalence between granular paths holds:*

$$\partial_v^Q(\Sigma, L) \;\cong\; \Big(\gamma_1 \partial_L^Q(P_1)^{\theta_1}\gamma_1^{-1}\Big)\;\Big(\gamma_2 \partial_L^Q(P_2)^{\theta_2}\gamma_2^{-1}\Big)\ldots\Big(\gamma_N \partial_L^Q(P_N)^{\theta_N}\gamma_N^{-1}\Big). \tag{17}$$

Here $\cong$ is the equivalence relation on granular paths in Def. 35. Hence we can pass from the left-hand-side of (17) to the right-hand-side by inserting or removing pairs $t^{\pm 1}t^{\mp 1}$ where t is a 1-cell of Σ.

Remark 67. Note that, by Rem. 64, equation (17) holds if, and only if, in $\pi_2(\Sigma, \Sigma^1, \Sigma^0)$ (cf. Def. 55, 31):

$$\iota_v(\Sigma, L) = ([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1}([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2}\ldots([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N}.$$

This is because by (16), equation above holds if, and only if, in $\pi_1(\Sigma^1, \Sigma^0)$ we have:

$$\begin{aligned}\partial\big([\gamma_1] \triangleright \iota_L(P_1)^{\theta_1}\;[\gamma_2] \triangleright \iota_L(P_2)^{\theta_2}\;\ldots\;[\gamma_N] \triangleright \iota_L(P_N)^{\theta_N}\big)\qquad&\\ = \big([\gamma_1]\partial_L(P_1)^{\theta_1}[\gamma_1^{-1}]\big)\;\big([\gamma_2]\partial_L(P_2)^{\theta_2}[\gamma_2^{-1}]\big)\;\ldots\big([\gamma_N]\partial_L(P_N)^{\theta_N}[\gamma_N^{-1}]\big) = \partial_v(\Sigma, L) = \partial\big(\iota_v(\Sigma, L)\big).&\end{aligned}$$

Example 68. In Fig. 5, we can put $N = 3$, $\gamma_1 = t_5t_1$, $\gamma_2 = t_6$ and $\gamma_3 = t_6t_8$. Also put $\theta_1 = 1$, $\theta_2 = -1$ and $\theta_3 = 1$. Then, as indicated in Fig. 5:

$$\partial_v^Q(\Sigma, L) \cong \big(\gamma_1 \partial_L^Q(P_1)^{\theta_1}\gamma_1^{-1}\big)\;\big(\gamma_2 \partial_L^Q(P_2)^{\theta_2}\gamma_2^{-1}\big)\;\big(\gamma_3 \partial_L^Q(P_3)^{\theta_3}\gamma_3^{-1}\big). \tag{18}$$

This follows from a simple calculation, which we recommend the reader to do. From (18) it follows that:

$$\iota_v(\Sigma, L) = \big([\gamma_1] \triangleright \iota_L(P_1)^{\theta_1}\big)\;\big(\gamma_2 \triangleright \iota_L(P_2)^{\theta_2}\gamma_2^{-1}\big)\;\big(\gamma_3\iota_L(P_3)^{\theta_3}\big).$$

Proof. **(Lemma 66)** Consider the fundamental crossed modules $\Pi_2(\Sigma, \Sigma^1, v) \subset \Pi_2(\Sigma, \Sigma^1, \Sigma^0)$, and the elements $\iota_v(\Sigma, L)$ and $\partial_v(\Sigma, L)$ of $\Pi_2(\Sigma, \Sigma^1, \Sigma^0)$; see Rem. 55. Recall $\partial\big(\iota_v(\Sigma, L)\big) = \partial_v(\Sigma, L)$ in $\pi_1(\Sigma^1, \Sigma^0)$.

Cf. Def. 30, Rem. 31 and Thm. 52. We know that $\Pi_2(\Sigma, \Sigma^1, \Sigma^0)$ is a free crossed module on the map $\partial_L \colon L^2 \to \pi_1(\Sigma^1, \Sigma^0)$, where $\pi_1(\Sigma^1, \Sigma^0)$ is the free groupoid on the 1-cells; see Thm. 39. Let us apply Rem. 53. Hence there exists a positive integer N, such that we can choose plaquettes $P_i, i = 1, 2, \ldots, N$, granular paths $\gamma_i, i = 1, 2, \ldots, N$, from the base-point v of Σ to the base point x_{P_i} of P_i, and integers $\theta_i \in \{\pm 1\}, i = 1, 2, \ldots, N$, such that, in $\pi_2(\Sigma, \Sigma^1, \Sigma^0)$:

$$\iota_v(\Sigma, L) = ([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1}([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2}\ldots([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N}.$$

By using the first Peiffer Law in Def. 2, and Rem. 30 and 31, it follows that in $\pi_1(\Sigma^1, \Sigma^0) \cong \mathsf{FG}\langle L^0, L^1\rangle$, and where [ ] means equivalence class of granular paths (Def. 35):

$$\begin{aligned}[\partial_v^Q(\Sigma, L)] &= \partial_v(\Sigma, L)\\ &= \partial\big(\iota_v(\Sigma, L)\big)\\ &= \partial\big(([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1}\;([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2}\;\ldots\;([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N}\big)\\ &= [\gamma_1]\partial(\iota_L(P_1))^{\theta_1}[\gamma_1^{-1}]\;[\gamma_2]\partial(\iota_L(P_2))^{\theta_2}[\gamma_2^{-1}]\ldots[\gamma_N]\partial(\iota_L(P_N))^{\theta_N}[\gamma_N^{-1}]\\ &= [\gamma_1]\partial_L(P_1)^{\theta_1}[\gamma_1^{-1}]\;[\gamma_2]\partial_L(P_2)^{\theta_2}[\gamma_2^{-1}]\;\ldots\;[\gamma_N]\partial_L(P_N)^{\theta_N}[\gamma_N^{-1}]\\ &= \big[\gamma_1\,\partial_L^Q(P_1)^{\theta_1}\,\gamma_1^{-1}\;\gamma_2\,\partial_L^Q(P_2)^{\theta_2}\,\gamma_2^{-1}\ldots\gamma_N\,\partial_L^Q(P_N)^{\theta_N}\,\gamma_N^{-1}\big].\end{aligned}$$

Hence we can go from $\partial_v^Q(\Sigma, L)$ to the granular path $\gamma_1\partial_L^Q(P_1)^{\theta_1}\gamma_1^{-1}\;\gamma_2\partial_L^Q(P_2)^{\theta_2}\gamma_2^{-1}\ldots\gamma_N\partial_L^Q(P_N)^{\theta_N}\gamma_N^{-1}$ in a finite number of steps by inserting, or removing pairs $t^{\pm 1}t^{\mp 1}$, where t is any 1-cell of Σ. ☐

Remark 69. The choice of a positive integer N and of an assignment:

$$\begin{aligned}&i \mapsto P_i, \text{ a plaquette,}\\ &i \mapsto \gamma_i, \text{ a granular path from } v \text{ to the base point } x_{P_i} \text{ of the plaquette } P_i,\\ &i \mapsto \theta_i \in \{\pm 1\},\end{aligned} \tag{19}$$

where $i \in \{1, \ldots, N\}$, such that we have an equivalence of granular paths:

$$\partial_v^Q(\Sigma, L) \cong \left(\gamma_1 \partial_L^Q(P_1)^{\theta_1} \gamma_1^{-1}\right) \left(\gamma_2 \partial_L^Q(P_2)^{\theta_2} \gamma_2^{-1}\right) \ldots \left(\gamma_N \partial_L^Q(P_N)^{\theta_N} \gamma_N^{-1}\right) \tag{20}$$

– equivalently (cf. Rem. 64) such that, in $\pi_2(\Sigma, \Sigma^1, v)$:

$$\iota_v(\Sigma, L) = ([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1} ([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2} \ldots ([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N} \tag{21}$$

or equivalently such that, in $\pi_1(\Sigma^1, v)$:

$$\partial_v(\Sigma, L) = \left([\gamma_1] \partial_L(P_1)^{\theta_1} [\gamma_1^{-1}]\right) \left([\gamma_2] \partial_L(P_2)^{\theta_2} [\gamma_2^{-1}]\right) \ldots \left([\gamma_N] \partial_L(P_N)^{\theta_N} [\gamma_N^{-1}]\right), \tag{22}$$

– is far from being unique.

3.5.2 A combinatorial description of the 2D holonomy along embedded 2-disks

Cf. [60]. Let (M, L) be a 2-lattice. Suppose that Σ is homeomorphic to the 2-disk D^2 and that L is a decomposition of the triple $(M, \Sigma, \mathrm{bd}(\Sigma))$; Def. 32. Fix an orientation on Σ. Choose a base-point $v \in \mathrm{bd}(\Sigma) \cap M^0$. Consider a fake-flat 2-gauge configuration $\mathcal{F} = (\mathcal{F}^2, \mathcal{F}^1)$ in (M, L). Let L_Σ be the induced 2-lattice decomposition of $(\Sigma, \mathrm{bd}(\Sigma)) \cong (D^2, S^1)$. Let $\partial_v^Q(\Sigma, L)$ be the granular boundary of Σ; Def. 65.

Cf. §3.5.1, choose a positive integer N and plaquettes $P_i \in L^2_\Sigma, i = 1, \ldots, N$ (plaquettes might be repeated), integers $\theta_i \in \{\pm 1\}$ (where $i = 1, \ldots, N$), and granular paths $\gamma_1, \ldots, \gamma_N$ in Σ^1, from v to the base point x_{P_i} of P_i, such that we have an equivalence of granular paths (cf. Def. 35, 38, 48): (*)

$$\partial_v^Q(\Sigma, L) \cong \left(\gamma_1 \partial_L^Q(P_1)^{\theta_1} \gamma_1^{-1}\right) \left(\gamma_2 \partial_L^Q(P_2)^{\theta_2} \gamma_2^{-1}\right) \ldots \left(\gamma_N \partial_L^Q(P_N)^{\theta_N} \gamma_N^{-1}\right). \tag{23}$$

Recall that by Rem. 64 and 69, equation (23) is the same as saying that in $\pi_2(\Sigma, \Sigma^1, \Sigma^0)$ (cf. Rem. 55):

$$\iota_L(D^2, L) = ([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1} ([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2} \ldots ([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N}.$$

Fix a crossed module $\mathcal{G} = (\partial_\mathcal{G} \colon E \to G, \triangleright)$. Recall the construction of the 2D holonomy $\mathrm{Hol}_v(\mathcal{F}, \Sigma, L)$ of a fake-flat 2-gauge configuration (cf. Def. 47) along $\Sigma \cong D^2$, with initial point v; Def. 57.

Theorem 70. *Suppose that L is a 2-lattice decomposition of $(M, \Sigma, \mathrm{bd}(\Sigma))$, where $\Sigma \cong D^2$ is a surface cellularly embedded in M. Let $v \in \mathrm{bd}(\Sigma) \cap M^0$. Let $i \in \{1, \ldots, N\} \mapsto (P_i, \theta_i, \gamma_i)$ be as in (*). If*

$$\mathcal{F} = (\mathcal{F}^1 \colon t \in L^1 \to g_t \in G\ ,\ \mathcal{F}^2 \colon P \in L^2 \mapsto e_P \in E)$$

is a fake-flat 2-gauge configuration on (M, L), then $\mathrm{Hol}_v(\mathcal{F}, \Sigma, L) \in E \times G$ can be calculated as:

$$\mathrm{Hol}_v(\mathcal{F}, \Sigma, L) = \left(\mathrm{Hol}_v^2(\mathcal{F}, \Sigma, L), \mathrm{Hol}_v^1(\mathcal{F}, \Sigma, L)\right) = \left(g_{\gamma_1} \triangleright e_{P_1}^{\theta_1}\ g_{\gamma_2} \triangleright e_{P_2}^{\theta_2}\ \ldots\ g_{\gamma_N} \triangleright e_{P_N}^{\theta_N},\ g_{\partial_v^Q(\Sigma, L)}\right). \tag{24}$$

Cf. Def. 42, here g_{γ_i} is the product of the elements of G assigned to the 1-cells of the granular path γ_i (or their inverses), and the same for $g_{\partial_v^Q(\Sigma,L)}$. In other words $g_{\gamma_i} = \Phi_{\mathcal{F}^1}([\gamma_i])$ and $g_{\partial_v^Q(\Sigma,L)} = \Phi_{\mathcal{F}^1}([\partial_v^Q(\Sigma, L)])$.

As an immediate consequence, we have the promised independence theorem of the 2D holonomy of a fake-flat 2-gauge configuration along a pointed 2-disk on the way we combine the group elements associated to the edges and plaquettes, as long as the rules of the assignment (*) are followed.

Theorem 71. *Fix $v \in \mathrm{bd}(\Sigma) \cap M^0$. The evaluation $\mathrm{Hol}_v(\mathcal{F}, \Sigma, L)$ in (24) does not depend on the assignment $i \mapsto (P_i, \theta_i, \gamma_i)$ as in (*) chosen; see Rem. 69. Moreover (13) holds, i.e. $\partial_\mathcal{G}\left(\mathrm{Hol}_v^2(\mathcal{F}, \Sigma, L)\right) = \mathrm{Hol}_v^1(\mathcal{F}, \Sigma, L)$.*

Example 72. Let us consider a fake-flat 2-gauge configuration on the 2-lattice decomposition of D^2 in Fig. 5. In the figure below, we put $g_i = g_{t_i} = \mathcal{F}^1(t_i) \in G$ and $e_i = e_{P_i} = \mathcal{F}^2(P_i) \in E$. For (*) to hold, we can put (see Ex. 68): $\gamma_1 = t_5 t_1$, $\gamma_2 = t_6$ and $\gamma_3 = t_6 t_8$; and $\theta_1 = 1$, $\theta_2 = -1$ and $\theta_3 = 1$. Hence:

$$\mathrm{Hol}_v(\mathcal{F}, D^2, L) = \left(\mathrm{Hol}_v^2(\mathcal{F}, D^2, L), \mathrm{Hol}_v^1(\mathcal{F}, D^2, L)\right) = \left(g_{\gamma_1} \triangleright e_1\ g_{\gamma_2} \triangleright e_2^{-1}\ g_{\gamma_3} \triangleright e_3\ ,\ g_5 g_1 g_2^{-1} g_3^{-1} g_4\right).$$

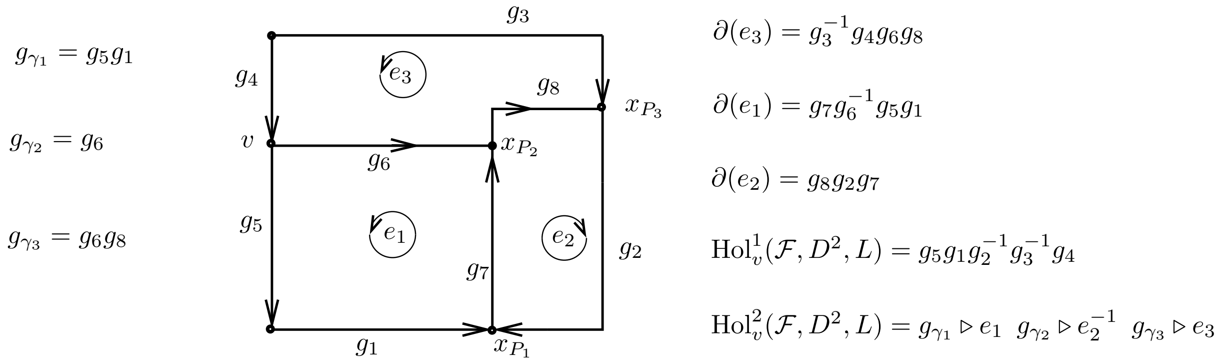


Figure6: A fake-flat configuration $\mathcal{F}$ on (D^2, L) and its 2-dimensional holonomy.

Proof. **(Theorem 70)** We use Rem. 67, 64. Given an assignment $i \in \{1, \ldots, N\} \mapsto (P_i, \gamma_i, \theta_i)$ as in $(*)$, then

$$[\gamma_1] \triangleright \iota_L(P_1)^{\theta_1} \ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2} \ldots [\gamma_N] \triangleright \iota_L(P_N)^{\theta_N} = \iota_v(\Sigma, L),$$

where this holds in $\pi_2(\Sigma, \Sigma^1, \Sigma^0)$, and, now in $\pi_1(\Sigma^1, \Sigma^0)$:

$$\partial\big([\gamma_1] \triangleright \iota_L(P_1)^{\theta_1} \ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2} \ldots [\gamma_N] \triangleright \iota_L(P_N)^{\theta_N}\big) =$$
$$= \big([\gamma_1]\partial_L(P_1)^{\theta_1}[\gamma_1^{-1}]\big) \ \big([\gamma_2]\partial_L(P_2)^{\theta_2}[\gamma_2^{-1}]\big) \ \ldots \big([\gamma_N]\partial_L(P_N)^{\theta_N}[\gamma_N^{-1}]\big) = \partial_v(\Sigma, L).$$

The restriction $\mathcal{F}_\Sigma$ of $\mathcal{F}$ to Σ gives a crossed module map $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma}) \colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$. Thus:

$$\begin{aligned}
\mathrm{Hol}_v^2(\mathcal{F}, \Sigma, L) &\doteq \Psi_{\mathcal{F}_\Sigma}(\iota_v(\Sigma, L)) \\
&= \Psi_{\mathcal{F}_\Sigma}\Big([\gamma_1] \triangleright \iota_L(P_1)^{\theta_1} \ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2} \ldots [\gamma_N] \triangleright \iota_L(P_N)^{\theta_N}\Big) \\
&= \Phi_{\mathcal{F}_\Sigma}([\gamma_1]) \triangleright \Psi_{\mathcal{F}_\Sigma}(\iota_L(P_1)^{\theta_1}) \ \Phi_{\mathcal{F}_\Sigma}([\gamma_2]) \triangleright \Psi_{\mathcal{F}_\Sigma}(\iota_L(P_2)^{\theta_2}) \ldots \Phi_{\mathcal{F}_\Sigma}([\gamma_N]) \triangleright \Psi_{\mathcal{F}_\Sigma}(\iota_L(P_N)^{\theta_N}) \\
&\doteq g_{\gamma_1} \triangleright e_{P_1}^{\theta_1} \ g_{\gamma_2} \triangleright e_{P_2}^{\theta_2} \ \ldots \ g_{\gamma_N} \triangleright e_{P_N}^{\theta_N},
\end{aligned}$$

where, N.B., firstly $\triangleright$ is in $\Pi_2(\Sigma, \Sigma^1, \Sigma^0)$ and then it is in $\mathcal{G}$. Analogously:

$$\mathrm{Hol}_v^1(\mathcal{F}, \Sigma, L) = \Psi_{\mathcal{F}_\Sigma}(\partial_v(\Sigma, L)) = g_{\partial_v^Q(\Sigma, L)}.$$

□

3.5.3 Algebraic topology preliminaries for the 2-sphere case

Let (Σ, L) be a 2-lattice. Suppose that Σ is an oriented surface homeomorphic to the 2-sphere. Choose a base-point $v \in \Sigma$, to be a 0-cell. Recall that the homotopy long exact sequence of (Σ, Σ^1) yields:

$$\pi_2(\Sigma^1, v) \cong \{0\} \to \pi_2(\Sigma, v) \cong \mathbb{Z} \xrightarrow{i} \pi_2(\Sigma, \Sigma^1, v) \xrightarrow{\partial} \pi_1(\Sigma^1, v) \to \pi_1(\Sigma, v) \cong \{1\} \tag{25}$$

(which is exact), and hence we have an isomorphism $i \colon \pi_2(\Sigma, v) \cong \mathbb{Z} \to \ker(\partial) \subset \pi_2(\Sigma, \Sigma^1, v)$.

Cf. Def. 59, 61, Rem. 31 and the construction in §3.5.2. In order to find a combinatorial definition of $\mathrm{Hol}_v^2(\mathcal{F}, \Sigma, L)$, we must express $i(\overline{\iota_v}(\Sigma)) \in \pi_2(\Sigma, \Sigma^1, v)$ (see Def. 59) as a product of terms like $[\gamma] \triangleright \iota_L(P)$, where γ is a granular path from v to the base point x_P of the plaquette P, and $[\gamma]$ is the element in $\pi_1(\Sigma^1, \Sigma^0)$ it yields; Def. 38 and 35. A crucial point in §3.5.1 is that the kernel of the boundary map $\partial \colon \pi_2(N, N^1, v) \to \pi_1(N^1, v)$ is trivial if $N \cong D^2$ (with any CW-decomposition), whereas if $\Sigma \cong S^2$ we have $\ker(\partial) = \pi_2(\Sigma, v) \cong \mathbb{Z}$; see (25). In order to identify $i(\overline{\iota_v}(\Sigma)) \in \pi_2(\Sigma, \Sigma^1, v)$, we will need to use the Hurewicz map between homotopy and homology long exact sequences of pairs; see [40, pp 374]. Our main tool is the fact that the Hurewicz map $h \colon \pi_2(\Sigma, v) \to H_2(\Sigma)$ is an isomorphism, if $\Sigma \cong S^2$.

Before continuing, let us define, given a plaquette $P \in L^2$, with characteristic map $\phi_P^2 \colon D^2 \to \overline{c_P^2} \subset \Sigma$:

$$\mathrm{sgn}(P) = \begin{cases} 1, & \text{if the restriction of } \phi_P^2 \colon D^2 \to \Sigma \text{ to } \mathrm{int}(D^2) \text{ is orientation preserving;} \\ -1, & \text{if the restriction of } \phi_P^2 \colon D^2 \to \Sigma \text{ to } \mathrm{int}(D^2) \text{ is orientation reversing.} \end{cases} \tag{26}$$

Lemma 73. *There exist a positive integer N, and an assignment $i \in \{1,\dots,N\} \to (P_i,\gamma_i,\theta_i)$, where $P_i \in L^2$, γ_i is a granular path from v to the base point x_{P_i} of the plaquette P_i, and $\theta_i \in \{\pm 1\}$, such that:*

1. *we have an equivalence of granular paths (Def. 38 and 35):*
$$\emptyset_v \cong \left(\gamma_1 \partial_L^Q(P_1)^{\theta_1}\gamma_1^{-1}\right)\ \left(\gamma_2 \partial_L^Q(P_2)^{\theta_2}\gamma_2^{-1}\right)\dots\left(\gamma_N \partial_L^Q(P_N)^{\theta_N}\gamma_N^{-1}\right),$$
2. *given any $P \in L^2$,* $$\sum_{i\in\{1,\dots,N\} \textit{ such that } P_i=P} \theta_i = \operatorname{sgn}(P).$$

Moreover, given a positive integer N and an assignment $i \in \{1,\dots,N\} \to (P_i,\gamma_i,\theta_i)$, where $P_i \in L^2$, γ_i is a granular path from v to the base point x_{P_i} of the plaquette P_i, and $\theta_i \in \{\pm 1\}$, then:
$$[\gamma_1] \triangleright \iota_L(P_1)^{\theta_1}\ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2}\dots[\gamma_N] \triangleright \iota_L(P_N)^{\theta_N} = i(\overline{\iota_v}(\Sigma)),$$
(cf. Def. 59 and Rem. 31) happens if, and only if, conditions 1. and 2. of the lemma are satisfied.

Proof. Cf. Rem. 30, Thm. 52 and Rem. 53. Since $\Pi_2(\Sigma,\Sigma^1,\Sigma^0)$ is the free crossed module on $\partial_L \colon L^2 \to \pi_1(\Sigma^1,\Sigma^0)$, there exist a positive integer N, and an assignment $i \in \{1,\dots,N\} \mapsto (P_i,\gamma_i,\theta_i)$, where $P_i \in L^2$, γ_i is a granular path from v to the base point x_{P_i} of the plaquette P_i, and $\theta_i \in \{\pm 1\}$, such that:
$$([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1}([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2}\dots([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N} = i\big(\overline{\iota_v}(\Sigma,L)\big).$$
We claim that $i \in \{1,\dots,N\} \mapsto (P_i,\gamma_i,\theta_i)$ satisfies items 1 and 2, of the statement of the lemma.
Item 1. Since $\partial(i\big(\overline{\iota_v}(\Sigma,L)\big)) = \emptyset_v$, combining with:
$$\begin{aligned}\partial\big(([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1}([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2}\dots([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N}\big)&\\ = [\gamma_1]\partial_L(P_1)^{\theta_1}[\gamma_1]^{-1}\ [\gamma_2]\partial_L(P_2))^{\theta_2}[\gamma_2]^{-1}\dots[\gamma_N]\partial_L(P_N))^{\theta_N}[\gamma_N]^{-1},&\end{aligned}$$
yields that $i \in \{1,\dots,N\} \mapsto (P_i,\gamma_i,\theta_i)$ satisfies item 1.
Item 2. Consider the map of exact sequences obtained from the Hurewicz map between homotopy and homology long exact sequences, [40, pp 374]:
$$\begin{array}{ccccccccc}\pi_2(\Sigma^1,v)\cong\{0\} & \longrightarrow & \mathbb{Z}\cong\pi_2(\Sigma,v) & \overset{i}{\hookrightarrow} & \pi_2(\Sigma,\Sigma^1,v) & \overset{\partial}{\longrightarrow} & \pi_1(\Sigma^1,v) & \overset{p}{\longrightarrow} & \pi_1(\Sigma,v)\cong\{1\}\\ \downarrow\cong & & \cong\downarrow h & & \downarrow h_r & & \downarrow h & & \downarrow\cong\\ H_2(\Sigma^1)\cong\{0\} & \longrightarrow & \mathbb{Z}\cong H_2(\Sigma) & \overset{i}{\hookrightarrow} & H_2(\Sigma,\Sigma^1) & \overset{\partial}{\longrightarrow} & H_1(\Sigma^1) & \overset{p}{\longrightarrow} & H_1(\Sigma)\cong\{0\}\end{array} \qquad (27)$$
The group $H_2(\Sigma,\Sigma^1)$ is the free abelian group on the relative homology classes $a(P) = h_r(\iota_L(P))$ determined by the plaquettes $P \in L^2$; see [40, pp 137]. Moreover $h_r(\gamma \triangleright \iota_L(P)) = a(P)$, for each plaquette P and each path $\gamma \in \pi_1(\Sigma^1,\Sigma^0)$ connecting v to the base-point of P. We let $K = h(\overline{\iota_v}(\Sigma)) \in H_2(\Sigma)$. Then K is the positive generator of $H_2(\Sigma) \cong \mathbb{Z}$. We now need the following claim:
Claim $i(K) = \sum_{P\in L^2} \operatorname{sgn}(P)a(P) \in H_2(\Sigma,\Sigma^1)$.
Proof of the claim (sketch) This is seemingly well-known, however we could not find a proof anywhere. Since $H_2(\Sigma,\Sigma^1)$ is the free abelian group on the $a(P) = h_r(\iota_L(P))$, we know that there exist unique $\lambda_P \in \mathbb{Z}$, where $P \in L^2$, such that $i(K) = \sum_{P\in L^2} \lambda_P\ a(P)$. We need to prove that $\lambda_P = \operatorname{sgn}(P)$, for each $P \in L^2$.

Let $P \in L^2$. Let x be an interior point of open cell c_P^2 corresponding to P. We have a commutative diagram (28), where all morphisms are induced by inclusion. The vertical line p_x corresponds to the identity map $\mathbb{Z} \to \mathbb{Z}$, in the sense that it sends the positive generator $K \in H_2(\Sigma) \cong \mathbb{Z}$ to the positive generator K_x of $H_2(\Sigma,\Sigma\setminus\{x\}) \cong \mathbb{Z}$. (Note Σ is oriented, so it makes sense to speak about those positive generators.)
$$\begin{array}{ccc}\mathbb{Z}\cong H_2(\Sigma)=H_2(\Sigma,\emptyset) & \overset{i}{\hookrightarrow} & H_2(\Sigma,\Sigma^1)\\ \downarrow p_x & \swarrow p'_x & \\ \mathbb{Z}\cong H_2(\Sigma,\Sigma\setminus\{x\}) & & \end{array} \qquad (28)$$

Then $p'_x(a(P)) = \mathrm{sgn}(P)K_x$, by definition of sgn(P). Whereas if $Q \in L^2$ is another plaquette, then since the corresponding closed 2-cell $\overline{e^2_Q}$ is contained in $\Sigma \setminus \{x\}$ it holds $p'_x(a(Q)) = 0$. Hence $\mathrm{sgn}(P) = \lambda_P$. **QED.**

Having proven the claim, item 2 of the statement of the lemma now follows from the fact that:

$$\begin{aligned}\sum_{P\in L^2} \mathrm{sgn}(P)a(P) = i(K) &= i \circ h\big(\overline{\iota_v}(\Sigma)\big)\\ &= h_r \circ i(\overline{\iota_v}(\Sigma))\\ &= h_r\left(([\gamma_1] \triangleright \iota_L(P_1))^{\theta_1}([\gamma_2] \triangleright \iota_L(P_2))^{\theta_2} \dots ([\gamma_N] \triangleright \iota_L(P_N))^{\theta_N}\right)\\ &= \sum_{i=1}^{N} \theta_i a(P_i).\end{aligned}$$

We note that $H_2(\Sigma, \Sigma^1)$ is the free abelian group on the $a(P)$, where $P \in L^2$.

To finalise, let us be given an assignment $i \in \{1, \dots, N\} \mapsto (P_i, \gamma_i, \theta_i)$, where $P_i \in L^2$, γ_i is a granular path from v to x_{P_i}, and $\theta_i \in \{\pm 1\}$. Let $A = [\gamma_1] \triangleright \iota_L(P_1)^{\theta_1}\ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2} \dots [\gamma_m] \triangleright \iota_L(P_m)^{\theta_m} \in \pi_2(\Sigma, \Sigma^1, v) \in \pi_2(\Sigma, \Sigma^1, \Sigma^0)$. From (27) it follows:

$$\begin{aligned}A = i(\overline{\iota_v}(\Sigma)) &\Leftrightarrow \partial(A) = \emptyset_v \text{ and } h_r(A) = (i \circ h)(\overline{\iota_v}(\Sigma))\\ &\Leftrightarrow \text{conditions of item 1 and item 2 each are satisfied.}\end{aligned}$$

□

3.5.4 A combinatorial description of the 2D holonomy along embedded 2-spheres

Let Σ be an oriented S^2 embedded in a manifold M. Let L be a 2-lattice decomposition of (M, Σ). Let $v \in \Sigma \cap M^0$. Let $\mathcal{F} = (\mathcal{F}^1 \colon t \in L^1 \to g_t \in G\ ,\ \mathcal{F}^2 \colon P \in L^2 \mapsto e_P \in E)$ be a fake-flat 2-gauge configuration in (M, L). Recall the definition of the 2D holonomy of $\mathcal{F}$ along Σ as $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) = \Psi_{\mathcal{F}_\Sigma}\big(i(\overline{\iota_v}(\Sigma, L)\big) \in \ker(\partial) \subset E$; Def. 61.

Theorem 74. *Let L be a 2-lattice decomposition of (M, Σ). Let L_Σ be the induced 2-lattice decomposition of Σ. Let $\mathcal{F}$ be a fake-flat gauge 2-configuration in (M, L). Let $\mathcal{F}_\Sigma$ be its restriction to L_Σ. Recall Lem. 73.*

Find a positive integer N, and, for each $i \in \{1, \dots, N\}$, a plaquette $P_i \in L^2_\Sigma$ (plaquettes might be repeated), an integer $\theta_i \in \{\pm 1\}$, and a granular path γ_i, connecting v to the base point x_{P_i} of P_i, such that:

1. *we have an equivalence of granular paths (Def. 38 and 35):*

$$\emptyset_v \cong \left(\gamma_1 \partial^Q_L(P_1)^{\theta_1} \gamma_1^{-1}\right)\ \left(\gamma_2 \partial^Q_L(P_2)^{\theta_2} \gamma_2^{-1}\right) \dots \left(\gamma_N \partial^Q_L(P_N)^{\theta_N} \gamma_N^{-1}\right), \tag{**}$$

2. *given any $P \in L^2$,* $\displaystyle\sum_{i\in\{1,\dots,N\} \text{ such that } P_i = P} \theta_i = \mathrm{sgn}(P)$*; see equation (26) for notation.*

Then we have the following combinatorial formula for $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) \in \ker(\partial) \subset E$

$$\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) = g_{\gamma_1} \triangleright e^{\theta_1}_{P_1}\ g_{\gamma_2} \triangleright e^{\theta_2}_{P_2}\ \dots\ g_{\gamma_N} \triangleright e^{\theta_N}_{P_N}. \tag{29}$$

Here g_{γ_i} is the product of the elements of G assigned to the 1-cells of the granular path γ_i (or their inverses).

In particular, fixing $v \in \Sigma$, to be a 0-cell of L, the expression (29) for $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L)$ does not depend on the assignment $i \in \{1, \dots, N\} \mapsto (P_i, \gamma_i, \theta_i)$ as in (**) chosen. Moreover, Lem. 63 holds mutatis mutandis.

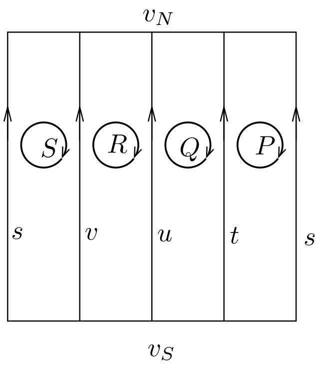
Proof. By Lem. 73, condition (**) is equivalent to: $[\gamma_1] \triangleright \iota_L(P_1)^{\theta_1}\ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2} \dots [\gamma_N] \triangleright \iota_L(P_N)^{\theta_N} = i\big(\overline{\iota_v}(\Sigma)\big)$. Hence:

$$\begin{aligned}\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) &= \Psi_{\mathcal{F}_\Sigma}\big(i(\overline{\iota_v}(\Sigma, L))\big) = \Psi_{\mathcal{F}_\Sigma}\Big([\gamma_1] \triangleright \iota_L(P_1)^{\theta_1}\ [\gamma_2] \triangleright \iota_L(P_2)^{\theta_2} \dots [\gamma_N] \triangleright \iota_L(P_N)^{\theta_N}\Big)\\ &= \Phi_{\mathcal{F}_\Sigma}([\gamma_1]) \triangleright \Psi_{\mathcal{F}_\Sigma}(\iota_L(P_1)^{\theta_1})\ \Phi_{\mathcal{F}_\Sigma}([\gamma_2]) \triangleright \Psi_{\mathcal{F}_\Sigma}(\iota_L(P_2)^{\theta_2}) \dots \Phi_{\mathcal{F}_\Sigma}([\gamma_N]) \triangleright \Psi_{\mathcal{F}_\Sigma}(\iota_L(P_N)^{\theta_N})\\ &\doteq g_{\gamma_1} \triangleright e^{\theta_1}_{P_1}\ g_{\gamma_2} \triangleright e^{\theta_2}_{P_2}\ \dots\ g_{\gamma_N} \triangleright e^{\theta_N}_{P_N},\end{aligned}$$

since $(\Psi_{\mathcal{F}_\Sigma}, \Phi_{\mathcal{F}_\Sigma}) \colon \Pi_2(\Sigma, \Sigma^1, \Sigma^0) \to \mathcal{G}$ is a crossed module map. □

Example 75. Consider the 2-lattice decomposition L_0 of the 2-sphere S^2 with a single 0-cell v and a single 2-cell P, whose characteristic map $\phi^2_P\colon D^2 \to S^2$ is positively oriented; cf. Ex. 29. The base point of P is v. A 2-gauge configuration $\mathcal{F}$ is simply an element $m_P \in E$, colouring its unique plaquette P. Fake-flatness imposes $m_P \in \ker(\partial)$. Let $\Sigma = S^2$, positively oriented. An assignment as in $(**)$ in Thm. 74 is such that $N = 1$, $P_1 = P$, $\gamma_1 = \emptyset_v$ and $\theta_1 = 1$. Hence $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) = m_P$, as it should.

Example 76. To facilitate drawing diagrams, let us now see the 2-sphere S^2 has being the square D^2, where we squash the upper edge and the lower edge to be single points (the north and south poles v_N and v_S), and we identify the left and right boundary edges. We give S^2 the reverse orientation to the one induced by $[0,1]^2$. Consider the 2-lattice decomposition L of the 2-sphere, with two zero cells, at v_N and v_S, and four one cells s, t, u, v, all connecting v_S to v_N. We have 2-cells P, Q, R, S, indicated in figure below. All plaquettes are based at the south pole. The characteristic map of each plaquette preserves orientation, so $\mathrm{sgn}(P), \mathrm{sgn}(Q), \mathrm{sgn}(R), \mathrm{sgn}(S) = 1$. The granular boundary of each plaquette is indicated in figure below.

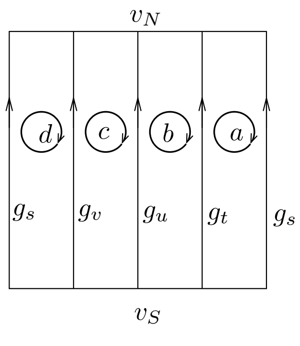


$$\partial^Q_L(P) = ts^{-1}$$
$$\partial^Q_L(Q) = ut^{-1}$$
$$\partial^Q_L(R) = vu^{-1}$$
$$\partial^Q_L(S) = sv^{-1}$$

$$\partial^Q_L(S)\ \partial^Q_L(R)\ \partial^Q_L(Q)\ \partial^Q_L(P) \cong \emptyset_{v_S}$$

Let $\Sigma = S^2$, with the same orientation as the ambient manifold S^2. Let $v = v_S$. We want to calculate the 2-dimensional hohonomy of a fake-flat gauge 2-configuration along Σ. An assignment $i \mapsto (P_i, \gamma_i, \theta_i)$ (for $N = 4$) satisfying $(**)$ in Thm. 74 can be $1 \mapsto (S, \emptyset_{v_s}, 1)$, $2 \mapsto (R, \emptyset_{v_s}, 1)$, $3 \mapsto (Q, \emptyset_{v_s}, 1)$ and $4 \mapsto (P, \emptyset_{v_s}, 1)$. Any cyclic permutation will also work.

A 2-gauge configuration $\mathcal{F}$ of the 2-sphere with this 2-lattice decomposition is given by elements $g_s, g_t, g_u, g_v \in G$, colouring the edges s, t, u, v, and elements $d, c, b, a \in E$ colouring the plaquettes S, R, Q, P, as indicated in the figure below, where the conditions for fake-flatness to hold are also made explicit.



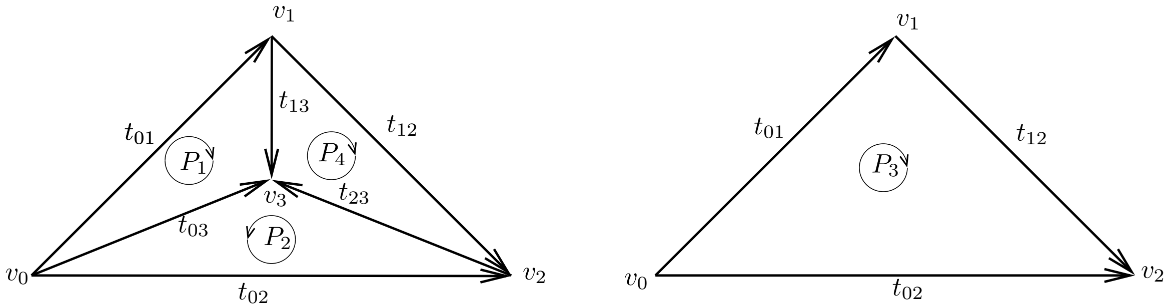
$$\begin{cases} \partial(a) = g_t g_v^{-1} \\ \partial(b) = g_u g_t^{-1} \\ \partial(c) = g_v g_u^{-1} \\ \partial(d) = g_s g_v^{-1} \end{cases}$$

$$\mathrm{Hol}^2_{v_S}(\mathcal{F}, D^2, L) = dcba$$

Hence $\mathrm{Hol}^2_{v_S}(\mathcal{F}, \Sigma, L) = dcba \in \ker(\partial) \subset E$.

By Thm. 74, cyclic permutations of $i \mapsto (P_i, \gamma_i, \theta_i)$ must yield the same value for $\mathrm{Hol}^2_{v_S}(\mathcal{F}, \Sigma, L)$, as (**) is still satisfied. The former can be directly proven: note $\partial(dcba) = 1_G$, thus $dcba$ is central, by the second Peiffer law of the definition of crossed modules (Def. 2). Hence $dcba = d^{-1}dcbad = cbad$.

Example 77. Consider the standard tetrahedron $T \subset \mathbb{R}^3$ displayed below. Hence the boundary Σ, of T, with the induced orientation, is given by the two triangles below identified along their boundaries. We give T a 2-lattice decomposition derived from the obvious triangulation of T. The granular boundary of each plaquette is indicated in the figure below. Note, P_1, P_2 and P_3 are based in v_0, whereas P_4 is based in v_1.

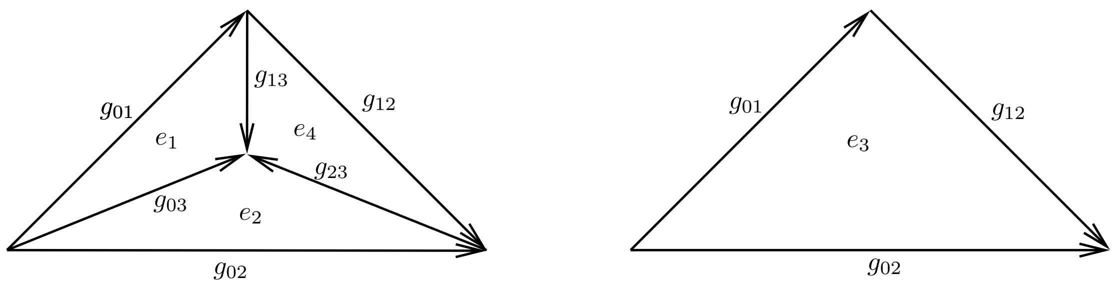


$$\partial_L^Q(P_1) = t_{01}t_{13}(t_{03})^{-1} \quad \partial_L^Q(P_4) = t_{12}t_{23}(t_{13})^{-1} \quad \partial_L^Q(P_2) = t_{02}t_{23}(t_{03})^{-1} \quad \partial_L^Q(P_3) = t_{01}t_{12}(t_{02})^{-1}$$

Let consider 2D holonomy along Σ based at v_0. An assignment satisfying (**) is such that $N = 4$, and:

$$1 \mapsto (P_1, \emptyset_{v_0}, 1), \qquad 2 \mapsto (P_2, \emptyset_{v_0}, -1),$$
$$3 \mapsto (P_3, \emptyset_{v_0}, -1), \qquad 4 \mapsto (P_4, t_{01}, 1).$$

The general form of a fake-flat 2-gauge configuration $\mathcal{F}$ of (T, L) is presented below:



$$\partial_{\mathcal{G}}(e_1) = g_{01}g_{13}(g_{03})^{-1} \quad \partial_{\mathcal{G}}(e_4) = g_{12}g_{23}(g_{13})^{-1} \quad \partial_{\mathcal{G}}(e_2) = g_{02}g_{23}(g_{03})^{-1} \quad \partial_{\mathcal{G}}(e_3) = g_{01}g_{12}(g_{02})^{-1}$$

Hence, by (29) it follows that: $\mathrm{Hol}^2_{v_0}(\mathcal{F}, \Sigma, L) = e_1 \; e_2^{-1} \; e_3^{-1} \; g_{01} \triangleright e_4$.

3.6 2-flat 2-gauge configurations

Let (M, L) be a 2-lattice. Let $b \in L^3$. The corresponding closed 3-cell (called a blob) is also denoted by $b = \overline{c_b^3}$. From the definition of 2-lattices (Def. 22), the attaching map $\psi_b^3 \colon S^2 \to M^2$ of b is an embedding and $\psi_b^3(S^2) = \mathrm{bd}(b) \cong S^2$ is a subcomplex of M^2, called the boundary of the blob b. Orient $\mathrm{bd}(b)$ by using $\psi_b^3 \colon S^2 \to \mathrm{bd}(b)$. Cf. Rem. 30, 31 and Def. 59, we have $\overline{\iota_v}\big(\mathrm{bd}(b)\big) = \partial_L(b)$, in $\pi_2(M^2, v) \subset \pi_2(M^2, M^1, v) \subset \pi_2(M^2, M^1, M^0)$, where $v = \psi_b^3(*)$ is the base-point of $b = \overline{c_b^3}$.

Definition 78 (2-flat 2-gauge configuration)**.** Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a crossed module. Consider a fake-flat 2-gauge configuration $\mathcal{F}$ based on (M, L). The boundary $\mathrm{bd}(b)$ of each blob $b \in L^3$ inherits a 2-lattice decomposition. The fake-flat 2-gauge configuration $\mathcal{F}$ is said to be 2-flat if for every blob b, we have:

$$\mathrm{Hol}^2_v(\mathcal{F}, \mathrm{bd}(b), L) = 1_E, \text{ where } 1_E \text{ is the identity of } E.$$

Recalling (Def. 47) that $\Theta(M, L, \mathcal{G})$ denotes the set of fake-flat 2-gauge configurations in (M, L), the set of 2-flat 2-gauge configurations is denoted $\Theta_{2\mathrm{flat}}(M, L, \mathcal{G})$.

More generally, a fake-flat 2-gauge configuration $\mathcal{F}$ is said to be 2-flat along a cellularly embedded 2-sphere $\Sigma \subset M$ if, for some $v \in \Sigma \cap M^0$, hence – by Lem. 63 – for all $v \in \Sigma \cap M^0$, it holds that $\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) = 1_E$.

Example 79. The fake-flat 2-gauge configuration Ω_1 from the end of §3.2.1 (the naive vacuum) is 2-flat.

Example 80. The fake-flat 2-gauge configurations in Ex. 77 is 2-flat if, and only if, $e_1 \; e_2^{-1} \; e_3^{-1} \; g_{01} \triangleright e_4 = 1_E$.

Let us provide an algebraic-topological interpretation of 2-flat 2-gauge configurations. Let $\mathcal{F}$ be a fake-flat 2-gauge configuration in (M, L). Cf. the construction of $\mathrm{Hol}^2_v(\mathcal{F}, \mathrm{bd}(b), L)$ in §3.4.2. Consider the discrete 2D parallel transport 2-functor $(\Psi_\mathcal{F}, \Phi_\mathcal{F})\colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ of $\mathcal{F}$; see Thm. 54. By the construction in §3.4.2 and §3.5.4, it holds that $\mathrm{Hol}^2_v(\mathcal{F}, \mathrm{bd}(b), L) = \Psi_\mathcal{F}(\partial_L(b))$. Recall that $\mathcal{F} \mapsto (\Psi_\mathcal{F}, \Phi_\mathcal{F})$ gives a one-to-one correspondence between fake-flat 2-gauge configurations and crossed module maps $\Pi_2(M^2, M^1, M^0) \to \mathcal{G}$.

The map of crossed modules induced by the inclusion $(M^2, M^1, M^0) \to (M^3, M^1, M^0)$ is denoted by $p_2\colon \Pi_2(M^2, M^1, M^0) \to \Pi_2(M^3, M^1, M^0)$. In components $p_2 = (p_2', \mathrm{id})$, where $p_2'\colon \pi_2(M^2, M^1, M^0) \to \pi_2(M^3, M^1, M^0)$ is a surjection and id is the identity $\pi_1(M^1, M^0) \to \pi_1(M^1, M^0)$.

Cf. Rem. 30. Given any 3-cell b, note that $p_2'(\partial_L(b)) = 1_{\pi_2(M^3, x_b)}$, the identity of $\pi_2(M^3, x_b)$, where x_b is the base-point of b. These are the only relations we need to impose in order to pass from $\pi_2(M^2, M^1, M^0)$ to $\pi_2(M^3, M^1, M^0) \cong \pi_2(M, M^1, M^0)$; what is meant by this is in Lem. 81. This follows from the long homotopy exact sequence of the triple (M^3, M^2, M^1), applied to each choice of base point $x \in M^0$, namely:

$$\pi_3(M^3, M^2, x) \to \pi_2(M^2, M^1, x) \xrightarrow{p_2'} \pi_2(M^3, M^1, x) \to \pi_2(M^3, M^2, x) \cong \{0\},$$

together with the fact that the group $\pi_3(M^3, M^2, x)$ is isomorphic to the free $\mathbb{Z}\big(\pi_1(M^1, x)\big)$-module on L^3; [40, Lemma 4.38].

Cf. the diagram below, a crossed module map $f\colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ is said to descend to $\Pi_2(M, M^1, M^0)$ if there exists a (necessarily unique) crossed module map $f^\flat\colon \Pi_2(M^3, M^1, M^0) \to \mathcal{G}$ such that $f^\flat \circ p_2 = f$.

$$\begin{array}{ccc} \Pi_2(M^2, M^1, M^0) & \xrightarrow{f} & \mathcal{G} \\ {\scriptstyle p_2}\big\downarrow & \nearrow{\scriptstyle f^\flat} & \\ \Pi_2(M^3, M^1, M^0) & & \end{array}$$

Lemma 81. *A crossed module map $f = (f_2, f_1)\colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ descends to $\Pi_2(M, M^1, M^0)$ if, and only if, for each blob $b \in L^3$ we have $f_2(\partial_L(b)) = 1_E$.*

Note that $\Pi_2(M, M^1, M^0) = \Pi_2(M^3, M^1, M^0)$, by the cellular approximation theorem.

Proof. As mentioned above, this follows from the long homotopy exact sequence of the triple (M^3, M^2, M^1), applied to each choice of base point $x \in M^0$; details can be found in [30], for the case of CW-complexes with a single base-point. Alternatively we can also use the higher-dimensional van Kampen theorem of Brown and Higgins; see [13, 14, 15, 11] and [18, §6], stating that (under mild conditions) the fundamental crossed module functor preserves colimits. Note that the conditions of 2-lattices (Def. 22) imply that for each $b \in L^3$, the corresponding closed 3-cell $\overline{c_b^3} = b$ is a subcomplex of M, homeomorphic to D^3. Moreover $b^2 = \mathrm{bd}(b)$, $b^1 = \mathrm{bd}(b)^1$ and $b^0 = \mathrm{bd}(b)^0$. From [11, §6.3], it follows that the diagram (30) below is a pushout diagram in the category of crossed modules of groupoids:

$$\begin{array}{ccc} \bigsqcup_{b \in L^3} \Pi_2(\mathrm{bd}(b), \mathrm{bd}(b)^1, \mathrm{bd}(b)^0) & \xrightarrow{p_1} & \bigsqcup_{b \in L^3} \Pi_2(b, b^1, b^0) \\ {\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\ \Pi_2(M^2, M^1, M^0) & \xrightarrow{p_2} & \Pi_2(M^3, M^1, M^0) \end{array} \qquad (30)$$

In the diagram (30) above, all arrows are induced by inclusions. Also, $\pi_1(b^1, b^0) = \pi_1(\mathrm{bd}(b)^1, \mathrm{bd}(b)^0)$. Given $x \in b^0 = \mathrm{bd}(b)^0$, we pass from $\pi_2(\mathrm{bd}(b), \mathrm{bd}(b)^1, x)$ to $\pi_2(b, b^1, x)$ by quotienting by the normal closure of $\overline{\iota_x}(\mathrm{bd}(b)) \in \pi_2(\mathrm{bd}(b), x) \subset \pi_2(\mathrm{bd}(b), \mathrm{bd}(b)^1, x)$; we are using the notation of Def 59. By inspecting (30), and applying the universal property of pushouts, it hence follows that a crossed module map $f\colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ descends to $\Pi_2(M, M^1, M^0)$ if, and only if, for each $b \in L^3$, and for each $x \in b^0$, it holds that $f_2(\overline{\iota_x}(\mathrm{bd}(b))) = 1_E$. by Lem. 63, in order for the latter to happen, for a blob b, it suffices to check that $f_2(\overline{\iota_v}(\mathrm{bd}(b))) = 1_E$, if v is the base-point of b, which is the same as saying that $f_2(\partial_L(b)) = 1_E$. ☐

Combining Lem. 81 with Thm. 54, yields the following interpretation of fake-flat 2-gauge configurations.

Theorem 82 (2-flat 2D parallel transport 2-functors)**.** *The bijection* $\mathcal{F} \in \Theta(M, L, \mathcal{G}) \mapsto (\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}})$ *of Thm. 54 restricts to a bijection between 2-flat configurations* $\mathcal{F} \in \Theta_{2\text{flat}}(M, L, \mathcal{G})$ *and crossed module maps* $\Pi_2(M, M^1, M^0) \to \mathcal{G}$*, from now on called* 2-flat 2D parallel transport 2-functors*. The correspondence sends* $\mathcal{F} \in \Theta_{2\text{flat}}(M, L, \mathcal{G})$ *to the crossed module map* $(\Psi^\flat_{\mathcal{F}}, \Phi^\flat_{\mathcal{F}})\colon \Pi_2(M, M^1, M^0) \to \mathcal{G}$ *that* $(\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}})$ *descends to.*

4 Gauge transformations

Throughout this section, we fix a crossed module $\mathcal{G} = (\partial_{\mathcal{G}}\colon E \to G, \triangleright)$ of groups §2.1 and a 2-lattice (M, L) §2.4. Recall that (L^0, L^1) has the structure of a directed graph $\sigma, \tau\colon L^1 \to L^0$, see §2.5. We denote the edges (1-cells) of L as $x \xrightarrow{t} y$, where $x = \sigma(t)$ and $y = \tau(t)$ are the source and target of t. (It may be that $x = y$.)

4.1 The group $\mathcal{T} = \mathcal{T}(M, L, \mathcal{G})$ of gauge operators

If G is a group and S is a set we put G^S to denote the group $\prod_{s\in S} G$, with pointwise multiplication.

Definition 83 (The group $\mathcal{T}(M, L, \mathcal{G})$ of gauge operators)**.** There is a left-action $\bullet$ of $\mathcal{V}(M, L, \mathcal{G}) = G^{L^0}$ on $\mathcal{E}(M, L, \mathcal{G}) = E^{L^1}$ by automorphisms. Given $\eta \in \mathcal{E}(M, L, \mathcal{G})$ and $u \in \mathcal{V}(M, L, \mathcal{G})$, the action $\bullet$ has the form:

$$(u \bullet \eta)\left(\sigma(t) \xrightarrow{t} \tau(t)\right) = u\big(\sigma(t)\big) \triangleright \left(\eta\left(\sigma(t) \xrightarrow{t} \tau(t)\right)\right), \tag{31}$$

for each $\sigma(t) \xrightarrow{t} \tau(t)$ in L^1. (Note that $\triangleright$ denotes the underlying action of G on E, which exists since $(\partial_{\mathcal{G}}\colon E \to G, \triangleright)$ is a crossed module; Def. 2.) We define the group $\mathcal{T} = \mathcal{T}(M, L, \mathcal{G})$ of gauge operators to be:

$$\mathcal{T}(M, L, \mathcal{G}) \;=\; \mathcal{E}(M, L, \mathcal{G}) \rtimes_\bullet \mathcal{V}(M, L, \mathcal{G}) \;=\; E^{L^1} \rtimes_\bullet G^{L^0}. \tag{32}$$

Here $\rtimes_\bullet$ denotes semidirect product. In particular we take:

$$(\eta, u)(\eta', u') \;=\; \big(\eta\, u \bullet \eta', u\, u'\big). \tag{33}$$

Example 84. Recall the 2-lattice decomposition L of S^2 from Fig. 2. We have a unique edge and a unique vertex. Hence $\mathcal{T}(S^2, L, \mathcal{G}) = E \rtimes_\triangleright G$. If we extend L to be a 2-lattice decomposition $L_{\mathfrak{g}}$ of S^3, by adding 3-cells (Ex. 29), it also holds that $\mathcal{T}(S^3, L_{\mathfrak{g}}, \mathcal{G}) = E \rtimes_\triangleright G$. This is because groups of gauge operators on 2-lattices depend only on 1-skeletons.

The group $\mathcal{T}(M, L, \mathcal{G})$ of gauge operators acts on the set $\Theta(M, L, \mathcal{G})$ of fake-flat 2-gauge configuration on (M, L) in a way such that the 2D holonomy is preserved; see §4.4, below. Moreover, this action restricts to an action of $\mathcal{T}(M, L, \mathcal{G})$ on the set $\Theta_{2\text{flat}}(M, L, \mathcal{G})$ of 2-flat configurations; Def. 78. In order to present the action, we now define a double groupoid $\mathcal{D}(\mathcal{G})$ out of the crossed module $\mathcal{G}$; see [18, §6.6], [13, 15] and [34].

4.2 The double groupoid $\mathcal{D}(\mathcal{G})$

Double groupoids (more precisely 'special double groupoids') [13, 21] form a category equivalent to the category of crossed modules, hence to the category of 2-groups. The languange of double groupoids is particularly useful for proving results concerning the fundamental crossed module of a pair of spaces, as shown in [18, Chapter 6] and [11, 13]. Double groupoids are also quite convenient for treating the 2-dimensional holonomy of a 2-bundle, in an differential geometric setting, and its behaviour under gauge transformations. This point of view was developed in [34]. Double groupoids can also be applied for formulating gauge transformations in discrete higher gauge theory, as we explain in this subsection.

The definition of a double groupoid appears e.g. in [18, §6.1], [11, §6.1], and [34, 65]. We explain the layers of structure inherent to a double groupoid as we elaborate how a crossed module $\mathcal{G} = (\partial_{\mathcal{G}}\colon E \to G, \triangleright)$ of groups gives rise to one, denoted by $\mathcal{D}(\mathcal{G})$; see [18, §6.6].

We have a unique object $*$, and sets $\mathcal{D}^1_H(\mathcal{G})$ and $\mathcal{D}^1_V(\mathcal{G})$ of horizontal and vertical 1-squares in $\mathcal{G}$; see [34]. These sets of horizontal and vertical 1-squares in $\mathcal{G}$ consist of diagrams of the form:

$$\left(* \xrightarrow{X} *\right) \quad \text{and} \quad \begin{pmatrix} * \\ Z\big\uparrow \\ * \end{pmatrix}, \text{ where } X, Z \in G.$$

Horizontal and vertical 1-squares in $\mathcal{G}$ are composed in the obvious way, here shown for horizontal 1-squares:

$$\big(* \xrightarrow{X} *\big) \circ \big(* \xrightarrow{Y} *\big) = \big(* \xrightarrow{XY} *\big), \text{ where } X, Y \in G.$$

We therefore have horizontal and vertical groupoids, also denoted $\mathcal{D}^1_H(\mathcal{G})$ and $\mathcal{D}^1_V(\mathcal{G})$, with a single object. Their sets of morphisms are in one-to-one correspondence with G. The groupoids $\mathcal{D}^1_H(\mathcal{G})$ and $\mathcal{D}^1_V(\mathcal{G})$ are isomorphic. An obvious isomorphism $\mathcal{D}^1_V(\mathcal{G}) \to \mathcal{D}^1_H(\mathcal{G})$ is obtained by clockwise rotation.

We have a set $\mathcal{D}^2(\mathcal{G})$ of squares in $\mathcal{G}$. This set consists of diagrams K of the form below:

$$K = \begin{array}{ccc} * & \xrightarrow{W} & * \\ {\scriptstyle Z}\uparrow & e & \uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{array} \qquad \text{where } X, Y, Z, W \in G \text{ and } e \in E \text{ are such that } \partial(e) = XYW^{-1}Z^{-1}. \tag{34}$$

Remark 85 (Squares in $\mathcal{G}$ and fake-flat 2-gauge configurations of $[0,1]^2$)**.** Elements $K \in \mathcal{D}^2(\mathcal{G})$ hence can be seen as fake-flat 2-gauge configurations on the obvious 2-lattice decomposition of $D^2 = [0,1]^2$; see Ex. 50.

Several maps exist connecting $\mathcal{D}^1_H(\mathcal{G})$, $\mathcal{D}^1_V(\mathcal{G})$ and $\mathcal{D}^2(\mathcal{G})$; see below (here K is as in (34)):

$$d_l(K) = \begin{pmatrix} * \\ {\scriptstyle Z}\uparrow \\ * \end{pmatrix}, \quad d_r(K) = \begin{pmatrix} * \\ {\scriptstyle Y}\uparrow \\ * \end{pmatrix}, \quad d_u(K) = \big(* \xrightarrow{W} *\big) \text{ and } d_d(K) = \big(* \xrightarrow{X} *\big). \tag{35}$$

$$\mathrm{id}_V\big(* \xrightarrow{X} *\big) = \begin{array}{ccc} * & \xrightarrow{X} & * \\ {\scriptstyle 1_G}\uparrow & 1_E & \uparrow{\scriptstyle 1_G} \\ * & \xrightarrow[X]{} & * \end{array} \quad \text{and} \quad \mathrm{id}_H \begin{pmatrix} * \\ {\scriptstyle Y}\uparrow \\ * \end{pmatrix} = \begin{array}{ccc} * & \xrightarrow{1_G} & * \\ {\scriptstyle Y}\uparrow & 1_E & \uparrow{\scriptstyle Y} \\ * & \xrightarrow[1_G]{} & * \end{array}. \tag{36}$$

Horizontal and vertical compositions of 2-squares can be done when squares match on the relevant sides:

$$\begin{array}{ccc} * & \xrightarrow{W} & * \\ {\scriptstyle Z}\uparrow & e & \uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{array} \begin{array}{ccc} * & \xrightarrow{W'} & * \\ {\scriptstyle Y}\uparrow & e' & \uparrow{\scriptstyle Y'} \\ * & \xrightarrow[X']{} & * \end{array} = \begin{array}{ccc} * & \xrightarrow{WW'} & * \\ {\scriptstyle Z}\uparrow & (X \triangleright e')e & \uparrow{\scriptstyle Y'} \\ * & \xrightarrow[XX']{} & * \end{array} \quad \text{and} \quad \begin{array}{ccc} * & \xrightarrow{W'} & * \\ {\scriptstyle Z'}\uparrow & e' & \uparrow{\scriptstyle Y'} \\ * & \xrightarrow[W]{} & * \\ * & \xrightarrow{W} & * \\ {\scriptstyle Z}\uparrow & e & \uparrow{\scriptstyle Y} \\ * & \xrightarrow[X]{} & * \end{array} = \begin{array}{ccc} * & \xrightarrow{W'} & * \\ {\scriptstyle ZZ'}\uparrow & eZ \triangleright e' & \uparrow{\scriptstyle YY'} \\ * & \xrightarrow[X]{} & * \end{array} \tag{37}$$

These compositions are associative. Hence the set $\mathcal{D}^2(\mathcal{G})$ of squares in $\mathcal{G}$ is the set of morphisms of two categories, called horizontal and vertical categories. The correspondent sets of objects are the sets of vertical and horizontal squares in $\mathcal{G}$, respectively. Source and target maps are in (35). Unit maps are in (36).

Remark 86 (Interchange law in $\mathcal{D}(\mathcal{G})$)**.** Horizontal and vertical compositions in $\mathcal{D}(\mathcal{G})$ satisfy the interchange law, which says that the composition indicated below does not depend on the order whereby it is done.

$$\begin{array}{ccc} * & \xrightarrow{W'} & * \\ {\scriptstyle Z'}\uparrow & f & \uparrow{\scriptstyle C} \\ * & \xrightarrow[W]{} & * \\ * & \xrightarrow{W} & * \\ {\scriptstyle Z}\uparrow & e & \uparrow{\scriptstyle B} \\ * & \xrightarrow[X]{} & * \end{array} \begin{array}{ccc} * & \xrightarrow{W''} & * \\ {\scriptstyle C}\uparrow & f' & \uparrow{\scriptstyle Y''} \\ * & \xrightarrow[W''']{} & * \\ * & \xrightarrow{W'''} & * \\ {\scriptstyle B}\uparrow & e' & \uparrow{\scriptstyle Y'''} \\ * & \xrightarrow[X']{} & * \end{array}$$

This means that we can either first perform horizontal compositions, and then vertical compositions, or vice-versa, yielding the same result. To prove the interchange law we must make explicit use of the 2nd Peiffer condition in Def. 2, for crossed modules of groups. (All other mentioned properties in $\mathcal{D}(\mathcal{G})$ follow from the 1st Peiffer relation in Def. 2, and the fact that G acts on E by automorphisms.)

The horizontal and vertical categories are both groupoids. Given $K \in \mathcal{D}^2(\mathcal{G})$, the inverses r_V and r_H of K, with respect to the vertical and horizontal compositions, called vertical and horizontal reverses of K, are given in (38), below. This finishes the construction of the double groupoid $\mathcal{D}(\mathcal{G})$.

$$r_V\left(\begin{array}{ccc} * & \xrightarrow{W} & * \\ Z\Big\uparrow & e & \Big\uparrow Y \\ * & \xrightarrow[X]{} & * \end{array}\right) = \begin{array}{ccc} * & \xrightarrow{X} & * \\ Z^{-1}\Big\uparrow & Z^{-1}\triangleright e^{-1} & \Big\uparrow Y^{-1} \\ * & \xrightarrow[W]{} & * \end{array}, \qquad r_H\left(\begin{array}{ccc} * & \xrightarrow{W} & * \\ Z\Big\uparrow & e & \Big\uparrow Y \\ * & \xrightarrow[X]{} & * \end{array}\right) = \begin{array}{ccc} * & \xrightarrow{W^{-1}} & * \\ Y\Big\uparrow & X^{-1}\triangleright e^{-1} & \Big\uparrow Z \\ * & \xrightarrow[X^{-1}]{} & * \end{array} \tag{38}$$

4.3 Full gauge transformations between fake-flat 2-gauge configurations

The action of the group $\mathcal{T}(M, L, \mathcal{G})$ §4.1 of gauge operators on the set $\Theta(M, L, \mathcal{G})$ §3.2.1 of fake flat 2-gauge configurations is given in §4.4. We still need some technicalities.

4.3.1 Groupoid $\Theta^{\#}(M, L, \mathcal{G})$ of fake-flat 2-gauge configurations and full gauge transformations

Definition 87 (Full gauge transformation)**.** A full gauge transformation $\mathcal{U} = (U_2, U_1)$, starting in the fake-flat 2-gauge configuration $\mathcal{F} = (\mathcal{F}^1 \colon L^1 \to G\ ,\ \mathcal{F}^2 \colon L^2 \to E)$, is given by a pair of maps: $U_2 : L^1 \to \mathcal{D}^2(\mathcal{G})$ and $U_1 \colon L^0 \to \mathcal{D}^1_V(\mathcal{G})$, such that:

- Let $v \in L^0$. Put: $U_1(v) = \left(\begin{array}{c} * \\ g_v\Big\uparrow \\ * \end{array}\right)$. Let E_v be the set of edges of L incident to v. If $t \in \mathsf{E}_v$ then:

$$\sigma(t) = v \implies d_l\big(U_2(t)\big) = \left(\begin{array}{c} * \\ g_v\Big\uparrow \\ * \end{array}\right) \text{ and } \tau(t) = v \implies d_r\big(U_2(t)\big) = \left(\begin{array}{c} * \\ g_v\Big\uparrow \\ * \end{array}\right). \tag{39}$$

  Hence if two edges t and t' share a vertex, the corresponding vertical sides of $U_2(t)$ and $U_2(t')$ match.

- For each edge $t \in L^1$, it holds that $d_u\big(U_2(t)\big) = \mathcal{F}^1(t)$. (For notation see (35).)

Definition 88. An example of a full gauge transformation starting in $\mathcal{F}$ is $\mathrm{id}_{\mathcal{F}}$. It is such that $g_v = 1_G$, for each $v \in L^0$, hence $U_1(v)$ is an identity vertical 1-square. $\mathrm{id}_{\mathcal{F}}$ assigns $\mathrm{id}_V(\mathcal{F}^1(t))$ in (36) to each $t \in L^1$.

Remark 89 (Full gauge transformations and crossed module homotopies)**.** In the language of [19, §2.1] and [20], full gauge transformations, starting at $\mathcal{F}$, boil down to crossed module homotopies starting on the associated crossed module map $(\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}}) \colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ of Thm. 54. (An explanation of crossed module homotopies in a language close to this paper's is in [35, 29, 32, 31].) In order to prove this fact, we must use the fact that the groupoid $\pi_1(M, M^0)$ is free on the set of 1-cells of M. Therefore, a crossed module homotopy starting on the crossed module map $(\Psi_{\mathcal{F}}, \Phi_{\mathcal{F}}) \colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$ can be arbitrarily (and uniquely) specified by its value on the set of 0 and 1-cells of M, yielding our full gauge transformations.

For an explanation of crossed module homotopy in the general framework of *crossed complexes*, see [18, §9] and [16].

A full gauge transformation $\mathcal{U}$, starting in the fake-flat 2-gauge configuration $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2)$, *transforms* $\mathcal{F}$ into another fake-flat 2-gauge configuration, denoted

$$\mathcal{U} \triangleright \mathcal{F} = (\mathcal{U} \triangleright \mathcal{F}^1, \mathcal{U} \triangleright \mathcal{F}^2).$$

The definition of $\mathcal{U} \triangleright \mathcal{F}$ is given below. [1]

We give a combinatorial explanation of $\mathcal{U} \triangleright \mathcal{F} = (\mathcal{U} \triangleright \mathcal{F}^1, \mathcal{U} \triangleright \mathcal{F}^2)$, based on the framework of this paper.

At the level of edge colourings, if $t \in L^1$ then $(\mathcal{U} \triangleright \mathcal{F}^1)(t)$ is the bottom colour of the square $U_2(t) \in \mathcal{D}^2(\mathcal{G})$, i.e. $(\mathcal{U} \triangleright \mathcal{F}^1)(t) = d_d(U_2(t))$; see (35). Let us describe $\mathcal{U} \triangleright \mathcal{F}^2(P)$, where P is a plaquette. We consider two cases, depending on the attaching map $\psi^2_P \colon S^1 \to M^1$ of the closed 2-cell $\overline{c^2_P}$ (also denoted P); cf. Def. 22.

1. If $\psi^2_P \colon S^1 \to M^1$ is constant, then $\psi^2_P(S^1) = \{v\}$, where v is a 0-cell of L. And we then put:
$$(\mathcal{U} \triangleright \mathcal{F}^2)(P) = g_v \triangleright \mathcal{F}^2(P),$$
where g_v is defined in Def. 87. Since $\partial_{\mathcal{G}}(\mathcal{F}^2(P)) = 1_G$ (cf. Prop. 49), $\mathcal{U} \triangleright \mathcal{F}$ is fake-flat at P, because:
$$\partial_{\mathcal{G}}\big((\mathcal{U} \triangleright \mathcal{F}^2)(P)\big) = \partial_{\mathcal{G}}(g_v \triangleright \mathcal{F}^2(P)) = g_v \partial_{\mathcal{G}}(\mathcal{F}^2(P)) g_v^{-1} = 1_G.$$

2. Otherwise, we now make explicit use of the fact that $\psi^2_P \colon S^1 \to M^1$ must then be an embedding; as such the characteristic map $\phi^2_P \colon D^2 \to \overline{e^2_P} = P$ of P is a homeomorphism. Consider $P \times [0,1] \subset M^2 \times [0,1]$, with the obvious product lattice decomposition, where $[0,1]$ has unique 0-cells at 0 and 1. The 2-dimensional lattice made out of the top and lateral sides of $P \times [0,1]$ can be given a fake-flat 2-gauge configuration, obtained by putting together $\mathcal{F}$ and $\mathcal{U}$. We refer to Fig. 7. It depicts a fake-flat 2-gauge configuration in the boundary of the cylinder $P \times [0,1]$. The base-point of plaquette P, whose attaching map is oriented counterclockwise, is $v = v_1$. The top $P \times \{1\}$ of the cylinder is coloured by the restriction of $\mathcal{F}$ to P. The sides of the cylinder are coloured by the 1 and 2-squares $U_1(x) \in \mathcal{D}^1_V(\mathcal{G})$ and $U_2(t) \in \mathcal{D}^2(\mathcal{G})$, where x is a vertex of $\mathrm{bd}(P)$ and t is an edge of $\mathrm{bd}(P)$. (Each $U_2(t)$ can be seen as a fake-flat 2-gauge configuration of the 2-disk $[0,1]^2$; see Rem. 85. The direction of the edges of $\mathrm{bd}(P)$ gives an unambiguous way to transport the fake-flat 2-gauge configuration $U_2(t)$ of $[0,1]^2$ onto the correspondent lateral square in Fig 7.) Finally, the bottom $P \times \{0\}$ of the cylinder $P \times [0,1]$ is coloured with $\mathcal{U} \triangleright \mathcal{F} = \mathcal{F}'$.

   By definition, the 'gauge-transformed' colour $e'_P = \mathcal{U} \triangleright \mathcal{F}^2(P)$ of the plaquette P (which is based at v) is the 2D holonomy, based at v', along the 2-disk consisting of the top and lateral sides of the cylinder in Fig. 7, with the fake-flat 2-gauge configuration obtained by putting together $\mathcal{F}$ and $\mathcal{U}$. By Thm. 71, the 2D holonomy along this 2-disk is well-defined. Also by (13) $\mathcal{U} \triangleright \mathcal{F}$ is fake-flat at P.

Remark 90. A more concrete expression for $e'_P = \mathcal{U} \triangleright \mathcal{F}^2(P)$ can be derived by using the double groupoid $\mathcal{D}^2(\mathcal{G})$. In the example in Fig. 7, we evaluate the following composition in $\mathcal{D}^2(\mathcal{G})$. (We note that the elements of E assigned to the three squares in the bottom right arise from the horizontal reverses of the squares $U_2(t_3)$, $U_2(t_2)$ and $U_2(t_1)$, above; cf. (38).) And then e'_P is the element of E assigned to the resulting square in $\mathcal{G}$.

$$\begin{array}{c|ccccc|c}
 & & & 1_G & & & \\
\hline
1_G & & & e_P & & & 1_G \\
 & g_{t_5} & g_{t_4} & g_{t_3}^{-1} & g_{t_2}^{-1} & g_{t_1}^{-1} & \\
\hline
u_1 & & u_5 & u_4 & u_3 & u_2 & u_1 \\
 & \eta(t_5) & \eta(t_4) & {g'_{t_3}}^{-1} \triangleright \eta(t_3)^{-1} & {g'_{t_2}}^{-1} \triangleright \eta(t_2)^{-1} & {g'_{t_1}}^{-1} \triangleright \eta(t_1)^{-1} & \\
\hline
 & g'_{t_5} & g'_{t_4} & {g'_{t_3}}^{-1} & {g'_{t_2}}^{-1} & {g'_{t_1}}^{-1} &
\end{array} \tag{40}$$

Remark 91 (Notation)**.** If $\mathcal{U}$ is a full gauge transformation starting at $\mathcal{F}$, and transforming $\mathcal{F}$ into $\mathcal{F}' = \mathcal{U} \triangleright \mathcal{F}$, we use the notation: $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}'$. By construction it clearly follows that $\mathcal{F} \xrightarrow{\mathrm{id}_{\mathcal{F}}} \mathcal{F}$; see Def. 88.

Lemma 92. *Consider a sequence of full gauge transformations $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}' \xrightarrow{\mathcal{U}'} \mathcal{F}''$. A full gauge transformation $\mathcal{U}' * \mathcal{U}$, starting in $\mathcal{F}$, can be defined. Its underlying 1 and 2-squares in $\mathcal{G}$ are obtained by vertically composing the 1- and 2-squares in $\mathcal{G}$ of $\mathcal{U}$ and $\mathcal{U}'$, in the obvious way. Therefore the squares in $\mathcal{G}$ making $\mathcal{U}'$ will be put under the squares in $\mathcal{G}$ making $\mathcal{U}$ in* (37)*. Then $\mathcal{U}' * \mathcal{U}$ connects $\mathcal{F}$ to $\mathcal{F}''$; i.e. $\mathcal{F} \xrightarrow{\mathcal{U}' * \mathcal{U}} \mathcal{F}''$.*

[1] This is a consequence of the general construction in [19, §2.1], [20] and [18, §9] of crossed module and crossed complex homotopy. Some explicit calculations are in [35, 31, 32].

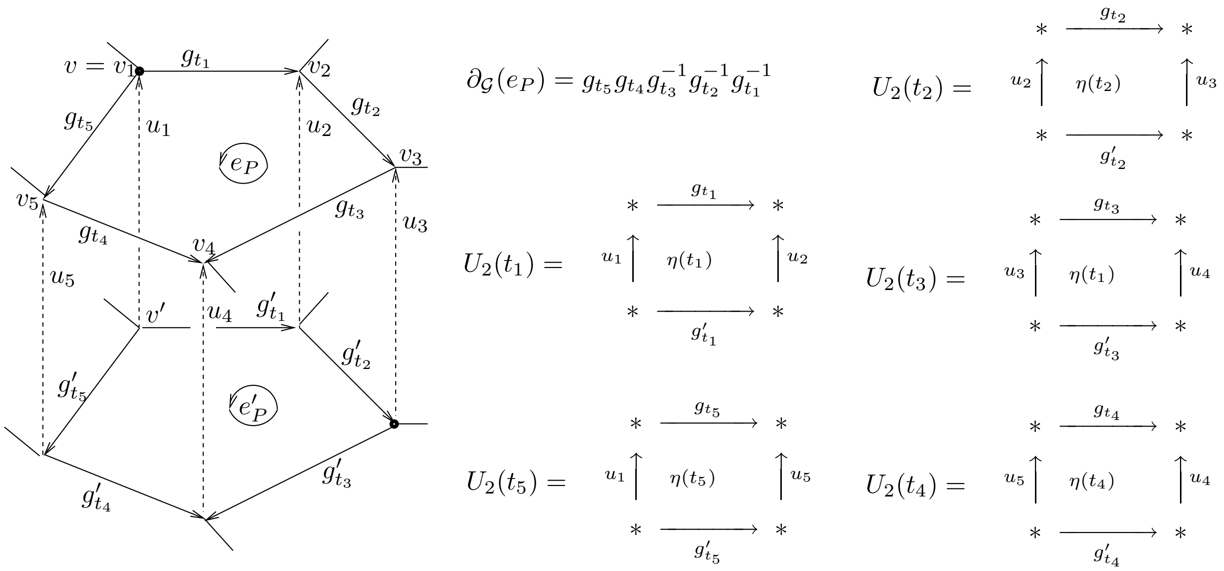


Figure 7: A full gauge transformation $\mathcal{U}$, transforming $\mathcal{F}$ into $\mathcal{U} \triangleright \mathcal{F} = \mathcal{F}'$, in the vicinity of a plaquette P. The 2-gauge configurations $\mathcal{F}$ and $\mathcal{F}' = \mathcal{U} \triangleright \mathcal{F}$ are (respectively) at the top and at the bottom of the cylinder $P \times I$. Note that, given an edge $t \in L^1$, the element of E associated to $U_2(t)$ is here denoted by $\eta(t)$. Also $u_1, \ldots, u_5 \in G$ are given by $g_{v_1}, \ldots g_{v_5}$. The squares in $\mathcal{G}$ on the right are used to give E labellings to the lateral squares of the cylinder on the left, in the obvious way.

Proof. **(Sketch)** We must prove that $\big((\mathcal{U}' * \mathcal{U}) \triangleright (\mathcal{F}^1)\big)(t) = \mathcal{F}''^1(t)$ for each $t \in L^1$, and that $\big((\mathcal{U}' * \mathcal{U}) \triangleright (\mathcal{F}^2)\big)(P) = \mathcal{F}''^2(P)$, for each $P \in L^2$. This is trivial to verify for edges, and for plaquettes attaching along constant maps. Otherwise, cf. Fig. 7. Put the squares of the full gauge transformation $\mathcal{U}'$ on the bottom of the ones of $\mathcal{U}$. This yields a fake-flat 2-gauge configuration $\mathcal{M}$, defined on the 2-disk Σ made out of the top and lateral faces of $P \times [0,2]$, where $[0,2]$ has 0-cells at $0, 1$ and 2. There are two different ways to explicitly compute the 2D holonomy of $\mathcal{M}$ along Σ. They must yield the same element of E; see Thm. 71. a) Either we firstly multiply the squares standing on top of each other, and then compose with the top 2-disk, and in this case the result will be $\big((\mathcal{U}' * \mathcal{U}) \triangleright (\mathcal{F}^2)\big)(P)$. Or b) compose $\mathcal{U}$ with $\mathcal{F}$ and only after that compose with $\mathcal{U}'$; and then the result will be $\big(\mathcal{U}' \triangleright (\mathcal{U} \triangleright \mathcal{F}^2)\big)(P) = \big(\mathcal{U}' \triangleright \mathcal{F}'^2\big)(P) = \mathcal{F}''^2(P)$.

Cf. Rem. 90, if we put the squares of $\mathcal{U}'$ under those of $\mathcal{U}$ in (40), then the statement of the lemma also follows from the interchange law for the vertical and horizontal compositions in $\mathcal{D}^2(\mathcal{G})$; see Rem. 86. ☐

Cf. (38). By using the vertical reverse of 2-squares in $\mathcal{G}$, we conclude that full gauge transformations can be reversed. Namely if we have $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}'$, then $\mathcal{U}^{-1}$, obtained by applying vertical reverses to the 1-squares $U_1(x)$, $x \in L^0$, and the 2-squares $U_2(t)$, $t \in L^1$, is such that $\mathcal{F}' \xrightarrow{\mathcal{U}^{-1}} \mathcal{F}$. Also $\mathcal{U} * \mathcal{U}^{-1} = \mathrm{id}_{\mathcal{F}'}$ and $\mathcal{U}^{-1} * \mathcal{U} = \mathrm{id}_{\mathcal{F}}$; see Def. 88. Therefore we have the following result.

Theorem 93 (The groupoid $\Theta^{\#}(M, L, \mathcal{G})$ of fake-flat 2-gauge configurations and full gauge transformations)**.** *Let (M, L) be a 2-lattice and $\mathcal{G}$ a crossed module. We have a groupoid $\Theta^{\#}(M, L, \mathcal{G})$, whose objects are the fake-flat 2-gauge configurations $\mathcal{F} \in \Theta(M, L, \mathcal{G})$. The morphisms are the full gauge transformations $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}'$.*

Remark 94 (Algebraic topological definition of $\Theta^{\#}(M, L, \mathcal{G})$ – following Brown and Higgins)**.** Recall (Thm. 54) that we have a bijection $\mathcal{F} \mapsto f_{\mathcal{F}}$, between fake-flat 2-gauge configurations $\mathcal{F}$ and crossed module maps $f_{\mathcal{F}} \colon \Pi_2(M^2, M^1, M^0) \to \mathcal{G}$. As mentioned in Rem. 89, there exists a relation of homotopy between crossed module maps $f, f' \colon \mathcal{G}' \to \mathcal{G}$, where $\mathcal{G}$ and $\mathcal{G}'$ are crossed modules of groupoids, discussed in [19, 20, 35]. This relation is a particular case of homotopy of crossed complex maps [18, §7.1.vii, §9.3], [16]. By using the notation in [18, §9.3.i], given crossed modules $\mathcal{G}'$ and $\mathcal{G}$, we have a groupoid $\mathrm{CRS}_1(\mathcal{G}', \mathcal{G})$ of crossed module maps $\mathcal{G}' \to \mathcal{G}$ and homotopies between them. When $\mathcal{G}' = \Pi_2(M^2, M^1, M^0)$, where M a CW-complex, and $\mathcal{G}$ is a crossed module of groups, a homotopy connecting f and f' boils down to a full gauge transformation $f \xrightarrow{\mathcal{U}} f'$. For discussion see [35, 29, 31]. Hence $\Theta^{\#}(M, L, \mathcal{G}) = \mathrm{CRS}_1(\Pi_2(M^2, M^1, M^0), \mathcal{G})$ in [18, §9.3.i].

Remark 95 (A 2-groupoid of fake-flat 2-gauge configurations, full gauge transformations and 2-fold gauge transformations)**.** The groupoid $\Theta^\sharp(M, L, \mathcal{G})$ is part of a more general construction. Let $\mathcal{G}, \mathcal{G}'$ be crossed module of groupoids. By considering 2-fold homotopies between crossed module homotopies (see [19, 20, 35] and [18, §9.3.i]), we can furthermore define a 2-groupoid $\mathrm{CRS}_2(\mathcal{G}', \mathcal{G})$, whose objects are crossed module maps $\mathcal{G}' \to \mathcal{G}$, 1-morphisms are homotopies between 2-crossed module maps, and 2-morphisms are 2-fold homotopies between homotopies. Explicit formulae are in [35, 29, 32, 31]. This leads to a notion of 2-fold gauge transformation between full gauge transformations, prominent in higher gauge theory [34, 5, 63]. These 2-fold gauge transformation between full gauge transformations do not appear to have large importance for this paper. However they have prime importance for addressing algebraic topology descriptions of higher gauge theory invariants of manifolds (namely Yetter invariant [76, 61]), as explained in [35, 29].

4.3.2 Full gauge transformations preserve 2D holonomy along embedded 2-spheres

Full gauge transformations preserve the 2D holonomy of fake-flat 2-gauge configurations along embedded surfaces. Let us address how to prove this using some basic algebraic topology, closely following the work of Brown and Higgins [16, 15, 13, 17]. Our proof is done in very identical lines to the one of [34], which was done for the case of differential-geometric 2-connections.

We temporarily denote the fundamental crossed module of a CW-complex X by $\Pi_2(X) = \Pi_2(X, X^1, X^0)$. If X is a CW-complex, let $X \times [0,1]$ be the product CW-complex, where $[0,1]$ is given the obvious CW-decomposition with 0-cells at 0 and 1. If $\mathcal{F}$ is a fake-flat 2-gauge configuration in a 2-lattice (M, L), we hence denote the (Thm. 54) 2D parallel transport 2-functor of $\mathcal{F}$ by $(\Psi_\mathcal{F}, \Phi_\mathcal{F})\colon \Pi_2(M^2) \to \mathcal{G}$.

The main result underpinning our discussion is the following interpretation of full gauge transformations between fake-flat 2-gauge configurations. It essentially appears in [18, §9.3.i, §9.7 and §9.8] and [16, 17], in the more general case of crossed complexes.

Lemma 96. *Let (M, L) be a 2-lattice. Let $\mathcal{G} = (\partial_\mathcal{G}\colon E \to G, \triangleright)$ be a group crossed module. Let $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}'$ be a full gauge transformation connecting $\mathcal{F}$ and $\mathcal{F}'$. We then have a crossed module map $H_\mathcal{U}\colon \Pi_2(M^2 \times [0,1]) \to \mathcal{G}$, making the diagram below (in the category of crossed modules) commute:*

$$
\begin{array}{ccc}
\Pi_2(M^2) & & \\
\big\downarrow{\scriptstyle i_1} & \searrow^{(\Psi_\mathcal{F}, \Phi_\mathcal{F})} & \\
\Pi_2\big(M^2 \times [0,1]\big) & \xrightarrow{\quad H_\mathcal{U} \quad} & \mathcal{G} \\
\big\uparrow{\scriptstyle i_0} & \nearrow_{(\Psi_{\mathcal{F}'}, \Phi_{\mathcal{F}'})} & \\
\Pi_2(M^2) & &
\end{array}
\tag{41}
$$

Here i_0 and i_1 are induced by $m \in M^2 \mapsto (m, 0) \in M^2 \times [0,1]$ and $m \in M^2 \mapsto (m, 1) \in M^2 \times [0,1]$.

Proof. Let us for simplicity assume that the product CW-decomposition of $M^2 \times [0,1]$ is a 2-lattice J. The general case in analogous. Recall the construction of the usual CW-decomposition of $M^2 \times [0,1]$, where $M^2 \times \{0\}$ and $M^2 \times \{1\}$ embed as subcomplexes, and we have an additional $(i+1)$-cell $c \times [0,1]$ of $M^2 \times [0,1]$ for each i-cell c of M^2. (See [40, Page 523].)

Cf. the discussion in §4.3.1, particularly the construction of $\mathcal{F}' = \mathcal{U} \triangleright \mathcal{F}$, and Fig. 7. The fake-flat 2-gauge configurations $\mathcal{F}$, $\mathcal{F}'$, and the full gauge transformation $\mathcal{U}$, can together be assembled to yield a fake-flat 2-gauge configuration $\mathcal{M}$ of $(M^2 \times [0,1], J)$. The restriction of $\mathcal{M}$ to $M \times \{1\}$ is obtained from $\mathcal{F}$, and the restriction of $\mathcal{M}$ to $M \times \{0\}$ is obtained from $\mathcal{F}'$. Finally the restriction of $\mathcal{M}$ to the 1- and 2-cells $v \times [0,1]$ (where $v \in L^0$) and $t \times [0,1]$ (where $t \in L^1$) is obtained from $U_1(v)$ and $U_2(t)$; for conventions on how to do this see the discussion in §4.3.1.

We have a 3-cell $P \times I$ of J for each 2-cell $P \in L^2$. And $\mathcal{M}$ is 2-flat along $P \times I$, given the explicit construction of the value of $\mathcal{F}' = \mathcal{U} \triangleright \mathcal{F}$ at P. Since there are no more 3-cells in $M^2 \times [0,1]$, we hence conclude that $\mathcal{M}$ is a 2-flat 2-gauge configuration in $(M^2 \times [0,1], J)$. We now just need to apply Thm 82 to $\mathcal{M}$. Clearly $H_\mathcal{U} = (\Psi_\mathcal{M}, \Phi_\mathcal{M})\colon \Pi_2\big(M^2 \times [0,1]\big) \to \mathcal{G}$ makes the diagram in (41) commute. ∎

Remark 97. A stronger result can be proved, and is implicit in [16] and [18, Chapter 9]. Namely, there is a one-to-one correspondence between full-gauge transformations $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}'$ and maps $H_{\mathcal{U}} \colon \Pi_2(M^2 \times [0,1]) \to \mathcal{G}$, making (41) commute. This can be inferred by combining the beginning of [18, §9.3.i] with Thm [18, 9.8.1].

We state the result concerning invariance of 2D holonomy under full gauge transformations for a 2-sphere cellularly embedded in a 2-lattice, only; see §3.4.2, §3.5.4. This is the case whose behaviour under full gauge transformations is the neatest and it is the generality needed to formulate higher Kitaev models in §5.1.

Theorem 98. *Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a crossed module of groups. Let (M, L) be a 2-lattice. Let Σ be a 2-sphere cellularly embedded in M. Let $\mathcal{F}$ be a fake-flat 2-gauge configuration on (M, L). Let $\mathcal{U}$ be a full gauge transformation starting in $\mathcal{F}$. Let $v \in \Sigma$, to be a 0-cell of M. Let $g_v \in G$ be the element of G associated to $U_1(v)$; see Def. 87. Then:*

$$\mathrm{Hol}^2_v(\mathcal{U} \triangleright \mathcal{F}, \Sigma, L) \;=\; g_v \triangleright \mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L).$$

Proof. We strongly use the previous lemma, and resume the notation there introduced.

Let $\mathcal{F}' = \mathcal{U} \triangleright \mathcal{F}$. Cf. (41), put $H_{\mathcal{U}} = (H^2_{\mathcal{U}}, H^1_{\mathcal{U}})$. Let $\overline{\iota_v}(\Sigma) \in \pi_2(M^2, v) \subset \pi_2(M^2, M^1, v)$ be as in Def. 59. By Def. 61, we have that:

$$\mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L) = \Psi_{\mathcal{F}}(\overline{\iota_v}(\Sigma)) = H^2_{\mathcal{U}}\big(i_1(\overline{\iota_v}(\Sigma))\big) \text{ and } \mathrm{Hol}^2_v(\mathcal{U} \triangleright \mathcal{F}, \Sigma, L) = \Psi_{\mathcal{F}'}(\overline{\iota_v}(\Sigma)) = H^2_{\mathcal{U}}\big(i_0(\overline{\iota_v}(\Sigma))\big).$$

Let γ_v be the following path in $M^2 \times [0,1]$, connecting $(v, 0)$ to $(v, 1)$:

$$t \in [0,1] \mapsto (v, t) \in M^2 \times [0,1].$$

Then, passing to the correspondent element $[\gamma_v]$ in the underlying groupoid $\pi_1\big((M^2 \times [0,1])^1, (M^2 \times [0,1])^0\big)$ of the crossed module $\Pi_2(M^2 \times [0,1])$, it holds that:

$$i_0(\overline{\iota_v}(\Sigma)) = [\gamma_v] \triangleright (i_1(\overline{\iota_v}(\Sigma))), \text{ in } \pi_2\big(M^2 \times [0,1], (M^2 \times [0,1])^1, (M^2 \times [0,1])^0\big).$$

By construction we have that $g_v = H^1_{\mathcal{U}}([\gamma_v])$. Cf. (41), it hence follows that:

$$\begin{aligned}
\mathrm{Hol}^2_v(\mathcal{F}', \Sigma, L) &= \Psi_{\mathcal{F}'}(\overline{\iota_v}(\Sigma)) = H^2_{\mathcal{U}}\big(i_0(\overline{\iota_v}(\Sigma))\big) \\
&= H^2_{\mathcal{U}}\big([\gamma_v] \triangleright (i_1(\overline{\iota_v}(\Sigma)))\big) \\
&= H^1_{\mathcal{U}}\big([\gamma_v]\big) \triangleright H^2_{\mathcal{U}}\big((i_1(\overline{\iota_v}(\Sigma)))\big) \\
&= g_v \triangleright \Psi_{\mathcal{F}}\big(\overline{\iota_v}(\Sigma)\big) = g_v \triangleright \mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L).
\end{aligned}$$

□

4.3.3 Groupoid $\Theta^{\#}_{\mathrm{flat}}(M, L, \mathcal{G})$ of 2-flat 2-gauge configurations and full gauge transformations

Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a crossed module of groups. Let (M, L) be a 2-lattice. Recall the definition of a 2-flat 2-gauge configuration in §3.6 and details therein. Let $\mathcal{F}$ be a fake-flat 2-gauge configuration. The 2D holonomy $\mathrm{Hol}^2_v(\mathcal{F}, \mathrm{bd}(b), L)$ of $\mathcal{F}$ along the boundary $\mathrm{bd}(b)$ of a 3-cell b is invariant under full gauge transformations, in the sense of Thm. 98. Suppose that $\mathcal{F}$ is 2-flat, hence that $\mathrm{Hol}^2_v(\mathcal{F}, \mathrm{bd}(b), L) = 1_E$, for each $b \in L^3$. Since G acts on E by automorphisms, if $\mathcal{U}$ is any full gauge transformation, starting in $\mathcal{F}$, it follows that $\mathrm{Hol}^2_v(\mathcal{U} \triangleright \mathcal{F}, \mathrm{bd}(b), L) = 1_E$, for each $b \in L^3$. Hence full gauge transformations transform 2-flat 2-gauge configurations into 2-flat 2-gauge configurations.

In particular, the groupoid $\Theta^\sharp(M, L, \mathcal{G})$ of fake-flat 2-gauge configurations and full gauge transformations of Thm 93 has a full subgroupoid $\Theta^\sharp_{\mathrm{flat}}(M, L, \mathcal{G})$, whose objects are the 2-flat 2-gauge transformations.

Remark 99 (Algebraic topological definition of $\Theta^{\#}_{\mathrm{flat}}(M, L, \mathcal{G})$ – following Brown and Higgins)**.** Cf. Rem. 94. Recall (Thm. 82) that we have a bijection $\mathcal{F} \mapsto f_{\mathcal{F}}$, between 2-flat 2-gauge configurations $\mathcal{F}$ and crossed module maps $f_{\mathcal{F}} \colon \Pi_2(M, M^1, M^0) \to \mathcal{G}$. Cf. [18, §9.3.i], given crossed modules $\mathcal{G}'$ and $\mathcal{G}$ we have a groupoid $\mathrm{CRS}_1(\mathcal{G}', \mathcal{G})$ of crossed module maps $\mathcal{G}' \to \mathcal{G}$ and homotopies between them. When $\mathcal{G}' = \Pi_2(M, M^1, M^0)$, where M a CW-complex, $\mathcal{G}$ is a group crossed module, and $\mathcal{F}, \mathcal{F}'$ are 2-flat 2-gauge configurations, a homotopy H, connecting $f_{\mathcal{F}}$ to $f_{\mathcal{F}'}$, boils down to a full gauge transformation $\mathcal{F} \xrightarrow{\mathcal{U}} \mathcal{F}'$. Hence $\Theta^{\#}_{\mathrm{flat}}(M, L, \mathcal{G}) \cong \mathrm{CRS}_1(\Pi_2(M, M^1, M^0), \mathcal{G})$ in [18, §9.3.i].

4.4 Gauge operators on fake-flat 2-gauge configurations

Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a crossed module of groups. We now finally define a left-action $\bullet$ of the group of gauge operators $\mathcal{T}(M, L, \mathcal{G})$ of §4.1 on the set of fake-flat 2-gauge configurations $\Theta(M, L, \mathcal{G})$. Our main tool is the groupoid $\Theta^\sharp(M, L, \mathcal{G})$ of fake-flat 2-gauge configurations and full gauge transformations between them; see Thm 93. We will define a left-action whose action-groupoid is isomorphic to $\Theta^\sharp(M, L, \mathcal{G})$.

The main observation is that given a fake-flat 2-gauge configuration $\mathcal{F} = (\mathcal{F}^2 \colon L^2 \to E, \mathcal{F}^1 \colon L^1 \to G)$, the set of full gauge transformations starting in $\mathcal{F}$ can be put in one-to-one correspondence with elements $(\eta, u) \in \mathcal{T}(M, L, \mathcal{G}) = \mathcal{E}(M, L, \mathcal{G}) \rtimes_\bullet \mathcal{V}(M, L, \mathcal{G})$. Given $\mathcal{F} \in \Theta(M, L, \mathcal{G})$ and $(\eta, u) \in \mathcal{T}(M, L, \mathcal{G})$, we have a full gauge transformation $\mathcal{U}_{(\eta,u,\mathcal{F})} = \mathcal{U} = (U_2, U_1)$, starting at $\mathcal{F}$, defined as:

$$
\begin{aligned}
v \in L^0 &\xmapsto{U_1} \begin{pmatrix} * \\ \uparrow u(v) \\ * \end{pmatrix} \\
\left(v \xrightarrow{t} v'\right) \in L^1 &\xmapsto{U_2} \begin{pmatrix} * & \xrightarrow{\mathcal{F}^1(t)} & * \\ u(v)\uparrow & \eta(t) & \uparrow u(v') \\ * & \xrightarrow[\mathcal{U} \triangleright \mathcal{F}^1(t)]{} & * \end{pmatrix}; \text{ for } \mathcal{U} \triangleright \mathcal{F}^1(t) = \partial\big(\eta(t)\big)\, u(v)\, \mathcal{F}^1(t)\, u(v')^{-1}.
\end{aligned}
\tag{42}
$$

Lemma 100. *Let $(\eta, u), (\eta', u') \in \mathcal{T}(M, L, \mathcal{G})$. Let $\mathcal{F} \in \Theta(M, L, \mathcal{G})$. We have:*

$$
\mathcal{U}_{\left((\eta,u)(\eta',u'),\mathcal{F}\right)} = \mathcal{U}_{\left(\eta\, u \bullet \eta',\, u\, u',\mathcal{F}\right)} = \mathcal{U}_{\left(\eta,u,\mathcal{U}_{(\eta',u',\mathcal{F})} \triangleright \mathcal{F}\right)} * \mathcal{U}_{\left(\eta',u',\mathcal{F}\right)}. \tag{43}
$$

N.B.: See §4.1 for conventions on the product $(\eta, u)(\eta', u') = (\eta\, u \bullet \eta', u\, u')$ of $(\eta, u), (\eta', u') \in \mathcal{T}(M, L, \mathcal{G})$. The composition of $*$ of full gauge transformation is made explicit in Lem. 92.

Proof. This follows by construction, by looking at the conventions (42) for $\mathcal{U}_{(\eta,u,\mathcal{F})}$. Just compare the explicit definition of the group operation in $\mathcal{T}(M, L, \mathcal{G})$ in (33) with the explicit form of the vertical composition of squares in $\mathcal{G}$ in (37). The latter yields the composition $*$ of full gauge transformations in Lem. 92. □

An operation $\bullet$ of the group $\mathcal{T}(M, L, \mathcal{G})$ on $\Theta(M, L, \mathcal{G})$ can then be defined by:

$$
(\eta, u) \bullet \mathcal{F} \; = \; \mathcal{U}_{(\eta,u,\mathcal{F})} \triangleright \mathcal{F}. \tag{44}
$$

By Lem. 100, $\bullet$ is indeed a left-action of $\mathcal{T}(M, L, \mathcal{G})$ on $\Theta(M, L, \mathcal{G})$ and $\Theta^\sharp(M, L, \mathcal{G})$ is its action groupoid.

By construction of $\mathcal{U}_{(\eta,u,\mathcal{F})}$ and Thm. 98, it follows that:

Theorem 101. *Let (M, L) be a 2-lattice. Let Σ be a 2-sphere cellularly embedded in M. Let $v \in \Sigma \cap M^0$. Let $\mathcal{F}$ be a fake-flat 2-gauge configuration in (M, L). Let $(\eta, u) \in \mathcal{T}(M, L, \mathcal{G})$. Then:*

$$
\mathrm{Hol}^2_v\big((\eta, u) \bullet \mathcal{F}, \Sigma, L\big) \; = \; u(v) \triangleright \mathrm{Hol}^2_v(\mathcal{F}, \Sigma, L).
$$

Cf. §4.3.3, in particular $\bullet$ restricts to an action of $\mathcal{T}(M, L, \mathcal{G})$ on the set of 2-flat 2-gauge configurations.

5 The Hamiltonian models

5.1 A Hamiltonian model for higher gauge theory

In this subsection, we fix a manifold M, a 2-lattice decomposition (M, L) of M (Def. 22), and a crossed module of groups $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$; Def. 2. We suppose that M is compact (thus that L is finite) and that $\mathcal{G}$ is finite, meaning that both G and E are finite groups. Hence the set $\Theta(M, L, \mathcal{G})$ of fake-flat 2-gauge configurations is finite. Note that $\Theta(M, L, \mathcal{G})$ is non-empty, as the naive vacuum is in $\Theta(M, L, \mathcal{G})$; see §3.2.1.

Recall from §4.1 the group $\mathcal{T}(M, L, \mathcal{G}) = \mathcal{E}(M, L, \mathcal{G}) \rtimes_\bullet \mathcal{V}(M, L, \mathcal{G})$ of gauge operators. We have §4.4 a left-action $\bullet$ of the group of gauge operators on $\Theta(M, L, \mathcal{G})$, preserving 2D holonomy, as in Thm. 101.

Definition 102 (Hilbert space $\mathcal{H}(M, L, \mathcal{G})$)**.** The Hilbert space for Hamiltonian higher gauge theory is the free vector space $\mathcal{H}(M, L, \mathcal{G}) = \mathbb{C}\Theta(M, L, \mathcal{G})$ on the set of fake-flat 2-gauge configurations, with the unique inner product $\langle -, - \rangle$ that renders different fake-flat 2-gauge configurations orthonormal.

The group algebra $\mathbb{C}\mathcal{T}(M, L, \mathcal{G})$ of $\mathcal{T}(M, L, \mathcal{G})$ has a representation on $\mathcal{H}(M, L, \mathcal{G})$, obtained by linearising the action of $\mathcal{T}(M, L, \mathcal{G})$ on $\Theta(M, L, \mathcal{G})$. Given $m \in \mathbb{C}\mathcal{T}(M, L, \mathcal{G})$, we denote the corresponding linear operator by $\widehat{m}\colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$. Thus $\widehat{m}(\mathcal{F}) = m \bullet \mathcal{F}$, for each fake-flat 2-gauge configuration $\mathcal{F}$.

By construction, if $m \in \mathcal{T}(M, L, \mathcal{G})$, then the operator $\widehat{m}\colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ is unitary, i.e. $\widehat{m}^\dagger = \widehat{m}^{-1} = \widehat{m^{-1}}$, where $\dagger$ denotes Hermitian adjoint.

5.1.1 Vertex and edge gauge spikes U_v^g and U_t^e

Definition 103 (Edge gauge spikes and vertex gauge spikes)**.** A vertex gauge spike, supported in $v \in L^0$, is a gauge operator $(\eta, u) \in \mathcal{T}(M, L, \mathcal{G})$ such that $\eta(t) = 1_E$ for each $t \in L^1$, and such that for each $w \in L^0$ it holds that $u(w) = 1_G$, unless $w = v$. Analogously, given an edge $t \in L^1$, a gauge spike supported in t is a gauge operator (η, u) such that $u(v) = 1_G$ for each $v \in L^0$, and such that for each $s \in L^1$, it holds that $\eta(s) = 1_E$, unless $s = t$.

For a vertex $v \in L^0$ and $g \in G$, we let U_v^g be the unique vertex gauge spike supported in v such that $u(v) = g$. For $t \in L^1$ and $e \in E$, we let U_t^e be the unique edge gauge spike, supported in t, such that $\eta(t) = e$.

The linear operators $\widehat{U_v^g}, \widehat{U_t^e}\colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ are also called vertex and edge gauge spikes.

Lemma 104. *(I) Vertex gauge spikes supported in different vertices commute, and edge gauge spikes supported in different edges commute. I.e., given $v, v' \in L^0$, with $v \neq v'$, and $t, t' \in L^1$, with $t \neq t'$ (including the case when t and t' share a vertex), it holds that, given $e, e' \in E$ and $g, g' \in G$:*

$$[U_v^g, U_{v'}^{g'}] = 0 \qquad \textit{and} \qquad [U_t^e, U_{t'}^{e'}] = 0.$$

(II) For each vertex $v \in L^0$ and each edge $t \in L^1$, then given any $g, h \in G$ and any $e, f \in E$, we have:

$$U_v^g U_v^h = U_v^{gh} \qquad \textit{and} \qquad U_t^e U_t^f = U_t^{ef}.$$

(III) If $t = \big(\sigma(t) \xrightarrow{t} \tau(t)\big) \in L^1$, including the case when $\sigma(t) = \tau(t)$, and $v \in L^0$ is such that $v \neq \sigma(t)$, then given $g \in G$ and $e \in E$, it follows that U_v^g and U_t^e commute: $[U_v^g, U_t^e] = 0$.
(IV) Given an edge $t = \big(\sigma(t) \xrightarrow{t} \tau(t)\big) \in L^1$, including the case $\sigma(t) = \tau(t)$, $g \in G$ and $e \in E$, we have:

$$U_t^e U_{\sigma(t)}^g = U_{\sigma(t)}^g U_t^{g^{-1} \triangleright e}.$$

(V) Moreover, any gauge operator can be obtained as a product of vertex and edge gauge spikes.

Proof. This all follows immediately from the definition of the group of gauge operators as a semidirect product in §4.1. Equations (31) and (33) translate exacty to the formulae in the lemma. □

Remark 105. Note that (I–IV) also hold for the associated operators $\widehat{U_v^g}, \widehat{U_t^e}\colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$, since they are constructed from a representation of the group $\mathcal{T}(M, L, \mathcal{G})$ on the Hilbert space $\mathcal{H}(M, L, \mathcal{G})$.

Let us now unpack the construction in §4.3 and §4.4, and give an explicit description of how vertex and edge gauge spikes act on $\mathcal{H}(M, L, \mathcal{G}) = \mathbb{C}\Theta(M, L, \mathcal{G})$. Let $\mathcal{F} = (\mathcal{F}^2\colon L^2 \to E\ ,\ \mathcal{F}^1\colon L^1 \to G)$ be a fake-flat 2-gauge configuration; Def. 47. If $m \in \mathbb{C}\mathcal{T}(M, L, \mathcal{G})$, recall $\widehat{m}\colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ is the operator, $\mathcal{F} \mapsto m \bullet \mathcal{F}$. Put $\widehat{m}(\mathcal{F}) = \big(\widehat{m}(\mathcal{F}^2)\colon L^2 \to E\ ,\ \widehat{m}(\mathcal{F}^1)\colon L^1 \to G\big)$. Recall (42):

Action of vertex gauge spikes: Let $v \in L^0$ and $g \in G$. Let $\big(\sigma(t) \xrightarrow{t} \tau(t)\big) \in L^1$. Then:

$$\big(\widehat{U_v^g}(\mathcal{F}^1)\big)(\sigma(t) \xrightarrow{t} \tau(t)) = \begin{cases} g\,\mathcal{F}^1\big(\sigma(t) \xrightarrow{t} \tau(t)\big), \text{ if } v = \sigma(t), \text{ and } \sigma(t) \neq \tau(t); \\ \mathcal{F}^1\big(\tau(t) \xrightarrow{t} \tau(t)\,\big)g^{-1}, \text{ if } v = \tau(t), \text{ and } \sigma(t) \neq \tau(t); \\ g\mathcal{F}^1\big(\tau(t) \xrightarrow{t} \tau(t)\,\big)g^{-1}, \text{ if } v = \tau(t) = \sigma(t); \\ \mathcal{F}^1\big(\sigma(t) \xrightarrow{t} \tau(t)\,\big), \text{ if } v \neq \sigma(t) \text{ and } v \neq \tau(t). \end{cases}$$

Let $P \in L^2$. Let $x_P \in \mathrm{bd}(P)$ be the base-point of P. Then:

$$\big(\widehat{U_v^g}(\mathcal{F}^2)\big)(P) = \begin{cases} g \triangleright \mathcal{F}^2\big(P\big), & \text{if } v = x_P; \\ \mathcal{F}^2\big(P\big), & \text{if } v \text{ is not the base-point } x_P \text{ of } P. \end{cases}$$

Action of edge gauge spikes: Let $(\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)) \in L^1$ and $e \in E$. Let $(\sigma(t) \xrightarrow{t} \tau(t)) \in L^1$, then, cf. (42):

$$\big(\widehat{U_\gamma^e}(\mathcal{F}^1)\big)(\sigma(t) \xrightarrow{t} \tau(t)) = \begin{cases} \partial(e)\, \mathcal{F}^1\big(\sigma(t) \xrightarrow{t} \tau(t)\big), & \text{if the edges } (\sigma(t) \xrightarrow{t} \tau(t)) \text{ and } (\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)) \text{ are the same;} \\ \mathcal{F}^1\big(\sigma(t) \xrightarrow{t} \tau(t)\,\big), & \text{if } (\sigma(t) \xrightarrow{t} \tau(t)) \neq (\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)). \end{cases}$$

Let $P \in L^2$. Let x_P be the base-point of P. Some bits of notation are necessary to describe $\big(\widehat{U_\gamma^e}(\mathcal{F}^1)\big)(P)$. Let $\partial_L^Q(P) = \big(x_P \xrightarrow{t_1^{\theta_1} t_2^{\theta_2} \dots t_n^{\theta_n}} x_P\big)$ be the granular boundary of P; Def. 48. Put $t_i = (x_i \xrightarrow{t_i} y_i)$, $i = 1, \dots, n$. Given that the attaching map $\psi_P^2\colon S^1 \to M^1$ of the 2-cell $\overline{c_P^2} = P$ is either constant or an embedding, an arbitrary edge $\gamma = (\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)) \in L^1$ can appear in the list $(x_i \xrightarrow{t_i} y_i)$, where $t \in \{1, \dots, n\}$, at most once.

Definition 106. Let $\partial_L^Q(P) = \big(x_P \xrightarrow{t_1^{\theta_1} t_2^{\theta_2} \dots t_n^{\theta_n}} x_P\big)$. We say that the edge $\gamma = (\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)) \in L^1$:

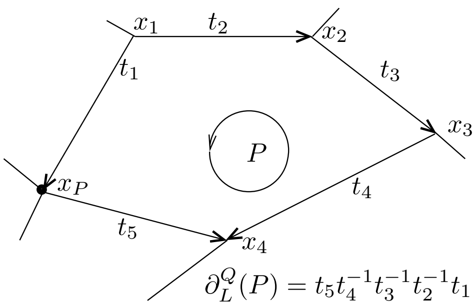
$$\text{is not incident to } \mathrm{bd}(P)\text{, if } (\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)) \text{ is not in the list } (x_i \xrightarrow{t_i} y_i), i \in \{1, \dots, n\};$$
$$\text{is positively incident to } \mathrm{bd}(P)\text{, if } \big(\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)\big) = (x_i \xrightarrow{t_i} y_i), \text{ for some } i \in \{1, \dots, n\}, \text{ and } \theta_i = 1;$$
$$\text{is negatively incident to } \mathrm{bd}(P)\text{, if } \big(\sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma)\big) = (x_i \xrightarrow{t_i} y_i), \text{ for some } i \in \{1, \dots, n\}, \text{ and } \theta_i = -1.$$

Suppose $\psi_P^2\colon S^1 \to M^1$ is an embedding. Given a vertex $x \in \mathrm{bd}(P)$, with $x \neq x_P$, there exists a unique granular path $x_P \xrightarrow{p^+(x)} x$ made from the $(x_i \xrightarrow{t_i} y_i)^{\theta^i}$ and contouring $\mathrm{bd}(P)$ in the positive direction (where orientation is provided by the attaching map of P). There is another granular path $x_P \xrightarrow{p^-(x)} x$ contouring $\mathrm{bd}(P)$ in the negative direction. We also put $p^+(x_P) = \emptyset_{x_P}$ and $p^-(x_P) = \emptyset_{x_P}$.

We let $g_{p^+(x)}$ and $g_{p^-(x)}$ be the 1D holonomy of $\mathcal{F}^1$ along $p^+(x)$ and $p^-(x)$; see Def. 42. And then:

$$\big(\widehat{U_\gamma^e(\mathcal{F}^2)}\big)(P) = \begin{cases} \mathcal{F}^2\big(P\big), & \text{if } P \text{ attaches along a constant map;} \\ \mathcal{F}^2\big(P\big), & \text{if } \sigma(\gamma) \xrightarrow{\gamma} \tau(\gamma) \text{ is not incident to } \mathrm{bd}(P); \\ \big(g_{p^+(\sigma(\gamma))} \triangleright e\big)\mathcal{F}^2\big(P\big), & \text{if } \gamma \text{ is positively incident to } \mathrm{bd}(P); \\ \mathcal{F}^2\big(P\big)\ \big(g_{p^-(\sigma(\gamma))} \triangleright e^{-1}\big), & \text{if } \gamma \text{ is negatively incident to } \mathrm{bd}(P). \end{cases}$$

Example 107. Consider a 2-lattice decomposition L that close to a plaquette P looks like figure 8, below:



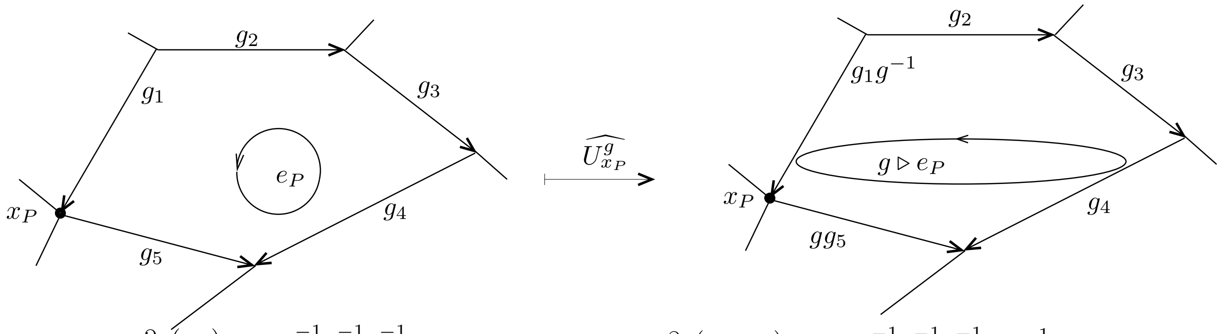
$$\partial_L^Q(P) = t_5 t_4^{-1} t_3^{-1} t_2^{-1} t_1$$

$p^+(x_P) = \emptyset_{x_P}$	$p^-(x_P) = \emptyset_{x_P}$
$p^+(x_1) = t_5 t_4^{-1} t_3^{-1} t_2^{-1}$	$p^-(x_1) = t_1^{-1}$
$p^+(x_2) = t_5 t_4^{-1} t_3^{-1}$	$p^-(x_2) = t_1^{-1} t_2$
$p^+(x_3) = t_5 t_4^{-1}$	$p^-(x_3) = t_1^{-1} t_2 t_3$
$p^+(x_4) = t_5$	$p^-(x_4) = t_1^{-1} t_2 t_3 t_4$

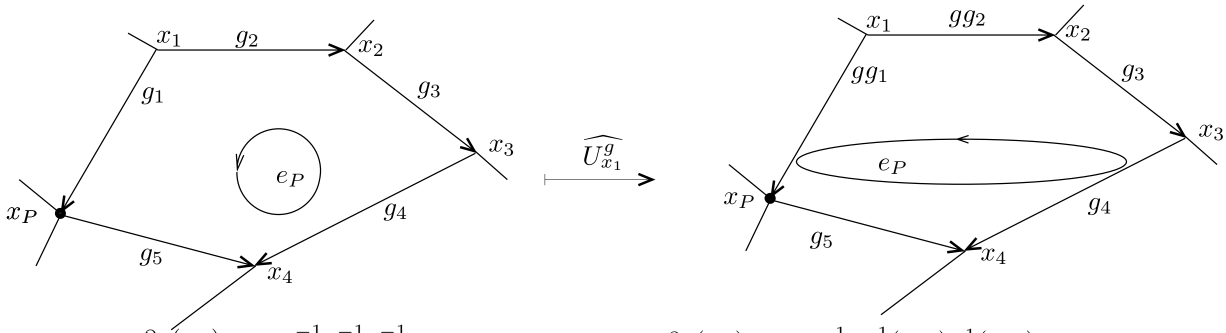
Figure 8: A plaquette P of a 2-lattice decomposition L. As indicated, the plaquette attaches counterclockwise. The base-point of P is x_P. We also show the granular paths $p^\pm(x_P), p^\pm(x_1), p^\pm(x_2), p^\pm, (x_3), p^\pm(x_4)$.

The only edge and vertex gauge spikes which have a non-trivial action on the restriction of a fake-flat 2-gauge configuration $\mathcal{F}$ to P are the vertex gauge spikes $\widehat{U_{x_P}^g}, \widehat{U_{x_1}^g}, \widehat{U_{x_2}^g}, \widehat{U_{x_3}^g}, \widehat{U_{x_4}^g}$, and the edge gauge spikes

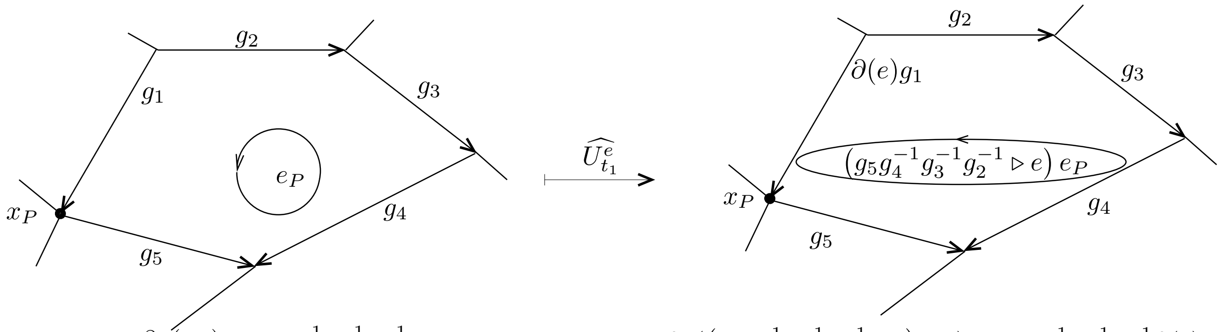
$\widehat{U^e_{t_1}}, \widehat{U^e_{t_2}}, \widehat{U^e_{t_3}}, \widehat{U^e_{t_4}}, \widehat{U^e_{t_5}}$, where $g \in G$ and $e \in E$ are arbitary. The actions of some of these on a fake-flat 2-gauge configuration at P are below. (We can also see that fake-flatness at P is preserved in these examples.)



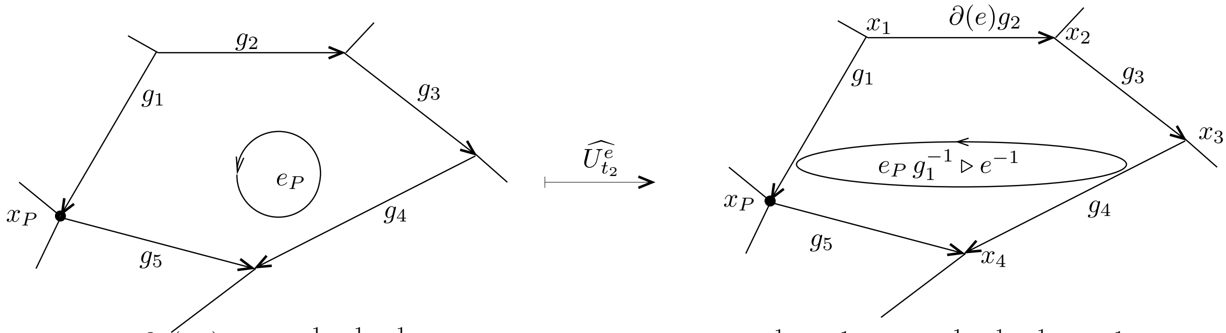
$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$ $\qquad$ $\partial_{\mathcal{G}}(g \triangleright e_P) = g g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1 g^{-1}$



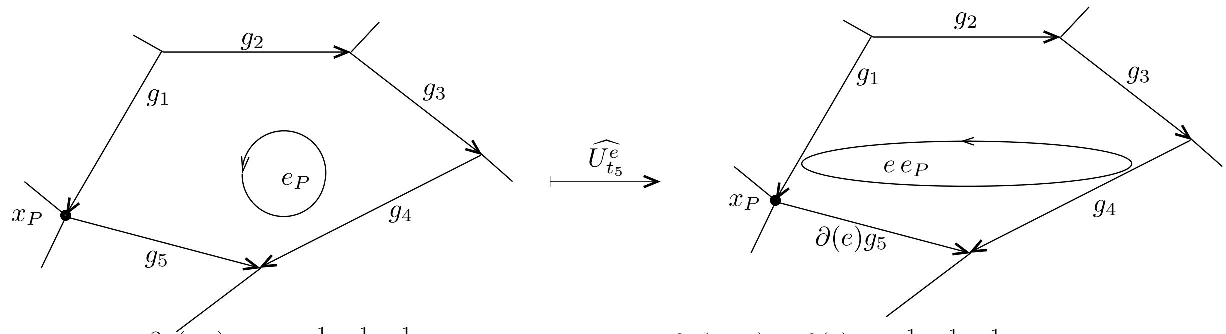
$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$ $\qquad$ $\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} (g g_2)^{-1} (g g_1)$



$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$ $\qquad$ $\partial_{\mathcal{G}}(\left(g_5 g_4^{-1} g_3^{-1} g_2^{-1} \triangleright e\right)\ e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} \partial(e) g_1$



$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$ $\qquad$ $\partial_{\mathcal{G}}(e_P\ g_1^{-1} \triangleright e^{-1}) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} \partial(e)^{-1} g_1$



$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1 \qquad \partial_{\mathcal{G}}(e\, e_P) = \partial(e) g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$

Remark 108. We note that if L is a triangulation of M, then the vertex and edge gauge spikes here defined coincide with the vertex and edge gauge transformations appearing in [23, III-A & III-B].

5.1.2 Vertex operators, edge operators and blob operators

Given a set X, we put $\#X$ to denote the cardinality of X.

Definition 109 (Vertex operators $\widehat{A_v}$ and edge operators $\widehat{B_t}$)**.** Let $v \in L^0$ be a vertex and $t \in L^1$ be an edge, of (M, L). The elements below of the group algebra $\mathbb{C}\mathcal{T}(M, L, \mathcal{G})$ are called vertex and edge operators:

$$A_v \doteq \frac{1}{\#G} \sum_{g \in G} U_v^g, \qquad\qquad B_t \doteq \frac{1}{\#E} \sum_{e \in E} U_t^e. \tag{45}$$

The corresponding operators $\widehat{A_v}, \widehat{B_t} \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ are also called vertex and edge operators.

Note that the operators $\widehat{A_v}, \widehat{B_t} \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ are all self-adjoint. This is because, if $v \in L^0$:

$$\widehat{A_v}^{\dagger} = \frac{1}{\#G} \sum_{g \in G} \widehat{U_v^g}^{\dagger} = \frac{1}{\#G} \sum_{g \in G} \widehat{U_v^{g^{-1}}} = \frac{1}{\#G} \sum_{g \in G} \widehat{U_v^g} = \widehat{A_v},$$

and analogously for $\widehat{B_t}$.

From Lem. 104 we have the following.

Lemma 110. *Given arbitrary vertices $u, v \in L^0$ and edges $s, t \in L^1$ we have:*

$$\begin{aligned} &A_v A_v = A_v, && B_t B_t = B_t, && \\ &[A_v, A_u] = 0, && [A_v, B_t] = 0, && [B_t, B_s] = 0. \end{aligned}$$

These relations also hold for $\widehat{A_v}$ and $\widehat{B_t}$, since $\mathbb{C}\mathcal{T}(M, L, \mathcal{G})$ acts on $\mathcal{H}(M, L, \mathcal{G})$. □

Let $b \in L^3$ be a 3-cell (i.e a blob; Rem. 21). Cf. §3.6, by definition of 2-lattices, $\mathrm{bd}(b)$ is a subcomplex of (M, L) homeomorphic to S^2, with a base-point $x_b = \beta_0(b)$, which is a 0-cell, and an orientation. We can thus consider the 2D holonomy $\mathrm{Hol}^2_{\beta_0(b)}(\mathrm{bd}(b), \mathcal{F}, L) \in \ker(\partial_{\mathcal{G}}) \subset E$, of $\mathcal{F} \in \Theta(M, L, \mathcal{G})$ along $\mathrm{bd}(b) \cong S^2$.

Definition 111 (Blob operator)**.** Let $b \in L^3$. Let also $a \in \ker(\partial_{\mathcal{G}}) \subset E$. The diagonal idempotent blob operator $C_b^a \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ is given by the formula below, for each basis element $\mathcal{F} \in \Theta(M, L, \mathcal{G})$:

$$C_b^a(\mathcal{F}) = \delta\Big(\mathrm{Hol}^2_{\beta_0(b)}\big(\mathcal{F}, \mathrm{bd}(b), L\big), a\Big)\mathcal{F}.$$

Here if $e, e' \in E$ we put $\delta(e', e) = 1$, if $e = e'$ and $\delta(e, e') = 0$, if $e \neq e'$.

See [23, III-B] and Ex. 77, for the explicit form of blob operators when our lattice L is a triangulation of M.

Since fake-flat 2-gauge configurations form an orthonormal basis of $\mathcal{H}(M, L, \mathcal{G})$, it is easy to see that each $C_b^a \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ is a self-adjoint operator.

As an immediate application of Thm. 101, follows:

Lemma 112. *Let $v \in L^0$, $t \in L^1$ and $b, b' \in L^3$ (it may be that $b = b'$). Let $g \in G$ and $e \in E$. Let $a, a' \in \ker(\partial_{\mathcal{G}}) \subset E$. We have:*

$$[\widehat{U_t^e}, C_b^a] = 0, \qquad [C_{b'}^{a'}, C_b^a] = 0, \qquad C_b^a C_b^{a'} = \delta(a, a') C_b^a,$$
$$v \neq \beta_0(b) \implies [\widehat{U_v^g}, C_b^a] = 0, \qquad v = \beta_0(b) \implies C_b^a\, \widehat{U_v^g} = \widehat{U_v^g}\, C_b^{g^{-1} \triangleright a}.$$

Hence edge gauge-spikes $\widehat{U_t^e}$ always commute with blob operators C_b^a, regardeless of t being an edge in b, or not. A vertex gauge-spike $\widehat{U_v^g}$ commutes with a blob operator C_b^a, unless v is the base point $\beta_0(b)$ of b.

5.1.3 The local operator algebra of higher lattice gauge theory

The algebra $\mathcal{OP}(M, L, \mathcal{G})$, which underpins the construction of the higher Kitaev model in 5.1.4, is our proposal for the local operator algebra of higher lattice gauge theory.

Definition 113 (Local operator algebra for higher lattice gauge theory)**.** Let (M, L) be a 2-lattice. Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a crossed module of finite groups. We define the $\mathbb{C}$-algebra $\mathcal{OP}(M, L, \mathcal{G})$ as formally generated by the

$$\widehat{U_v^g}, \quad v \in L^0, \quad g \in G;$$
$$\widehat{U_t^e}, \quad t = \big(\sigma(t) \xrightarrow{t} \tau(t)\big) \in L^1, \quad e \in E;$$
$$C_b^a, \quad b \in L^3, \quad a \in \ker(\partial_{\mathcal{G}});$$

imposing the relations appearing in Lem. 104 and 112.

Note that $\mathcal{OP}(M, L, \mathcal{G})$ is a *-algebra, where:

$$\widehat{U_v^g}^\dagger = \widehat{U_v^{g^{-1}}}, \qquad \widehat{U_t^e}^\dagger = \widehat{U_t^{e^{-1}}}, \qquad (C_b^a)^\dagger = C_b^a.$$

Given the discussion in 5.1.1 and 5.1.2, we hence have a unitary representation of $\mathcal{OP}(M, L, \mathcal{G})$ on the Hilbert space $\mathcal{H}(M, L, \mathcal{G})$ of higher lattice gauge theory.

5.1.4 The higher Kitaev model for (3+1)-dimensional topological phases

We now propose a higher gauge theory version (the "higher Kitaev model") of Kitaev quantum-double model for (2+1)-dimensional topological phases of matter [47, 42]. This higher Kitaev model for (3+1)-dimensional topological phases is formulated for manifolds M, of any dimension, with a 2-lattice decomposition L; see Def. 22. For a description of higher Kitaev model in the particular case of triangulated manifolds we refer the reader to [23], and to [73], in a more general context. Topological phases protected by higher gauge symmetry are also proposed in [44].

Definition 114 (Higher Kitaev model)**.** (Cf. the notation in §5.1.2). Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a finite crossed module of groups. Let M be a compact topological manifold, of any dimension, with a 2-lattice decomposition L. Our proposal for a totally solvable (the sum of mutually commuting projection operators) higher lattice gauge theory Hamiltonian, which we call the "higher Kitaev model":

$$H_L \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$$

(where $\mathcal{H}(M, L, \mathcal{G})$ is as in Def. 102), with respect to the 2-lattice (M, L) is (where 1_E is the identity of E):

$$\begin{aligned} H_L &= \sum_{v \in L^0} \Big(\mathrm{id} - \widehat{A_v}\Big) + \sum_{t \in L^1} \Big(\mathrm{id} - \widehat{B_t}\Big) + \sum_{b \in L^3} \big(\mathrm{id} - C_b^{1_E}\big) \\ &= \sum_{v \in L^0} \mathcal{A}_v + \sum_{t \in L^1} \mathcal{B}_t + \sum_{b \in L^3} \mathcal{C}_b \\ &= \mathcal{A} + \mathcal{B} + \mathcal{C}. \end{aligned} \tag{46}$$

The commutation relations of Lem. 104, 110 and 112, ensure that, if $u, v \in L^0$, $t, s \in L^1$ and $b, c \in L^3$:

$$\begin{aligned} &\mathcal{A}_v\mathcal{A}_v = \mathcal{A}_v, && \mathcal{B}_t\mathcal{B}_t = \mathcal{B}_t, && \mathcal{C}_b\mathcal{C}_b = \mathcal{C}_b, \\ &[\mathcal{A}_v, \mathcal{A}_u] = 0, && [\mathcal{B}_t, \mathcal{B}_s] = 0, && [\mathcal{C}_b, \mathcal{C}_c] = 0, \\ &[\mathcal{A}_v, \mathcal{B}_t] = 0, && [\mathcal{A}_v, \mathcal{C}_b] = 0, && [\mathcal{B}_t, \mathcal{C}_b] = 0, \end{aligned} \tag{47}$$

(Observe that these relations also hold for the $\widehat{A_v}$, $\widehat{B_t}$ and $C_b^{1_E}$.) And moreover we have that:

$$\begin{aligned} &[\mathcal{A}, \mathcal{B}] = 0, && [\mathcal{A}, \mathcal{C}] = 0, && [\mathcal{B}, \mathcal{C}] = 0. \\ &\mathcal{A}^2 = \mathcal{A}, && \mathcal{B}^2 = \mathcal{B} && \mathcal{C}^2 = \mathcal{C}. \end{aligned} \tag{48}$$

Note that by construction each term $\mathcal{A}_v, \mathcal{B}_t, \mathcal{C}_b \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ is Hermitian, hence so is H_L.

Typically M will be a 3-dimensional manifold, and the higher Kitaev model should be considered to be a model for (3+1)-dimensional topological phases [68, 73, 69, 23, 67, 49, 24, 44, 52]. The higher Kitaev model also makes sense if M is a surface, but in this case blob operators $C_b^{1_E}$ will not appear in the model.

Note that vertex and edge operators, which implement gauge invariance at vertices and edges of a 2-lattice, and the blob operators, which enforce 2-flatness at a blob, are very different in nature.

Lemma 115. *Let (M, L) be a 2-lattice. Let $\mathcal{G} = (\partial_{\mathcal{G}} \colon E \to G, \triangleright)$ be a finite crossed module of groups. Let $\mathcal{F} \in \Theta(M, L, \mathcal{G})$. Let*

$$\zeta_L(\mathcal{F}) = \#\{b \in L^3 : \mathrm{Hol}^2_{\beta_0(b)}\big(\mathcal{F}, \mathrm{bd}(b), L\big) \neq 1_E\} \in \mathbb{Z}_0^+.$$

Then $\zeta(\mathcal{F})$ is invariant under full gauge transformations and in particular it is invariant under the action of vertex and edge gauge operators. Moreover $\mathcal{C}(\mathcal{F}) = \zeta(\mathcal{F})\mathcal{F}$.

Proof. The first bits follow from Thm. 101: given $b \in L^3$, then $\mathrm{Hol}^2_{\beta_0(b)}\big(\mathcal{F}, \mathrm{bd}(b), L\big)$ is invariant under the action of gauge operators, up to acting by an element of G, which acts on E by automorphisms. On the other hand, the fact that $\mathcal{C}(\mathcal{F}) = \zeta(\mathcal{F})\mathcal{F}$ follows from the definition of $\mathcal{C} \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$. □

5.1.5 Example: higher gauge theory in the 3-sphere

Let us give an explicit description of the higher Kitaev model, if the underlying manifold is S^3. We consider two different 2-lattice decompositions of S^3. Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module of groups.

First case: (S^3, L_0)
Cf. Ex. 29 and 75. Consider the 3-sphere S^3 with the lattice decomposition L_0, with a unique 0-cell, no 1-cells, one 2-cell (thus the 2-skeleton is S^2) and two blobs, attaching on each side of the 2-sphere. By Ex. 75, the Hilbert space $\mathcal{H}(S^3, L_0, \mathcal{G})$ is thus isomorphic to $\mathbb{C}\ker(\partial)$, the vector space generated by the orthonormal basis $\ker(\partial) \subset E$. The 2D holonomy along the 2-sphere of a fake-flat 2-gauge configuration associated with $m \in \ker(\partial)$ is m itself: $\mathrm{Hol}^2_v(m, S^2, L_0) = m$.

For this 2-lattice we have no edge operators. The higher Kitaev Hamiltonian $H_{L_0} \colon \mathcal{H}(S^3, L_0, \mathcal{G}) \to \mathcal{H}(S^3, L_0, \mathcal{G})$ therefore has the form: $H_{L_0} = \mathcal{A}_{L_0} + \mathcal{C}_{L_0}$, where:

$$\mathcal{A}_{L_0} m = m - \frac{1}{\#G} \sum_{a \in G} (a \triangleright m), \tag{49}$$

$$\mathcal{C}_{L_0} m = m - \delta(m, 1_E)\, m. \tag{50}$$

Second case: $(S^3, L_{\mathfrak{g}})$
For a more substantial example, let us give S^3 the 2-lattice decomposition: $L_{\mathfrak{g}} = (\{v\}, \{t\}, \{P, P'\}, \{b, b'\})$ of Ex. 29. A 2-gauge configuration is given by a $g = g_t \in G$ and a pair $(e = e_P, f = e_{P'}) \in E \times E$. The fake flateness condition enforces that $\partial(e) = \partial(f) = g$. Therefore, we have:

$$\mathcal{H}(S^3, L_{\mathfrak{g}}, \mathcal{G}) \;=\; \mathbb{C}\{(g, e, f) \in G \times E \times E \;:\; \partial(e) = g \text{ and } \partial(f) = g\}.$$

The 2D holonomy of a configuration $\mathcal{F} = (g, e, f)$, along the 2-sphere $S^2 \subset S^3$, based at v, is:

$$\mathrm{Hol}^2_v(\mathcal{F}, S^2, L_\mathfrak{g}) = e^{-1}f \in \ker(\partial).$$

The vertex and edge gauge spikes on v and t, and the blob operators along b and b', have the form:

$$\widehat{U^a_v}(g, e, f) = \left(aga^{-1}, a \triangleright e, a \triangleright f\right), \tag{51}$$

$$\widehat{U^k_t}(g, e, f) = (\partial(k)g, ke, kf), \tag{52}$$

$$C^{k'}_b(g, e, f) = \delta(k', e^{-1}f)(g, e, f), \tag{53}$$

$$C^{k'}_{b'}(g, e, f) = \delta(k', e^{-1}f)(g, e, f). \tag{54}$$

(Cf. §5.1.1.) Here $a \in G$, $k \in E$ and $k' \in \ker(\partial) \subset E$. Note that $C^{k'}_b = C^{k'}_{b'}$, for each $k' \in \ker(\partial)$.

The commutation relations of Lem. 104 and 112 here boil down to:

$$\begin{aligned}
\widehat{U^a_v}\widehat{U^{a'}_v} &= \widehat{U^{aa'}_v}, && \text{where} \quad a, a' \in G;\\
\widehat{U^k_t}\widehat{U^l_t} &= \widehat{U^{kl}_t}, && \text{where} \quad k, l \in E;\\
\widehat{U^k_t}\widehat{U^a_v} &= \widehat{U^a_v}\widehat{U^{a^{-1}\triangleright e}_t}, && \text{where} \quad k \in E,\ a \in G;\\
C^{k'}_b\widehat{U^a_v} &= \widehat{U^a_v}C^{a^{-1}\triangleright k'}_b, && \text{where} \quad k' \in \ker(\partial),\ a \in G;\\
C^{k'}_b\widehat{U^l_t} &= \widehat{U^l_t}C^{k'}_b, && \text{where} \quad l \in E,\ k' \in \ker(\partial);\\
C^{k'}_b C^{k''}_b &= \delta(k', k'')C^{k'}_b, && \text{where} \quad k', k'' \in \ker(\partial).
\end{aligned}$$

This gives the local operator algebra $\mathcal{OP}(S^3, L_\mathfrak{g}, \mathcal{G})$; see Def. 113.

The higher Kitaev hamiltonian $H_{L_\mathfrak{g}} : \mathcal{H}(S^3, L_\mathfrak{g}, \mathcal{G}) \to \mathcal{H}(S^3, L_\mathfrak{g}, \mathcal{G})$ has the form $H_{L_\mathfrak{g}} = \mathcal{A}+\mathcal{B}+\mathcal{C}$, where:

$$\begin{aligned}
\mathcal{A}(g, e, f) &= (g, e, f) - \frac{1}{\#G}\sum_{a\in G}(aga^{-1}, a \triangleright e, a \triangleright f),\\
\mathcal{B}(g, e, f) &= (g, e, f) - \frac{1}{\#E}\sum_{k\in E}(\partial(k)g, ke, kf),\\
\mathcal{C}(g, e, f) &= (g, e, f) - \delta(e^{-1}f, 1_E)\,(g, e, f).
\end{aligned}$$

5.1.6 Comparison with the Kitaev model

Though constructed in a similar way, the higher Kitaev model and the Kitaev model [47] (also known as Kitaev quantum double model) are subtly different constructions. In the following, we will demonstrate that a subspace of the Kitaev model is equivalent to a class of higher Kitaev models, while differing in the whole Hilbert space.

In the language of this paper, the Kitaev model takes as input: a 2-lattice (M, L) and a finite group G, realising a lattice model with local operator algebra, which is at the base point of each plaquette isomorphic to $\mathcal{D}(G)$, the quantum double of G [47]. The Hilbert space $\mathcal{H}_K\big(M, L, G\big)$ of the Kitaev model is the free vector space on the set of gauge configurations $\mathcal{F}^1 : L^1 \to G$; see Def. 40. Considering the group G as the crossed module, $(1 \to G)$ (see Ex. 5), it follows:

$$\mathcal{H}_K\big(M, L, G\big) = \mathcal{H}\big(M^1, L, (1 \to G)\big).$$

Here M^1 is the 1-skeleton of (M, L). Note the use of M^1 as opposed to M, so that the fake-flatness condition becomes void. The Kitaev model is defined by the Hamiltonian

$$H^K_L = \mathcal{A} + \mathcal{D} : \mathcal{H}_K\big(M, L, G\big) \to \mathcal{H}_K\big(M, L, G\big).$$

Here the operator $\mathcal{A} = \sum_{v\in L^0}(\mathrm{id} - \widehat{A_v})$ is as in 5.1.2, defined for the crossed module $(1 \to G)$, with action on $\mathcal{H}_K\big(M, L, G\big)$ given by the action on $\mathcal{H}\big(M^1, L, (1 \to G)\big)$. Whereas, $\mathcal{D} = \sum_{P\in L^2}(\mathrm{id} - D^{1_E}_P)$ is defined from a new type of self-adjoint operators D_P^g, which act on gauge configurations $\mathcal{F}^1 : L^1 \to G$ as follows (for notation see Def. 43):

$$D_P^g(\mathcal{F}^1) = \delta\Big(\mathrm{Hol}^1_{\beta_0(P)}\big(\mathcal{F}^1, \mathrm{bd}(P), L\big), g\Big)\mathcal{F}^1, \text{ where } \beta_0(P) \text{ is the basepoint of } P.$$

Lemma 116. *Given $v, v' \in L^0$ and $P, P' \in L^2$ the following relations hold:*

$$\begin{aligned} [\mathcal{A}_v, \mathcal{A}_{v'}] &= 0, & \mathcal{A}_v\mathcal{A}_v &= \mathcal{A}_v; \\ [D_P^{1_E}, D_{P'}^{1_E}] &= 0, & D_P^{1_E} D_P^{1_E} &= D_P^{1_E}; \\ [\mathcal{A}_v, D_P^{1_E}] &= 0. \end{aligned}$$

Hence H_L^K is a sum of mutually commuting projection operators.

We now compare the Kitaev model with group G to the higher Kitaev model with crossed module $(1 \to G)$, with fixed 2-lattice (M, L). We begin by defining the flat sub-Hilbert space of the Kitaev model $\mathcal{H}_K^{\mathrm{flat}}(M, L, G) \subsetneq \mathcal{H}_K(M, L, G)$:

$$\mathcal{H}_K^{\mathrm{flat}}(M, L, G) = \{\mathcal{F}^1 \in \mathcal{H}_K(M, L, G) \quad | \quad \prod_{P \in L^2} D_P^{1_E}(\mathcal{F}^1) = \mathcal{F}^1\}.$$

It is straightforward to show

$$\mathcal{H}_K^{\mathrm{flat}}(M, L, G) = \mathcal{H}(M, L, (1 \to G)) \subsetneq \mathcal{H}_K(M, L, G).$$

This is due to the requirement of fake-flat 2-gauge configurations of $(1 \to G)$ on (M, L) being an equivalent condition to requiring $\prod_{P \in L^2} D_P^{1_E}(\mathcal{F}^1) = \mathcal{F}^1$.

For the crossed module $(1 \to G)$, both the blob and edge operators act as the identity. In this way the higher Kitaev Hamiltonian reduces to:

$$H_L = \mathcal{A}\colon \mathcal{H}\big(M, L, (1 \to G)\big) \to \mathcal{H}\big(M, L, (1 \to G)\big).$$

This model is equivalent to the Kitaev model defined on the flat sub-Hilbert space $\mathcal{H}_K^{\mathrm{flat}}\big(M, L, G\big)$:

$$H_K = \mathcal{A} + \mathcal{D}\colon \mathcal{H}_K^{\mathrm{flat}}\big(M, L, G\big) \to \mathcal{H}_K^{\mathrm{flat}}\big(M, L, G\big).$$

This is because, by definition, the operator $\mathcal{D}$ has trivial action on $\mathcal{H}_K^{\mathrm{flat}}\big(M, L, G\big) = \mathcal{H}\big(M, L, (1 \to G)\big)$, while the $\mathcal{A}$ operator has the same action on both Hilbert spaces. In this way we can identify the higher Kitaev model for $(1 \to G)$ with the Kitaev model defined on the sub-Hilbert space $\mathcal{H}_K^{\mathrm{flat}}\big(M, L, G\big)$. However, the Kitaev model diverges from the higher Kitaev model for $(1 \to G)$ outside of the sub-space $\mathcal{H}_K^{\mathrm{flat}}\big(M, L, G\big)$ due to the presence of non-fake-flat configurations in $H_K\big(M, L, G\big)$.

5.2 Ground state degeneracy

Let M be a compact manifold and $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite group crossed module. Let L be a 2-lattice decomposition of M. Hence $\mathcal{H}(M, L, \mathcal{G})$ is a finite dimensional Hilbert space, which explicitly depends on the 2-lattice decomposition L of M. In this subsection, we prove that the dimension of the ground state space $GS(M, L, \mathcal{G})$ of the higher Kitaev model $H_L\colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ in 5.1.4 is a topological invariant of M, meaning that $\dim GS(M, L, \mathcal{G})$ depends only on M alone. Specifically, we will show that $GS(M, L, \mathcal{G})$ has a basis in canonical one-to-one correspondence with the set of homotopy classes of maps from M to the classifying space $B_{\mathcal{G}}$ of the crossed module $\mathcal{G}$. (Classifying spaces of crossed modules are defined in [18, §2.4] and [17, 11, 35, 29].) It therefore follows that the dimension of the ground state space $GS(M, L, \mathcal{G})$ is a homotopy invariant of manifolds, as expected given the relation [23] of our model to Yetter's invariant of manifolds [76, 61]. Yetter's invariant was proven in [35, 29] to be a homotopy invariant of manifolds.

This subsection is less self-contained than the remainder of the paper. Cf. Rems. 94 and 95. Let X be a CW-complex. We use deep results of Brown and Higgins on the description of the weak homotopy type of the function space $TOP(X, B_{\mathcal{X}})$, where $\mathcal{X}$ is a crossed complex (a generalisation of crossed modules) and

$B_{\mathcal{X}}$ is its classifying space; the bits we need can be found in [17, Thm. A] and [18, Thm. 11.4.19]. Let $\Pi(X)$ denote the fundamental crossed complex of X. The main tool we use is the fact that the weak homotopy type of $TOP(X, B_{\mathcal{X}})$, with the k-ification of the compact-open topology on the space of continuous maps $X \to B_{\mathcal{X}}$, is represented by the crossed complex $\mathrm{CRS}(\Pi(X), \mathcal{X})$; an explanation of this is in [35, §2.6.1]. The crossed complex $\mathrm{CRS}(\Pi(X), \mathcal{X})$ is made of crossed complex maps $\Pi(X) \to \mathcal{X}$ and n-fold homotopies, $n \in \mathbb{N}$.

For a crossed module $\mathcal{G}$, the underlying groupoid of the crossed complex $\mathrm{CRS}(\Pi(X), \mathcal{G})$ is the groupoid $\mathrm{CRS}_1(\Pi_2(X, X^1, X^0), \mathcal{G})$ of crossed module maps $\Pi_2(X, X^1, X^0) \to \mathcal{G}$ and their homotopies, referred to in Rem 89, 94 and 99; see [18, §7.1.vii and §9.3.i]. Combining with Thm 82, it hence follows that if (M, L) is a 2-lattice, then the underlying groupoid of $\mathrm{CRS}(\Pi(X), \mathcal{G})$ is isomorphic to the groupoid $\Theta^{\#}_{\text{flat}}(M, L, \mathcal{G})$ of 2-flat 2-gauge configurations and full gauge transformations between them: see 4.3.3.

Given a 2-lattice (M, L), the results of [17] and [18, §11.4] hence tell us that path-connected components of the function space $TOP(M, B_{\mathcal{G}})$ – i.e. homotopy classes of maps $M \to B_{\mathcal{G}}$ – are in canonical one-to-one correspondence with connected components of the groupoid $\Theta^{\#}_{\text{flat}}(M, L, \mathcal{G})$ of 2-flat 2-gauge configurations and full gauge transformations between them; see §4.3.3. This will be the main tool used in this subsection.

Let us now connect the discussion in the previous paragraphs with the ground state degeneracy of the higher Kitaev model. We start by looking at the expression (46) for $H_L \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$:

$$H_L = \sum_{v \in L^0} \left(\mathrm{id} - \widehat{A_v}\right) + \sum_{t \in L^1} \left(\mathrm{id} - \widehat{B_t}\right) + \sum_{b \in L^3} \left(\mathrm{id} - C_b^{1_E}\right).$$

Each of the operators $\mathrm{id} - \widehat{A_v}$, where $v \in L^0$ is a vertex; $\mathrm{id} - \widehat{B_t}$, where $t \in L^1$ is an edge; and $\mathrm{id} - C_b^{1_E}$, where $b \in L^3$ is a blob, is a Hermitian projector. All of those projectors commute. We can choose an eigenspace decomposition $\mathcal{H}(M, L, \mathcal{G}) = \sum_i^{\perp} \mathcal{H}_i(M, L, \mathcal{G})$ with respect to which all those projectors are diagonal.

If we apply $\widehat{A_v}$ or $\widehat{B_t}$ to $\mathcal{F} \in \Theta(M, L, \mathcal{G})$, we get a non-zero linear combination of fake-flat 2-gauge configurations with non-negative coefficients. Let us write $\mathcal{H}^+$ for the subset of non-zero $\mathbb{R}_{\geq 0}$-linear combinations in $\mathcal{H} = \mathcal{H}(M, L, \mathcal{G})$. Recall the naive vacuum Ω_1 is given by $\mathcal{F}(x) = 1$ for all cells x; see §3.2.1 and Ex. 79. Since $\widehat{A_v}$ and $\widehat{B_t}$ both take an $\mathcal{F}$ to an element of $\mathcal{H}^+$, and indeed take an element of $\mathcal{H}^+$ to an element of $\mathcal{H}^+$, we have that $\Psi_0 = \prod_v \widehat{A_v} \prod_t \widehat{B_t} \Omega_1 \in \mathcal{H}$ is non-zero. Since $C_b^{1_E} \Omega_1 = \Omega_1$, for each $b \in L^3$, it follows that there is an H_L-eigenspace of $\mathcal{H}$ (containing Ψ_0) with eigenvalue 0. (Note that Ψ_0 is in the kernel of all of the $\mathrm{id} - \widehat{A_v}$, $\mathrm{id} - \widehat{B_t}$ and $\mathrm{id} - C_b^{1_E}$, where $v \in L^0$, $t \in L^1$ and $b \in L^3$, given the commutation relations in (47).)

Now projectors have eigenvalues 0 or 1, thus the ground state has energy zero, meaning:

$$GS(M, L, \mathcal{G}) = \{\Psi \in \Theta(M, L, \mathcal{G}) : H_L \Psi = 0\}.$$

And, furthermore, a vector belongs to the ground state $GS(M, L, \mathcal{G})$ if, and only if, it is in the kernel of all of the projectors $\mathrm{id} - \widehat{A_v}$, $\mathrm{id} - \widehat{B_t}$ and $\mathrm{id} - C_b^{1_E}$, where $v \in L^0$, $t \in L^1$ and $b \in L^3$.

Lemma 117. *A state* $\Psi = \sum_{\mathcal{F} \in \Theta(M, L, \mathcal{G})} \lambda_{\mathcal{F}} \mathcal{F} \in \mathcal{H}(M, L, \mathcal{G})$*, where* $\lambda_{\mathcal{F}} \in \mathbb{C}$*, is in* $GS(M, L, \mathcal{G})$ *if and only if:*

(i) unless $\mathcal{F}$ *is 2-flat then* $\lambda_{\mathcal{F}} = 0$*; see Def. 78 for the definition of a 2-flat configuration;*

(ii) given any $g \in G$*, any vertex* $v \in L^0$ *and any* $\mathcal{F} \in \Theta(M, L, \mathcal{G})$*, it holds* $\lambda_{\mathcal{F}} = \lambda_{\widehat{U_v^g}(\mathcal{F})}$*;*

(iii) given any $e \in E$*, any edge* $t \in L^1$ *and any* $\mathcal{F} \in \Theta(M, L, \mathcal{G})$*, it holds* $\lambda_{\mathcal{F}} = \lambda_{\widehat{U_t^e}(\mathcal{F})}$*.*

Proof. First the 'only if' part. Cf. the discussion just before the Lemma. In order that $\Psi \in GS(M, L, \mathcal{G})$ it must be that (i) $C_b^{1_E}(\Psi) = \Psi, \forall b \in L^3$; that (ii) $\widehat{A_v}\Psi = \Psi, \forall v \in L^0$; and that (iii) $\widehat{B_t}\Psi = \Psi, \forall t \in L^1$.

(i) For each $b \in L^3$, $C_b^{1_E}(\Psi) = \sum_{\mathcal{F} \in \Theta(M, L, \mathcal{G})} \lambda_{\mathcal{F}} \delta\Big(\mathrm{Hol}_v^2(\mathcal{F}, \mathrm{bd}(b)), 1_E\Big)\mathcal{F}$. In order that $C_b^{1_E}(\Psi) = \Psi, \forall b \in L^3$, it must be that whenever $\lambda_{\mathcal{F}} \neq 0$: $\delta\Big(\mathrm{Hol}_v^2(\mathcal{F}, \mathrm{bd}(b)), 1_E\Big) = 1, \forall b \in L^3$. Hence $\lambda_{\mathcal{F}} \neq 0 \implies \mathcal{F}$ is 2-flat.

(ii) Suppose that $\widehat{A_v}(\Psi) = \Psi$, for all $v \in L^0$. Then:

$$\Psi = \sum_{\mathcal{F}' \in \Theta(M,L,\mathcal{G})} \lambda_{\mathcal{F}'} \mathcal{F}' = \widehat{A_v}(\Psi) = \frac{1}{\#G} \sum_{\mathcal{F}' \in \Theta(M,L,\mathcal{G})} \Big(\sum_{h \in G} \lambda_{\mathcal{F}'} \widehat{U_v^h}(\mathcal{F}') \Big).$$

Now apply $\langle \mathcal{F}, - \rangle$. And since $\widehat{U_v^{gh}}(\mathcal{F}) = \widehat{U_v^g}\big(\widehat{U_v^h}(\mathcal{F})\big)$, we have for each $v \in L^0$ and any $g \in G$:

$$\lambda_{\mathcal{F}} = \frac{1}{\#G} \sum_{h \in G} \lambda_{\widehat{U_v^{h^{-1}}}(\mathcal{F})} = \frac{1}{\#G} \sum_{h \in G} \lambda_{\widehat{U_v^{h^{-1}}}(\widehat{U_v^g}(\mathcal{F}))} = \lambda_{\widehat{U_v^g}(\mathcal{F})}.$$

(iii) Analogously, it order that Ψ be in the kernel of all $(\mathrm{id} - \widehat{B_t})$, then $\lambda_{\mathcal{F}} = \lambda_{\widehat{U_t^e}(\mathcal{F})}$.

Conversely, if Ψ satisfies (i),(ii) and (iii), then Ψ will be in the kernel of all operators $\mathrm{id} - \widehat{A_v}$, $\mathrm{id} - \widehat{B_t}$ and $\mathrm{id} - C_b^{1_E}$, where $v \in L^0, t \in L^1$ and $b \in L^3$. As such Ψ will be in the ground state $GS(M, L, \mathcal{G})$. ☐

Let $\Theta_{\mathrm{2flat}}(M, L, \mathcal{G})$ be the set of 2-flat 2-gauge configurations in (M, L); Def. 78. We say that $\mathcal{F}$ and $\mathcal{F}'$ in $\Theta_{\mathrm{2flat}}(M, L, \mathcal{G})$ are equivalent ($\mathcal{F} \cong \mathcal{F}'$) if we can go from $\mathcal{F}$ to $\mathcal{F}'$ through acting by a sequence of vertex and edge gauge spikes; see 5.1.1. By Lem. 117: we hence have:

$$GS(M, L, \mathcal{G}) = \Big\{ \sum_{\mathcal{F} \in \Theta_{\mathrm{2flat}}(M,L,\mathcal{G})} \lambda_{\mathcal{F}} \mathcal{F} \in \mathcal{H}(M, L, \mathcal{G}) \ \Big| \ \forall \mathcal{F}, \mathcal{F}' \in \Theta_{\mathrm{2flat}}(M, L, \mathcal{G}) : \mathcal{F} \cong \mathcal{F}' \implies \lambda_{\mathcal{F}} = \lambda_{\mathcal{F}'} \Big\}.$$

The composite of gauge spikes is a full gauge transformation and, Lem. 104 (V), any full gauge transformation is the composition of gauge spikes; as such to say that $\mathcal{F}$ and $\mathcal{F}'$ in $\Theta_{\mathrm{2flat}}(M, L, \mathcal{G})$ are equivalent is to say that they are connected by a full gauge transformation. In other words $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if, and only if, they can be connected by a morphism in the groupoid $\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})$, of 2-flat 2-gauge configurations and full gauge transformations between them; see §4.3.3.

The set of connected components $[\mathcal{F}']$ of the groupoid $\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})$ is denoted by $\pi_0\big(\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})\big)$. I.e. $\pi_0\big(\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})\big)$ is the set of equivalence classes of objects of $\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})$, where two 2-flat 2-gauge configurations are equivalent if a full gauge transformation connects the two. We have a basis $\mathsf{B}_0(M, L, \mathcal{G})$ for the ground space $GS(M, L, \mathcal{G})$, in one-to-one correspondence with $\pi_0\big(\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})\big)$, namely:

$$\mathsf{B}_0(M, L, \mathcal{G}) = \Big\{ \sum_{\mathcal{F} \in \Theta_{\mathrm{2flat}}(M,L,\mathcal{G}) \text{ such that } \mathcal{F} \in [\mathcal{F}']} \mathcal{F} \ \Big| [\mathcal{F}'] \in \pi_0\big(\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})\big) \Big\}. \tag{55}$$

As explained in [17, Thm. A] or [18, Thm. 11.4.19] (in the general case of crossed complexes), there is a natural bijection between elements of $\pi_0\big(\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})\big) = \pi_0\big(\mathrm{CRS}_1\big(\Pi_2(M, M^1, M^0), \mathcal{G}\big)\big)$ and homotopy classes of maps $M \to B_{\mathcal{G}}$; cf. Rems. 94 and 99. (For an explanation of this in the crossed module case see [35, 29].) In particular the cardinality of the set $\pi_0\big(\Theta^\sharp_{\mathrm{2flat}}(M, L, \mathcal{G})\big)$ does not depend on L.

Theorem 118. *Let M be a compact manifold. Let L be a 2-lattice decomposition M. Let $\mathcal{G}$ be a finite crossed module. Consider the higher Kitaev Hamiltonian $H_L \colon \mathcal{H}(M, L, \mathcal{G}) \to \mathcal{H}(M, L, \mathcal{G})$ of Def. 114. Then the ground state $GS(M, L, \mathcal{G})$ of H_L has a basis in canonical one-to-one correspondence with the set of homotopy classes of maps $f \colon M \to B_{\mathcal{G}}$, where $B_{\mathcal{G}}$ is the classifying space of the crossed module $\mathcal{G}$. Hence $\dim GS(M, L, \mathcal{G})$ depends only on the topology of M and not on the chosen 2-lattice decomposition L of M.*

Proof. Compare the preceding observation with (55). ☐

Remark 119. It was proven in [23] that the dimension of the ground state $GS(M, L, \mathcal{G})$ coincides with Yetter invariant [76, 61] $Y(M \times S^1)$ of $M \times S^1$. We note that the Yetter invariant of $M \times S^1$ is not quite the same as $\dim GS(M \times S^1, L, \mathcal{G})$ since $Y(M \times S^1)$ uses finer information on the space of functions $M \times S^1 \to B_{\mathcal{G}}$ than its number of connected components; see [35].

Remark 120. We can write $\dim GS(M, L, \mathcal{G})$ as $\dim GS(M, \mathcal{G})$, since it does not depend on L, and only on M. Indeed $\dim GS(M, \mathcal{G})$ depends only on the homotopy type of M and the weak homotopy type of the crossed module $\mathcal{G}$ [35], since the homotopy type of $B_{\mathcal{G}}$ depends only on the weak homotopy type of $\mathcal{G}$.

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