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MATRIX SEMIGROUPS OVER SEMIRINGS

VICTORIA GOULD, MARIANNE JOHNSON, AND MUNAZZA NAZ

ABSTRACT. We study properties determined by idempotents in the following families of matrix semigroups over a semiring S : the full matrix semigroup $M_n(S)$, the semigroup $UT_n(S)$ consisting of upper triangular matrices, and the semigroup $U_n(S)$ consisting of all unitriangular matrices. It has been shown that (for $n > 2$) the semigroup $M_n(S)$ is regular if and only if S is a regular *ring*. We show that $UT_n(S)$ is regular if and only if $n = 1$ and the multiplicative semigroup of S is regular. The notions of being abundant or Fountain (formerly, weakly abundant) are weaker than being regular but are also defined in terms of idempotents, namely, every class of certain equivalence relations must contain an idempotent. Each of $M_n(S)$, $UT_n(S)$ and $U_n(S)$ admits a natural anti-isomorphism allowing us to characterise abundance and Fountainicity in terms of the left action of idempotent matrices upon column spaces. In the case where the semiring is exact, we show that $M_n(S)$ is abundant if and only if it is regular.

Our main interest is in the case where S is an idempotent semifield, our motivating example being that of the tropical semiring \mathbb{T} . We prove that certain subsemigroups of $M_n(S)$, including several generalisations of well-studied monoids of binary relations (Hall relations, reflexive relations, unitriangular Boolean matrices), are Fountain. We also consider the subsemigroups $UT_n(S^*)$ and $U_n(S^*)$ consisting of those matrices of $UT_n(S)$ and $U_n(S)$ having all elements on and above the leading diagonal non-zero. We prove the idempotent generated subsemigroup of $UT_n(S^*)$ is $U_n(S^*)$. Further, $UT_n(S^*)$ and $U_n(S^*)$ are families of Fountain semigroups with interesting and unusual properties. In particular, every $\tilde{\mathcal{R}}$ -class and $\tilde{\mathcal{L}}$ -class contains a *unique* idempotent, where $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ are the relations used to define Fountainicity, but yet the idempotents do not form a semilattice.

1. INTRODUCTION

The ideal structure of the multiplicative semigroup $M_n(F)$ of all $n \times n$ matrices over a field F is simple to understand. The semigroup $M_n(F)$ is regular (that is, for each $A \in M_n(F)$ there is a $B \in M_n(F)$ such that $A = ABA$) and possesses precisely $n + 1$ ideals. These are the principal ideals $I_k = \{A \in M_n(F) : \text{rank } A \leq k\}$ where $0 \leq k \leq n$. Clearly $\{0\} = I_0 \subset I_1 \subset \dots \subset I_n = M_n(F)$; the resulting (Rees) quotients I_k/I_{k-1} for $1 \leq k \leq n$ have a particularly pleasing structure, as we now explain. To do so, we use the language of Green's relations \mathcal{L} , \mathcal{R} and \mathcal{J} on a semigroup S , where two elements are \mathcal{L} -related (\mathcal{R} -related, \mathcal{J} -related) if and only if they generate the same principal left (right, two-sided) ideal; correspondingly, the \mathcal{L} -classes, (\mathcal{R} -classes, \mathcal{J} -classes) are partially ordered by inclusion of left (right, two-sided) ideals. We give more details of Green's relations in Section 2.

The matrices of each fixed rank k form a single \mathcal{J} -class, so that the \mathcal{J} -order corresponds to the natural order on ranks, and the \mathcal{L} - and \mathcal{R} - orders correspond to containment of row and column spaces, respectively. We have that \mathcal{J} is the join \mathcal{D} of \mathcal{L} and \mathcal{R} , so

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that, since $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ for any semigroup, it follows that the non-zero elements of the quotients I_k/I_{k-1} (for $1 \leq k \leq n$) may be co-ordinatised by their \mathcal{L} -class, their \mathcal{R} -class and a maximal subgroup of rank k matrices. Moreover, each such subgroup is isomorphic to the general linear group $\mathrm{GL}_k(F)$; (see [34] for further details and results).

The subset $UT_n(F)$ of all upper triangular matrices (that is, those with all entries below the main diagonal equal to 0) forms a submonoid of $M_n(F)$. This submonoid is not regular, but satisfies a weaker regularity property called *abundance*. The subset $U_n(F)$ of unitriangular matrices (that is, those upper triangular matrices with all diagonal entries equal to 1) forms a subgroup of the group of units $\mathrm{GL}_n(F)$ of $M_n(F)$. In the case where F is an algebraically closed field, the semigroups $M_n(F)$, $UT_n(F)$ and $U_n(F)$ are examples of *linear algebraic monoids* [36, 38] and have been extensively studied. Certain important examples of linear algebraic monoids, including $M_n(F)$, have the property that the subsemigroup of singular elements is idempotent generated [7, 36].

Motivated by the above, in this paper we consider $n \times n$ matrices with entries in a *semiring* S , with a focus on *idempotent semifields*. The operation of matrix multiplication, defined in the usual way with respect to the operations of S , yields semigroups $M_n(S)$, $UT_n(S)$ and $U_n(S)$ analogous to those above. Our motivating examples are that of the Boolean semiring \mathbb{B} and the tropical semiring \mathbb{T} . In the first case, it is well known that $M_n(\mathbb{B})$ is isomorphic to the monoid of all binary relations on an n -element set, under composition of relations. In the second case, $M_n(\mathbb{T})$ is the monoid of $n \times n$ *tropical matrices*, which are a source of significant interest (see, for example, [10, 21, 22]). Many of the tools which apply in the field case, such as arguments involving rank and invertible matrices, do not immediately carry over to the more general setting of semirings, and we are required to develop largely new strategies. However, since $M_n(S)$ can be identified with the semigroup of endomorphisms of the free module S^n , one can phrase several of its structural properties in terms of finitely generated submodules of S^n . For a general (semi)ring S , the relationships between such modules can be much more complicated than the corresponding situation for fields (where it is easy to reason with finite dimensional vector spaces), and as a result one finds that the structure of $M_n(S)$ can be highly complex.

It is known that the semigroup $M_n(R)$ over a ring R is regular if and only if R is von Neumann regular¹ (see [4, 42]). Il'in [20] has generalised this result to the setting of semirings, providing a necessary and sufficient condition for $M_n(S)$ to be regular; for all $n \geq 3$ the criterion is simply that S is a von Neumann regular *ring*. Given that our work concerns semigroups of matrices over a *semiring* S , we consider two natural generalisations of regularity, namely abundance and “Fountainicity” (also known as weak abundance, or semi-abundance; the term Fountain having been recently introduced by Margolis and Steinberg [32] in honour of John Fountain’s work in this area). We do this first in the context of the semigroups $M_n(S)$, $UT_n(S)$ and $U_n(S)$. Later, in the case where S is an idempotent semifield, we make a careful study of these properties for the semigroups $UT_n(S^*)$ and $U_n(S^*)$, where these are the subsemigroups of $UT_n(S)$ and $U_n(S)$, respectively, in which all the entries above the leading diagonal are non-zero. Abundance and Fountainicity are determined by properties of the relations \mathcal{L}^* and \mathcal{R}^* (for abundance) and $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ (for Fountainicity), where \mathcal{L}^* is a natural extension of \mathcal{L} , and $\tilde{\mathcal{L}}$ a natural extension of \mathcal{L}^* , similar statements being true for the dual relations. It is worth remarking that the property of abundance may be phrased in terms of projectivity of monogenic acts over monoids [29].

For an idempotent semifield S , we describe two functions $(+): M_n(S) \rightarrow E(M_n(S))$ and $(*): M_n(S) \rightarrow E(M_n(S))$ which map $A \in M_n(S)$ to left and right identities $A^{(+)}$ and $A^{(*)}$ for A , respectively. These maps are used to show that certain subsemigroups of

¹That is, the multiplicative semigroup of R is regular.

$M_n(S)$, including several generalisations of well-studied monoids of binary relations (Hall relations, reflexive relations, unitriangular Boolean matrices), are Fountain. Many of the semigroups under consideration turn out to have the stronger property that each $\tilde{\mathcal{R}}$ -class and each $\tilde{\mathcal{L}}$ -class contains a *unique* idempotent. Fountain semigroups whose idempotents commute are known to have this property, as does any semigroup $\mathcal{P}(G)$ formed by taking subsets of a finite group G under multiplication inherited from G . In general, however, examples of Fountain semigroups with non-commuting idempotents are elusive. In this paper we provide examples of such semigroups which occur naturally in our setting, and give a detailed study of some particularly interesting families of these.

In Section 2 we provide the necessary background on semirings, semimodules and generalised regularity conditions, including the details of Green's relations and their extensions mentioned above. In Section 3 we show that each of the semigroups $M_n(S)$, $UT_n(S)$ and $U_n(S)$ admits a natural left-right symmetry and provide module theoretic characterisations of \mathcal{R} , \mathcal{L} , \mathcal{R}^* , \mathcal{L}^* , $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ and hence of regularity, abundance and Fountainicity in the case of $M_n(S)$. In contrast to Il'in's result that $M_n(S)$ is regular if and only if S is a regular ring [20] we show that $UT_n(S)$ is regular if and only if $n = 1$ and the multiplicative semigroup of S is a regular semigroup (**Proposition 3.4**). Many of the semirings we study in this article are exact [46]; in the case where S is exact, $M_n(S)$ is abundant if and only if it is regular (**Theorem 3.5**). Thus in large part in this article we concentrate on Fountainicity.

From Section 4 onwards we focus our attention solely on the case where S is an idempotent semifield. We make use of a particular idempotent matrix construction to prove that several natural subsemigroups of $M_n(S)$, including $U_n(S)$, (**Corollary 4.18**) are Fountain. Specifically, for every $A \in M_n(S)$ we associate idempotents $A^{(+)}$ and $A^{(*)}$ in $M_n(S)$; these idempotents, in certain situations, play the role of AB and BA for a regular matrix A with $A = ABA$. Section 4 ends by considering the behaviour of idempotents in $UT_n(S^*)$. Idempotent generated semigroups are of fundamental importance in algebraic semigroup theory (see, for example, [17, 33, 36]). We show that the idempotent generated subsemigroup of $UT_n(S^*)$ is precisely $U_n(S^*)$ (**Corollary 4.25**).

Every idempotent semifield can be constructed from a lattice ordered abelian group \mathfrak{L}^* , by adjoining a zero element and taking addition to be least upper bound: the resulting semifield is denoted by \mathfrak{L} and (with some abuse of notation) we use \mathfrak{L}^* to denote its non-zero elements. In Section 5 we specialise to the case where the natural partial order on \mathfrak{L} is total. For the Boolean semiring \mathbb{B} we provide a complete description of the generalised regularity properties of $M_n(\mathbb{B})$, $UT_n(\mathbb{B})$ and $U_n(\mathbb{B})$, making use of our idempotent construction (**Theorem 5.2**), and provide a partial description for $M_n(\mathfrak{L})$, $UT_n(\mathfrak{L})$ and $U_n(\mathfrak{L})$.

For the remainder of the paper we consider one family of Fountain subsemigroups of $M_n(S)$ in detail, namely the semigroups $UT_n(\mathfrak{L}^*)$ where \mathfrak{L} is the semiring associated with a linearly ordered abelian group. This family has some unusual, but striking, properties as Fountain semigroups. It is known that if the idempotents of a Fountain semigroup S commute, then every element is $\tilde{\mathcal{R}}$ -related and $\tilde{\mathcal{L}}$ -related to a unique idempotent. Examples of Fountain semigroups in which the latter behaviour holds without the idempotents commuting are hard to find: see [1] in the case where $\tilde{\mathcal{L}} = \mathcal{L}^*$ and $\tilde{\mathcal{R}} = \mathcal{R}^*$. On the other hand, by this stage we have seen that every element A of $UT_n(\mathfrak{L}^*)$ is $\tilde{\mathcal{R}}$ -related to a unique idempotent, the idempotent $A^{(+)}$ described above, and dually, A is $\tilde{\mathcal{L}}$ -related to a unique idempotent, $A^{(*)}$ (**Corollary 4.17**), but the idempotents of $UT_n(\mathfrak{L}^*)$ do not commute for $n \geq 3$.

Much of the literature dealing with Fountain semigroups considers only those for which $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ are, respectively, left and right compatible with multiplication, a property held by \mathcal{R}^* and \mathcal{L}^* , and by \mathcal{R} and \mathcal{L} . For such semigroups the $\tilde{\mathcal{H}} := \tilde{\mathcal{R}} \cap \tilde{\mathcal{L}}$ -classes of

idempotents are subsemigroups, indeed, *unipotent monoids* (monoids possessing a single idempotent). A substantial theory exists for Fountain semigroups in which unipotent monoids, arising from the $\tilde{\mathcal{H}}$ -classes of idempotents, or as quotients, play the role held by maximal subgroups in the theory of regular semigroups (see, for example, [13]). In Section 6 we see that not only are $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ not left and right compatible, respectively, but are as far from being so as possible, in a sense we will make precise (**Proposition 6.10**). We show that $UT_n(\mathfrak{L}^*)$ is regular if and only if it is abundant, and characterise the generalised regularity properties of $UT_n(\mathfrak{L}^*)$ and $U_n(\mathfrak{L}^*)$ (**Corollary 6.5**). In particular, $UT_n(\mathfrak{L}^*)$ and $U_n(\mathfrak{L}^*)$ are not regular, but are Fountain, for $n \geq 3$. The maximal subgroups of $UT_n(\mathfrak{L}^*)$ are isomorphic to the underlying abelian group \mathfrak{L}^* (**Theorem 6.6**).

Finally, in Section 7 we continue the theme of Section 6 by carefully analysing the structure of $\tilde{\mathcal{R}}$ -, $\tilde{\mathcal{L}}$ - and $\tilde{\mathcal{H}}$ -classes in $UT_n(\mathfrak{L}^*)$. We use the notion of defect, introduced in [40] in the special case of the tropical semiring, to determine the behaviour of the $\tilde{\mathcal{R}}$ - and $\tilde{\mathcal{L}}$ -classes, and hence the $\tilde{\mathcal{H}}$ -classes, in $UT_n(\mathfrak{L}^*)$. For $n \leq 4$ the $\tilde{\mathcal{H}}$ -classes of idempotents are always subsemigroups, and in certain cases for $n \geq 5$, depending on properties related to defect, that we refer to as being tight or loose. However, we are also able to show that for every $n \geq 5$ there is an idempotent in $UT_n(\mathfrak{L}^*)$ such that its $\tilde{\mathcal{H}}$ -class is not a subsemigroup (**Proposition 7.16**). These properties of tightness and looseness derive from the fact that the conditions for a matrix $E \in UT_n(\mathfrak{L}^*)$ to be idempotent correspond to a certain set of inequalities holding between products of the entries of E ; we say that E is tight in a particular product, if the inequality corresponding to that product is tight. In the tropical case, these inequalities are a classical linear system of inequalities in $\mathbb{R}^{n(n+1)/2}$, and hence describe a polyhedron of idempotents; tightness of an idempotent matrix E in a specified product therefore corresponds to the point of $\mathbb{R}^{n(n+1)/2}$ corresponding to E lying on the hyperplane specified by that product.

Throughout this article we pose a series of Open Questions concerning abundance and Fountainicity of matrix semigroups over semirings. Indeed, many of these questions remain unanswered even in the case of rings. We hope our article provides a catalyst for other investigators.

2. PRELIMINARIES

In order to keep our paper self-contained, we briefly recall some key definitions and results. For further information on semirings the reader may consult [11].

2.1. Semirings. A semiring is a commutative monoid $(S, +, 0_S)$ with an associative (but not necessarily commutative) multiplication $S \times S \rightarrow S$ that distributes over addition from both sides, where the additive identity 0_S is assumed to be an absorbing element for multiplication (that is, $0_S a = a 0_S = 0_S$ for all $a \in S$). Thus a ring is a semiring in which $(S, +, 0_S)$ is an abelian group. *Throughout this paper we shall assume that S is unital (i.e. contains a multiplicative identity element, denoted 1_S) and commutative.* A (semi)field is a (semi)ring in which $S \setminus \{0_S\}$ is an abelian group under multiplication; we denote this group by S^* . We say that a semiring is *idempotent* if the addition is idempotent, and *anti-negative* if $a + b = 0_S$ implies $a = b = 0_S$. It is easy to show that every idempotent semiring is anti-negative and that idempotent semirings are endowed with a natural partial order structure given by $a \leq b$ if and only if $a + b = b$.

It may help the reader to keep the following examples in mind:

1. The set of non-negative integers \mathbb{N}_0 with the usual operations of addition and multiplication.
2. The Boolean semiring $\mathbb{B} = \{0, 1\}$ with idempotent addition $1 + 1 = 1$.
3. The tropical semiring $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ with addition \oplus given by taking the maximum (where $-\infty$ is the least element and hence plays the role of the additive

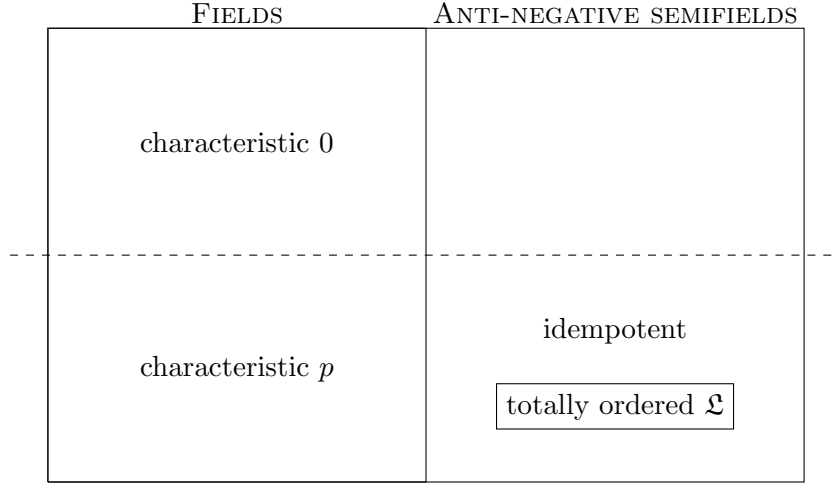


FIGURE 1. A rough guide to semifields.

identity) and multiplication \otimes given by usual addition of real numbers, together with the rule $-\infty \otimes a = a \otimes -\infty = -\infty$.

4. More generally, if G is lattice ordered abelian group, one may construct an idempotent semifield from G by adjoining a minimal element, to be treated as zero, to G and taking addition to be least upper bound. (In fact, every idempotent semifield arises in this way – see Lemma 2.1 below.)

For ease of reference we record (with no claim of originality) a number of facts about semifields, which have been observed by many authors (see for example [15, 19, 44]):

Lemma 2.1. *Let S be a semifield.*

- (i) *Either S is a field or S is anti-negative.*
- (ii) *If S is anti-negative, then the multiplicative group S^* is torsion-free.*
- (iii) *If S is finite, then S is a finite field or the Boolean semiring.*
- (iv) *If $x + x = x$ for some $x \in S^*$, then S is idempotent.*
- (v) *If S is idempotent, then S^* is a lattice ordered abelian group.*

Examples 2 and 3 above are both idempotent semifields, in which the natural partial order is total. The respective multiplicative groups are the trivial group, and the group of real numbers with the usual order. In general, we shall denote an idempotent semifield in which the natural partial order is total by \mathfrak{L} , since the corresponding multiplicative group \mathfrak{L}^* is a linearly ordered abelian group.

2.2. Modules and matrices. A (left) S -module is a commutative monoid $(X, +, 0_X)$ together with a left action $S \times X \rightarrow X$ satisfying for all $s, t \in S$, and all $x, y \in X$:

$$\begin{aligned} 1_S \cdot x &= x, & s \cdot (t \cdot x) &= st \cdot x, & s \cdot 0_X &= 0_X = 0_S \cdot x, \\ s \cdot (x + y) &= s \cdot x + s \cdot y, & (s + t) \cdot x &= s \cdot x + t \cdot x. \end{aligned}$$

It is clear that S itself is a left S -module with left action given by multiplication within S . Let I be a non-empty index set, and consider the set S^I of all functions $I \rightarrow S$, together with the operation of pointwise addition $(f + g)(i) = f(i) + g(i)$, and zero map $0(i) = 0_S$ for all $i \in I$. This forms an S -module with action given by $(s \cdot f)(i) = sf(i)$, for all $i \in I$. For $i \in I$ we write δ_i for the element of S^I defined by $\delta_i(j) = 1_S$ if $i = j$ and 0_S otherwise. The set $S^{(I)}$ consisting of all functions with finite support is a free (left) S -module with basis $\{\delta_i : i \in I\}$. Thus every finitely generated free S -module is isomorphic to a module of the form S^n . We shall also write $S^{m \times n}$ to denote the set of all $m \times n$ matrices over S , which forms an S -module in the obvious way.

Given $A \in M_n(S)$ we write $A_{i,*} \in S^{1 \times n}$ to denote the i th row of A and $A_{*,i} \in S^{n \times 1}$ to denote the i th column. The column space

$$\text{Col}(A) = \left\{ \sum_{i=1}^n A_{*,i} \lambda_i : \lambda_i \in S \right\}$$

is the (right) S -submodule of $S^{n \times 1}$ generated by the columns of A . Dually, we define $\text{Row}(A)$ to be the (left) S -submodule of $S^{n \times 1}$ generated by the rows of A .

Since an S -module over an arbitrary semiring has no underlying group structure, we must form quotients by considering congruences per se (rather than the congruence class of a particular element). Here by a congruence we mean an equivalence relation that is compatible with addition and the left action of S . The kernel of a set $X \subseteq S^{n \times 1}$ of column vectors is the left congruence on $S^{1 \times n}$ defined by

$$\text{Ker}(X) = \{(v, v') \in S^{1 \times n} \times S^{1 \times n} : vx = v'x \text{ for all } x \in X\}.$$

Given a matrix $A \in M_n(S)$, it is easy to see that the kernel of the set of columns of $\{A_{*,1}, \dots, A_{*,n}\}$ is equal to the kernel of the column space $\text{Col}(A)$; this is the set-theoretic kernel of the surjective left linear function $S^{1 \times n} \rightarrow \text{Row}(A)$ given by $v \mapsto vA$. Thus $S^{1 \times n} / \text{Ker}(\text{Col}(A)) \cong \text{Row}(A)$ as (left) S -modules.

For $A \in M_n(S)$ we also define

$$\text{ColStab}(A) = \{E \in M_n(S) : E^2 = E, EA = A\} \subseteq M_n(S),$$

that is, the set of idempotents which act as left identities for A (hence stabilising the column space of A) and

$$\text{ColFix}(A) = \bigcap_{F \in \text{ColStab}(A)} \text{Col}(F) \subseteq S^{n \times 1}$$

to denote the intersection of all the column spaces of elements of $\text{ColStab}(A)$.

2.3. Left/right relations, regularity and generalisations. We briefly outline the basic ideas required from semigroup theory; for further reference, the reader is referred to [18]. Let T be a semigroup. Several equivalence relations on T are defined using properties of the left and right actions of T upon itself. We write T^1 to denote the monoid obtained by adjoining, if necessary, an identity element to T ; $E(T)$ for the set of idempotent elements of T (that is, those $e \in T$ for which $e^2 = e$); and U for an arbitrary (but fixed) subset of $E(T)$. For $a, b \in T$, we say that:

$$\begin{aligned} a \mathcal{R} b & \text{ if } aT^1 = bT^1; & a \leq_{\mathcal{R}} b & \text{ if } aT^1 \subseteq bT^1; \\ a \mathcal{L} b & \text{ if } T^1a = T^1b; & a \leq_{\mathcal{L}} b & \text{ if } T^1a \subseteq T^1b; \end{aligned}$$

$$\begin{aligned} a \mathcal{R}^* b & \text{ if for all } x, y \in T^1: xa = ya \text{ if and only if } xb = yb; \\ a \mathcal{L}^* b & \text{ if for all } x, y \in T^1: ax = ay \text{ if and only if } bx = by; \end{aligned}$$

$$\begin{aligned} a \tilde{\mathcal{R}}_U b & \text{ if for all } e \in U: ea = a \text{ if and only if } eb = b; \\ a \tilde{\mathcal{L}}_U b & \text{ if for all } e \in U: ae = a \text{ if and only if } be = b. \end{aligned}$$

When $U = E(T)$, we write simply $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ in place of $\tilde{\mathcal{R}}_{E(T)}$ and $\tilde{\mathcal{L}}_{E(T)}$. It is easily verified that the relations $\mathcal{R}, \mathcal{L}, \mathcal{R}^*, \mathcal{L}^*, \tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ are equivalence relations on T . The two relations \mathcal{R} and \mathcal{L} are the familiar Green's relations (see for example [18]), whilst the remaining relations are much-studied [6, 8, 31, 35] extensions of these, in the sense that:

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}} \subseteq \tilde{\mathcal{R}}_U \quad \text{and} \quad \mathcal{L} \subseteq \mathcal{L}^* \subseteq \tilde{\mathcal{L}} \subseteq \tilde{\mathcal{L}}_U.$$

The relations \mathcal{R} and \mathcal{L} clearly encapsulate notions of (one-sided) divisibility. The relation corresponding to the obvious two-sided version is denoted by \mathcal{J} . The meet of \mathcal{L} and \mathcal{R}

is denoted by \mathcal{H} , whilst their join is denoted by \mathcal{D} and the latter turns out to be equal to both $\mathcal{L} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{L}$. Starting with \mathcal{L}^* and \mathcal{R}^* (respectively, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$) one may define analogous relations \mathcal{H}^* and \mathcal{D}^* (respectively, $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{D}}$), although the relations \mathcal{L}^* and \mathcal{R}^* (respectively, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$) need not commute in general. For $a \in T$ and an equivalence relation \mathcal{K} on T we write K_a to denote the equivalence class containing a .

It follows from [35] that $a\mathcal{R}^*b$ in T may be alternatively characterised as $a\mathcal{R}b$ in an oversemigroup $T' \supseteq T$, and dually for the relation \mathcal{L}^* . Whilst the relations \mathcal{R} and \mathcal{R}^* (respectively, \mathcal{L} and \mathcal{L}^*) are well-known to be left (respectively, right) congruences on S , in general $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}_U$ (respectively, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}_U$) need not be.

We say that a semigroup T is *regular* if for each element $a \in T$ there exists $x \in T$ such that $axa = a$. Equivalently, T is regular if every \mathcal{R} -class (or every \mathcal{L} -class) of T contains an idempotent. The latter characterisation in terms of the ‘abundance’ of idempotents in T , has given rise to the following generalisations:

- T is *abundant* if and only if every \mathcal{R}^* -class and every \mathcal{L}^* -class of T contains an idempotent;
- T is *Fountain* (or weakly abundant, or semi-abundant) if and only if every $\tilde{\mathcal{R}}$ -class and every $\tilde{\mathcal{L}}$ -class of T contains an idempotent;
- T is *U -Fountain* (or weakly U -abundant, or U semi-abundant) if and only if every $\tilde{\mathcal{R}}_U$ -class and every $\tilde{\mathcal{L}}_U$ -class of T contains an idempotent.

The term “Fountain” was coined in [32] to highlight the contribution of John Fountain to the study of such semigroups.

The following lemma (which provides a useful characterisation of the elements that are $\tilde{\mathcal{R}}_U$ -related to an idempotent of U) is well known to specialists (see for example [12]), however we record a brief proof for completeness.

Lemma 2.2. *Let $a, e, f \in T$ with $e, f \in U \subseteq E(T)$. Then $a\tilde{\mathcal{R}}_U e$ if and only if $ea = a$ and $e \leq_{\mathcal{R}} f$ for all idempotents $f \in U$ with $fa = a$. In particular, $e\tilde{\mathcal{R}}_U f$ if and only if $e\mathcal{R}f$.*

Proof. Suppose first that $a\tilde{\mathcal{R}}_U e$. Then (by definition) for all $f \in U$ we have $fa = a$ if and only if $fe = e$. Since $ee = e \in U$, the converse direction of the previous statement yields that $ea = a$. The forward direction states that an idempotent $f \in U$ with $fa = a$ must satisfy $fe = e$. Thus we have shown that $a\tilde{\mathcal{R}}_U e$ implies that $ea = a$, and $e \leq_{\mathcal{R}} f$ for all $f \in U$ with $fa = a$.

Now suppose that $ea = a$, and $e \leq_{\mathcal{R}} f$ for all $f \in U$ with $fa = a$. We show that $a\tilde{\mathcal{R}}_U e$. If $g \in U$ with $ga = a$, then (by our assumption $e \leq_{\mathcal{R}} g$) we have $e = gx$ for some $x \in T$. From this and the fact that g is idempotent we obtain $ge = g(gx) = (gg)x = gx = e$. Thus if $g \in U$ with $ga = a$, then $ge = e$. Conversely, if $g \in U$ with $ge = e$, then (by our assumption $ea = a$) we obtain $ga = g(ea) = (ge)a = ea = a$.

Finally, if $e, f \in U$, then the argument of the first paragraph shows that $e\tilde{\mathcal{R}}_U f$ implies that $e \leq_{\mathcal{R}} f$ and $f \leq_{\mathcal{R}} e$. In other words, $e\tilde{\mathcal{R}}_U f$ implies $e\mathcal{R}f$. That $e\mathcal{R}f$ implies $e\tilde{\mathcal{R}}_U f$ is immediate from the definitions. \square

2.4. Involutory anti-automorphisms. Let T be a semigroup. We say that a bijective map $\varphi : T \rightarrow T$ is an *involutory anti-automorphism* if $\varphi^{-1} = \varphi$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in T$. For example, in an inverse semigroup the map sending an element to its inverse is such a map. The existence of such a map ensures left-right symmetry in the structure of T .

Lemma 2.3. *Let T be a semigroup with involutory anti-automorphism φ .*

- T is abundant if and only if each \mathcal{R}^* -class contains an idempotent.*
- T is Fountain if and only if each $\tilde{\mathcal{R}}$ -class contains an idempotent.*

Further, if $U \subseteq E(T)$ with $\varphi(U) = U$, then:

(iii) T is U -Fountain if and only if each $\tilde{\mathcal{R}}_U$ -class contains an idempotent.

Proof. (i) Suppose that each \mathcal{R}^* -class contains at least one idempotent. Choose one idempotent element of each \mathcal{R}^* -class as a representative and define a function $\varepsilon : T \rightarrow E(T)$ mapping each element to the idempotent representative of its \mathcal{R}^* -class. We claim that for all $a \in T$, the element $\varphi(\varepsilon(\varphi(a)))$ is an idempotent which is \mathcal{L}^* -related to a . First, since φ is an anti-automorphism and $\varepsilon(\varphi(a))$ is an idempotent we clearly have

$$\varphi(\varepsilon(\varphi(a))) \cdot \varphi(\varepsilon(\varphi(a))) = \varphi(\varepsilon(\varphi(a)) \cdot \varepsilon(\varphi(a))) = \varphi(\varepsilon(\varphi(a))).$$

Let $a, x, y \in T$. Then, by applying φ , we see that $ax = ay$ if and only if $\varphi(x)\varphi(a) = \varphi(y)\varphi(a)$. Since $\varphi(a) \mathcal{R}^* \varepsilon(\varphi(a))$ the latter is equivalent to $\varphi(x)\varepsilon(\varphi(a)) = \varphi(y)\varepsilon(\varphi(a))$. Applying φ once more then yields $ax = ay$ if and only if $\varphi(\varepsilon(\varphi(a)))x = \varphi(\varepsilon(\varphi(a)))y$. Thus $a \mathcal{L}^* \varphi(\varepsilon(\varphi(a)))$ as required. This shows that if each \mathcal{R}^* -class contains an idempotent, then each \mathcal{L}^* -class also contains an idempotent, and hence T is abundant.

(ii) Suppose that each $\tilde{\mathcal{R}}$ -class contains at least one idempotent. Choose one idempotent element of each $\tilde{\mathcal{R}}$ -class as a representative and define a function $\gamma : T \rightarrow E(T)$ mapping each element to the idempotent representative of its $\tilde{\mathcal{R}}$ -class. As above, it is easy to see that $a \in T$ is $\tilde{\mathcal{L}}$ -related to the idempotent $\varphi(\gamma(\varphi(a)))$, and so T is Fountain.

(iii) Since $\varphi(U) = U$, arguing as above gives the desired result. \square

Following the convention in [15], we say that a map $\varphi : T \rightarrow T$ *exchanges* two binary relations τ and ρ on T if:

$$x \tau y \Rightarrow \varphi(x) \rho \varphi(y) \text{ and } x \rho y \Rightarrow \varphi(x) \tau \varphi(y).$$

We say that φ *strongly exchanges* τ with ρ if

$$x \tau y \Leftrightarrow \varphi(x) \rho \varphi(y) \text{ and } x \rho y \Leftrightarrow \varphi(x) \tau \varphi(y).$$

Lemma 2.4. *Let T be a semigroup and $\varphi : T \rightarrow T$ an involutory anti-automorphism. Then:*

- (i) φ strongly exchanges \mathcal{L} and \mathcal{R} .
- (ii) φ strongly exchanges \mathcal{L}^* and \mathcal{R}^* .
- (iii) φ strongly exchanges $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$.
- (iv) φ acts on the \mathcal{K} , \mathcal{K}^* , and $\tilde{\mathcal{K}}$ -classes for $\mathcal{K} \in \{\mathcal{H}, \mathcal{D}\}$.

Further, if $U \subseteq E(T)$ with $\varphi(U) = U$, then:

- (v) φ strongly exchanges $\tilde{\mathcal{L}}_U$ and $\tilde{\mathcal{R}}_U$, and φ acts on the $\tilde{\mathcal{H}}_U$ and $\tilde{\mathcal{D}}_U$ -classes.

Proof. We may extend φ to an involutory anti-automorphism $T^1 \rightarrow T^1$ by setting $\varphi(1) = 1$. By an abuse of notation, we denote this map also by φ .

(i) Let $a, b \in T$ and $x, y \in T^1$. By applying φ it is easy to see that $a = xb$ and $b = ya$ if and only if $\varphi(a) = \varphi(b)\varphi(x)$ and $\varphi(b) = \varphi(a)\varphi(y)$. Since φ is bijective, it is clear that $a \mathcal{L} b$ if and only if $\varphi(a) \mathcal{R} \varphi(b)$. Since φ is an involution we must also have $a \mathcal{R} b$ if and only if $\varphi(a) \mathcal{L} \varphi(b)$.

(ii) Suppose that $a \mathcal{L}^* b$. Then for all $x, y \in T^1$, we have $ax = ay$ if and only if $bx = by$. Applying φ gives that $\varphi(x)\varphi(a) = \varphi(y)\varphi(a)$ if and only if $\varphi(x)\varphi(b) = \varphi(y)\varphi(b)$. Since φ is bijective, this shows that $a \mathcal{L}^* b$ if and only if $\varphi(a) \mathcal{R}^* \varphi(b)$. Since φ is an involution we must also have $a \mathcal{R}^* b$ if and only if $\varphi(a) \mathcal{L}^* \varphi(b)$.

(iii) For $x \in T$ it is easy to see that x is idempotent if and only if $\varphi(x)$ is idempotent. Arguing as in parts (i) and (ii) one then finds that $a \tilde{\mathcal{L}} b$ if and only if $\varphi(a) \tilde{\mathcal{R}} \varphi(b)$ and vice versa.

(iv) This follows easily from parts (i)-(iii) together with the innate left-right symmetry of the definitions.

(v) Repeating the argumentation of part (iii) for $e \in U$, noting that $\varphi(e) \in U$, gives the desired result. \square

3. GENERALISED REGULARITY CONDITIONS FOR MATRIX SEMIGROUPS

In this section we consider the generalised regularity properties of the semigroups $M_n(S)$, $UT_n(S)$ and $U_n(S)$ over a general semiring S . In the case where S is idempotent, we shall see that for all $n \geq 4$, the semigroups $M_n(S)$ and $UT_n(S)$ are not Fountain.

Reflecting along the diagonals illuminates left-right symmetry in the ideal structures of $M_n(S)$, $UT_n(S)$ and $U_n(S)$.

Lemma 3.1. *Let S be a semiring.*

- (i) *The transpose map is an involutory anti-automorphism of $M_n(S)$.*
- (ii) *The map $\Delta : UT_n(S) \rightarrow UT_n(S)$ obtained by reflecting along the anti-diagonal is an involutory anti-automorphism of $UT_n(S)$.*
- (iii) *Restricting Δ to $U_n(S)$ yields an involutory anti-automorphism of $U_n(S)$.*

Proof. We give the details of part (ii) only, part (i) being well known. First notice that for all $1 \leq i \leq j \leq n$ we have $\Delta(X)_{i,j} = X_{n-j+1,n-i+1}$. Then, since

$$(AB)_{i,j} = \sum_{i \leq k \leq j} A_{i,k} \cdot B_{k,j}$$

we obtain

$$[\Delta(AB)]_{i,j} = (AB)_{n-j+1,n-i+1} = \sum_{n-j+1 \leq k \leq n-i+1} A_{n-j+1,k} \cdot B_{k,n-i+1}.$$

On the other hand,

$$\begin{aligned} [\Delta(B)\Delta(A)]_{i,j} &= \sum_{i \leq p \leq j} \Delta(B)_{i,p} \cdot \Delta(A)_{p,j} = \sum_{i \leq p \leq j} B_{n-p+1,n-i+1} \cdot A_{n-j+1,n-p+1} \\ &= \sum_{n-j+1 \leq q \leq n-i+1} A_{n-j+1,q} \cdot B_{q,n-i+1} = [\Delta(AB)]_{i,j}. \end{aligned}$$

(iii) Clearly $\Delta(U_n(S)) = U_n(S)$. \square

Thus it follows from Lemma 2.3 that in investigating generalised regularity properties of $M_n(S)$, $UT_n(S)$ and $U_n(S)$ (and many other subsemigroups of $M_n(S)$) it suffices to consider the relations \mathcal{R} , \mathcal{R}^* and $\tilde{\mathcal{R}}$ only.

3.1. Green's relations. The monoid $M_n(S)$ is isomorphic to the monoid of endomorphisms of the free S -module S^n . The relations \mathcal{R} , \mathcal{R}^* , $\tilde{\mathcal{R}}$ on $M_n(S)$ can be readily phrased in terms of certain submodules and congruences on S^n (in the case of \mathcal{R} this is well known— see [34, 30, 24] for example); for the reader's convenience we briefly outline the details.

Lemma 3.2. *Let S be a semiring and let $A, B \in M_n(S)$. Then*

- (i) *$A \mathcal{R} B$ if and only if $\text{Col}(A) = \text{Col}(B)$;*
- (ii) *$A \mathcal{R}^* B$ if and only if $\text{Ker}(\text{Col}(A)) = \text{Ker}(\text{Col}(B))$;*
- (iii) *$A \tilde{\mathcal{R}} B$ if and only if $\text{ColStab}(A) = \text{ColStab}(B)$;*

Proof. (i) Suppose that $A = BX$ for some $X \in M_n(S)$. Then we can write each column of A as a right S -linear combination of the columns of B . It then follows (from distributivity of the action) that if $x \in \text{Col}(A)$ we may write x first as a right S -linear combination of the columns of A , and hence then as a right S -linear combination of the columns of B , showing that $\text{Col}(A) \subseteq \text{Col}(B)$. On the other hand, suppose that $\text{Col}(A) \subseteq \text{Col}(B)$. Using the fact that S has identity elements 0_S and 1_S which act in the appropriate manner,

it is clear that each column of A is an element of the column space $\text{Col}(A)$ and so can be written as a right S -linear combination of the columns of B . Taking X to be the coefficient matrix of these combinations, we see that $A = BX$. It then follows from the above that $A \mathcal{R} B$ if and only if the two column spaces are equal.

(ii) Suppose that whenever $XA = YA$ for $X, Y \in M_n(S)$ we also have that $XB = YB$. Let $(x, y) \in \text{Ker}(\text{Col}(A)) \subseteq S^{1 \times n} \times S^{1 \times n}$, and let X be the matrix with all rows equal to x and Y the matrix with all rows equal to y . Then it follows that $XA = YA$ and hence $XB = YB$, whence $xB = yB$. This shows that $\text{Ker}(\text{Col}(A)) \subseteq \text{Ker}(\text{Col}(B))$. It then follows that $A \mathcal{R}^* B$ implies that the two column kernels agree. Conversely, suppose that the column kernels of A and B agree. If $XA = YA$ for some $X, Y \in M_n(S)$, then for all $1 \leq i, j \leq n$ we have $X_{i,*}A_{*,j} = Y_{i,*}A_{*,j}$. Thus for each $i = 1, \dots, n$ we have $(X_{i,*}, Y_{i,*}) \in \text{Ker}(\text{Col}(A)) = \text{Ker}(\text{Col}(B))$, whence $XB = YB$.

(iii) This is immediate from the definition of $\text{ColStab}(A)$. \square

The relations \mathcal{L} , \mathcal{L}^* and $\tilde{\mathcal{L}}$ can be dually characterised in terms of row spaces. It is clear from the above proof that the pre-order $\leq_{\mathcal{R}}$ corresponds to containment of column spaces. Lemma 3.2 and its dual enable one description of regularity, abundance and Fountainicity, since it allows us to describe the conditions under which $A \in M_n(S)$ is related to an idempotent $E \in M_n(S)$ under the relevant relations.

We now give the first in a series of results examining the relationship between generalisations of Green's relations in $M_n(S)$ and certain natural subsemigroups, such as $UT_n(S)$.

Lemma 3.3. *Let S be a semiring and let $A, B \in UT_n(S)$. Then $A \mathcal{R}^* B$ in $UT_n(S)$ if and only if $A \mathcal{R}^* B$ in $M_n(S)$.*

Proof. If $A \mathcal{R}^* B$ in $M_n(S)$ then by definition we have that for all $X, Y \in M_n(S)$ the equality $XA = YA$ holds if and only if the equality $XB = YB$ holds. Specialising to $X, Y \in UT_n(S)$ then yields $A \mathcal{R}^* B$ in $UT_n(S)$. Conversely, suppose that $A \mathcal{R}^* B$ in $UT_n(S)$. We show that $(x, y) \in \text{Ker}(\text{Col}(A))$ if and only if $(x, y) \in \text{Ker}(\text{Col}(B))$. Let X and Y be the upper triangular matrices with first row x and y respectively, and all remaining rows zero. Since $A \mathcal{R}^* B$ in $UT_n(S)$, we have $XA = YA$ if and only if $XB = YB$, and hence in particular (by looking at the content of the first row of these two products), for each column $A_{*,j}$ we have $xA_{*,j} = yA_{*,j}$ if and only if $xB_{*,j} = yB_{*,j}$, thus giving the desired result. \square

3.2. Regularity. The question of which monoids $M_n(S)$ are regular has been answered by Il'in [20, Theorem 4], who has shown that for $n \geq 3$, $M_n(S)$ is regular if and only if S is a von-Neumann regular ring.

On the other hand, for an arbitrary semiring S the upper triangular monoids $UT_n(S)$ are not regular for all $n \geq 2$.

Proposition 3.4. *Let S be a semiring. The monoid $UT_n(S)$ is regular if and only if $n = 1$ and the multiplicative reduct (S, \cdot) is regular.*

Proof. The monoid $UT_1(S)$ is clearly isomorphic to (S, \cdot) . Suppose then that $n \geq 2$. Consider the upper triangular matrix A with entries $A_{1,j} = 1$ for $2 \leq j \leq n$, and all other entries equal to 0. We first show that an upper triangular matrix B is \mathcal{R} -related to A in $UT_n(S)$ if and only if:

- (i) All non-zero entries of B lie in the first row;
- (ii) $B_{1,1} = 0$; and
- (iii) $B_{1,2}$ has a right inverse $b' \in S$.

It is straightforward to verify that each such B is \mathcal{R} -related to A ; taking X to be the diagonal matrix with $X_{j,j} = B_{1,j}$ for all $j \in [n]$ yields $AX = B$, whilst taking Y to be the matrix with $Y_{2,j} = b'$ for all $j \geq 2$ and all other entries equal to 0 yields $BY = A$.

Suppose then that $C \mathcal{R} A$. Thus $C = AU$ and $A = CV$ for some $U, V \in UT_n(S)$. Since rows 2 to n of A are all zero, it is clear that all non-zero entries of $C = AU$ must lie in the first row. Moreover, since U is triangular, we see that $A_{1,1} = 0$ also forces $C_{1,1} = A_{1,1}U_{1,1} = 0$. Since $CV = A$, then we must also have $C_{1,2}V_{2,2} = A_{1,2} = 1$. Thus we have shown that each matrix \mathcal{R} -related to A must satisfy conditions (i)-(iii) above.

Notice that, since the diagonal entries of each any matrix in this \mathcal{R} -class are 0, these elements are nilpotent. Since each such matrix is non-zero, this \mathcal{R} -class does not contain an idempotent. Thus $UT_n(S)$ is not regular. \square

When S is a commutative ring, it is easy to see (e.g. by performing invertible row operations of the form $r_i \mapsto r_i + \lambda r_k$ for $1 \leq i \leq k \leq n$) that the unitriangular monoid $U_n(S)$ is a group, and hence in particular regular. At the other extreme, when S is an anti-negative semiring it is easy to see that $U_n(S)$ is not regular for all $n \geq 3$, since for example it is straightforward to show that the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

has no solution.

Thus in the case where S is a semifield, it follows from the above discussion and Lemma 2.1 that $U_n(S)$ is regular if and only if (i) $n = 1$, or (ii) $n = 2$ and for each $a \in S$ there exists x such that $a + x + a = a$, or (iii) $n \geq 3$ and S is a field.

3.3. Abundance. Likewise, one can ask under what conditions on S and n do we have that $M_n(S)$ (respectively, $UT_n(S)$ or $U_n(S)$) is abundant?

Recall from [46, Theorem 3.2] that a semiring S is said to be *exact* (or FP-injective) if for all $A \in S^{m \times n}$:

- (F1) For each $B \in S^{p \times n}$ we have $\text{Ker}(\text{Row}(A)) \subseteq \text{Ker}(\text{Row}(B))$ if and only if $\text{Row}(B) \subseteq \text{Row}(A)$.
- (F2) For each $B \in S^{m \times q}$ we have $\text{Ker}(\text{Col}(A)) \subseteq \text{Ker}(\text{Col}(B))$ if and only if $\text{Col}(B) \subseteq \text{Col}(A)$.

When S is exact, abundance coincides with regularity for $M_n(S)$:

Theorem 3.5. *Let S be an exact semiring, $n \in \mathbb{N}$. Then $\mathcal{R} = \mathcal{R}^*$ in the semigroup $M_n(S)$. Thus $M_n(S)$ is abundant if and only if it is regular.*

Proof. It is clear from the definitions that $\mathcal{R} \subseteq \mathcal{R}^*$. Suppose that $A, B \in M_n(S)$ are \mathcal{R}^* -related. By Lemma 3.2 we see that $\text{Ker}(\text{Col}(A)) = \text{Ker}(\text{Col}(B))$. By the exactness of S , this yields $\text{Col}(A) = \text{Col}(B)$, and hence $A \mathcal{R} B$, by Lemma 3.2 again. \square

Together with [20, Theorem 4], this gives the following:

Corollary 3.6. *Let S be an exact semiring and let $n \geq 3$. The semigroup $M_n(S)$ is abundant if and only if the semiring S is a von-Neumann regular ring.*

Theorem 3.5 also provides a simple characterisation of the relation \mathcal{R}^* on $UT_n(S)$ in the case that S is exact.

Corollary 3.7. *Let S be an exact semiring. Then $A \mathcal{R}^* B$ in $UT_n(S)$ if and only if $\text{Col}(A) = \text{Col}(B)$.*

Proof. By Lemma 3.3 we have that $A \mathcal{R}^* B$ in $UT_n(S)$ if and only if $A \mathcal{R}^* B$ in $M_n(S)$. Theorem 3.5 says that $A \mathcal{R}^* B$ in $M_n(S)$ if and only if $A \mathcal{R} B$ in $M_n(S)$, which by Theorem 3.2 happens precisely when $\text{Col}(A) = \text{Col}(B)$. \square

As mentioned earlier, by [35] one has that $a \mathcal{R}^* b$ in a semigroup T if and only if $a \mathcal{R} b$ in *some* oversemigroup T' of T ; Corollary 3.7 says that for $T = UT_n(S)$ over an exact semiring S it suffices to consider the oversemigroup $M_n(S)$.

Notice however that the \mathcal{R} -relation on $UT_n(S)$ is not characterised by equality of column spaces; for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ possess the same column space but are not \mathcal{R} -related in $UT_2(S)$. One can formulate an alternative characterisation in terms of upper triangular column operations (see for example [28], noting that the notation used there conflicts with our own).

The class of exact semirings is known to contain all fields, *proper* quotients of principal ideal domains, matrix rings and finite group rings over the above, the Boolean semiring \mathbb{B} , the tropical semiring \mathbb{T} , and some generalisations of these (see [46, 45] for details). If S is a semifield, Shitov has shown that S is exact if and only if either S is a field or S is an idempotent semifield [39]. In light of Figure 1 one may wonder if abundance and regularity coincide in $M_n(S)$ for all semifields S , however, this is not the case as the following example illustrates.

Example 3.8. Let $\mathbb{Q}_{\geq 0}$ denote the semifield of non-negative rational numbers, and for $n \geq 2$ consider the block diagonal matrix

$$A = \left(\begin{array}{cc|c} 2 & 1 & \\ 1 & 3 & \\ \hline & & I_{n-2} \end{array} \right) \in M_n(\mathbb{Q}_{\geq 0}),$$

where I_{n-2} denotes the identity matrix of $M_{n-2}(\mathbb{Q}_{\geq 0})$ and omitted entries are zero. It is straightforward to verify that A is not regular in $M_n(\mathbb{Q}_{\geq 0})$. However, since A is invertible in $M_n(\mathbb{Q})$ one has that $A \mathcal{R}^* I_n$ within $M_n(\mathbb{Q}_{\geq 0})$. Thus $\mathcal{R} \neq \mathcal{R}^*$ in $M_n(\mathbb{Q}_{\geq 0})$.

It follows easily from the above results that the semigroup of upper triangular matrices over any field is abundant:

Corollary 3.9. *If S is a field, then $UT_n(S)$ is abundant.*

Proof. For each $A \in UT_n(S)$, let \bar{A} denote the upper triangular matrix obtained from the reduced column echelon form of A by permuting the columns of A to put all leading ones on the diagonal. Thus $\bar{A} = AX$ for some $X \in GL_n(S)$ and hence $A \mathcal{R} \bar{A}$ in $M_n(S)$. It is straightforward to check that \bar{A} is an idempotent, and so Lemma 3.3 yields that $A \mathcal{R}^* \bar{A}$ in $UT_n(S)$. \square

Over idempotent semifields the monoids $UT_n(S)$ and $U_n(S)$ are typically not abundant.

Proposition 3.10. *Let S be an idempotent semiring in which the only multiplicative idempotents are 0 and 1 (for example, an idempotent semifield), and let $n \geq 3$. Then the monoids $UT_n(S)$ and $U_n(S)$ are not abundant.*

Proof. Let A be the matrix with top left corner:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and zeros elsewhere. We shall show that A is not \mathcal{R}^* -related to any idempotent in $UT_n(S)$.

For $M \in UT_n(S)$ define $K(M) = \{(X, Y) : XM = YM\} \subseteq UT_n(S) \times UT_n(S)$, and suppose for contradiction that E is an idempotent \mathcal{R}^* -related to A in $UT_n(S)$, so that $K(A) = K(E)$.

From the definition of A , it is easy to see that $(X, Y) \in K(A)$ if and only if:

- (i) $X_{i,i} = Y_{i,i}$ for $i = 1, 2, 3$;

- (ii) $X_{1,1} + X_{1,2} = Y_{1,1} + Y_{1,2}$;
- (iii) $X_{2,2} + X_{2,3} = Y_{2,2} + Y_{2,3}$;
- (iv) $X_{1,2} + X_{1,3} = Y_{1,2} + Y_{1,3}$.

It follows easily from condition (i) that we must have $E_{i,i} \neq 0$ for $i = 1, 2, 3$. (Otherwise, $(E, I_n) \in K(E) \setminus K(A)$.) Since E is assumed to be idempotent and upper triangular, the condition that $E_{i,i}^2 = E_{i,i}$ then yields $E_{i,i} = 1$.

By considering the pair $(P, Q) \in K(A)$ with $P_{1,1} = P_{1,2} = Q_{1,1} = Q_{1,3} = 1$ and all other entries equal to zero, we note that in position $(1, 2)$ the equality $PE = QE$ yields:

$$\begin{aligned} (PE)_{1,2} = P_{1,1}E_{1,2} + P_{1,2}E_{2,2} &= Q_{1,1}E_{1,2} + Q_{1,2}E_{2,2} = (QE)_{1,2} \\ E_{1,2} + 1 &= E_{1,2} + 0 \\ 1 &\leq E_{1,2}. \end{aligned}$$

Similarly, by considering the pair $(P, Q) \in K(A)$ with $P_{1,3} = P_{2,2} = P_{2,3} = Q_{2,2} = Q_{1,3} = 1$ and all other entries equal to zero, we note that in position $(2, 3)$ the equality $PE = QE$ yields:

$$\begin{aligned} (PE)_{2,3} = P_{2,2}E_{2,3} + P_{2,3}E_{3,3} &= Q_{2,2}E_{2,3} + Q_{2,3}E_{3,3} = (QE)_{2,3} \\ E_{2,3} + 1 &= E_{2,3} + 0 \\ 1 &\leq E_{2,3}. \end{aligned}$$

Since E was assumed to be idempotent we must have

$$E_{1,3} = (E^2)_{1,3} = E_{1,1}E_{1,3} + E_{1,2}E_{2,3} + E_{1,3}E_{3,3} \geq E_{1,3} + 1.$$

But now, one can check that taking V to be the matrix with $V_{1,3} = V_{i,i} = 1$ for all i and all remaining entries equal to zero we have $(I_n, V) \in K(E) \setminus K(A)$, hence giving the desired contradiction.

Finally, if $A \in U_n(S)$ is \mathcal{R}^* -related to an idempotent in $U_n(S)$, then arguing exactly as above, observing that in this case $X_{i,i} = Y_{i,i} = A_{i,i} = E_{i,i} = 1$ for all $i \in [n]$, we arrive at the same contradiction. \square

For general semirings, a characterisation of abundance remains open.

Question 3.11. What conditions on S and n are necessary and sufficient for $M_n(S)$ (respectively, $UT_n(S)$, $U_n(S)$) to be abundant?

3.4. Fountainicity.

Proposition 3.12. *The monoid $M_n(S)$ is Fountain if and only if for each $A \in M_n(S)$ there exists an idempotent E such that $\text{ColFix}(A) = \text{Col}(E)$.*

Proof. Let $A, E \in M_n(S)$ with $E^2 = E$. By Lemma 2.2, $A \tilde{\mathcal{R}} E$ if and only if $EA = A$ and $E \leq_{\mathcal{R}} F$ for any idempotent $F \in M_n(S)$ satisfying $FA = A$. From earlier remarks this is equivalent to $E \in \text{ColStab}(A)$ and $\text{Col}(E) \subseteq \text{Col}(F)$ for all $F \in \text{ColStab}(A)$. Thus if $A \tilde{\mathcal{R}} E$ we have $\text{Col}(E) = \text{ColFix}(A)$ from the definition. Conversely, if $\text{Col}(E) = \text{ColFix}(A)$ for some $E = E^2$, then since $\text{Col}(A) \subseteq \text{Col}(F)$ for any $F \in \text{ColStab}(A)$, we have $\text{Col}(A) \subseteq \text{Col}(E)$, whence $A \leq_{\mathcal{R}} E$ so that certainly $EA = A$, and by definition of $\text{ColFix}(A)$, $E \leq_{\mathcal{R}} F$ for any $F = F^2$ with $FA = A$. \square

We note that the column space of any idempotent (or indeed regular) matrix E is a retract of S^n , and hence is a finitely (at most n) generated projective S -submodule of S^n (see for example [11, Example 17.15 and Proposition 17.16]). Thus the previous proposition indicates that understanding the conditions under which $M_n(S)$ is Fountain boils down to understanding properties of intersections of certain projective S -submodules of S^n .

The monoids $M_n(S)$ and $UT_n(S)$ over any idempotent semiring are seldom Fountain, as the following proposition illustrates.

Proposition 3.13. *Let S be an idempotent semiring and let $n \geq 4$. Then $M_n(S)$ and $UT_n(S)$ are not Fountain.*

Proof. Consider first the case $n = 4$. We shall show that the matrix

$$A := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

is not $\tilde{\mathcal{R}}$ -related to any idempotent in $M_4(S)$.

If $X \in M_4(S)$, then by using the anti-negativity of S it is easy to see that satisfies $XA = A$ if and only if X has the form

$$\begin{pmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1)$$

where $a, b, c, d, e \in S$ satisfy

$$a + b = a + b + c = a + c + d = 1 + e = 1. \quad (2)$$

Consider the matrices

$$F_1 := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, F_2 := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which by above satisfy $F_1 A = F_2 A = A$. It is straightforward to check that $F_1^2 = F_1$ and $F_2^2 = F_2$.

Now suppose for contradiction that $E^2 = E \tilde{\mathcal{R}} A$ in $M_4(S)$. Since $EA = A$, E must have the form (1) for some $a, b, c, d, e, f \in S$ satisfying (2). Since F_1 and F_2 are idempotents fixing A , we must also have $F_1 E = E$ (which implies that $a = 0$, $d = e$ and $b = c = 1$) and $F_2 E = E$ (which implies that $a = c = 0$ and $b = d = 1$), hence giving a contradiction. We conclude that $M_4(S)$ is not Fountain. Noting that $M_4(S)$ naturally embeds into $M_n(S)$ for all $n \geq 4$, it is easy to see that the matrix with top right hand corner A is not $\tilde{\mathcal{R}}$ -related to any idempotent of $M_n(S)$. Moreover, since all matrices involved in the above reasoning were in fact upper triangular, the same argument applies to show that $UT_n(S)$ is not Fountain. \square

Question 3.14. What conditions on S and n are necessary and sufficient for $M_n(S)$ to be Fountain?

In the following sections, we shall show that for an idempotent semifield S each of the monoids $U_n(S)$ is Fountain. We also investigate the $\tilde{\mathcal{R}}$ -classes of $M_n(S)$, $UT_n(S)$ and $U_n(S)$ in this case. We begin by considering certain idempotent matrix constructions, which in turn allow us to show that several interesting subsemigroups of $M_n(S)$, including $U_n(S)$, are Fountain.

4. MATRIX SEMIGROUPS OVER IDEMPOTENT SEMIFIELDS

Throughout this section let S be an idempotent semifield. We recall from Lemma 2.1 that every idempotent semifield S arises by adjoining a minimal element $\mathbf{0}$ to a lattice ordered abelian group $(S^*, \cdot, \mathbf{1})$, with addition corresponding to least upper bound in S . We write \wedge and \vee to denote the operations of greatest lower bound and least upper bound in S . (Note that taking the S^* to be the trivial group yields the Boolean semiring \mathbb{B} , whilst taking S^* to be the real numbers under addition with respect to the usual total

order yields the tropical semifield \mathbb{T} . Since the *multiplicative* identity here is the real number 0, to avoid confusion the bottom element is usually denoted by $-\infty$. It would do no harm to first think of this example in what follows, keeping in mind these conventions.)

In the case where $S = \mathbb{B}$, it is well known that the monoid $M_n(\mathbb{B})$ is isomorphic to the monoid \mathcal{B}_n of all binary relations on $[n]$ under relational composition, via the map sending a relation $\alpha \subseteq [n] \times [n]$ to the matrix $A \in M_n(\mathbb{B})$ whose (i, j) th entry is 1 if and only if $(i, j) \in \alpha$. With this in mind, for $A \in M_n(S)$ we shall write $\text{dom}(A)$ to denote the subset of $[n] := \{1, \dots, n\}$ indexing the non-zero rows of A , and $\text{im}(A)$ to denote the subset of $[n]$ indexing the non-zero columns of A . For $A, B \in M_n(S)$ we also write $A \preceq B$ if $A_{i,j} \leq B_{i,j}$ for all $i, j \in [n]$, with respect to the partial order on S , noting that in the Boolean case the order \preceq corresponds to containment of relations. We make use of the following simple observations several times in our arguments.

Lemma 4.1. *Let S be an idempotent semifield and let $A, B, C \in M_n(S)$.*

- (i) *If $A \preceq B$, then $CA \preceq CB$ and $AC \preceq BC$.*
- (ii) *If $C_{i,i} = \mathbf{1}$ for all $i \in [n]$, then $A \preceq CA$ and $A \preceq AC$.*

Proof. (i) For all $i, j \in [n]$ we have

$$\begin{aligned} (CA)_{i,j} &= \bigvee_{k=1}^n C_{i,k} A_{k,j} \leq \bigvee_{k=1}^n C_{i,k} B_{k,j} = (CB)_{i,j}, \\ (AC)_{i,j} &= \bigvee_{k=1}^n A_{i,k} C_{k,j} \leq \bigvee_{k=1}^n B_{i,k} C_{k,j} = (BC)_{i,j}. \end{aligned}$$

- (ii) If $C_{i,i} = \mathbf{1}$ for all $i \in [n]$, then for all $i, j \in [n]$ we have

$$(CA)_{i,j} = \bigvee_{k=1}^n C_{i,k} A_{k,j} \geq C_{i,i} A_{i,j} = A_{i,j}, \quad (AC)_{i,j} = \bigvee_{k=1}^n A_{i,k} C_{k,j} \geq A_{i,j} C_{j,j} = A_{i,j}.$$

□

(Observe that Lemma 4.1 holds for any idempotent semiring.)

4.1. An idempotent construction. Let \bar{S} denote the idempotent semiring² obtained from our idempotent semifield S by adjoining a top element satisfying $a \vee \mathbf{T} = \mathbf{T} = \mathbf{T} \vee a$ for all $a \in S$, $\mathbf{T} \cdot a = \mathbf{T} = a \cdot \mathbf{T}$ for all $a \in \bar{S} \setminus \mathbf{0}$ and $\mathbf{T} \cdot \mathbf{0} = \mathbf{0} \cdot \mathbf{T} = \mathbf{0}$. In the language of [5], the commutative semiring \bar{S} is *residuated*, meaning that for every pair $a, b \in \bar{S}$, the set $\{x \in \bar{S} : ax \leq b\} = \{x \in \bar{S} : xa \leq b\}$ admits a maximal element $a \setminus b$, given by

$$a \setminus b = \begin{cases} \mathbf{T} & \text{if } a = \mathbf{0} \text{ or } a = b = \mathbf{T}, \\ ba^{-1} & \text{if } \mathbf{0} < a < \mathbf{T}, \\ \mathbf{0} & \text{if } a = \mathbf{T} \text{ and } b \neq \mathbf{T}. \end{cases}$$

This idea can be extended to matrices, as follows.

Lemma 4.2. [5, Proposition 2 and Theorem 14]. *Let S be an idempotent semifield, $A, X, Y \in M_n(\bar{S})$ and define*

$$(A \setminus X)_{i,j} = \bigwedge_k A_{k,i} \setminus X_{k,j}, \quad (X/A)_{i,j} = \bigwedge_l A_{j,l} \setminus X_{i,l}, \quad (X \setminus A/Y)_{i,j} = \bigwedge_{k,l} Y_{j,l} \setminus (X_{k,i} \setminus A_{k,l}).$$

Then

- (i) $X \leq_{\mathcal{R}} A$ in $M_n(\bar{S})$ if and only if $A(A \setminus X) = X$.
- (ii) $X \leq_{\mathcal{L}} A$ in $M_n(\bar{S})$ if and only if $(X/A)A = X$.
- (iii) $A(A \setminus A/Y)A \preceq A$.

²This is not a semifield, since the element \mathbf{T} adjoined does not have a multiplicative inverse.

- (iv) If A is regular, then $A(A \setminus A/A)A = A$, and $AXA = A \Rightarrow X \preceq A \setminus A/A$.
- (v) $X \setminus A/Y = (X \setminus A)/Y = (X \setminus A)/Y$.

We require similar constructions lying *within* $M_n(S)$. For $x, y \in S^n$ let $\text{Supp}(x)$ denote the subset of $[n]$ indexing the non-zero positions of x and define

$$\langle x|y \rangle := \begin{cases} \bigwedge_{i \in \text{Supp}(x)} y_i x_i^{-1} & \text{if } \emptyset \neq \text{Supp}(x) \subseteq \text{Supp}(y) \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The operation $\langle \cdot | \cdot \rangle$ is a modification of similar constructions present in the tropical literature (see for example [5, 24, 25]), modified to our purpose, where we allow for the semiring to contain a bottom element, but do not assume that it contains a top element.

Remark 4.3. We record some properties of $\langle \cdot | \cdot \rangle$:

- (i) It is immediate from the definition that $\langle x|y \rangle \in S$ and this is non-zero if and only if $\emptyset \neq \text{Supp}(x) \subseteq \text{Supp}(y)$. In particular, whenever $\langle x|y \rangle$ is non-zero, both $\text{Supp}(x)$ and $\text{Supp}(y)$ are non-empty, i.e. x and y must be non-zero vectors.
- (ii) For each non-zero vector x , it is easy to see that $\langle x|x \rangle = \mathbf{1}$.
- (iii) Let $x, y, z \in S^n$. From the above observations $\langle x|y \rangle \langle y|z \rangle \neq \mathbf{0}$ implies that $\emptyset \neq \text{Supp}(x) \subseteq \text{Supp}(y) \subseteq \text{Supp}(z)$. In this case, for each $i \in \text{Supp}(x)$ we have $\langle x|y \rangle \langle y|z \rangle \leq y_i x_i^{-1} z_i y_i^{-1} = z_i x_i^{-1}$, and hence

$$\langle x|y \rangle \langle y|z \rangle \leq \langle x|z \rangle.$$

- (iv) Notice that for all $i \in [n]$ we have $\langle x|y \rangle x_i \leq y_i$. That is, $\langle x|y \rangle x \preceq y$.

For each $A \in M_n(S)$ define $A^{(+)}, A^{(*)} \in M_n(S)$ via:

$$\begin{aligned} (A^{(+)})_{i,j} &= \langle A_{j,\star} | A_{i,\star} \rangle \\ &= \begin{cases} \bigwedge \{ A_{i,k} A_{j,k}^{-1} : k \in \text{Supp}(A_{j,\star}) \} & \text{if } \emptyset \neq \text{Supp}(A_{j,\star}) \subseteq \text{Supp}(A_{i,\star}) \\ \mathbf{0} & \text{otherwise} \end{cases} \end{aligned} \quad (3)$$

and

$$\begin{aligned} (A^{(*)})_{i,j} &= \langle A_{\star,i} | A_{\star,j} \rangle \\ &= \begin{cases} \bigwedge \{ A_{k,j} A_{k,i}^{-1} : k \in \text{Supp}(A_{\star,i}) \} & \text{if } \emptyset \neq \text{Supp}(A_{\star,i}) \subseteq \text{Supp}(A_{\star,j}) \\ \mathbf{0} & \text{otherwise.} \end{cases} \end{aligned} \quad (4)$$

By the previous remark, if $(A^{(+)})_{i,j}$ is non-zero, then $i, j \in \text{dom}(A)$. Likewise, if $(A^{(*)})_{i,j}$ is non-zero, then $i, j \in \text{im}(A)$.

Proposition 4.4. Let S be an idempotent semifield and $A \in M_n(S)$. Then

- (i) $A^{(+)} A^{(+)} = A^{(+)}$.
- (ii) $A^{(+)} A = A$.
- (iii) If $\text{dom}(A) = [n]$ and $BA = A$, then $B \preceq A^{(+)}$ and $BA^{(+)} \preceq A^{(+)}$.
- (iv) $A^{(*)} A^{(*)} = A^{(*)}$.
- (v) $AA^{(*)} = A$.
- (vi) If $\text{im}(A) = [n]$ and $AB = A$, then $B \preceq A^{(*)}$ and $A^{(*)} B \preceq A^{(*)}$.

Proof. We prove parts (i)-(iii) only. Since $A^{(*)} = ((A^T)^{(+)})^T$, parts (iv)-(vi) follow from the fact that the transpose map is an anti-isomorphism of $M_n(S)$ (see Lemma 3.1).

- (i) First note that for all $i, j, k \in [n]$ we have:

$$(A^{(+)})_{i,j} = \langle A_{j,\star} | A_{i,\star} \rangle \geq \langle A_{j,\star} | A_{k,\star} \rangle \langle A_{k,\star} | A_{i,\star} \rangle = (A^{(+)})_{k,j} (A^{(+)})_{i,k},$$

and so

$$(A^{(+)}_{i,j}) \geq \bigvee_{k=1}^n (A^{(+)}_{i,k} (A^{(+)}_{k,j}) = (A^{(+)} A^{(+)}_{i,j}).$$

If $i \notin \text{dom}(A)$, then row i of $A^{(+)}$ is zero, and so too is row i of the product $A^{(+)} A^{(+)}$. If $i \in \text{dom}(A)$, then $(A^{(+)}_{i,i}) = \langle A_{i,\star} | A_{i,\star} \rangle = \mathbf{1}$, and it follows that for all $j \in [n]$ we have

$$(A^{(+)} A^{(+)}_{i,j}) = \bigvee_{k=1}^n (A^{(+)}_{i,k} (A^{(+)}_{k,j}) \geq (A^{(+)}_{i,i} (A^{(+)}_{i,j})) = (A^{(+)}_{i,j}).$$

(ii) Noting that $\langle A_{k,\star} | A_{i,\star} \rangle A_{k,\star} \preceq A_{i,\star}$ yields

$$(A^{(+)} A)_{i,j} = \bigvee_{k=1}^n (A^{(+)}_{i,k} A_{k,j}) = \bigvee_{k=1}^n \langle A_{k,\star} | A_{i,\star} \rangle A_{k,j} \leq A_{i,j}.$$

If $i \notin \text{dom}(A)$, then for all j we have $A_{i,j} = (A^{(+)}_{i,j}) = \mathbf{0}$. If $i \in \text{dom}(A)$, then $(A^{(+)} A)_{i,j} \geq (A^{(+)}_{i,i} A_{i,j}) = A_{i,j}$ for all $j \in [n]$.

(iii) Suppose that $BA = A$. Then for all i, j, k we have $B_{i,k} A_{k,j} \leq A_{i,j}$, and hence for all i, k we have

$$B_{i,k} \leq \bigwedge_{j: A_{k,j} \neq 0} A_{i,j} A_{k,j}^{-1} = (A^{(+)}_{k,j}),$$

where the equality follows from the fact that $\text{dom}(A) = [n]$. This shows that $B \preceq A^{(+)}$, and hence $BA^{(+)} \preceq A^{(+)} A^{(+)} = A^{(+)}$. \square

Parts (i), (ii), (iv) and (v) of the previous proposition say that $A^{(+)}$ is a left identity for A , whilst $A^{(*)}$ is a right identity. In general, $A^{(+)}$ need not be a right identity for A , nor $A^{(*)}$ a left identity. For example, consider

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, A^{(+)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A^{(*)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, in the case where A has no zero rows Part (iii) of the previous result says that $A^{(+)}$ is the the least upper bound with respect to \preceq of the set of elements which fix A under left multiplication, whilst Part (vi) says that $A^{(*)}$ is the the least upper bound with respect to \preceq of the set of elements which fix A under right multiplication.

As noted in the proof, it follows easily from (3) and (4) that $A^{(*)} = ((A^T)^{(+)})^T$ for all $A \in M_n(S)$. Likewise, one has $A^{(*)} = \Delta((\Delta(A))^{(+)})$ for all $A \in M_n(S)$, where Δ is the map from Lemma 3.1. We shall see shortly, the map $A \mapsto A^{(+)}$ (along with a corresponding left-right dual map $A \mapsto A^{(*)}$) allows us to show that certain subsemigroups of $M_n(S)$ are Fountain. The notation $A^{(+)}$ and $A^{(*)}$ stems from the theory of regular semigroups and their generalisations; we use brackets in the superscript to distinguish these from the *Kleene plus/star operations* as applied to matrices, which arise naturally in applications of Boolean and tropical matrices, for example.

Proposition 4.5. *Let U denote the set of idempotents of $M_n(S)$ with all diagonal entries equal to $\mathbf{1}$. If $A, B \in M_n(S)$ with $\text{dom}(A) = \text{dom}(B) = [n]$, then*

$$A \tilde{\mathcal{R}} B \Rightarrow A^{(+)} = B^{(+)}, \text{ and } A \tilde{\mathcal{R}}_U B \Leftrightarrow A^{(+)} = B^{(+)}.$$

Proof. By Proposition 4.4, we see that $A^{(+)}$ is an idempotent acting as a left identity for A , and $B^{(+)}$ is an idempotent acting as a left identity for B . It then follows from the definition of $\tilde{\mathcal{R}}$ that if $A \tilde{\mathcal{R}} B$, then $A^{(+)} B = B$ and $B^{(+)} A = A$. Applying Part (iii) of Proposition 4.4 twice then gives $A^{(+)} = B^{(+)}$. Noting that $\text{dom}(A) = \text{dom}(B) = [n]$ ensures that $A^{(+)}, B^{(+)} \in U$, the same argument shows that $A \tilde{\mathcal{R}}_U B$ implies $A^{(+)} = B^{(+)}$.

Suppose that $A^{(+)} = B^{(+)}$. If $X \in U$ with $XA = A$, then by Proposition 4.4(iii), $X \preceq A^{(+)} = B^{(+)}$. It then follows from Lemma 4.1 (i) and Proposition 4.4(ii) that $XB \preceq B^{(+)}B = B$, whilst $B \preceq XB$ by Lemma 4.1 (ii). \square

The matrices A for which $\text{dom}(A) = [n]$ are precisely those for which the matrix (A/A) of [5] is an idempotent of $M_n(S)$ acting as a left identity for A :

Lemma 4.6. *Let S be an idempotent semifield, and $A \in M_n(S)$. Then the following are equivalent:*

- (i) $(A/A) \in M_n(S)$;
- (ii) $\text{dom}(A) = [n]$;
- (iii) $(A/A) = A^{(+)}$.

Proof. By definition, if $(A/A) \in M_n(S)$, then for all i, j there exists k such that $A_{i,k} \setminus A_{j,k} \neq \mathbf{1}$, or in other words, every row of A is non-zero. In this case, it is easy to see that the (i, j) th entry of A/A is given by

$$(A/A)_{i,j} = \bigwedge_k (A_{j,k} \setminus A_{i,k}) = \bigwedge_{k:A_{j,k} \neq 0} (A_{j,k} \setminus A_{i,k}) = \bigwedge_{k:A_{j,k} \neq 0} (A_{i,k} A_{j,k}^{-1}) = (A^{(+)}_{i,j})$$

This shows that (i) implies (ii) and (ii) implies (iii). Since by definition $A^{(+)} \in M_n(S)$, that (iii) implies (i) is trivial. \square

Theorem 4.7. *Let S be an idempotent semifield, T a subsemigroup of $M_n(S)$, U the set of all idempotents of $M_n(S)$ whose diagonal entries are all equal to $\mathbf{1}$, and write $V = T \cap U$. Assume that $\text{dom}(A) = [n]$ and $A^{(+)} \in T$ for all $A \in T$. Then for all $A, B \in T$ we have*

- (i) $A \tilde{\mathcal{R}}_V A^{(+)}$ in T ;
- (ii) if $A \in V$, then $A^{(+)} = A$; and
- (iii) $A \tilde{\mathcal{R}}_V B$ in T if and only if $A^{(+)} = B^{(+)}$.

In particular, if T admits an involutory anti-isomorphism φ with $\varphi(V) = V$, then T is V -Fountain, with each $\tilde{\mathcal{R}}_V$ -class and each $\tilde{\mathcal{L}}_V$ -class containing a unique idempotent of V .

Proof. (i) Let $A \in T$. By Proposition 4.4 we know that $A^{(+)}$ is an idempotent which acts as a left identity on A . By assumption, $A^{(+)} \in T$, and since $\text{dom}(A) = [n]$, it follows from the definition that all diagonal entries are equal to $\mathbf{1}$. Thus $A^{(+)} \in V$ and, by Lemma 2.2, it suffices to show that for all idempotents $F \in V$ with $FA = A$ we have $FA^{(+)} = A^{(+)}$. Since $\text{dom}(A) = [n]$, we have $FA^{(+)} \preceq A^{(+)}$ by Proposition 4.4(iii). Since $F \in V$, we have $F_{i,i} = \mathbf{1}$ for all $i \in [n]$, and hence $A^{(+)} \preceq FA^{(+)}$ by Lemma 4.1. This shows that $A \tilde{\mathcal{R}}_V A^{(+)}$ in T .

(ii) Since A is idempotent, Proposition 4.4 (iii) gives $A \preceq A^{(+)}$. But since all diagonal entries of A are equal to $\mathbf{1}$, Proposition 4.4 and Lemma 4.1, give $A^{(+)} \preceq A^{(+)}A = A$. Thus $A^{(+)} = A$ for all $A \in V$.

(iii) If $A^{(+)} = B^{(+)}$, then by Part (i) we have $A \tilde{\mathcal{R}}_V A^{(+)} = B^{(+)} \tilde{\mathcal{R}}_V B$. Suppose that $A \tilde{\mathcal{R}}_V B$ in T . Then in particular, $A^{(+)}B = B$ and $B^{(+)}A = A$. Applying Proposition 4.4 (iii) then gives $A^{(+)} = B^{(+)}$.

In the case where T admits an involutory anti-isomorphism fixing V , that T is V -Fountain now follows from Part (i) together with Lemma 2.3. If $E, F \in V$ are $\tilde{\mathcal{R}}_V$ -related then Parts (ii) and (iii) yield $E = E^{(+)} = F^{(+)} = F$. \square

From now on let U denote the set of idempotents of $M_n(S)$ having all diagonal entries equal to $\mathbf{1}$. In the following subsections we apply Theorem 4.7 to exhibit several V -Fountain subsemigroups of $M_n(S)$, for an appropriate set of idempotents $V \subseteq U$.

4.2. The semigroup without zero rows or zero columns is U -Fountain. Let $W_n(S)$ denote the set of all matrices $A \in M_n(S)$ with $\text{dom}(A) = \text{im}(A) = [n]$. It is readily verified that $W_n(S)$ is a submonoid of $M_n(S)$ containing U . (In the case where $S = \mathbb{B}$, the semigroup $W_n(\mathbb{B})$ corresponds to the monoid of binary relations that are both left- and right-total.)

Corollary 4.8. *Let S be an idempotent semifield. The monoid $W_n(S)$ is U -Fountain. Each $\tilde{\mathcal{R}}_U$ -class and each $\tilde{\mathcal{L}}_U$ -class contains a unique idempotent of U , given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. The transpose map restricts to give an involutory anti-automorphism of $W_n(S)$, mapping U to itself. Thus in order to show that $W_n(S)$ is U -Fountain, by Lemma 2.3 it suffices to show that each $\tilde{\mathcal{R}}_U$ -class of $W_n(S)$ contains an idempotent of U . By definition, $\text{dom}(A) = [n]$ for all $A \in W_n(S)$, and hence $A^{(+)} \in U \subseteq W_n(S)$. Theorem 4.7 with $T = W_n(S)$, $V = U$ and φ the transpose map now yields that each $\tilde{\mathcal{R}}_U$ -class contains a unique idempotent of U , specified by the map $A \mapsto A^{(+)}$. Dually, via the transpose map, each $\tilde{\mathcal{L}}_U$ -class contains a unique idempotent of U , giving the desired result. \square

Question 4.9. Is $W_n(S)$ is K -Fountain for some $K \supset U$? Indeed, is $W_n(S)$ Fountain?

By an abuse of notation, let us denote by $M_n(S^*)$ the subsemigroup of $M_n(S)$ whose entries are all non-zero.

Corollary 4.10. *Let S be an idempotent semifield. The semigroup $M_n(S^*)$ is V -Fountain, where V is the set of all idempotents of $M_n(S^*)$ with all diagonal entries equal to $\mathbf{1}$. Each $\tilde{\mathcal{R}}_V$ -class and each $\tilde{\mathcal{L}}_V$ -class contains a unique idempotent of V , given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. The transpose map is an involutory anti-isomorphism on $M_n(S^*)$, which maps V bijectively to itself. Thus by Theorem 4.7 we just need to show that if $A \in M_n(S^*)$ then so is $A^{(+)}$. This is clearly the case, by (3). \square

Remark 4.11. In the case where $S^* = (\mathbb{R}, +)$ we give an alternative geometric proof of the previous result. We first recall that each matrix $A \in M_n(\mathbb{T}^*)$ has unique tropical eigenvalue, and that if this value is a non-positive real number, then the max-plus Kleene star (formed by taking the component-wise maximum of all powers of A together with identity matrix of $M_n(\mathbb{T})$), is a well-defined³ idempotent (see for example [16]). The set V in Corollary 4.10 is precisely the set of all Kleene star matrices in $M_n(\mathbb{T}^*)$. (Whilst $A^{(+)} \in V$, in general $A^{(+)}$ is *not* equal to the Kleene star of A ; the latter need not be a left identity for A .)

Continuing with this alternative viewpoint, to show that $M_n(\mathbb{T}^*)$ is V -Fountain, by arguing as in Proposition 3.12 it suffices to show that the intersection:

$$\text{ColFix}_V(A) := \bigcap_{\substack{A \leq_{\mathcal{R}} F, \\ F \in V}} \text{Col}(F)$$

is equal to the column space of an idempotent in V . The column space of a matrix $A \in M_n(\mathbb{T}^*)$ may be naturally identified with a subset of \mathbb{R}^n , by removing the single element at $-\infty$ which does not lie in this set. The projectivisation map $\mathcal{P} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ given by $(x_1, \dots, x_n) \mapsto (x_1 - x_n, \dots, x_{n-1} - x_n)$ identifies those elements of $\text{Col}(A)$ which are tropical scalings of each other, and the image of $\text{Col}(A)$ under this map is a compact subset of \mathbb{R}^{n-1} . It follows from [26, Theorem A and Theorem B] that for all $F \in V$, $\text{Col}(F)$ is max-plus convex, min-plus convex and Euclidean convex in \mathbb{R}^n . Thus $\text{ColFix}_V(A)$ also

³If the eigenvalue is positive, then the powers of A do not stabilise. However, in many situations it suffices to consider the Kleene star of a suitable scaling of A .

has these three properties. Moreover, since this intersection has compact projectivisation, [26, Theorem A] then gives that $\text{ColFix}_V(A)$ is the max-plus column space of a max-plus Kleene star.

Question 4.12. Is $M_n(S^*)$ is Fountain?

Let $\text{Sym}(n)$ denote the symmetric group acting on the set $[n]$. It is easy to see that

$$H_n(S) = \{A \in M_n(S) : \exists \sigma \in \text{Sym}(n), \forall i \in [n], A_{i,\sigma(i)} \neq \mathbf{0}\}$$

is a submonoid of $W_n(S)$.

Corollary 4.13. *Let S be an idempotent semifield. The monoid $H_n(S)$ is U -Fountain. Each $\tilde{\mathcal{R}}_U$ -class and each $\tilde{\mathcal{L}}_U$ -class contains a unique idempotent of U , given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. The transpose map is an involutory anti-isomorphism of $H_n(S)$, since taking transpose exchanges σ with σ^{-1} . The set U is contained in $H_n(S)$, since for each $E \in U$ we can take σ to be the identity permutation, and U is clearly closed under transpose. Noting that if $A \in H_n(S) \subseteq W_n(S)$, we automatically have $A^{(+)} \in U$, the result then follows immediately from Theorem 4.7. \square

Remark 4.14. Suppose that E is an idempotent of $H_n(\mathbb{B})$. There there exists $\sigma \in \text{Sym}(n)$ such that $E_{i,\sigma(i)} = \mathbf{1}$ for all $i \in [n]$. If $\sigma^k(i) = i$, we obtain

$$E_{i,i} = (E^k)_{i,i} \geq E_{i,\sigma(i)} E_{\sigma(i),\sigma^2(i)} \cdots E_{\sigma^{k-1}(i),i} = \mathbf{1},$$

showing that each idempotent of $H_n(\mathbb{B})$ lies in U , and hence $H_n(\mathbb{B})$ is Fountain. (For $S \neq \mathbb{B}$, this argument only shows that the diagonal elements of each idempotent are non-zero; the idempotents of $H_n(S)$ do not lie in U in general.) Alternatively, we note that the monoid $H_n(\mathbb{B})$ may be identified with the monoid of all Hall relations (those relations containing a perfect matching) on a set of cardinality n . Since the latter is known to be a finite block group, it is Fountain by [32, Corollary 3.2]. Another very natural question is whether $H_n(S)$ is Fountain.

Let $R_n([0, 1])$ denote the set of $n \times n$ matrices with all diagonal entries equal to $\mathbf{1}$, with all remaining entries lying in the interval $[0, 1]$. Noting that $[0, 1]$ is a subsemiring of the semifield S , it is easy to see that $R_n([0, 1])$ is a submonoid of $W_n(S)$.

Corollary 4.15. *Let S be a linearly ordered idempotent semifield. The monoid $R_n([0, 1])$ is Fountain. Each $\tilde{\mathcal{R}}$ -class and each $\tilde{\mathcal{L}}$ -class contains a unique idempotent, given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. First notice that, by definition, each of the diagonal entries of $R_n([0, 1])$ is equal to $\mathbf{1}$. The idempotent elements of this submonoid of $M_n(S)$ therefore form a subset of the set of all idempotents of $M_n(S)$ with $\mathbf{1}$'s on their diagonal. The latter is the set of idempotents U from Theorem 4.7. The transpose map is an involutory anti-isomorphism of $R_n([0, 1])$. It therefore remains to show that for $A \in R_n([0, 1])$ we have $A^{(+)} \in R_n([0, 1])$. Given $A \in R_n([0, 1])$ it is clear that the diagonal entries of $A^{(+)}$ are all equal to $\mathbf{1}$. Suppose for contradiction that $(A^{(+)})_{i,j} > \mathbf{1}$ for some $i, j \in [n]$. Then

$$(A^{(+)}A)_{i,j} \geq (A^{(+)})_{i,j}A_{j,j} = (A^{(+)})_{i,j} > \mathbf{1},$$

contradicting that $A^{(+)}A = A$. Since S is assumed to be linearly ordered, we must then have $(A^{(+)})_{i,j} \leq \mathbf{1}$ for all i, j and hence $A^{(+)} \in R_n([0, 1])$. \square

Remark 4.16. In the Boolean case, notice that the monoid $R_n([0, 1])$ is the subsemigroup of $M_n(\mathbb{B})$ defined by the single condition that all diagonal entries are equal to $\mathbf{1}$, and hence may be identified with the monoid of all reflexive binary relations on a set of cardinality n . Since the latter is a finite \mathcal{J} -trivial monoid (see for example, [41]), the fact that it is Fountain can be seen as a consequence of [32, Corollary 3.2].

4.3. Triangular matrix semigroups with full diagonal are Fountain. Proposition 3.13 gives an indication that the failure of $UT_n(S)$ to be Fountain lies in the degenerate behaviour of matrices with diagonal entries equal to $\mathbf{0}$. For each subset $J \subseteq [n]$ write $UT_n^J(S)$ to denote the subset of $UT_n(S)$ whose non-zero diagonal entries lie precisely in positions (j, j) for $j \in J$. Since S has no zero divisors, it is easy to see that $UT_n^J(S)$ is a subsemigroup of $UT_n(S)$. It is clear from the definitions that $UT_n(S)$ is the meet semilattice of component semigroups $UT_n^J(S)$, with partial order corresponding to inclusion of subsets. (Specifically, if $A \in UT_n^J(S)$ and $B \in UT_n^K(S)$ it is easy to see that $AB, BA \in UT_n^{J \cap K}(S)$.) These components are not usually Fountain. For example, arguing as in the proof of Proposition 3.13 the component $UT_n^J(S)$ is not Fountain for $n \geq 4$ and $\{2, 3, 4\} \subseteq J \subsetneq [n]$.

From now on, let $\overline{UT}_n(S)$ denote the component of $UT_n(S)$ consisting of those upper triangular matrices A with $A_{i,i} \neq \mathbf{0}$ for all $i \in [n]$. We say that the elements of $\overline{UT}_n(S)$ have “full diagonal”. Suppose that $A \in \overline{UT}_n(S)$ is idempotent. Thus $A_{i,i}^2 = A_{i,i}$, and since $A_{i,i} \neq \mathbf{0}$, we must have $A_{i,i} = \mathbf{1}$ for all $i \in [n]$, giving $E(\overline{UT}_n(S)) = U \cap U_n(S) = E(U_n(S))$.

Corollary 4.17. *Let S be an idempotent semifield. Then $\overline{UT}_n(S)$ is Fountain and each $\tilde{\mathcal{R}}$ -class and each $\tilde{\mathcal{L}}$ -class contains a unique idempotent, given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. Since $\text{dom}(A) = [n]$, we have $(A^{(+)})_{i,i} = \mathbf{1}$. Moreover, it follows easily from (3) that for all $i > j$, $(A^{(+)})_{i,j} = \mathbf{0}$, since in this case $\text{Supp}(A_{j,\star}) \not\subseteq \text{Supp}(A_{i,\star})$. Thus $A^{(+)} \in \overline{UT}_n(S)$. As observed above, $E(\overline{UT}_n(S)) \subseteq U$. The result now follows from Theorem 4.7 via the anti-isomorphism Δ , which clearly restricts to an anti-isomorphism of $\overline{UT}_n(S)$, mapping $E(\overline{UT}_n(S))$ to itself. \square

Corollary 4.18. *Let S be an idempotent semifield. Then the monoid of unitriangular matrices $U_n(S)$ is Fountain and each $\tilde{\mathcal{R}}$ -class and each $\tilde{\mathcal{L}}$ -class contains a unique idempotent, given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. We consider the setup of Theorem 4.7 with $T = U_n(S)$, $\varphi = \Delta$ and $V = E(U_n(S))$. Clearly $\Delta(E(U_n(S))) = E(U_n(S))$, and the argument of Corollary 4.17 shows that if $A \in U_n(S)$, then $A^{(+)} \in U_n(S)$. \square

Remark 4.19. In the case where S^* is the trivial group, we obtain the monoid of all unitriangular Boolean matrices. This is a finite \mathcal{J} -trivial monoid (see for example, [41]), and so the fact that it is Fountain can be seen as a consequence of [32, Corollary 3.2].

By an abuse of notation, let us write $UT_n(S^*)$ to denote the subsemigroup of $UT_n(S)$ consisting of those upper triangular matrices whose entries on and above the diagonal are non-zero. Likewise, we write $U_n(S^*)$ to denote the subsemigroup of $U_n(S)$ consisting of those upper triangular matrices with all diagonal entries equal to $\mathbf{1}$ and whose entries above the diagonal are non-zero. (In the case where $S = \mathbb{B}$, notice that $UT_n(S^*) = U_n(S^*)$ is the trivial group.)

Corollary 4.20. *Let S be an idempotent semifield. Then the semigroups $UT_n(S^*)$ and $U_n(S^*)$ are Fountain. Each $\tilde{\mathcal{R}}$ -class and each $\tilde{\mathcal{L}}$ -class $UT_n(S^*)$ (respectively, $U_n(S^*)$) contains a unique idempotent, given by the maps $A \mapsto A^{(+)}$ and $A \mapsto A^{(*)}$.*

Proof. Consider the setup of Theorem 4.7 with $T = UT_n(S^*)$, $\varphi = \Delta$ and $V = E(UT_n(S^*)) = E(U_n(S^*))$. Clearly $\Delta(V) = V$, so it suffices to show that if $A \in UT_n(S^*)$, then $A^{(+)} \in UT_n(S^*)$. Arguing as before, we see that $(A^{(+)})_{i,i} = \mathbf{1}$ for all i and if $j < i$, then $(A^{(+)})_{i,j} = \mathbf{0}$. On the other hand, for $j > i$ we have $\emptyset \neq \text{Supp}(A_{j,\star}) \subseteq \text{Supp}(A_{i,\star})$,

so that $(A^{(+)})_{i,j}$ is the least upper bound of a finite set of elements of S^* and hence lies in S^* .

Finally, consider the setup of Theorem 4.7 with $T = U_n(S^*) \subseteq \overline{UT}_n(S)$, $\varphi = \Delta$ and $V = E(U_n(S^*))$. Arguing as above we find that $A^{(+)} \in U_n(S^*)$ for all $A \in U_n(S^*)$. \square

We write $D_n(S^*)$ to denote the set of invertible diagonal matrices. For any $A \in \overline{UT}_n(S)$, we write D_A to denote the element of $D_n(S^*)$ with (i, i) entry equal to $A_{i,i}$ for all i , and write A^\diamond and A^\bullet to denote the unique unitriangular matrices satisfying $A = A^\diamond D_A$ and $A = D_A A^\bullet$.

Theorem 4.21. *Let $A, B \in \overline{UT}_n(S)$. Then the following are equivalent:*

- (i) $A \tilde{\mathcal{R}} B$ in $\overline{UT}_n(S)$;
- (ii) $A^\diamond \tilde{\mathcal{R}} B^\diamond$ in $\overline{UT}_n(S)$;
- (iii) $A^\diamond \tilde{\mathcal{R}} B^\diamond$ in $U_n(S)$;
- (iv) $A^{(+)} = B^{(+)}$;
- (v) $(A^\diamond)^{(+)} = (B^\diamond)^{(+)}$.

Proof. The equivalence of (i) and (iv), and of (ii) and (v) is given by Corollary 4.17. The equivalence of (iii) and (v) is given by Corollary 4.18. Let $A \in \overline{UT}_n(S)$. To complete the proof, we show that $A^{(+)} = (A^\diamond)^{+}$, hence giving that (iv) and (v) are equivalent (indeed, the same statement). Since $A = A^\diamond D_A$, for all $i, j \in [n]$ we have $A_{i,j}^\diamond = A_{i,j} A_{j,j}^{-1}$, and so by definition

$$((A^\diamond)^{+})_{i,j} = \bigwedge_{j \leq k \leq n} A_{i,k}^\diamond (A_{j,k}^\diamond)^{-1} = \bigwedge_{j \leq k \leq n} A_{i,k} A_{k,k}^{-1} (A_{j,k} A_{k,k}^{-1})^{-1} = \bigwedge_{j \leq k \leq n} A_{i,k} A_{j,k}^{-1} = (A^{+})_{i,j}.$$

\square

Similarly, using Corollary 4.20 in place of Corollary 4.17 and Corollary 4.18, one obtains:

Theorem 4.22. *Let $A, B \in UT_n(S^*)$. Then the following are equivalent:*

- (i) $A \tilde{\mathcal{R}} B$ in $UT_n(S^*)$;
- (ii) $A^\diamond \tilde{\mathcal{R}} B^\diamond$ in $UT_n(S^*)$;
- (iii) $A^\diamond \tilde{\mathcal{R}} B^\diamond$ in $U_n(S^*)$;
- (iv) $A^{(+)} = B^{(+)}$;
- (v) $(A^\diamond)^{(+)} = (B^\diamond)^{(+)}$.

Corollary 4.23. *There is a one-one correspondence between:*

- (1) $\overline{UT}_n(S)/\tilde{\mathcal{R}}$ and $E(U_n(S))$;
- (2) $U_n(S)/\tilde{\mathcal{R}}$ and $E(U_n(S))$;
- (3) $UT_n(S^*)/\tilde{\mathcal{R}}$ and $E(U_n(S^*))$;
- (4) $U_n(S)/\tilde{\mathcal{R}}$ and $E(U_n(S^*))$.

4.4. The idempotent generated subsemigroups of $\overline{UT}_n(S)$ and $UT_n(S^*)$. First observe that

$E \in \overline{UT}_n(S)$ is idempotent if and only if for all $i, j, k \in [n]$, $E_{i,i} = \mathbf{1}$ and $E_{i,k} E_{k,j} \leq E_{i,j}$.

Theorem 4.24. *Let S be an idempotent semifield. The semigroup $U_n(S)$ is the idempotent generated subsemigroup of $\overline{UT}_n(S)$. Every element of $U_n(S)$ can be written as a product of at most $n - 1$ idempotent elements.*

Proof. The result is trivial for $n = 1$, since in this case both semigroups are equal to the trivial group. Thus we shall assume that $n \geq 2$. As we have already observed, the idempotents of $\overline{UT}_n(S)$ form a subset of $U_n(S)$. Thus it suffices to prove the second statement.

Given $X \in U_n(S)$, let \mathfrak{X} denote the matrix with (i, j) entry given by $(X^{(+)})_{i,j} \wedge (X^{(*)})_{i,j}$, where $X^{(*)}$ denotes the right identity of X dual to $X^{(+)}$ under the involution Δ . Since $X^{(*)}, X^{(+)} \in U_n(S)$, we see that $\mathfrak{X} \in U_n(S)$. Since all diagonal entries of \mathfrak{X} are equal to $\mathbf{1}$, for all $i, j \in [n]$ we have $(\mathfrak{X}\mathfrak{X})_{i,j} \geq \mathfrak{X}_{i,i}\mathfrak{X}_{i,j} = \mathfrak{X}_{i,j}$. In fact, \mathfrak{X} is idempotent since

$$\begin{aligned} (\mathfrak{X}\mathfrak{X})_{i,j} &= \bigvee_{k=1}^n ((X^{(+)})_{i,k} \wedge (X^{(*)})_{i,k}) ((X^{(+)})_{k,j} \wedge (X^{(*)})_{k,j}) \\ &\leq \bigvee_{k=1}^n ((X^{(+)})_{i,k} (X^{(+)})_{k,j} \wedge (X^{(*)})_{i,k} (X^{(*)})_{k,j}) \leq \bigvee_{k=1}^n ((X^{(+)})_{i,j} \wedge (X^{(*)})_{i,j}) = \mathfrak{X}_{i,j}, \end{aligned}$$

where the last inequality follows from the fact that $X^{(+)}$ and $X^{(*)}$ are idempotents. Moreover, since $X^{(+)}$ (respectively, $X^{(*)}$) is a left (respectively, right) identity for X , it follows that for all $i, j, k \in [n]$ we also have

$$\mathfrak{X}_{i,k} X_{k,j} = ((X^{(+)})_{i,k} \wedge (X^{(*)})_{i,k}) X_{k,j} \leq (X^{(+)})_{i,k} X_{k,j} \leq X_{i,j}, \quad (5)$$

$$X_{i,k} \mathfrak{X}_{k,j} = X_{i,k} ((X^{(+)})_{k,j} \wedge (X^{(*)})_{k,j}) \leq X_{i,k} (X^{(*)})_{k,j} \leq X_{i,j}. \quad (6)$$

It follows from (5) and (6) that $\mathfrak{X}X \preceq X$ and $X\mathfrak{X} \preceq X$. Moreover, since $\mathfrak{X}_{i,i} = \mathbf{1}$ for all i , Lemma 4.1 gives $X \preceq \mathfrak{X}X$ and $X \preceq X\mathfrak{X}$. Thus $\mathfrak{X}X = X = X\mathfrak{X}$. By taking $k = j$ in (5) we also have $\mathfrak{X} \preceq X$.

Now, for each $h \in [n]$ let $X(h) \in U_n(S)$ be the matrix with entries given by

$$X(h)_{i,j} = \begin{cases} X_{i,j} & \text{if } i < h \leq j \\ \mathfrak{X}_{i,j} & \text{otherwise.} \end{cases}$$

In other words, $X(1) = \mathfrak{X}$, and for $h \geq 2$ we have

$$X(h) = \left(\begin{array}{cccc|cccc} \mathbf{1} & \mathfrak{X}_{1,2} & \cdots & \mathfrak{X}_{1,(h-1)} & X_{1,h} & X_{1,(h+1)} & \cdots & X_{1,n} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathfrak{X}_{2,(h-1)} & X_{2,h} & X_{2,(h+1)} & \cdots & X_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} & X_{h-1,h} & X_{h-1,h+1} & \cdots & X_{h-1,n} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathfrak{X}_{h,(h+1)} & \cdots & \mathfrak{X}_{h,n} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdots & \mathfrak{X}_{h+1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{array} \right)$$

We shall show that for each $h \in [n]$ the matrix $X(h)$ is idempotent, and moreover that $X = X(n)X(n-1) \cdots X(2)$. To this end for $j, h \in [n]$ let us consider the following elements of S^n :

$$x(j, h) = \begin{pmatrix} X_{1,j} \\ \vdots \\ \frac{X_{h-1,j}}{\mathfrak{X}_{h,j}} \\ \vdots \\ \mathfrak{X}_{n,j} \end{pmatrix}, \quad y(j, h) = \begin{pmatrix} \mathfrak{X}_{1,j} \\ \vdots \\ \frac{\mathfrak{X}_{h-1,j}}{\mathbf{0}} \\ \vdots \\ \mathbf{0} \end{pmatrix}.$$

It is easy to see that $x(j, n)$ is the j th column of X (since the final row of \mathfrak{X} agrees with the final row of X). We claim that the following equations hold:

$$X(h)y(j, h) = y(j, h), \quad \text{for all } 1 \leq j \leq h-1 \leq n-1, \quad (7)$$

$$X(h)x(j, h) = x(j, h), \quad \text{for all } 2 \leq h \leq j \leq n, \quad (8)$$

$$X(h)x(j, h-1) = x(j, h), \quad \text{for all } 2 \leq h \leq n, 1 \leq j \leq n. \quad (9)$$

Before proving this, let us show how these equations can be used to give the desired result. Notice that (7) and (8) yield that $X(h)$ is idempotent, since $y(1, h), \dots, y(h-1, h), x(h, h), \dots, x(n, h)$ are precisely the columns of $X(h)$. Since $X_{1,1} = \mathfrak{X}_{1,1} = \mathbf{1}$, we note that the j th column of $X(2)$ is equal to $x(j, 2)$. Thus repeated application of (9) shows that the j th column of the product $X(h) \cdots X(2)$ is equal to $x(j, h)$ for all $j \in [n]$. Taking $h = n$ and recalling that $x(j, n)$ is the j th column of X then yields $X = X(n) \cdots X(2)$.

To prove that (7) holds, let $1 \leq j \leq h-1$ and consider the i th entry of the product $X(h)y(j, h)$, given by

$$\bigvee_{k=1}^n X(h)_{i,k} y(j, h)_k = \bigvee_{k=1}^{h-1} X(h)_{i,k} \mathfrak{X}_{k,j}.$$

If $i \geq h$, then since the first $h-1$ entries in row i of $X(h)$ are zero, the supremum above is clearly equal to $\mathbf{0}$. If $i \leq h-1$, then the supremum above becomes $\bigvee_{k=1}^{h-1} \mathfrak{X}_{i,k} \mathfrak{X}_{k,j} = \bigvee_{i \leq k \leq j} \mathfrak{X}_{i,k} \mathfrak{X}_{k,j} = \mathfrak{X}_{i,j}$, since \mathfrak{X} is upper triangular and idempotent. Thus in both cases we obtain the i th entry of $y(j, h)$.

To prove that (8) holds, let $h \leq j \leq n$ and consider the i th entry of the product $X(h)x(j, h)$ given by

$$\bigvee_{k=1}^n X(h)_{i,k} x(j, h)_k = \bigvee_{k=1}^{h-1} X(h)_{i,k} X_{k,j} \vee \bigvee_{k=h}^j X(h)_{i,k} \mathfrak{X}_{k,j}.$$

If $i \leq h-1$ then this becomes $\bigvee_{k=1}^{h-1} \mathfrak{X}_{i,k} X_{k,j} \vee \bigvee_{k=h}^j X_{i,k} \mathfrak{X}_{k,j} \leq X_{i,j}$, where the inequality follows from (5) and (6); in fact the term corresponding to $k = i$ yields equality. On the other hand, if $i \geq h$, then the supremum above becomes $\bigvee_{k=h}^n \mathfrak{X}_{i,k} \mathfrak{X}_{k,j} = \bigvee_{i \leq k \leq j} \mathfrak{X}_{i,k} \mathfrak{X}_{k,j} = \mathfrak{X}_{i,j}$, as before. Thus in both cases we obtain the i th entry of $x(j, h)$.

Finally, to prove that (9) holds, consider the i th entry of the product $X(h)x(j, h-1)$ given by

$$\begin{aligned} \bigvee_{k=1}^n X(h)_{i,k} x(j, h-1)_k &= \bigvee_{k=1}^{h-2} \mathfrak{X}_{i,k} X_{k,j} \vee (\mathfrak{X}_{i,h-1} \mathfrak{X}_{h-1,j}) \vee \bigvee_{k=h}^n X(h)_{i,k} \mathfrak{X}_{k,j} \quad (10) \\ &\leq X_{i,j}, \end{aligned}$$

where the inequality follows from (5) and (6), together with the facts that \mathfrak{X} is idempotent satisfying $\mathfrak{X} \preceq X$, and for each i, k the entry $X(h)_{i,k}$ is equal to one of $X_{i,k}$ or $\mathfrak{X}_{i,k}$. We show that the supremum in (10) is equal to the i th entry of $x(j, h)$. That is, for $i \leq h-1$, we show that the inequality above is an equality, whilst for $i \geq h$ we show that the supremum is equal to $\mathfrak{X}_{i,j}$.

If $i \leq h-2$, then setting $k = i$ in the supremum in (10) yields equality.

If $i = h-1$, then (10) becomes

$$\mathfrak{X}_{h-1,h-1} \mathfrak{X}_{h-1,j} \vee \bigvee_{k=h}^n X(h)_{h-1,k} \mathfrak{X}_{k,j} \leq X_{h-1,j}.$$

If $j \geq h$, then the term corresponding to $k = j$ is $X_{h-1,j}$ and so we have equality. If $j = h-1$, then $\mathfrak{X}_{h-1,h-1} \mathfrak{X}_{h-1,h-1} = \mathbf{1} = X_{h-1,h-1}$, and we again have equality. If $j < h-1$, then $X_{h-1,j}$ is zero and hence so is the supremum in (10).

If $i \geq h$, then (10) becomes

$$\bigvee_{k=h}^n X(h)_{i,k} \mathfrak{X}_{k,j} = \bigvee_{k=h}^n \mathfrak{X}_{i,k} \mathfrak{X}_{k,j} \leq \mathfrak{X}_{i,j},$$

where the inequality follows from the fact that \mathfrak{X} is idempotent. If $j \geq h$, then the term corresponding to $k = j$ is $\mathfrak{X}_{i,j}$ and so we have equality. If $i \geq h > j$ then $\mathfrak{X}_{i,j} = \mathbf{0}$ is zero and hence so is the supremum in (10).

Thus in all cases we find that the i th entry of the product $X(h)x(j, h-1)$ is equal to the i th entry of $x(j, h)$. This completes the proof. \square

Corollary 4.25. *Let S be an idempotent semifield. The semigroup $U_n(S^*)$ is the idempotent generated subsemigroup of $UT_n(S^*)$. Every element of $U_n(S^*)$ can be written as a product of at most $n-1$ idempotent elements.*

Proof. The idempotents of $UT_n(S^*)$ form a subset of $U_n(S^*)$. For $X \in U_n(S^*)$, notice that each of the idempotents $X(h)$ constructed in proof of the previous proposition lie in $U_n(S^*)$. \square

Corollary 4.26. *Every element of $\overline{UT}_n(S)$ (respectively, $UT_n(S^*)$) is a product of a diagonal matrix in $D_n(S^*)$ and $n-1$ idempotents of $\overline{UT}_n(S)$ (respectively, $UT_n(S^*)$). Moreover, we have the following decompositions as semidirect products of semigroups:*

$$\begin{aligned}\overline{UT}_n(S) &\simeq U_n(S) * D_n(S^*) \simeq (E(\overline{UT}_n(S)))^{n-1} * D_n(S^*), \\ UT_n(S^*) &\simeq U_n(S^*) * D_n(S^*) \simeq (E(UT_n(S^*)))^{n-1} * D_n(S^*).\end{aligned}$$

Proof. Since each $A \in \overline{UT}_n(S)$ may be written as $A = A^\diamond D_A$ where $A^\diamond \in U_n(S)$ and $D_A \in D_n(S^*)$, the first statement follows from Theorem 4.24 (respectively, Corollary 4.25). Recall that the semidirect product $M * N$ of (not necessarily commutative) semigroups $(M, +)$ and (N, \boxplus) with respect to a left action $\cdot : N \times M \rightarrow M$ is the set $M \times N$ with multiplication given by $(m_1, n_1)(m_2, n_2) = (m_1 + n_1 \cdot m_2, n_1 \boxplus n_2)$. The elements of $\overline{UT}_n(S)$ are easily seen to be in one to one correspondence with the elements of $U_n(S) \times D_n(S^*)$ via the map identifying A with the pair (A^\diamond, D_A) . For $A, B \in \overline{UT}_n(S)$ one then has that $AB = A^\diamond D_A B^\diamond D_B = (A^\diamond D_A B^\diamond D_A^{-1})(D_A D_B)$, showing that $\overline{UT}_n(S)$ is isomorphic to the semidirect product defined by the conjugation action of $D_n(S^*)$ on $U_n(S)$. Similarly, $UT_n(S^*)$ is isomorphic to the semidirect product defined by the conjugation action of $D_n(S^*)$ on $U_n(S^*)$. \square

Question 4.27. What is the idempotent generated subsemigroup of $M_n(S), UT_n(S), U_n(S), UT_n(S^*)$ and $U_n(S^*)$ for an arbitrary (semi)ring?

Both $U_n(S)$ and $U_n(S^*)$ are \mathcal{J} -trivial (see for example, [23, Lemma 4.1]). Our results then say that the idempotents of the Fountain semigroup $\overline{UT}_n(S)$ (respectively, $UT_n(S^*)$) generate the \mathcal{J} -trivial semigroup $U_n(S)$ (respectively, $U_n(S^*)$). Moreover, each of these Fountain semigroups turns out to be the semidirect product of the corresponding \mathcal{J} -trivial semigroup just mentioned with a group acting by conjugation. Our results for these (in general, infinite) semigroups therefore bring to mind a known result about finite monoids: any finite monoid whose idempotents generate a \mathcal{J} -trivial monoid is Fountain [32, Corollary 3.2]. A source of examples of such finite monoids is to take the semidirect product of a finite \mathcal{J} -trivial monoid with a finite group acting by automorphisms.

Remark 4.28. If $X \in U_n(S)$, then it is easy to see that for all $s \in \mathbb{N}$ and all $i, j \in [n]$ we have

$$(X^s)_{i,j} = \bigvee_{i \leq r_1 \leq \dots \leq r_{s-1} \leq j} X_{i,r_1} X_{r_1,r_2} \cdots X_{r_{s-1},j} = X_{i,j} \vee \bigvee_{t=1}^m \bigvee_{i < p_1 < \dots < p_t < j} X_{i,p_1} X_{p_1,p_2} \cdots X_{p_t,j},$$

where $m = \min(|i-j|-1, s-1)$. It follows from this that $X \preceq X^2 \preceq \dots \preceq X^{n-1} = X^n$, and hence $U_n(S)$ is aperiodic.

Proposition 4.29. *Let $E, F \in \overline{UT}_n(S)$ be idempotents. Then for all $m \in \mathbb{N}$ such that $2m \geq n + 1$ one has:*

$$(EF)^m = (EF)^m E = E(FE)^m = (FE)^m F = F(EF)^m = (FE)^m.$$

Proof. Let $m \in \mathbb{N}$, $1 \leq i \leq j \leq n$ and set

$$\begin{aligned} \Sigma_1 &= ((EF)^m)_{i,j} = \bigvee_{i \leq r_1 \leq \dots \leq r_{2m-1} \leq j} E_{i,r_1} F_{r_1,r_2} \cdots F_{r_{2m-1},j}, \\ \Sigma_2 &= ((EF)^m E)_{i,j} = \bigvee_{i \leq r_1 \leq \dots \leq r_{2m} \leq j} E_{i,r_1} F_{r_1,r_2} \cdots F_{r_{2m-1},r_{2m}} E_{r_{2m},j}, \\ \Sigma_3 &= ((FE)^m)_{i,j} = \bigvee_{i \leq r_1 \leq \dots \leq r_{2m-1} \leq j} F_{i,r_1} E_{r_1,r_2} \cdots E_{r_{2m-1},j}, \quad \text{and} \\ \Sigma_4 &= ((FE)^m F)_{i,j} = \bigvee_{i \leq r_1 \leq \dots \leq r_{2m} \leq j} F_{i,r_1} E_{r_1,r_2} \cdots E_{r_{2m-1},r_{2m}} F_{r_{2m},j}. \end{aligned}$$

Now let t denote a term in the supremum Σ_1 . Noting that $t = tE_{j,j}$, by setting $r_{2m} = j$ in the expression for Σ_2 above we see that t is also a term in the supremum Σ_2 . This shows that $\Sigma_1 \leq \Sigma_2$. Since $(FE)^m E = F(EF)^m$ and $(EF)^m E = E(FE)^m$, similar arguments show that $\Sigma_1 \leq \Sigma_4$ and $\Sigma_3 \leq \Sigma_2, \Sigma_4$.

Conversely, suppose that s is a term of the supremum Σ_2 . If $2m \geq n$, then for $1 \leq i \leq r_1 \leq \dots \leq r_{2m} \leq j \leq n$, we must have that $r_a = r_{a+1}$ for at least two distinct a in the range $0 \leq a \leq 2m$ (with $r_0 = i$ and $r_{2m+1} = j$) so as not to contradict $2m + 2 \geq n + 2$. At least one of these values of a must satisfy $a > 0$. If $a = 2m$, then clearly s is a term of Σ_1 , and we obtain that $\Sigma_1 = \Sigma_2$. Suppose then that $0 < a < 2m$, then one of $E_{r_a,r_{a+1}}$ or $F_{r_a,r_{a+1}}$ is a factor of s . By idempotency of E and F together with the fact that $E_{r_a,r_{a+1}} = F_{r_a,r_{a+1}} = 1$ one has

$$\begin{aligned} F_{r_{a-1},r_a} E_{r_a,r_{a+1}} F_{r_{a+1},r_{a+2}} &= F_{r_{a-1},r_a} F_{r_a,r_{a+1}} F_{r_{a+1},r_{a+2}} \leq F_{r_{a-1},r_{a+2}}, \quad \text{and} \\ E_{r_{a-1},r_a} F_{r_a,r_{a+1}} E_{r_{a+1},r_{a+2}} &= E_{r_{a-1},r_a} E_{r_a,r_{a+1}} E_{r_{a+1},r_{a+2}} \leq E_{r_{a-1},r_{a+2}}, \end{aligned}$$

from which it is then easy to see that $s \leq \Sigma_1$ and hence $\Sigma_1 = \Sigma_2$. Switching the roles of E and F in the above yields $\Sigma_3 = \Sigma_4$, and hence $\Sigma_1 \leq \Sigma_4 = \Sigma_3 \leq \Sigma_2 = \Sigma_1$. \square

In the following section we consider in more detail the generalised regularity properties of the semigroups $M_n(S)$, $UT_n(S)$ and $U_n(S)$ in the case where the idempotent semifield S has a linear order.

5. MATRICES OVER LINEARLY ORDERED IDEMPOTENT SEMIFIELDS

Throughout this section let \mathfrak{L} be a linearly ordered idempotent semifield.

5.1. Monoids of binary relations. Consider first the case where the underlying group is trivial, that is, where \mathfrak{L} is the Boolean semifield $\mathbb{B} = \{0, 1\}$. As noted in Lemma 2.1, \mathbb{B} is the unique idempotent semifield containing finitely many elements. We recall from [30, Theorem 1.1.1] that each submodule M of \mathbb{B}^n has a unique minimal generating set (basis), consisting of those non-zero elements $x \in M$ which cannot be expressed as a Boolean sum of elements of M occurring beneath x (with respect to the obvious partial order on Boolean vectors).

Proposition 5.1. *If every non-zero row of $A \in M_n(\mathbb{B})$ is contained in the unique basis of $\text{Row}(A)$, then $A \tilde{\mathcal{R}} A^{(+)}$.*

Proof. Since $A^{(+)}A = A$, from Lemma 2.2 it suffices to show that for any idempotent F satisfying $FA = A$ we also have $FA^{(+)} = A^{(+)}$. In fact, we shall show that for any $X \in M_n(\mathbb{B})$ with $XA = A$ we have $XA^{(+)} = A^{(+)}$. For all $i, j \in [n]$ we have

$$(XA^{(+)})_{i,j} = \bigvee_{k=1}^n X_{i,k}(A^{(+)})_{k,j} = \bigvee_{k: X_{i,k}=1} (A^{(+)})_{k,j}, \quad (11)$$

where the final equality follows from the fact that \mathbb{B} contains only $\mathbf{0}$ and $\mathbf{1}$.

We shall first show that if $XA = A$, then $XA^{(+)} \preceq A^{(+)}$. Since \mathbb{B} contains just two elements, it is clear that $XA^{(+)} \preceq A^{(+)}$ if and only if for all $i, j \in [n]$, $A_{i,j}^{(+)} = \mathbf{0}$ implies $(XA^{(+)})_{i,j} = \mathbf{0}$. From (3) it is easy to see that $A_{i,j}^{(+)} = \mathbf{0}$ if and only if either (i) $\emptyset = \text{Supp}(A_{j,\star})$ or (ii) $\emptyset \neq \text{Supp}(A_{j,\star}) \not\subseteq \text{Supp}(A_{i,\star})$. We consider the two cases separately.

(i) $\emptyset = \text{Supp}(A_{j,\star})$: It follows from (3) that column j of $A^{(+)}$ is zero, and hence so too is column j of $XA^{(+)}$. Thus $(XA^{(+)})_{i,j} = A_{i,j}^{(+)} = \mathbf{0}$.

(ii) $\emptyset \neq \text{Supp}(A_{j,\star}) \not\subseteq \text{Supp}(A_{i,\star})$: It follows from $XA = A$ that if $X_{i,k} = \mathbf{1}$ then for all t we have $A_{k,t} = X_{i,k}A_{k,t} \leq A_{i,t}$. In other words, if $X_{i,k} = \mathbf{1}$ then $\text{Supp}(A_{k,\star}) \subseteq \text{Supp}(A_{i,\star})$. Now, if $(XA^{(+)})_{i,j} = \mathbf{1}$, then $X_{i,k} = (A^{(+)})_{k,j} = \mathbf{1}$ for some k , in which case we would obtain

$$\text{Supp}(A_{j,\star}) \subseteq \text{Supp}(A_{k,\star}) \subseteq \text{Supp}(A_{i,\star}),$$

contradicting our assumption. So we must have that the supremum in (11) is equal to zero. This completes the proof that $XA^{(+)} \preceq A^{(+)}$.

We now show that the i th row of $XA^{(+)}$ is equal to the i th row of $A^{(+)}$ for all i . In the case where row i of A is zero, (3) yields that row i of $A^{(+)}$ is zero, and since $XA = A$ we see that whenever $X_{i,k} = \mathbf{1}$ we must also have that row k of A is zero. This in turn yields that whenever $X_{i,k} = \mathbf{1}$, row k of $A^{(+)}$ is zero, and so the supremum in (11) is equal to zero. Thus the i th row of $A^{(+)}$ and $XA^{(+)}$ clearly agree in this case.

Now suppose that row i of A is non-zero. Then, by assumption, $A_{i,\star}$ is contained in the unique basis for $\text{Row}(A)$. Since $XA = A$ we have in particular that the i th row of XA is equal to the i th row of A . The i th row of XA is a linear combination of the rows of A , namely, $(XA)_{i,\star} = \bigvee_{k=1}^n X_{i,k}A_{k,\star}$. Since this linear combination is equal to the basis element $A_{i,\star}$ of $\text{Row}(A)$, it follows in particular that there exists $s \in [n]$ with $X_{i,s} = \mathbf{1}$ and $A_{s,\star} = A_{i,\star}$. By formula (3), $(A^{(+)})_{i,s} = (A^{(+)})_{s,i} = \mathbf{1}$. But then for all j we have:

$$(XA^{(+)})_{i,j} = \bigvee_{k: X_{i,k}=1} (A^{(+)})_{k,j} \geq (A^{(+)})_{s,j} \geq (A^{(+)})_{s,i}(A^{(+)})_{i,j} = (A^{(+)})_{i,j},$$

where the final inequality is due to the fact that $A^{(+)}$ is idempotent. Since we have already proved that $XA^{(+)} \preceq A^{(+)}$, it now follows that the two matrices must agree in row i . \square

Theorem 5.2. *The generalised regularity properties of the monoids $M_n(\mathbb{B})$, $UT_n(\mathbb{B})$ and $U_n(\mathbb{B})$ can be summarised as follows:*

	$n = 1$	$n = 2$	$n = 3$	$n \geq 4$
$M_n(\mathbb{B})$	<i>Regular (band)</i>	<i>Regular</i>	<i>Fountain</i>	<i>Not Fountain</i>
$UT_n(\mathbb{B})$	<i>Regular (band)</i>	<i>Abundant</i>	<i>Fountain</i>	<i>Not Fountain</i>
$U_n(\mathbb{B})$	<i>Regular (group)</i>	<i>Regular (band)</i>	<i>Fountain</i>	<i>Fountain</i>

Proof. We consider each family of monoids: (i) $M_n(\mathbb{B})$; (ii) $UT_n(\mathbb{B})$; and (iii) $U_n(\mathbb{B})$, in turn.

(i) The monoid $M_n(\mathbb{B})$, or equivalently the monoid \mathcal{B}_n of binary relations on an n element set, is well known to be regular if and only if $n \leq 2$; for $n = 1$ this is isomorphic to the multiplicative monoid of \mathbb{B} consisting of two idempotents. Since \mathbb{B} is exact, it follows from Theorem 3.5 above, that $M_n(\mathbb{B})$ is also not abundant for $n \geq 3$. Moreover,

Proposition 3.13 shows that \mathcal{B}_n is not Fountain for all $n \geq 4$. Thus it remains to show that $M_3(\mathbb{B})$ is Fountain. By Lemma 3.1, the transpose map is an involutory anti-automorphism of $M_3(\mathbb{B})$. By Lemma 2.3 it will therefore suffice to show that every $\tilde{\mathcal{R}}$ -class of $M_3(\mathbb{B})$ contains an idempotent. Proposition 5.1 tells us that if $A \in M_3(\mathbb{B})$ is such that every non-zero row of A is contained in the unique basis of $\text{Row}(A)$, then the $\tilde{\mathcal{R}}$ -class of A contains the idempotent $A^{(+)}$. Since a submodule of \mathbb{B}^3 generated by fewer than three non-zero elements must contain each of these elements in its basis, it follows that if A contains a zero row or a repeated row, then A is $\tilde{\mathcal{R}}$ -related to an idempotent. Suppose then that $\text{dom}(A) = 3$ and all rows of A are distinct. If the three rows of A form a basis of $\text{Row}(A)$, then Proposition 5.1 shows that A is $\tilde{\mathcal{R}}$ -related to an idempotent. Thus it suffices to consider the case where $A_{\sigma(1),*} = A_{\sigma(2),*} \vee A_{\sigma(3),*}$ for some permutation σ of $\{1, 2, 3\}$. In this case one finds that A is $\tilde{\mathcal{R}}$ -related to the matrix with rows $\sigma(2)$ and $\sigma(3)$ equal to the corresponding rows of the identity matrix, and row $\sigma(1)$ equal to the join of these two.

(ii) Since $UT_1(\mathbb{B}) = M_1(\mathbb{B})$, it is clearly regular. Proposition 3.4 shows that $UT_n(\mathbb{B})$ is not regular for $n \geq 2$, Proposition 3.10 shows that $UT_n(\mathbb{B})$ is not abundant for $n \geq 3$ and Proposition 3.13 shows that $UT_n(\mathbb{B})$ is not Fountain for $n \geq 4$. This leaves only the cases $n = 2$ and $n = 3$ to consider in more detail.

The monoid $UT_2(\mathbb{B})$ is easily seen to be abundant; seven of the eight elements are actually idempotents. The remaining element is the non-regular element considered in the proof of Proposition 3.4, and it is easily verified that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \mathcal{R}^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathcal{L}^* \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^2.$$

For $n = 3$, first notice that we cannot hope to use the idempotent construction $A \mapsto A^{(+)}$ to show that $UT_3(\mathbb{B})$ is Fountain, since this map need not map an upper triangular matrix to an upper triangular matrix. For example,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in UT_3(\mathbb{B}), \quad A^{(+)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \notin UT_3(\mathbb{B}).$$

By Lemma 3.1, there is an involutory anti-automorphism of $UT_3(\mathbb{B})$, and so by Lemma 2.3 it will suffice to show that every $\tilde{\mathcal{R}}$ -class of $UT_3(\mathbb{B})$ contains an idempotent. Any matrix \mathcal{R}^* -related to an idempotent is also $\tilde{\mathcal{R}}$ -related to an idempotent, so we shall begin by showing that all but four elements of $UT_3(\mathbb{B})$ are \mathcal{R}^* -related to an idempotent. By direct computation we find that forty-one of the sixty-four elements of $UT_3(\mathbb{B})$ are idempotents. They have the form

$$\begin{aligned} & \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & y \\ 0 & 0 & z \\ 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 1 & u & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & v \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & u & uv \\ 0 & 1 & v \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $u, v, w, x, y, z \in \mathbb{B}$ with $xz \leq y$.

Since \mathbb{B} is exact, it now follows from Corollary 3.7 that $A \in UT_3(\mathbb{B})$ is \mathcal{R}^* -related to an idempotent if and only if the column space of A is equal to the column space of one of the idempotent matrices listed above. In this way one finds that all but the following

four matrices are \mathcal{R}^* -related to an idempotent:

$$X_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is then straightforward to verify that:

$$X_1 \tilde{\mathcal{R}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathcal{R}} X_2 \quad \text{and} \quad X_3 \tilde{\mathcal{R}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{\mathcal{R}} X_4.$$

(iii) Since $U_1(\mathbb{B})$ is the trivial group, it is obviously regular. The monoid $U_2(\mathbb{B})$ consists of two idempotent elements, and hence is also regular. Proposition 3.10 shows that for all $n \geq 3$, $U_n(\mathbb{B})$ is not abundant, whilst Theorem 4.18 shows that $U_n(\mathbb{B})$ is Fountain for all n . \square

Remark 5.3. The representation theory of monoid algebras KM of certain⁴ finite (right) Fountain monoids M over a field K is elucidated in [32], where descriptions of the projective indecomposables and quiver of such a monoid algebra are given. Thus the results of [32] cannot be applied to the monoid of binary relations for $n > 3$, although there is scope for these results to apply for $n \leq 3$. A preliminary study of the (rational) representation theory in the case $n = 3$ can already be found in [3].

5.2. Infinite linearly ordered semifields. We recall from [43] that $V = \{v_1, \dots, v_k\} \subseteq \mathfrak{L}^n$ is said to be *linearly independent* if

$$v_i \notin \left\{ \bigvee_{j \neq i} \lambda_j v_j : \lambda_j \in S \right\} \text{ for } 1 \leq i \leq k,$$

and that a *basis* for an \mathfrak{L} -semimodule M is a linearly independent generating set for M . By [43, Corollary 4.7 and Theorem 5], every finitely generated \mathfrak{L} -submodule $M \subseteq \mathfrak{L}^n$ admits a basis, and moreover bases are unique up to scaling (that is, if $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_j\}$ are two bases for M , then $k = j$ and for $i = 1, \dots, j$ there exists $\lambda_i \in \mathfrak{L}^*$ such that $w_i = \lambda_i v_i$). In particular, the column space (respectively, row space) of $A \in M_n(\mathfrak{L})$ admits a unique up to scaling basis; by the proof of [43, Theorem 6] this can be taken to be a subset of the set of columns (respectively, rows) of A .

Proposition 5.4. *Let $A, B \in M_n(\mathfrak{L})$. Then:*

- (i) $(A^{(+)})_{i,i} = \mathbf{1}$ for all $i \in \text{dom}(A)$.
- (ii) $A^{(+)(+)} = A^{(+)}$.
- (iii) *Suppose that $\text{dom}(A) = [n]$ and $BA = A$. If $A_{j,\star}$ is contained in a basis of $\text{Row}(A)$, then $(BA^{(+)})_{j,\star} = (A^{(+)})_{j,\star}$.*

Proof. (i) This is immediate from the definition.

(ii) We shall show first that $A^{(+)} \preceq A^{(+)(+)}$. If $A_{i,j}^{(+)} = \mathbf{0}$, then it is clear that $A_{i,j}^{(+)} \leq A_{i,j}^{(+)(+)}$, since $\mathbf{0}$ is the least element of \mathfrak{L} . Suppose then that $A_{i,j}^{(+)} \neq \mathbf{0}$. By formula (3) it is easy to see that this is the case if and only if $\emptyset \neq \text{Supp}(A_{j,\star}) \subseteq \text{Supp}(A_{i,\star})$.

Since $\emptyset \neq \text{Supp}(A_{j,\star})$ we have $j \in \text{dom}(A)$. By Part (i), $A_{j,j}^{(+)} = \mathbf{1}$ giving $\emptyset \neq \text{Supp}(A_{j,\star}^{(+)})$. We also have $\text{Supp}(A_{j,\star}^{(+)}) \subseteq \text{Supp}(A_{i,\star}^{(+)})$, since if $A_{j,k}^{(+)} \neq \mathbf{0}$ we have

$$\emptyset \neq \text{Supp}(A_{k,\star}^{(+)}) \subseteq \text{Supp}(A_{j,\star}^{(+)}) \subseteq \text{Supp}(A_{i,\star}^{(+)})$$

⁴Namely, those for which all sandwich matrices indexed by idempotents are right invertible over the group algebra of the corresponding group \mathcal{H} -class.

hence giving $A_{i,k}^{(+)} \neq \mathbf{0}$ too.

It follows from the above that whenever $A_{i,j}^{(+)} \neq \mathbf{0}$ we also have $A_{i,j}^{(+)(+)} \neq \mathbf{0}$. Now suppose for contradiction that $A_{i,j}^{(+)} > A_{i,j}^{(+)(+)}$. Since $A_{i,j}^{(+)} \neq \mathbf{0}$ we also have $A_{i,j}^{(+)(+)} \neq \mathbf{0}$. By (3) we have $A_{i,j}^{(+)(+)} = A_{i,s}^{(+)}(A_{j,s}^{(+)})^{-1}$ for some s with $A_{i,s}^{(+)} \neq \mathbf{0} \neq A_{j,s}^{(+)}$. Together with our assumption this gives

$$A_{i,j}^{(+)} > A_{i,s}^{(+)}(A_{j,s}^{(+)})^{-1}$$

and hence

$$A_{i,j}^{(+)} A_{j,s}^{(+)} > A_{i,s}^{(+)}.$$

This contradicts that $A^{(+)}$ is idempotent. Thus we must have $A_{i,j}^{(+)} \leq A_{i,j}^{(+)(+)}$. This completes the proof that $A^{(+)} \preceq A^{(+)(+)}$.

Next suppose for contradiction that $A_{i,j}^{(+)} < A_{i,j}^{(+)(+)}$ for some i, j . Since $A_{i,j}^{(+)(+)} \neq \mathbf{0}$ we deduce from (3) that $A_{j,p}^{(+)} \neq \mathbf{0}$ for some p , and hence also $A_{j,t} \neq \mathbf{0}$ for some t . But then Part (i) gives $A_{j,j}^{(+)} = \mathbf{1}$ and so the equation $A^{(+)(+)}A^{(+)} = A^{(+)}$ of Proposition 4.4 implies that

$$A_{i,j}^{(+)(+)} = A_{i,j}^{(+)(+)} A_{j,j}^{(+)} \leq A_{i,j}^{(+)},$$

contradicting our assumption.

(iii) Suppose that $A_{j,\star}$ is contained in a basis for $\text{Row}(A)$. Since $BA = A$ we require in particular that the j th row of BA is equal to the j th row of A . The j th row of BA is a linear combination of the rows of A , namely, $(BA)_{j,\star} = \bigvee_{k=1}^n B_{j,k} A_{k,\star}$. Since this linear combination is equal to a basis element $A_{j,\star}$ of $\text{Row}(A)$, it follows in particular that there exists $s \in [n]$ with $B_{j,s} = \mathbf{1}$ and $A_{s,\star} = A_{j,\star}$. Thus $B_{j,s} = \mathbf{1}$ and $(A^{(+)})_{j,s} = (A^{(+)})_{s,j} = \mathbf{1}$. But then for all i we have:

$$(BA^{(+)})_{j,i} = \bigvee_{k=1}^n B_{j,k} (A^{(+)})_{k,i} \geq (A^{(+)})_{s,i} \geq (A^{(+)})_{s,j} (A^{(+)})_{j,i} = (A^{(+)})_{j,i},$$

where the final inequality is due to the fact that $A^{(+)}$ is idempotent. By Proposition 4.4 we have $BA^{(+)} \preceq A^{(+)}$, and so it follows that the two matrices agree in row j . \square

Theorem 5.5. *The generalised regularity properties of the monoids $M_n(\mathfrak{L})$, $UT_n(\mathfrak{L})$ and $U_n(\mathfrak{L})$ can be summarised as follows:*

	$n = 1$	$n = 2$	$n = 3$	$n \geq 4$
$M_n(\mathfrak{L})$	<i>Regular</i>	<i>Regular</i>	<i>Not Abundant</i>	<i>Not Fountain</i>
$UT_n(\mathfrak{L})$	<i>Regular</i>	<i>Abundant</i>	<i>Not Abundant</i>	<i>Not Fountain</i>
$U_n(\mathfrak{L})$	<i>Regular (group)</i>	<i>Regular (band)</i>	<i>Fountain</i>	<i>Fountain</i>

Proof. We consider each family of monoids: (i) $M_n(\mathfrak{L})$; (ii) $UT_n(\mathfrak{L})$; and (iii) $U_n(\mathfrak{L})$, in turn.

(i) For $n = 1$, $M_n(\mathfrak{L})$ is isomorphic to the multiplicative monoid of \mathfrak{L} ; a group with zero adjoined, which is plainly regular. For $a \in \mathfrak{L}$ define $\bar{a} = a^{-1}$ if $a \neq \mathbf{0}$ and $\bar{a} = \mathbf{0}$ otherwise. Thus for all $a \in \mathfrak{L}$ we have $a\bar{a}a = a$, and $a\bar{a} \leq \mathbf{1}$ with equality if $a \neq \mathbf{0}$. From this it is clear that

$$\begin{pmatrix} \mathbf{0} & b \\ c & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \bar{c} \\ \bar{b} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & b \\ c & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & b \\ c & \mathbf{0} \end{pmatrix}.$$

Suppose now that $a, b, c, d \in \mathfrak{L}$ with at least one of a or d non-zero. If $ad \geq bc$ we have $\mathbf{1} \geq bc\bar{a}\bar{d}$, $a \geq bcd$ and $d \geq bc\bar{a}$. From this one readily verifies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & b\bar{a}\bar{d} \\ c\bar{a}\bar{d} & \bar{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Likewise, if $bc \geq ad$ one finds that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{c} & d\bar{c}\bar{b} \\ a\bar{c}\bar{b} & \bar{b} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus we conclude that $M_2(\mathfrak{L})$ is regular. Since \mathfrak{L} is exact, it follows from Corollary 3.5 that $M_n(\mathfrak{L})$ is not abundant for $n \geq 3$. Moreover, Proposition 3.13 shows that $M_n(\mathfrak{L})$ is not Fountain for all $n \geq 4$.

(ii) Since $UT_1(\mathfrak{L}) = M_1(\mathfrak{L})$, it is clearly regular. Propositions 3.4, 3.10 and 3.13 show that $UT_n(\mathfrak{L})$ is not regular for $n \geq 2$, not abundant for $n \geq 3$ and not Fountain for $n \geq 4$. We show that the monoid $UT_2(\mathfrak{L})$ is abundant. The idempotents of $UT_2(\mathfrak{L})$ are precisely the elements:

$$\begin{pmatrix} \mathbf{1} & a \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{1} & a \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & a \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \text{ where } a \in \mathfrak{L}.$$

The zero matrix is clearly idempotent. For each non-zero matrix $A \in UT_2(\mathfrak{L})$ we construct an idempotent E with the same column space as A . If $A \in UT_2(\mathfrak{L})$ has at least one non-zero entry on the diagonal, then let E be the matrix obtained by scaling the columns of A to replace any non-zero entries on the diagonal with $\mathbf{1}$. Otherwise, let E be the matrix obtained by first permuting the two columns of A , and then scaling the columns to replace any non-zero entries on the diagonal with $\mathbf{1}$. In both cases, E is an idempotent with the same column space as A . Since \mathfrak{L} is exact, Corollary 3.7 now applies to give that $UT_2(\mathfrak{L})$ is abundant.

(iii) By Theorem 4.18, $U_n(\mathfrak{L})$ is Fountain for all n . For $n = 1$ this is clearly the trivial group. For $n = 2$, it is a band. For $n \geq 3$, Proposition 3.10 says that $U_n(\mathfrak{L})$ is not abundant. □

Note that the previous result leaves open the question of whether $M_3(\mathfrak{L})$ and $UT_3(\mathfrak{L})$ are Fountain. By analogy with Theorem 5.2, we make the following conjecture.

Conjecture 5.6. The monoids $M_3(\mathfrak{L})$ and $UT_3(\mathfrak{L})$ are Fountain.

6. THE SEMIGROUPS $UT_n(\mathfrak{L}^*)$

For the remainder of the paper we focus on the Fountain semigroups $UT_n(\mathfrak{L}^*)$, for $\mathfrak{L} \neq \mathbb{B}$. This family of semigroups turns out to have a particularly interesting structure. In the case where $\mathfrak{L} = \mathbb{T}$, these semigroups have been studied by [40]. (For $\mathfrak{L} = \mathbb{B}$ these are, of course, trivial groups.)

Recall that $D_n(\mathfrak{L}^*)$ denotes the set of invertible diagonal matrices and that for any $A \in UT_n(\mathfrak{L}^*)$, we write $A = A^\diamond D_A$ and $A = D_A A^\bullet$ where D_A is the element of $D_n(\mathfrak{L}^*)$ with diagonal entries equal to those of A , and $A^\diamond, A^\bullet \in U_n(\mathfrak{L}^*)$.

Theorem 6.1. Let $A, B \in UT_n(\mathfrak{L}^*)$. Then the following are equivalent:

- (i) $A \mathcal{R}^* B$ in $UT_n(\mathfrak{L}^*)$;
- (ii) $A \mathcal{R}^* B$ in $UT_n(\mathfrak{L})$;
- (iii) $A \mathcal{R}^* B$ in $M_n(\mathfrak{L})$;
- (iv) $A \mathcal{R} B$ in $M_n(\mathfrak{L})$;
- (v) $A \mathcal{R} B$ in $UT_n(\mathfrak{L})$;
- (vi) $A \mathcal{R} B$ in $UT_n(\mathfrak{L}^*)$;
- (vii) $A = BX$ for some $X \in D_n(\mathfrak{L}^*)$;
- (viii) $A^\diamond = B^\diamond$.

Proof. Parts (ii) and (iii) are equivalent by Lemma 3.3. Since \mathfrak{L} is exact, parts (iii) and (iv) are equivalent by Theorem 3.5. Anti-negativity of \mathfrak{L} means that the i th column of A (respectively, B) cannot be expressed as a linear combination of columns $i + 1, \dots, n$ of

B (respectively, A). Thus if $A = BX$ and $B = AY$, then we must have $X, Y \in UT_n(\mathfrak{L})$. This shows that (iv) and (v) are equivalent. The equivalence of (vii) and (viii), that (vi) implies (i), and that (vii) implies (iv) are obvious. To complete the proof we shall show that (iv) implies (vii), that (v) implies (vi) and that (i) implies (ii).

To see that (iv) implies (vii), first note that each column of A cannot be expressed as a linear combination of the remaining columns. This shows that every column of A is contained in a basis of $\text{Col}(A)$. Since $\text{Col}(A) = \text{Col}(B)$ and bases are unique up to permutation and scaling, we must have $A = BX$ where $X \in D_n(\mathfrak{L}^*)$.

Suppose that (v) holds, thus $A = BX$ for some $X \in UT_n(\mathfrak{L})$. For each $i \leq j$ let $m(i, j) = \{s : i \leq s \leq j, A_{i,j} = B_{i,s}X_{s,j}\}$. Since $A_{i,j} \neq \mathbf{0}$ for all $i \leq j$, we must have $X_{s,j} \neq \mathbf{0}$ for all $s \in m(i, j)$. Let X' be the matrix obtained by replacing all of the zero entries of X lying on or above the diagonal by some element $g \in \mathfrak{L}^*$ such that for all $i \leq j$ and all t one has $A_{i,j} > B_{i,t}g$. Then it is straightforward to check that

$$(BX')_{i,j} = \bigvee_{i \leq s \leq j} B_{i,s}X'_{s,j} = \bigvee_{s \in m(i,j)} B_{i,s}X_{s,j} = A_{i,j}.$$

To prove that (i) implies (ii), suppose for contradiction that $A \mathcal{R}^* B$ in $UT_n(\mathfrak{L}^*)$ but not in $UT_n(\mathfrak{L})$. By symmetry of the relation, it suffices to consider the case where for some $X, Y \in UT_n(\mathfrak{L})$ we have $XA = YA$, but $XB \neq YB$. Without loss of generality let us further suppose that p, q, r, s are such that

$$(XB)_{r,s} = X_{r,p}B_{p,s} > Y_{r,q}B_{q,s} = (YB)_{r,s}. \quad (12)$$

Construct a new pair of matrices X', Y' by replacing all of the $\mathbf{0}$ entries of X and Y by an element $g \in \mathfrak{L}^*$ satisfying:

- (1) $g \leq \bigwedge \{A_{i,j}A_{k,j}^{-1}, B_{i,j}B_{k,j}^{-1} : i \leq k \leq j\}$ and
- (2) $X_{r,p}B_{p,s} > gB_{t,s}$ for all $t = 1, \dots, n$.

Notice that in doing so, terms corresponding to those zero terms that did not contribute to the maximum in any non-zero entry of the products (XA, YA, XB, YB) also do not contribute to the corresponding maximum in any of the new products $(X'A, Y'A, X'B, Y'B)$. In particular, if $(XA)_{i,j} = (YA)_{i,j}$ is non-zero, then we find $(X'A)_{i,j} = (XA)_{i,j} = (YA)_{i,j} = (Y'A)_{i,j}$.

On the other hand, if $(XA)_{i,j} = (YA)_{i,j} = 0$, then it follows that $X_{i,k} = Y_{i,k} = 0$ for all $i \leq k \leq j$. Replacing each of these 0's by g then gives $(X'A)_{i,j} = g \bigvee_{i \leq k \leq j} A_{k,j} = (Y'A)_{i,j}$. This shows that we have $X'A = Y'A$.

Finally,

$$(X'B)_{r,s} \geq X_{r,p}B_{p,s} > \bigvee_{r \leq t \leq s} Y'_{r,t}B_{t,s} = (Y'B)_{r,s}.$$

To see that the first inequality holds, note that (12) implies that $X_{r,p}$ is non-zero, and hence $X'_{r,p} = X_{r,p}$. The second inequality holds by (12) and our choice of g satisfying (2). This gives $X'B \neq Y'B$. Since $X', Y' \in UT_n(\mathfrak{L}^*)$ this gives the desired contradiction. \square

Together with its obvious left-right dual, the previous result yields:

Corollary 6.2. *Let $A, B \in UT_n(\mathfrak{L}^*)$. Then $A \mathcal{D} B$ in $UT_n(\mathfrak{L}^*)$ if and only if there exists $C \in UT_n(\mathfrak{L}^*)$ such that $A^\diamond = C^\diamond$ and $C^\bullet = B^\bullet$.*

Theorem 6.3. *Let $A, B \in U_n(\mathfrak{L}^*)$. Then the following are equivalent:*

- (i) $A \mathcal{R}^* B$ in $U_n(\mathfrak{L}^*)$;
- (ii) $A \mathcal{R}^* B$ in $U_n(\mathfrak{L})$;
- (iii) $A \mathcal{J} B$ in $U_n(\mathfrak{L}^*)$;
- (iv) $A \mathcal{J} B$ in $U_n(\mathfrak{L})$;
- (v) any of the equivalent conditions of Theorem 1;

(vi) $A = B$.

Proof. Since $A = A^\diamond$ for all $A \in U_n(\mathfrak{L}^*)$, it is clear that (v) implies (vi). Hence it suffices to show that each of (i)-(iv) implies (v). It is clear from the definitions that (iii) implies (iv), which by [23, Lemma 4.1] implies (v). Since (ii) implies condition (i), it remains to show that (i) implies (v).

Suppose that $A \mathcal{R}^* B$ in $U_n(\mathfrak{L}^*)$. Let $X, Y \in UT_n(\mathfrak{L}^*)$ with $XA = YA$. Then $D_X X^\bullet A = D_Y Y^\bullet A$. By comparing the diagonal entries of XA and YA one finds that $D_X = D_Y$, and hence by cancellation $X^\bullet A = Y^\bullet A$. Since $A \mathcal{R}^* B$ in $U_n(\mathfrak{L}^*)$ this gives $X^\bullet B = Y^\bullet B$. Left multiplication by D_X then yields $XB = YB$. By symmetry of assumption, this shows that $A \mathcal{R}^* B$ in $UT_n(\mathfrak{L}^*)$, as required. \square

Corollary 6.4. *There is a one-one correspondence between $UT_n(\mathfrak{L}^*)/\mathcal{R}$ and $U_n(\mathfrak{L}^*)$.*

Proof. Consider the mapping $\diamond : UT_n(\mathfrak{L}^*) \rightarrow U_n(\mathfrak{L}^*)$ where $A \mapsto A^\diamond$. By the definition of \diamond , $(A^\diamond)^\diamond = A^\diamond$ and if $A \in U_n(\mathfrak{L}^*)$ then $A = A^\diamond$. For any $A, B \in UT_n(\mathfrak{L}^*)$, $A^\diamond = B^\diamond$ implies that

$$A = A^\diamond D_A = B^\diamond D_A = B(D_B)^{-1} D_A.$$

From Theorem 6.1 we see that $A \mathcal{R} B$ in $UT_n(\mathfrak{L}^*)$. Conversely, if $A \mathcal{R} B$ in $UT_n(\mathfrak{L}^*)$ then $A^\diamond \mathcal{R} A \mathcal{R} B \mathcal{R} B^\diamond$, which implies that $A^\diamond \mathcal{R} B^\diamond$. From Theorem 6.3, it follows that $A^\diamond = B^\diamond$. Hence \mathcal{R} is the kernel of the map \diamond , which assures the one-one correspondence. \square

Corollary 6.5. *For $\mathfrak{L} \neq \mathbb{B}$, the generalised regularity properties of $UT_n(\mathfrak{L}^*)$ and $U_n(\mathfrak{L}^*)$ can be summarised as follows:*

	$n = 1$	$n = 2$	$n \geq 3$
$UT_n(\mathfrak{L}^*)$	<i>Regular (group)</i>	<i>Regular (inverse)</i>	<i>Fountain</i>
$U_n(\mathfrak{L}^*)$	<i>Regular (group)</i>	<i>Regular (semilattice)</i>	<i>Fountain</i>

Proof. That $UT_n(\mathfrak{L}^*)$ and $U_n(\mathfrak{L}^*)$ are Fountain for all n follows from Corollary 4.20. It is clear that $UT_1(\mathfrak{L}^*)$ is isomorphic to the linearly ordered abelian group (\mathfrak{L}^*, \cdot) , whilst $U_1(\mathfrak{L}^*)$ is isomorphic to the trivial group.

For $A \in UT_2(\mathfrak{L}^*)$, setting

$$\bar{A} = \begin{pmatrix} (A_{1,1})^{-1} & A_{1,2}(A_{1,1})^{-1}(A_{2,2})^{-1} \\ \mathbf{0} & (A_{2,2})^{-1} \end{pmatrix},$$

yields that $A\bar{A}A = A$ and $\bar{A}A\bar{A} = \bar{A}$. Moreover, it is straightforward to check that if $X \in UT_2(\mathfrak{L}^*)$ is a matrix satisfying $AXA = A$, then $X \preceq \bar{A}$, whilst if $XAX = X$, then $\bar{A} \preceq X$. This shows that $UT_2(\mathfrak{L}^*)$ is an inverse semigroup. The semilattice of idempotents of $UT_2(\mathfrak{L}^*)$ is precisely $U_2(\mathfrak{L}^*)$, which as a semilattice is order isomorphic to \mathfrak{L}^* .

It remains to show that $UT_n(\mathfrak{L}^*)$ and $U_n(\mathfrak{L}^*)$ are not abundant for $n \geq 3$. To this end, let $n \geq 3$ and consider the unitriangular matrix $G \in U_n(\mathfrak{L}^*)$ with all entries above the diagonal equal to $g > \mathbf{1}$. By Theorem 6.1, the matrices \mathcal{R}^* -related to G in $UT_n(\mathfrak{L}^*)$ are those of the form GX where $X \in D_n(\mathfrak{L}^*)$. Since the idempotent elements of $UT_n(\mathfrak{L}^*)$ have all diagonal entries equal to $\mathbf{1}$, it follows that the only element of this \mathcal{R}^* -class that could be idempotent is G itself. A straightforward calculation reveals that G is not idempotent, and hence $UT_n(\mathfrak{L}^*)$ is not abundant. By Theorem 6.3, the only matrix \mathcal{R}^* -related to G in $U_n(\mathfrak{L}^*)$ is G itself, and hence $U_n(\mathfrak{L}^*)$ is not abundant. \square

Theorem 6.6. *Each maximal subgroup in $UT_n(\mathfrak{L}^*)$ is isomorphic to (\mathfrak{L}^*, \cdot) .*

Proof. Let $A, E \in UT_n(\mathfrak{L}^*)$ be such that $E^2 = E$ and $A \mathcal{H} E$. Then by Theorem 6.1, $A^\diamond = E = A^\bullet$. Therefore

$$A = A^\diamond D_A = E D_A \text{ and } A = D_A A^\bullet = D_A E.$$

For each $i \in [n]$, $A_{1,i} = E_{1,i}A_{i,i} = A_{1,1}E_{1,i}$ gives $A_{i,i} = A_{1,1} = \lambda$. Then we have $A = \lambda E$. Conversely, every $A \in UT_n(\mathfrak{L}^*)$ such that $A = \lambda E$ is \mathcal{R} and \mathcal{L} -related with E . Hence

$$H_E = \{\lambda E : \lambda \in \mathfrak{L}^*\}.$$

Since \mathcal{H} -classes of idempotents are precisely the maximal subgroups of $UT_n(\mathfrak{L}^*)$, every maximal subgroup is isomorphic to (\mathfrak{L}^*, \cdot) . \square

Theorem 6.1 does not generalise to give similar characterisations of \mathcal{R} and \mathcal{R}^* for $UT_n(\mathfrak{L})$, as the following example illustrates.

Example 6.7. Let S be an arbitrary semiring.

(i) For $n \geq 2$, let $A, B \in UT_n(S)$ be the matrices with $A_{1,n} = A_{1,n-1} = B_{1,n} = \mathbf{1}$ and all other entries equal to $\mathbf{0}$, we see that $A \mathcal{R} B$ in $M_n(S)$. However, in order for $A = BX$ to hold, we require

$$\mathbf{1} = A_{1,n-1} = \sum_{k=1}^n B_{1,k} X_{k,n-1} = X_{n,n-1},$$

so that X is not upper triangular. Thus the \mathcal{R} -relation on $UT_n(S)$ is not the restriction of the \mathcal{R} -relation on $M_n(S)$.

(i) For $n \geq 3$, let $A, B \in UT_n(S)$ be the matrices with $A_{1,n-1} = A_{2,n} = B_{1,n} = B_{2,n-1} = \mathbf{1}$ and all other entries equal to $\mathbf{0}$. It is clear that these matrices have identical row spaces and identical column spaces, and hence are \mathcal{H} -related in $M_n(S)$. However, in order for $A = XB$ to hold, we require

$$\mathbf{1} = A_{2,n} = \sum_{k=1}^n X_{2,k} B_{k,n} = X_{2,1},$$

so that X is not upper triangular. Thus the \mathcal{H} -relation on $UT_n(S)$ is not the restriction of the \mathcal{H} -relation on $M_n(S)$.

Example 6.8. Let S be an anti-negative semiring, and $n \geq 2$. Take $A, B \in UT_n(S)$ to be the matrices with $A_{2,n} = B_{1,n-1} = \mathbf{1}$, and all other entries equal to $\mathbf{0}$. Then it is clear that $A \mathcal{L} C \mathcal{R} B$ in $M_n(S)$, where C is the matrix with a single non-zero entry in position $(1, n)$. We show that there does not exist an upper triangular matrix X simultaneously satisfying $A \mathcal{L} X$ and $X \mathcal{R} B$ in $UT_n(S)$.

If $X \mathcal{R} B$ in $UT_n(S)$, then we would have $XQ = B$ for some $Q \in UT_n(S)$, giving

$$B_{i,j} = \sum_{k=i}^j X_{i,k} Q_{k,j} \text{ for all } i \leq j.$$

Since $B_{1,n-1} = \mathbf{1}$ we note that there must exist $h < n$ such that $X_{1,h} \neq \mathbf{0} \neq Q_{h,n-1}$. But now, the first row of X contains a non-zero entry in position $h < n$, whilst each element of $\text{Row}(A)$ does not. This tells us that X cannot be written in the form RA for *any* $R \in M_n(S)$, thus contradicting that $X \mathcal{L} A$ in $UT_n(S)$. Hence the \mathcal{D} relation in $UT_n(S)$ is *not* the restriction of the \mathcal{D} relation in $M_n(S)$.

At this point it is useful to have a simplified formulation for $A^{(+)}$ and $B^{(*)}$, where $A, B \in UT_n(\mathfrak{L}^*)$. By the formulae given in (3) and (4), for all $1 \leq i \leq j \leq n$ we have

$$A_{i,j}^{(+)} := \bigwedge \{A_{i,k}(A_{j,k})^{-1} : j \leq k \leq n\}, \quad (13)$$

$$B_{i,j}^{(*)} := \bigwedge \{B_{k,j}(B_{k,i})^{-1} : 1 \leq k \leq i\}. \quad (14)$$

For $X, Y \in UT_n(\mathfrak{L})$ we write $X \circ Y$ to denote the *Hadamard product* of X and Y ; this is the matrix whose (i, j) th entry is equal to $X_{i,j}Y_{i,j}$ for all $i, j \in [n]$. Since it is crucial to our later work, we record the following observation as a lemma.

Lemma 6.9. *Let $A, B, E \in UT_n(\mathfrak{L}^*)$ with $A^{(+)} = E = B^{(*)}$. Then*

(i) *There exist unique $\widehat{A} = (\alpha_{i,j}), \widehat{B} = (\beta_{i,j}) \in UT_n(\mathfrak{L}^*)$ such that*

$$A = (\widehat{A} \circ E)D_A, \quad \text{and} \quad B = D_B(E \circ \widehat{B}).$$

(ii) *For all $1 \leq i \leq j \leq n$ we have $\alpha_{i,j} \geq \mathbf{1}, \beta_{i,j} \geq \mathbf{1}$ with $\alpha_{i,n} = \alpha_{i,i} = \beta_{i,i} = \beta_{1,j} = \mathbf{1}$.*

(iii) *Moreover, $\widehat{A^\diamond} = \widehat{A}$ and $\widehat{B^\bullet} = \widehat{B}$.*

Proof. (i) Define $\alpha_{i,j}, \beta_{i,j}$ as follows

$$\alpha_{i,j} := A_{i,j}E_{i,j}^{-1}A_{j,j}^{-1}, \quad (15)$$

$$\beta_{i,j} := B_{i,i}^{-1}E_{i,j}^{-1}B_{i,j}. \quad (16)$$

Then it is easy to see that

$$A = (\widehat{A} \circ E)D_A, \quad B = D_B(\widehat{B} \circ E)$$

where $\widehat{A}_{i,j} := \alpha_{i,j}$ and $\widehat{B}_{i,j} := \beta_{i,j}$. This proves the existence of matrices \widehat{A} and \widehat{B} in Part (i). For uniqueness, note that if $X, Y \in UT_n(\mathfrak{L}^*)$ with $A = (X \circ E)D_A$ and $B = D_B(E \circ Y)$, then for all $1 \leq i \leq j \leq n$ we have

$$\begin{aligned} \alpha_{i,j}E_{i,j}A_{j,j} &= X_{i,j}E_{i,j}A_{j,j}, \\ B_{i,i}E_{i,j}\beta_{i,j} &= B_{i,i}E_{i,j}Y_{i,j}. \end{aligned}$$

Since the entries of A, B, E appearing above equation are all invertible, we may conclude that $\widehat{A} = X$ and $\widehat{B} = Y$.

(ii) Since E is an idempotent in $UT_n(\mathfrak{L}^*)$ such that $A^{(+)} = E = B^{(*)}$, we have in particular that $EA = A$ and $BE = B$. Thus for all $1 \leq i \leq j \leq n$, we have $E_{i,j}A_{j,j} \leq A_{i,j}$ and $B_{i,i}E_{i,j} \leq B_{i,j}$ and hence $\alpha_{i,j} \geq \mathbf{1}, \beta_{i,j} \geq \mathbf{1}$.

Since $E_{i,i} = \mathbf{1}$ for all i , setting $i = j$ in formulae (15) and (16) yields $\alpha_{i,i} = \mathbf{1} = \beta_{i,i}$. Since $E = A^{(+)}$, formula (13) gives $E_{i,n} = A_{i,n}(A_{n,n})^{-1}$ for all $1 \leq i \leq n$. Thus by (15) we also have $\alpha_{i,n} = \mathbf{1}$ for all $1 \leq i \leq n$. Similarly, formula (14) gives $E_{1,j} = B_{1,j}B_{1,1}^{-1}$ for all $1 \leq j \leq n$ and hence (16) gives $\beta_{1,j} = \mathbf{1}$ for all $1 \leq j \leq n$. Thus Part (ii) holds.

For Part (iii), first note that formulae (13) and (14) yield

$$(A^\diamond)^{(+)} = A^{(+)} = E = B^{(*)} = (B^\bullet)^{(*)}.$$

Since the diagonal entries of A^\diamond and B^\bullet are all equal to $\mathbf{1}$, Part (i) says that there exist unique matrices $\widehat{A^\diamond}$ and $\widehat{B^\bullet}$ satisfying $A^\diamond = \widehat{A^\diamond} \circ E$ and $B^\bullet = \widehat{B^\bullet} \circ E$. Right multiplying the first equation by D_A and left multiplying the second equation by D_B then gives $A = A^\diamond D_A = (\widehat{A^\diamond} \circ E)D_A$ and $B = D_B B^\bullet = D_B(\widehat{B^\bullet} \circ E)$. Thus by uniqueness in Part (i) we conclude that $\widehat{A^\diamond} = \widehat{A}$ and $\widehat{B^\bullet} = \widehat{B}$. \square

In many of the contexts in which Fountain semigroups have arisen, such as that of Ehresmann and adequate semigroups [27, 2], restriction and ample semigroups [9], and wider classes [14] $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ are, respectively, left and right congruences. We now show that not only is this not true of the relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ in $UT_n(\mathfrak{L}^*)$ but that it fails in the most extreme manner possible, as we now state and prove.

Proposition 6.10. *Let $A, B \in UT_n(\mathfrak{L}^*)$. Then $CA \widetilde{\mathcal{R}} CB$ for all $C \in UT_n(\mathfrak{L}^*)$ exactly if $A \mathcal{R} B$.*

Proof. Let $A, B \in UT_n(\mathfrak{L}^*)$. It is clear that if $A \mathcal{R} B$ then $CA \mathcal{R} CB$ since \mathcal{R} is a left congruence and hence $CA \widetilde{\mathcal{R}} CB$. We prove the converse. Suppose $CA \widetilde{\mathcal{R}} CB$ for all $C \in UT_n(\mathfrak{L}^*)$. Let $C \in UT_n(\mathfrak{L}^*)$ be given by:

$$C_{i,j} = (A^{(+)}_{i,j} \wedge (B^{(+)}_{i,j})).$$

Then, $C \preceq A^{(+)}$ and $C \preceq B^{(+)}$. So $C_{i,k}A_{k,j} \leq A_{i,k}^{(+)}A_{k,j} \leq A_{i,j}$ and $C_{i,k}B_{k,j} \leq B_{i,k}^{(+)}B_{k,j} \leq B_{i,j}$ for all $1 \leq i \leq k \leq j \leq n$. Since $C_{i,i} = \mathbf{1}$ so $(CA)_{i,j} = A_{i,j}$ and $(CB)_{i,j} = B_{i,j}$. By our assumption, we have $A \tilde{\mathcal{R}} B$ hence $A^{(+)} = B^{(+)}$. Let us write $E = A^{(+)} = B^{(+)}$. By Lemma 6.9, we can write

$$A_{i,j} = A_{j,j}E_{i,j}\alpha_{i,j} \text{ and } B_{i,j} = B_{j,j}E_{i,j}\gamma_{i,j} \quad (17)$$

where $\alpha_{i,j}, \gamma_{i,j} \geq \mathbf{1}$. For each $i \in [n]$, $\alpha_{i,i} = \alpha_{i,n} = \mathbf{1}$ and $\gamma_{i,i} = \gamma_{i,n} = \mathbf{1}$.

Claim: We claim that $\alpha_{i,j} = \gamma_{i,j}$ for each $1 < i < j < n$.

Proof of Claim: Fix $i, \ell \in [n]$ where $i < \ell < n$. Suppose contrary that $\alpha_{i,\ell} \neq \gamma_{i,\ell}$ given $\alpha_{i,s} = \gamma_{i,s}$ for all s such that $i < \ell < s \leq n$. Then formulae in (17) tell us that

$$A_{i,s} = E_{i,s}A_{s,s}\alpha_{i,s} = E_{i,s}\gamma_{i,s}A_{s,s} = B_{i,s}(B_{s,s})^{-1}A_{s,s}$$

so each column vector $A_{*,s}$ is a scalar multiple of the corresponding vector $B_{*,s}$ where $\ell < s \leq n$.

Now $\alpha_{i,\ell} \neq \gamma_{i,\ell}$ implies that either $\gamma_{i,\ell} > \alpha_{i,\ell}$ or $\alpha_{i,\ell} > \gamma_{i,\ell}$. Without loss of generality, assume that $\gamma_{i,\ell} > \alpha_{i,\ell}$ and let $C \in UT_n(\mathfrak{L}^*)$ with its i th row as follows:

$$C_{i,j} = \begin{cases} E_{i,j}(\gamma_{i,\ell})^{-1} & \text{if } i \leq j < \ell \\ E_{i,\ell}(\gamma_{i,\ell})^{-1}\alpha_{i,\ell} & \text{if } j = \ell \\ E_{i,j}\gamma_{i,\ell} \dots \gamma_{i,j} & \text{if } \ell < j \leq n \end{cases}$$

while all other rows are the same as those of E .

Let $X \in M_n(\mathfrak{L}^*)$ and suppose that $EX = X$. Since all rows of C except row i agree with the rows of E , we see that all rows of CX except the i th row will be equal to the corresponding row of X .

For $1 \leq j \leq n$, the (i, j) th entry of CX is given by $(CX)_{i,j} = \bigvee_{i \leq k \leq n} C_{i,k}X_{k,j}$. Using the definition of the i th row of C and the fact that $E_{i,i} = 1$, we see that $(CX)_{i,j}$ is given by

$$\begin{aligned} & C_{i,i}X_{i,j} \vee \left(\bigvee_{i < k < \ell} C_{i,k}X_{k,j} \right) \vee C_{i,\ell}X_{\ell,j} \vee \left(\bigvee_{\ell < k \leq n} C_{i,k}X_{k,j} \right) \\ &= \gamma_{i,\ell}^{-1}X_{i,j} \vee \left(\bigvee_{i < k < \ell} E_{i,k}\gamma_{i,\ell}^{-1}X_{k,j} \right) \vee E_{i,\ell}\gamma_{i,\ell}^{-1}\alpha_{i,\ell}X_{\ell,j} \vee \left(\bigvee_{\ell < k \leq n} E_{i,k}(\gamma_{i,\ell} \dots \gamma_{i,k})X_{k,j} \right) \\ &= \gamma_{i,\ell}^{-1} \left[X_{i,j} \vee \left(\bigvee_{i < k < \ell} E_{i,k}X_{k,j} \right) \vee E_{i,\ell}X_{\ell,j}\alpha_{i,\ell} \right] \vee \left(\bigvee_{\ell < k \leq n} E_{i,k}(\gamma_{i,\ell}\gamma_{i,\ell+1} \dots \gamma_{i,k})X_{k,j} \right). \end{aligned}$$

Since $EX = X$, we have $\bigvee_{i < k < \ell} E_{i,k}X_{k,j} \leq X_{i,j}$, and hence the expression in the square brackets rearranges and simplifies to give:

$$(CX)_{i,j} = \gamma_{i,\ell}^{-1} [E_{i,\ell}X_{\ell,j}\alpha_{i,\ell} \vee X_{i,j}] \vee \left(\bigvee_{\ell < k \leq n} E_{i,k}(\gamma_{i,\ell}\gamma_{i,\ell+1} \dots \gamma_{i,k})X_{k,j} \right).$$

Consider the effect of setting $X = A$ and $X = B$ in the above. We shall show that, in each case, the right-most non-zero term of this summation dominates. There are therefore three cases to consider:

First, if $i \leq j < \ell$ we have $A_{k,j} = B_{k,j} = 0$ for all $k \geq \ell$. Thus

$$(CA)_{i,j} = \gamma_{i,\ell}^{-1}A_{i,j} \text{ and } (CB)_{i,j} = \gamma_{i,\ell}^{-1}B_{i,j}.$$

Second, if $j = l$ we have $A_{k,l} = 0$ and $B_{k,l} = 0$ for all $k > l$. Thus

$$\begin{aligned}(CA)_{i,l} &= \gamma_{i,l}^{-1} [E_{i,l}A_{l,l}\alpha_{i,l} \vee A_{i,l}] = \gamma_{i,l}^{-1}A_{i,l}, \\ (CB)_{i,l} &= \gamma_{i,l}^{-1} [E_{i,l}B_{l,l}\alpha_{i,l} \vee B_{i,l}] = \gamma_{i,l}^{-1}B_{i,l},\end{aligned}$$

where right-hand equalities follow from the fact that $A_{l,l}E_{i,l}\alpha_{i,l} = A_{i,l}$ and $B_{l,l}E_{i,l}\alpha_{i,l} \leq B_{i,l}$.

Third, if $l < j \leq n$ we have $A_{k,j} = 0 = B_{k,j}$ for all $k > j$. Thus for $X = A$ or $X = B$ we have

$$(CX)_{i,j} = \gamma_{i,l}^{-1} [E_{i,l}X_{l,j}\alpha_{i,l} \vee X_{i,j}] \vee \left(\bigvee_{l < k \leq j} E_{i,k}(\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,k})X_{k,j} \right)$$

We show that the term $E_{i,j}(\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j})X_{j,j}$ dominates all others in the supremum in round brackets. First note that by assumption $\alpha_{i,j} = \gamma_{i,j}$; let us call this common value $\delta_{i,j}$. Then, by commutativity, this final term can be rewritten as $(X_{j,j}E_{i,j}\delta_{i,j})(\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j-1})$. Since $X = A$ or $X = B$, by (17) we see that the latter is equal to $X_{i,j}(\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j-1})$. Thus we seek to prove that each term in the supremum is dominated by $X_{i,j}\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j-1}$.

To see this, we recall that:

- (i) the γ 's were chosen so that $\gamma_{i,q} \geq \mathbf{1} \geq \gamma_{i,q}^{-1}$ for all q ;
- (ii) since $EX = X$ we have $E_{i,k}X_{k,j} \leq X_{i,j}$ for all k ;
- (iii) for the fixed values i and l we have assumed that $\alpha_{i,l} < \gamma_{i,l}$.

From this one sees that for all $l < k < j$ we have

$$\begin{aligned}X_{i,j}\gamma_{i,l}^{-1} &\leq X_{i,j} \leq X_{i,j}\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j-1}, \\ \gamma_{i,l}^{-1}E_{i,l}X_{l,j}\alpha_{i,l} &\leq X_{i,j}\gamma_{i,l} \leq X_{i,j}\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j-1}, \text{ and} \\ E_{i,k}\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,k}X_{k,j} &\leq X_{i,j}\gamma_{i,l}\gamma_{i,l+1} \cdots \gamma_{i,j-1}.\end{aligned}$$

Now, by our assumption $CA\tilde{\mathcal{R}}CB$, by Corollary 4.20 we must have $(CA^{(+)})_{i,\ell} = (CB^{(+)})_{i,\ell}$. For $X = A, B$ we therefore have

$$\begin{aligned}(CX)_{i,k}((CX)_{\ell,k})^{-1} &= \begin{cases} X_{i,\ell}(\gamma_{i,\ell})^{-1}(X_{\ell,\ell})^{-1} & \text{if } k = \ell \\ X_{i,k}\gamma_{i,\ell} \cdots \gamma_{i,\ell-1}(X_{\ell,k})^{-1} & \text{if } \ell < k \leq n \end{cases} \\ &= \begin{cases} X_{i,\ell}(X_{\ell,\ell})^{-1}(\gamma_{i,\ell})^{-1} & \text{if } k = \ell \\ X_{i,k}(X_{\ell,k})^{-1}\gamma_{i,\ell} \cdots \gamma_{i,\ell-1} & \text{if } \ell < k \leq n \end{cases}\end{aligned}$$

Since $E_{i,\ell} \leq A_{i,k}(A_{\ell,k})^{-1}$ for all k , where $\ell \leq k \leq n$ and $\alpha_{i,\ell}(\gamma_{i,\ell})^{-1} < \mathbf{1}$, we have $A_{i,l}(A_{l,l})^{-1}(\gamma_{i,l})^{-1} = E_{i,\ell}\alpha_{i,\ell}(\gamma_{i,\ell})^{-1} < A_{i,k}(A_{\ell,k})^{-1}\gamma_{i,\ell} \cdots \gamma_{i,\ell-1}$. Thus

$$(CA^{(+)})_{i,\ell} = \bigwedge \{(CA)_{i,k}((CA)_{\ell,k})^{-1} : \ell \leq k \leq n\} = E_{i,\ell}\alpha_{i,\ell}(\gamma_{i,\ell})^{-1}.$$

On the other hand, since $B_{i,l}(B_{l,l})^{-1}(\gamma_{i,l})^{-1} = E_{i,\ell} \leq B_{i,k}(B_{\ell,k})^{-1}$, we find

$$(CB^{(+)})_{i,\ell} = \bigwedge \{(CB)_{i,k}((CB)_{\ell,k})^{-1} : \ell \leq k \leq n\} = E_{i,\ell}.$$

But now we must have $E_{i,\ell}\alpha_{i,\ell}(\gamma_{i,\ell})^{-1} = E_{i,\ell}$, which is true if and only if $\alpha_{i,\ell} = \gamma_{i,\ell}$, and hence providing the desired contradiction.

Dually, if we assume $\alpha_{i,\ell} > \gamma_{i,\ell} \geq \mathbf{1}$ and reverse their roles in the formation of matrix C , we also arrive at a contradiction. So our supposition is wrong and $\alpha_{i,\ell} = \gamma_{i,\ell}$ for all $1 \leq i < \ell \leq n$. Thus we conclude that

$$A_{i,j} = A_{j,j}(E_{i,j}\alpha_{i,j}) = A_{j,j}(E_{i,j}\gamma_{i,j}) = A_{j,j}(B_{i,j}(B_{j,j})^{-1}) = \mu_j B_{i,j}$$

for every i and fixed j , where $1 \leq i \leq j \leq n$, and hence $A\mathcal{R}B$. \square

7. STRUCTURE OF $\tilde{\mathcal{R}}$ - AND $\tilde{\mathcal{L}}$ -CLASSES IN $UT_n(\mathfrak{L}^*)$

The aim of this section is to make a careful analysis of the relations $\tilde{\mathcal{R}}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{H}}$ on the semigroups $UT_n(\mathfrak{L}^*)$, and establish some facts concerning $\tilde{\mathcal{D}}$, building on the work of Section 6. The reader may wonder why we do not also consider an analogue $\tilde{\mathcal{J}}$ of \mathcal{J} ; such a relation certainly exists (see, for example [37]). However, since $\mathcal{J} \subseteq \tilde{\mathcal{J}}$, the following result tells us that $UT_n(\mathfrak{L}^*)$ is a single $\tilde{\mathcal{J}}$ -class.

Theorem 7.1. [40, Theorem 2.3.11] *The semigroup $UT_n(\mathfrak{L}^*)$ is simple for all $n \in \mathbb{N}$.*

We note that the proof of the above theorem in [40] is written for the special case where \mathfrak{L}^* is \mathbb{R} under $+$, so that the corresponding semiring \mathfrak{L} is the tropical semiring \mathbb{T} , but carries over without significant adjustment to our more general context.

7.1. Deficiency, tightness and looseness. The notion of *deficiency* was introduced in [40], again in the case where \mathfrak{L} is the tropical semiring. For $n \in \mathbb{N}$ we denote by $\Phi[n]$ the directed graph with vertices the elements of $[n]$ and an edge (i, j) , denoted $i \rightarrow j$, if and only if $i \leq j$. We refer to a sequence of edges $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$ in $\Phi[n]$ as a *path of length $k - 1$ in $\Phi[n]$* . If $i_1 < i_2 < \cdots < i_k$ we shall say that the path is *simple*.

Definition 7.2. Let $2 \leq k \leq n$, $A \in UT_n(\mathfrak{L}^*)$ and let γ denote a path $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$ in $\Phi[n]$ of length $k - 1$. The *deficiency of γ in A* , denoted $\text{Def}_A(\gamma)$, is defined to be

$$\text{Def}_A(\gamma) = A_{i_1, i_k} \left(\prod_{2 \leq t \leq k} A_{i_{t-1}, i_t} \right)^{-1}.$$

In the case where $M, N \in U_n(\mathfrak{L}^*)$ and $\mathfrak{L} = \mathbb{T}$, our next result forms part of [40, Theorem 2.3.6]. We will complete the generalisation of the latter in Theorem 7.4.

Theorem 7.3. *Let $M, N \in UT_n(\mathfrak{L}^*)$. Then the following are equivalent:*

- (i) *for all paths γ in $\Phi[n]$, we have $\text{Def}_M(\gamma) = \text{Def}_N(\gamma)$;*
- (ii) *for all paths γ of length 2 in $\Phi[n]$, we have $\text{Def}_M(\gamma) = \text{Def}_N(\gamma)$;*
- (iii) *for all paths γ in $\Phi[n]$ of the form $1 \rightarrow i \rightarrow j$, we have $\text{Def}_M(\gamma) = \text{Def}_N(\gamma)$.*

Proof. Clearly (i) \Rightarrow (ii) \Rightarrow (iii). It remains to show that (iii) \Rightarrow (i). Let γ denote the path $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$ in $\Phi[n]$, and let $M \in UT_n(\mathfrak{L}^*)$. Then

$$\begin{aligned} \text{Def}_M(\gamma) &= [M_{i_1, i_k}] [(M_{i_1, i_2})^{-1}] \cdots [(M_{i_{k-1}, i_k})^{-1}] \\ &= [M_{i_1, i_k} M_{1, i_k} (M_{1, i_k})^{-1}] [M_{1, i_1} (M_{1, i_1} M_{i_1, i_2})^{-1}] \cdots [M_{1, i_{k-1}} (M_{1, i_{k-1}} M_{i_{k-1}, i_k})^{-1}] \\ &= (M_{i_1, i_k} M_{1, i_1} (M_{i_1, i_k})^{-1}) (M_{1, i_2} (M_{1, i_1} M_{i_1, i_2})^{-1}) \cdots (M_{1, i_k} (M_{1, i_{k-1}} M_{i_{k-1}, i_k})^{-1}) \\ &= (\text{Def}_M(1 \rightarrow i_1 \rightarrow i_k))^{-1} \text{Def}_M(1 \rightarrow i_1 \rightarrow i_2) \cdots \text{Def}_M(1 \rightarrow i_{k-1} \rightarrow i_k), \end{aligned}$$

from which it follows that if (iii) holds, then so will (i). \square

We now recall from [40, Theorem 2.3.6] that deficiency characterises \mathcal{D} -class of unitriangular matrices in $UT_n(\mathbb{T}^*)$. This is true in general for $UT_n(\mathfrak{L}^*)$ and follows from Theorem 7.3 and following characterisation of the \mathcal{D} -related unitriangular matrices in $UT_n(\mathfrak{L}^*)$:

Theorem 7.4. (c.f. [40, Theorem 2.3.6]) *Let $A, B \in U_n(\mathfrak{L}^*)$. Then $A \mathcal{D} B$ in $UT_n(\mathfrak{L}^*)$ exactly if $\text{Def}_A(\gamma) = \text{Def}_B(\gamma)$ for all paths γ of length 2 in $\Phi[n]$.*

Proof. Suppose that $A, B \in U_n(\mathfrak{L}^*)$ and $A \mathcal{D} B$ in $UT_n(\mathfrak{L}^*)$. By Corollary 6.2, there exists $C \in UT_n(\mathfrak{L}^*)$ such that $C^\diamond = A^\diamond$ and $C^\bullet = B^\bullet$. Since $A, B \in U_n(\mathfrak{L}^*)$ we have

$A = A^\diamond = C^\diamond$ and $B = B^\bullet = C^\bullet$ giving $A = C(D_C)^{-1} = D_C C^\bullet (D_C)^{-1} = D_C B (D_C)^{-1}$. Thus for all $i \leq k \leq j$ one has

$$\begin{aligned} \text{Def}_A(i \rightarrow k \rightarrow j) &= A_{i,j}(A_{i,k})^{-1}(A_{k,j})^{-1} \\ &= C_{i,i}B_{i,j}(C_{j,j})^{-1}(C_{i,i})^{-1}(B_{i,k})^{-1}C_{k,k}(C_{k,k})^{-1}(B_{k,j})^{-1}C_{j,j} \\ &= B_{i,j}(B_{i,k}B_{k,j})^{-1} = \text{Def}_B(\gamma) \end{aligned}$$

Conversely, suppose that $\text{Def}_A(\gamma) = \text{Def}_B(\gamma)$ for all paths γ of length 2 in $\Phi[n]$. Let $G \in D_n(\mathfrak{L}^*)$ be the diagonal matrix with $G_{i,i} = A_{1,i}(B_{1,i})^{-1}$ for all $i \in [n]$. Then for all $i, j \in [n]$ where $i \leq j$, we have

$$\begin{aligned} (GAG^{-1})_{i,j} &= G_{i,i}A_{i,j}(G_{j,j})^{-1} = (A_{1,i}B_{1,i}^{-1})A_{i,j}A_{1,j}^{-1}B_{1,j} = (A_{1,i}A_{i,j}A_{1,j}^{-1})(B_{1,j}B_{1,i}^{-1}) \\ &= (\text{Def}_A(1 \rightarrow i \rightarrow j))^{-1}(B_{1,j}B_{1,i}^{-1}) = (\text{Def}_B(1 \rightarrow i \rightarrow j))^{-1}(B_{1,j}B_{1,i}^{-1}) \\ &= (B_{1,j}^{-1}B_{1,i}B_{i,j})(B_{1,j}B_{1,i}^{-1}) = B_{i,j}. \end{aligned}$$

and hence $GAG^{-1} = B$. Now by Theorem 6.1 we have $A \mathcal{L} GA \mathcal{R} GAG^{-1} = B$ in $UT_n(\mathfrak{L}^*)$. Hence $A \mathcal{D} B$. \square

In order to examine the relations $\tilde{\mathcal{R}}, \tilde{\mathcal{L}}, \tilde{\mathcal{H}}$ and $\tilde{\mathcal{D}}$ in $UT_n(\mathfrak{L}^*)$ we build on the notion of deficiency for an idempotent $E \in UT_n(\mathfrak{L}^*)$. Recall that, for such an element, and any path $i \rightarrow j \rightarrow k$, we have that $\mathbf{1} \leq \text{Def}_E(i \rightarrow j \rightarrow k)$; indeed, this condition is necessary and sufficient for $E = E^2$ in $UT_n(\mathfrak{L}^*)$.

Definition 7.5. Let $E \in UT_n(\mathfrak{L}^*)$ (equivalently, $U_n(\mathfrak{L}^*)$) be idempotent. For any path γ in $\Phi[n]$, we say that, E is *tight in* γ if $\text{Def}_E(\gamma) = \mathbf{1}$ and *loose in* γ if $\text{Def}_E(\gamma) > \mathbf{1}$.

It is clear that all paths of the form $i \rightarrow i \rightarrow j$ or $i \rightarrow j \rightarrow j$ are tight in every idempotent. In what follows we consider how the different configurations of tightness and looseness of the remaining (simple) paths of length 2 impact upon the tilde classes of an idempotent.

Consider the set of upper triangular matrices defined by

$$\mathcal{O}_n([\mathbf{0}, \mathbf{1}]) = \{A \in UT_n(\mathfrak{L}^*) : \forall i \in [n], A_{i,i} \leq A_{i,i+1} \leq \dots \leq A_{i,n} \leq \mathbf{1} \geq A_{1,i} \geq A_{2,i} \geq \dots \geq A_{i,i}\}.$$

Notice that if $n = 1$ then

$$\mathcal{O}_n([\mathbf{0}, \mathbf{1}]) = \{\lambda \in \mathfrak{L}^* : \lambda \leq \mathbf{1}\} := \mathfrak{L}_{\leq \mathbf{1}}^*.$$

Clearly, $\mathfrak{L}_{\leq \mathbf{1}}^*$ is a commutative, cancellative submonoid of \mathfrak{L}^* , but (except in the trivial case) is not a subgroup.

Lemma 7.6. *The set $\mathcal{O}_n([\mathbf{0}, \mathbf{1}])$ forms a subsemigroup of $UT_n(\mathfrak{L}^*)$.*

Proof. If $A, B \in \mathcal{O}_n([\mathbf{0}, \mathbf{1}])$ then for $1 \leq i \leq j \leq n$ we have $(AB)_{i,j} = \bigvee_{i \leq k \leq j} A_{i,k}B_{k,j} \leq \mathbf{1}$. Moreover, since for each fixed i, k with $i < n$ we have that $A_{i,k} \geq A_{i+1,k}$, it follows that $(AB)_{i,j} \geq (AB)_{i+1,j}$. Similarly, if $j < n$, then $(AB)_{i,j+1} \geq (AB)_{i,j}$. Thus $AB \in \mathcal{O}_n([\mathbf{0}, \mathbf{1}])$. \square

Proposition 7.7. *Let $n > 2$ and let E be an idempotent of $UT_n(\mathfrak{L}^*)$. If E is tight in all paths in $\Phi[n]$ then \tilde{H}_E is a subsemigroup isomorphic to $\mathfrak{L}^* \times \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$.*

Proof. For $G \in \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$ let \overline{G} denote the upper triangular matrix constructed by extending G as follows

$$\overline{G} = \begin{pmatrix} \mathbf{1} & \dots & \mathbf{1} \\ & G & \vdots \\ & & \mathbf{1} \end{pmatrix} \in \mathcal{O}_n([\mathbf{0}, \mathbf{1}]).$$

Observe that, since all entries of $G, H \in \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$ are no greater than $\mathbf{1}$, we have $\overline{G}\overline{H} = \overline{GH}$. We shall show that $\tilde{H}_E = \{\lambda(\overline{G} \circ E) : \lambda \in \mathfrak{L}^*, G \in \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])\} \cong \mathfrak{L}^* \times \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$.

Consider $C = \overline{G} \circ E$. Then, since E is tight in all paths, for any $1 \leq i \leq j \leq n$ we have

$$(C^{(+)})_{i,j} = \bigwedge_{j \leq k \leq n} C_{i,k} C_{j,k}^{-1} = \bigwedge_{j \leq k \leq n} E_{i,k} \overline{G}_{i,k} (E_{j,k} \overline{G}_{j,k})^{-1} = \bigwedge_{j \leq k \leq n} E_{i,j} \overline{G}_{i,k} \overline{G}_{j,k}^{-1} = E_{i,j},$$

where the final equality follows from the fact that $\overline{G} \in \mathcal{O}_n([\mathbf{0}, \mathbf{1}])$ with all entries in the final column equal to $\mathbf{1}$. This shows that $C^{(+)} = E$; a dual argument yields $C^{(*)} = E$ and so $(\lambda C)^+ = (\lambda C)^* = E$, for any $\lambda \in \mathfrak{L}^*$.

Conversely, we must show that any matrix in \tilde{H}_E is a scalar multiple of a matrix of the form $\overline{G} \circ E$, where $G \in \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$. Let $A \in \tilde{H}_E$. Then $(A^\diamond)^{(+)} = A^{(+)} = E$ and $(A^\bullet)^{(*)} = A^{(*)} = E$. By Lemma 6.9 we have $A_{i,j}^\diamond = E_{i,j} \alpha_{i,j}$ and $A_{i,j}^\bullet = E_{i,j} \beta_{i,j}$ where $\alpha_{i,j}, \beta_{i,j} \geq \mathbf{1}$ for $i < j$ and $\alpha_{i,i} = \alpha_{i,n} = \beta_{i,i} = \beta_{1,i} = \mathbf{1}$. Calculating $(A^\diamond)^{(+)}$ using Formula (3) and the fact that E is tight for all paths (giving $E_{i,j} E_{j,k} = E_{i,k}$ for all $i \leq j \leq k$), we obtain:

$$E_{i,j} = \bigwedge_{j \leq k \leq n} E_{i,k} \alpha_{i,k} (E_{j,k} \alpha_{j,k})^{-1} = \bigwedge_{j \leq k \leq n} E_{i,j} \alpha_{i,k} \alpha_{j,k}^{-1},$$

from which it follows that $\alpha_{j,k} \leq \alpha_{i,k}$ for all $i \leq j \leq k$. Dually, $\beta_{k,i} \leq \beta_{k,j}$ for all $1 \leq k \leq i$.

Since $A^\diamond D_A = A = D_A A^\bullet$, we then have

$$E_{i,j} \alpha_{i,j} A_{j,j} = A_{i,i} E_{i,j} \beta_{i,j} \quad (18)$$

so that $A_{j,j} \alpha_{i,j} = A_{i,i} \beta_{i,j}$ for all $i \leq j$. Since $\beta_{1,j} = \mathbf{1}$, we see that $A_{j,j} \alpha_{1,j} = A_{1,1}$ and so

$$A_{j,j} = A_{1,1} \alpha_{1,j}^{-1}. \quad (19)$$

Now, since $\alpha_{1,n} = \mathbf{1}$, we then obtain $A_{1,1} = A_{n,n}$. Using (18) and (19) we calculate

$$\beta_{i,j} = A_{j,j} \alpha_{i,j} A_{i,i}^{-1} = A_{1,1} \alpha_{1,j}^{-1} \alpha_{i,j} (A_{1,1} \alpha_{1,i}^{-1})^{-1} = \alpha_{1,i} \alpha_{i,j} \alpha_{1,j}^{-1}. \quad (20)$$

For $1 \leq i \leq j \leq n$ we now let $\gamma_{i,j} = \alpha_{i,j} \alpha_{1,j}^{-1}$ so that $\gamma_{i,j} \leq \mathbf{1}$. Using the facts that $\alpha_{j,k} \leq \alpha_{i,k}$ and $\beta_{k,i} \leq \beta_{k,j}$ for all allowable subscripts, together with (20), we see that $(\gamma_{i,j}) = \overline{G(A)}$ for some $G(A) \in \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$. Moreover, $A_{i,j} = E_{i,j} \gamma_{i,j} A_{1,1}$, and so $A = A_{1,1} (\overline{G(A)} \circ E)$, as required.

Given that the decomposition of A as $A = \lambda D$ where $D_{1,1} = \mathbf{1}$ is perforce unique, the above establishes that $\theta : \tilde{H}_E \rightarrow \mathfrak{L}^* \times \mathcal{O}_{n-2}([\mathbf{0}, \mathbf{1}])$ given by $\theta(A) = (A_{1,1}, G(A))$, is a bijection. It remains to show that θ is an isomorphism. To this end, suppose now that $A, B \in \tilde{H}_E$, so that $A = A_{1,1} (\overline{G(A)} \circ E)$ and $B = B_{1,1} (\overline{G(B)} \circ E)$. Clearly, $AB = A_{1,1} B_{1,1} (\overline{G(A)} \circ E) (\overline{G(B)} \circ E)$. Let $i \leq j$; using the fact that E is tight in all paths, we calculate

$$((\overline{G(A)} \circ E) (\overline{G(B)} \circ E))_{i,j} = \bigvee_{i \leq k \leq j} E_{i,k} \overline{G(A)}_{i,k} E_{k,j} \overline{G(B)}_{k,j} = E_{i,j} \bigvee_{i \leq k \leq j} \overline{G(A)}_{i,k} \overline{G(B)}_{k,j}.$$

Bearing in mind the remark made at the beginning of this proof, we have

$$\bigvee_{i \leq k \leq j} \overline{G(A)}_{i,k} \overline{G(B)}_{k,j} = (\overline{G(A)} \overline{G(B)})_{i,j} = \overline{(G(A) G(B))}_{i,j}.$$

Thus $AB = A_{1,1} B_{1,1} (\overline{G(A) G(B)} \circ E)$ and it follows (by restricting the range of i, j) that

$$\theta(AB) = (A_{1,1} B_{1,1}, G(A) G(B)) = \theta(A) \theta(B),$$

so that θ is an isomorphism as required. \square

At the other extreme we have:

Proposition 7.8. *Let $E, A \in UT_n(\mathfrak{L}^*)$. Suppose that E is an idempotent that is loose in all simple paths $i \rightarrow k \rightarrow j$ of length 2 in $\Phi[n]$. Then $A \tilde{\mathcal{R}} E$ ($A \tilde{\mathcal{L}} E$) if and only if $A \mathcal{R} E$ ($A \mathcal{L} E$). Thus \tilde{H}_E is a group isomorphic to \mathfrak{L}^* .*

Proof. Suppose that $A \tilde{\mathcal{R}} E$. Then $E = A^{(+)}$ and so for all $i \in \{1, \dots, n\}$, $E_{i,n} = A_{i,n}(A_{n,n})^{-1}$ by formula (13). Suppose for finite induction that $1 < \ell \leq n$ and for all t with $\ell \leq t \leq n$ and for all $i \in \{1, \dots, n\}$ with $i \leq t$ we have $E_{i,t} = A_{i,t}(A_{t,t})^{-1}$.

Let $i \in \{1, \dots, n\}$ with $i \leq \ell - 1$. Then by formula (13) and our inductive hypothesis we see that

$$\begin{aligned} E_{i,\ell-1} &= \bigwedge_{\ell-1 \leq k \leq n} A_{i,k}(A_{\ell-1,k})^{-1} = A_{i,\ell-1}(A_{\ell-1,\ell-1})^{-1} \wedge \bigwedge_{\ell \leq k \leq n} A_{i,k}(A_{\ell-1,k})^{-1} \\ &= A_{i,\ell-1}(A_{\ell-1,\ell-1})^{-1} \wedge \bigwedge_{\ell \leq k \leq n} E_{i,k}(E_{\ell-1,k})^{-1}. \end{aligned}$$

By our assumption on tightness, we have $E_{i,\ell-1} < E_{i,k}(E_{\ell-1,k})^{-1}$ for all $i < \ell - 1 < k$. Since \mathfrak{L}^* is linearly ordered we deduce that $E_{i,\ell-1} = A_{i,\ell-1}(A_{\ell-1,\ell-1})^{-1}$ for all $i < \ell - 1$, whilst for $i = \ell - 1$ we have $\mathbf{1} = E_{i,\ell-1} = A_{i,\ell-1}A_{\ell-1,\ell-1}^{-1}$. Finite induction now yields that $E = A(D_A)^{-1}$ and hence, by Theorem 6.1, $A \mathcal{R} E$. The converse is clear.

It follows from the above that $\tilde{H}_E = H_E$, which by Theorem 6.6 is isomorphic to \mathfrak{L}^* . \square

The previous two results hint towards the properties of deficiency, tightness and looseness being important for determination of the $\tilde{\mathcal{R}}$ - $\tilde{\mathcal{L}}$ - and $\tilde{\mathcal{H}}$ -classes in $UT_n(\mathfrak{L}^*)$; our subsequent investigations confirm that this is indeed the case. We begin by examining the notion of tightness in more detail.

Lemma 7.9. *Let $E \in UT_n(\mathfrak{L}^*)$ be such that $E^2 = E$.*

- (i) *If E is tight in the path $1 \rightarrow 2 \rightarrow \dots \rightarrow n$ then E is tight in all paths $\gamma \in \Phi[n]$.*
- (ii) *If E is tight in all paths $i \rightarrow u \rightarrow j$ of length 2, then E is tight in all paths $\gamma \in \Phi[n]$.*
- (iii) *If E is tight in the paths $i \rightarrow u \rightarrow j$ and $u \rightarrow v \rightarrow j$ then E is tight in the paths $i \rightarrow v \rightarrow j$ and $i \rightarrow u \rightarrow v$.*
- (iv) *For any $i < u < v < j$, if E is tight in three out of the four paths $i \rightarrow u \rightarrow v$, $i \rightarrow u \rightarrow j$, $i \rightarrow v \rightarrow j$ and $u \rightarrow v \rightarrow j$, then E is tight in the fourth.*

Proof. Let $E \in UT_n(\mathfrak{L}^*)$ such that $E^2 = E$, so that for all $i, k, j \in \{1, \dots, n\}$ we have $E_{i,k}E_{k,j} \leq E_{i,j}$. As already observed, if $i = k$ or $k = j$, then this inequality is tight.

- (i) Suppose that E is tight in the path $1 \rightarrow 2 \rightarrow \dots \rightarrow n$ and $1 \leq i < j < k \leq n$. Then

$$\begin{aligned} E_{1,n} &= E_{1,2} \cdots E_{i-1,i} E_{i,i+1} \cdots E_{j-1,j} E_{j,j+1} \cdots E_{k-1,k} E_{k,k+1} \cdots E_{n-1,n} \\ &\leq E_{1,i} E_{i,j} E_{j,k} E_{k,n} \leq E_{1,i} E_{i,k} E_{k,n} \leq E_{1,n}. \end{aligned}$$

Notice that if E is loose in the path $i \rightarrow j \rightarrow k$, then the second inequality would be strict, giving a contradiction.

- (ii) If E is tight in all paths of length 2 in $\Phi[n]$, then

$$E_{1,2} E_{2,3} E_{3,4} \cdots E_{n-1,n} = E_{1,3} E_{3,4} \cdots E_{n-1,n} = \cdots = E_{1,n},$$

and the result follows from part (i).

- (iii) If $E_{i,u}E_{u,j} = E_{i,j}$ and $E_{u,v}E_{v,j} = E_{u,j}$ then

$$E_{i,j} = E_{i,u}E_{u,v}E_{v,j} \leq E_{i,v}E_{v,j} \leq E_{i,j}.$$

This gives $E_{i,j} = E_{i,v}E_{v,j}$ and hence

$$E_{i,u}E_{u,v} = (E_{i,j}(E_{u,j})^{-1})(E_{u,j}(E_{v,j})^{-1}) = E_{i,j}(E_{v,j})^{-1} = E_{i,v}.$$

- (iv) This is a matter of checking the remaining possibilities, making use of part (iii). \square

7.2. Detailed calculations for $n < 5$. We have already shown in Corollary 6.5 that $UT_n(\mathfrak{L}^*)$ is regular for $n = 1$ and $n = 2$ and so, in these cases, $\tilde{\mathcal{R}} = \mathcal{R}^* = \mathcal{R}$ and $\tilde{\mathcal{L}} = \mathcal{L}^* = \mathcal{L}$. In this subsection, we perform a full analysis for $n = 3$ and $n = 4$. In view of Corollary 6.4 it is sufficient to characterise the equivalence classes in $UT_n(\mathfrak{L}^*)$ by computing the $\tilde{\mathcal{R}}$ - and $\tilde{\mathcal{L}}$ -classes of idempotents in $U_n(\mathfrak{L}^*)$. We begin with the case where $n = 3$. To this end, we introduce some notation to allow us to express our results compactly. For $\alpha \in \mathfrak{L}^*$ and $1 \leq i \leq j \leq n$, let $[\alpha]_{i,j}$ denote the element of $UT_n(\mathfrak{L})$ with (i, j) th entry equal to α and all other entries equal to $\mathbf{1}$. Recall that for $A, B \in UT_n(\mathfrak{L}^*)$ we write $A \circ B$ to denote the Hadamard product of A and B .

Proposition 7.10. *Let E be an idempotent of $UT_3(\mathfrak{L}^*)$.*

(i) *If E is loose in the path $1 \rightarrow 2 \rightarrow 3$ then*

$$\tilde{R}_E = ED_n(\mathfrak{L}^*) = R_E, \quad \tilde{L}_E = D_n(\mathfrak{L}^*)E = L_E, \quad \text{and} \quad \tilde{H}_E = H_E \cong \mathfrak{L}^*.$$

(ii) *Otherwise*

$$\begin{aligned} \tilde{R}_E &= \{([\alpha]_{1,2} \circ E)G : \alpha^{-1} \in \mathfrak{L}_{\leq 1}^*, G \in D_n(\mathfrak{L}^*)\} \\ \tilde{L}_E &= \{G([\beta]_{2,3} \circ E) : \beta^{-1} \in \mathfrak{L}_{\leq 1}^*, G \in D_n(\mathfrak{L}^*)\} \\ \tilde{H}_E &\cong \mathfrak{L}^* \times \mathfrak{L}_{\leq 1}^*. \end{aligned}$$

Proof. Part (i) follows immediately from Proposition 7.8, Theorem 6.1 (and its left-right dual statement), and Theorem 6.6.

Now let $E \in UT_3(\mathfrak{L}^*)$ be an idempotent which is tight in the path $1 \rightarrow 2 \rightarrow 3$. Thus $E_{1,3} = E_{1,2}E_{2,3}$. If $A \in UT_3(\mathfrak{L}^*)$ with $A \tilde{\mathcal{R}} E$, then (by the equivalence of parts (i) and (v) of Theorem 4.22 together with the fact that $E = E^\diamond = E^{(+)} = E^{(*)}$) we have $E = (A^\diamond)^{(+)}$. Thus we may apply Lemma 6.9 to obtain $A = ([\alpha]_{1,2} \circ E)D_A$ for some $\alpha \geq \mathbf{1}$.

Conversely, suppose that $A = ([\alpha]_{1,2} \circ E)G$ for some $\alpha \geq \mathbf{1}$ and $G \in D_n(\mathfrak{L}^*)$. Since $[\alpha]_{1,2} \circ E$ is unitriangular, notice that we must have $A^\diamond = [\alpha]_{1,2} \circ E$ and hence

$$(A^\diamond)^{(+)} = \begin{pmatrix} \mathbf{1} & E_{12}\alpha \wedge E_{1,3}E_{2,3} & E_{1,3} \\ \mathbf{0} & \mathbf{1} & E_{2,3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & E_{12}\alpha \wedge E_{1,2} & E_{1,3} \\ \mathbf{0} & \mathbf{1} & E_{2,3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} = E,$$

using the fact that E is tight in the path $1 \rightarrow 2 \rightarrow 3$, and $\alpha \geq \mathbf{1}$. This shows that the $\tilde{\mathcal{R}}$ -class of E is as required. A similar argument holds for the $\tilde{\mathcal{L}}$ -class.

Suppose now that $A \tilde{\mathcal{H}} E$ in $UT_3(\mathfrak{L}^*)$. Then $A = ([\alpha]_{1,2} \circ E)D_A = D_A([\beta]_{2,3} \circ E)$, where $\alpha, \beta \geq \mathbf{1}$, giving

$$\begin{pmatrix} A_{1,1} & \alpha E_{1,2}A_{2,2} & E_{1,3}A_{3,3} \\ \mathbf{0} & A_{2,2} & E_{2,3}A_{3,3} \\ \mathbf{0} & \mathbf{0} & A_{3,3} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,1}E_{1,2} & A_{1,1}E_{1,3} \\ \mathbf{0} & A_{2,2} & A_{2,2}\beta E_{2,3} \\ \mathbf{0} & \mathbf{0} & A_{3,3} \end{pmatrix}.$$

Comparing entries at the position $(1, 3)$ yields $A_{1,1} = A_{3,3}$, whilst comparing entries at positions $(1, 2)$ and $(2, 3)$ gives $A_{2,2}\alpha = A_{1,1}$ and $A_{2,2}\beta = A_{3,3}$. Thus it follows that $\alpha = \beta$ and together with our tightness assumption this gives

$$A = A_{1,1}([\alpha^{-1}]_{2,2} \circ E), \quad \text{where } \alpha^{-1} \leq \mathbf{1}, A_{1,1} \in \mathfrak{L}^*. \quad (21)$$

In fact, for any matrix of the form given in (21), it is straightforward to check that $A^{(+)} = E = A^{(*)}$ in $UT_3(\mathfrak{L}^*)$. Hence

$$\tilde{H}_E = \{\lambda([\mu]_{2,2} \circ E) : \lambda, \mu \in \mathfrak{L}^*, \mu \leq \mathbf{1}\}.$$

Also, note that for $\lambda, \lambda', \mu, \mu' \in \mathfrak{L}^*$ with $\mu, \mu' \leq \mathbf{1}$ we have $\mu\mu' \leq \mathbf{1}$ and

$$\lambda([\mu]_{2,2} \circ E) \cdot \lambda'([\mu']_{2,2} \circ E) = \lambda\lambda' \begin{pmatrix} \mathbf{1} & E_{1,2} \vee E_{12}\mu' & E_{1,3} \\ \mathbf{0} & \mu\mu' & E_{2,3}\mu \vee E_{2,3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} = \lambda\lambda'([\mu\mu']_{2,2} \circ E)$$

Thus \tilde{H}_E is closed under multiplication and, moreover, it follows from the form of this product that, $\tilde{H}_E \cong \mathfrak{L}^* \times \mathfrak{L}_{\leq 1}^*$. \square

We know that \mathcal{R} and \mathcal{L} commute in any semigroup so that in a regular semigroup (where $\mathcal{R} = \tilde{\mathcal{R}}$ and $\mathcal{L} = \tilde{\mathcal{L}}$), certainly $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ commute. We have remarked in Corollary 6.5 that $UT_3(\mathfrak{L}^*)$ is not regular. Nevertheless, we can show that, in this semigroup, $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ commute.

Proposition 7.11. (i) For each $A \in UT_3(\mathfrak{L}^*)$ we have that $A^{(+)} \mathcal{D} A^{(*)}$.

(ii) The relations $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ commute on $UT_3(\mathfrak{L}^*)$, that is, we have $\tilde{\mathcal{L}} \circ \tilde{\mathcal{R}} = \tilde{\mathcal{R}} \circ \tilde{\mathcal{L}}$. Consequently, $\tilde{\mathcal{D}} = \tilde{\mathcal{L}} \circ \tilde{\mathcal{R}} = \tilde{\mathcal{R}} \circ \tilde{\mathcal{L}}$.

(iii) Let E, F be idempotents in $UT_3(\mathfrak{L}^*)$. Then $E \mathcal{D} F$ if and only if $E \tilde{\mathcal{D}} F$.

Proof. (i) Let $A \in UT_3(\mathfrak{L}^*)$. Direct calculation gives

$$A^{(+)} = \begin{pmatrix} \mathbf{1} & A_{1,2}A_{2,2}^{-1} \wedge A_{1,3}A_{2,3}^{-1} & A_{1,3}A_{3,3}^{-1} \\ \mathbf{0} & \mathbf{1} & A_{2,3}A_{3,3}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad A^{(*)} = \begin{pmatrix} \mathbf{1} & A_{1,2}A_{1,1}^{-1} & A_{1,3}A_{1,1}^{-1} \\ \mathbf{0} & \mathbf{1} & A_{1,3}A_{1,2}^{-1} \wedge A_{2,3}A_{2,2}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Noting that

$$A_{1,2}^{(+)} A_{2,3}^{(+)} (A_{1,3}^{(+)})^{-1} = A_{1,2}^{(*)} A_{2,3}^{(*)} (A_{1,3}^{(*)})^{-1} = \mathbf{1} \wedge A_{1,2} A_{2,3} A_{2,2}^{-1} A_{1,3}^{-1},$$

it follows from Theorem 7.4 that $A^{(+)} \mathcal{D} A^{(*)}$ in $UT_3(\mathfrak{L}^*)$.

(ii) Let $A, B \in UT_3(\mathfrak{L}^*)$ be such that $A(\tilde{\mathcal{R}} \circ \tilde{\mathcal{L}})B$. Then there exists $X \in UT_3(\mathfrak{L}^*)$ with $A \tilde{\mathcal{R}} X \tilde{\mathcal{L}} B$. It then follows from Theorem 4.22 and the above observation that

$$A^{(*)} \mathcal{D} A^{(+)} = X^{(+)} \mathcal{D} X^{(*)} = B^{(*)} \mathcal{D} B^{(+)},$$

and so there is a $Y \in UT_3(\mathfrak{L}^*)$ with

$$A \tilde{\mathcal{L}} A^{(*)} \mathcal{L} Y \mathcal{R} B^{(+)} \tilde{\mathcal{R}} B$$

giving $A \tilde{\mathcal{L}} Y \tilde{\mathcal{R}} B$, as required. The dual argument now completes the proof.

(iii) Suppose that E and F are idempotents with $E \tilde{\mathcal{D}} F$. By part (ii) there exists $X \in UT_3(\mathfrak{L}^*)$ such that $E \tilde{\mathcal{R}} X \tilde{\mathcal{L}} F$. Since each $\tilde{\mathcal{R}}$ -class and each $\tilde{\mathcal{L}}$ -class contains a unique idempotent, we must have $E = X^{(+)}$ and $F = X^{(*)}$. But now part (i) gives $E \mathcal{D} F$. \square

We will see that the very regular behaviour of the \sim -relations in $UT_n(\mathfrak{L}^*)$ for $n = 1, 2, 3$ does not persist for larger n . In fact, even for $n = 4$, we see that the characterisations of $\tilde{\mathcal{R}}$ -, $\tilde{\mathcal{L}}$ - and $\tilde{\mathcal{H}}$ -classes become more complex, and $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ no longer commute. However, we still have the property that \tilde{H}_E for an idempotent $E \in UT_4(\mathfrak{L}^*)$ is a subsemigroup: its nature depends on the tightness or otherwise of E in the paths of length 2 in $\Phi[4]$.

In the rest of this subsection we carefully analyse the $\tilde{\mathcal{R}}$ -, $\tilde{\mathcal{L}}$ - and $\tilde{\mathcal{H}}$ -classes of $E \in UT_4(\mathfrak{L}^*)$, depending on the tightness and looseness patterns, as indicated. We proceed along the lines of that of Propositions 7.10, but with inevitably more complex arguments.

We observe that there are four simple paths of length 2 in $\Phi[4]$ namely, $1 \rightarrow 2 \rightarrow 3$, $1 \rightarrow 2 \rightarrow 4$, $1 \rightarrow 3 \rightarrow 4$ and $2 \rightarrow 3 \rightarrow 4$. We explicitly consider the $\tilde{\mathcal{R}}$ -, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{H}}$ -classes of idempotents in $UT_4(\mathfrak{L}^*)$ depending on the tightness and looseness patterns of the four simple paths of lengths 2 in $\Phi[4]$. Apriori there are 16 possible tightness patterns; by Lemma 7.9 six of these do not arise, since any idempotent which is tight both $1 \rightarrow 2 \rightarrow 3$ and $1 \rightarrow 3 \rightarrow 4$ (or dually in both $1 \rightarrow 2 \rightarrow 4$ and $2 \rightarrow 3 \rightarrow 4$) is tight in all four paths. This leaves ten cases to consider. Moreover, since the involutory anti-automorphism Δ strongly exchanges the relations $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$, it is easy to see that there will be a duality between the following pairs of cases:

- ‘ E is tight in only $1 \rightarrow 2 \rightarrow 3$ ’ is dual to ‘ E is tight in only $2 \rightarrow 3 \rightarrow 4$ ’;

- ‘ E is tight in only $1 \rightarrow 2 \rightarrow 4$ ’ is dual to ‘ E is tight in only $1 \rightarrow 3 \rightarrow 4$ ’;
- ‘ E is tight in only $1 \rightarrow 2 \rightarrow 3$ and $1 \rightarrow 2 \rightarrow 4$ ’ is dual to ‘ E is tight in only $2 \rightarrow 3 \rightarrow 4$ and $1 \rightarrow 3 \rightarrow 4$ ’;

while remaining four cases are self-dual. This reduces the task to consideration of seven patterns of tightness and looseness of idempotents in $UT_4(\mathfrak{L}^*)$. The following table gives the $\tilde{\mathcal{R}}$ -, $\tilde{\mathcal{L}}$ - and $\tilde{\mathcal{H}}$ -classes corresponding to each of these 7 cases obtained by direct calculations (see below) and calling upon Proposition 7.8 for the case where E is loose in all paths.

1. E is loose in all four paths (self-dual)
$\tilde{R}_E = ED_4(\mathfrak{L}^*) = R_E$, $\tilde{L}_E = D_4(\mathfrak{L}^*)E = L_E$, $\tilde{H}_E = H_E$
2. E is tight in only $1 \rightarrow 2 \rightarrow 3$ (dual case: tight in only $2 \rightarrow 3 \rightarrow 4$)
$\tilde{R}_E = \{[\alpha]_{1,2} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\} D_4(\mathfrak{L}^*)$ $\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\text{Def}_E(2 \rightarrow 3 \rightarrow 4)]_{2,3} \circ [\alpha]_{3,4} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\}$ $\cup D_4(\mathfrak{L}^*) \{[\alpha]_{2,3} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha \leq \text{Def}_E(2 \rightarrow 3 \rightarrow 4)\}$ $\tilde{H}_E = H_E$
3. E is tight in only $1 \rightarrow 2 \rightarrow 4$ (dual case: tight in only $1 \rightarrow 3 \rightarrow 4$)
$\tilde{R}_E = \{[\alpha]_{1,2} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\} D_4(\mathfrak{L}^*)$ $\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\alpha]_{2,4} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\}$ $\tilde{H}_E = H_E$
4. E is tight in $1 \rightarrow 3 \rightarrow 4$, $2 \rightarrow 3 \rightarrow 4$ (dual case: tight in $1 \rightarrow 2 \rightarrow 4$, $1 \rightarrow 2 \rightarrow 3$)
$\tilde{R}_E = \{[\alpha]_{1,3} \circ [\beta]_{2,3} \circ E : \alpha, \beta \in \mathfrak{L}^*, 1 \leq \alpha, 1 \leq \beta \leq \text{Def}_E(1 \rightarrow 2 \rightarrow 3)\alpha\} D_4(\mathfrak{L}^*)$ $\cup \{[\alpha]_{1,2} \circ [\beta]_{1,3} \circ [\text{Def}_E(1 \rightarrow 2 \rightarrow 3)\beta]_{2,3} \circ E : \alpha, \beta \in \mathfrak{L}^*, 1 \leq \alpha, \beta\} D_4(\mathfrak{L}^*)$ $\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\alpha]_{3,4} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\}$ $\tilde{H}_E = \{\lambda([\mu]_{3,3} \circ E) : \lambda, \mu \in \mathfrak{L}^*, \mu \leq 1\}$
5. E is tight in $1 \rightarrow 2 \rightarrow 3$, $2 \rightarrow 3 \rightarrow 4$ (self dual)
$\tilde{R}_E = \{[\alpha]_{1,2} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\} D_4(\mathfrak{L}^*)$ $\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\alpha]_{3,4} \circ E : \alpha \in \mathfrak{L}^*, 1 \leq \alpha\}$ $\tilde{H}_E = H_E$
6. E is tight in $1 \rightarrow 2 \rightarrow 4$, $1 \rightarrow 3 \rightarrow 4$ (self dual)
$\tilde{R}_E = \{[\alpha]_{1,2} \circ [\beta]_{1,3} \circ E : \alpha, \beta \in \mathfrak{L}^*, 1 \leq \alpha, \beta\} D_4(\mathfrak{L}^*)$ $\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\alpha]_{2,4} \circ [\beta]_{3,4} \circ E : \alpha, \beta \in \mathfrak{L}^*, 1 \leq \alpha, \beta\}$ $\tilde{H}_E = \{\lambda([\mu]_{2,2} \circ [\mu]_{2,3} \circ [\mu]_{3,3} \circ E) : \lambda, \mu \in \mathfrak{L}^*, \mu \leq 1\}$
7. E is tight in all four paths (self dual)
$\tilde{R}_E = \{[\alpha]_{1,2} \circ [\beta]_{1,3} \circ [\gamma]_{2,3} \circ E : \alpha, \beta, \gamma \in \mathfrak{L}^*, 1 \leq \alpha, 1 \leq \gamma \leq \beta\} D_4(\mathfrak{L}^*)$ $\tilde{L}_E = \{[\alpha]_{2,3} \circ [\beta]_{2,4} \circ [\gamma]_{3,4} \circ E : \alpha, \beta, \gamma \in \mathfrak{L}^*, 1 \leq \gamma, 1 \leq \alpha \leq \beta\} D_4(\mathfrak{L}^*)$ $\tilde{H}_E = \{\lambda([\alpha]_{2,2} \circ [\beta]_{2,3} \circ [\gamma]_{3,3} \circ E : \lambda, \alpha, \beta, \gamma \in \mathfrak{L}^*, \alpha \vee \gamma \leq \beta \leq 1\}$

TABLE 1. Table of $\tilde{\mathcal{R}}$ -, $\tilde{\mathcal{L}}$ - and $\tilde{\mathcal{H}}$ -classes in $UT_4(\mathfrak{L}^*)$

Notice that from the above table one can easily compute the dual results. For example, if E is tight only in the simple path $1 \rightarrow 2 \rightarrow 3$, we find that $\tilde{R}_E = \{[\alpha]_{1,2} \circ E : \mathbf{1} \leq \alpha\} D_4(\mathfrak{L}^*)$, and hence by duality if E is tight only in the simple path $2 \rightarrow 3 \rightarrow 4$, then $\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\beta]_{3,4} \circ E : \mathbf{1} \leq \beta\}$.

Let $E, A \in UT_4(\mathfrak{L}^*)$ such that $E^2 = E$. If $A \tilde{R} E$, then by Lemma 6.9 we may write

$$A^\diamond = [\alpha_{1,2}]_{1,2} \circ [\alpha_{1,3}]_{1,3} \circ [\alpha_{2,3}]_{2,3} \circ E,$$

where $\alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3} \geq \mathbf{1}$. Since $A \tilde{R} E$ if and only if $(A^\diamond)^{(+)} = E$, a description of the \tilde{R} -class of E can be found by computing the solutions of the following system of equations:

$$E_{1,2} = \alpha_{1,2} E_{1,2} \wedge \alpha_{1,3} E_{1,3} \alpha_{2,3}^{-1} E_{2,3}^{-1} \wedge E_{1,4} E_{2,4}^{-1}, \quad (22)$$

$$E_{1,3} = \alpha_{1,3} E_{1,3} \wedge E_{1,4} E_{3,4}^{-1}, \quad (23)$$

$$E_{2,3} = \alpha_{2,3} E_{2,3} \wedge E_{2,4} E_{3,4}^{-1}. \quad (24)$$

Notice that equation (22) holds if and only if $\alpha_{2,3} \leq \text{Def}_E(1 \rightarrow 2 \rightarrow 3) \alpha_{1,3}$ and either: E is tight in the path $1 \rightarrow 2 \rightarrow 4$; or $\alpha_{1,2} = \mathbf{1}$; or $\alpha_{2,3} = \text{Def}_E(1 \rightarrow 2 \rightarrow 3) \alpha_{1,3}$. Equation (23) holds if and only if either E is tight in the path $1 \rightarrow 3 \rightarrow 4$ or $\alpha_{1,3} = \mathbf{1}$; whilst (24) holds if and only if either E is tight in the path $2 \rightarrow 3 \rightarrow 4$ or $\alpha_{2,3} = \mathbf{1}$.

Similarly, if $A \tilde{L} E$, then by Lemma 6.9 we may write $A^\bullet = [\beta_{2,3}]_{2,3} \circ [\beta_{2,4}]_{2,4} \circ [\beta_{3,4}]_{3,4} \circ E$, where $\beta_{2,3}, \beta_{2,4}, \beta_{3,4} \geq \mathbf{1}$. Since $A \tilde{L} E$ if and only if $(A^\bullet)^{(*)} = E$, the \tilde{L} class of E can be found by computing the solutions to:

$$E_{3,4} = \beta_{3,4} E_{3,4} \wedge \beta_{2,4} E_{2,4} \beta_{2,3}^{-1} E_{2,3}^{-1} \wedge E_{1,4} E_{1,3}^{-1}, \quad (25)$$

$$E_{2,3} = \beta_{2,3} E_{2,3} \wedge E_{1,3} E_{1,2}^{-1}, \quad (26)$$

$$E_{2,4} = \beta_{2,4} E_{2,4} \wedge E_{1,4} E_{1,2}^{-1}. \quad (27)$$

Equation (25) holds if and only if $\beta_{2,3} \leq \text{Def}_E(2 \rightarrow 3 \rightarrow 4) \beta_{2,4}$ and either: E is tight in the path $1 \rightarrow 3 \rightarrow 4$; or $\beta_{3,4} = \mathbf{1}$; or $\beta_{2,3} = \text{Def}_E(2 \rightarrow 3 \rightarrow 4) \beta_{2,4}$. Similarly, (26) holds if and only if either E is tight in the path $1 \rightarrow 2 \rightarrow 3$ or $\beta_{2,3} = \mathbf{1}$; whilst (27) holds if and only if either E is tight in the path $1 \rightarrow 2 \rightarrow 4$ or $\beta_{2,4} = \mathbf{1}$.

Suppose now that equations (22)-(27) hold, and additionally that A is equal to:

$$\begin{pmatrix} A_{1,1} & \alpha_{1,2} E_{1,2} A_{2,2} & \alpha_{1,3} E_{1,3} A_{3,3} & E_{1,4} A_{4,4} \\ \mathbf{0} & A_{2,2} & \alpha_{2,3} E_{2,3} A_{3,3} & E_{2,4} A_{4,4} \\ \mathbf{0} & \mathbf{0} & A_{3,3} & E_{3,4} A_{4,4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{4,4} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,1} E_{1,2} & A_{1,1} E_{1,3} & A_{1,1} E_{1,4} \\ \mathbf{0} & A_{2,2} & A_{2,2} E_{2,3} \beta_{2,3} & A_{2,2} E_{2,4} \beta_{2,4} \\ \mathbf{0} & \mathbf{0} & A_{3,3} & A_{3,3} E_{3,4} \beta_{3,4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{4,4} \end{pmatrix}.$$

Notice that the latter equation can be summarised as:

$$A_{1,1} = \alpha_{1,2} A_{2,2} = \alpha_{1,3} A_{3,3} = A_{4,4} = A_{2,2} \beta_{2,4} = A_{3,3} \beta_{3,4}, \quad (28)$$

$$\alpha_{2,3} A_{3,3} = A_{2,2} \beta_{2,3}, \quad (29)$$

and hence that equations (22)-(29) determine the $\tilde{\mathcal{H}}$ -class of E .

Using the above equations it is now straightforward to analyse the required cases.

1. If E is loose in all four paths: By Proposition 7.8, $A \tilde{R} E$ exactly if $A \mathcal{R} E$, and dually $A \tilde{L} E$ if and only if $A \mathcal{L} E$. It then follows from Theorem 6.1 that

$$\tilde{R}_E = E D_4(\mathfrak{L}^*) = R_E; \quad \tilde{L}_E = D_4(\mathfrak{L}^*) E = L_E; \quad \tilde{H}_E = H_E.$$

2. If E is tight in only $1 \rightarrow 2 \rightarrow 3$: It follows from the observations above that in order to satisfy equations (23) and (24) we must have $\alpha_{1,3} = \alpha_{2,3} = \mathbf{1}$. Since $\text{Def}_E(1 \rightarrow 2 \rightarrow 3) = \mathbf{1}$, it then follows that (22) holds. Thus

$$\tilde{R}_E = \{[\alpha]_{1,2} \circ E : \mathbf{1} \leq \alpha\} D_4(\mathfrak{L}^*).$$

Since the path $1 \rightarrow 2 \rightarrow 3$ is tight, equation (26) holds. In order to satisfy equation (27) we must have $\beta_{2,4} = \mathbf{1}$. There are then two possibilities which lead to the satisfaction of (25); either $\beta_{3,4} = \mathbf{1}$ and $\beta_{2,3} \leq \text{Def}_E(2 \rightarrow 3 \rightarrow 4)$, or $\beta_{2,3} = \text{Def}_E(2 \rightarrow 3 \rightarrow 4)$. This gives

$$\begin{aligned} \tilde{L}_E = & D_4(\mathfrak{L}^*) \{ [\text{Def}_E(2 \rightarrow 3 \rightarrow 4)]_{2,3} \circ [\alpha]_{3,4} \circ E : \mathbf{1} \leq \alpha \} \\ & \cup D_4(\mathfrak{L}^*) \{ [\alpha]_{2,3} \circ E : \mathbf{1} \leq \alpha \leq \text{Def}_E(2 \rightarrow 3 \rightarrow 4) \}. \end{aligned}$$

Finally, $A \in \tilde{H}_E$ if and only if (22)-(29) hold simultaneously. This is the case if and only if $A_{1,1} = A_{2,2} = A_{3,3} = A_{4,4}$ and all other parameters are equal to $\mathbf{1}$. Hence $\tilde{H}_E = \{ \lambda E : \lambda \in \mathfrak{L}^* \} = H_E$. Dually, if E is tight in only $2 \rightarrow 3 \rightarrow 4$ then

$$\begin{aligned} \tilde{R}_E = & \{ [\text{Def}_E(1 \rightarrow 2 \rightarrow 3)]_{2,3} \circ [\alpha]_{1,2} \circ E : \mathbf{1} \leq \alpha \} D_4(\mathfrak{L}^*) \\ & \cup \{ [\alpha]_{2,3} \circ E : \mathbf{1} \leq \alpha \leq \text{Def}_E(1 \rightarrow 2 \rightarrow 3) \} D_4(\mathfrak{L}^*) \\ \tilde{L}_E = & D_4(\mathfrak{L}^*) \{ [\alpha]_{3,4} \circ E : \mathbf{1} \leq \alpha \}, \quad \tilde{H}_E = H_E. \end{aligned}$$

3. If E is tight in only $1 \rightarrow 2 \rightarrow 4$: As in the previous case, in order to satisfy equations (23) and (24) we must have $\alpha_{1,3} = \alpha_{2,3} = \mathbf{1}$, whence (22) holds, giving

$$\tilde{R}_E = \{ [\alpha]_{1,2} \circ E : \mathbf{1} \leq \alpha \} D_4(\mathfrak{L}^*).$$

Since the path $1 \rightarrow 2 \rightarrow 4$ is tight, equation (27) holds. In order to satisfy equation (26) we must have $\beta_{2,3} = \mathbf{1}$. Since E is loose in the path $2 \rightarrow 3 \rightarrow 4$, and $\beta_{2,4} \geq \mathbf{1}$ we note that $\mathbf{1} < \text{Def}_E(2 \rightarrow 3 \rightarrow 4)\beta_{2,4}$. Thus in order for (25) to be satisfied we require $\beta_{3,4} = \mathbf{1}$, giving

$$\tilde{L}_E = D_4(\mathfrak{L}^*) \{ [\alpha]_{2,4} \circ E : \mathbf{1} \leq \alpha \}.$$

Now $A \in \tilde{H}_E$ if and only if (22)-(29) hold simultaneously, and this is the case if and only if $A_{1,1} = A_{2,2} = A_{3,3} = A_{4,4}$ and all other parameters are equal to $\mathbf{1}$. Thus $\tilde{H}_E = \{ \lambda E : \lambda \in \mathfrak{L}^* \} = H_E$. Dually, if E is tight in only $1 \rightarrow 3 \rightarrow 4$ then,

$$\tilde{R}_E = \{ [\alpha]_{1,3} \circ E : \mathbf{1} \leq \alpha \} D_4(\mathfrak{L}^*) \quad \tilde{L}_E = D_4(\mathfrak{L}^*) \{ [\alpha]_{3,4} \circ E : \mathbf{1} \leq \alpha \} \quad \tilde{H}_E = H_E.$$

4. If E is tight in only $1 \rightarrow 3 \rightarrow 4$ and $2 \rightarrow 3 \rightarrow 4$: In this case (23) and (24) are satisfied. For (22) to hold we need $\mathbf{1} \leq \alpha_{2,3} \leq \text{Def}_E(1 \rightarrow 2 \rightarrow 3)\alpha_{1,3}$ and either $\alpha_{1,2} = \mathbf{1}$ or $\alpha_{2,3} = \text{Def}_E(1 \rightarrow 2 \rightarrow 3)\alpha_{1,3}$. Thus

$$\begin{aligned} \tilde{R}_E = & \{ [\alpha]_{1,3} \circ [\beta]_{2,3} \circ E : \mathbf{1} \leq \alpha, \mathbf{1} \leq \beta \leq \text{Def}_E(1 \rightarrow 2 \rightarrow 3)\alpha \} D_4(\mathfrak{L}^*) \\ & \cup \{ [\alpha]_{1,2} \circ [\beta]_{1,3} \circ [\text{Def}_E(1 \rightarrow 2 \rightarrow 3)\beta]_{2,3} \circ E : \mathbf{1} \leq \alpha, \beta \} D_4(\mathfrak{L}^*). \end{aligned}$$

For equations (26) and (27) to hold we must have $\beta_{2,3} = \beta_{2,4} = \mathbf{1}$. Since E is tight in $1 \rightarrow 3 \rightarrow 4$, it then follows that (25) is also satisfied, giving

$$\tilde{L}_E = D_4(\mathfrak{L}^*) \{ [\alpha]_{3,4} \circ E : \mathbf{1} \leq \alpha \}.$$

Combining our previous observations with (28) and (29) yields $A_{1,1} = A_{4,4} = A_{2,2} = \alpha_{2,3}A_{3,3}$, $\beta_{2,3} = \beta_{2,4} = \alpha_{1,2} = \mathbf{1}$ and $\alpha_{2,3} = \alpha_{1,3} = \beta_{3,4}$. Hence

$$\tilde{H}_E = \{ \lambda([\mu]_{3,3} \circ E) : \lambda, \mu \in \mathfrak{L}^*, \mu \leq \mathbf{1} \}.$$

Dually, if E is tight only in $1 \rightarrow 2 \rightarrow 3$, $1 \rightarrow 2 \rightarrow 4$ then

$$\begin{aligned} \tilde{R}_E = & \{ [\alpha]_{3,4} \circ E : \mathbf{1} \leq \alpha \} D_4(\mathfrak{L}^*) \\ \tilde{L}_E = & D_4(\mathfrak{L}^*) \{ [\alpha]_{2,4} \circ [\beta]_{2,3} \circ E : \mathbf{1} \leq \alpha, \mathbf{1} \leq \beta \leq \text{Def}_E(2 \rightarrow 3 \rightarrow 4)\alpha \} \\ & \cup D_4(\mathfrak{L}^*) \{ [\alpha]_{3,4} \circ [\beta]_{2,4} \circ [\text{Def}_E(2 \rightarrow 3 \rightarrow 4)\beta]_{2,3} \circ E : \mathbf{1} \leq \alpha, \beta \} \\ \tilde{H}_E = & \{ \lambda([\mu]_{2,2} \circ E) : \lambda, \mu \in \mathfrak{L}^*, \mu \leq \mathbf{1} \}. \end{aligned}$$

5. If E is tight only in $1 \rightarrow 2 \rightarrow 3$, $2 \rightarrow 3 \rightarrow 4$: In this case (24) is satisfied, and for (23) to hold requires $\alpha_{1,3} = \mathbf{1}$. Since E is tight in $1 \rightarrow 2 \rightarrow 3$, we find that the only way that (22) can hold is if $\alpha_{2,3} = \mathbf{1}$. Thus

$$\tilde{R}_E = \{[\alpha]_{1,2} \circ E : \mathbf{1} \leq \alpha\} D_4(\mathfrak{L}^*).$$

Similarly, (26) is satisfied and for (27) to hold we must have $\beta_{2,4} = \mathbf{1}$. Since E is tight in $2 \rightarrow 3 \rightarrow 4$, it then follows that the only way that (25) can hold is if $\beta_{2,3} = \mathbf{1}$. Thus

$$\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\alpha]_{3,4} \circ E : \mathbf{1} \leq \alpha\}.$$

Combining the previous observations with (28) and (29) then yields that $A_{1,1} = A_{4,4} = A_{3,3} = A_{4,4}$, and all other parameters are equal to $\mathbf{1}$, giving $\tilde{H}_E = H_E$.

6. If E is tight in only $1 \rightarrow 2 \rightarrow 4$, $1 \rightarrow 3 \rightarrow 4$: In this case (23) is satisfied, and for (24) to hold requires $\alpha_{2,3} = \mathbf{1}$. But then, since E is tight in $1 \rightarrow 2 \rightarrow 4$, we see that (22) also holds, giving

$$\tilde{R}_E = \{[\alpha]_{1,2} \circ [\beta]_{1,3} E : \mathbf{1} \leq \alpha, \beta\} D_4(\mathfrak{L}^*).$$

Likewise, (27) is satisfied and for (26) to hold we must have $\beta_{2,3} = \mathbf{1}$. Since E is tight in $2 \rightarrow 3 \rightarrow 4$, we then have that (25) holds, giving

$$\tilde{L}_E = D_4(\mathfrak{L}^*) \{[\alpha]_{2,4} \circ [\beta]_{3,4} \circ E : \mathbf{1} \leq \alpha, \beta\}.$$

Finally, $A \in \tilde{H}_E$ if and only if (22)-(29) hold simultaneously, which is the case if $A_{1,1} = A_{4,4} = \alpha_{12}A_{2,2}$, $A_{2,2} = A_{3,3}$, $\alpha_{1,2} = \alpha_{1,3} = \beta_{2,4} = \beta_{3,4}$ and $\alpha_{2,3} = \beta_{2,3} = \mathbf{1}$. It follows from this that

$$\tilde{H}_E = \{\lambda([\mu]_{2,2} \circ [\mu]_{2,3} \circ [\mu]_{3,3} \circ E) : \lambda, \mu, \in \mathfrak{L}^*, \mu \leq \mathbf{1}\}.$$

7. If E is tight in all four paths : Equations (23), (24), (26) and (27) are satisfied. Since all paths are tight (and hence all deficiencies are equal to $\mathbf{1}$) it is clear that (22) holds if and only if $\alpha_{2,3} \leq \alpha_{1,3}$, whilst (25) holds if and only if $\beta_{2,3} \leq \beta_{2,4}$, giving

$$\begin{aligned} \tilde{R}_E &= \{[\alpha]_{1,2} \circ [\beta]_{1,3} \circ [\gamma]_{2,3} \circ E : \mathbf{1} \leq \alpha, \mathbf{1} \leq \gamma \leq \beta\} D_4(\mathfrak{L}^*) \\ \tilde{L}_E &= D_4(\mathfrak{L}^*) \{[\alpha]_{2,3} \circ [\beta]_{2,4} \circ [\gamma]_{3,4} \circ E : \mathbf{1} \leq \gamma, \mathbf{1} \leq \alpha \leq \beta\}. \end{aligned}$$

Combining these observations with (28) and (29) then yields

$$\tilde{H}_E = \{\lambda([\alpha]_{2,2} \circ [\beta]_{2,3} \circ [\gamma]_{3,3} \circ E) : \lambda, \alpha, \beta, \gamma \in \mathfrak{L}^*, \alpha \vee \gamma \leq \beta \leq \mathbf{1}\}.$$

We now set out to confirm the fact stated earlier, that for any idempotent $E \in UT_4(\mathfrak{L}^*)$ we have that \tilde{H}_E is a subsemigroup.

Proposition 7.12. *For any idempotent $E \in UT_4(\mathfrak{L}^*)$, the $\tilde{\mathcal{H}}$ -class \tilde{H}_E of E is a subsemigroup.*

Proof. Let $E \in UT_4(\mathfrak{L}^*)$ such that $E^2 = E$. From Table 1, there are four different patterns for \tilde{H}_E , depending on the tightness patterns of E in paths of length 2. In the case where $\tilde{H}_E = H_E$, clearly \tilde{H}_E is a monoid (it is a maximal subgroup). In each of the remaining three cases, the elements of \tilde{H}_E have the form $\lambda([\alpha]_{2,2} \circ [\beta]_{2,3} \circ [\gamma]_{3,3} \circ E)$, where certain restrictions are placed on the parameters α, β, γ . Noting that

$\lambda([\alpha]_{2,2} \circ [\beta]_{2,3} \circ [\gamma]_{3,3} \circ E) \cdot \lambda'([\alpha']_{2,2} \circ [\beta']_{2,3} \circ [\gamma']_{3,3} \circ E) = \lambda\lambda'([\alpha\alpha']_{2,2} \circ [x]_{2,3} \circ [\gamma\gamma']_{3,3} \circ E)$, where $x = \alpha\beta' \vee \beta\gamma'$, it then suffices to show that in each case the restrictions placed on the parameters are preserved by this product.

Case 4. If E is tight in only $1 \rightarrow 3 \rightarrow 4$, $2 \rightarrow 3 \rightarrow 4$:

In this case we have $\alpha = \beta = \mathbf{1}$ and $\gamma \leq \mathbf{1}$. Since for $\gamma, \gamma' \leq \mathbf{1}$, we have $\gamma\gamma' \leq \mathbf{1}$, it follows from the form of the product above that, $\tilde{H}_E \cong \mathfrak{L}^* \times \mathfrak{L}_{\leq \mathbf{1}}^*$.

Case 6. If E is tight in only $1 \rightarrow 2 \rightarrow 4$, $1 \rightarrow 3 \rightarrow 4$:

In this case we have $\alpha = \beta = \gamma \leq 1$, and as before, since $\gamma, \gamma' \leq 1$ implies that $\gamma\gamma' \leq 1$, it follows that $\tilde{H}_E \cong \mathfrak{L}^* \times \mathfrak{L}_{\leq 1}^*$.

Case 7. If E is tight in all four paths:

In this case we have $\alpha \vee \gamma \leq \beta \leq 1$. Since $\alpha' \leq \alpha' \vee \gamma' \leq \beta'$ and $\gamma \leq \alpha \vee \gamma \leq \beta$ we have

$$\alpha\alpha' \vee \gamma\gamma' \leq \alpha\beta' \vee \beta\gamma' = x,$$

and hence \tilde{H}_E is closed under multiplication. Indeed, by the previous lemma we have $\tilde{H}_E \cong \mathfrak{L}^* \times \mathcal{O}_2([0, 1])$. In general this semigroup, it is not (unlike in the previous cases) commutative. For example, taking $\mathfrak{L}^* = \mathbb{R}$, and tuples $(\alpha_1, \beta_1, \gamma_1) = (0, 1, 2)$ and $(\alpha_2, \beta_2, \gamma_2) = (1, 3, 1)$, we see that $(\alpha_1 + \beta_2) \oplus (\beta_1 + \gamma_2) = 3$ but $(\alpha_2 + \beta_1) \oplus (\beta_2 + \gamma_1) = 4$, so that \tilde{H}_E is not commutative. \square

7.3. More complex behaviour persists as n increases. Let $UT_n^{1,n}(\mathfrak{L}^*)$ denote the set of all matrices in $UT_n(\mathfrak{L}^*)$ with 1 at $(1, 1)$ and (n, n) positions. Then $UT_n^{(1,n)}(\mathfrak{L}^*)$ is a subsemigroup of $UT_n(\mathfrak{L}^*)$ and $U_n(\mathfrak{L}^*) \subseteq UT_n^{(1,n)}(\mathfrak{L}^*)$. Thus $UT_n^{(1,n)}(\mathfrak{L}^*)$ is a Fountain semigroup. In order to use facts concerning the behaviour of Green's \sim -relations in $UT_n(\mathfrak{L}^*)$ to inform their behaviour in $UT_{n+1}(\mathfrak{L}^*)$, we first show that $UT_n^{1,n}(\mathfrak{L}^*)$ may be embedded in $UT_{n+1}^{1,n+1}(\mathfrak{L}^*)$ in a way that preserves the unary operations of $(+)$ and $(*)$, as well as the semigroup multiplication⁵.

Definition 7.13. Let $n \in \mathbb{N}$. We define $\theta_n : UT_n(\mathfrak{L}^*) \rightarrow UT_{n+1}(\mathfrak{L}^*)$ by the rule

$$A \mapsto \left(\begin{array}{c|c} A & A_{*,n} \\ \hline \mathbf{0}_{1 \times n} & \mathbf{1} \end{array} \right).$$

Clearly θ_n is an injective map; θ_n is particularly well behaved when restricted to $UT_n^{(1,n)}(\mathfrak{L}^*)$.

Lemma 7.14. For all $n \in \mathbb{N}$ the map

$$\theta_n \upharpoonright_{UT_n^{(1,n)}(\mathfrak{L}^*)} : UT_n^{(1,n)}(\mathfrak{L}^*) \rightarrow UT_{n+1}^{(1,n+1)}(\mathfrak{L}^*)$$

is an $\tilde{\mathcal{R}}$ - and $\tilde{\mathcal{L}}$ -preserving homomorphism. Consequently, θ_n maps idempotents to idempotents.

Proof. Let $A, B \in UT_n^{(1,n)}(\mathfrak{L}^*)$. If $1 \leq i \leq k \leq j \leq n$ then $\theta_n(A)_{i,k} \theta_n(B)_{k,j} = A_{i,k} B_{k,j}$. So

$$(\theta_n(A) \theta_n(B))_{i,j} = \bigvee_{i \leq k \leq j} \theta_n(A)_{i,k} \theta_n(B)_{k,j} = \bigvee_{i \leq k \leq j} A_{i,k} B_{k,j} = (AB)_{i,j} = \theta_n(AB)_{i,j}.$$

It remains to check the case where $i \leq j = n+1$. Here we have

$$\begin{aligned} (\theta_n(A) \theta_n(B))_{i,n+1} &= \bigvee_{i \leq k \leq n+1} \theta_n(A)_{i,k} \theta_n(B)_{k,n+1} \\ &= \left(\bigvee_{i \leq k \leq n} A_{i,k} B_{k,n} \right) \vee (\theta_n(A))_{i,n+1} (\theta_n(B))_{n+1,n+1} \\ &= \left(\bigvee_{i \leq k \leq n} A_{i,k} B_{k,n} \right) \vee A_{i,n} \mathbf{1} \\ &= \left(\bigvee_{i \leq k \leq n} A_{i,k} B_{k,n} \right) \\ &= (AB)_{i,n} \end{aligned}$$

where the last step uses the fact that $A_{i,n} = A_{i,n} \mathbf{1} = A_{i,n} B_{n,n}$. Thus $\theta_n(AB) = (\theta_n(A) \theta_n(B))$.

To see that θ_n preserves $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$, we show that $\theta_n(A)^{(*)} = \theta_n(A^{(*)})$ and $\theta_n(A)^{(+)} = \theta_n(A^{(+)})$ for all $A \in UT_n^{(1,n)}(\mathfrak{L}^*)$.

⁵Formally, the embedding is of algebras of signature type $(2, 1, 1)$.

For $i \leq j < n + 1$,

$$(\theta_n(A)^{(*)})_{i,j} = \bigwedge_{1 \leq k \leq i} \theta_n(A)_{k,j} (\theta_n(A_{k,i}))^{-1} = \bigwedge_{1 \leq k \leq i} A_{k,j} (A_{k,i})^{-1} = A_{i,j}^{(*)}.$$

On the other hand, for $i < n + 1$,

$$(\theta_n(A)^{(*)})_{i,n+1} = \bigwedge_{1 \leq k \leq i} \theta_n(A)_{k,n+1} (\theta_n(A_{k,i}))^{-1} = \bigwedge_{1 \leq k \leq i} A_{k,n} (A_{k,i})^{-1} = A_{i,n}^{(*)}.$$

Finally, for $i = n + 1, j = n + 1$, we have $(\theta_n(A)^{(*)})_{n+1,n+1} = \mathbf{1} = (\theta_n(A^{(*)}))_{n+1,n+1}$. Hence $\theta_n(A)^{(*)} = \theta_n(A^{(*)})$.

Turning our attention to $(+)$, we have that if $j < n + 1$ then

$$\begin{aligned} (\theta_n(A)^{(+)})_{i,j} &= \bigwedge_{j \leq k \leq n+1} \theta_n(A)_{i,k} (\theta_n(A_{j,k}))^{-1} \\ &= \left(\bigwedge_{j \leq k \leq n} \theta_n(A)_{i,k} (\theta_n(A_{j,k}))^{-1} \right) \wedge \theta_n(A)_{i,n+1} (\theta_n(A)_{j,n+1})^{-1} \\ &= \left(\bigwedge_{j \leq k \leq n} A_{i,k} (A_{j,k})^{-1} \right) \wedge A_{i,n} (A_{j,n})^{-1} \\ &= \bigwedge_{j \leq k \leq n} A_{i,k} (A_{j,k})^{-1} = A_{i,j}^{(+)} . \end{aligned}$$

For $j = n + 1$, recalling that $(A^{(+)})_{i,n} = A_{i,n} (A_{n,n})^{-1} = A_{i,n}$ as $A_{n,n} = \mathbf{1}$, we have

$$(\theta_n(A)^{(+)})_{i,n+1} = \theta_n(A)_{i,n+1} (\theta_n(A)_{n+1,n+1})^{-1} = A_{i,n} = (A^{(+)})_{i,n}.$$

Hence $\theta_n(A)^{(+)} = \theta_n(A^{(+)})$. \square

Our first use of Lemma 7.14 is to show that the analogue of Proposition 7.11 does not hold for $n \geq 4$.

Proposition 7.15. *Let $n \geq 4$, and suppose that \mathfrak{L}^* is non-trivial. Then*

- (i) *there is an $A \in UT_n(\mathfrak{L}^*)$ such that $A^{(*)}$ and $A^{(+)}$ are not \mathcal{D} -related;*
- (ii) *the relations $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{L}}$ do not commute in $UT_n(\mathfrak{L}^*)$.*

Proof. We begin with $n = 4$. Let $g \in \mathfrak{L}^*$ with $g > \mathbf{1}$ and let

$$A = \begin{pmatrix} \mathbf{1} & g & \mathbf{1} & g^2 \\ \mathbf{0} & \mathbf{1} & g & g \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ so that } A^{(+)} = \begin{pmatrix} \mathbf{1} & g^{-1} & \mathbf{1} & g^2 \\ \mathbf{0} & \mathbf{1} & g & g \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ and } A^{(*)} = \begin{pmatrix} \mathbf{1} & g & \mathbf{1} & g^2 \\ \mathbf{0} & \mathbf{1} & g^{-1} & g \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Notice that $\text{Def}_{A^{(+)}}(1 \rightarrow 2 \rightarrow 4) = g^2(g^{-1}g)^{-1} = g^2$ whereas $\text{Def}_{A^{(*)}}(1 \rightarrow 2 \rightarrow 4) = g^2(gg)^{-1} = \mathbf{1}$. By Theorem 7.4, $A^{(+)}$ and $A^{(*)}$ are not \mathcal{D} -related in $UT_4(\mathfrak{L}^*)$.

By using Lemma 7.14, for any $n \geq 5$, setting $\phi_n := \theta_{n-1}\theta_{n-2}\cdots\theta_4$ we have that $\phi_n(A^{(+)}) = (\phi_n(A))^+$ and $\phi_n(A^{(*)}) = (\phi_n(A))^*$; but the same argument using deficiencies as in the case $n = 4$ gives that $(\phi_n(A))^+$ and $(\phi_n(A))^*$ are not \mathcal{D} -related in $UT_n(\mathfrak{L}^*)$. This proves that (i) holds.

Notice that the above shows that Proposition 7.11 (i) and (iii) no longer holds for $3 < n$. Also note that, $\phi_n(A^{(+)}) \tilde{\mathcal{R}} \circ \tilde{\mathcal{L}} \phi_n(A^{(*)})$ in $UT_n(\mathfrak{L}^*)$ for $n \geq 4$. If there exists a matrix $B \in UT_n(\mathfrak{L}^*)$ such that $\phi_n(A^{(+)}) \tilde{\mathcal{L}} B \tilde{\mathcal{R}} \phi_n(A^{(*)})$ then, by the equivalence of the first and last parts of Theorem 4.22 and its left-right dual we have $(B^\bullet)^{(*)} = B^{(*)} = \phi_n(A^{(+)})$ and $(B^\diamond)^{(+)} = B^{(+)} = \phi_n(A^{(*)})$. By Lemma 6.9 we may write

$$B^\bullet = \left(\begin{array}{cccc|ccc} \mathbf{1} & g^{-1} & \mathbf{1} & g^2 & g^2 & \dots & g^2 \\ \mathbf{0} & \mathbf{1} & g\beta_{2,3} & g\beta_{2,4} & g\beta_{2,5} & \dots & g\beta_{2,n} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \beta_{3,4} & \beta_{3,5} & \dots & \beta_{3,n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \beta_{4,5} & \dots & \beta_{4,n} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{1} \end{array} \right)$$

and

$$B^\diamond = \left(\begin{array}{cccc|ccc} 1 & g\alpha_{1,2} & \alpha_{1,3} & g^2\alpha_{1,4} & \dots & g^2\alpha_{1,n-1} & g^2 \\ 0 & 1 & g^{-1}\alpha_{2,3} & g\alpha_{2,4} & \dots & g\alpha_{2,n-1} & g \\ 0 & 0 & 1 & \alpha_{3,4} & \dots & \alpha_{3,n-1} & \alpha_{3,n} \\ 0 & 0 & 0 & 1 & \dots & \alpha_{4,n-1} & \alpha_{4,n} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{array} \right)$$

where $1 \leq \alpha_{i,j}, \beta_{i,j}$. By comparing at positions $(2, j)$, where $4 \leq j \leq n$, we have

$$g = \phi_n(A^{(*)})_{2,j} = (B^\bullet)_{2,j}^{(*)} = B_{1,j}^\bullet (B_{1,2}^\bullet)^{-1} \wedge B_{2,j}^\bullet = g^2(g^{-1})^{-1} \wedge g\beta_{2,j} = g\beta_{2,j}.$$

Hence $\beta_{2,j} = 1$ for all $4 \leq j \leq n$. Now $B = D_B B^\bullet = B^\diamond D_B$ and then by comparison at positions $(1, n)$ $(2, n)$, we must have $B_{1,1}g^2 = B_{1,n} = B_{n,n}g^2$, $gB_{2,2} = B_{2,n} = B_{n,n}g$ respectively. So $B_{1,1} = B_{n,n} = B_{2,2}$. Again comparison at positions $(1, 2)$ yields $B_{1,1}g^{-1} = B_{1,2} = B_{2,2}g\alpha_{1,2}$. So that, $\alpha_{1,2} = g^{-2} < 1$, which is a contradiction! Thus our supposition is wrong and no such B exists. So $\tilde{\mathcal{R}} \circ \tilde{\mathcal{L}} \neq \tilde{\mathcal{L}} \circ \tilde{\mathcal{R}}$ in $UT_n(\mathfrak{L}^*)$ where $n \geq 4$. \square

Proposition 7.16. *We know from Proposition 7.12 that \tilde{H}_E is a monoid for each idempotent $E \in UT_4(\mathfrak{L}^*)$. For $n \geq 5$, however, this is not the case, as the following example illustrates.*

Proof. Let $1 < g \in \mathfrak{L}^*$ and $E, A \in UT_5(\mathfrak{L}^*)$ be such that

$$E = \begin{pmatrix} 1 & 1 & g^2 & g^2 & g^2 \\ 0 & 1 & g & g^2 & g^2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 1 & 1 & g^2 & g^2 & g^2 \\ 0 & g^{-2} & g^{-1} & g & g^2 \\ 0 & 0 & g^{-3} & 1 & 1 \\ 0 & 0 & 0 & g^{-3} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then it is straightforward to verify that $A^{(+)} = E = A^{(*)}$ and hence $A \tilde{\mathcal{H}} E$. But

$$A^2 = \begin{pmatrix} 1 & 1 & g^2 & g^2 & g^2 \\ 0 & g^{-4} & g^{-3} & g^{-1} & g^2 \\ 0 & 0 & g^{-6} & g^{-3} & 1 \\ 0 & 0 & 0 & g^{-6} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } (A^2)^{(+)} = \begin{pmatrix} 1 & 1 & g^2 & g^2 & g^2 \\ 0 & 1 & g^2 & g^2 & g^2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly $(A^2)^{(+)} \neq A^{(+)}$ and by uniqueness of $(A^2)^{(+)}$ it follows that $A^2 \notin \tilde{H}_E$ and hence \tilde{H}_E is not a subsemigroup of $UT_5(\mathfrak{L}^*)$.

We can extend this example to any $n > 5$ by using the map $\phi_n = \theta_{n-1} \dots \theta_5$. Since $A \in \tilde{H}_E$ and $A \in UT_5^{(1,5)}(\mathfrak{L}^*)$, Lemma 7.14 yields that $\phi_n(A) \in \tilde{H}_{\phi_n(E)}$. Now $A^2 \in UT_5^{(1,5)}(\mathfrak{L}^*)$, and so $\phi_n((A^2))^{(+)} = \phi_n((A^2)^{+})$ and then by the injectivity of the maps θ_k it follows that $\phi_n((A^2)^{+})$ is not equal to $\phi_n(E)$. Hence $\phi_n(A^2) \notin \tilde{H}_{\phi_n(E)}$ and $\tilde{H}_{\phi_n(E)}$ is not a subsemigroup of $UT_n(\mathfrak{L}^*)$ for all $n > 5$. \square

The above considerations prompt us to end with the following.

Question 7.17. Determine the $\tilde{\mathcal{D}}$ -classes of $UT_n(\mathfrak{L}^*)$ for $n \geq 3$. In particular, are they determined by tightness and looseness properties of idempotents?

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