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LARGE DEVIATIONS FOR STOCHASTIC NEMATIC LIQUID CRYSTALS DRIVEN BY MULTIPLICATIVE GAUSSIAN NOISE

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Abstract. We study a stochastic two-dimensional nematic liquid crystal model with multiplicative Gaussian noise. We prove the Wentzell-Freidlin type large deviations principle for the small noise asymptotic of solutions using weak convergence method.

1. Introduction

The dissimilarity between the states of matter arises due to the degree and type of ordering, the molecules of the matter display with respect to their neighbours. In between solid and liquid states, there lives an intermediate state, which displays long-range orientational order. Liquid crystals are such phases containing molecules with high shape-anisotropy. At elevated temperatures, the axes of the liquid crystal molecules orient in a random manner. On cooling, the phase to evolve first is the nematic phase. The molecules in this phase exhibit orientational order but have no positional order. On average, the molecules in the nematic phase align parallel to a well-defined spatial direction which is denoted by the unit vector $\mathbf{d}$, known as the director.

To model the dynamics of the nematic liquid crystals, most of the scientists bank on the continuum theory developed by Ericksen [15] and Leslie [18] in the 1960's. Stimulated by this theory, Lin and Liu [19] established the most elementary form of dynamical system representing the motion of nematic liquid crystals. This system can be derived as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = -\lambda\nabla\cdot\big(\nabla\mathbf{d}\odot\nabla\mathbf{d}\big), \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$

$$\frac{\partial \mathbf{d}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{d} = \gamma\left(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\right), \tag{1.3}$$

$$|\mathbf{d}|^2 = 1. \tag{1.4}$$

This holds in $\mathcal{O}_T := (0,T]\times\mathcal{O}$, where $\mathcal{O}\subset\mathbb{R}^2$ and $0<T<\infty$. The vector field $\mathbf{u} : [0,T)\times\mathcal{O}\to\mathbb{R}^2$ denotes the velocity of the fluid, $\mathbf{d} : [0,T)\times\mathcal{O}\to\mathbb{R}^3$ is the director field that represents the macroscopic molecular orientation of the liquid crystal material, $p : [0,T)\times\mathcal{O}\to\mathbb{R}$ denotes the pressure function. The constants μ, λ and γ are positive constants that represent viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time for the molecular orientation field. $\nabla\cdot$ denotes the divergence operator. The

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symbol $\nabla\mathbf{d}\odot\nabla\mathbf{d}$ is the 2×2-matrix with the entries

$$\big[\nabla\mathbf{d}\odot\nabla\mathbf{d}\big]_{i,j}=\sum_{k=1}^{3}\partial_{x_i}\mathbf{d}^{(k)}\partial_{x_j}\mathbf{d}^{(k)},\quad i,j=1,2.$$

We equip the system with the initial and boundary conditions respectively as follows

$$\mathbf{u}(0)=\mathbf{u}_0 \quad \text{with} \quad \nabla\cdot\mathbf{u}_0=0, \quad \text{and} \;\; \mathbf{d}(0)=\mathbf{d}_0, \tag{1.5}$$

$$\mathbf{u}\big|_{\partial\mathcal{O}}=0 \;\; \text{and} \;\; \frac{\partial\mathbf{d}}{\partial n}\Big|_{\partial\mathcal{O}}=0, \tag{1.6}$$

where the vector field n is the outward unit normal vector to $\partial\mathcal{O}$, i.e., $n(x)$ is perpendicular to the tangent space of $\partial\mathcal{O}$ at each point $x\in\mathcal{O}$, of length 1 and facing outside of $\mathcal{O}$.

Although (1.1)-(1.4) is a simpler model than the original system studied by Ericksen and Leslie, it conserves many essential physical properties of the nematic liquid crystals. However, due to the presence of the term $\nabla\cdot\big(\nabla\mathbf{d}\odot\nabla\mathbf{d}\big)$, we have high nonlinearity and due to $\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}=\mathbf{d}\times\big(\Delta\mathbf{d}\times\mathbf{d}\big)$, we have non-parabolicity. Thus the above system of equations forms a fully nonlinear system of partial differential equations with constraints. In 1995, Lin and Liu [19] proposed a corresponding system to subdue the difficulty caused by gradient nonlinearity $|\nabla\mathbf{d}|^2\mathbf{d}$, where they replaced this term by the Ginzburg-Landau bounded function $\chi_{|\mathbf{d}|\le 1}(|\mathbf{d}|^2-1)\mathbf{d}$. In this paper, we consider a general polynomial function $f(\mathbf{d})$ as a substitute for the Ginzburg-Landau function. Our aim is to study the following system:

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\Delta\mathbf{u}+\nabla p=-\lambda\nabla\cdot\big(\nabla\mathbf{d}\odot\nabla\mathbf{d}\big) \;\; \text{in} \;\; (0,T]\times\mathcal{O}, \tag{1.7}$$

$$\nabla\cdot\mathbf{u}=0 \;\; \text{in} \;\; [0,T]\times\mathcal{O}, \tag{1.8}$$

$$\frac{\partial\mathbf{d}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{d}=\gamma\left(\Delta\mathbf{d}-\frac{1}{\delta^2}f(\mathbf{d})\right) \;\; \text{in} \;\; (0,T]\times\mathcal{O}, \tag{1.9}$$

with the same initial and boundary conditions as in (1.5)-(1.6). Here $\delta>0$ is an arbitrary constant. We assume that the initial value of the director field $\mathbf{d}$ satisfies the saturation condition

$$|\mathbf{d}_0(x)|_{\mathbb{R}^3}\equiv 1, \quad \text{for all}\, x\in\mathcal{O}. \tag{1.10}$$

In the present paper, we shall consider a stochastically perturbed nematic liquid crystal system in a two-dimensional bounded region $\mathcal{O}$, with smooth boundary $\partial\mathcal{O}$. The noise is multiplicative Gaussian and is incorporated in the Itô sense in the velocity equation (1.7) and in the Stratonovich sense in the director field equation (1.9). Since $\mathbf{d}$ solves equation (1.9) (for $\delta=1$), this saturation condition is not satisfied for $t>0$. However, our larger aim of this project (which will be addressed in subsequent works) is to study the problem with equation (1.3) as a limit, as $\delta\searrow$, of the Ginzburg-Landau approximations (1.9) and then to show that the saturation condition is also satisfied for all $t>0$. For that purpose the Stratonovich form of the Gaussian noise (see e.g. [3, 4]) (or Marcus canonical form for a more general Lévy noise, see [8]) will prove essential and is the key motivation for considering the director field equation (1.9) to be perturbed by noise in these special forms.

In this work, we aim to establish large deviations principle (LDP) for the small noise asymptotic of solutions of the following system as the parameter $\varepsilon \to 0$.

$$d\mathbf{u}(t) + \big[(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) - \Delta\mathbf{u}(t) + \nabla p\big]dt = -\nabla\cdot\big(\nabla\mathbf{d}(t)\odot\nabla\mathbf{d}(t)\big)dt + \sqrt{\varepsilon}\,\sigma(\mathbf{u}(t))\,dW_1(t), \tag{1.11}$$

$$\nabla\cdot\mathbf{u}(t) = 0, \tag{1.12}$$

$$d\mathbf{d}(t) + \big[(\mathbf{u}(t)\cdot\nabla)\mathbf{d}(t)\big]dt = (\Delta\mathbf{d}(t) - f(\mathbf{d}(t)))\,dt + \sqrt{\varepsilon}\,(\mathbf{d}(t)\times\mathbf{h})\circ dW_2(t). \tag{1.13}$$

Here $\mathbf{h}$ is a given bounded function. $W = (W_1, W_2)$ is an $(\mathrm{H}\times\mathbb{R})$-valued Wiener process. We provide more details about the noise in the next section. The initial, boundary and constraint conditions are respectively given by (1.5), (1.6) and (1.10). Here we have taken, without any loss of generality, $\mu = \lambda = \gamma = \delta = 1$.

We believe, based on the LDP developed in this work, we will be able to study the probability of switching of the director from the vicinity of one stationary solution to the another one and prove that the lower bound of the switching probability is strictly positive (see [2], by the first named author, for similar interesting ideas on magnetisation reversal for the stochastic Landau-Lifshitz-Gilbert equation). This result might give a mathematical justification for a well-known physical phenomena called *Fréedericksz transition.* Thus our current work may be viewed as an important step towards the understanding of director reversal phenomena, which we propose to take further in a subsequent work.

We now provide brief details of relevant literature pertaining to the stochastic liquid crystal model and LDP of other closely related physical models. Brzeźniak *et al.*, in [3], first initiated the study on the Ginzburg-Landau approximation of (1.7)-(1.9) under the effect of fluctuating external forces. In this work, the authors used fixed point argument to prove the existence and uniqueness of local maximal solution in both two and three dimensions. Furthermore, they proved the existence of global strong solution to the problem in two dimensions. In a follow-up paper Brzeźniak *et al.* [4] (see also [5, 6, 7]), the authors considered the same model with multiplicative Gaussian noise and established the existence of global weak solution and proved pathwise uniqueness of the solution in two dimensions. Recently, in [8], authors of this paper have studied the above system (1.7)-(1.9) driven by jump processes and proved existence of weak solutions in two and three dimensions.

There is no work on the LDP for the solutions of stochastic nematic liquid crystal model other than a very recent paper by Zhang and Zhou [26], where noise, in the pure jump form, appears only in the velocity field equation. Although this paper [26] is based on an existence theory developed in our earlier work [8], in our opinion, the study of LDP lacks physical motivation (as described above), and is closely related to the study of LDP of Navier-Stokes equation in terms of technicality (see e.g. [25]). Nematic liquid crystal model is structurally close to other relevant physical models, e.g. Landau-Lifshitz-Gilbert equations, Harmonic map flow etc., and LDP of such models are rarely studied in the literature. The (possibly) only relevant work in this direction is [2], where the first named author of this paper and his collaborators have studied LDP for an one-dimensional stochastic Landau-Lifshitz-Gilbert equations using the weak convergence approach and have further shown that noise can induce magnetisation reversal. Our current work is inspired by [2] and is built on the theory of weak convergence approach to LDP due to Budhiraja and Dupuis [10].

The organisation of this paper is as follows. In Section 2, we will recall basic functional spaces and define some operators and its properties. In Section 3, we will state the assumptions on the general polynomial, noise and its coefficients. In Section 4, we discuss the existence of various forms of solutions and uniqueness. In Section 5, we will first recall the general criteria for a large deviation principle obtained in Budhiraja and Dupuis [10]. Then we introduce two crucial systems, i.e. stochastic control equations and deterministic control equations (termed as skeleton equation) and discuss about their existence results. The proof of the stochastic system is based on the Girsanov theorem and that of the deterministic system is based on the classical Faedo-Galerkin approximations and compactness arguments, and is placed in Appendix A. In Section 6, we state two sufficient conditions for establishing an LDP, which are proved in the next two sections, i.e. in Sections 7 and 8. Finally, we will prove the main result in Section 9.

2. Functional Setting of the Model

2.1. **Basic Definitions and Functional Spaces.** Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\mathcal{O}$. For any $p \in [1,\infty)$ and $k \in \mathbb{N}$, $L^p(\mathcal{O})$ and $\mathrm{W}^{k,p}(\mathcal{O})$ are well-known Lebesgue and Sobolev spaces of $\mathbb{R}$-valued functions respectively. For $p = 2$, put $\mathrm{W}^{k,2} = H^k$.

For instance, $H^1(\mathcal{O};\mathbb{R}^2)$ is the Sobolev space of all $\mathbf{u} \in L^2(\mathcal{O};\mathbb{R}^2)$, for which there exist weak derivatives $\frac{\partial \mathbf{u}}{\partial x_i} \in L^2(\mathcal{O};\mathbb{R}^2), i = 1,2$. It is a Hilbert space with the scalar product given by

$$(\mathbf{u},\mathbf{v})_{H^1} := (\mathbf{u},\mathbf{v})_{L^2} + (\nabla\mathbf{u},\nabla\mathbf{v})_{L^2}, \quad \mathbf{u},\mathbf{v} \in H^1(\mathcal{O},\mathbb{R}^2).$$

Let us define the following spaces

$$\mathcal{V} := \{\mathbf{u} \in \mathcal{C}_c^\infty(\mathcal{O};\mathbb{R}^2) : div\ \mathbf{u} = 0\},$$
$$\mathrm{H} := \text{the closure of } \mathcal{V} \text{ in } L^2(\mathcal{O};\mathbb{R}^2),$$
$$\mathrm{V} := \text{the closure of } \mathcal{V} \text{ in } H^1(\mathcal{O};\mathbb{R}^2).$$

One can use also an equivalent characterisation of these two spaces based on the trace (or Stokes) Theorem [24, Theorem I.1.2], see Theorems I.1.4 and I.1.6 therein.

In the space H we consider the scalar product and the norm inherited from $L^2(\mathcal{O};\mathbb{R}^2)$ and denote them by $(\cdot,\cdot)_{\mathrm{H}}$ and $|\cdot|_{\mathrm{H}}$, respectively, i.e.,

$$(\mathbf{u},\mathbf{v})_{\mathrm{H}} := (\mathbf{u},\mathbf{v})_{L^2}, \qquad |\mathbf{u}|_{\mathrm{H}} := |\mathbf{u}|_{L^2} := |\mathbf{u}|, \qquad \mathbf{u},\mathbf{v} \in \mathrm{H}.$$

In the space V we consider the scalar product inherited from the Sobolev space $H^1(\mathcal{O};\mathbb{R}^2)$ i.e.,

$$(\mathbf{u},\mathbf{v})_{\mathrm{V}} := (\mathbf{u},\mathbf{v})_{L^2} + ((\mathbf{u},\mathbf{v})),$$

where

$$((\mathbf{u},\mathbf{v})) := (\nabla\mathbf{u},\nabla\mathbf{v})_{L^2} = \sum_{i=1}^{2} \int_{\mathcal{O}} \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{v}}{\partial x_i}\, dx, \qquad \mathbf{u},\mathbf{v} \in \mathrm{V}. \tag{2.1}$$

and the norm

$$|\mathbf{u}|_{\mathrm{V}}^2 := |\mathbf{u}|_{\mathrm{H}}^2 + \|\mathbf{u}\|^2,$$

where

$$\|\mathbf{u}\|^2 := |\nabla\mathbf{u}|_{L^2}^2. \tag{2.2}$$

Note that since $\mathcal{O}$ is a bounded domain, the Poincaré inequality holds on it, and therefore the norms $|\cdot|_{\mathrm{V}}$ and $\|\cdot\|$ are equivalent (on V).

It is also known that V is dense in H and the embedding is continuous. We have

$$\mathrm{V} \hookrightarrow \mathrm{H} \cong \mathrm{H}' \hookrightarrow \mathrm{V}'.$$

The above spaces are the most used spaces to describe the fluid's velocity. To describe the fluid's director field, we will use spaces

$$L^2 := L^2(\mathcal{O}, \mathbb{R}^3), \quad H^1 := H^1(\mathcal{O}, \mathbb{R}^3) \text{ and } H^2 := H^2(\mathcal{O}, \mathbb{R}^3).$$

Note that elements of these spaces take values in the three-dimensional Euclidean space $\mathbb{R}^3$, irrespective of the spatial dimension, which is 2 in this work.

2.2. **Bilinear Operators.** Let us consider the following trilinear form, see Temam [24],

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \mathbf{u}^{(i)} \partial_{x_i} \mathbf{v}^{(j)} \mathbf{w}^j \, dx, \quad \mathbf{u} \in L^p, \mathbf{v} \in W^{1,q}, \mathbf{w} \in L^r, \tag{2.3}$$

where $p, q, r \in [1, \infty]$ satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1. \tag{2.4}$$

We will recall the fundamental properties of the form b that are valid for both bounded and unbounded domains. By the Sobolev embedding Theorem, see Adams [1], and the Hölder inequality, there exists a positive constant c such that

$$|b(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq c |\mathbf{u}|_{\mathrm{V}} |\mathbf{w}|_{\mathrm{V}} |\mathbf{v}|_{\mathrm{V}}, \qquad \mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathrm{V}.$$

The form b is continuous on V. In particular, we define a bilinear map B by $B(\mathbf{u}, \mathbf{w}) := b(\mathbf{u}, \mathbf{w}, \cdot)$, then we infer that $B(\mathbf{u}, \mathbf{w}) \in \mathrm{V}'$ for all $\mathbf{u}, \mathbf{w} \in \mathrm{V}$ and the following inequality holds

$$|B(\mathbf{u}, \mathbf{w})|_{\mathrm{V}'} \leq c \, |\mathbf{u}|_{\mathrm{V}} |\mathbf{w}|_{\mathrm{V}}, \qquad \mathbf{u}, \mathbf{w} \in \mathrm{V}.$$

Moreover, the mapping $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$ is bilinear and continuous. The form b also has the following properties, see [24],

$$b(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -b(\mathbf{u}, \mathbf{v}, \mathbf{w}), \qquad \mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathrm{V}.$$

In particular,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \qquad \mathbf{u}, \mathbf{v} \in \mathrm{V}.$$

Hence

$$\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v} \rangle = -\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle, \quad \mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathrm{V}$$

and

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0, \qquad \mathbf{u}, \mathbf{v} \in \mathrm{V}.$$

Moreover, for all $\mathbf{u} \in \mathrm{V}, \mathbf{v} \in H^1$, using the notation (2.2), we have

$$|B(\mathbf{u}, \mathbf{v})|_{\mathrm{V}'} \leq c \, |\mathbf{u}|_{\mathrm{H}}^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{1}{2}} |\mathbf{v}|_{L^2}^{\frac{1}{2}} |\nabla \mathbf{v}|_{L^2}^{\frac{1}{2}}. \tag{2.5}$$

For the proof, we refer to Section 1.2 of Temam [24].

We will use the following notation, $B(\mathbf{u}) := B(\mathbf{u},\mathbf{u})$. Also note that the map $B : \mathrm{V} \to \mathrm{V}'$ is Lipschitz continuous on balls.
One can define a bilinear map $\tilde{B}$ defined on $\mathrm{V} \times H^1$ with values in $(H^1)'$ such that [1]

$$\langle \tilde{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) \qquad \mathbf{u},\mathbf{v},\mathbf{w} \in H^1$$

With an abuse of notation, we again denote by $\tilde{B}(\cdot,\cdot)$ the restriction of $\tilde{B}(\cdot,\cdot)$ to $\mathrm{V} \times H^2$, which maps continuously $\mathrm{V} \times H^2$ into L^2. Using the Gagliardo-Nirenberg inequalities one can show there exists a positive constant C such that

$$\left|\tilde{B}(\mathbf{u},\mathbf{d})\right| \le C\, |\mathbf{u}|_{\mathrm{H}}^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}|\nabla \mathbf{d}|_{L^2}^{\frac{1}{2}}|\Delta \mathbf{d}|_{L^2}^{\frac{1}{2}}, \qquad \mathbf{u} \in \mathrm{V}, \mathbf{d} \in H^2. \tag{2.6}$$

Moreover, using Young's inequality one can get

$$\left|\tilde{B}(\mathbf{u},\mathbf{d})\right|_{L^2} \le C\, \|\mathbf{u}\||\mathbf{d}|_{H^2},$$

We also have

$$\langle \tilde{B}(\mathbf{u},\mathbf{d}),\mathbf{d}\rangle = 0, \qquad \mathbf{u} \in \mathrm{V}, \mathbf{d} \in H^2.$$

For the proof, we refer to Section 1.2 of Temam[24].

Let m be the trilinear form defined by

$$m(\mathbf{d}_1,\mathbf{d}_2,\mathbf{u}) = -\sum_{i,j=1}^{2}\sum_{k=1}^{3}\int_{\mathcal{O}} \partial_{x_i}\mathbf{d}_1^{(k)}\partial_{x_j}\mathbf{d}_2^{(k)}\partial_{x_j}\mathbf{u}^{(i)}\,dx, \quad \mathbf{d}_1 \in W^{1,p}, \mathbf{d}_2 \in W^{1,q}, \mathbf{u} \in W^{1,r},$$

with $p,q,r \in (1,\infty)$ satisfying condition (2.4). The above integral is well defined when $\mathbf{d}_1,\mathbf{d}_2 \in H^2$ and $\mathbf{u} \in \mathrm{V}$. We also have the following Lemma, where we use the notation (2.2).

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$|m(\mathbf{d}_1,\mathbf{d}_2,\mathbf{u})|_{L^2} \le C\, |\nabla \mathbf{d}_1|_{L^2}^{\frac{1}{2}}|\nabla^2 \mathbf{d}_1|_{L^2}^{\frac{1}{2}}|\nabla \mathbf{d}_2|_{L^2}^{\frac{1}{2}}|\nabla^2 \mathbf{d}_2|_{L^2}^{\frac{1}{2}}\|\mathbf{u}\|, \quad \mathbf{d}_1,\mathbf{d}_2 \in H^2, \mathbf{u} \in \mathrm{V}.$$

For proof see [4]. Now we state the following Lemma.

Lemma 2.2. *There exists a bilinear operator $M : H^2 \times H^2 \to \mathrm{V}'$ such that*

$$\left\langle M(\mathbf{d}_1,\mathbf{d}_2),\mathbf{u}\right\rangle = m(\mathbf{d}_1,\mathbf{d}_2,\mathbf{u}), \quad \mathbf{d}_1,\mathbf{d}_2 \in H^2, \mathbf{u} \in \mathrm{V}.$$

Furthermore, there exists $C > 0$ such that

$$|M(\mathbf{d}_1,\mathbf{d}_2)|_{\mathrm{V}'} \le C\, |\nabla \mathbf{d}_1|_{L^2}^{\frac{1}{2}}|\nabla^2 \mathbf{d}_1|_{L^2}^{\frac{1}{2}}|\nabla \mathbf{d}_2|_{L^2}^{\frac{1}{2}}|\nabla^2 \mathbf{d}_2|_{L^2}^{\frac{1}{2}}, \quad \mathbf{d}_1,\mathbf{d}_2 \in H^2. \tag{2.7}$$

For a proof we refer to [4]. We will use the following notation, $M(\mathbf{d}) := M(\mathbf{d},\mathbf{d})$.

2.3. **Linear Operators, Its Properties and Important Embeddings.** Now we will recall operators and their properties used in [9]. Consider the natural embedding $j : \mathrm{V} \hookrightarrow \mathrm{H}$ and its adjoint $j' : \mathrm{H} \hookrightarrow \mathrm{V}$. Since the range of j is dense in H, the map j' is one-to-one. Let us put

$$\mathscr{A}\mathbf{u} := ((\mathbf{u},\cdot)), \qquad \mathbf{u} \in \mathrm{V},$$

where $((\cdot,\cdot))$ is defined in (2.1). If $\mathbf{u} \in \mathrm{V}$, then $\mathscr{A}\mathbf{u} \in \mathrm{V}'$. Since we have the following inequalities

$$|((\mathbf{u},\mathbf{w}))| \le \|\mathbf{u}\| \cdot \|\mathbf{w}\| \le \|\mathbf{u}\|(\|\mathbf{w}\|^2 + |\mathbf{w}|_{\mathrm{H}}^2)^{\frac{1}{2}} = \|\mathbf{u}\| \cdot |\mathbf{w}|_{\mathrm{V}}, \quad \mathbf{w} \in \mathrm{V}.$$

[1] To be precise, the form b should be replaced by a form $\tilde{b}$ defined by formula (2.3) but for $\mathbb{R}^2$-valued vector field $\mathbf{u}$, and $\mathbb{R}^3$-valued vector fields $\mathbf{v}$ and $\mathbf{w}$.

we infer that

$$|\mathscr{A}\mathbf{u}|_{\mathrm{V}'} \leq \|\mathbf{u}\|, \qquad \mathbf{u} \in \mathrm{V}.$$

The Neumann Laplacian acting on $\mathbb{R}^3$-valued function $\mathbf{d}$ will be denoted by $\mathcal{A}$, i.e.,

$$D(\mathcal{A}) := \left\{ \mathbf{d} \in H^2 : \frac{\partial \mathbf{d}}{\partial n} = 0 \ on \ \partial \mathcal{O} \right\},$$

$$\mathcal{A}\mathbf{d} := -\sum_{i=1}^{2} \frac{\partial^2 \mathbf{d}}{\partial x_i^2}, \quad \mathbf{d} \in D(\mathcal{A}).$$

It is known that $\mathcal{A}$ is a non-negative self-adjoint operator in L^2. As we are working on a bounded domain, $\mathcal{A}$ has compact resolvent.

2.4. **Spaces involving time.** Let $1 \leq p \leq \infty$. Let us define the space

$$L^p([0,T]; X),$$

consisting of all measurable functions $\phi : [0,T] \to X$ with

$$|\phi|_{L^p([0,T];X)} := \Big(\int_0^T |\phi(t)|_X^p \, dt \Big)^{1/p} < \infty,$$

for $1 \leq p < \infty$ and for $p = \infty$ we define the norm

$$|\phi|_{L^\infty([0,T];X)} := \operatorname*{ess\,sup}_{0 \leq t \leq T} |\phi(t)|_X < \infty.$$

Let m be a positive integer and $1 \leq p < \infty$. We define the Sobolev space

$$\mathrm{W}^{m,p}([0,T]; X) := \big\{ \phi \in L^p([0,T]; X) \mid D^\alpha \phi \in L^p([0,T]; X), \text{ for all } |\alpha| \leq m \big\}.$$

We endow it with the norm

$$|\phi|_{\mathrm{W}^{m,p}([0,T];X)} = \Bigg(\sum_{|\alpha| \leq m} |D^\alpha \phi|^p_{L^p([0,T];X)} \Bigg)^{1/p}.$$

We define the space

$$\mathbb{C}([0,T]; X)$$

comprises all continuous functions $\phi : [0,T] \to X$ with

$$|\phi|_{\mathbb{C}([0,T];X)} := \max_{0 \leq t \leq T} |\phi(t)|_X < \infty.$$

The following result is similar to the compactness criteria proved in Temam [24], Section 13.3 and Lions [20], Section 5, Chapter I.

Lemma 2.3. *Let $X \subset Y \subset Z$ be Banach spaces, X and Z reflexive, with compact embedding of X in Y. Let $p \in (1, \infty)$ and $k \in (0,1)$ be given. Let Λ be the space*

$$\Lambda := L^p([0,T]; X) \cap \mathrm{W}^{k,p}([0,T]; Z)$$

endowed with the natural norm. Then the embedding of Λ in $L^p([0,T]; Y)$ is compact.

3. Hypotheses

3.1. **Assumption on the noise and its coefficients.** Let W_2 be $\mathbb{R}$-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and W_1 is an H-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the RKHS (i.e. Cameron-Martin space) H_0, where H_0 is a Hilbert space such that $\mathrm{H}_0 \hookrightarrow \mathrm{H}$ and the natural embedding $i : \mathrm{H}_0 \to \mathrm{H}$ is Hilbert-Schmidt. We assume that W_1 and W_2 are independent.

Denote by $\mathrm{L}_2 = \mathrm{L}_2(\mathrm{H}_0, \mathrm{H})$ be the space of linear operators S which are Hilbert-Schmidt operator from H_0 to H. The norm in the space L_2 is defined by $|S|_{\mathrm{L}_2}$. The noise intensity $\sigma : [0,T] \times \mathrm{H} \to \mathrm{L}_2(\mathrm{H}_0, \mathrm{H})$ is assumed to satisfy the following:

(1) There exists positive constant K such that
$$\left|\sigma(t,\mathbf{u})\right|^2_{\mathrm{L}_2} \le K(1 + |\mathbf{u}|^2_{\mathrm{H}}), \quad \forall\, t \in [0,T], \ \forall\, \mathbf{u} \in \mathrm{H}.$$

(2) There exists positive constant L such that
$$\left|\sigma(t,\mathbf{u}_1) - \sigma(t,\mathbf{u}_2)\right|^2_{\mathrm{L}_2} \le L\,|\mathbf{u}_1 - \mathbf{u}_2|^2_{\mathrm{H}}, \quad \forall\, t \in [0,T], \ \forall\, \mathbf{u}_1, \mathbf{u}_2 \in \mathrm{H}.$$

To ease the notation, we will suppose that $\sigma(t,\mathbf{u}) := \sigma(\mathbf{u})$.
We assume $\mathbf{h} \in L^\infty$ is fixed. We define a bounded linear operator G from L^2 into itself by
$$G : L^2 \ni \mathbf{d} \mapsto \mathbf{d} \times \mathbf{h} \in L^2.$$

It is straight forward to check that
(1) $|G(\mathbf{d})|_{L^2} \le |\mathbf{h}|_{L^\infty} |\mathbf{d}|_{L^2}$.
(2) $|G(\mathbf{d}_1) - G(\mathbf{d}_2)|_{L^2} \le |\mathbf{h}|_{L^\infty} |\mathbf{d}_1 - \mathbf{d}_2|_{L^2}$.

We have the following form between Stratonovich and Itô's integrals
$$G(\mathbf{d}) \circ dW_2 = \frac{1}{2} G^2(\mathbf{d})\, dt + G(\mathbf{d})\, dW_2,$$
where $G^2 = G \circ G$ and defined by
$$G^2(\mathbf{d}) = G \circ G(\mathbf{d}) = (\mathbf{d} \times \mathbf{h}) \times \mathbf{h}, \text{ for any } \mathbf{d} \in L^2.$$

3.2. **Assumption on the general polynomial.** Let $N \in \mathbb{N} := \{1, 2, 3, \cdots\}$. Let us assume that a function $\tilde{F} : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree $N+1$ with the leading coefficient a_{N+1} being strictly positive and such that $\tilde{F}(0) = 0$. We put $\tilde{f}(r) = \tilde{F}'(r)$, $r \in \mathbb{R}$. We define a map $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ by
$$f(\mathbf{d}) = \tilde{f}(|\mathbf{d}|^2)\mathbf{d}, \quad \mathbf{d} \in \mathbb{R}^3. \tag{3.1}$$
Let $\mathrm{F} : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by
$$\mathrm{F}(\mathbf{d}) = \frac{1}{2} \tilde{F}(|\mathbf{d}|^2), \quad \mathbf{d} \in \mathbb{R}^3,$$
Let us note that F Fréchét differentiable and that for any $\mathbf{d} \in \mathbb{R}^3$ and $\mathbf{g} \in \mathbb{R}^3$
$$\mathrm{F}'(\mathbf{d})[\mathbf{g}] = f(\mathbf{d}) \cdot \mathbf{g}$$

Remark 3.1. *Let the polynomial function f be defined as in* (3.1)*.*

(1) *There exist constants $l_1, l_2 > 0$ such that*
$$|\tilde{f}(r)| \le l_1(1 + r^N) \ \text{ and } \ |\tilde{f}'(r)| \le l_2(1 + r^{N-1}), \quad r > 0.$$

(2) *There exist constants $c, \tilde{c} > 0$ such that*
$$|f(\mathbf{d})|_{\mathbb{R}^3} \le c\, \left(1 + \left|\mathbf{d}\right|^{2N+1}_{\mathbb{R}^3}\right) \ \text{ and } \ |f'(\mathbf{d})|_{\mathbb{R}^3} \le \tilde{c}\, \left(1 + \left|\mathbf{d}\right|^{2N}_{\mathbb{R}^3}\right), \quad \mathbf{d} \in \mathbb{R}^3.$$

(3) *Since* $H^1 \subset L^{4N+2}$ *for any* $N \in \mathbb{N}$. *From previous results we infer that* $\mathbf{d} \in H^2 \subset L^\infty$ *whenever* $\mathbf{d} \in H^1$ *and* $f(\mathbf{d}) + \mathcal{A}\mathbf{d} \in L^2$.

Now, using the operators introduced in previous sections, we can rewrite the system (1.11)-(1.13) as

$$d\mathbf{u}(t) + \big[\mathscr{A}\mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) + M(\mathbf{d}(t))\big]\, dt = \sqrt{\varepsilon}\, \sigma(\mathbf{u}(t))\, dW_1(t), \tag{3.2}$$

$$d\mathbf{d}(t) + \big[\mathcal{A}\mathbf{d}(t) + \tilde{B}(\mathbf{u}(t), \mathbf{d}(t)) + f(\mathbf{d}(t))\big]\, dt + \sqrt{\varepsilon}\, G(\mathbf{d}(t)) \circ dW_2(t). \tag{3.3}$$

4. The Existence of Solutions

We will consider the following stochastic integral equation form of the problem (3.2)-(3.3):

$$\mathbf{u}(t) = \mathbf{u}_0 - \int_0^t \big[\mathscr{A}\mathbf{u}(s) + B(\mathbf{u}(s), \mathbf{u}(s)) + M(\mathbf{d}(s))\big]\, ds + \sqrt{\varepsilon} \int_0^t \sigma(\mathbf{u}(s))\, dW_1(s), \tag{4.1}$$

$$\begin{aligned} \mathbf{d}(t) = \mathbf{d}_0 - \int_0^t \big[\mathcal{A}\mathbf{d}(s) + \tilde{B}(\mathbf{u}(s), \mathbf{d}(s)) + f(\mathbf{d}(s))\big] ds + \frac{\varepsilon}{2} \int_0^t G^2(\mathbf{d}(s))\, ds \\ + \sqrt{\varepsilon} \int_0^t G(\mathbf{d}(s))\, dW_2(s). \end{aligned} \tag{4.2}$$

In this section, we will provide the existence results of the problem (4.1)-(4.2).

Definition 4.1. *Let the assumptions stated earlier hold. A* **weak martingale solution** *to* (4.1)-(4.2) *is a system*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \mathbf{u}, \mathbf{d})$$

consisting of a filtered probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$, *of a canonical* $\mathrm{H}_0 \times \mathbb{R}$*-cylindrical Wiener process* $W = (W_1(t), W_2(t))_{t\in[0,T]}$ *and* $\mathrm{V} \times H^2$*-valued progressively measurable processes* $(\mathbf{u}(t), \mathbf{d}(t))_{t\in[0,T]}$ *such that:*

(1) $(\mathbf{u}, \mathbf{d})$ *is* $\mathrm{H} \times H^1$*-valued continuous processes with*

$$\mathbb{E} \sup_{s\in[0,T]} \Big[|\mathbf{u}(s)|_{\mathrm{H}} + |\nabla \mathbf{d}(s)|_{L^2}\Big] + \mathbb{E} \int_0^T \Big(\big|\nabla \mathbf{u}(s)\big|_{L^2}^2 + |\mathcal{A}\mathbf{d}(s)\big|_{L^2}^2\Big)\, ds < \infty,$$

(2) *for each* $(\varphi, \psi) \in \mathrm{V} \times L^2$ *we have, for all* $t \in [0, T]$, $\mathbb{P}$*-a.s.*

$$\begin{aligned} &\langle \mathbf{u}(t) - \mathbf{u}_0, \varphi\rangle + \int_0^t \big\langle \mathscr{A}\mathbf{u}(s) + B(\mathbf{u}(s), \mathbf{u}(s)) + M(\mathbf{d}(s)), \varphi\big\rangle_{\mathrm{V}',\mathrm{V}}\, ds \\ &= \sqrt{\varepsilon} \int_0^t \langle \sigma(\mathbf{u}(s)), \varphi\rangle\, dW_1(s), \end{aligned}$$

and

$$\begin{aligned} &\langle \mathbf{d}(t) - \mathbf{d}_0, \psi\rangle + \int_0^t \big\langle \mathcal{A}\mathbf{d}(s) + \tilde{B}(\mathbf{u}(s), \mathbf{d}(s)) + f(\mathbf{d}(s)) - \frac{\varepsilon}{2}\, G^2(\mathbf{d}(s)), \psi\big\rangle_{L^2,L^2}\, ds \\ &= \sqrt{\varepsilon} \int_0^t \langle G(\mathbf{d}(s)), \psi\rangle\, dW_2(s). \end{aligned}$$

Theorem 4.2. (Existence of weak martingale solution) *Let the assumptions stated earlier hold. Assume that* $(\mathbf{u}_0, \mathbf{d}_0) \in \mathrm{H} \times H^1$ *and* $\mathbf{h} \in \mathrm{W}^{1,3} \cap L^\infty$. *Then the system* (4.1)-(4.2) *has weak martingale solution in the sense of Definition 4.1 in two dimensions.*

Proof. We refer to Brzeźniak *et al.* [4] for the proof. It has been proved for $\varepsilon = 1$ there in Subsections 3.1-3.3. $\square$

Theorem 4.3. (Pathwise Uniqueness) *Assume that* $(\mathbf{u}_i(t), \mathbf{d}_i(t)), i = 1, 2$ *are two weak martingale solutions of* (4.1)-(4.2) *defined on the same stochastic system* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W_1, W_2)$ *and with the same initial condition* $(\mathbf{u}_0, \mathbf{d}_0) \in \mathrm{H} \times H^1$. *Then for any* $t \in (0, T]$, $\mathbb{P}$-*a.s.*

$$(\mathbf{u}_1(t), \mathbf{d}_1(t)) = (\mathbf{u}_2(t), \mathbf{d}_2(t)).$$

Proof. For proof see Subsection 3.4 in [4]. $\square$

In what follows we will denote by $\mathcal{K}_T$ the Banach space

$$\mathcal{K}_T := \mathcal{K}_{1,T} \times \mathcal{K}_{2,T} := \mathbb{C}([0, T]; \mathrm{H}) \cap L^2([0, T]; \mathrm{V}) \times \mathbb{C}([0, T]; H^1) \cap L^2([0, T]; D(\mathcal{A})).$$

We will need the Banach space

$$\mathcal{Y}_T := {}_0\mathbb{C}([0, T]; \mathrm{H}) \times {}_0\mathbb{C}([0, T]; \mathbb{R}),$$

where

$${}_0\mathbb{C}([0, T]; \mathrm{H}) := \{\omega \in \mathbb{C}([0, T]; \mathrm{H}) : \omega(0) = 0\},$$

endowed with the standard sup norm. Similarly we define ${}_0\mathbb{C}([0, T]; \mathbb{R})$.

The pathwise uniqueness and the existence of weak solutions imply uniqueness in law and the existence of a strong solution (See infinite-dimensional version of the Yamada and Watanabe Theorem in Ondreját [22], Theorem 12.1 and 13.2), see also Theorem 5.6 in [2].

Theorem 4.4. *Let the assumptions of Theorem 4.2 hold and let* $\varepsilon > 0$. *Then uniqueness in law and the existence of a strong solution hold for the system* (4.1)-(4.2) *in the following sense.*

(1) If $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \mathbf{u}, \mathbf{d})$ *and* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{u}}, \tilde{\mathbf{d}})$ *are two weak martingale solutions to problem* (4.1)-(4.2) *such that* $(\mathbf{u}, \mathbf{d})$ *and* $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}})$ *are* $\mathcal{K}_T$*-valued random variables, then* $(\mathbf{u}, \mathbf{d})$ *and* $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}})$ *have the same laws on* $\mathcal{K}_T$.

(2) There exists a Borel measurable function $\mathcal{J}^\varepsilon : \mathcal{Y}_T \to \mathcal{K}_T$ *such that the following holds. If* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is an arbitrary filtered probability space with filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $W = (W_1(t), W_2(t))_{t\in[0,T]}$ *is an arbitrary* $(\mathrm{H} \times \mathbb{R})$*-valued Wiener processes on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *and* $X^\varepsilon := (\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon) = \mathcal{J}^\varepsilon \circ W$, *i.e.*

$$X^\varepsilon : \Omega \ni \omega \mapsto \mathcal{J}^\varepsilon(W(\omega)) \in \mathcal{K}_T,$$

then the system $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, X^\varepsilon)$ *is a weak martingale solution to problem* (4.1)-(4.2).

5. The Large Deviation Principle

5.1. Basic Definitions and Properties.

Definition 5.1. *Let Z be a Polish space. A function $I : Z \to [0, \infty]$ is called a* **rate function** *if I is lower semicontinuous. A rate function I is a* **good rate function** *if for arbitrary $M \in [0, \infty)$, the level set $K_M = \{x : I(x) \leq M\}$ is compact in Z.*

Definition 5.2. *We say that a family of probability measures $\{\mathbb{P}_\varepsilon : \varepsilon > 0\}$ satisfies the* **large deviation principle** (*LDP*) *on Z with a good rate function $I : Z \to [0, \infty]$ satisfying,*

(1) for each closed set $F_1 \subset Z$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_\varepsilon(F_1) \leq -\inf_{x\in F_1} I(x),$$

(2) for each open set $F_2 \subset Z$

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_\varepsilon(F_2) \geq - \inf_{x\in F_2} I(x).$$

We consider the following random perturbations of nematic liquid crystals system

$$d\mathbf{u}^\varepsilon(t) + \big[\mathscr{A}\mathbf{u}^\varepsilon(t) + B(\mathbf{u}^\varepsilon(t)) + M(\mathbf{d}^\varepsilon(t))\big]\, dt = \sqrt{\varepsilon}\,\sigma(\mathbf{u}^\varepsilon(t))\, dW_1(t), \tag{5.1}$$

$$d\mathbf{d}^\varepsilon(t) = -\big[\mathcal{A}\mathbf{d}^\varepsilon(t) + \tilde{B}(\mathbf{u}^\varepsilon(t), \mathbf{d}^\varepsilon(t)) + f(\mathbf{d}^\varepsilon(t))\big]dt + \sqrt{\varepsilon}\, G(\mathbf{d}^\varepsilon(t)) \circ dW_2(t), \tag{5.2}$$

with initial and boundary conditions as below

$$\mathbf{u}^\varepsilon(0) = \mathbf{u}_0 \in \mathrm{H} \ \text{ and } \ \mathbf{d}^\varepsilon(0) = \mathbf{d}_0 \in H^1,$$
$$\mathbf{u}^\varepsilon = 0 \ \text{ and } \ \frac{\partial \mathbf{d}^\varepsilon}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{O}.$$

In this section we will construct a large deviation principle for the family of laws of the solutions $(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ of equations (5.1)-(5.2) with parameter $\varepsilon \in (0, 1]$ tending to zero. The main result of our paper is as follows:

Theorem 5.3. *The family of laws $\{\mathscr{L}(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon) : \varepsilon \in (0, 1]\}$ on $\mathcal{K}_T$ satisfies the large deviation principle with rate function I defined below in* (5.12)*.*

Before the proof of the above main result, we will provide some required background. In particular, we will formulate important Lemmata 6.1 and 6.2. In order to prove Theorem 5.3, we formulate two conditions which are sufficient to establish an LDP for the system (5.1)-(5.2). At first we will formulate these two conditions, which will be aftermath of Lemmata 6.1 and 6.2.
Recall the space

$$\mathcal{Y}_T := {}_0\mathbb{C}([0, T]; \mathrm{H}) \times {}_0\mathbb{C}([0, T]; \mathbb{R}).$$

Let $\big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}\big)$, be the classical Wiener space, i.e.

$$\begin{aligned}
\Omega \ &= \ \mathcal{Y}_T \\
\mathbb{P} \ & \quad \text{is the Wiener measure on } \Omega, \\
W \ &= \ \big(W(t)\big)_{t\in[0,T]} \text{ is the canonical } \mathrm{H}\times\mathbb{R}\text{-valued Wiener process on } (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \\
\mathbb{F} \ &= \ \big(\mathcal{F}_t\big)_{t\in[0,T]} \text{ is the } \mathbb{P}\text{-completion of the natural filtration } \mathbb{F}^0 = \big(\mathcal{F}^0_t\big)_{t\in[0,T]} \text{ generated by } W.
\end{aligned}$$

By part(2) of Theorem 4.4 for every $\varepsilon > 0$ there exists a Borel map

$$J^\varepsilon : \mathcal{Y}_T \to \mathcal{K}_T$$

such that a system $\Big(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, M^\varepsilon\Big)$, with $M^\varepsilon = J^\varepsilon \circ W$, is a weak martingale solution to problem (5.1)-(5.2).
By $\mathbb{E}$ we will denote the integration with respect to the measure $\mathbb{P}$.
We will denote by $\mathcal{U}$ the set of all $\mathrm{H}_0 \times \mathbb{R}$-valued $\mathbb{F}$-predictable process

$$q = (\theta, \rho) : [0, T] \times \mathcal{Y}_T \to \mathrm{H}_0 \times \mathbb{R}$$

satisfying the following condition

$$\begin{aligned}\|q\|_T^2 &= \operatorname*{ess\,sup}_{\omega\in\Omega} \int\limits_0^T |q(t,\omega)|^2_{\mathrm{H}_0\times\mathbb{R}}\,dt \\ &= \operatorname*{ess\,sup}_{\omega\in\Omega} \int_0^T \left(\left|\theta(t,\omega)\right|^2_{\mathrm{H}_0} + \left|\rho(t,\omega)\right|^2_{\mathbb{R}}\right) dt < \infty.\end{aligned}$$

Recall $\mathcal{K}_T$, the Banach space defined by

$$\mathcal{K}_T := \mathcal{K}_{1,T}\times\mathcal{K}_{2,T} := \mathbb{C}([0,T];\mathrm{H})\cap L^2([0,T];\mathrm{V})\times\mathbb{C}([0,T];H^1)\cap L^2([0,T];D(\mathcal{A})).$$

5.2. **A General Criteria.** In this subsection, we follow the criteria for a large deviation principle established in Budhiraja and Dupuis [10]. Let $\{\mathcal{J}^\varepsilon\}_{\varepsilon>0}$ be a family of measurable maps from $\mathcal{Y}_T$ to $\mathcal{K}_T$. In Section 6, we suggest sufficient conditions for LDP to hold for the family $\mathcal{J}^\varepsilon(\sqrt{\varepsilon}\,W)$ as $\varepsilon\to 0$,

For $\alpha\in\mathbb{N}$ define,

$$S_1^\alpha := \left\{\theta\in L^2([0,T];\mathrm{H}_0) : \int_0^T \left|\theta(t)\right|^2_{\mathrm{H}_0} dt \le \alpha\right\},$$

$$S_2^\alpha := \left\{\rho\in L^2([0,T];\mathbb{R}) : \int_0^T \left|\rho(t)\right|^2_{\mathbb{R}} dt \le \alpha\right\}.$$

and set $S^\alpha = S_1^\alpha\times S_2^\alpha$. On S^α we consider the topology induced by the weak topology on the Hilbert space $L^2(0,T;\mathrm{H}_0\times\mathbb{R})$. Note that this topology is metrizable.

Define $\mathbb{S} := \bigcup_{\alpha\ge1} S^\alpha$, and let

$$\mathcal{U}^\alpha = \left\{q = (\theta,\rho)\in\mathcal{U} : q(\omega)\in S^\alpha \text{ for } \mathbb{P}-\text{a.e. } \omega\in\Omega\right\}. \tag{5.3}$$

Let us note that $\mathcal{U} = \bigcup_{\alpha\ge1}\mathcal{U}^\alpha$.

5.3. **Stochastic Control Equation.** In this section we introduce the stochastic control equation. Given $(\theta,\rho)\in\mathcal{U}$, we consider the following system:

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u}_0 &- \int_0^t \left[\mathscr{A}\mathbf{u}(s) + B(\mathbf{u}(s),\mathbf{u}(s)) + M(\mathbf{d}(s),\mathbf{d}(s))\right] ds + \int_0^t \sigma(\mathbf{u}(s))\,\theta(s)\,ds \\ &+ \sqrt{\varepsilon}\int_0^t \sigma(\mathbf{u}(s))\,dW_1(s), \qquad t\in[0,T],\end{aligned} \tag{5.4}$$

$$\begin{aligned}\mathbf{d}(t) = \mathbf{d}_0 &- \int_0^t \left[\mathcal{A}\mathbf{d}(s) + \tilde{B}(\mathbf{u}(s),\mathbf{d}(s)) + f(\mathbf{d}(s))\right] ds + \frac{\varepsilon}{2}\int_0^t G^2(\mathbf{d}(s))\,ds \\ &+ \int_0^t G(\mathbf{d}(s))\,\rho(s)\,ds + \sqrt{\varepsilon}\int_0^t G(\mathbf{d}(s))\,dW_2(s), \qquad t\in[0,T].\end{aligned} \tag{5.5}$$

We show existence and uniqueness of this system using the Girsanov Theorem (see [13]).

Theorem 5.4. *Assume that $(\theta,\rho)\in\mathcal{U}$ and $\varepsilon\in(0,1]$. Then there exists a process $\tilde{X} := (\tilde{\mathbf{u}},\tilde{\mathbf{d}}) = (\mathbf{u}^{\theta,\varepsilon},\mathbf{d}^{\rho,\varepsilon})$ such that the system $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W,\tilde{X})$ is a weak martingale solution of* *the problem* (5.4)-(5.5) *and satisfy*

$$\mathbb{E}\sup_{t\in[0,T]}\Big[\big|\tilde{\mathbf{u}}(t)\big|_{\mathrm{H}}^{p}+\big|\tilde{\mathbf{d}}(t)\big|_{H^1}^{p}\Big]<\infty,$$

$$\mathbb{E}\bigg[\int_0^T\big|\tilde{\mathbf{u}}(t)\big|_{\mathrm{V}}^{2}\,dt+\int_0^T\big|\tilde{\mathbf{d}}(t)\big|_{D(\mathcal{A})}^{2}\,dt\bigg]^{p}<\infty.$$

Proof. **Existence:**
Let us fix $\varepsilon>0$. For any $q:=(\theta,\rho)\in\mathcal{U}$, let us put

$$\gamma_q:=(\gamma_\theta,\gamma_\rho),$$

where

$$\gamma_\theta=\exp\bigg(-\frac{1}{\sqrt{\varepsilon}}\int_0^T\theta(s)\,dW_1(s)-\frac{1}{2\varepsilon}\int_0^T\big|\theta(s)\big|_{\mathrm{H}_0}^{2}\,ds\bigg),$$
$$\gamma_\rho=\exp\bigg(-\frac{1}{\sqrt{\varepsilon}}\int_0^T\rho(s)\,dW_2(s)-\frac{1}{2\varepsilon}\int_0^T\big|\rho(s)\big|_{\mathbb{R}}^{2}\,ds\bigg)$$

and we put

$$W_q(t):=(W_\theta(t),W_\rho(t)),$$

where

$$W_\theta(t)=W_1(t)+\frac{1}{\sqrt{\varepsilon}}\int_0^t\theta(s)\,ds,$$
$$W_\rho(t)=W_2(t)+\frac{1}{\sqrt{\varepsilon}}\int_0^t\rho(s)\,ds.$$

Since $q\in\mathcal{U}$, we observe that,

$$\mathbb{E}(\gamma_q)^{-1}<\infty. \tag{5.6}$$

Therefore there exists a probability measure $\mathbb{P}_q$ on $\mathcal{F}_T$ such that

$$\frac{d\,\mathbb{P}_q}{d\,\mathbb{P}}=\gamma_q.$$

Using the Girsanov Theorem we observe that the process W_q is a Wiener process on probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}_q)$. Therefore, by Theorem 4.4, if the process $\tilde{X}:=(\tilde{\mathbf{u}},\tilde{\mathbf{d}})$ is defined by

$$\tilde{X}:\Omega\ni\omega\mapsto\mathcal{J}^{\varepsilon}\big(W_q(\omega)\big)\in\mathcal{K}_T,$$

then the system

$$(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}_q,W_q,\tilde{X})$$

is a martingale solution of problem (4.1)-(4.2). By Propositions C.1 and C.2 of [4], for $p\geq 1$ we infer

$$\mathbb{E}_q\bigg[\sup_{t\in[0,T]}\big|\tilde{\mathbf{u}}(t)\big|_{\mathrm{H}}^{2p}+\sup_{t\in[0,T]}\big|\tilde{\mathbf{d}}(t)\big|_{H^1}^{2p}\bigg]<\infty,$$

$$\mathbb{E}_q\bigg[\int_0^T\big|\tilde{\mathbf{u}}(t)\big|_{\mathrm{V}}^{2}\,dt+\int_0^T\big|\tilde{\mathbf{d}}(t)\big|_{D(\mathcal{A})}^{2}\,dt\bigg]^{2p}<\infty.$$

On the other hand, since $q \in \mathcal{U}$, from (5.6) we observe that

$$\mathbb{E}_q(\gamma_q)^{-2} = \int_\Omega (\gamma_q)^{-2}\, d\,\mathbb{P}_q = \int_\Omega (\gamma_q)^{-2} \frac{d\,\mathbb{P}_q}{d\,\mathbb{P}}\, d\,\mathbb{P} = \int_\Omega (\gamma_q)^{-1}\, d\,\mathbb{P} = \mathbb{E}(\gamma_q)^{-1} < \infty. \tag{5.7}$$

Therefore, $\mathbb{P}$ is absolutely continuous w.r.t. $\mathbb{P}_q$ and

$$\frac{d\,\mathbb{P}}{d\,\mathbb{P}_q} = \gamma_q^{-1}.$$

Similarly as in (5.7) and applying Hölder's inequality for any $p \geq 1$, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\tilde{\mathbf{u}}(t)\big|_{\mathrm{H}}^p + \sup_{t\in[0,T]} \big|\tilde{\mathbf{d}}(t)\big|_{H^1}^p\Big] < \infty,$$
$$\mathbb{E}\Big[\int_0^T \big|\tilde{\mathbf{u}}(t)\big|_{\mathrm{V}}^2\, dt + \int_0^T \big|\tilde{\mathbf{d}}(t)\big|_{D(\mathcal{A})}^2\, dt\Big]^p < \infty.$$

By a standard argument, we infer that the system

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \tilde{X})$$

is a martingale solution of the problem (5.4)-(5.5) (see Appendix A of [12]).

Uniqueness:
The proof of uniqueness is similar to the proof of Lemma 6.2 (which will be proved later) with minor modifications and hence is omitted.

□

Having this done, we will go back to our fixed probability space with a fixed Wiener process.

5.4. **Skeleton equation.** In this section we consider the deterministic control equation. We call it the skeleton equation. Recall the spaces

$$\mathcal{Y}_T := {}_0\mathbb{C}([0,T];\mathrm{H}) \times {}_0\mathbb{C}([0,T];\mathbb{R})$$

and

$$\mathcal{K}_T := \mathcal{K}_{1,T} \times \mathcal{K}_{2,T} := \mathbb{C}([0,T];\mathrm{H}) \cap L^2([0,T];\mathrm{V}) \times \mathbb{C}([0,T];H^1) \cap L^2([0,T];D(\mathcal{A})).$$

Let us define a Borel map

$$\mathcal{J}^0 : \mathcal{Y}_T \to \mathcal{K}_T.$$

For $q := (\theta, \rho) \in \mathbb{S}$, consider the following deterministic control equation:

$$\tilde{\mathbf{u}}^q(t) = \tilde{\mathbf{u}}_0 - \int_0^t \big[\mathscr{A}\tilde{\mathbf{u}}^q(s) + B(\tilde{\mathbf{u}}^q(s)) + M(\tilde{\mathbf{d}}^q(s))\big]\, ds + \int_0^t \sigma(\tilde{\mathbf{u}}^q(s))\,\theta(s)\, ds, \tag{5.8}$$
$$\tilde{\mathbf{d}}^q(t) = \tilde{\mathbf{d}}_0 - \int_0^t \big[\mathcal{A}\tilde{\mathbf{d}}^q(s) + \tilde{B}(\tilde{\mathbf{u}}^q(s), \tilde{\mathbf{d}}^q(s)) + f(\tilde{\mathbf{d}}^q(s))\big] ds + \int_0^t G(\tilde{\mathbf{d}}^q(s))\,\rho(s)\, ds. \tag{5.9}$$

The existence result of the above **skeleton equation** (5.8)-(5.9) is proved later in the Appendix A.

Theorem 5.5. *Let* $(\tilde{\mathbf{u}}_0, \tilde{\mathbf{d}}_0) \in \mathrm{H} \times H^1$ *and* $q = (\theta, \rho) \in \mathbb{S}$. *Suppose the assumptions hold. Then there exists a unique solution* $(\tilde{\mathbf{u}}^q, \tilde{\mathbf{d}}^q) \in \mathbb{C}([0,T]; \mathrm{H}) \cap L^2([0,T]; \mathrm{V}) \times \mathbb{C}([0,T]; H^1) \cap L^2([0,T]; D(\mathcal{A}))$ *to* (5.8)-(5.9). *Moreover, for fixed* $\alpha \in \mathbb{N}$, *there exist* $C_{1,\alpha}, C_{2,\alpha} > 0$ *such that*

$$\sup_{q \in S^\alpha} \left(\sup_{t \in [0,T]} \left| \tilde{\mathbf{u}}^q(t) \right|_{\mathrm{H}}^2 + \int_0^T \left| \tilde{\mathbf{u}}^q(t) \right|_{\mathrm{V}}^2 dt \right) \le C_{1,\alpha}, \tag{5.10}$$

$$\sup_{q \in S^\alpha} \left(\sup_{t \in [0,T]} \left| \tilde{\mathbf{d}}^q(t) \right|_{H^1}^2 + \int_0^T \left| \tilde{\mathbf{d}}^q(t) \right|_{D(\mathcal{A})}^2 dt \right) \le C_{2,\alpha}. \tag{5.11}$$

We will follow classical approach to prove the existence and uniqueness (see Appendix A). For $q = (\theta, \rho) \in \mathbb{S}$, we put

$$\mathcal{J}^0 \Big(\int_0^{\cdot} q(s)\, ds \Big) := (\tilde{\mathbf{u}}^q, \tilde{\mathbf{d}}^q).$$

It is easy to see that $\mathcal{J}^0$ is Borel measurable. Now we are ready to define the rate function. For $v \in \mathcal{K}_T$, define $\mathbb{S}_v = \{q = (\theta, \rho) \in \mathbb{S} : v = \mathcal{J}^0(\int_0^{\cdot} q(s)\, ds)\}$. Define the **rate function** $I : \mathcal{K}_T \to [0, \infty]$ by

$$I(v) = \inf_{q \in \mathbb{S}_v} \left\{ \int_0^T \Big(\left| \theta(s) \right|_{\mathrm{H}_0}^2 + \left| \rho(s) \right|_{\mathbb{R}}^2 \Big) ds \right\}, \qquad v \in \mathcal{K}_T. \tag{5.12}$$

By convention, $I(v) = \infty$ if $\mathbb{S}_v$ is an empty set.

6. Sufficient conditions for LDP

To prove the main result of our paper, i.e. Theorem 5.3, we proceed by using the method of weak convergence as in Budhiraja and Dupuis [10] (see also Chueshov *et al.* [11], Manna *et al.* [21], Sritharan *et al.* [23]). We need to show the following two conditions hold true for establishing a LDP for the family $\mathcal{J}^\varepsilon(\sqrt{\varepsilon}\, W)$.

Condition 1. *For each* $\alpha > 0$, *the set* $\{(\tilde{\mathbf{u}}^q, \tilde{\mathbf{d}}^q) : q \in S^\alpha\}$ *is a compact subset of* $\mathcal{K}_T$, *where* $S^\alpha \subset L^2(0,T; \mathrm{H}_0) \times L^2(0,T; \mathbb{R})$ *is the centered closed ball of radius* α *endowed with the weak topology. In other words, for all* $\alpha \in \mathbb{N}$, *let* $q_n := (\theta_n, \rho_n), q := (\theta, \rho) \in S^\alpha$ *with* $q_n \to q$, *i.e. weakly in* $L^2(0,T; \mathrm{H}_0 \times \mathbb{R})$. *Then the map* $\mathcal{J}^0 : \mathcal{Y}_T \to \mathcal{K}_T$ *satisfy*

$$\mathcal{J}^0 \Big(\int_0^{\cdot} q_n(s)\, ds \Big) \to \mathcal{J}^0 \Big(\int_0^{\cdot} q(s)\, ds \Big) \quad \textit{in } \ \mathcal{K}_T.$$

Condition 2. *Assume that* $\alpha \in \mathbb{N}$, *that* (ε_n) *is an* $(0,1]$*-valued sequence convergent to 0. Let* $q_n := (\theta_n, \rho_n), q := (\theta, \rho) \in \mathcal{U}^\alpha$ *such that the laws* $\mathscr{L}(q_n)$ *converges weakly to the law* $\mathscr{L}(q)$. *Then the processes*

$$\mathcal{Y}_T \ni W \mapsto \mathcal{J}^{\varepsilon_n} \left(W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^{\cdot} q_n(s)\, ds \right) \in \mathcal{K}_T$$

converge in law on $\mathcal{K}_T$ *to* $\mathcal{J}^0(\int_0^{\cdot} q(s)\, ds)$.

Before proving both these conditions, we will need the following results.

Let us recall that $\mathbb{S} = L^2(0,T; \mathrm{H}_0 \times \mathbb{R})$.

Lemma 6.1. *Assume that* $q_n := (\theta_n, \rho_n)$ *is an* $\mathbb{S}$*-valued sequence such that*

$$q_n \to q := (\theta, \rho) \ \textit{weakly in } L^2(0,T; \mathrm{H}_0 \times \mathbb{R}). \tag{6.1}$$

Then the sequence $\mathcal{J}^0\Big(\int_0^\cdot q_n(s)\,ds\Big)$ *converges strongly to* $\mathcal{J}^0\Big(\int_0^\cdot q(s)\,ds\Big)$ *in* $\mathcal{K}_T$.
In particular, for every $\alpha > 0$, *the mapping*

$$S^\alpha \ni q \mapsto \mathcal{J}^0\Big(\int_0^\cdot q(s)\,ds\Big) \in \mathcal{K}_T$$

is Borel. In particular if q *and* $\tilde{q} \in S^\alpha$, *possibly defined on different probability spaces, with same laws, then the laws of*

$$\Omega \ni \omega \mapsto \mathcal{J}^0\Big(\int_0^\cdot q(s,\omega)\,ds\Big) \in \mathcal{K}_T \quad \textit{and} \quad \tilde{\Omega} \ni \tilde{\omega} \mapsto \mathcal{J}^0\Big(\int_0^\cdot \tilde{q}(s,\tilde{\omega})\,ds\Big) \in \mathcal{K}_T$$

are also equal.

Let us note that the assumption implies that there exists $\alpha > 0$ such that $q_n \in S^\alpha$ for all n and $q_n \to q$ in $\mathbb{S}$.

Lemma 6.2. *Assume* $\alpha > 0$ *and* (ε_n) *be a* $(0,1]$*-valued sequence converging to 0. Let* $q_n := (\theta_n, \rho_n) \in \mathcal{U}^\alpha$ *with* $\mathscr{L}(q_n)$ *converges to* $\mathscr{L}(q)$ *weakly on* S^α. *Then the sequence of random variables*

$$\mathcal{Y}_T \ni W \mapsto \mathcal{J}^{\varepsilon_n}\Big(\sqrt{\varepsilon_n}\, W + \int_0^\cdot q_n(s)\,ds\Big) - \mathcal{J}^0\Big(\int_0^\cdot q_n(s)\,ds\Big) \in \mathcal{K}_T$$

converges in probability to 0.

Condition 1 will be direct consequence of Lemma 6.1. The proof of Condition 2 requires Lemma 6.2. We will prove Lemma 6.1 and 6.2 later in Section 7 and Subsection 8.1 respectively. Now the remaining part of the paper is devoted to the proof of the sufficient conditions for a LDP, stated earlier.

It is easy to see that the proof of Condition 1 follows from the first part of Lemma 6.1. So we will prove Lemma 6.1.

7. Proof of Lemma 6.1

. Assume that $q = (\theta, \rho) \in \mathbb{S}$ and $q_n := (\theta_n, \rho_n)$ is an $\mathbb{S}$-valued sequence such that condition (6.1) is satisfied. We define the solution of the skeleton equation (5.8)-(5.9) by $\mathcal{J}^0\big(\int_0^\cdot q(s)\,ds\big) := (\tilde{\mathbf{u}}^q, \tilde{\mathbf{d}}^q) := \tilde{X}^q$ and $\mathcal{J}^0(\int_0^\cdot q_n(s)\,ds) := (\tilde{\mathbf{u}}^{q_n}, \tilde{\mathbf{d}}^{q_n}) := \tilde{X}^{q_n}$. For simplicity, we denote $X_n = (\mathbf{u}_n, \mathbf{d}_n) = \tilde{X}^{q_n}$, which satisfy the equations

$$\mathbf{u}_n(t) = \mathbf{u}_0 - \int_0^t \left[\mathscr{A}\mathbf{u}_n(s) + B(\mathbf{u}_n(s)) + M(\mathbf{d}_n(s))\right] ds + \int_0^t \sigma(\mathbf{u}_n(s))\,\theta(s)\,ds, \tag{7.1}$$

$$\mathbf{d}_n(t) = \mathbf{d}_0 - \int_0^t \left[\mathcal{A}\mathbf{d}_n(s) + \tilde{B}(\mathbf{u}_n(s), \mathbf{d}_n(s)) + f(\mathbf{d}_n(s))\right] ds + \int_0^t G(\mathbf{d}_n(s))\,\rho(s)\,ds. \tag{7.2}$$

Our aim is to show $X_n \to \tilde{X}^q$.

From Propositions A.2, A.3 and A.4, for $\beta \in (0, \frac{1}{2})$, there exist positive constants $C_1, C_2, C_3(\beta), C_4(\beta)$ such that we infer

$$\sup_{s\in[0,T]} \big|\mathbf{u}_n(s)\big|^2_{\mathrm{H}} + \int_0^T \big|\mathbf{u}_n(s)\big|^2_{\mathrm{V}}\, ds \le C_1, \tag{7.3}$$

$$\sup_{s\in[0,T]} \big|\mathbf{d}_n(s)\big|^2_{H^1} + \int_0^T \big|\mathbf{d}_n(s)\big|^2_{D(\mathcal{A})}\, ds \le C_2, \tag{7.4}$$

$$\sup_{n\in\mathbb{N}} \big|\mathbf{u}_n\big|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{V}')} \le C_3(\beta) \quad \text{and} \quad \sup_{n\in\mathbb{N}} \big|\mathbf{d}_n\big|^2_{\mathrm{W}^{\beta,2}([0,T];(D(\mathcal{A}))')} \le C_4(\beta). \tag{7.5}$$

This guarantees the existence of a subsequence $(\mathbf{u}_{m'}, \mathbf{d}_{m'})$ and an element $X := (\mathbf{u}, \mathbf{d})$ which lies in the space

$$L^2([0,T];\mathrm{V}) \cap L^\infty([0,T];\mathrm{H}) \times L^2([0,T];D(\mathcal{A})) \cap L^\infty([0,T];H^1)$$

such that as $m' \to \infty$ and using Lemma 2.3 we have

$$\begin{cases} \mathbf{u}_{m'} \to \mathbf{u} & \text{in} \quad L^2([0,T];\mathrm{V}) \quad \text{weakly}, \\ \mathbf{u}_{m'} \to \mathbf{u} & \text{in} \quad L^\infty([0,T];\mathrm{H}) \quad \text{weak-star}, \\ \mathbf{d}_{m'} \to \mathbf{d} & \text{in} \quad L^2([0,T];D(\mathcal{A})) \quad \text{weakly}, \\ \mathbf{d}_{m'} \to \mathbf{d} & \text{in} \quad L^\infty([0,T];H^1) \quad \text{weak-star}, \\ \mathbf{u}_{m'} \to \mathbf{u} & \text{in} \quad L^2([0,T];\mathrm{H}) \quad \text{strongly}, \\ \mathbf{d}_{m'} \to \mathbf{d} & \text{in} \quad L^2([0,T];H^1) \quad \text{strongly}. \end{cases}$$

Now we need to show $X := (\mathbf{u}, \mathbf{d}) = \tilde{X}^q$. Let $\{\varrho_j\}_{j=1}^\infty$ be the orthonormal basis of H composed of eigenfunctions of the stokes operator $\mathscr{A}$. Let $\{\varsigma_j\}_{j=1}^\infty$ be the orthonormal basis of L^2 consisting of the eigenfunctions of the Neumann Laplacian $\mathcal{A}$. Let ϕ be a continuously differentiable function on $[0,T]$ with $\phi(T) = 0$.

We multiply (7.1) by $\phi(s)\,\varrho_j$ to get,

$$\begin{aligned} \big\langle \mathbf{u}_n(s), \phi(s)\,\varrho_j \big\rangle_{\mathrm{H,H}} = \big\langle \mathbf{u}_0, \phi(0)\,\varrho_j \big\rangle_{\mathrm{H,H}} &- \int_0^T \big\langle \mathscr{A}\mathbf{u}_n(s), \phi(s)\,\varrho_j \big\rangle_{\mathrm{H,H}}\, ds \\ &- \int_0^T \big\langle B(\mathbf{u}_n(s)), \phi(s)\,\varrho_j \big\rangle_{\mathrm{V}',\mathrm{V}}\, ds - \int_0^T \big\langle M(\mathbf{d}_n(s)), \phi(s)\,\varrho_j \big\rangle_{\mathrm{V}',\mathrm{V}}\, ds \\ &+ \int_0^T \big\langle \sigma(\mathbf{u}_n(s))\,\theta_n(s), \phi(s)\,\varrho_j \big\rangle_{\mathrm{H,H}}\, ds, \end{aligned} \tag{7.6}$$

and multiply (7.2) by $\phi(s)\,\varsigma_j$ to get,

$$\begin{aligned}\big\langle \mathbf{d}_n(s), \phi(s)\,\varsigma_j\big\rangle_{L^2,L^2} = \big\langle \mathbf{d}_0, \phi(0)\,\varsigma_j\big\rangle_{L^2,L^2} - \int_0^T \big\langle \mathcal{A}\mathbf{d}_n(s), \phi(s)\,\varsigma_j\big\rangle_{L^2,L^2}\,ds \\ - \int_0^T \big\langle \tilde{B}(\mathbf{u}_n(s), \mathbf{d}_n(s)), \phi(s)\,\varsigma_j\big\rangle_{(H^1)',H^1}\,ds \\ - \int_0^T \big\langle f(\mathbf{d}_n(s)), \phi(s)\,\varsigma_j\big\rangle_{(H^1)',H^1}\,ds \\ + \int_0^T \big\langle G(\mathbf{d}_n(s))\,\rho_n(s), \phi(s)\,\varsigma_j\big\rangle_{L^2,L^2}\,ds.\end{aligned} \tag{7.7}$$

Here we only show the convergence of terms involving control parameters, i.e. the last terms of (7.6) and (7.7). For all other linear and nonlinear terms, we follow our earlier work (see [8]). Now split the term

$$\begin{aligned}&\int_0^T \big\langle \sigma(\mathbf{u}_{m'}(s))\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds - \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds \\ &= \int_0^T \big\langle \sigma(\mathbf{u}_{m'}(s))\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds + \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds \\ &\quad - \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds - \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds.\end{aligned} \tag{7.8}$$

Since by assumption (6.1) $\theta_{m'} \to \theta$ weakly in $L^2(0,T;\mathrm{H}_0)$, we infer

$$\begin{aligned}&\left| \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds - \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds\right| \\ &= \left| \int_0^T \big\langle \sigma(\mathbf{u}(s))\,(\theta_{m'}(s) - \theta(s)), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds\right| \\ &= \left| \int_0^T \big\langle (\theta_{m'}(s) - \theta(s)), \sigma^*(\mathbf{u}(s))\,\phi(s)\,\varrho_j\big\rangle_{\mathrm{H}_0}\,ds\right| \longrightarrow 0 \quad \text{as} \quad n \to \infty,\end{aligned} \tag{7.9}$$

where $\sigma^*(\mathbf{u}) = \big(\sigma(\mathbf{u})\big)^* \in \mathrm{L}_2(\mathrm{H},\mathrm{H}_0)$ for $\mathbf{u} \in \mathrm{H}$.

Since $\theta_{m'} \in S^\alpha$ and $\mathbf{u}_{m'} \to \mathbf{u}$ in $L^2([0,T];\mathrm{H})$ strongly, we deduce

$$\begin{aligned}&\left| \int_0^T \big\langle \big(\sigma(\mathbf{u}_{m'}(s)) - \sigma(\mathbf{u}(s))\big)\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds\right| \\ &\leq \int_0^T |\sigma(\mathbf{u}_{m'}(s)) - \sigma(\mathbf{u}(s))|_{\mathrm{L}_2}\,|\theta_{m'}(s)|_{\mathrm{H}}\,|\phi(s)\,\varrho_j|_{\mathrm{H}}\,ds \\ &\leq C_T \int_0^T \sqrt{L}\,|\mathbf{u}_{m'}(s) - \mathbf{u}(s)|_{\mathrm{H}}\,|\theta_{m'}(s)|_{\mathrm{H}_0}\,ds \\ &\leq C_T\sqrt{L}\,\big|\mathbf{u}_{m'} - \mathbf{u}\big|^2_{L^2([0,T];\mathrm{H})}\,\big|\theta_{m'}\big|^2_{L^2([0,T];\mathrm{H}_0)} \longrightarrow 0 \ \text{ as } m' \to \infty.\end{aligned} \tag{7.10}$$

Finally from (7.8), (7.9) and (7.10) we have,

$$\lim_{m'\to\infty} \int_0^T \big\langle \sigma(\mathbf{u}_{m'}(s))\,\theta_{m'}(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds = \int_0^T \big\langle \sigma(\mathbf{u}(s))\,\theta(s), \phi(s)\,\varrho_j\big\rangle_{\mathrm{H}}\,ds. \tag{7.11}$$

Similarly it can be proved that

$$\lim_{m'\to\infty}\int_0^T \big\langle G(\mathbf{d}_{m'}(s))\,\rho_{m'}(s), \phi(s)\,\varsigma_j\big\rangle_{L^2}\,ds = \int_0^T \big\langle G(\mathbf{d}(s))\,\rho(s), \phi(s)\,\varsigma_j\big\rangle_{L^2}\,ds.$$

Finally using the similar arguments as in the proof of Theorem 3.1 in Temam [24], Section 3, Chapter III, we conclude $X := (\mathbf{u}, \mathbf{d})$ satisfy the skeleton equation (5.8)-(5.9).

Next we will prove $X_n \to X$ in $\mathcal{K}_T$. For that we need to show $\mathbf{u}_n \to \mathbf{u}$ in $\mathcal{K}_{1,T}$ and $\mathbf{d}_n \to \mathbf{d}$ in $\mathcal{K}_{2,T}$. For that denote, $\xi_n := \mathbf{u}_n - \mathbf{u}$ and $\mu_n := \mathbf{d}_n - \mathbf{d}$. Then these processes satisfy the following equations:

$$\begin{aligned} &d\xi_n(t) + \Big(\mathscr{A}\xi_n(t) + B(\xi_n(t), \mathbf{u}_n(t)) + B(\mathbf{u}(t), \xi_n(t))\Big)\,dt \\ &\quad = -\Big(M(\mu_n(t), \mathbf{d}_n(t)) + M(\mathbf{d}(t), \mu_n(t))\Big)\,dt + \Big(\sigma(\mathbf{u}_n(t))\,\theta_n(t) - \sigma(\mathbf{u}(t))\,\theta(t)\Big)\,dt, \end{aligned} \tag{7.12}$$

and

$$\begin{aligned} &d\mu_n(t) + \Big(\mathcal{A}\mu_n(t) + \tilde{B}(\xi_n(t), \mathbf{d}_n(t)) + \tilde{B}(\mathbf{u}(t), \mu_n(t))\Big)\,dt \\ &\quad = -\Big(f(\mathbf{d}_n(t)) - f(\mathbf{d}(t))\Big)\,dt + \Big(G(\mathbf{d}_n(t))\,\rho_n(t) - G(\mathbf{d}(t))\,\rho(t)\Big)\,dt. \end{aligned} \tag{7.13}$$

From (2.5), (2.6), (2.7) and using Poincaré and Young's inequalities, we obtain for any $\alpha_1, \alpha_2, \cdots, \alpha_6 > 0$, there exist $C(\alpha_1), C(\alpha_2, \alpha_3), C(\alpha_7, \alpha_4)$ and $C(\alpha_5) > 0$ such that

$$\begin{aligned} |\langle B(\xi_n(t), \mathbf{u}_n(t)), \xi_n(t)\rangle| &\le \alpha_1 \big|\nabla\xi_n(t)\big|_{\mathrm{H}}^2 + C(\alpha_1)\big|\mathbf{u}_n(t)\big|_{\mathrm{H}}^2\,\big|\nabla\mathbf{u}_n(t)\big|_{\mathrm{H}}^2\,\big|\xi_n(t)\big|_{\mathrm{H}}^2, \\ |\langle M(\mathbf{d}(t), \mu_n(t)), \xi_n(t)\rangle| &\le \alpha_2 \big|\nabla\xi_n(t)\big|_{\mathrm{H}}^2 + \alpha_3\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2 \\ &\quad + C(\alpha_2, \alpha_3)\big|\nabla\mathbf{d}(t)\big|_{L^2}^2\,\big|\mathcal{A}\mathbf{d}(t)\big|_{L^2}^2\,\big|\nabla\mu_n(t)\big|_{L^2}^2, \\ |\langle M(\mu_n(t), \mathbf{d}_n(t)), \xi_n(t)\rangle| &\le \alpha_7 \big|\nabla\xi_n(t)\big|_{\mathrm{H}}^2 + \alpha_4\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2 \\ &\quad + C(\alpha_7, \alpha_4)\big|\nabla\mathbf{d}_n(t)\big|_{L^2}^2\,\big|\mathcal{A}\mathbf{d}_n(t)\big|_{L^2}^2\,\big|\nabla\mu_n(t)\big|_{L^2}^2, \\ |\langle \tilde{B}(\mathbf{u}(t), \mu_n(t)), \mathcal{A}\mu_n(t)\rangle| &\le \alpha_5\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2 + C(\alpha_5)\big|\mathbf{u}(t)\big|_{\mathrm{H}}^2\,\big|\nabla\mathbf{u}(t)\big|_{\mathrm{H}}^2\,\big|\nabla\mu_n(t)\big|_{L^2}^2, \\ |\langle \tilde{B}(\xi_n(t), \mathbf{d}_n(t)), \mu_n(t)\rangle| &\le \alpha_6\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2 + C(\alpha_6)\big|\xi_n(t)\big|_{\mathrm{H}}^2\,\big|\nabla\mathbf{d}_n(t)\big|_{L^2}^2. \end{aligned}$$

And for any $\alpha_8, \alpha_9 > 0$ there exist $C(\alpha_8), C_1(\alpha_9), C_2(\alpha_9) > 0$ such that (see Appendix of [8])

$$\begin{aligned} &|\langle f(\mathbf{d}_n(t)) - f(\mathbf{d}(t)), \mathbf{d}_n(t) - \mathbf{d}(t)\rangle| \le \alpha_8\big|\nabla\mathbf{d}_n(t) - \nabla\mathbf{d}(t)\big|_{L^2}^2 \\ &\qquad\qquad + C(\alpha_8)\big|\mathbf{d}_n(t) - \mathbf{d}(t)\big|_{L^2}^2\,\varphi(\mathbf{d}_n(t), \mathbf{d}(t)), \\ &|\langle f(\mathbf{d}_n(t)) - f(\mathbf{d}(t)), \mathcal{A}\mathbf{d}_n(t) - \mathcal{A}\mathbf{d}(t)\rangle| \\ &\le \alpha_9\big|\mathcal{A}\mathbf{d}_n(t) - \mathcal{A}\mathbf{d}(t)\big|_{L^2}^2 + \big[C_1(\alpha_9)\big|\nabla\mathbf{d}_n(t) - \nabla\mathbf{d}(t)\big|_{L^2}^2\,\varphi(\mathbf{d}_n(t), \mathbf{d}(t)) \\ &\quad + C_2(\alpha_9)\big|\mathbf{d}_n(t) - \mathbf{d}(t)\big|_{L^2}^2\,\varphi(\mathbf{d}_n(t), \mathbf{d}(t))\big]. \end{aligned}$$

where $\varphi(\mathbf{d}_n(t), \mathbf{d}(t))$ is defined as

$$\varphi(\mathbf{d}_n(t), \mathbf{d}(t)) := C\big(1 + |\mathbf{d}_n(t)|_{L^{4N+2}}^{2N} + |\mathbf{d}(t)|_{L^{4N+2}}^{2N}\big)^2.$$

Now consider

$$\Psi(t) := \exp\Big(- \int_0^t (\psi_1(s) + \psi_2(s) + \psi_3(s))\, ds \Big), \quad \text{for any } t > 0.$$

where

$$\begin{aligned}
\psi_1(s) &:= C(\alpha_1)\big|\mathbf{u}_n(s)\big|_{\mathrm{H}}^2 \big|\nabla \mathbf{u}_n(s)\big|_{\mathrm{H}}^2 + C(\alpha_6)\big|\nabla \mathbf{d}_n(s)\big|_{L^2}^2, \\
\psi_2(s) &:= C(\alpha_2,\alpha_3)\big|\nabla \mathbf{d}(s)\big|_{L^2}^2\big|\mathcal{A}\mathbf{d}(s)\big|_{L^2}^2 + C(\alpha_7,\alpha_4)\big|\nabla \mathbf{d}_n(s)\big|_{L^2}^2\big|\mathcal{A}\mathbf{d}_n(s)\big|_{L^2}^2 \\
&\qquad + C(\alpha_5)\big|\mathbf{u}(s)\big|_{\mathrm{H}}^2\big|\nabla \mathbf{u}(s)\big|_{\mathrm{H}}^2 + C_1(\alpha_9)\varphi(\mathbf{d}_n(s),\mathbf{d}(s)), \\
\psi_3(s) &:= \big[C(\alpha_8) + C_2(\alpha_9)\big]\varphi(\mathbf{d}_n(s),\mathbf{d}(s)).
\end{aligned}$$

Now multiply $\Psi(t)$ with (7.13) and taking inner product with $\mu_n(t)$,

$$\begin{aligned}
d\big[\Psi(t)\,\big|\mu_n(t)\big|_{L^2}^2\big] &= -2\Psi(t)\,\big|\nabla\mu_n(t)\big|_{L^2}^2\,dt - 2\Psi(t)\,\big\langle \tilde{B}(\xi_n(t),\mathbf{d}_n(t)),\mu_n(t)\big\rangle\,dt \\
&\quad - 2\Psi(t)\,\big\langle f(\mathbf{d}_n(t)) - f(\mathbf{d}(t)),\mu_n(t)\big\rangle\,dt \\
&\quad + \underbrace{2\Psi(t)\,\big\langle (G(\mathbf{d}_n(t))\,\rho_n(t) - G(\mathbf{d}(t))\,\rho(t)),\mu_n(t)\big\rangle}_{J_{1,n}(t)}\,dt + \Psi'(t)\,\big|\mu_n(t)\big|_{L^2}^2\,dt
\end{aligned} \tag{7.14}$$

Multiply $\Psi(t)$ with (7.13) and taking inner product with $\mathcal{A}\mu_n(t)$,

$$\begin{aligned}
d\big[\Psi(t)\,\big|\nabla\mu_n(t)\big|_{L^2}^2\big] &= 2\Psi(t)\big(\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2 + \big\langle \tilde{B}(\xi_n(t),\mathbf{d}_n(t)) + \tilde{B}(\mathbf{u}(t),\mu_n(t)),\mathcal{A}\mu_n(t)\big\rangle\big)\,dt \\
&\quad - 2\Psi(t)\,\big\langle f(\mathbf{d}_n(t)) - f(\mathbf{d}(t)),\mathcal{A}\mu_n(t)\big\rangle\,dt \\
&\quad + \underbrace{2\Psi(t)\,\big\langle (G(\mathbf{d}_n(t))\,\rho_n(t) - G(\mathbf{d}(t))\,\rho(t)),\mathcal{A}\mu_n(t)\big\rangle}_{J_{2,n}(t)}\,dt \\
&\quad + \Psi'(t)\,\big|\nabla\mu_n(t)\big|_{L^2}^2\,dt
\end{aligned} \tag{7.15}$$

Multiplying $\Psi(t)$ with (7.12) and taking inner product with $\xi_n(t)$,

$$\begin{aligned}
d\big[\Psi(t)\,\big|\xi_n(t)\big|_{\mathrm{H}}^2\big] &= -2\Psi(t)\big|\nabla\xi_n(t)\big|_{\mathrm{H}}^2\,dt \\
&\quad - 2\Psi(t)\,\big\langle B(\xi_n(t),\mathbf{u}_n(t)) + M(\mu_n(t),\mathbf{d}_n(t)) + M(\mathbf{d}(t),\mu_n(t)),\xi_n(t)\big\rangle\,dt \\
&\quad + \underbrace{2\Psi(t)\,\big\langle (\sigma(\mathbf{u}_n(t))\,\theta_n(t) - \sigma(\mathbf{u}(t))\,\theta(t)),\xi_n(t)\big\rangle}_{J_{3,n}(t)}\,dt + \Psi'(t)\,\big|\xi_n(t)\big|_{L^2}^2\,dt
\end{aligned} \tag{7.16}$$

Adding (7.14), (7.15) and (7.16), then rearranging we get,

$$
\begin{aligned}
&d\big[\Psi(t)\big(\big|\xi_n(t)\big|_{\mathrm{H}}^2+\big|\mu_n(t)\big|_{L^2}^2+\big|\nabla\mu_n(t)\big|_{L^2}^2\big)\big]+2\Psi(t)\big[\big|\nabla\xi_n(t)\big|_{\mathrm{H}}^2+\big|\nabla\mu_n(t)\big|_{L^2}^2+\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2\big]\,dt\\
&\leq 2\Psi(t)\bigg[\Big(\alpha_9+\sum_{i=3}^{6}\alpha_i\Big)\,\big|\mathcal{A}\mu_n(t)\big|_{L^2}^2+(\alpha_1+\alpha_2+\alpha_7)\big|\nabla\xi_n(t)\big|_{\mathrm{H}}^2+\alpha_8\big|\nabla\mu_n(t)\big|_{L^2}^2\bigg]\,dt\\
&\quad+2\Psi(t)\bigg[\psi_1(t)\,\big|\xi_n(t)\big|_{\mathrm{H}}^2+\psi_2(t)\,\big|\nabla\mu_n(t)\big|_{L^2}^2+\psi_3(t)\big|\mu_n(t)\big|_{L^2}^2\bigg]\,dt\\
&\quad+\Psi'(t)\Big(\big|\xi_n(t)\big|_{\mathrm{H}}^2+\big|\mu_n(t)\big|_{L^2}^2+\big|\nabla\mu_n(t)\big|_{L^2}^2\Big)\,dt\\
&\quad+2\Psi(t)\bigg[C\big(\big|\xi_n(t)\big|_{\mathrm{H}}^2+\big|\mu_n(t)\big|_{L^2}^2+\big|\nabla\mu_n(t)\big|_{L^2}^2\big)\bigg]\,dt+\big[J_{1,n}(t)+J_{2,n}(t)+J_{3,n}(t)\big]\,dt.
\end{aligned}
\tag{7.17}
$$

By the choice of Ψ we have,

$$
\begin{aligned}
2\Psi(t)&\bigg[\psi_1(t)\,\big|\xi_n(t)\big|_{\mathrm{H}}^2+\psi_2(t)\,\big|\nabla\mu_n(t)\big|_{L^2}^2+\psi_3(t)\big|\mu_n(t)\big|_{L^2}^2\bigg]\\
&+\Psi'(t)\Big(\big|\xi_n(t)\big|_{\mathrm{H}}^2+\big|\mu_n(t)\big|_{L^2}^2+\big|\nabla\mu_n(t)\big|_{L^2}^2\Big)\leq 0.
\end{aligned}
$$

First we consider $J_{3,n}(t):=2\Psi(t)\,\big\langle(\sigma(\mathbf{u}_n(t))\,\theta_n(t)-\sigma(\mathbf{u}(t))\,\theta(t)),\xi_n(t)\big\rangle$. We split it as:

$$
\begin{aligned}
2\Psi(t)\,&\bigg[\big\langle(\sigma(\mathbf{u}_n(t))\,\theta(t)-\sigma(\mathbf{u}(t))\,\theta(t)),\mathbf{u}_n(t)-\mathbf{u}(t)\big\rangle\\
&+\big\langle(\sigma(\mathbf{u}_n(t))\,\theta_n(t)-\sigma(\mathbf{u}_n(t))\,\theta(t)),\mathbf{u}_n(t)-\mathbf{u}(t)\big\rangle\bigg]=2\Psi(t)\,\big[J'_{3,n}(t)+J''_{3,n}(t)\big].
\end{aligned}
$$

Now using energy estimates, the Lipschitz and linear growth property of σ we get,

$$
\begin{aligned}
|J'_{3,n}(t)|&=\big|\big\langle(\sigma(\mathbf{u}_n(t))\,\theta(t)-\sigma(\mathbf{u}(t))\,\theta(t)),\mathbf{u}_n(t)-\mathbf{u}(t)\big\rangle\big|\\
&\leq\big|\sigma(\mathbf{u}_n(t))-\sigma(\mathbf{u}(t))\big|_{\mathrm{L}_2}\,|\theta(t)|_{\mathrm{H}}\,\big|\mathbf{u}_n(t)-\mathbf{u}(t)\big|_{\mathrm{H}}\\
&\leq\sqrt{L}\,|\theta(t)|_{\mathrm{H}}\,\big|\mathbf{u}_n(t)-\mathbf{u}(t)\big|_{\mathrm{H}}^2\leq\big(L+\big|\theta(t)\big|_{\mathrm{H}_0}^2\big)\big|\xi_n(t)\big|_{\mathrm{H}}^2.
\end{aligned}
$$

Using the above estimate we obtain,

$$
|J_{3,n}(t)|\leq 2\Psi(t)\Big[\big(L+\big|\theta(t)\big|_{\mathrm{H}_0}^2\big)\big|\xi_n(t)\big|_{\mathrm{H}}^2+|J''_{3,n}(t)|\Big].
$$

Similarly calculating the term $J_{1,n}(t)$ and using the properties of G we get,

$$
\begin{aligned}
|J_{1,n}(t)|&=2\Psi(t)\,\big|\big\langle(G(\mathbf{d}_n(t))\,\rho_n(t)-G(\mathbf{d}(t))\,\rho(t)),\mu_n(t)\big\rangle\big|\\
&\leq 2\Psi(t)\Big[\big(C_{\mathbf{h}}+\big|\rho(t)\big|_{\mathbb{R}}^2\big)\big|\mu_n(t)\big|_{L^2}^2+|J''_{1,n}(t)|\Big],
\end{aligned}
$$

where $J''_{1,n}(t):=\big\langle\rho_n(t)-\rho(t),G(\mathbf{d}_n(t))\,\mu_n(t)\big\rangle$.

Now consider $J_{2,n}(t) := 2\Psi(t)\left\langle (G(\mathbf{d}_n(t))\,\rho_n(t) - G(\mathbf{d}(t))\,\rho(t)), \mathcal{A}\mu_n(t)\right\rangle$. Using integration byparts we have,

$$\begin{aligned}
J_{2,n}(t) &= 2\Psi(t)\left\langle \nabla\big(G(\mathbf{d}_n(t))\big)\,\rho_n(t) - \nabla\big(G(\mathbf{d}(t))\big)\,\rho(t), \nabla(\mathbf{d}_n(t)-\mathbf{d}(t))\right\rangle\\
&= 2\Psi(t)\Big[\left\langle \nabla\big(G(\mathbf{d}_n(t))\big)\,\rho(t) - \nabla\big(G(\mathbf{d}(t))\big)\,\rho(t), \nabla(\mathbf{d}_n(t)-\mathbf{d}(t))\right\rangle\\
&\qquad + \left\langle \nabla\big(G(\mathbf{d}_n(t))\big)\,\rho_n(t) - \nabla\big(G(\mathbf{d}_n(t))\big)\,\rho(t), \nabla(\mathbf{d}_n(t)-\mathbf{d}(t))\right\rangle\Big]\\
&:= 2\Psi(t)[J'_{2,n}(t) + J''_{2,n}(t)].
\end{aligned}$$

Now using the properties of G we obtain,

$$|J'_{2,n}(t)| \le \sqrt{C_{\mathbf{h}}}\,|\rho(t)|_{\mathbb{R}}\,\big|\nabla\big(G(\mathbf{d}_n(t) - G(\mathbf{d}(t))\big|_{L^2}\,|\nabla\mu_n(t)|_{L^2} \le \big(C_{\mathbf{h}} + \big|\rho(t)\big|^2_{\mathbb{R}}\big)\big|\nabla\mu_n(t)\big|^2_{L^2}.$$

and So we get,

$$|J_{2,n}(t)| \le 2\Psi(t)\big[\big(C_{\mathbf{h}} + \big|\rho(t)\big|^2_{\mathbb{R}}\big)\big|\nabla\mu_n(t)\big|^2_{L^2} + |J''_{2,n}(t)|\big].$$

In the equation (7.17), choose $\alpha_9 = \alpha_i = \frac{1}{10}$ for $i = 3,4,5,6$, $\alpha_1 = \alpha_2 = \alpha_7 = \frac{1}{6}$ and $\alpha_8 = \frac{1}{2}$. Then using the estimates for $J_{1,n}$, $J_{2,n}$ and $J_{3,n}$ and rearranging we obtain,

$$\begin{aligned}
&d\big[\Psi(t)\big(\big|\xi_n(t)\big|^2_{\mathrm{H}} + \big|\mu_n(t)\big|^2_{L^2} + \big|\nabla\mu_n(t)\big|^2_{L^2}\big)\big] + \Psi(t)\big[\big|\nabla\xi_n(t)\big|^2_{\mathrm{H}} + \big|\nabla\mu_n(t)\big|^2_{L^2} + \big|\mathcal{A}\mu_n(t)\big|^2_{L^2}\big]\,dt\\
&\le 2\Psi(t)\Big[\big(C + K + \big|\theta(t)\big|^2_{\mathrm{H}_0} + C_2\big)\big|\xi_n(t)\big|^2_{\mathrm{H}}\\
&\qquad + \big(C + L + \big|\rho(t)\big|^2_{\mathbb{R}} + C_4\big)\big|\mu_n(t)\big|^2_{L^2} + \big(C + L + \big|\rho(t)\big|^2_{\mathbb{R}} + C_6\big)\big|\nabla\mu_n(t)\big|^2_{L^2}\Big]\,dt\\
&\quad + 2\Psi(t)\,\big[|J''_{1,n}(t)| + |J''_{2,n}(t)| + |J''_{3,n}(t)|\big]\,dt.
\end{aligned}\tag{7.18}$$

Now choose

$$M = \max\{C + K + C_2, C + L + C_4, C + L + C_6\},\ \ \beta(t) = M + \big|\theta(t)\big|^2_{\mathrm{H}_0} + 2\big|\rho(t)\big|^2_{\mathbb{R}}$$

and

$$\lambda_n(t) = 2\Psi(t)\,\big[|J''_{1,n}(t)| + |J''_{2,n}(t)| + |J''_{3,n}(t)|\big].$$

Then writing (7.18) in integral form we get,

$$\begin{aligned}
&\sup_{t\in[0,T]}\Big[\Psi(t)\big(\big|\xi_n(t)\big|^2_{\mathrm{H}} + \big|\mu_n(t)\big|^2_{L^2} + \big|\nabla\mu_n(t)\big|^2_{L^2}\big)\Big]\\
&\qquad + \int_0^T \Psi(t)\Big(\big|\nabla\xi_n(t)\big|^2_{\mathrm{H}} + \big|\nabla\mu_n(t)\big|^2_{L^2} + \big|\mathcal{A}\mu_n(t)\big|^2_{L^2}\Big)\,dt\\
&\qquad \le \int_0^T \lambda_n(t)\,dt + \int_0^T \beta(t)\,\Psi(t)\,\big(\big|\xi_n(t)\big|^2_{\mathrm{H}} + \big|\mu_n(t)\big|^2_{L^2} + \big|\nabla\mu_n(t)\big|^2_{L^2}\big)\,dt.
\end{aligned}\tag{7.19}$$

Now dropping the second term of left hand side and applying Gronwall lemma we obtain,

$$\sup_{t\in[0,T]}\Big[\Psi(t)\big(\big|\xi_n(t)\big|^2_{\mathrm{H}} + \big|\mu_n(t)\big|^2_{L^2} + \big|\nabla\mu_n(t)\big|^2_{L^2}\big)\Big] \le \int_0^T \lambda_n(t)\,dt\cdot\exp\Big(\int_0^T \beta(t)\,dt\Big).\tag{7.20}$$

Now we show $\int_0^T \lambda_n(t)\,dt := 2\int_0^T \Psi(t)\left[|J''_{1,n}(t)| + |J''_{2,n}(t)| + |J''_{3,n}(t)|\right] dt$ goes to 0 as $n \to \infty$. Let us first consider one of the terms of the form $\int_0^T |J''_{i,n}(t)|\,dt$, say, $\int_0^T |J''_{3,n}(t)|\,dt$.

Using the fact that $\mathbf{u}_n$ converges to $\mathbf{u}$ strongly in $L^2([0,T];\mathrm{H})$ and using linear growth property of σ we infer,

$$\begin{aligned}
\int_0^T |J''_{3,n}(t)|\,dt &= \int_0^T \left|\left\langle(\sigma(\mathbf{u}_n(t))\,\theta_n(t) - \sigma(\mathbf{u}_n(t))\,\theta(t)), \mathbf{u}_n(t) - \mathbf{u}(t)\right\rangle\right| dt \\
&\leq \int_0^T |\theta_n(t) - \theta(t)|_{\mathrm{H}_0}\,|\sigma(\mathbf{u}_n(t))|_{\mathrm{L}_2}\,|\mathbf{u}_n(t) - \mathbf{u}(t)|_{\mathrm{H}}\,dt \\
&\leq \Big(\int_0^T |\theta_n(t) - \theta(t)|^2_{\mathrm{H}_0}\,|\sigma(\mathbf{u}_n(t))|^2_{\mathrm{L}_2}\,dt\Big)^{\frac{1}{2}}\Big(\int_0^T |\mathbf{u}_n(t) - \mathbf{u}(t)|^2_{\mathrm{H}}\,dt\Big)^{\frac{1}{2}} \\
&\leq \Big(K\int_0^T (|\theta_n(t)|^2_{\mathrm{H}_0} + |\theta(t)|^2_{\mathrm{H}_0})(1 + |\mathbf{u}_n(t)|^2_{\mathrm{H}})\,dt\Big)^{\frac{1}{2}}\Big(\int_0^T |\mathbf{u}_n(t) - \mathbf{u}(t)|^2_{\mathrm{H}}\,dt\Big)^{\frac{1}{2}} \\
&\leq (2\alpha K)^{\frac{1}{2}}\Big(T + \sup_{t\in[0,T]} |\mathbf{u}_n(t)|^2_{\mathrm{H}}\Big)^{\frac{1}{2}} |\mathbf{u}_n - \mathbf{u}|_{L^2([0,T];\mathrm{H})} \\
&\leq C(\alpha, K, T)\,|\mathbf{u}_n - \mathbf{u}|_{L^2([0,T];\mathrm{H})} \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Using the similar arguments for $\int_0^T |J''_{1,n}(t)|\,dt$ and $\int_0^T |J''_{2,n}(t)|\,dt$ and using the fact that $\sup_{t\in[0,T]} \Psi(t) < \infty$, we infer

$$\int_0^T \lambda_n(t)\,dt \longrightarrow 0 \quad \text{as } n \to \infty.$$

Since $(\theta, \rho) \in S^\alpha$ we infer,

$$\int_0^T \beta(t)\,dt < \infty.$$

Since $\Psi(t)$ is a positive term, from (7.20) as $n \to \infty$ we get,

$$\sup_{t\in[0,T]} \Big(\big|\mathbf{u}_n(t) - \mathbf{u}(t)\big|^2_{\mathrm{H}} + \big|\mathbf{d}_n(t) - \mathbf{d}(t)\big|^2_{L^2} + \big|\nabla\big(\mathbf{d}_n(t) - \mathbf{d}(t)\big)\big|^2_{L^2}\Big) \longrightarrow 0. \tag{7.21}$$

Now using this and going back to equation (7.19), as $n \to \infty$ we obtain,

$$\int_0^T \Big(\big|\nabla\big(\mathbf{u}_n(t) - \mathbf{u}(t)\big)\big|^2_{\mathrm{H}} + \big|\nabla\big(\mathbf{d}_n(t) - \mathbf{d}(t)\big)\big|^2_{L^2} + \big|\mathcal{A}\big(\mathbf{d}_n(t) - \mathbf{d}(t)\big)\big|^2_{L^2}\Big)\,dt \longrightarrow 0. \tag{7.22}$$

From (7.21) and (7.22) we conclude, $\mathbf{u}_n \to \mathbf{u}$ in $\mathcal{K}_{1,T}$ and $\mathbf{d}_n \to \mathbf{d}$ in $\mathcal{K}_{2,T}$. So we have $X_n \to X$ in $\mathcal{K}_T$. Therefore,

$$\mathcal{J}^0\Big(\int_0^{\cdot} q_n(s)\,ds\Big) \to \mathcal{J}^0\Big(\int_0^{\cdot} q(s)\,ds\Big) \quad \text{in } \mathcal{K}_T.$$

This proves the Condition 1.

$\square$

8. Proof of Condition 2

Let us recall the statement of Condition 2 for the convenience of the reader. Assume that $\alpha \in \mathbb{N}$, and $\varepsilon_n \to 0$. Let $q_n := (\theta_n, \rho_n), q := (\theta, \rho) \in \mathcal{U}^\alpha$ such that the laws $\mathscr{L}(q_n)$ converges weakly to $\mathscr{L}(q)$. Then the processes

$$\mathcal{Y}_T \ni W \mapsto \mathcal{J}^{\varepsilon_n}\Big(W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^\cdot q_n(s)\, ds\Big) \in \mathcal{K}_T$$

converge in law to $\mathcal{J}^0(\int_0^\cdot q(s)\, ds)$ on $\mathcal{K}_T$.

For simplicity of the notation, denote

$$\begin{cases} \mathcal{J}^{\varepsilon_n}\Big(\sqrt{\varepsilon}_n\, W + \int_0^\cdot q_n(s)\, ds\Big) := \Gamma_W^{\varepsilon_n}(q_n) := \Big(\Gamma_{W_1}^{\varepsilon_n}(\theta_n), \Gamma_{W_2}^{\varepsilon_n}(\rho_n)\Big), \\ \mathcal{J}^0\Big(\int_0^\cdot q_n(s)\, ds\Big) := \Gamma_W^0(q_n) := \Big(\Gamma_{W_1}^0(\theta_n), \Gamma_{W_2}^0(\rho_n)\Big), \\ \Gamma_{W_1}^{\varepsilon_n}(\theta_n) := Z_1^n,\ \Gamma_{W_2}^{\varepsilon_n}(\rho_n) := Z_2^n \ \text{ and } \ \Gamma_{W_1}^0(\theta_n) := z_1^n,\ \Gamma_{W_2}^0(\rho_n) := z_2^n. \end{cases} \tag{8.1}$$

Now for each $n \in \mathbb{N}$, define $(\mathcal{F}_t)$- stopping times ,

$$\begin{aligned} \tau_1^n &:= \inf\{t > 0 : |Z_1^n(t)|_{\mathrm{H}} \geq N\}, \\ \tau_2^n &:= \inf\{t > 0 : |\nabla Z_2^n(t)|_{L^2} \geq N\}, \\ \tau^n &= \tau_1^n \wedge \tau_2^n \wedge T. \end{aligned} \tag{8.2}$$

In order to prove the Condition 2 we split the proof to several lemmata 6.2 and 8.1 (which is stated below).

Lemma 8.1. *For τ^n defined in (8.2) we have*

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \Big(\big|Z_1^n(t\wedge\tau^n) - z_1^n(t\wedge\tau^n)\big|_{\mathrm{H}}^2 + \big|Z_2^n(t\wedge\tau^n) - z_2^n(t\wedge\tau^n)\big|_{L^2}^2 + \big|\nabla Z_2^n(t\wedge\tau^n) - \nabla z_2^n(t\wedge\tau^n)\big|_{L^2}^2\Big)\Big] = 0,$$

and

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^{\tau^n} \Big(\big|\nabla\big(Z_1^n(t) - z_1^n(t)\big)\big|_{\mathrm{H}}^2 + \big|\nabla\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 + \big|\mathcal{A}\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2\Big)\, dt\Big] = 0.$$

At first, we will prove the convergence in probability (i.e. Lemma 6.2). For that we will require the pointwise convergence in $\mathcal{K}_T$ i.e. Lemma 8.1, which we will prove at the end of this section. The results of Lemma 6.2 and Lemma 8.1 will be used to prove Condition 2.

8.1. **Proof of Lemma 6.2.**

Proof. Let us recall the statement of the Lemma 6.2 for the convenience of the reader. Assume $\alpha > 0$ and (ε_n) be a $(0,1]$-valued sequence converging to 0. Let $q_n, q \in \mathcal{U}^\alpha$ with $\mathscr{L}(q_n) \to \mathscr{L}(q)$ on S^α. Then the sequence

$$\mathcal{Y}_T \ni W \mapsto \mathcal{J}^{\varepsilon_n}\Big(\sqrt{\varepsilon_n}\, W + \int_0^\cdot q_n(s)\, ds\Big) - \mathcal{J}^0\Big(\int_0^\cdot q_n(s)\, ds\Big) \in \mathcal{K}_T$$

converges in probability to 0.

Let $\lambda > 0$ and $\delta > 0$. From Theorem 4.2 and part(1) of Definition 4.1, there exists $N > |\mathbf{u}_0|_{\mathrm{H}} + |\nabla \mathbf{d}_0|_{L^2}$ such that

$$\frac{1}{N} \sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \big(|Z_1^n(t)|_{\mathrm{H}} + |\nabla Z_2^n(t)|_{L^2}\big)\Big] < \frac{\delta}{2}.$$

Then using Lemma 8.1 (which we are proving in the end of this section), for all n sufficiently large we infer,

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{t\in[0,T]} \Big[\big|Z_1^n(t) - z_1^n(t)\big|_{\mathrm{H}}^2 + \big|Z_2^n(t) - z_2^n(t)\big|_{L^2}^2 + \big|\nabla Z_2^n(t) - \nabla z_2^n(t)\big|_{L^2}^2 \Big] \\
&\quad + \int_0^T \Big[\big|\nabla\big(Z_1^n(t) - z_1^n(t)\big)\big|_{\mathrm{H}}^2 + \big|\nabla\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 + \big|\mathcal{A}\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 \Big]\, dt \geq \lambda \bigg) \\
&\leq \mathbb{P}\bigg(\sup_{t\in[0,T]} \Big[\big|Z_1^n(t\wedge\tau^n) - z_1^n(t\wedge\tau^n)\big|_{\mathrm{H}}^2 \\
&\quad + \big|Z_2^n(t\wedge\tau^n) - z_2^n(t\wedge\tau^n)\big|_{L^2}^2 + \big|\nabla Z_2^n(t\wedge\tau^n) - \nabla z_2^n(t\wedge\tau^n)\big|_{L^2}^2 \Big] \\
&\quad + \int_0^{\tau^n} \Big[\big|\nabla\big(Z_1^n(t) - z_1^n(t)\big)\big|_{\mathrm{H}}^2 + \big|\nabla\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 + \big|\mathcal{A}\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 \Big]\, dt \geq \lambda, \tau^n = T \bigg) \\
&\quad + \mathbb{P}\bigg(\sup_{t\in[0,T]} \Big[|Z_1^n(t)|_{\mathrm{H}} + |\nabla Z_2^n(t)|_{L^2} \Big] \geq N \bigg) \\
&\leq \frac{1}{\lambda}\, \mathbb{E}\bigg(\sup_{t\in[0,T]} \Big[\big|Z_1^n(t\wedge\tau^n) - z_1^n(t\wedge\tau^n)\big|_{\mathrm{H}}^2 \\
&\quad + \big|Z_2^n(t\wedge\tau^n) - z_2^n(t\wedge\tau^n)\big|_{L^2}^2 + \big|\nabla Z_2^n(t\wedge\tau^n) - \nabla z_2^n(t\wedge\tau^n)\big|_{L^2}^2 \Big] \\
&\quad + \int_0^{\tau^n} \Big[\big|\nabla\big(Z_1^n(t) - z_1^n(t)\big)\big|_{\mathrm{H}}^2 + \big|\nabla\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 + \big|\mathcal{A}\big(Z_2^n(t) - z_2^n(t)\big)\big|_{L^2}^2 \Big]\, dt \bigg) \\
&\quad + \frac{1}{N}\, \mathbb{E} \sup_{t\in[0,T]} \Big(|Z_1^n(t)|_{\mathrm{H}} + |\nabla Z_2^n(t)|_{L^2} \Big) \\
&< \delta.
\end{aligned}$$

This proves the convergence in probability as required.

□

Now we are ready to prove the Condition 2.

8.2. **Proof of Condition 2.**

Proof. Recall the notations in (8.1). Let us fix $\alpha > 0, \varepsilon_n$ be a $(0,1]$-valued sequence converging to 0. Let $q_n := (\theta_n, \rho_n) \in \mathcal{U}^\alpha$ which converges in law to $q := (\theta, \rho)$ on S^α. Then the following claims hold true:

(1) The $\mathcal{K}_T$-valued random variables $\Gamma_W^{\varepsilon_n}(q_n)$ converges in probability to $\Gamma_W^0(q_n)$.
(2) $\Gamma_W^0(q_n)$ converges in law to $\Gamma_W^0(q)$ on $\mathcal{K}_T$.

Claim(1) follows from Lemma 6.2.
Now we need to prove Claim(2). Recall that S^α is a separable metric space. The assumption

says, the sequence of laws $(\mathscr{L}(q_n))$ converges weakly to $\mathscr{L}(q)$. Then by the Skorokhod Theorem (See [17], Theorem 4.30), there exists a new probability space $\big(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\big)$, and on that probability space, there exist random variables $\tilde q_n$ and $\tilde q$, which have same laws as q_n and q respectively with $\tilde q_n \to \tilde q$ in S^α, pointwise on $\tilde\Omega$. Using this and from Lemma 6.1 we obtain,

$$\Gamma^0_W(\tilde q_n) \to \Gamma^0_W(\tilde q) \ \text{ pointwise on } \ \tilde\Omega \ \text{ in } \ \mathcal{K}_T.$$

Moreover, from the second part of Lemma 6.1, we infer that the laws

$$\mathscr{L}\big(\Gamma^0_W(\tilde q_n)\big) = \mathscr{L}\big(\Gamma^0_W(q_n)\big) \ \text{ and } \ \mathscr{L}\big(\Gamma^0_W(\tilde q)\big) = \mathscr{L}\big(\Gamma^0_W(q)\big).$$

Note that pointwise convergence implies almost sure convergence, which implies convergence in probability and it implies convergence in law. So it proves Claim(2).

Since in Lemma 6.2 we proved the convergence in probability, we can choose a subsequence (keeping the same notation) such that

$$\text{the sequence of } \ \mathcal{K}_T\text{-valued random variables } \ \Gamma^{\varepsilon_n}_W(\tilde q_n) - \Gamma^0_W(\tilde q_n) \ \text{ converges to } 0, \ \tilde{\mathbb{P}} - \text{a.s.}$$

Now we can see for any bounded and globally Lipschitz continuous function $g : \mathcal{K}_T \to \mathbb{R}$ (See Dudley [14], Theorem 11.3.3) we obtain,

$$\begin{aligned}
&\left| \int_{\mathcal{K}_T} g(x)\, d\mathscr{L}\big(\Gamma^{\varepsilon_n}_W(q_n)\big) - \int_{\mathcal{K}_T} g(x)\, d\mathscr{L}\big(\Gamma^0_W(q)\big) \right| \\
&= \left| \int_{\mathcal{K}_T} g(x)\, d\mathscr{L}\big(\Gamma^{\varepsilon_n}_W(\tilde q_n)\big) - \int_{\mathcal{K}_T} g(x)\, d\mathscr{L}\big(\Gamma^0_W(\tilde q)\big) \right| = \left| \int_{\tilde\Omega} g\big(\Gamma^{\varepsilon_n}_W(\tilde q_n)\big)\, d\tilde{\mathbb{P}} - \int_{\tilde\Omega} g\big(\Gamma^0_W(\tilde q)\big)\, d\tilde{\mathbb{P}} \right| \\
&\le \int_{\tilde\Omega} \big| g\big(\Gamma^{\varepsilon_n}_W(\tilde q_n)\big) - g\big(\Gamma^0_W(\tilde q_n)\big) \big|\, d\tilde{\mathbb{P}} + \int_{\tilde\Omega} \big| g\big(\Gamma^0_W(\tilde q_n)\big) - g\big(\Gamma^0_W(\tilde q)\big) \big|\, d\tilde{\mathbb{P}}.
\end{aligned} \tag{8.3}$$

Since g is a bounded continuous function and $\Gamma^0_W(\tilde q_n) \to \Gamma^0_W(\tilde q)$ a.s., the second term of right hand side goes to 0.
Using the properties of g we can show,

$$\int_{\tilde\Omega} \big| g\big(\Gamma^{\varepsilon_n}_W(\tilde q_n)\big) - g\big(\Gamma^0_W(\tilde q_n)\big) \big|\, d\tilde{\mathbb{P}} \le C_g \int_{\tilde\Omega} \big| \Gamma^{\varepsilon_n}_W(\tilde q_n) - \Gamma^0_W(\tilde q_n) \big|\, d\tilde{\mathbb{P}}.$$

Since $\Gamma^{\varepsilon_n}_W(\tilde q_n) - \Gamma^0_W(\tilde q_n)$ is $\tilde{\mathbb{P}}$-a.s. convergent to 0, the first term of right hand side in (8.3) also goes to 0. This proves the Condition 2.

Finally, as promised, we will prove Lemma 8.1.

8.3. **Proof of Lemma 8.1.**

Proof. Recall the notations in (8.1). Let $\xi^n = Z_1^n - z_1^n$ and $\mu^n = Z_2^n - z_2^n$. These processes satisfy,

$$\begin{aligned}
&d\xi^n(t) + \Big(\mathscr{A}\xi^n(t) + B(\xi^n(t), Z_1^n(t)) + B(z_1^n(t), \xi^n(t)) \Big)\, dt \\
&= -\Big(M(\mu^n(t), Z_2^n(t)) + M(z_2^n(t), \mu^n(t)) \Big)\, dt + \Big[\big(\sigma(Z_1^n(t)) - \sigma(z_1^n) \big)\, \theta_n(t) \Big]\, dt \\
&\quad + \sqrt{\varepsilon_n}\, \sigma(Z_1^n(t))\, dW_1(t),
\end{aligned}$$

and

$$
\begin{aligned}
&d\mu^n(t) + \Big(\mathcal{A}\mu^n(t) + \tilde{B}(\xi^n(t), Z_2^n(t)) + \tilde{B}(z_1^n(t), \mu^n(t))\Big)\, dt \\
&= -\Big(f(Z_2^n(t)) - f(z_2^n(t))\Big)\, dt + \frac{\varepsilon_n}{2}\, G^2(Z_2^n(t))\, dt + \Big[\big(G(Z_2^n(t)) - G(z_2^n(t))\big)\, \rho_n(t)\Big]\, dt \\
&\quad + \sqrt{\varepsilon_n}\, G(Z_2^n(t))\, dW_2(t).
\end{aligned}
$$

From (2.5), (2.6), (2.7) and using Poincaré and Young's inequalities, we obtain for any $\alpha_1, \alpha_2, \cdots, \alpha_6 > 0$, there exist $C(\alpha_1), C(\alpha_2, \alpha_3), C(\alpha_7, \alpha_4)$ and $C(\alpha_5) > 0$ such that

$$
\begin{aligned}
&|\langle B(\xi^n(t), Z_1^n(t)), \xi^n\rangle| \le \alpha_1 \big|\nabla \xi^n(t)\big|_{\mathrm{H}}^2 + C(\alpha_1)\big|Z_1^n(t)\big|_{\mathrm{H}}^2 \big|\nabla Z_1^n(t)\big|_{\mathrm{H}}^2 \big|\xi^n(t)\big|_{\mathrm{H}}^2, \\
&|\langle M(z_2^n(t), \mu^n(t)), \xi^n(t)\rangle| \le \alpha_2 \big|\nabla \xi^n(t)\big|_{\mathrm{H}}^2 + \alpha_3 \big|\mathcal{A}\mu^n(t)\big|_{L^2}^2 \\
&\qquad\qquad\qquad\qquad + C(\alpha_2, \alpha_3)\big|\nabla z_2^n(t)\big|_{L^2}^2 \big|\mathcal{A} z_2^n(t)\big|_{L^2}^2 \big|\nabla \mu^n(t)\big|_{L^2}^2, \\
&|\langle M(\mu^n(t), Z_2^n(t)), \xi^n\rangle| \le \alpha_7 \big|\nabla \xi^n(t)\big|_{\mathrm{H}}^2 + \alpha_4 \big|\mathcal{A}\mu^n(t)\big|_{L^2}^2 \\
&\qquad\qquad\qquad\qquad + C(\alpha_7, \alpha_4)\big|\nabla Z_2^n(t)\big|_{L^2}^2 \big|\mathcal{A} Z_2^n(t)\big|_{L^2}^2 \big|\nabla \mu^n(t)\big|_{L^2}^2, \\
&|\langle \tilde{B}(z_1^n(t), \mu^n(t)), \mathcal{A}\mu^n(t)\rangle| \le \alpha_5 \big|\mathcal{A}\mu^n(t)\big|_{L^2}^2 + C(\alpha_5)\big|z_1^n(t)\big|_{\mathrm{H}}^2 \big|\nabla z_1^n(t)\big|_{\mathrm{H}}^2 \big|\nabla \mu^n(t)\big|_{L^2}^2, \\
&|\langle \tilde{B}(\xi^n(t), Z_2^n(t)), \mu^n(t)\rangle| \le \alpha_6 \big|\mathcal{A}\mu^n(t)\big|_{L^2}^2 + C(\alpha_6)\big|\xi^n(t)\big|_{\mathrm{H}}^2 \big|\nabla Z_2^n(t)\big|_{L^2}^2.
\end{aligned}
$$

Also using the properties of G we have,

$$
\begin{aligned}
\big|\nabla G(Z_2^n(t))\big|_{L^2}^2 &\le C(\mathbf{h})\big(\big|\nabla Z_2^n(t)\big|_{L^2}^2 + \big|Z_2^n(t)\big|_{L^2}^2\big), \\
\big|\nabla G^2(Z_2^n(t))\big|_{L^2}^2 &\le C(\mathbf{h})\big(\big|\nabla Z_2^n(t)\big|_{L^2}^2 + \big|Z_2^n(t)\big|_{L^2}^2\big).
\end{aligned}
$$

And for any $\alpha_8, \alpha_9 > 0$ there exist $C(\alpha_8), C_1(\alpha_9), C_2(\alpha_9) > 0$ such that (see Appendix of [8])

$$
\begin{aligned}
&|\langle f(Z_2^n(t)) - f(z_2^n(t)), Z_2^n(t) - z_2^n(t)\rangle|_{L^2} \le \alpha_8 \big|\nabla Z_2^n(t) - \nabla z_2^n(t)\big|_{L^2}^2 \\
&\qquad\qquad\qquad\qquad + C(\alpha_8)\big|Z_2^n(t) - z_2^n(t)\big|_{L^2}^2\, \varphi(Z_2^n(t), z_2^n(t)), \\
&|\langle f(Z_2^n(t)) - f(z_2^n(t)), \mathcal{A} Z_2^n(t) - \mathcal{A} z_2^n(t)\rangle|_{L^2} \\
&\le \alpha_9 \big|\mathcal{A} Z_2^n(t) - \mathcal{A} z_2^n(t)\big|_{L^2}^2 + \big[C_1(\alpha_9)\big|\nabla Z_2^n(t) - \nabla z_2^n(t)\big|_{L^2}^2\, \varphi(Z_2^n(t), z_2^n(t)) \\
&\quad + C_2(\alpha_9)\big|Z_2^n(t) - z_2^n(t)\big|_{L^2}^2\, \varphi(Z_2^n(t), z_2^n(t))\big].
\end{aligned}
$$

where $\varphi(Z_2^n(t), z_2^n(t))$ is defined as

$$
\varphi(Z_2^n(t), z_2^n(t)) := C\Big(1 + \big|Z_2^n(t)\big|_{L^{4N+2}}^{2N} + \big|z_2^n(t)\big|_{L^{4N+2}}^{2N}\Big)^2.
$$

Now consider

$$
\Psi(t) := \exp\bigg(- \int_0^t (\psi_1(s) + \psi_2(s) + \psi_3(s))\, ds\bigg), \quad \text{for any } t > 0.
$$

where

$$
\begin{aligned}
\psi_1(s) &:= C(\alpha_1)\big|Z_1^n(s)\big|_{\mathrm{H}}^2\,\big|\nabla Z_1^n(s)\big|_{\mathrm{H}}^2 + C(\alpha_6)\big|\nabla Z_2^n(s)\big|_{L^2}^2,\\
\psi_2(s) &:= C(\alpha_2,\alpha_3)\big|\nabla z_2^n(s)\big|_{L^2}^2\big|\mathcal{A}z_2^n(s)\big|_{L^2}^2 + C(\alpha_7,\alpha_4)\big|\nabla Z_2^n(s)\big|_{L^2}^2\big|\mathcal{A}Z_2^n(s)\big|_{L^2}^2\\
&\qquad + C(\alpha_5)\big|z_1^n(s)\big|_{\mathrm{H}}^2\,\big|\nabla z_1^n(s)\big|_{\mathrm{H}}^2 + C_1(\alpha_9)\varphi(Z_2^n(s), z_2^n(s)),\\
\psi_3(s) &:= \big[C(\alpha_8) + C_2(\alpha_9)\big]\varphi(Z_2^n(s), z_2^n(s)).
\end{aligned}
$$

Now applying Itô's formula to $\Psi(t)\big|\mu^n(t)\big|_{L^2}^2$,

$$
\begin{aligned}
d\Big[\Psi(t)\big|\mu^n(t)\big|_{L^2}^2\Big] &= -2\Psi(t)\Big[\big|\nabla\mu^n(t)\big|_{L^2}^2 + \big\langle \tilde{B}(\xi^n(t), Z_2^n(t))\\
&\quad + \big(f(Z_2^n(t)) - f(z_2^n(t))\big), \mu^n(t)\big\rangle\Big]\,dt\\
&\quad + \underbrace{2\Psi(t)\,\big\langle\big(G(Z_2^n(t)) - G(z_2^n(t))\big)\rho_n(t), \mu^n(t)\big\rangle\,dt}_{I_{n,1}(t)}\\
&\quad + \underbrace{\Psi(t)\,\varepsilon_n\left(\big\langle G^2(Z_2^n(t)), \mu^n(t)\big\rangle + \big|G(Z_2^n(t))\big|_{L^2}^2\right)dt}_{I_{n,2}(t)}\\
&\quad + \underbrace{2\Psi(t)\,\sqrt{\varepsilon_n}\,\big\langle G(Z_2^n(t)), \mu^n(t)\big\rangle\,dW_2(t)}_{I_{n,3}(t)} + \Psi'(t)\,\big|\mu^n(t)\big|_{L^2}^2\,dt.
\end{aligned}
\tag{8.4}
$$

Again applying Itô's formula to $\Psi(t)\big|\xi^n(t)\big|_{\mathrm{H}}^2$,

$$
\begin{aligned}
d\Big[\Psi(t)\,\big|\xi^n(t)\big|_{\mathrm{H}}^2\Big] &= -2\Psi(t)\Big[\big|\nabla\xi^n(t)\big|_{\mathrm{H}}^2 + \big\langle B(\xi^n(t), Z_1^n(t))\\
&\quad + M(\mu^n(t), Z_2^n(t)) + M(z_2^n(t), \mu^n(t)), \xi^n(t)\big\rangle\Big]\,dt\\
&\quad + \underbrace{2\Psi(t)\,\big\langle\big(\sigma(Z_1^n(t)) - \sigma(z_1^n(t))\big)\,\theta_n(t), \xi^n(t)\big\rangle\,dt}_{I_{n,4}(t)}\\
&\quad + \underbrace{\Psi(t)\,\varepsilon_n\big|\sigma(Z_1^n(t))\big|_{\mathrm{H}}^2\,dt}_{I_{n,5}(t)}\\
&\quad + \underbrace{2\Psi(t)\,\sqrt{\varepsilon_n}\,\big\langle\sigma(Z_1^n(t)), \xi^n(t)\big\rangle\,dW_1(t)}_{I_{n,6}(t)} + \Psi'(t)\,\big|\xi^n(t)\big|_{\mathrm{H}}^2\,dt
\end{aligned}
\tag{8.5}
$$

Now applying Itô's formula to $\Psi(t)\big|\nabla\mu^n(t)\big|^2_{L^2}$,

$$
\begin{aligned}
d\Big[\Psi(t)\big|\nabla\mu^n(t)\big|^2_{L^2}\Big] = {} & 2\Psi(t)\Big[-\big|\mathcal{A}\mu^n(t)\big|^2_{L^2} \\
& - \big\langle \tilde{B}(\xi^n(t), Z_2^n(t)) + \tilde{B}(z_1^n(t), \mu^n(t)), \mathcal{A}\mu^n(t)\big\rangle\Big]\, dt \\
& - 2\Psi(t)\left\langle f(Z_2^n(t)) - f(z_2^n(t)), \mathcal{A}\mu^n(t)\right\rangle dt \\
& + \underbrace{2\Psi(t)\left\langle \big(G(Z_2^n(t)) - G(z_2^n(t))\big)\rho_n(t), \mathcal{A}\mu^n(t)\right\rangle dt}_{I_{n,7}(t)} \\
& + \underbrace{\Psi(t)\,\varepsilon_n\left(\left\langle \nabla G^2(Z_2^n(t)), \nabla\mu^n(t)\right\rangle + \big|G(Z_2^n(t))\big|^2_{L^2}\right) dt}_{I_{n,8}(t)} \\
& + \underbrace{2\Psi(t)\sqrt{\varepsilon_n}\left\langle \nabla G(Z_2^n(t)), \nabla\mu^n(t)\right\rangle dW_2(t)}_{I_{n,9}(t)} + \Psi'(t)\,\big|\nabla\mu^n(t)\big|^2_{L^2}\, dt.
\end{aligned}
\tag{8.6}
$$

Now adding (8.4), (8.5), (8.6) and using the inequalities stated earlier we have,

$$
\begin{aligned}
& d\Big[\Psi(t)\big(\big|\xi^n(t)\big|^2_{\mathrm{H}} + \big|\mu^n(t)\big|^2_{L^2} + \big|\nabla\mu^n(t)\big|^2_{L^2}\big)\Big] \\
& \quad + 2\Psi(t)\Big[\big|\nabla\xi^n(t)\big|^2_{\mathrm{H}} + \big|\nabla\mu^n(t)\big|^2_{L^2} + \big|\mathcal{A}\mu^n(t)\big|^2_{L^2}\Big]\, dt \\
& \le 2\Psi(t)\Bigg[\Big(\alpha_9 + \sum_{i=3}^{6}\alpha_i\Big)\,\big|\mathcal{A}\mu^n(t)\big|^2_{L^2} + (\alpha_1 + \alpha_2 + \alpha_7)\big|\nabla\xi^n(t)\big|^2_{\mathrm{H}} + \alpha_8\big|\nabla\mu^n(t)\big|^2_{L^2}\Bigg] \\
& \quad + 2\Psi(t)\Big[\psi_1(t)\,\big|\xi^n(t)\big|^2_{\mathrm{H}} + \psi_2(t)\,\big|\nabla\mu^n(t)\big|^2_{L^2} + \psi_3(t)\big|\mu^n(t)\big|^2_{L^2}\Big]\, dt \\
& \quad + \Psi'(t)\Big(\big|\xi^n(t)\big|^2_{\mathrm{H}} + \big|\mu^n(t)\big|^2_{L^2} + \big|\nabla\mu^n(t)\big|^2_{L^2}\Big)\, dt + \sum_{j=1}^{9} I_{n,j}(t).
\end{aligned}
$$

Now we estimate $I_{n,j}$, for $j = 1, \cdots, 9$.

$$
\begin{aligned}
|I_{n,1}(t)| &= \big|2\Psi(t)\left\langle \big(G(Z_2^n(t)) - G(z_2^n(t))\big)\rho_n(t), \mu^n(t)\right\rangle dt\big| \\
&\le 2\Psi(t)\,\big|G(Z_2^n(t)) - G(z_2^n(t))\big|_{L^2}\,|\rho_n(t)|_{\mathbb{R}}\,|\mu^n(t)|_{L^2}\, dt \\
&\le 2\Psi(t)\sqrt{L}\,|Z_2^n(t) - z_2^n(t)|_{L^2}\,|\rho_n(t)|_{\mathbb{R}}\,|\mu^n(t)|_{L^2}\, dt \\
&\le \Psi(t)\,\big(L + \big|\rho_n(t)\big|^2_{\mathbb{R}}\big)\big|\mu^n(t)\big|^2_{L^2}\, dt.
\end{aligned}
$$

$$
\begin{aligned}
|I_{n,2}(t)| &= \left|\Psi(t)\,\varepsilon_n\left(\left\langle G^2(Z_2^n(t)), \mu^n(t)\right\rangle + \big|G(Z_2^n(t))\big|^2_{L^2}\right) dt\right| \\
&\le \Psi(t)\,\varepsilon_n\left(|Z_2^n(t)|_{L^2}\,\big|\mathbf{h}\big|^2_{L^\infty}\,|\mu^n(t)|_{L^2} + 1 + \big|Z_2^n(t)\big|^2_{L^2}\right) dt \\
&\le \Psi(t)\,\varepsilon_n\left(1 + \tilde{C}\,\big|Z_2^n(t)\big|^2_{L^2} + C(\mathbf{h})\,\big|\mu^n(t)\big|^2_{L^2}\right) dt.
\end{aligned}
$$

Similar calculations yield,

$$
\begin{aligned}
|I_{n,4}(t)| &\le \Psi(t)\left(K + \left|\theta_n(t)\right|_{\mathbb{R}}^2\right)\left|\xi^n(t)\right|_{\mathrm{H}}^2\,dt,\\
|I_{n,5}(t)| &\le \Psi(t)\,\varepsilon_n\left(1 + \left|Z_1^n(t)\right|_{\mathrm{H}}^2\right)dt.
\end{aligned}
$$

Using integration by parts we obtain,

$$
\begin{aligned}
|I_{n,7}(t)| &= |2\Psi(t)\left\langle \left(G(Z_2^n(t)) - G(z_2^n(t))\right)\rho_n(t), \Delta\mu^n(t)\right\rangle dt|\\
&\le 2\Psi(t)\left|\left\langle \nabla\left(G(Z_2^n(t)) - G(z_2^n(t))\right)\rho_n(t), \nabla\mu^n(t)\right\rangle\right| dt\\
&\le 2\Psi(t)\sqrt{L}\,|\rho_n(t)|_{\mathbb{R}}\left|\nabla\big(Z_2^n(t) - z_2^n(t)\big)\right|_{L^2}|\nabla\mu^n(t)|_{L^2}\,dt\\
&\le 2\Psi(t)\left(L + \left|\rho_n(t)\right|_{\mathbb{R}}^2\right)\left|\nabla\mu^n(t)\right|_{L^2}^2 dt
\end{aligned}
$$

$$
\begin{aligned}
|I_{n,8}(t)| &= \left|\Psi(t)\,\varepsilon_n\left(\left\langle \nabla G^2(Z_2^n(t)), \nabla\mu^n(t)\right\rangle + \left|G(Z_2^n(t))\right|_{L^2}^2\right)dt\right|\\
&\le \Psi(t)\,\varepsilon_n\Big(|\nabla G^2(Z_2^n(t))|_{L^2}|\nabla\mu^n(t)|_{L^2} + 1 + \left|Z_2^n(t)\right|_{L^2}^2\Big)\,dt\\
&\le \Psi(t)\,\varepsilon_n\Big[C(\mathbf{h})\big(\left|\nabla Z_2^n(t)\right|_{L^2}^2 + \left|Z_2^n(t)\right|_{L^2}^2\big) + C\left|\nabla\mu^n(t)\right|_{L^2}^2 + 1 + \left|Z_2^n(t)\right|_{L^2}^2\Big]\,dt
\end{aligned}
$$

By the choice of Ψ we have,

$$
\begin{aligned}
2\Psi(t)&\Big[\psi_1(t)\left|\xi^n(t)\right|_{\mathrm{H}}^2 + \psi_2(t)\left|\nabla\mu^n(t)\right|_{L^2}^2 + \psi_3(t)\left|\mu^n(t)\right|_{L^2}^2\Big]\\
&+ \Psi'(t)\Big(\left|\xi^n(t)\right|_{\mathrm{H}}^2 + \left|\mu^n(t)\right|_{L^2}^2 + \left|\nabla\mu^n(t)\right|_{L^2}^2\Big) \le 0.
\end{aligned}
$$

In the equation (7.17), choose $\alpha_9 = \alpha_i = \frac{1}{10}$ for $i = 3, 4, 5, 6$, $\alpha_1 = \alpha_2 = \alpha_7 = \frac{1}{6}$ and $\alpha_8 = \frac{1}{2}$. Then using previous estimates and rearranging we obtain,

$$
\begin{aligned}
&d\big[\Psi(t)\big(\left|\xi^n(t)\right|_{\mathrm{H}}^2 + \left|\mu^n(t)\right|_{L^2}^2 + \left|\nabla\mu^n(t)\right|_{L^2}^2\big)\big]\\
&\quad + \Psi(t)\big[\left|\nabla\xi^n(t)\right|_{\mathrm{H}}^2 + \left|\nabla\mu^n(t)\right|_{L^2}^2 + \left|\mathcal{A}\mu^n(t)\right|_{L^2}^2\big]\,dt\\
&\le 2\Psi(t)\Big[\big(K + \left|\theta_n(t)\right|_{\mathrm{H}_0}^2\big)\left|\xi^n(t)\right|_{\mathrm{H}}^2 + \big(C(\mathbf{h})\varepsilon_n + L + \left|\rho_n(t)\right|_{\mathbb{R}}^2\big)\left|\mu^n(t)\right|_{L^2}^2\\
&\quad + \big(C\varepsilon_n + L + \left|\rho_n(t)\right|_{\mathbb{R}}^2\big)\left|\nabla\mu^n(t)\right|_{L^2}^2\Big]\,dt\\
&\quad + \Psi(t)\,\varepsilon_n\left(3 + \bar{C}\left|Z_2^n(t)\right|_{L^2}^2 + C(\mathbf{h})\left|\nabla Z_2^n(t)\right|_{L^2}^2 + \left|Z_1^n(t)\right|_{\mathrm{H}}^2\right)dt\\
&\quad + I_{n,3}(t) + I_{n,6}(t) + I_{n,9}(t).
\end{aligned}
\tag{8.7}
$$

Define

$$
\begin{aligned}
x_n(t) &= \Psi(t)\,\varepsilon_n\left(3 + \bar{C}\left|Z_2^n(t)\right|_{L^2}^2 + C(\mathbf{h})\left|\nabla Z_2^n(t)\right|_{L^2}^2 + \left|Z_1^n(t)\right|_{\mathrm{H}}^2\right),\\
y_n(t) &= 2L + K + \tilde{C}\varepsilon_n + \left|\theta_n(t)\right|_{\mathrm{H}_0}^2 + 2\left|\rho_n(t)\right|_{\mathbb{R}}^2.
\end{aligned}
$$

Then we write (8.7) in integral form as:

$$\begin{aligned}
&\Psi(t)\big[\big|\xi^n(t)\big|_{\mathrm{H}}^2+\big|\mu^n(t)\big|_{L^2}^2+\big|\nabla\mu^n(t)\big|_{L^2}^2\big] \\
&\quad+\int_0^t \Psi(s)\big[\big|\nabla\xi^n(s)\big|_{\mathrm{H}}^2+\big|\nabla\mu^n(s)\big|_{L^2}^2+\big|\mathcal{A}\mu^n(s)\big|_{L^2}^2\big]\,ds \\
&\leq \int_0^t x_n(s)\,ds+\int_0^t y_n(s)\,\Psi(s)\left(\big|\xi^n(s)\big|_{\mathrm{H}}^2+\big|\mu^n(s)\big|_{L^2}^2+\big|\nabla\mu^n(s)\big|_{L^2}^2\right)ds \\
&\quad+\int_0^t [I_{n,3}(s)+I_{n,6}(s)+I_{n,9}(s)]\,ds.
\end{aligned} \tag{8.8}$$

Drop the second term in the left hand side of (8.8). Define

$$\Xi(t):=\Psi(t)\big[\big|\xi^n(t)\big|_{\mathrm{H}}^2+\big|\mu^n(t)\big|_{L^2}^2+\big|\nabla\mu^n(t)\big|_{L^2}^2\big].$$

Using the definition of τ^n, we can write (8.8) further as

$$\begin{aligned}
\Xi_{t\wedge\tau^n} \leq& \int_0^{t\wedge\tau^n} x_n(s)\,ds \\
&+\int_0^{t\wedge\tau^n} y_n(s)\,\Psi(s)\Big(\big|\xi^n(s)\big|_{\mathrm{H}}^2+\big|\mu^n(s)\big|_{L^2}^2+\big|\nabla\mu^n(s)\big|_{L^2}^2\Big)\,ds \\
&+2\sqrt{\varepsilon_n}\left|\int_0^{t\wedge\tau^n} \Psi(s)\,\big\langle G(Z_2^n(s)),\mu^n(s)\big\rangle\,dW_2(s)\right| \\
&+2\sqrt{\varepsilon_n}\left|\int_0^{t\wedge\tau^n} \Psi(s)\,\big\langle \sigma(Z_1^n(s)),\xi^n(s)\big\rangle\,dW_1(s)\right| \\
&+2\sqrt{\varepsilon_n}\left|\int_0^{t\wedge\tau^n} \Psi(s)\,\big\langle \nabla G(Z_2^n(s)),\nabla\mu^n(s)\big\rangle\,dW_2(s)\right|.
\end{aligned}$$

Now taking supremum over all $s\leq t$ and taking expectation we infer that,

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}\Xi_{s\wedge\tau^n} \leq& \mathbb{E}\bigg[\int_0^{t\wedge\tau^n} x_n(s)\,ds \\
&+\int_0^{t\wedge\tau^n} y_n(s)\,\Psi(s)\Big(\big|\xi^n(s)\big|_{\mathrm{H}}^2+\big|\mu^n(s)\big|_{L^2}^2+\big|\nabla\mu^n(s)\big|_{L^2}^2\Big)\,ds\bigg] \\
&+\mathbb{E}\bigg[\sup_{0\leq s\leq t} 2\sqrt{\varepsilon_n}\left|\int_0^{s\wedge\tau^n} \Psi(r)\,\big\langle G(Z_2^n(r)),\mu^n(r)\big\rangle\,dW_2(r)\right|\bigg] \\
&+\mathbb{E}\bigg[\sup_{0\leq s\leq t} 2\sqrt{\varepsilon_n}\left|\int_0^{s\wedge\tau^n} \Psi(r)\,\big\langle \sigma(Z_1^n(r)),\xi^n(r)\big\rangle\,dW_1(r)\right|\bigg] \\
&+\mathbb{E}\bigg[\sup_{0\leq s\leq t} 2\sqrt{\varepsilon_n}\left|\int_0^{s\wedge\tau^n} \Psi(r)\,\big\langle \nabla G(Z_2^n(r)),\nabla\mu^n(r)\big\rangle\,dW_2(r)\right|\bigg].
\end{aligned} \tag{8.9}$$

Consider the third term of right hand side. Applying Burkhölder-Davis-Gundy inequality, using properties of σ, definition of τ^n and using the energy estimates we obtain,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t} 2\sqrt{\varepsilon_n}\Big|\int_0^{s\wedge\tau^n}\Psi(r)\left\langle\sigma(Z_1^n(r)),\xi^n(r)\right\rangle dW_1(r)\Big|\Big]\\
&\le C\,\mathbb{E}\Big[\sqrt{\varepsilon_n}\Big\{\int_0^{s\wedge\tau^n}|\Psi(r)|^2\left|\sigma(Z_1^n(r))\right|^2_{\mathrm{L}_2}\left|\xi^n(r)\right|^2_{\mathrm{H}}dr\Big\}^{\frac12}\Big]\\
&\le C\,\mathbb{E}\Big[\sqrt{\varepsilon_n}\Big\{(s\wedge\tau^n)\sup_{r\in[0,s\wedge\tau^n]}|\Psi(r)|^2\cdot\sup_{r\in[0,s\wedge\tau^n]}\left(1+\left|Z_1^n(r)\right|^2_{\mathrm{H}}\right)\\
&\qquad\qquad\cdot\sup_{r\in[0,s\wedge\tau^n]}\left(\left|Z_1^n(r)\right|^2_{\mathrm{H}}+\left|z_1^n(r)\right|^2_{\mathrm{H}}\right)\Big\}^{\frac12}\Big]\\
&\le\sqrt{C_N}\sqrt{\varepsilon_n}\,\mathbb{E}\Big[\sup_{s\in[0,t]}\left|\Psi(s\wedge\tau^n)\right|\Big]\le C_1(N)\sqrt{\varepsilon_n}.
\end{aligned}\tag{8.10}$$

Similar calculations show there exist $C_2(N),C_3(N)>0$ such that

$$\mathbb{E}\Big[\sup_{0\le s\le t} 2\sqrt{\varepsilon_n}\Big|\int_0^{s\wedge\tau^n}\Psi(r)\left\langle G(Z_2^n(r)),\mu^n(r)\right\rangle dW_2(r)\Big|\Big]\le C_2(N)\sqrt{\varepsilon_n},$$
$$\mathbb{E}\Big[\sup_{0\le s\le t} 2\sqrt{\varepsilon_n}\Big|\int_0^{s\wedge\tau^n}\Psi(r)\left\langle\nabla G(Z_2^n(r)),\nabla\mu^n(r)\right\rangle dW_2(r)\Big|\Big]\le C_3(N)\sqrt{\varepsilon_n}.$$

Now dropping the second term in the definition of $\Xi_{s\wedge\tau^n}$, using previous estimates, there exists $\bar{C}_N>0$ such that we write (8.9) as,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t}\Big\{\Psi(s\wedge\tau^n)\Big(\left|\xi^n(s\wedge\tau^n)\right|^2_{\mathrm{H}}+\left|\mu^n(s\wedge\tau^n)\right|^2_{L^2}+\left|\nabla\mu^n(s\wedge\tau^n)\right|^2_{L^2}\Big)\Big\}\Big]\\
&\le\bar{C}_N\sqrt{\varepsilon_n}+\int_0^t\mathbb{E}\Big[\sup_{r\le s}x_n(r\wedge\tau^n)\Big]ds+\int_0^t\mathbb{E}\Big[\sup_{r\le s}\Big\{\Psi(r\wedge\tau^n)\Big(\left|\xi^n(r\wedge\tau^n)\right|^2_{\mathrm{H}}\\
&\quad+\left|\mu^n(r\wedge\tau^n)\right|^2_{L^2}+\left|\nabla\mu^n(r\wedge\tau^n)\right|^2_{L^2}\Big)y_n(r\wedge\tau^n)\Big\}\Big]ds.
\end{aligned}$$

Applying Gronwall lemma we get,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,T]}\Big\{\Psi(s\wedge\tau^n)\Big(\left|\xi^n(s\wedge\tau^n)\right|^2_{\mathrm{H}}+\left|\mu^n(s\wedge\tau^n)\right|^2_{L^2}+\left|\nabla\mu^n(s\wedge\tau^n)\right|^2_{L^2}\Big)\Big\}\Big]\\
&\le\Big\{\bar{C}_N\sqrt{\varepsilon_n}+\int_0^T\mathbb{E}\Big[\sup_{r\le s}x_n(r\wedge\tau^n)\Big]ds\Big\}\cdot\exp\Big(\int_0^T y_n(r\wedge\tau^n)\,dr\Big).
\end{aligned}$$

Since $\theta_n,\rho_n\in\mathcal{U}^\alpha$, as $\varepsilon_n\to 0$, from the definition of y_n we have

$$\int_0^T y_n(r\wedge\tau^n)dr=\int_0^T\left(2L+K+C\varepsilon_n+\left|\theta_n(r\wedge\tau^n)\right|^2_{\mathrm{H}_0}+2\left|\rho_n(r\wedge\tau^n)\right|^2_{\mathbb{R}}\right)dr<\infty.$$

Using the definition of τ^n, as $\varepsilon_n \to 0$ we infer that,

$$\begin{aligned}&\bar{C}_N \sqrt{\varepsilon_n} + \int_0^T \mathbb{E}\Big[\sup_{r\le s} \; x_n(r\wedge\tau^n)\Big]\, ds \\ &= \bar{C}_N \sqrt{\varepsilon_n} + \int_0^T \mathbb{E}\Big[\sup_{r\le s}\Big\{\Psi(r\wedge\tau^n)\,\varepsilon_n \Big(3+\bar{C}\left|Z_2^n(r\wedge\tau^n)\right|_{L^2}^2 \\ &\quad + C(\mathbf{h})\left|\nabla Z_2^n(r\wedge\tau^n)\right|_{L^2}^2 + \left|Z_1^n(r\wedge\tau^n)\right|_{\mathrm{H}}^2\Big)\Big\}\Big]\, ds \longrightarrow 0 \quad \text{as } n\to\infty.\end{aligned}$$

Since Ψ is a positive term, finally we obtain,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\left|\xi^n(t\wedge\tau^n)\right|_{\mathrm{H}}^2 + \left|\mu^n(t\wedge\tau^n)\right|_{L^2}^2 + \left|\nabla\mu^n(t\wedge\tau^n)\right|_{L^2}^2\Big)\Big] \to 0 \quad \text{as } n\to\infty.$$

Using this result in (8.9) we obtain,

$$\mathbb{E}\Big[\int_0^{\tau^n}\Big(\left|\nabla\xi^n(t)\right|_{\mathrm{H}}^2 + \left|\nabla\mu^n(t)\right|_{L^2}^2 + \left|\mathcal{A}\mu^n(t)\right|_{L^2}^2\Big)\, dt\Big] \to 0 \quad \text{as } n\to\infty.$$

So this proves the desired result.

□

9. Proof of the main result, Theorem 5.3.

The proofs of Condition 1 and Condition 2 in previous sections are sufficient to claim that the family of laws $\{\mathscr{L}(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon) : \varepsilon \in (0,1]\}$ on $\mathcal{K}_T$ satisfies the large deviation principle with rate function I defined in (5.12).

□

Appendix A. Proof of Theorem 5.5

This theorem speaks about the existence and uniqueness of the skeleton equation (5.8)-(5.9). We will use the classical Faedo-Galerkin approximation to prove the existence results.

A.1. **Faedo-Galerkin approximation.** Our proof of existence of the skeleton equation depends on the Galerkin approximation method. Let $\{\varrho_i\}_{i=1}^\infty$ be the orthonormal basis of H composed of eigenfunctions of the stokes operator $\mathscr{A}$. Let $\{\varsigma_i\}_{i=1}^\infty$ be the orthonormal basis of L^2 consisting of the eigenfunctions of the Neumann Laplacian $\mathcal{A}$. Let us define the following finite dimensional spaces for any $n\in\mathbb{N}$

$$\begin{aligned}\mathrm{H}_n &:= Linspan\{\varrho_1,\dots,\varrho_n\},\\ \mathbb{L}_n &:= Linspan\{\varsigma_1,\dots,\varsigma_n\}.\end{aligned}$$

Our aim is to derive uniform estimates for the solution of the projection of (5.8)-(5.9) onto the finite dimensional space $\mathrm{H}_n\times\mathbb{L}_n$, i.e., its Galerkin approximation. For this let us denote by P_n the projection from H onto H_n and $\tilde{P}_n$ be the projection from L^2 onto $\mathbb{L}_n$. We consider the following locally Lipschitz mappings:

$$B_n : \mathrm{H}_n \ni \mathbf{u} \mapsto P_n B(\mathbf{u}, \mathbf{u}) \in \mathrm{H}_n,$$
$$M_n : \mathbb{L}_n \ni \mathbf{d} \mapsto P_n M(\mathbf{d}) \in \mathrm{H}_n,$$
$$f_n : \mathbb{L}_n \ni \mathbf{d} \mapsto \tilde{P}_n f(\mathbf{d}) \in \mathbb{L}_n,$$
$$\tilde{B}_n : \mathrm{H}_n \times \mathbb{L}_n \ni (\mathbf{u}, \mathbf{d}) \mapsto \tilde{P}_n \tilde{B}(\mathbf{u}, \mathbf{d}) \in \mathbb{L}_n.$$

Let $P_n \mathbf{u}_0 = \mathbf{u}_n(0) := \mathbf{u}_{0n}$ and $\tilde{P}_n \mathbf{d}_0 = \mathbf{d}_n(0) := \mathbf{d}_{0n}$.
So the Galerkin approximation of the problem is:

$$d\mathbf{u}_n(t) + \left[\mathscr{A}\mathbf{u}_n(t) + B_n(\mathbf{u}_n(t)) + M_n(\mathbf{d}_n(t))\right] dt = P_n \sigma(\mathbf{u}_n(t))\, \theta(t)\, dt, \tag{A.1}$$

$$d\mathbf{d}_n(t) + \left[\mathcal{A}\mathbf{d}_n(t) + \tilde{B}_n(\mathbf{u}_n(t), \mathbf{d}_n(t)) + f_n(\mathbf{d}_n(t))\right] dt = \tilde{P}_n G(\mathbf{d}_n(t))\, \rho(t)\, dt. \tag{A.2}$$

Lemma A.1. *For each $n \in \mathbb{N}$, the problem* (A.1)-(A.2) *has a unique global solution.*

Proof. We have the existence result due to [19]. □

The processes $\big(\mathbf{u}_n\big)_{n\in\mathbb{N}}$ and $\big(\mathbf{d}_n\big)_{n\in\mathbb{N}}$ satisfy the following estimates.

Proposition A.2. *For any $p \geq 2$, there exists a positive constant $\tilde{C} = \tilde{C}_p$, independent of ρ such that*

$$\sup_{n\in\mathbb{N}} \left[\sup_{s\in[0,T]} \big|\mathbf{d}_n(s)\big|_{L^2}^p + \int_0^T \big|\mathbf{d}_n(s)\big|_{L^2}^{p-2} \big(|\nabla \mathbf{d}_n(s)|_{L^2}^2 + |\mathbf{d}_n(s)|_{L^{2N+2}}^{2N+2}\big)\, ds \right] \leq \tilde{C},$$

where $\tilde{C} := |\mathbf{d}_0|_{L^2}^p \big(1 + CTe^{CT}\big)$.

Proof. The proof is similar to the proof in [4]. In our case we will be using the fact that $\langle G(\mathbf{d}_n(s)), \mathbf{d}_n(s)\rangle = 0$. Then the other steps of the proof will follow as in [4] (see also [19]). □

Proposition A.3. *There exists a positive constant $\bar{C}$ depending on $K, T, \mathbf{h}, |\rho|_{L^2(0,T;\mathbb{R})}, |\theta|_{L^2(0,T;\mathrm{H}_0)}$ such that*

$$\sup_{n\geq 1} \left[\sup_{s\in[0,T]} \big|\mathbf{u}_n(s)\big|_{\mathrm{H}}^2 + \Psi(\mathbf{d}_n(s)) + \int_0^T \Big(\big\|\mathbf{u}_n(s)\big\| + \frac{1}{2} \big|f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)\big|_{L^2}^2 \Big)\, ds \right] \leq \bar{C},$$

where $\Psi(z) := \frac{1}{2}|\nabla z|^2 + \frac{1}{2}\int_{\mathcal{O}} \tilde{\mathrm{F}}(|z|^2)\, dx$ and K is the linear growth coefficient for σ.

Proof. Consider the approximated system (A.1)-(A.2). Now take the inner product of (A.1) with $\mathbf{u}_n$ and writing in integral form we obtain,

$$\begin{aligned}
&\big|\mathbf{u}_n(t)\big|_{\mathrm{H}}^2 - \big|\mathbf{u}_{0n}\big|_{\mathrm{H}}^2 \\
&= -\int_0^t \big\langle \mathscr{A}\mathbf{u}_n(s) + B_n(\mathbf{u}_n(s)) + M_n(\mathbf{d}_n(s)), \mathbf{u}_n(s)\big\rangle\, ds + \int_0^t \big\langle P_n\sigma(\mathbf{u}_n(s))\, \theta(s), \mathbf{u}_n(s)\big\rangle\, ds \\
&= -\int_0^t |\nabla \mathbf{u}_n(s)|^2\, ds - \int_0^t \big\langle M_n(\mathbf{d}_n(s)), \mathbf{u}_n(s)\big\rangle\, ds + \int_0^t \big\langle P_n\sigma(\mathbf{u}_n(s))\, \theta(s), \mathbf{u}_n(s)\big\rangle\, ds
\end{aligned} \tag{A.3}$$

Now consider the map

$$\Psi(z) := \frac{1}{2}|\nabla z|^2 + \frac{1}{2}\int_{\mathcal{O}} \tilde{\mathrm{F}}(|z|^2)\, dx.$$

The first Fréchet derivative is: $\Psi'(z)[g] = \langle \nabla z, \nabla g\rangle + \langle f(z), g\rangle = \langle \mathcal{A}z + f(z), g\rangle$.

From (A.2) we get,

$$
\begin{aligned}
&\Psi(\mathbf{d}_n(t)) - \Psi(\mathbf{d}_n(0)) \\
&= -\int_0^t \Psi'(\mathbf{d}_n(s))\big[\mathcal{A}\mathbf{d}_n(s) + \tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s)) + f_n(\mathbf{d}_n(s))\big]\,ds \\
&\quad + \int_0^t \Psi'(\mathbf{d}_n(s))\big[\tilde{P}_n G(\mathbf{d}_n(s))\,\rho(s)\big]\,ds \\
&= \int_0^t -|f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)|^2\,ds - \big\langle \tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s)), f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)\big\rangle\,ds \\
&\quad + \int_0^t \big\langle \tilde{P}_n G(\mathbf{d}_n(s))\,\rho(s), f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)\big\rangle\,ds.
\end{aligned}
\tag{A.4}
$$

From the proof of Proposition 5.5 in [8] we have,

$$
\begin{aligned}
&\big\langle \tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s)), f_n(\mathbf{d}_n(s))\big\rangle = 0, \\
&\text{and} \\
&\big\langle \tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s)), \mathcal{A}\mathbf{d}_n(s)\big\rangle = \big\langle M_n(\mathbf{d}_n(s)), \mathbf{u}_n(s)\big\rangle.
\end{aligned}
$$

Now adding (A.3) and (A.4) and rearranging we obtain,

$$
\begin{aligned}
&|\mathbf{u}_n(t)|^2 + \Psi(\mathbf{d}_n(t)) + \int_0^t \big(|\nabla \mathbf{u}_n(s)|^2 + |f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)|^2\big)\,ds \\
&= |\mathbf{u}_{0n}|^2 + \Psi(\mathbf{d}_{0n}) + \int_0^t \big\langle P_n\sigma(\mathbf{u}_n(s))\,\theta(s), \mathbf{u}_n(s)\big\rangle\,ds \\
&\quad + \int_0^t \big\langle \tilde{P}_n G(\mathbf{d}_n(s))\,\rho(s), f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)\big\rangle\,ds
\end{aligned}
\tag{A.5}
$$

We will estimate the term $\int_0^t \big|\big\langle P_n\sigma(\mathbf{u}_n(s))\,\theta(s), \mathbf{u}_n(s)\big\rangle\big|\,ds$. Using the linear growth property of σ (with growth constant K), the Cauchy-Schwarz inequality, the embedding of $\mathrm{H}_0 \hookrightarrow \mathrm{H}$ and the fact that $a \le \sqrt{1+a^2} \le 1+a^2$, for $a>0$ we infer,

$$
\begin{aligned}
&\int_0^t \big|\big\langle P_n\sigma(\mathbf{u}_n(s))\,\theta(s), \mathbf{u}_n(s)\big\rangle\big|\,ds \le \int_0^t |\sigma(\mathbf{u}_n(s))|_{\mathrm{L}_2}\,|\theta(s)|_{\mathrm{H}}\,|\mathbf{u}_n(s)|_{\mathrm{H}}\,ds \\
&\le \sqrt{K}\int_0^t \sqrt{1+|\mathbf{u}_n(s)|_{\mathrm{H}}^2}\;|\theta(s)|_{\mathrm{H}}\,|\mathbf{u}_n(s)|_{\mathrm{H}}\,ds \le \sqrt{K}\int_0^t \big(1+|\mathbf{u}_n(s)|_{\mathrm{H}}^2\big)\,|\theta(s)|_{\mathrm{H}_0} \\
&\le \sqrt{K}\int_0^t \big(1+|\mathbf{u}_n(s)|_{\mathrm{H}}^2\big)\big(1+|\theta(s)|_{\mathrm{H}_0}^2\big)\,ds \\
&= \int_0^t \sqrt{K}\,ds + \sqrt{K}\int_0^t |\theta(s)|_{\mathrm{H}_0}^2\,ds + \sqrt{K}\int_0^t |\mathbf{u}_n(s)|_{\mathrm{H}}^2\,\big(1+|\theta(s)|_{\mathrm{H}_0}^2\big)\,ds.
\end{aligned}
\tag{A.6}
$$

Similar procedure will give,

$$\begin{aligned}&\int_0^t \left|\left\langle \tilde{P}_n G(\mathbf{d}_n(s))\,\rho(s), f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)\right\rangle\right| ds\\&\le \int_0^t |G(\mathbf{d}_n(s))|_{L^2}\,|\rho(s)|_{\mathbb{R}}\,|f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)|_{L^2}\,ds\\&\le \frac{C(\mathbf{h})}{2}\int_0^t |\mathbf{d}_n(s)|^2_{L^2}\,|\rho(s)|^2_{\mathbb{R}}\,ds + \frac{1}{2}\int_0^t |f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)|^2_{L^2}\,ds.\end{aligned}\tag{A.7}$$

Using (A.6) and (A.7) in (A.5) we obtain,

$$\begin{aligned}&|\mathbf{u}_n(t)|^2 + \Psi(\mathbf{d}_n(t)) + \int_0^t \Big(|\nabla \mathbf{u}_n(s)|^2 + \frac{1}{2}\,|f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)|^2\Big)\,ds\\&\le |\mathbf{u}_{0n}|^2 + \Psi(\mathbf{d}_{0n}) + \tilde{C}(K,T) + \sqrt{K}\int_0^t |\mathbf{u}_n(s)|^2_{\mathrm{H}}\big(1 + |\theta(s)|^2_{\mathrm{H}_0}\big)\,ds\\&\quad + \frac{C(\mathbf{h})}{2}\sup_{s\in[0,T]} |\mathbf{d}_n(s)|^2_{L^2}\int_0^t |\rho(s)|^2_{\mathbb{R}}\,ds.\end{aligned}\tag{A.8}$$

Now using Proposition A.2 (for $p = 2$) in (A.8) we finally get,

$$\begin{aligned}\big|\mathbf{u}_n(t)\big|^2_{\mathrm{H}} &\le |\mathbf{u}_{0n}|^2 + \Psi(\mathbf{d}_{0n}) + C(K,T) + C(\mathbf{h}, T, |\rho|_{L^2(0,T;\mathbb{R})})\\&\quad + \sqrt{K}\int_0^t \big|\mathbf{u}_n(s)\big|^2_{\mathrm{H}}\big(1 + \big|\theta(s)\big|^2_{\mathrm{H}_0}\big)\,ds.\end{aligned}\tag{A.9}$$

Using the Gronwall lemma we infer from (A.9),

$$\sup_{s\in[0,T]} |\mathbf{u}_n(s)|^2_{\mathrm{H}} \le \big[|\mathbf{u}_{0n}|^2 + \Psi(\mathbf{d}_{0n}) + C(K, \mathbf{h}, |\rho|_{L^2(0,T;\mathbb{R})}, T)\big] \times \exp\Big\{\sqrt{K}\big(1 + |\theta|^2_{L^2(0,T;\mathrm{H}_0)}\big)\Big\}.\tag{A.10}$$

Since $(\theta, \rho) \in L^2(0, T; \mathrm{H}_0 \times \mathbb{R})$, finally using (A.10) we obtain,

$$\sup_{n\ge 1}\left[\sup_{s\in[0,T]} |\mathbf{u}_n(s)|^2_{\mathrm{H}} + \Psi(\mathbf{d}_n(s)) + \int_0^T \Big(\|\mathbf{u}_n(s)\|^2 + \frac{1}{2}\,|f_n(\mathbf{d}_n(s)) + \mathcal{A}\mathbf{d}_n(s)|^2_{L^2}\Big)\,ds\right] \le \bar{C},$$

where the positive constant

$$\bar{C} := \big[|\mathbf{u}_{0n}|^2 + \Psi(\mathbf{d}_{0n}) + C(K, \mathbf{h}, |\rho|_{L^2(0,T;\mathbb{R})}, T)\big]\exp\big\{C(K, |\theta|_{L^2(0,T;\mathrm{H}_0)}, T)\big\}.$$

$\square$

Proposition A.4. *Let $\beta \in (0, \frac{1}{2})$. Then there exist positive constants $C = C_\beta$ and $\tilde{C} = \tilde{C}_\beta$ such that*

$$\sup_{n\ge 1}\big|\mathbf{u}_n\big|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{V}')} \le C \qquad \textit{and} \qquad \sup_{n\ge 1}\big|\mathbf{d}_n\big|^2_{\mathrm{W}^{\beta,2}([0,T];(D(\mathcal{A}))')} \le \tilde{C}.$$

Proof. We write the approximated system as:

$$\begin{aligned}\mathbf{u}_n(t) &= \mathbf{u}_{0n} - \int_0^t \mathscr{A}\,\mathbf{u}_n(s)\,ds - \int_0^t B_n(\mathbf{u}_n(s))\,ds - \int_0^t M_n(\mathbf{d}_n(s))\,ds + \int_0^t P_n\sigma(\mathbf{u}_n(s))\,\theta(s)\,ds,\\&:= I_n^1 + I_n^2(t) + I_n^3(t) + I_n^4(t) + I_n^5(t).\end{aligned}$$

and

$$\mathbf{d}_n(t) = \mathbf{d}_{0n} - \int_0^t \mathcal{A}\mathbf{d}_n(s)\,ds - \int_0^t \tilde{B}_n(\mathbf{u}_n(s),\mathbf{d}_n(s))\,ds - \int_0^t f_n(\mathbf{d}_n(s))\,ds \\ + \int_0^t \tilde{P}_n G(\mathbf{d}_n(s))\,\rho(s)\,ds := J_n^1 + J_n^2(t) + J_n^3(t) + J_n^4(t) + J_n^5(t). \tag{A.11}$$

Using the same arguments as in Theorem 3.1 of Flandoli and Gatarek [16], we obtain

$$\left|I_n^1\right|^2_{\mathrm{H}} \le k_1, \quad \left|I_n^2\right|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{V}')} \le k_2, \quad \left|I_n^3\right|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{V}')} \le k_3.$$

For $t > s$, using the Cauchy-Schwarz and Young's inequality and the property of σ, we get

$$\left|I_n^5(t) - I_n^5(s)\right|^2_{\mathrm{H}} = \left|\int_s^t P_n\sigma(\mathbf{u}_n(l))\,\theta(l)\,dl\right|^2_{\mathrm{H}} \le \left(\int_s^t |P_n\sigma(\mathbf{u}_n(l))\,\theta(l)|_{\mathrm{H}}\,dl\right)^2 \\ \le \left(\int_s^t \sqrt{K}\sqrt{1+|\mathbf{u}_n(l)|^2_{\mathrm{H}}}\,|\theta(l)|_{\mathrm{H}}\,dl\right)^2 \le c\Big(1+\sup_{l\in[0,T]}|\mathbf{u}_n(l)|^2_{\mathrm{H}}\Big)\int_s^t K\,dl\int_s^t|\theta(l)|^2_{\mathrm{H}_0}\,dl. \tag{A.12}$$

Taking $s=0$, then integrating from 0 to T we obtain,

$$\int_0^T \left|I_n^5(l)\right|^2_{\mathrm{H}}\,dl \le C(T)\Big(1+\sup_{l\in[0,T]}|\mathbf{u}_n(l)|^2_{\mathrm{H}}\Big)\int_0^T K\,dl\int_0^T|\theta(l)|^2_{\mathrm{H}_0}\,dl, \tag{A.13}$$

and

$$\int_0^T\int_0^T \frac{\left|I_n^5(t)-I_n^5(s)\right|^2_{\mathrm{H}}}{|t-s|^{1+2\beta}}\,dt\,ds \le c\Big(1+\sup_{l\in[0,T]}|\mathbf{u}_n(l)|^2_{\mathrm{H}}\Big)\int_0^T K\,dl\int_0^T\int_0^T\int_s^t \frac{\left|\theta(l)\right|^2_{\mathrm{H}_0}}{|t-s|^{1+2\beta}}\,dl\,dt\,ds. \tag{A.14}$$

Using Fubini's theorem for $\beta\in(0,\frac{1}{2})$,

$$\int_0^T\int_0^T\int_s^t \frac{\left|\theta(l)\right|^2_{\mathrm{H}}}{|t-s|^{1+2\beta}}\,dl\,dt\,ds \le \tilde{c}\int_0^T \left|\theta(l)\right|^2_{\mathrm{H}_0}\,dl \le C. \tag{A.15}$$

Using Proposition A.3, inequalities (A.12), (A.13), (A.14) and (A.15) we get,

$$\left|I_n^5\right|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{H})} \le k_5.$$

Now we calculate for $t>s$,

$$\left|I_n^4(t)-I_n^4(s)\right|^2_{\mathrm{V}'} = \left|\int_s^t M_n(\mathbf{d}_n(l))\,dl\right|^2_{\mathrm{V}'} \le \left(\int_s^t |M_n(\mathbf{d}_n(l))|_{\mathrm{V}'}\,dl\right)^2 \\ \le \left(\int_s^t |\nabla\mathbf{d}_n(l))|_{L^2}\,|\mathcal{A}\mathbf{d}_n(l))|_{L^2}\,dl\right)^2 \le C_T\sup_{l\in[0,T]}|\nabla\mathbf{d}_n(l))|^2_{L^2}\int_s^t|\mathcal{A}\mathbf{d}_n(l))|^2_{L^2}\,dl.$$

Now using the similar procedure as done to get inequalities (A.12), (A.13), (A.14), (A.15) with the help of Proposition A.2 and Proposition A.3 we get,

$$\left|I_n^4\right|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{V}')} \le k_4.$$

Now consider (A.11). From Theorem 3.1 of Flandoli and Gatarek [16] we get,

$$\left|J_n^1\right|^2_{H^1} \le C_1, \qquad \left|J_n^2\right|^2_{\mathrm{W}^{1,2}([0,T];(D(\mathcal{A}))')} \le C_2.$$

Again for $t > s$, from (2.6) we have,

$$
\begin{aligned}
&\left|J_n^3(t) - J_n^3(s)\right|^2_{(D(\mathcal{A}))'} = \left|\int_s^t \tilde{B}_n(\mathbf{u}_n(l), \mathbf{d}_n(l))\, dl\right|^2_{(D(\mathcal{A}))'} \leq \left(\int_s^t |\tilde{B}_n(\mathbf{u}_n(l), \mathbf{d}_n(l))|_{L^2}\, dl\right)^2 \\
&\leq c\left(\int_s^t \left\{|\mathbf{u}_n(l)|\,|\nabla \mathbf{u}_n(l)| + |\nabla \mathbf{d}_n(l)|\,|\mathcal{A}\mathbf{d}_n(l)|\right\} dl\right)^2 \\
&\leq C_T \int_s^t \left\{|\mathbf{u}_n(l)|^2\,|\nabla \mathbf{u}_n(l)|^2 + |\nabla \mathbf{d}_n(l)|^2\,|\mathcal{A}\mathbf{d}_n(l)|^2\right\} dl \\
&\leq C_T\left[\sup_{l\in[0,T]} |\mathbf{u}_n(l)|^2_{\mathrm{H}} \int_s^t \|\mathbf{u}_n(l)\|^2\, dl + \sup_{l\in[0,T]} |\nabla \mathbf{d}_n(l)|^2_{L^2} \int_s^t |\mathcal{A}\mathbf{d}_n(l)|^2_{L^2}\, dl\right].
\end{aligned}
$$

Now using Proposition A.2 and Proposition A.3, following the same technique as (A.12)-(A.15), for $\beta \in (0, \frac{1}{2})$ we obtain

$$
\left|J_n^3\right|^2_{\mathrm{W}^{\beta,2}([0,T];(D(\mathcal{A}))')} \leq C_3.
$$

Following Lemma 6.1 (equation (6.26)) in [8] and similar procedure as (A.12)-(A.15) will give for $\beta \in (0, \frac{1}{2})$,

$$
\left|J_n^4\right|^2_{\mathrm{W}^{\beta,2}([0,T];(D(\mathcal{A}))')} \leq C_4.
$$

Now for J_n^5, since $\rho \in S_2^{\alpha}$, using the property of G, Proposition A.2 and similar technique as (A.12)-(A.15) we conclude for $\beta \in (0, \frac{1}{2})$,

$$
\left|J_n^5\right|^2_{\mathrm{W}^{\beta,2}([0,T];L^2)} \leq C_5.
$$

Since $\mathrm{H} \hookrightarrow \mathrm{V}'$ and $L^2 \hookrightarrow (D(\mathcal{A}))'$, from above estimates for $\beta \in (0, \frac{1}{2})$ we get

$$
\sup_{n\geq 1} \left|\mathbf{u}_n\right|^2_{\mathrm{W}^{\beta,2}([0,T];\mathrm{V}')} \leq C_\beta \qquad \text{and} \qquad \sup_{n\geq 1} \left|\mathbf{d}_n\right|^2_{\mathrm{W}^{\beta,2}([0,T];(D(\mathcal{A}))')} \leq \tilde{C}_\beta
$$

□

Finally we are ready to prove Theorem 5.5.

A.2. **Proof of Theorem 5.5.**

Proof. **Existence:**
Let us choose and fix $\beta \in (0, \frac{1}{2})$. From Proposition A.2, A.3 and A.4 we deduce that, there exist positive constants $C_1, C_2, C_3(\beta), C_4(\beta)$ such that (7.3), (7.4) and (7.5) hold. From the above estimates and Lemma 2.3, we infer that there exists a subsequence $(\mathbf{u}_{m'}, \mathbf{d}_{m'})$ and an element

$$
(\mathbf{u}, \mathbf{d}) \in L^2([0,T]; \mathrm{V}) \cap L^\infty([0,T]; \mathrm{H}) \times L^2([0,T]; D(\mathcal{A})) \cap L^\infty([0,T]; H^1)
$$

such that as $m' \to \infty$ we have,

$$\begin{cases} \mathbf{u}_{m'} \to \mathbf{u} & \text{in} \quad L^2([0,T];\mathrm{V}) \quad \text{weakly}, \\ \mathbf{u}_{m'} \to \mathbf{u} & \text{in} \quad L^\infty([0,T];\mathrm{H}) \quad \text{weak-star}, \\ \mathbf{d}_{m'} \to \mathbf{d} & \text{in} \quad L^2([0,T];D(\mathcal{A})) \quad \text{weakly}, \\ \mathbf{d}_{m'} \to \mathbf{d} & \text{in} \quad L^\infty([0,T];H^1) \quad \text{weak-star}. \\ \mathbf{u}_{m'} \to \mathbf{u} & \text{in} \quad L^2([0,T];\mathrm{H}) \quad \text{strongly}, \\ \mathbf{d}_{m'} \to \mathbf{d} & \text{in} \quad L^2([0,T];H^1) \quad \text{strongly}. \end{cases}$$

Finally, we show $(\mathbf{u}, \mathbf{d})$ is the unique solution to (5.8)-(5.9). For this we will argue similarly as in the proof of Theorem 3.1 in Temam [24], Section 3.

The similar technique to prove (7.11) (in the proof of Condition 1), gives us the convergence as $m' \to \infty$, for individual terms involving control parameters. For all other linear and nonlinear terms, we follow the calculations of our earlier work (see [8]).

Using similar arguments as in the proof of Theorem 3.1 in Temam [24], Section 3, Chapter III, we infer that $(\mathbf{u}, \mathbf{d})$ is the desired solution.
Equations (5.10), (5.11) can be proved similarly as equations (7.3) and (7.4). Now using the similar arguments as in the proof of Theorem 3.2 in Temam [24], Section 3, Chapter III, we also have

$$\frac{d\mathbf{u}}{dt} \in L^2([0,T];\mathrm{V}') + L^1([0,T];\mathrm{H}) \quad \text{and} \quad \frac{d\mathbf{d}}{dt} \in L^2([0,T];(D(\mathcal{A}))') + L^1([0,T];H^1).$$

Due to Lemma 1.2, (1.84) and (1.85) in Temam [24], Chapter III, we infer that

$$\mathbf{u} \in \mathbb{C}([0,T];\mathrm{H}) \quad \text{and} \quad \mathbf{d} \in \mathbb{C}([0,T];H^1).$$

Uniqueness:
The uniqueness will follow the same technique required to get (7.21) and (7.22) in the proof of Condition 1.

□

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