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Optimal investment and contingent claim valuation with exponential disutility under proportional transaction costs

Alet Roux* and Zhikang Xu†

Abstract. We consider indifference pricing of contingent claims consisting of payment flows in a discrete time model with proportional transaction costs and under exponential disutility. This setting covers utility maximisation as a special case. A dual representation is obtained for the associated disutility minimisation problem, together with a dynamic procedure for solving it. This leads to an efficient and convergent numerical procedure for indifference pricing which applies to a wide range of payoffs, a large range of time steps and all magnitudes of transaction costs.

Key words. transaction costs, option pricing, utility maximisation, entropy, indifference pricing, generalised convex hull, dynamic programming

AMS subject classifications. 91G20, 91G60

1. Introduction.

The price of a contingent claim in a complete market is uniquely determined by the principle of replication: it is the discounted expectation of the claim price under the (unique) martingale measure. However, the presence of transaction costs can lead to the curious contradiction that superreplicating a claim may involve less trading (and lower transaction costs) than exact replication, and therefore be less expensive, so that the replication price can in fact lead to arbitrage. Furthermore, financial markets with transaction costs generally admit many different martingale measures, leading to intervals of no-arbitrage claim prices. This means that subjective factors, such as an investor's risk appetite, come into play when determining the price of a claim. The indifference principle offers a compelling alternative to replication and arbitrage pricing: it states that the seller of a claim will charge (at least) a price that will allow him to sell the claim without increasing the risk of his existing financial position. This is called the *indifference price*. As a special case, the *reservation price* is a price that would have allowed the seller to cover a claim at an acceptable level of risk, had their existing position been zero (in other words, not taking it into account). This is often associated with the terms "economic capital" in banking, and "technical provisions" or "reserving" in insurance.

Indifference pricing based on utility maximisation has been well studied in the literature on proportional transaction costs. Work in continuous time has mostly focused on adapting stochastic optimal control and other techniques from friction-free models (such as the Black-Scholes model), and in recent years have led to numerical approximation and asymptotics for small transaction costs; see [2, 7, 8, 17, 21, 23, 24, 38], for example. Results obtained in continuous time models typically assume continuous trading, which limits their applicability in realistic settings [12], hence motivating the need for continued theoretical and numerical work in the discrete time setting. The literature on indifference pricing in discrete time models

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with proportional transaction costs is nevertheless very sparse.

The present paper is motivated by the work of Pennanen [26], who studied indifference pricing in a very general discrete time setting, including proportional transaction costs. In view of the fact that financial liabilities in banking and insurance often consist of sequences of payment streams, such as swaps, coupon paying bonds, insurance premia, etc, the classical utility maximisation framework, which focuses on the expected disutility of hedging shortfall at the expiration date of the liability faced by an investor (and insists on self-financing trading at other times), is extended in [26] to a more flexible framework which allows hedging to fall short at intermediate steps too, takes into account the expected total disutility of hedging shortfall at all steps, and presents theoretical results for contingent claims consisting of cash payment streams and a very general class of disutility functions.

The present paper specialises the setting of [26] to exponential utility and proportional transaction costs, which allows the use of powerful dual methods, and finite state space, motivated by the need for numerical results. Our results apply to contingent claims with physical delivery (in other words, streams of portfolios rather than just cash). We propose a backward recursive procedure that can be used to solve the utility maximisation problem and compute indifference prices, together with an efficient and convergent numerical approximation method (with error bounds). Our results apply to all magnitudes of transaction costs, and our numerical methods work for a large range of time steps; see [39] for more demanding numerical results that have not been included in this paper for lack of space.

The results reveal interesting features of disutility minimisation problems and indifference prices. In particular, because asset holdings in our model can be carried over between different time periods, the value of the disutility minimisation problem of an investor faced with delivering a portfolio stream depends only on the total payment involved in the stream (suitably discounted), which implies that indifference prices also depend only on the total payment due. Nevertheless, the additional flexibility offered by allowing hedging to fall short at time periods other than the final time leads to smaller spreads in indifference prices, when compared to utility indifference pricing spreads. Our numerical results further suggest that there is a complex relationship between disutility indifference prices and the real-world measure.

The results in this paper extend and complement the limited number of results that have already been reported in the literature for discrete time models with proportional transaction costs. The results on disutility minimisation generalise the results reported in [4] in a one-step binomial model with proportional transaction costs. To put the power of the numerical methods into context, previously reported numerical results are limited to European put options in a 3-step Cox-Ross-Rubinstein binomial model with convex transaction costs and exponential utility ([5]), utility indifference prices of a European call option under exponential utility in a binomial tree model with 6 steps and proportional transaction costs ([29]), and numerical solution of utility maximisation problems under power utility with multiple assets and proportional transaction costs ([3]).

Whilst we restrict our attention to indifference prices (payable at time 0 in cash) rather than indifference swap rates (used in [26]) for brevity, we believe that the extension is straightforward (with preliminary work reported in [39]). We believe that our work can be generalised to include measuring hedging shortfall in terms of portfolios rather than just cash; this is the subject of ongoing research, as is application of these methods to other classes of utility functions and multi-asset models.

The paper is arranged as follows. Background information on arbitrage and superhedging in discrete time models with proportional transaction costs is collected in section 2. The disutility minimisation problem that forms the basis of the indifference pricing framework is introduced in section 3; this includes utility maximisation as a special case. In section 4 we derive a Lagrangian dual formulation for the disutility minimisation problem, which leads to a dynamic procedure for solving it, presented in section 6. Indifference prices are introduced in section 5, together with arbitrage pricing bounds. A number of illustrative numerical examples are reported in section 7. Appendix A is devoted to the study of a number of properties of a generalisation of the convex hull of convex functions that appears in the dynamic procedure of section 6; this includes a numerical approximation by piecewise linear functions, complete with error bound.

2. Preliminaries.

2.1. Discrete-time model with proportional transaction costs. In this paper we consider a discrete-time financial market model with a finite time horizon $T \in \mathbb{N}$ and trading dates $t = 0, \ldots, T$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t=0}^T$. We assume without loss of generality that $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. For each t, the collection of atoms of $\mathcal{F}_t$ is denoted by Ω_t. The elements of Ω_t are called the *nodes* of the model at time t, and they form a partition of Ω. For each $\omega \in \Omega$ and $t = 0, \ldots, T$, denote by ω_t the unique node $\nu \in \Omega_t$ such that $\omega \in \nu$. A node $\nu \in \Omega_{t+1}$ is said to be a *successor* of a node $\mu \in \Omega_t$ if $\nu \subseteq \mu$. For each $t < T$, denote the collection of successors of any given node $\mu \in \Omega_t$ by μ^+, and define the transition probability from μ to any successor node $\nu \in \mu^+$ by $p_{t+1}^\nu := \frac{\mathbb{P}(\nu)}{\mathbb{P}(\mu)}$.

For each t and $d \in \mathbb{N}$, let $\mathcal{L}_t^d$ be the space of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random variables. Every random variable $x \in \mathcal{L}_t^d$ satisfies $x(\omega) = x(\omega')$ for all $\omega, \omega' \in \nu$ on every node $\nu \in \Omega_t$, and we will sometimes denote this common value by x^ν. A similar convention will apply to $\mathcal{F}_t$-measurable random functions $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d'}$ (where $d' \in \mathbb{N}$). Let $\mathcal{N}^d$ be the space of adapted $\mathbb{R}^d$-valued processes. We write $\mathcal{L}_t = \mathcal{L}_t^1$ and $\mathcal{N} = \mathcal{N}^1$ for convenience.

The financial market model consists of a risky and risk-free asset. The price of the risk-free asset, *cash*, is constant and equal to 1 at all times. This is equivalent to assuming that interest rates are zero, or that asset prices are discounted. Trading in the risky asset, the *stock*, is subject to proportional transaction costs. At any time step t, a share of the stock can be bought for the *ask price* S_t^a and sold for the *bid price* S_t^b, where $S_t^a \geq S_t^b > 0$. We assume that $(S_t^a)_{t=0}^T \in \mathcal{N}$ and $(S_t^b)_{t=0}^T \in \mathcal{N}$.

The cost of creating a portfolio $x = (x^b, x^s) \in \mathcal{L}_t^2$ at any time t is

$$\phi_t(x) := x^b + x_+^s S_t^a - x_-^s S_t^b, \tag{2.1}$$

where $z_+ := \max\{z, 0\}$ and $z_- := -\min\{z, 0\}$ for all $z \in \mathbb{R}$. The liquidation value of the portfolio x is

$$x^b + x_+^s S_t^b - x_-^s S_t^a = -\phi_t(-x).$$

Define the *solvency cone* $\mathcal{K}_t$ at any time t as the collection of portfolios that can be liquidated into a nonnegative cash amount, in other words,

$$\mathcal{K}_t := \left\{x \in \mathcal{L}_t^2 : -\phi_t(-x) \geq 0\right\} = \left\{(x^b, x^s) \in \mathcal{L}_t^2 : x^b + x^s S_t^b \geq 0, x^b + x^s S_t^a \geq 0\right\}.$$

A trading strategy is an adapted sequence of portfolios, denoted $(y_t)_{t=-1}^T$, where $y_{-1} \in \mathcal{L}_0^2$ denotes the initial endowment at time 0, the portfolio $y_t \in \mathcal{L}_t^2$ is held between time steps t and $t+1$ for $t = 0, \ldots, T-1$, and $y_T \in \mathcal{L}_T^2$ is the terminal portfolio created at time T. Denote the collection of trading strategies by $\mathcal{N}^{2\prime}$, and define

$$\Delta y_t := y_t - y_{t-1} \text{ for all } t = 0, \ldots, T.$$

A trading strategy $(y_t)_{t=-1}^T$ is called *self-financing* if $-\Delta y_t \in \mathcal{K}_t$ for all $t = 0, \ldots, T$. The collection of self-financing trading strategies is defined as

$$\Phi := \left\{(y_t)_{t=-1}^T \in \mathcal{N}^{2\prime} : -\Delta y_t \in \mathcal{K}_t \ \forall t = 0, \ldots, T\right\}.$$

We will also frequently consider the class of trading strategies that start and end with zero holdings (and are not necessarily self-financing). This class of trading strategies is denoted by

$$\Psi := \left\{(y_t)_{t=-1}^T \in \mathcal{N}^{2\prime} : y_{-1} = 0, y_T = 0\right\}.$$

2.2. Arbitrage and duality. There is a connection between the absence of arbitrage and the existence of classes of objects that appear in the study of disutility minimisation problems. To this end, define

$$\begin{aligned} \bar{\mathcal{P}} &:= \left\{(\mathbb{Q}, S) : \mathbb{Q} \ll \mathbb{P}, S \text{ a } \mathbb{Q}\text{-martingale}, S_t^b \leq S_t \leq S_t^a \, \forall t\right\}, \\ \mathcal{P} &:= \left\{(\mathbb{Q}, S) : \mathbb{Q} \sim \mathbb{P}, S \text{ a } \mathbb{Q}\text{-martingale}, S_t^b \leq S_t \leq S_t^a \, \forall t\right\}. \end{aligned} \tag{2.2}$$

We shall refer to the elements of $\bar{\mathcal{P}}$ ($\mathcal{P}$) as (*equivalent*) *martingale pairs*. Observe that $\mathcal{P} \subseteq \bar{\mathcal{P}}$.

The *no-arbitrage condition* is equivalent to the existence of a martingale pair. The definition (2.3) is consistent with that in [37, Def. 1.6] and equivalent, though formally different, to the notion of weak no-arbitrage in [20].

Proposition 2.1 ([20, Theorem 1]). *The no-arbitrage condition*

$$\left\{y_T : (y_t)_{t=-1}^T \in \Phi, y_{-1} = 0\right\} \cap \left\{z \in \mathcal{L}_T^2 : z \geq 0\right\} = \{0\} \tag{2.3}$$

holds if and only if $\mathcal{P} \neq \emptyset$.

We will assume a stronger condition in this paper, namely *robust no-arbitrage* [37, Def. 1.9], which ensures existence of a solution to the disutility minimisation problem. It is characterised as follows.

Proposition 2.2 ([37, Theorem 1.7]). *The robust no-arbitrage condition holds if and only if there exists an equivalent martingale pair $(\mathbb{Q}, S) \in \mathcal{P}$ such that*

$$S_t \in \operatorname{ri}[S_t^b, S_t^a] \text{ for all } t = 0, \ldots, T. \tag{2.4}$$

We assume throughout the rest of this paper that the model satisfies the robust no-arbitrage condition (2.4). Here ri denotes *relative interior*, so that

$$\mathrm{ri}[S_t^{b\omega}, S_t^{a\omega}] = \begin{cases} \{S_t^{b\omega}\} & \text{if } S_t^{b\omega} = S_t^{a\omega}, \\ (S_t^{b\omega}, S_t^{a\omega}) & \text{if } S_t^{b\omega} < S_t^{a\omega} \end{cases}$$

for all $t = 0, \dots, T$ and $\omega \in \Omega$.

We conclude this section by introducing some notation that will be useful when working with martingale pairs. For every $\mathbb{Q} \ll \mathbb{P}$, we write

$$\Lambda_t^{\mathbb{Q}} := \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] \text{ for all } t = 0, \dots, T, \tag{2.5}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$. As Ω is finite it follows that

$$\Lambda_t^{\mathbb{Q}\nu} = \frac{\mathbb{Q}(\nu)}{\mathbb{P}(\nu)} \text{ for all } \nu \in \Omega_t,\ t = 0, \dots, T. \tag{2.6}$$

Define also for all $t = 0, \dots, T$

$$\Omega_t^{\mathbb{Q}} := \{\nu \in \Omega_t : \mathbb{Q}(\nu) > 0\}$$

as the collection of nodes in Ω_t with positive probability under $\mathbb{Q}$. Moreover, for every $t = 0, \dots, T-1$ and $\mu \in \Omega_t^{\mathbb{Q}}$, denote the transition probability from μ to any successor node $\nu \in \mu^+$ by $q_{t+1}^\nu := \frac{\mathbb{Q}(\nu)}{\mathbb{Q}(\mu)}$. Simple rearrangement of (2.6) then gives

$$\Lambda_{t+1}^{\mathbb{Q}\nu} = \frac{\mathbb{Q}(\mu) q_{t+1}^\nu}{\mathbb{P}(\mu) p_{t+1}^\nu} = \Lambda_t^{\mathbb{Q}\mu} \frac{q_{t+1}^\nu}{p_{t+1}^\nu} \text{ for all } \mu \in \Omega_t,\ t = 0, \dots, T-1, \nu \in \mu^+. \tag{2.7}$$

2.3. Superhedging. If the seller of a claim is completely risk-averse, then he would charge (at least) the *superhedging price*, which is the lowest amount that the seller of a claim can charge that will allow him to sell the claim without taking any risk. Such prices are usually lower than the cost of replication (see, for example, [1]), and have been well studied for European options offering a payoff at a single expiration date; for a selection of contributions at a similar technical level to the current paper, see [10, 11, 13, 19, 20, 22, 28, 34, 35].

In this subsection we generalise the theory slightly to the case of payment streams of the form $c = ((c_t^b, c_t^s))_{t=0}^T \in \mathcal{N}^2$, consisting of sequences of payments $c_t = (c_t^b, c_t^s)$ to be made at all trading dates t. A trading strategy $(y_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ is said to *superhedge* such a payment stream c if it allows a trader to deliver c without risk, in other words,

$$-\Delta y_t - c_t \in \mathcal{K}_t \text{ for all } t, \qquad y_T = 0.$$

The *seller's superhedging price* of the payment stream c is defined as the smallest cash endowment that is sufficient to superhedge c, in other words,

$$\pi^a(c) := \inf\left\{x \in \mathbb{R} : \exists (y_t)_{t=-1}^T \in \mathcal{N}^{2\prime} \text{ superhedging } c \text{ with } y_0 = (x, 0)\right\}.$$

The *buyer's superhedging price* of c is defined as

$$\begin{aligned} \pi^b(c) &:= \sup\left\{x \in \mathbb{R} : \exists (y_t)_{t=-1}^T \in \mathcal{N}^{2\prime} \text{ superhedging } -c \text{ with } y_0 = (-x, 0)\right\} \\ &= -\pi^a(-c). \end{aligned} \tag{2.8}$$

It is the largest cash amount that can be raised without risk by using the payoff of c as collateral. The superhedging prices admit the following dual representation.

Proposition 2.3. *For every $c=((c^b_t,c^s_t))_{t=0}^T\in\mathcal{N}^2$ we have*

$$\pi^a(c)=\sup_{(\mathbb{Q},S)\in\mathcal{P}}\sum_{t=0}^T\mathbb{E}_{\mathbb{Q}}\left[c^b_t+c^s_tS_T\right]=\max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}}\sum_{t=0}^T\mathbb{E}_{\mathbb{Q}}\left[c^b_t+c^s_tS_T\right], \tag{2.9}$$

$$\pi^b(c)=\inf_{(\mathbb{Q},S)\in\mathcal{P}}\sum_{t=0}^T\mathbb{E}_{\mathbb{Q}}\left[c^b_t+c^s_tS_T\right]=\min_{(\mathbb{Q},S)\in\bar{\mathcal{P}}}\sum_{t=0}^T\mathbb{E}_{\mathbb{Q}}\left[c^b_t+c^s_tS_T\right]. \tag{2.10}$$

Proof. Observe that $(y_t)_{t=-1}^T\in\mathcal{N}^{2\prime}$ superhedges c if and only if $y_T=0$ and the trading strategy $(x_t)_{t=-1}^T\in\mathcal{N}^{2\prime}$ defined as

$$x_{-1}\coloneqq y_{-1}, \qquad\qquad x_t\coloneqq y_t+\sum_{k=0}^t c_k \text{ for all } t\geq 0$$

satisfies $-\Delta x_t\in\mathcal{K}_t$ for all t. The result then follows from [35, Theorem 4.4] and (2.8). ■

The collection of payment streams that can be superhedged from zero will play an important role in the next section. Proposition 2.3 gives that

$$\begin{aligned}\mathcal{Z}&\coloneqq\left\{c\in\mathcal{N}^2:\exists(y_t)_{t=-1}^T\in\Psi\text{ superhedging }c\right\} &(2.11)\\ &=\left\{c\in\mathcal{N}^2:\pi^a(c)\leq 0\right\}\\ &=\left\{(c^b_t,c^s_t)_{t=0}^T\in\mathcal{N}^2:\sum_{t=0}^T\mathbb{E}_{\mathbb{Q}}\left[c^b_t+c^s_tS_T\right]\leq 0\ \forall(\mathbb{Q},S)\in\bar{\mathcal{P}}\right\}. &(2.12)\end{aligned}$$

It is self-evident from the representation (2.12) that $\mathcal{Z}$ is a convex cone.

2.4. Convex sets and convex functions. This brief section contains a collection of the notation and terminology regarding convex sets and convex functions that will be used throughout the paper.

Let $A\subseteq\mathbb{R}^n$ be a set. The *convex hull* $\operatorname{conv}A$ of A is the smallest convex set containing A. The *convex cone generated by* A is

$$\operatorname{cone}A\coloneqq\{\lambda x:\lambda\geq 0,x\in A\},$$

and the *closure* $\operatorname{cl}A$ of A is the smallest closed set containing A. The *recession cone* of A is

$$0^+A\coloneqq\{x\in\mathbb{R}^n:A+\lambda x\subseteq A\text{ for all }\lambda\geq 0\}.$$

The *effective domain* of a convex function $f:\mathbb{R}\to\mathbb{R}\cup\{-\infty,\infty\}$ is

$$\operatorname{dom}f\coloneqq\{x\in\mathbb{R}:f(x)<\infty\}.$$

The function f is called *proper* if $f(x)>-\infty$ for all $x\in\mathbb{R}$ and $\operatorname{dom}f\neq\emptyset$. Its *epigraph* is

$$\operatorname{epi}f\coloneqq\{(x,y)\in\mathbb{R}^2:y\geq f(x)\}.$$

3. Disutility minimisation problem. The ability to manage investments in such a way that their proceeds cover an investor's liabilities as well as possible, is of fundamental importance in financial economics, and has therefore been well studied in the literature; see, for example, [7, 9, 16, 18] and the references therein. The purpose of this section is to formulate an optimal investment problem in the model with proportional transaction costs, which will form the basis of the indifference prices that will be studied in section 5.

Consider an investor who faces the liability of a given payment stream $u = (u_t)_{t=0}^T = ((u_t^b, u_t^s))_{t=0}^T \in \mathcal{N}^2$. The investor can create a trading strategy $(y_t)_{t=-1}^T \in \Psi$ in cash and stock, and is additionally allowed to inject (invest) cash on every trading date in a given set $\mathcal{I} \subseteq \{0, \ldots, T\}$. At each trading date $t \in \mathcal{I}$, in order to manage his position, the investor needs to inject $\phi_t(\Delta y_t + u_t)$ in cash in order to manage his position. At trading dates $t \notin \mathcal{I}$, the investor is required to manage his position in a self-financing manner, in other words, $\phi_t(\Delta y_t + u_t) \leq 0$. Denote the number of elements of $\mathcal{I}$ by $|\mathcal{I}|$ and, for simplicity of exposition, assume that $|\mathcal{I}| \neq 0$.

The objective of the investor is to choose $(y_t)_{t=-1}^T$ in such a way as to minimise the sum of expected disutility of the cash injections over all the trading dates in $\mathcal{I}$, using for each time step $t \in \mathcal{I}$ the risk-averse exponential disutility (regret) function

$$v_t(x) := e^{\alpha_t x} - 1 \text{ for all } x \in \mathbb{R}$$

with deterministic risk aversion parameter $\alpha_t \in (0, \infty)$. Define for every $t \notin \mathcal{I}$

$$v_t(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ \infty & \text{if } x > 0. \end{cases}$$

The investor's objective can then be written as the unconstrained optimisation problem

$$\text{minimise } \sum_{t=0}^{T} \mathbb{E}[v_t(\phi_t(\Delta y_t + u_t))] \text{ over } y \in \Psi. \tag{3.1}$$

The value function V of (3.1) is defined as

$$V(u) := \inf_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}[v_t(\phi_t(\Delta y_t + u_t))]. \tag{3.2}$$

The value of $V(u)$ is finite because v_t is bounded from below for all t.

Remark 3.1. In the special case where $\mathcal{I} = \{T\}$ and $u_t = 0$ for all $t < T$, the problem (3.1) becomes

$$\text{maximise } \mathbb{E}\big[1 - e^{-\alpha_T(-\phi_T(-y_{T-1}+u_T))}\big] \text{ over } y \in \Psi, -\Delta y_t \in \mathcal{K}_t \ \forall t = 0, \ldots, T-1. \tag{3.3}$$

Noting that $-\phi_T(-y_{T-1} + u_T)$ is the liquidation value of the portfolio $y_{T-1} - u_T$, this is the classical utility maximisation problem of an investor facing a liability of u_T at time T.

It is possible to rewrite (3.1) directly in terms of the cash injections. This reduces the dimensionality of the controlled process from two to one, and will aid in the study of the dual problem in the next section. Combining the fact that v_t is nondecreasing for all t with (2.11), we obtain

$$\begin{aligned} V(u) &= \inf\left\{\sum_{t=0}^{T}\mathbb{E}[v_t(x_t)] : (x,y)\in\mathcal{N}\times\Psi, x_t\geq\phi_t(\Delta y_t+u_t)\ \forall t\right\} \\ &= \inf\left\{\sum_{t=0}^{T}\mathbb{E}[v_t(x_t)] : (x,y)\in\mathcal{N}\times\Psi, -\Delta y_t-u_t+(x_t,0)\in\mathcal{K}_t\ \forall t\right\} \quad (3.4)\\ &= \inf\left\{\sum_{t=0}^{T}\mathbb{E}[v_t(x_t)] : (x,y)\in\mathcal{N}\times\Psi, y \text{ superhedges } (u^b_t-x_t,u^s_t)_{t=0}^T\right\} \\ &= \inf\left\{\sum_{t=0}^{T}\mathbb{E}[v_t(x_t)] : x\in\mathcal{N}, (u^b_t-x_t,u^s_t)_{t=0}^T\in\mathcal{Z}\right\} \\ &= \inf_{x\in\mathcal{A}_u}\sum_{t=0}^{T}\mathbb{E}[v_t(x_t)], \quad (3.5)\end{aligned}$$

where

$$\mathcal{A}_u := \left\{(x_t)_{t=0}^T\in\mathcal{N} : (u^b_t-x_t,u^s_t)_{t=0}^T\in\mathcal{Z}\right\}. \tag{3.6}$$

In conclusion, the problem (3.1) has the same value function as the optimisation problem

$$\text{minimise } \sum_{t=0}^{T}\mathbb{E}[v_t(x_t)] \text{ over } x\in\mathcal{A}_u. \tag{3.7}$$

We conclude this section by presenting a few key properties of V.

Theorem 3.2. *The function V is convex and lower semicontinuous on $\mathcal{N}^2$, and the infima in (3.2) and (3.5) are attained for every $u\in\mathcal{N}^2$.*

Proof. The main argument is analogous to existing results (see [26, Theorem 5.1], for example) and is therefore presented in outline only. Observe first from (3.4) that

$$V(u) = \inf_{x\in\mathcal{N},y\in\mathcal{N}^{2'}}\mathbb{E}[f(x,y,u)],$$

where $f:\Omega\times\mathbb{R}^{T+1}\times\mathbb{R}^{2(T+2)}\times\mathbb{R}^{2(T+1)}\to\mathbb{R}\cup\{\infty\}$ is defined as

$$f^\omega(x,y,u) := \begin{cases}\sum_{t=0}^T v_t(x_t) & \text{if } (x,y,u)\in\mathcal{B}^\omega,\\ \infty & \text{if } (x,y,u)\notin\mathcal{B}^\omega,\end{cases}$$

where $x=(x_0,\ldots,x_T)$, $y=(y_{-1},\ldots,y_T)$, $u=(u_0,\ldots,u_T)$, and where

$$\begin{aligned}\mathcal{B}^\omega &:= \left\{(x,y,u)\in\mathbb{R}^{T+1}\times\mathbb{R}^{2(T+2)}\times\mathbb{R}^{2(T+1)} : y_{-1}=y_T=0, -\Delta y_t-u_t+(x_t,0)\in\mathcal{K}^\omega_t\ \forall t\right\},\\ \mathcal{K}^\omega_t &:= \left\{z^\omega\in\mathbb{R}^2 : z\in\mathcal{K}_t\right\} = \left\{(z^b,z^s)\in\mathbb{R}^2 : x^b+x^sS^{b\omega}_t\geq 0, x^b+x^sS^{a\omega}_t\geq 0\right\}.\end{aligned}$$

For each $\omega \in \Omega$ the set $\mathcal{B}^\omega$ is a closed convex cone containing the origin $(0,0,0)$. The regret functions $(v_t)_{t=0}^T$ are convex, lower semicontinuous and bounded from below, and so is $(x,y,u) \mapsto f^\omega(x,y,u)$ [31, Theorems 5.2, 9.3]. In particular, f is a normal integrand [32, Def. 14.27] satisfying $f(0,0,0)=0$.

The convexity of V follows from the convexity of $(x,y,u) \mapsto \mathbb{E}(f(x,y,u))$ [30, Theorem 1]. The remainder of the claim follows from [27, Theorem 2], provided that

$$\mathcal{M} := \left\{(x,y) \in \mathcal{N} \times \mathcal{N}^{2'} : f^{\omega\infty}(x^\omega, y^\omega, 0) \leq 0 \ \forall \omega \in \Omega\right\}$$

is a linear space, where for every ω the recession function $f^{\omega\infty}$ of f^ω is given by

$$f^{\omega\infty}(x,y,u) = \lim_{\lambda \downarrow 0} f(\lambda x, \lambda y, \lambda u) = \begin{cases} 0 & \text{if } (x,y,u) \in B^\omega, x_t \leq 0 \ \forall t, \\ \infty & \text{otherwise} \end{cases}$$

[31, Corollary 8.5.2].

The proof is therefore complete upon showing that

$$\mathcal{M} = \left\{\left((x_t)_{t=0}^T, (y_t)_{t=-1}^T\right) \in \mathcal{N} \times (\Phi \cap \Psi) : -\Delta y_t + (x_t, 0) \in \mathcal{K}_t, x_t \leq 0 \ \forall t\right\}$$

is linear. The robust no-arbitrage condition implies that $\Phi \cap \Psi$ is linear [37, Lemma 2.6], and so it suffices to show that if $((x_t)_{t=0}^T, (y_t)_{t=-1}^T) \in \mathcal{M}$, then $x_t = 0$ for all t. To this end, assume by contradiction that $\{x_{t^*} < 0\} \neq \emptyset$ for some t^* and define $z = (z_t)_{t=-1}^T \in \mathcal{N}^{2'}$ as

$$z_{-1} := 0, \qquad z_t := y_t - \sum_{s=0}^t (x_s, 0) \text{ for all } t = 0, \dots, T.$$

Then

$$\Delta z_t = \Delta y_t - (x_t, 0) \in -\mathcal{K}_t \text{ for all } t = 0, \dots, T,$$

so that $z \in \Phi$. It further follows from $y_T = 0$ that

$$z_T = -\sum_{t=0}^T (x_t, 0) \neq 0,$$

and hence z violates the no-arbitrage condition (2.3). This contradiction permits us to conclude that $x_t = 0$ for all $t = 0, \dots, T$. ■

4. Dual formulation.

It is possible to obtain a Lagrangian dual formulation for the optimisation problem (3.7). For every $u = (u_t)_{t=0}^T \in \mathcal{N}^2$, define the Lagrangian $L_u : \mathcal{N} \times [0,\infty) \times \bar{\mathcal{P}} \to \mathbb{R} \cup \{\infty\}$ as

$$L_u(x, \lambda, (\mathbb{Q}, S)) := \sum_{t=0}^T \left(\mathbb{E}[v_t(x_t)] + \lambda \mathbb{E}_{\mathbb{Q}}\left[u_t^b + u_t^s S_T - x_t\right]\right). \tag{4.1}$$

The formulation of L_u is motivated by [36, (74)] (in the context of utility maximisation in incomplete market models without transaction costs). The coefficient of λ encapsulates the constraints in (3.7); see (2.12).

The following strong duality result holds.

Theorem 4.1. *For all $u \in \mathcal{N}^2$, we have*

$$V(u) = \inf_{x\in\mathcal{N}} \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)) = \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)). \tag{4.2}$$

Proof. For any $x = (x_t)_{t=0}^T \in \mathcal{N}$, there are two possibilities for the second term in the Lagrangian L_u. If $x \in \mathcal{A}_u$, then the coefficient of λ must be nonpositive, and by taking $\lambda = 0$ we obtain

$$\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)) = \sum_{t=0}^T \mathbb{E}[v_t(x_t)].$$

If $x \notin \mathcal{A}_u$, then there exists some $(\mathbb{Q},S) \in \bar{\mathcal{P}}$ for which the second term is positive whenever $\lambda > 0$, and by taking λ arbitrarily large we obtain

$$\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)) = \infty.$$

This means that

$$\inf_{x\in\mathcal{N}} \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)) = \inf_{x\in\mathcal{A}_u} \sum_{t=0}^T \mathbb{E}[v_t(x_t)] = V(u)$$

due to (3.5).

Since the function V is lower semicontinuous and convex on $\mathcal{N}^2$, it follows that

$$V(u) = \sup_{z\in\mathcal{N}^2} \left\{ \sum_{t=0}^T \mathbb{E}[u_t \cdot z_t] - V^*(z) \right\} \text{ for all } u = (u_t)_{t=0}^T \in \mathcal{N}^2 \tag{4.3}$$

[30, Theorem 5], where the conjugate function V^* of V is defined as

$$V^*(z) := \sup_{u\in\mathcal{N}^2} \left\{ \sum_{t=0}^T \mathbb{E}[u_t \cdot z_t] - V(u) \right\} \text{ for all } z = (z_t)_{t=0}^T \in \mathcal{N}^2.$$

For every $z = (z_t)_{t=0}^T \in \mathcal{N}^2$, it follows from (3.4) that

$$V^*(z) = \sup \left\{ \sum_{t=0}^T \mathbb{E}[z_t \cdot u_t - v_t(x_t)] : (x,y,u) \in \mathcal{N} \times \Psi \times \mathcal{N}^2, \Delta y_t + u_t - (x_t,0) \in -\mathcal{K}_t \,\forall t \right\}.$$

This optimization problem can be decoupled into three optimization problems over x, y and the transformed variable $w = (w_t)_{t=0}^T \in \mathcal{N}^2$, defined as

$$w_t := \Delta y_t + u_t - (x_t,0) \text{ for all } t = 0,\dots,T.$$

Observing that

$$z_t \cdot u_t - v_t(x_t) = z_t \cdot (w_t - \Delta y_t + (x_t,0)) - v_t(x_t) = z_t \cdot w_t - z_t \cdot \Delta y_t + z_t^b x_t - v_t(x_t)$$

for all t, it follows that

$$V^*(z) = \sup\left\{\sum_{t=0}^{T} \mathbb{E}[z_t \cdot w_t] : w \in \mathcal{N}^2, w_t \in -\mathcal{K}_t \,\forall t\right\} - \inf_{y\in\Psi} \sum_{t=0}^{T} \mathbb{E}[z_t \cdot \Delta y_t] + \sup_{x\in\mathcal{N}} \sum_{t=0}^{T} \mathbb{E}\big[z^b x_t - v_t(x_t)\big]. \tag{4.4}$$

For the first term on the right hand side of (4.4), define the positive polar of the solvency cone $\mathcal{K}_t$ for every $t = 0, \dots, T$ as

$$\mathcal{K}_t^+ := \left\{y \in \mathcal{L}_t^2 : y \cdot x \geq 0 \text{ for all } x \in \mathcal{K}_t\right\}.$$

Then

$$\sup\left\{\sum_{t=0}^{T} \mathbb{E}[z_t \cdot w_t] : w \in \mathcal{N}^2, w_t \in -\mathcal{K}_t \,\forall t\right\} = \begin{cases} 0 & \text{if } z_t \in \mathcal{K}_t^+ \,\forall t, \\ \infty & \text{otherwise} \end{cases} \tag{4.5}$$

because

$$\sup_{w_t \in -\mathcal{K}_t} \mathbb{E}[z_t \cdot w_t] = \begin{cases} 0 & \text{if } z_t \in \mathcal{K}_t^+, \\ \infty & \text{otherwise} \end{cases}$$

for all $t = 0, \dots, T$. For the second term, using the property $y_{-1} = y_T = 0$ and rearrangement leads to

$$\sum_{t=0}^{T} z_t \cdot \Delta y_t = -\sum_{t=0}^{T-1} \Delta z_{t+1} \cdot y_t \text{ for all } y = (y_t)_{t=-1}^{T} \in \Psi.$$

Moreover, for all $t = 0, \dots, T-1$, the tower property gives

$$\sup_{y_t \in \mathcal{L}_t^2} \mathbb{E}[\Delta z_{t+1} \cdot y_t] = \sup_{y_t \in \mathcal{L}_t^2} \mathbb{E}[\mathbb{E}[\Delta z_{t+1} | \mathcal{F}_t] \cdot y_t] = \begin{cases} 0 & \text{if } \mathbb{E}[\Delta z_{t+1} | \mathcal{F}_t] = 0, \\ \infty & \text{otherwise,} \end{cases}$$

which implies that

$$\inf_{y\in\Psi} \sum_{t=0}^{T} \mathbb{E}[z_t \cdot \Delta y_t] = -\sum_{t=0}^{T-1} \sup_{y_t \in \mathcal{L}_t^2} \mathbb{E}[\Delta z_{t+1} \cdot y_t] = \begin{cases} 0 & \text{if } z \text{ is a martingale,} \\ -\infty & \text{otherwise.} \end{cases} \tag{4.6}$$

Combining (4.4)–(4.6), we obtain

$$V^*(z) = \begin{cases} \sup_{x\in\mathcal{N}} \sum_{t=0}^{T} \mathbb{E}\big[z^b x_t - v_t(x_t)\big] & \text{if } z \in \bar{\mathcal{C}}, \\ \infty & \text{otherwise,} \end{cases} \tag{4.7}$$

where

$$\bar{\mathcal{C}} := \left\{z \in \mathcal{N}^2 : z \text{ a martingale}, z_t \in \mathcal{K}_t^+ \,\forall t\right\} = \left\{\big(\lambda(1, S_t)\Lambda_t^{\mathbb{Q}}\big)_{t=0}^{T} : \lambda \geq 0, \, (\mathbb{Q}, S) \in \bar{\mathcal{P}}\right\}, \tag{4.8}$$

and where the final equality follows by straightforward adaptation of the arguments of [37, pp. 24-25]. Substituting (4.7) into (4.3) gives, for all $u = (u_t)_{t=0}^T \in \mathcal{N}^2$,

$$\begin{aligned} V(u) &= \sup_{z\in\bar{\mathcal{C}}} \left\{ \sum_{t=0}^T \mathbb{E}[u_t \cdot z_t] - \sup_{x\in\mathcal{N}} \sum_{t=0}^T \mathbb{E}\big[z^b x_t - v_t(x_t)\big] \right\} \\ &= \sup_{z\in\bar{\mathcal{C}}} \inf_{x\in\mathcal{N}} \sum_{t=0}^T \mathbb{E}\big[v_t(x_t) + u_t \cdot z_t - z_t^b x_t\big]. \end{aligned}$$

The representation (4.8) then leads to

$$\begin{aligned} V(u) &= \sup_{\lambda\geq 0, (\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} \sum_{t=0}^T \mathbb{E}\big[v_t(x_t) + \lambda\big(u_t^b + u_t^s S_t - x_t\big)\Lambda_t^{\mathbb{Q}}\big] \\ &= \sup_{\lambda\geq 0, (\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} \sum_{t=0}^T \big(\mathbb{E}[v_t(x_t)] + \lambda\mathbb{E}_{\mathbb{Q}}\big[u_t^b + u_t^s S_t - x_t\big]\big) \\ &= \sup_{\lambda\geq 0, (\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} \sum_{t=0}^T \big(\mathbb{E}[v_t(x_t)] + \lambda\mathbb{E}_{\mathbb{Q}}\big[u_t^b + u_t^s S_T - x_t\big]\big) \\ &= \sup_{\lambda\geq 0, (\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)), \end{aligned}$$

by the tower property of conditional expectation in conjunction with (2.5) and the martingale property of S. ■

The strong duality established in Theorem 4.1 suggests that further study of the *dual problem*

$$\text{maximise } \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \text{ over } (\lambda, (\mathbb{Q}, S)) \in [0, \infty) \times \bar{\mathcal{P}} \tag{4.9}$$

of (3.7) would be profitable. It turns out that there is an explicit formula for the value of the inner optimisation problem over x. Note that in this paper we adopt the convention $0 \ln 0 = 0$.

Proposition 4.2. *For any $u \in \mathcal{N}^2$ and $(\lambda, (\mathbb{Q}, S)) \in [0, \infty) \in \bar{\mathcal{P}}$, we have*

$$\begin{aligned} \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) = &- \sum_{t\in\mathcal{I}} \tfrac{\lambda}{\alpha_t} \mathbb{E}\big[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\big] + \lambda \sum_{t=0}^T \mathbb{E}_{\mathbb{Q}}\big[u_t^b + u_t^s S_T\big] \\ &- \sum_{t\in\mathcal{I}} \tfrac{\lambda}{\alpha_t} \left(\ln \tfrac{\lambda}{\alpha_t} - 1\right) - |\mathcal{I}|. \end{aligned} \tag{4.10}$$

Proof. Fix any $\lambda \geq 0$ and $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, and observe from (4.1), the definition of $\mathcal{N}$ and the finiteness of Ω that

$$\begin{aligned}\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)) &= -\sup_{x\in\mathcal{N}}\sum_{t=0}^{T}\mathbb{E}\big[\lambda\Lambda_t^{\mathbb{Q}}x_t - v_t(x_t)\big] + \lambda\sum_{t=0}^{T}\mathbb{E}_{\mathbb{Q}}\big[u_t^b + u_t^s S_T\big]\\ &= -\sum_{t=0}^{T}\sup_{x_t\in\mathcal{L}_t}\mathbb{E}\big[\lambda\Lambda_t^{\mathbb{Q}}x_t - v_t(x_t)\big] + \lambda\sum_{t=0}^{T}\mathbb{E}_{\mathbb{Q}}\big[u_t^b + u_t^s S_T\big]\\ &= -\sum_{t=0}^{T}\mathbb{E}\big[v_t^*\big(\lambda\Lambda_t^{\mathbb{Q}}\big)\big] + \lambda\sum_{t=0}^{T}\mathbb{E}_{\mathbb{Q}}\big[u_t^b + u_t^s S_T\big],\end{aligned}$$

where

$$v_t^*(z) := \sup_{y\in\mathbb{R}}\{zy - v_t(y)\} \text{ for all } z\in\mathbb{R}$$

denotes the convex conjugate of v_t for all $t = 0,\dots,T$. It is straightforward to derive

$$v_t^*(z) = \begin{cases}\frac{z}{\alpha_t}\ln\frac{z}{\alpha_t} - \frac{z}{\alpha_t} + 1 & \text{if } t\in\mathcal{I},\\ 0 & \text{if } t\notin\mathcal{I}\end{cases}$$

whenever $z \geq 0$. The result then follows after observing that, for each $t\in\mathcal{I}$,

$$\begin{aligned}\mathbb{E}\big[v_t^*\big(\lambda\Lambda_t^{\mathbb{Q}}\big)\big] &= \tfrac{\lambda}{\alpha_t}\mathbb{E}\big[\Lambda_t^{\mathbb{Q}}\ln\Lambda_t^{\mathbb{Q}}\big] + \tfrac{\lambda}{\alpha_t}\Big(\ln\tfrac{\lambda}{\alpha_t} - 1\Big)\mathbb{E}\big[\Lambda_t^{\mathbb{Q}}\big] + 1\\ &= \tfrac{\lambda}{\alpha_t}\mathbb{E}\big[\Lambda_t^{\mathbb{Q}}\ln\Lambda_t^{\mathbb{Q}}\big] + \tfrac{\lambda}{\alpha_t}\Big(\ln\tfrac{\lambda}{\alpha_t} - 1\Big) + 1.\end{aligned}$$ ■

In the representation (4.10), the joint dependence on λ and $(\mathbb{Q},S)$ is very simple: the two terms on the right hand side that depend on $(\mathbb{Q},S)$, both contain λ only as a nonnegative linear coefficient. This suggests that it should be possible to rewrite the outer maximisation in the dual problem (4.9) as a two-step maximisation, in other words, maximising first over $(\mathbb{Q},S)$, and then over λ.

The solution to the first step, maximisation over $(\mathbb{Q},S)$, will be the subject of section 6. In the remainder of this section, we introduce some notation in order to capture the two-step nature of the maximisation, and then show that the maximisation problem over λ has a unique closed form solution. To this end, for any $X = (X^b, X^s)\in\mathcal{L}_T^2$, define

$$H((\mathbb{Q},S);X) := \sum_{t\in\mathcal{I}}\tfrac{1}{\alpha_t}\mathbb{E}\big[\Lambda_t^{\mathbb{Q}}\ln\Lambda_t^{\mathbb{Q}}\big] + \mathbb{E}_{\mathbb{Q}}\big[X^b + X^s S_T\big] \text{ for all } (\mathbb{Q},S)\in\bar{\mathcal{P}}, \tag{4.11}$$

$$K(X) := \inf_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} H((\mathbb{Q},S);X). \tag{4.12}$$

Notice that $K(X)$ is finite because the values of the mapping $x\mapsto x\ln x$ are finite and bounded from below on $[0,\infty)$. Combining this notation with (4.2) and (4.10), we obtain, for all $u\in\mathcal{N}^2$,

$$\begin{aligned}V(u) &= \sup_{\lambda\geq 0}\left\{-\lambda K\left(-\sum_{t=0}^{T}u_t\right) - \sum_{t\in\mathcal{I}}\tfrac{\lambda}{\alpha_t}\Big(\ln\tfrac{\lambda}{\alpha_t} - 1\Big) - |\mathcal{I}|\right\}\\ &= -\inf_{\lambda\geq 0}\left\{\lambda K\left(-\sum_{t=0}^{T}u_t\right) + \sum_{t\in\mathcal{I}}\tfrac{\lambda}{\alpha_t}\Big(\ln\tfrac{\lambda}{\alpha_t} - 1\Big)\right\} - |\mathcal{I}|. \qquad (4.13)\end{aligned}$$

The following result concludes this section.

Theorem 4.3. *For any $u \in \mathcal{N}^2$, the minimal disutility is*

$$V(u) = \hat{\lambda}_u \sum_{t\in\mathcal{I}} \tfrac{1}{\alpha_t} - |\mathcal{I}|, \tag{4.14}$$

where

$$\hat{\lambda}_u := \exp\left\{ \frac{1}{\sum_{t\in\mathcal{I}} \frac{1}{\alpha_t}} \left(\sum_{t\in\mathcal{I}} \frac{\ln \alpha_t}{\alpha_t} - K\left(-\sum_{t=0}^T u_t\right)\right)\right\} > 0 \tag{4.15}$$

is the unique value attaining the infimum in (4.13).

Proof. Define

$$f(\lambda) := \lambda K\left(-\sum_{t=0}^T u_t\right) + \sum_{t\in\mathcal{I}} \tfrac{\lambda}{\alpha_t}\left(\ln \tfrac{\lambda}{\alpha_t} - 1\right) \text{ for all } \lambda \geq 0.$$

The function f is convex and twice continuously differentiable, and in fact

$$f'(\lambda) = K\left(-\sum_{t=0}^T u_t\right) + \sum_{t\in\mathcal{I}} \tfrac{1}{\alpha_t} \ln \tfrac{\lambda}{\alpha_t}, \qquad f''(\lambda) = \tfrac{1}{\lambda}\sum_{t\in\mathcal{I}} \tfrac{1}{\alpha_t}$$

for all $\lambda > 0$. The first derivative f' is increasing, whilst being negative for small λ and positive for λ large enough. This means that f attains its minimum at the point which is the unique solution $\lambda \in (0, \infty)$ to the equation $f'(\lambda) = 0$. It is straightforward to verify that this solution is indeed given by (4.15). The formula (4.14) is obtained by substituting (4.15) into (4.13). ∎

Note that Theorem 4.3 implies that $\hat{\lambda}_u$, and hence $V(u)$, depend on u only through $\sum_{t=0}^T u_t$. This is perhaps surprising in view of the definition (3.2) of $V(u)$. The reason for this comes from the dual formulation and the nature of the dual objects in models with proportional transaction costs: for example, it can be seen in (2.12) that whether a payment stream can be superhedged from zero depends only on its total payoff. This is the reason why the Lagrangian L_u depends linearly on $\sum_{t=0}^T u_t$, which in turn leads directly into the dual formulation of $V(u)$.

5. Indifference pricing. In this section we consider an investor trading in cash and shares and who is entitled to receive a given portfolio $w_t \in \mathcal{L}_t^2$ at each time step $t = 0, \ldots, T$. We refer to the payment stream $w = (w_t)_{t=0}^T$ as the *endowment* of the investor (though it may in fact represent a liability if negative). The minimal disutility of the investor in this situation is $V(-w)$.

Indifference pricing provides a way for such an investor to determine the value of derivatives, or payment streams. We will introduce disutility indifference prices for the seller and buyer of a payment stream $c = (c_t)_{t=0}^T \in \mathcal{N}^2$. Consider the situation where the investor is selling the payment stream c. He receives a single payment of $\delta \in \mathbb{R}$ in cash at time 0, and then delivers the portfolio c_t at each time step $t = 0, \ldots, T$. After selling c, the investor's minimum disutility becomes $V(c - \delta\mathbb{1} - w)$, where the process $\mathbb{1} = (\mathbb{1}_t)_{t=0}^T$ is defined as

$$\mathbb{1}_t := \begin{cases} (1,0) & \text{if } t = 0, \\ (0,0) & \text{if } t = 1, \ldots, T. \end{cases}$$

The *seller's disutility indifference price* $\pi^{ai}(c; w)$ of c is defined as the lowest price for which he could sell c without increasing his minimal disutility, in other words,

$$\pi^{ai}(c; w) := \inf\{\delta \in \mathbb{R} : V(c - \delta\mathbb{1} - w) \leq V(-w)\}. \tag{5.1}$$

The *buyer's disutility indifference price* $\pi^{bi}(c; w)$ is similarly defined as the highest price at which the investor could buy the payment stream (and receive c_t at each time step $t = 0, \ldots, T$) without increasing his minimal disutility, in other words,

$$\begin{aligned} \pi^{bi}(c; w) &:= \sup\{\delta \in \mathbb{R} : V(-c + \delta\mathbb{1} - w) \leq V(-w)\} \\ &= -\inf\{\delta \in \mathbb{R} : V(-c - \delta\mathbb{1} - w) \leq V(-w)\} \\ &= -\pi^{ai}(-c; w). \end{aligned} \tag{5.2}$$

The following theorem establishes formulae for computing the buyer's and seller's indifference prices. These pricing formulae resemble existing formulae for utility indifference prices in friction-free models under exponential utility, in particular those obtained in [9] and [33] in general continuous-time market models without transaction costs, and [25] in a discrete time friction-free model with a non-traded asset.

Observe that, to determine the buyer's and seller's indifference prices of a payment stream, it is sufficient to be able to determine the value of K for three different random variables.

Theorem 5.1. *For any $c, w \in \mathcal{N}^2$, we have*

$$\pi^{ai}(c; w) = K\left(\sum_{t=0}^T w_t\right) - K\left(\sum_{t=0}^T (w_t - c_t)\right), \tag{5.3}$$

$$\pi^{bi}(c; w) = K\left(\sum_{t=0}^T (w_t + c_t)\right) - K\left(\sum_{t=0}^T w_t\right). \tag{5.4}$$

Proof. Observe first that (5.4) follows directly from (5.2) and (5.3). Define

$$\hat{\pi} := K\left(\sum_{t=0}^T w_t\right) - K\left(\sum_{t=0}^T (w_t - c_t)\right).$$

As $\hat{\pi}$ is deterministic, we have

$$K\left((\hat{\pi}, 0) + \sum_{t=0}^T (w_t - c_t)\right) = \hat{\pi} + K\left(\sum_{t=0}^T (w_t - c_t)\right) = K\left(\sum_{t=0}^T w_t\right)$$

by (4.11) and (4.12). It then follows from (4.15) that

$$\hat{\lambda}_{c-\hat{\pi}\mathbb{1}-w} = \exp\left\{\frac{1}{\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}}\left(\sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t} - K\left((\hat{\pi},0)+\sum_{t=0}^{T}(w_t-c_t)\right)\right)\right\} = \hat{\lambda}_{-w}, \tag{5.5}$$

and from (4.14) that

$$V(c-\hat{\pi}\mathbb{1}-w) = \hat{\lambda}_{c-\hat{\pi}\mathbb{1}-w}\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} - |\mathcal{I}| = \hat{\lambda}_{-w}\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} - |\mathcal{I}| = V(-w).$$

This permits us to conclude that $\pi^{ai}(c;w) \leq \hat{\pi}$.

In order to establish (5.3), it suffices to show that $V(c-\pi\mathbb{1}-w) > V(c-\hat{\pi}\mathbb{1}-w)$ for any $\pi < \hat{\pi}$. For every $\pi < \hat{\pi}$, there exists a process $x^\pi = (x_t^\pi)_{t=0}^T$ such that $x^\pi \in \mathcal{A}_{c-\pi\mathbb{1}-w}$ and

$$V(c-\pi\mathbb{1}-w) = \sum_{t=0}^{T}\mathbb{E}[v_t(x_t^\pi)].$$

by Theorem 3.2. Define a new process $x^{\hat{\pi}} = (x_t^{\hat{\pi}})_{t=0}^T \in \mathcal{N}$ as

$$x_t^{\hat{\pi}} := \begin{cases} x_t^\pi + \frac{1}{|\mathcal{I}|}(\pi-\hat{\pi}) & \text{if } t\in\mathcal{I},\\ x_t^\pi & \text{otherwise.}\end{cases}$$

Then

$$\sum_{t=0}^{T}(c_t-\hat{\pi}\mathbb{1}_t-w_t-(x_t^{\hat{\pi}},0)) = \sum_{t=0}^{T}(c_t-w_t-(x_t^\pi,0))-(\pi,0) = \sum_{t=0}^{T}(c_t-\pi\mathbb{1}_t-w_t-(x_t^\pi,0)),$$

and so it follows from (3.6) that $x^{\hat{\pi}} \in \mathcal{A}_{c-\hat{\pi}\mathbb{1}-w}$. Furthermore, for every $t\in\mathcal{I}$ we have $v_t(x_t^\pi) > v_t(x_t^{\hat{\pi}})$ so that

$$V(c-\pi\mathbb{1}-w) = \sum_{t=0}^{T}\mathbb{E}[v_t(x_t^\pi)] > \sum_{t=0}^{T}\mathbb{E}[v_t(x_t^{\hat{\pi}})] \geq V(c-\hat{\pi}\mathbb{1}-w)$$

by (3.5), as required. ■

The following one-step toy model demonstrates the calculation of the indifference prices using (5.3) and (5.4).

Example 5.2. Let $T=1$ and $\Omega=\{\mathrm{u},\mathrm{d}\}$, and take any probability measure $\mathbb{P}$ with $p := \mathbb{P}(\mathrm{u}) \in (0,1)$. Suppose furthermore that the bid and ask prices in this model satisfy

$$S_1^{b\mathrm{d}} \leq S_1^{a\mathrm{d}} < S_0^b = \bar{S}_0 = S_0^a < S_1^{b\mathrm{u}} \leq S_1^{a\mathrm{u}}. \tag{5.6}$$

The mid-price process $\bar{S} = (\bar{S}_0,\bar{S}_1) \in \mathcal{N}$ with $\bar{S}_1 := \frac{1}{2}(S_1^a+S_1^b)$ together with the unique probability measure $\mathbb{Q}$ with $\mathbb{Q}(\mathrm{u}) = \frac{\bar{S}_0-\bar{S}_1^{\mathrm{d}}}{\bar{S}_1^{\mathrm{u}}-\bar{S}_1^{\mathrm{d}}}$ satisfies the robust no-arbitrage condition in Proposition 2.2.

Every probability measure $\mathbb{Q}$ in this model can be characterised uniquely by $\mathbb{Q}(\mathrm{u})$. It follows from (5.6) and straightforward calculation that

$$\begin{aligned}\mathcal{Q} &:= \left\{\mathbb{Q}(\mathrm{u}) : (\mathbb{Q}, S) \in \bar{\mathcal{P}}\right\} \\ &= \left\{q \in [0,1] : qx^{\mathrm{u}} + (1-q)x^{\mathrm{d}} = \bar{S}_0 \text{ for some } x^{\mathrm{u}} \in \left[S_1^{b\mathrm{u}}, S_1^{a\mathrm{u}}\right], x^{\mathrm{d}} \in \left[S_1^{b\mathrm{d}}, S_1^{a\mathrm{d}}\right]\right\} \\ &= \left\{\tfrac{\bar{S}_0 - x^{\mathrm{d}}}{x^{\mathrm{u}} - x^{\mathrm{d}}} : x^{\mathrm{u}} \in \left[S_1^{b\mathrm{u}}, S_1^{a\mathrm{u}}\right], x^{\mathrm{d}} \in \left[S_1^{b\mathrm{d}}, S_1^{a\mathrm{d}}\right]\right\} = \left[\tfrac{\bar{S}_0 - S_1^{a\mathrm{d}}}{S_1^{a\mathrm{u}} - S_1^{a\mathrm{d}}}, \tfrac{\bar{S}_0 - S_1^{b\mathrm{d}}}{S_1^{b\mathrm{u}} - S_1^{b\mathrm{d}}}\right] =: [q_{\min}, q_{\max}].\end{aligned}$$

Observe in particular that $\mathcal{Q} \subset (0,1)$ by (5.6).

Let $\mathcal{I} := \{0,1\}$ and $\alpha_0 = \alpha_1 = \alpha > 0$, and set the investor's endowment $w = (w_0, w_1) \in \mathcal{N}^2$ to be zero, in other words, $w_0 = w_1 = (0,0)$. It is possibible to derive explicit formulae for the buyer's and seller's disutility indifference prices of a derivative security with cash payoff $D \in \mathcal{L}_1$ at time 1. This corresponds to the payment stream $c = (c_0, c_1) \in \mathcal{N}^2$ satisfying $c_0 = (0,0)$ and $c_1 = (D,0)$. From (5.3) and (5.4), these prices involve terms of the form $K((Y,0))$ where $Y \in \mathcal{L}_1$. For any such Y and any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, combining (2.6) and (4.11) gives

$$H((\mathbb{Q}, S); (Y,0)) = \tfrac{1}{\alpha}\mathbb{E}\left[\Lambda_1^{\mathbb{Q}} \ln \Lambda_1^{\mathbb{Q}}\right] + \mathbb{E}_{\mathbb{Q}}[Y] = f_Y(\mathbb{Q}(\mathrm{u})),$$

where

$$f_Y(q) := \tfrac{1}{\alpha}\left(q \ln \tfrac{q}{p} + (1-q)\ln \tfrac{1-q}{1-p}\right) + qY^{\mathrm{u}} + (1-q)Y^{\mathrm{d}} \text{ for all } q \in [0,1].$$

It is easily verified that f_Y is continuous and convex on $[0,1]$, and that it reaches its minimum at

$$\hat{q}_Y := \frac{pe^{-\alpha Y^{\mathrm{u}}}}{pe^{-\alpha Y^{\mathrm{u}}} + (1-p)e^{-\alpha Y^{\mathrm{d}}}} \in (0,1).$$

It then follows from (4.12) that

$$K((Y,0)) = \inf_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} H((\mathbb{Q}, S); (Y,0)) = \inf_{q\in[q_{\min}, q_{\max}]} f_Y(q) = f_Y(q_Y), \tag{5.7}$$

where

$$q_Y := \min\{\max\{\hat{q}_Y, q_{\min}\}, q_{\max}\}.$$

After substituting (5.7) into (5.3) and (5.4), the buyer's and seller's disutility indifference prices of c become

$$\begin{aligned}\pi^{ai}(c;0) &= K((0,0)) - K((-D,0)) = f_0(q_0) - f_{-D}(q_{-D}), \\ \pi^{bi}(c;0) &= K((D,0)) - K((0,0)) = f_D(q_D) - f_0(q_0).\end{aligned}$$

We conclude this section by presenting a key property of disutility indifference prices, namely that they produce smaller bid-ask intervals than superhedging prices.

Theorem 5.3. *We have for any $c, w \in \mathcal{N}^2$ that*

$$\pi^b(c) \leq \pi^{bi}(c;w) \leq \pi^{ai}(c;w) \leq \pi^a(c).$$

Moreover, the mapping $u \mapsto \pi^{ai}(u;w)$ is convex, and $u \mapsto \pi^{bi}(u;w)$ is concave.

Proof. We first show that

$$\pi^{ai}(c;w) \leq \pi^a(c) \text{ for all } c,w \in \mathcal{N}^2. \tag{5.8}$$

Note first that $c - \pi^a(c)\mathbb{1} \in \mathcal{Z}$ from (2.9) and (2.12). Furthermore, for any $x \in \mathcal{A}_{-w}$, we have $-w-(x_t,0)_{t=0}^T \in \mathcal{Z}$, and since $\mathcal{Z}$ is a convex cone, it follows that $c-\pi^a(c)\mathbb{1}-w-(x_t,0)_{t=0}^T \in \mathcal{Z}$, so that finally $x \in \mathcal{A}_{c-\pi^a(c)\mathbb{1}-w}$. Thus $\mathcal{A}_{-w} \subseteq \mathcal{A}_{c-\pi^a(c)\mathbb{1}-w}$, so that $V(c-\pi^a(c)\mathbb{1}-w) \leq V(-w)$ by (3.5). This in turn implies that $\pi^{ai}(c;w) \leq \pi^a(c)$ by (5.1).

Combining (5.8) with (2.8) and (5.2) immediately gives that

$$\pi^{bi}(c;w) = -\pi^{ai}(-c;w) \geq -\pi^a(-c) = \pi^b(c)$$

for all $c,w \in \mathcal{N}^2$.

The remainder of the proof is devoted to showing the convexity of $u \mapsto \pi^{ai}(u;w)$. Once established, it immediately gives that $u \mapsto \pi^{bi}(u;w)$ is concave by (5.2). Moreover, combining the convexity with (5.3) gives for all $c,w \in \mathcal{N}^2$ that

$$0 = \pi^{ai}(0;w) \leq \tfrac{1}{2}\pi^{ai}(c;w) + \tfrac{1}{2}\pi^{ai}(-c;w),$$

whence

$$\pi^{bi}(c;w) = -\pi^{ai}(-c;w) \leq \pi^{ai}(c;w).$$

To establish the convexity, fix $w \in \mathcal{N}^2$ and note that

$$C := \{x \in \mathcal{N}^2 : V(x-w) \leq V(-w)\}$$

is convex because, for all $x,y \in C$ and $\lambda \in [0,1]$ we have

$$V(\lambda x + (1-\lambda)y - w) \leq \lambda V(x-w) + (1-\lambda)V(y-w) \leq V(w)$$

by the convexity of V (Theorem 3.2). For any $c,d \in \mathcal{N}^2$ and $\lambda \in [0,1]$ we have

$$\begin{aligned}\lambda\pi^{ai}(c;w) + (1-\lambda)\pi^{ai}(d;w) &= \lambda \inf\{\gamma : c-\gamma\mathbb{1} \in C\} + (1-\lambda)\inf\{\delta : d-\delta\mathbb{1} \in C\} \\ &= \inf\{\lambda\gamma + (1-\lambda)\delta : c-\gamma\mathbb{1} \in C, d-\delta\mathbb{1} \in C\}.\end{aligned}$$

By the convexity of C, the conditions $c-\gamma\mathbb{1} \in C, d-\delta\mathbb{1} \in C$ imply that

$$\lambda c + (1-\lambda)d - (\lambda\gamma + (1-\lambda)\delta)\mathbb{1} = \lambda(c-\gamma\mathbb{1}) + (1-\lambda)(d-\delta\mathbb{1}) \in C,$$

which permits us to conclude that

$$\begin{aligned}\lambda\pi^{ai}(c;w) + (1-\lambda)\pi^{ai}(d;w) &\geq \inf\{\varepsilon : \lambda c + (1-\lambda)d - \varepsilon\mathbb{1} \in C\} \\ &= \pi^{ai}(\lambda c + (1-\lambda)d;w).\end{aligned}$$

This establishes the convexity of $u \mapsto \pi^{ai}(u;w)$ and completes the proof. ■

6. Solving the dual problem. It was shown in section 4 that solving the disutility minimisation problem (3.1) amounts to computing the value of $K(X)$, defined in (4.12), for suitably chosen X (see Theorem 4.3). The same holds true for determining the buyer's and seller's indifference prices in section 5 (see Theorem 5.1). In this section, we propose a dynamic procedure for determining $K(X)$ for any $X = (X^b, X^s) \in \mathcal{L}_T^2$. We also present a dynamic procedure for constructing a pair $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ such that

$$K(X) = H((\hat{\mathbb{Q}}, \hat{S}); X) = \sum_{t \in \mathcal{I}} \tfrac{1}{\alpha_t} \mathbb{E}\big[\Lambda_t^{\hat{\mathbb{Q}}} \ln \Lambda_t^{\hat{\mathbb{Q}}}\big] + \mathbb{E}_{\hat{\mathbb{Q}}}\big[X^b + X^s \hat{S}_T\big]. \tag{6.1}$$

Remark 6.1. The dynamic procedure can also be used to find the *minimal entropy martingale measure* (see [14, 15]). This is the measure $\hat{\mathbb{Q}}$ satisfying

$$K(0) = \mathbb{E}\big[\Lambda_T^{\hat{\mathbb{Q}}} \ln \Lambda_T^{\hat{\mathbb{Q}}}\big] = \mathbb{E}\left[\tfrac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \ln \tfrac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right],$$

in the special case when $\mathcal{I} = \{T\}$ and there are no transaction costs (in other words, $\hat{S} = S^b = S^a$).

The ability to construct a solution by dynamic programming follows from the following representation for H in terms of transition probabilities. The notation

$$a_t := \sum_{k \in \mathcal{I}, k \geq t} \tfrac{1}{\alpha_k} \text{ for all } t = 0, \ldots, T$$

will be used throughout this section for brevity.

Proposition 6.2. *For all $X = (X^b, X^s) \in \mathcal{L}_T^2$ and $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, we have*

$$H((\mathbb{Q}, S); X) = \sum_{t=0}^{T-1} a_{t+1} \sum_{\mu \in \Omega_t^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_{t+1}^\nu \ln \tfrac{q_{t+1}^\nu}{p_{t+1}^\nu} + \sum_{\mu \in \Omega_{T-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_T^\nu \big(X^{b\nu} + X^{s\nu} S_T^\nu\big). \tag{6.2}$$

Proof. For every $t = 1, \ldots, T$, observe from (2.7) that

$$\sum_{\nu \in \mu^+} q_t^\nu \ln \Lambda_t^{\mathbb{Q}\nu} = \ln \Lambda_{t-1}^{\mathbb{Q}\mu} + \sum_{\nu \in \mu^+} q_t^\nu \ln \tfrac{q_t^\nu}{p_t^\nu} \text{ for all } \mu \in \Omega_{t-1}^{\mathbb{Q}}, \nu \in \mu^+.$$

Using the nodes in Ω_{t-1} to partition Ω, and noting that $\mathbb{Q}$ and $\Lambda_t^{\mathbb{Q}}$ are nonzero only on the nodes in $\Omega_{t-1}^{\mathbb{Q}}$, leads to

$$\begin{aligned}
\mathbb{E}\big[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\big] = \mathbb{E}_{\mathbb{Q}}\big[\ln \Lambda_t^{\mathbb{Q}}\big] &= \sum_{\mu \in \Omega_{t-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_t^\nu \ln \Lambda_t^{\mathbb{Q}\nu} \\
&= \sum_{\mu \in \Omega_{t-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \ln \Lambda_{t-1}^{\mathbb{Q}\mu} + \sum_{\mu \in \Omega_{t-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_t^\nu \ln \tfrac{q_t^\nu}{p_t^\nu} \\
&= \mathbb{E}\big[\Lambda_{t-1}^{\mathbb{Q}} \ln \Lambda_{t-1}^{\mathbb{Q}}\big] + \sum_{\mu \in \Omega_{t-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_t^\nu \ln \tfrac{q_t^\nu}{p_t^\nu}.
\end{aligned}$$

Observing that $\mathbb{E}\big[\Lambda_0^{\mathbb{Q}} \ln \Lambda_0^{\mathbb{Q}}\big] = \mathbb{E}[1 \ln 1] = 0$, and introducing a telescoping sum, we obtain

$$\mathbb{E}\big[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\big] = \sum_{k=1}^{t} \left(\mathbb{E}\big[\Lambda_k^{\mathbb{Q}} \ln \Lambda_k^{\mathbb{Q}}\big] - \mathbb{E}\big[\Lambda_{k-1}^{\mathbb{Q}} \ln \Lambda_{k-1}^{\mathbb{Q}}\big]\right) = \sum_{k=1}^{t} \sum_{\mu \in \Omega_{k-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_k^\nu \ln \tfrac{q_k^\nu}{p_k^\nu}.$$

Then, after collecting like terms, it follows that

$$\sum_{t \in \mathcal{I}} \tfrac{1}{\alpha_t} \mathbb{E}\big[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\big] = \sum_{t \in \mathcal{I} \setminus \{0\}} \tfrac{1}{\alpha_t} \mathbb{E}\big[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\big] = \sum_{t=0}^{T-1} a_{t+1} \sum_{\mu \in \Omega_t^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_{t+1}^\nu \ln \tfrac{q_{t+1}^\nu}{p_{t+1}^\nu}.$$

The result follows from (4.11) after using the nodes in Ω_{T-1} to partition Ω and observing that

$$\mathbb{E}_{\mathbb{Q}}\big[X^b + X^s S_T\big] = \sum_{\mu \in \Omega_{T-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu \in \mu^+} q_T^\nu \big(X^{b\nu} + X^{s\nu} S_T^\nu\big).$$

■

The representation in Proposition 6.2 suggests that it is possible to construct a sequence $(\hat{q}_t)_{t=1}^T$ of transition probabilities, from which then to assemble the probability measure $\hat{\mathbb{Q}}$. The following construction provides a sequence of auxiliary functions to achieve this aim.

Construction 6.3. *For given $X = (X^b, X^s) \in \mathcal{L}_T^2$, construct two adapted sequences of random functions $(f_t)_{t=0}^{T-1}$ and $(J_t)_{t=0}^T$ by backward induction. Define $J_T : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as*

$$J_T^\nu(x) := \begin{cases} X^{b\nu} + x X^{s\nu} & \text{if } x \in \big[S_T^{b\nu}, S_T^{a\nu}\big], \\ \infty & \text{otherwise.} \end{cases} \tag{6.3}$$

for all $\nu \in \Omega_T$. For every $t = 0, \dots, T-1$, assume that J_{t+1} has already been constructed, and define

$$f_t^\mu(x) := \inf \left\{ \sum_{\nu \in \mu^+} q^\nu \left(J_{t+1}^\nu(x^\nu) + a_{t+1} \ln \tfrac{q^\nu}{p_{t+1}^\nu} \right) \right. \tag{6.4}$$
$$\left. : q^\nu \in [0,1], x^\nu \in \operatorname{dom} J_{t+1}^\nu \ \forall \nu \in \mu^+, \sum_{k=1}^m q^\nu = 1, \sum_{k=1}^m q^\nu x^\nu = x \right\},$$

$$J_t^\mu(x) := \begin{cases} f_t^\mu(x) & \text{if } x \in \big[S_t^{b\nu}, S_t^{a\nu}\big], \\ \infty & \text{otherwise.} \end{cases} \tag{6.5}$$

for all $\mu \in \Omega_t$ and $x \in \mathbb{R}$.

The definition (6.4) of f_t^ν is reminiscent of that of the convex hull of the collection $\{J_{t+1}^\nu\}_{\nu \in \mu^+}$ of convex functions, if the term involving the logarithm is disregarded; cf. [31, Theorem 5.6]. The following result summarises the main properties of $(J_t)_{t=0}^T$, with some of the technical arguments of the generalised convex hull deferred to Appendix A. Recall that the $\mathcal{F}_0$ is trivial, and therefore J_0 is a deterministic function.

Proposition 6.4. *Fix any $X \in \mathcal{L}^2_T$ and let $(J_t)_{t=0}^T$ be the sequence of functions from Construction 6.3. Then for each $t = 0, \dots, T$ and $\nu \in \Omega_t$, the function J^ν_t is convex, bounded from below, continuous on its closed effective domain $\operatorname{dom} J^\nu_t \subseteq [S^{b\nu}_t, S^{a\nu}_t]$ and the infimum in (6.4) is attained whenever it is finite. Moreover,*

$$J_0(S_0) = \inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}},\bar{S}_0=S_0} H((\bar{\mathbb{Q}},\bar{S});X) \text{ for all } (\mathbb{Q},S)\in\bar{\mathcal{P}}. \tag{6.6}$$

Proof. The properties of the J_t's are proved by backward induction. The convexity, continuity and boundedness properties of J^ν_T is self-evident from (6.3). For every $t = 0, \dots, T-1$, suppose that J^ν_t is convex, bounded from below and continuous on its effective domain $\operatorname{dom} J^\nu_t \subseteq [S^{b\nu}_t, S^{a\nu}_t]$ for all $\nu \in \Omega_{t+1}$. Define

$$g^\nu(q) := \begin{cases} a_{t+1} q \ln \frac{q}{p^\nu_{t+1}} & \text{if } q \in [0,1], \\ \infty & \text{otherwise} \end{cases}$$

for all $\nu \in \Omega_{t+1}$; then g^ν is convex, bounded from below and continuous on its effective domain $\operatorname{dom} g^\nu = [0,1]$. Propositions A.1 and A.4 then give that f^μ_t is convex, bounded from below and continuous on its effective domain for every $\mu \in \Omega_t$, and that the infimum in (6.4) is attained for all $x \in \operatorname{dom} f^\mu_t$. It is then clear from (6.5) that J^μ_t has the properties claimed. This concludes the inductive step.

To establish (6.6), fix any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$. We show first by backward induction that

$$\begin{aligned}\inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}}_{t+1}(\mathbb{Q},S)} H((\bar{\mathbb{Q}},\bar{S});X) &= \sum_{k=0}^{t} a_{k+1} \sum_{\mu\in\Omega^{\mathbb{Q}}_k} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q^\nu_{k+1} \ln \frac{q^\nu_{k+1}}{p^\nu_{k+1}} \\ &\quad + \sum_{\mu\in\Omega^{\mathbb{Q}}_t} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q^\nu_{t+1} J^\nu_{t+1}(S^\nu_{t+1})\end{aligned} \tag{6.7}$$

for all $t = 0, \dots, T-1$, where

$$\bar{\mathcal{P}}_t(\mathbb{Q},S) := \{(\bar{\mathbb{Q}},\bar{S}) \in \bar{\mathcal{P}} : \bar{\mathbb{Q}} = \mathbb{Q} \text{ on } \mathcal{F}_t, \bar{S}_k = S_k \, \forall k = 0,\dots,t\} \tag{6.8}$$

is the collection of martingale pairs that coincide with $(\mathbb{Q}, S)$ up to time $t = 0, \dots, T$. When $t = T-1$, we have $\bar{\mathcal{P}}_T(\mathbb{Q}, S) = \{(\mathbb{Q}, S)\}$, so that (6.7) follows from (6.2) and (6.3). Assume now that (6.7) holds for some $t = 1, \dots, T-1$. Rearrangement gives

$$\begin{aligned}\inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}}_{t+1}(\mathbb{Q},S)} H((\bar{\mathbb{Q}},\bar{S});X) &= \sum_{k=0}^{t-1} a_{k+1} \sum_{\mu\in\Omega^{\mathbb{Q}}_k} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q^\nu_{k+1} \ln \frac{q^\nu_{k+1}}{p^\nu_{k+1}} \\ &\quad + \sum_{\mu\in\Omega^{\mathbb{Q}}_t} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q^\nu_{t+1} \left(a_{t+1} \ln \frac{q^\nu_{t+1}}{p^\nu_{t+1}} + J^\nu_{t+1}(S^\nu_{t+1})\right),\end{aligned}$$

after which we obtain from (2.2), (6.5), and (6.8) that

$$\begin{aligned}\inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}}_t(\mathbb{Q},S)} & H((\bar{\mathbb{Q}},\bar{S});X) \\ &= \sum_{k=0}^{t-1} a_{k+1} \sum_{\mu\in\Omega_k^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q_{k+1}^\nu \ln \tfrac{q_{k+1}^\nu}{p_{k+1}^\nu} + \sum_{\mu\in\Omega_t^{\mathbb{Q}}} \mathbb{Q}(\mu) J_t^\mu(S_t^\mu) \\ &= \sum_{k=0}^{t-1} a_{k+1} \sum_{\mu\in\Omega_k^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q_{k+1}^\nu \ln \tfrac{q_{k+1}^\nu}{p_{k+1}^\nu} + \sum_{\mu\in\Omega_{t-1}^{\mathbb{Q}}} \mathbb{Q}(\mu) \sum_{\nu\in\mu^+} q_t^\nu J_t^\mu(S_t^\mu).\end{aligned}$$

This concludes the inductive step.

Finally, when $t=0$, the equation (6.7) reduces to

$$\inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}}_1(\mathbb{Q},S)} H((\bar{\mathbb{Q}},\bar{S});X) = a_1 \sum_{\nu\in\Omega_1} q_1^\nu \ln \tfrac{q_1^\nu}{p_1^\nu} + \sum_{\nu\in\Omega_1} q_1^\nu J_1^\nu(S_1^\nu),$$

and again combining (2.2), (6.5), and (6.8) yields

$$\inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}},\bar{S}_0=S_0} H((\bar{\mathbb{Q}},\bar{S});X) = \inf_{(\bar{\mathbb{Q}},\bar{S})\in\bar{\mathcal{P}}_0(\mathbb{Q},S)} H((\bar{\mathbb{Q}},\bar{S});X) = J_0(S_0).$$

This completes the proof. ■

The following construction uses the sequence $(J_t)_{t=0}^T$ of Construction 6.3 to produce a pair $(\hat{\mathbb{Q}},\hat{S})$ satisfying (6.1). It will be shown in Theorem 6.6 below that this indeed produces a solution to (4.12).

Construction 6.5. *For given $X=(X^b,X^s)\in\mathcal{L}_T^2$ and associated sequence $(J_t)_{t=0}^T$ from Construction 6.3, construct two adapted processes $(\hat{S}_t)_{t=0}^T$ and $(\hat{q}_t)_{t=0}^T$ by induction, as follows. First, choose any $\hat{S}_0$ satisfying*

$$J_0(\hat{S}_0) = \min_{x\in[S_0^b,S_0^a]} J_0(x). \tag{6.9}$$

For each $t=0,\dots,T-1$ and $\mu\in\Omega_t$, assume that $\hat{S}_t^\mu\in[S_t^{b\mu},S_t^{a\mu}]$ has already been defined, and choose $\hat{q}_{t+1}^\nu\in[0,1]$, $\hat{S}_{t+1}^\nu\in\left[S_{t+1}^{b\nu},S_{t+1}^{a\nu}\right]$ for all $\nu\in\mu^+$ such that

$$J_t^\mu(\hat{S}_t^\mu) = \sum_{\nu\in\mu^+} \hat{q}_{t+1}^\nu \left(a_{t+1}\ln\tfrac{\hat{q}_{t+1}^\nu}{p_{t+1}^\nu} + J_{t+1}^\nu(\hat{S}_{t+1}^\nu)\right), \tag{6.10}$$

$$\hat{S}_t^\mu = \sum_{\nu\in\mu^+} \hat{q}_{t+1}^\nu \hat{S}_{t+1}^\nu, \tag{6.11}$$

$$1 = \sum_{\nu\in\mu^+} \hat{q}_{t+1}^\nu. \tag{6.12}$$

Finally, define $\hat{\mathbb{Q}}:\mathcal{F}\to\mathbb{R}$ as

$$\hat{\mathbb{Q}}(A) := \sum_{\omega\in A}\prod_{t=1}^T \hat{q}_t^{\omega_t} \text{ for all } A\in\mathcal{F},$$

where the value of an empty summation is taken to be 0.

Construction 6.5 produces a well-defined pair $(\hat{\mathbb{Q}}, \hat{S})$. This is because the existence of $\hat{S}_0$ is assured by the continuity of J_0, and the infimum in (6.4) is attained whenever finite. It is, however, worth noting that the pair is not unique in general, because the solutions to (6.9) and (6.10)–(6.12) may not be unique.

The following theorem is the main result of this section. It establishes that Construction 6.5 indeed produces a solution to the optimization problem (4.12), as claimed at the start of the section.

Theorem 6.6. *For $X = (X^b, X^s) \in \mathcal{L}^2_T$ given, let $(J_t)_{t=0}^T$ and $(\hat{\mathbb{Q}}, \hat{S}) = (\hat{\mathbb{Q}}, (\hat{S}_t)_{t=0}^T)$ be the objects from Constructions 6.3 and 6.5. Then $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ and*

$$\begin{aligned} K(X) = J_0(\hat{S}_0) &= \min_{x \in [S_0^b, S_0^a]} J_0(x) \\ &= H((\hat{\mathbb{Q}}, \hat{S}); X) = \min_{(\mathbb{Q},S) \in \bar{\mathcal{P}}} H((\mathbb{Q}, S); X) \\ &= \min_{(\mathbb{Q},S) \in \bar{\mathcal{P}}} \sum_{t \in \mathcal{I}} \tfrac{1}{\alpha_t} \mathbb{E}\left[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\right] + \mathbb{E}_{\mathbb{Q}}\left[X^b + X^s S_T\right]. \end{aligned}$$

Proof. It is straightforward to show that $\hat{\mathbb{Q}}$ is a probability measure by standard arguments; cf. [6, Theorem 5.25]. Likewise, the process $\hat{S}$ is a martingale under $\hat{\mathbb{Q}}$ by (6.11), whence $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$. Furthermore, by recursive expansion of (6.10), we easily obtain

$$J_0(\hat{S}_0) = \sum_{t=0}^{T-1} a_{t+1} \sum_{\mu \in \Omega_t^{\hat{\mathbb{Q}}}} \hat{\mathbb{Q}}(\mu) \sum_{\nu \in \mu^+} \hat{q}_{t+1}^\nu \ln \tfrac{\hat{q}_{t+1}^\nu}{p_{t+1}^\nu} + \sum_{\mu \in \Omega_{T-1}^{\hat{\mathbb{Q}}}} \hat{\mathbb{Q}}(\mu) \sum_{\nu \in \mu^+} \hat{q}_T^\nu J_T^\nu(\hat{S}_T^\nu) = H((\hat{\mathbb{Q}}, \hat{S}); X)$$

from (4.11) and (6.3). Then (6.9), Proposition 6.4 and (4.12) combine to give

$$J_0(\hat{S}_0) = \min_{(\mathbb{Q},S) \in \bar{\mathcal{P}}} H((\mathbb{Q}, S); X) = K(X).$$

It remains to show that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$. Suppose by contradiction that $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}} \backslash \mathcal{P}$, in other words, $\Lambda_t^{\hat{\mathbb{Q}}}(\omega) = 0$ for some $t = 0, \dots, T$ and $\omega \in \Omega$. Fix any $(\mathbb{Q}, S) \in \mathcal{P}$, and define

$$\epsilon := \tfrac{1}{2} \exp \left\{ \frac{H((\hat{\mathbb{Q}}, \hat{S}); X) - H((\mathbb{Q}, S); X)}{\sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{Q}(\Lambda_t^{\hat{\mathbb{Q}}} = 0)} \right\}.$$

Observe that $\epsilon \in [0, 1)$ because

$$H((\hat{\mathbb{Q}}, \hat{S}); X) = J_0(\hat{S}_0) \leq J_0(S_0) \leq H((\mathbb{Q}, S); X).$$

Define a new probability measure $\mathbb{Q}^\epsilon : \mathcal{F} \to [0, 1]$ and stochastic process $S^\epsilon = (S_t^\epsilon)_{t=0}^T \in \mathcal{N}$ as

$$\mathbb{Q}^\epsilon := \epsilon \mathbb{Q} + (1 - \epsilon) \hat{\mathbb{Q}}, \qquad S_t^\epsilon := \epsilon S_t \tfrac{d\mathbb{Q}}{d\mathbb{Q}^\epsilon} + (1 - \epsilon) \hat{S}_t \tfrac{d\hat{\mathbb{Q}}}{d\mathbb{Q}^\epsilon} \text{ for all } t = 0, \dots, T.$$

Then $(\mathbb{Q}^\epsilon, S^\epsilon) \in \mathcal{P}$ [34, Lemma 7.2], after which (4.11) gives

$$\begin{aligned}(6.13)\quad & H((\mathbb{Q}^\epsilon, S^\epsilon); X) - H((\hat{\mathbb{Q}}, \hat{S}); X) \\ &= \sum_{t\in\mathcal{I}} \tfrac{1}{\alpha_t}\mathbb{E}\left[\Lambda_t^{\mathbb{Q}^\epsilon}\ln\Lambda_t^{\mathbb{Q}^\epsilon} - \Lambda_t^{\hat{\mathbb{Q}}}\ln\Lambda_t^{\hat{\mathbb{Q}}}\right] + \epsilon\left(\mathbb{E}_{\mathbb{Q}}\left[X^b + X^s S_T\right] - \mathbb{E}_{\hat{\mathbb{Q}}}\left[X^b + X^s\hat{S}_T\right]\right).\end{aligned}$$

The mapping $x \mapsto x\ln x$ is convex on $[0,\infty)$, and so, for all $t = 0,\dots,T$,

$$\Lambda_t^{\mathbb{Q}^\epsilon}\ln\Lambda_t^{\mathbb{Q}^\epsilon} - \Lambda_t^{\hat{\mathbb{Q}}}\ln\Lambda_t^{\hat{\mathbb{Q}}} \leq \epsilon\left(\Lambda_t^{\mathbb{Q}}\ln\Lambda_t^{\mathbb{Q}} - \Lambda_t^{\hat{\mathbb{Q}}}\ln\Lambda_t^{\hat{\mathbb{Q}}}\right).$$

Furthermore, on the set $\left\{\Lambda_t^{\hat{\mathbb{Q}}} = 0\right\}$, and recalling the convention $0\ln 0 = 0$, we have

$$\Lambda_t^{\mathbb{Q}^\epsilon}\ln\Lambda_t^{\mathbb{Q}^\epsilon} - \Lambda_t^{\hat{\mathbb{Q}}}\ln\Lambda_t^{\hat{\mathbb{Q}}} = \epsilon\Lambda_t^{\mathbb{Q}}\ln\epsilon\Lambda_t^{\mathbb{Q}} = \epsilon\left(\Lambda_t^{\mathbb{Q}}\ln\Lambda_t^{\mathbb{Q}} - \Lambda_t^{\hat{\mathbb{Q}}}\ln\Lambda_t^{\hat{\mathbb{Q}}}\right) + \epsilon\Lambda_t^{\mathbb{Q}}\ln\epsilon.$$

Substituting this into (6.13) gives

$$\begin{aligned}& H((\mathbb{Q}^\epsilon, S^\epsilon); X) - H((\hat{\mathbb{Q}}, \hat{S}); X) \\ &\qquad \leq \epsilon\left(H((\mathbb{Q}, S); X) - H((\hat{\mathbb{Q}}, \hat{S}); X) + \ln\epsilon\sum_{t\in\mathcal{I}}\tfrac{1}{\alpha_t}\mathbb{Q}\left(\Lambda_t^{\hat{\mathbb{Q}}} = 0\right)\right).\end{aligned}$$

The choice of ϵ implies that $H((\mathbb{Q}^\epsilon, S^\epsilon); X) < H((\hat{\mathbb{Q}}, \hat{S}); X)$, which is a contradiction. Hence $\hat{\mathbb{Q}}(\omega) > 0$ for all $\omega \in \Omega$, so that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$. ■

7. Numerical examples.

Consider a friction-free binomial tree model with $T = 52$ steps representing one year in real time with weekly rehedging, where the stock price $S = (S_0)_{t=0}^T$ satisfies $S_0 = 100$ and

$$S_{t+1} = \begin{cases} e^{\sigma\sqrt{1/52}}S_t & \text{with probability } p, \\ e^{-\sigma\sqrt{1/52}}S_t & \text{with probability } 1-p \end{cases}$$

for all $t = 0,\dots,51$. Here $\sigma = 0.2$ is the annual volatility of the return on stock, and the model is assumed to have an annual effective interest rate of $r_e = 0.02$. Define the bid and ask prices of the stock as

$$S_t^a \coloneqq (1+k)S_t, \qquad\qquad S_t^b \coloneqq (1-k)S_t$$

for $t = 1,\dots,52$, where k is the proportional transaction cost parameter. We assume that there are no transaction costs at time 0, in other words $S_0^a \coloneqq S_0^b \coloneqq S_0 = 100$.

The numerical results in this section have been obtained by applying the approximation methods introduced in Appendix A.2 for the generalised convex hull. Each of these methods allow us to construct a sequence of random piecewise linear functions approximating the sequence $(J_t)_{t=0}^{52}$ of Construction 6.3, starting from the final value J_{52}. This leads naturally to an approximation for K via Theorem 6.6, and $\pi^{ai}(c;w)$ and $\pi^{bi}(c;w)$ via Theorem 5.1. Superhedging bid and ask prices are also provided for the purposes of comparison; these have been calculated using the methods described in [34].

Table 1

Indifference prices by approximation method (Example 7.1)

n	20	50	100	150	200	300
Upper approximation method						
$\pi^{bi}(C;0)$	8.5759	8.5673	8.5658	8.5655	8.5654	8.5654
$\pi^{ai}(C;0)$	9.1596	9.1672	9.1684	9.1687	9.1687	9.1688
Lower approximation method						
$\pi^{bi}(C;0)$	8.4974	8.5533	8.5633	8.5647	8.5652	8.5653
$\pi^{ai}(C;0)$	9.2357	9.1797	9.171	9.1692	9.1690	9.1690

Throughout this section we assume that the investor's endowment is $w = 0$, and that the risk aversion coefficient is constant, in other words, $\alpha_t = \alpha$ for all $t \in \mathcal{I}$. We will consider a call option with expiry one year, strike 100 and physical delivery (based on the underlying). This corresponds to the payment stream $C = (C_t)_{t=0}^{52}$ where $C_t = 0$ for all $t < 52$ and

$$C_{52} = (-100, 1)\mathbb{1}_{\{S_{52}>100\}}.$$

We first demonstrate the accuracy of the numerical approximation.

Example 7.1. Table 1 contains approximate indifference prices for the seller and buyer of the call option in the case where $p = 0.5$, $k = 0.005$, $\mathcal{I} = \{0, \ldots, 52\}$ and $\alpha = 0.1$, as computed by both the upper and lower approximation methods described in Appendix A.2. In each case, the approximation is obtained by dividing each (discounted) bid-ask interval into n subintervals of equal length.

It is evident from Table 1 that the upper approximation converges much faster than the lower approximation. The two approximation methods are also consistent in that they appear to converge to the same limit. The results suggest that taking $n = 150$ results in accuracy up to 3 decimal places, which is perfectly adequate for graphical representation.

It is also interesting to note that the indifference pricing spread (between the seller's and buyer's indifference prices) is considerably smaller than the (superhedging) bid-ask spread; note that the ask and bid prices in this case are $\pi^a(C) = 10.4788$ and $\pi^b(C) = 6.9694$.

In each of the examples below we consider different possibilities for the set $\mathcal{I}$ of dates on which injection is allowed. In particular, the case $\mathcal{I} = \{52\}$ corresponds to the classical utility indifference pricing framework, where the "injection" at time 52 reflects the hedging shortfall at the expiration date of the option under exponential utility.

Example 7.2. Continuing with the case where $k = 0.005$ and $p = 0.5$, we now consider seller's and buyer's indifference prices for a range of values of the risk aversion coefficient α; see Figure 1. Observe that the indifference pricing spread (between the seller's and buyer's indifference prices) is smaller for disutility pricing than for utility indifference pricing. This is because being able to inject cash at different time steps introduces considerable flexiblity, which in turn results in decreased hedging costs.

As expected, indifference pricing spreads increase as the risk aversion coefficient increases. It does however appear that the indifference pricing spread remains well within the superhedging bid-ask spread for a large range of values of the risk aversion coefficient.

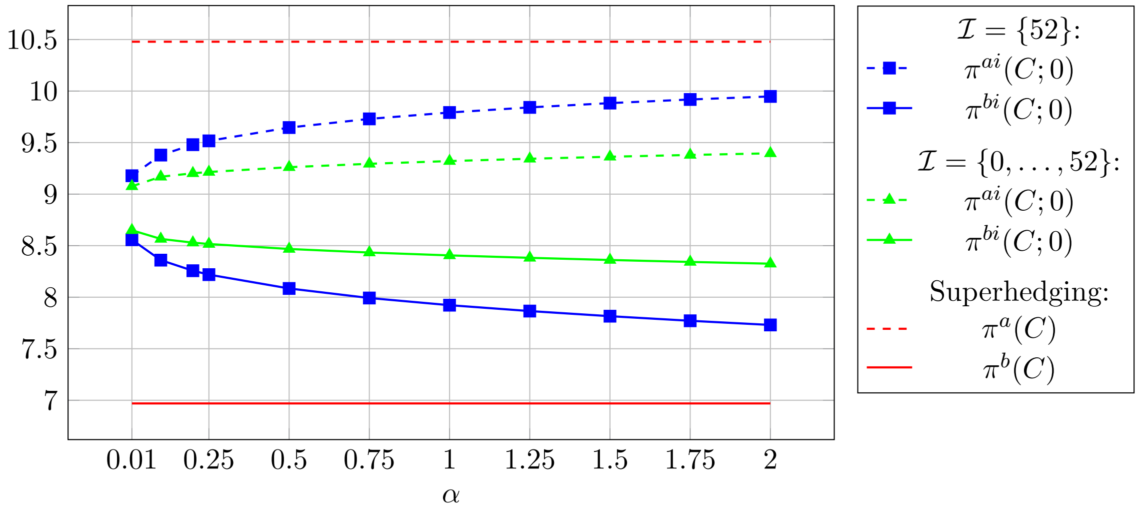

Figure 1. *Indifference prices and risk aversion (Example 7.2)*

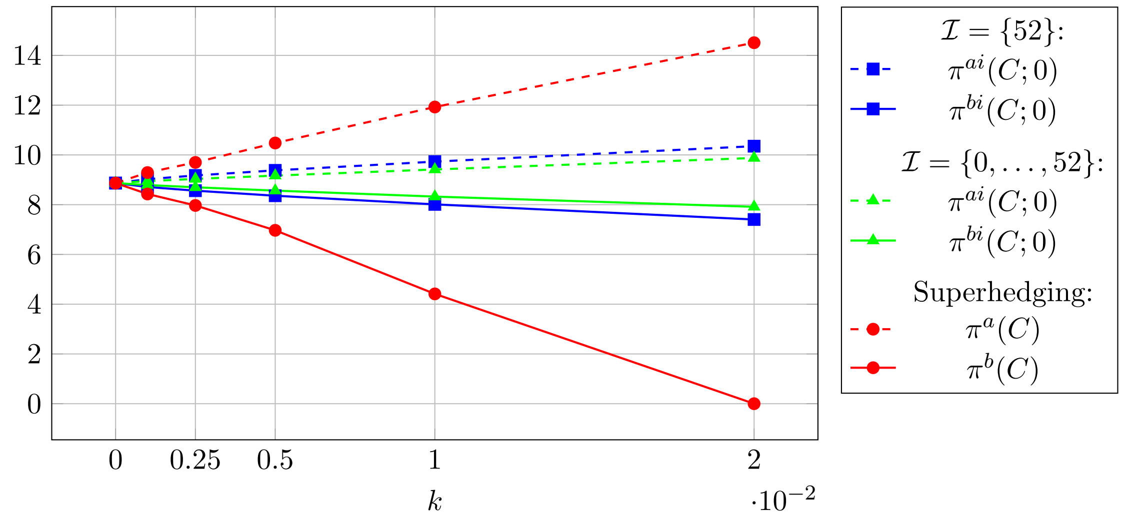

Figure 2. *Indifference prices and transaction costs (Example 7.3)*

Example 7.3. Figure 2 contains prices for a range of values of the transaction costs parameter k in the case where $p = 0.5$ and $\alpha = 0.1$. Indifference pricing spreads increase with k, the reason being that increased transaction costs results in an expansion of the set $\bar{\mathcal{P}}$, and hence tends to lead to lower values for $K(C_{52})$ and $K(-C_{52})$. At the same time, the value of $K(0)$ appears to be less sensitive to changes in k; in fact, for each of the data points in Figure 2 we have $K(0) = 0$, so that $\pi^{bi}(C;0) = K(C_{52})$ and $\pi^{ai}(C;0) = -K(-C_{52})$. Observe finally that the indifference pricing spreads remain well within the superhedging bid-ask spread for all values of k, and also expand slower as k increases.

Example 7.4. Buyer's and seller's indifference prices for a range of values of the market

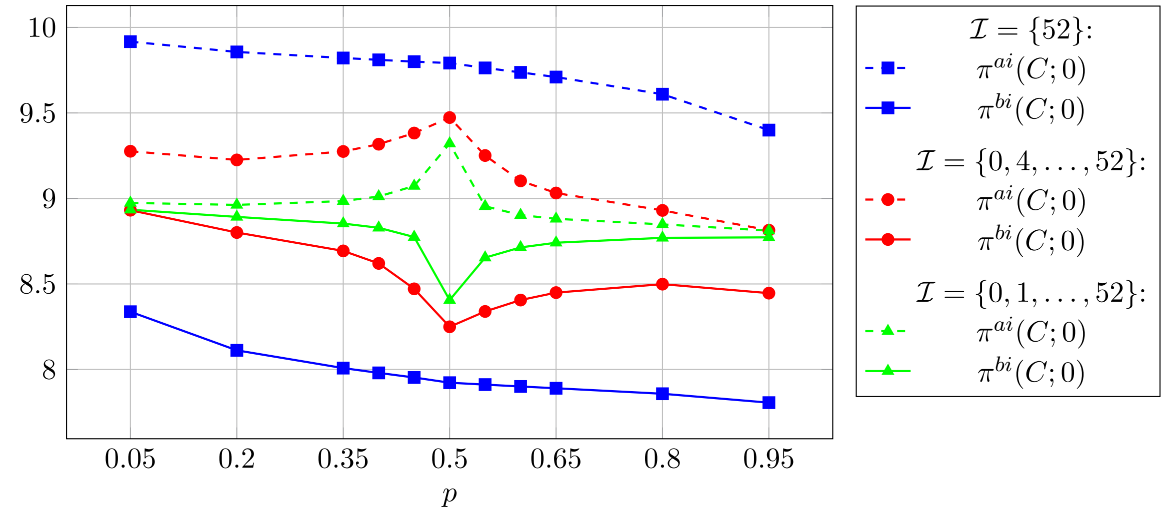

Figure 3. *Indifference prices and market probability (Example 7.4)*

probability parameter p in the case where $k = 0.005$ and $\alpha = 0.1$ are illustrated in Figure 3. It appears that indifference pricing spreads tend to be at their largest when p is close to the value of the friction-free risk-neutral probability in this model, which is

$$q = \frac{(1 + r_e)^{1/52} - e^{-\sigma\sqrt{1/52}}}{e^{\sigma\sqrt{1/52}} - e^{-\sigma\sqrt{1/52}}} \approx 0.4999.$$

The effect is more pronounced when injection is allowed at more trading dates. One possible explanation for this might be found upon examining the behaviour of $K(0)$, $K(C_{52})$ and $K(-C_{52})$ for different values of p; see Figure 4. Whilst the dependence of these values on p appear to be convex, they vary in steepness, both within groups associated with the same choice and $\mathcal{I}$, and between groups associated with different choices of $\mathcal{I}$. This then has consequences for the vertical differences $\pi^{bi}(C;0) = K(C_{52}) - K(0)$ and $\pi^{ai}(C;0) = K(0) - K(-C_{52})$.

A large number of numerical examples, for a selection of options with cash and physical delivery, and for a range of values of r_e and T, can be found in Section 5.5 of [39].

Appendix A. Generalised convex hull.

The constructions in section 6 involve a generalisation of the convex hull of convex functions. This appendix outlines the main properties used in this paper in an abstract setting.

For $k = 1, \dots, m$, let $f_k, g_k : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be proper convex functions that are continuous on their effective domains, and such that $\operatorname{dom} f_k = [b_k, a_k]$ for some $b_k, a_k \in \mathbb{R}$, $\operatorname{dom} g_k = [0, 1]$ and

$$g_k(0) = 0 \tag{A.1}$$

for all $k = 1, \dots, m$. Define the *generalised convex hull* $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ of $f_1, \dots, f_m$ and

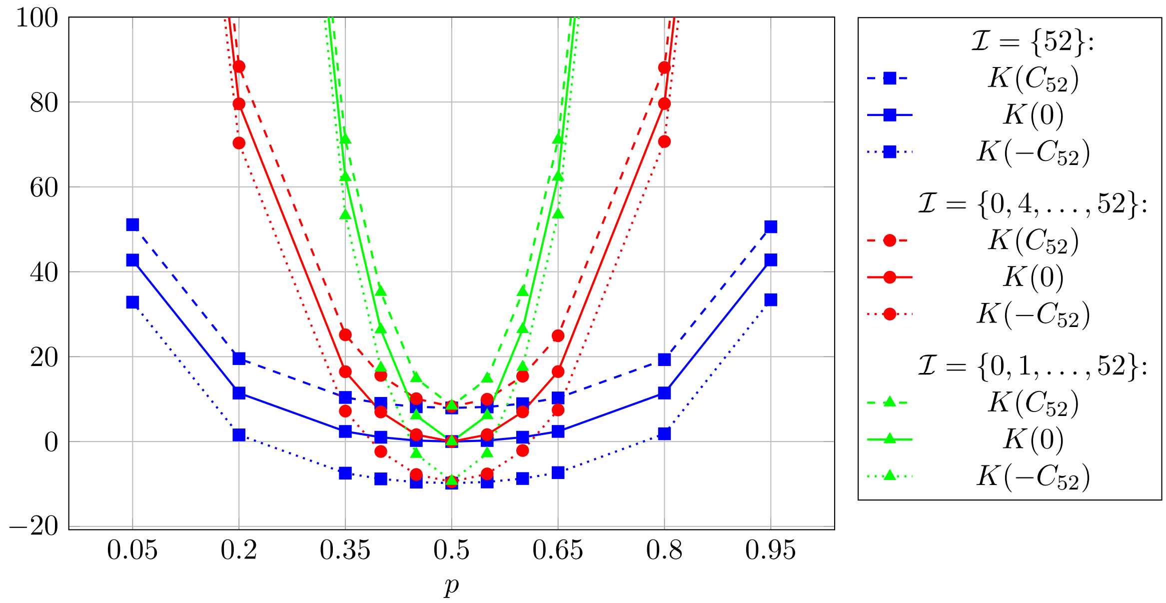

Figure 4. *Values of K and market probability (Example 7.4)*

$g_1, \dots g_m$ as

$$f(x) := \inf \left\{ \sum_{k=1}^{m} (q_k f_k(x_k) + g_k(q_k)) : q_k \in [0,1], x_k \in [b_k, a_k] \text{ for all } k = 1, \dots, m, \right.$$
$$\left. \sum_{k=1}^{m} q_k = 1, \sum_{k=1}^{m} q_k x_k = x \right\}. \tag{A.2}$$

A.1. General properties. The main aim of this section is to establish the key properties needed in section 6, namely, that f is convex, bounded from below, continuous on its effective domain, which is compact, and that the infimum in (A.2) is attained whenever it is finite. Further detail on the arguments below, in a slightly more general setting, can be found in [39, Chapter 4].

Most of the desired properties are straightforward, and collected in the following result.

Proposition A.1. *The function f in (A.2) is proper, convex, and its effective domain*

$$\operatorname{dom} f = \operatorname{conv} \bigcup_{k=1}^{m} [b_k, a_k] = \left[\min_{k=1,\dots,m} b_k, \max_{k=1,\dots,m} a_k \right] \tag{A.3}$$

is compact.

Proof. Much of the proof is straightforward, hence omitted. The compactness of $\operatorname{dom} f$ comes from [31, Corollary 9.8.2]. The properness of f follows from the fact that continuous proper convex functions with compact domains are bounded from below. To show that f is convex, fix any $y, z \in \operatorname{dom} f$ and $\lambda \in (0,1)$. By (A.3) there exists $(q_k^y, y_k)_{k=1}^m$ and $(q_k^z, z_k)_{k=1}^m$ such that $q_k^y, q_k^z \geq 0$ for all $k = 1, \ldots, m$ and

$$\sum_{k=1}^m q_k^y = 1, \qquad \sum_{k=1}^m q_k^z = 1, \qquad \sum_{k=1}^m q_k^y y_k = y, \qquad \sum_{k=1}^m q_k^z z_k = z.$$

Define now

$$q_k := \lambda q_k^y + (1-\lambda) q_k^z, \qquad x_k := \begin{cases} y_k & \text{if } q_k = 0, \\ \frac{1}{q_k}\left(\lambda q_k^y y_k + (1-\lambda) q_k^z z_k\right) & \text{if } q_k > 0 \end{cases}$$

for all $k = 1, \ldots, m$; then it is straightforward to verify that

$$q_k \geq 0 \text{ for all } k = 1, \ldots, m, \qquad \sum_{k=1}^m q_k = 1, \qquad \sum_{k=1}^m q_k x_k = \lambda y + (1-\lambda) z.$$

It then follows from (A.2) and the convexity of $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$ that

$$\begin{aligned} f(\lambda y + (1-\lambda) z) &\leq \sum_{k=1}^m (q_k f_k(x_k) + g_k(q_k)) \\ &\leq \lambda \sum_{k=1}^m \left(q_k^y f_k(y_k) + g_k(q_k^y)\right) + (1-\lambda) \sum_{k=1}^m \left(q_k^z f_k(z_k) + g_k(q_k^z)\right). \end{aligned}$$

Taking the infimum in both terms on the right hand side gives

$$f(\lambda y + (1-\lambda) z) \leq \lambda f(y) + (1-\lambda) f(z)$$

as required. ■

The remainder of this section is devoted to establishing the closedness of the epigraph of f. This then allows us to establish the desired properties; see Proposition A.4 at the end of the appendix. In order to prepare for this result, we present a number of technical results. Define

$$A_k^g := \{(q, qx, qy + g_k(q)) : q \in [0,1], (x,y) \in \operatorname{epi} f_k\} \tag{A.4}$$

for all $k = 1, \ldots, m$, and

$$U := \{(0,0,b) \in \mathbb{R}^3 : b \geq 0\}.$$

Observe immediately that if $q = 0$, then $(q,a,b) \in A_k^g$ if and only if $a = b = 0$. This also implies that $A_k^g \neq \emptyset$. Moreover, if $(q,a,b) \in A_k^g$ satisfies $q > 0$, then $(q,a,b) + U \subset A_k^g$. The following result establishes a number of properties of A_k^g that will be used in Proposition A.3.

Proposition A.2. *For any $k = 1, \ldots, m$, the following holds true for the set A_k^g in (A.4):*

1. *A_k^g is convex.*
2. $\operatorname{cl} A_k^g = U \cup A_k^g$.
3. $0^+(\operatorname{cl} A_k^g) = U$.

Proof. Item 1: Fix any $\lambda \in (0,1)$, $q_1, q_2 \in [0,1]$ and $(x_1,y_1),(x_2,y_2) \in \operatorname{epi} f_k$ and define

$$\begin{aligned} q &:= \lambda q_1 + (1-\lambda)q_2, \\ z &:= (q, \lambda q_1 x_1 + (1-\lambda)q_2x_2, \lambda(q_1y_1 + g_k(q_1)) + (1-\lambda)(q_2y_2 + g_k(q_2))). \end{aligned}$$

If $q = 0$, then $q_1 = q_2 = 0$, after which $x_1 = y_1 = x_2 = y_2 = 0$ by the observation above, so that $z = 0 \in A_k^g$. If $q > 0$, then define

$$\varepsilon := \lambda g_k(q_1) + (1-\lambda)g_k(q_2) - g_k(q), \quad (x,y) := \tfrac{1}{q}(\lambda q_1(x_1,y_1) + (1-\lambda)q_2(x_2,y_2)) + \tfrac{1}{q}(0,\varepsilon).$$

Then $\varepsilon \geq 0$ because g_k is convex and $(x,y) \in \operatorname{epi} f_k$ because $\operatorname{epi} f_k$ is convex and unbounded from above. This permits us to conclude that $z = (q, qx, qy + g_k(q)) \in A_k^g$, so that A_k^g is convex.

Item 2: Define $A_k := \operatorname{cone}(\{1\} \times \operatorname{epi} f_k)$; then $\operatorname{cl} A_k = U \cup A_k$ due to the compactness of $\operatorname{dom} f_k$ [31, Theorem 8.2]. For every $(0,0,b) \in U \subset \operatorname{cl} A_k$ there exist $(q_n)_{n\geq 1}$ in $[0,1]$ and $(x_n,y_n)_{n\geq 1}$ in $\operatorname{epi} f_k$ such that

$$(0,0,b) = \lim_{n\to\infty} q_n(1,x_n,y_n) = \lim_{n\to\infty} q_n(1,x_n,y_n + g_k(q_n)),$$

with the last equality due to (A.1) and the continuity of g_n. Thus $(0,0,b) \in \operatorname{cl} A_k^g$. Combining this with $A_k^g \subseteq \operatorname{cl} A_k^g$ permits us to conclude that $U \cup A_k^g \subseteq \operatorname{cl} A_k^g$.

To establish the opposite inclusion, suppose that $(q,a,b) \in \operatorname{cl} A_k^g$. Then there exist $(q_n)_{n\geq 1}$ in $[0,1]$ and $(x_n,y_n)_{n\geq 1}$ in $\operatorname{epi} f_k$ such that

$$(q,a,b) = \lim_{n\to\infty}(q_n, q_nx_n, q_ny_n + g_k(q_n)).$$

Observe that $\lim_{n\to\infty} g_k(q_n) = g_k(q)$ by the continuity of g_k, so that

$$b - g_k(q) = \lim_{n\to\infty} q_ny_n.$$

Moreover, since $q_n(1,x_n,y_n) \in A_k$ for all $n \in \mathbb{N}$ it follows that

$$(q,a,b-g_k(q)) = \lim_{n\to\infty} q_n(1,x_n,y_n) \in \operatorname{cl} A_k = U \cup A_k.$$

There are now two possibilities. If $(q,a,b-g_k(q)) \in U$, then $q = 0$ and so $(q,a,b) \in U$ by (A.1). If $(q,a,b-g_k(q)) \in A_k$ then there exist $(x,y) \in \operatorname{epi} f_k$ such that $(q,a,b-g_k(q)) = q(1,x,y)$, in other words, $(q,a,b) = (q,qx,qy+g_k(q)) \in A_k^g$.

Item 3: The comments just before this proposition together with Item 2 gives that $U \subseteq 0^+(\operatorname{cl} A_k^g)$. For the opposite inclusion, take any $(q,a,b) \in 0^+(\operatorname{cl} A_k^g)$. Since $0 \in \operatorname{cl} A_k^g$, this implies that

$$\lambda(q,a,b) = 0 + \lambda(q,a,b) \in \operatorname{cl} A_k^g = U \cup A_k^g \text{ for all } \lambda > 0.$$

It then follows from (A.4) and the comments following it that $q = a = 0$, whence $(q,a,b) \in U$. ■

Define

$$E_f := \left\{ \sum_{k=1}^m (q_kx_k, q_ky_k + g_k(q_k)) : q_k \in [0,1], (x_k,y_k) \in \operatorname{epi} f_k \ \forall k, \sum_{k=1}^m q_k = 1 \right\}; \tag{A.5}$$

then

$$E_f = \left\{ (a, b) : (1, a, b) \in \sum_{k=1}^{m} A_k^g \right\}. \tag{A.6}$$

It will be shown in the proof of Proposition A.4 that $E_f = \operatorname{epi} f$. The following result is the first step towards establish this, together with the desired closedness property.

Proposition A.3. *The set E_f in (A.5) is closed.*

Proof. We first show that

$$\{1\} \times E_f = M \cap \sum_{k=1}^{m} \operatorname{cl} A_k^g, \tag{A.7}$$

where

$$M := \{1\} \times \mathbb{R}^2.$$

Equation (A.6) immediately gives that $\{1\} \times E_f \subseteq M \cap \sum_{k=1} \operatorname{cl} A_k^g$. To establish the opposite inclusion, fix any $(q, a, b) \in M \cap \sum_{k=1} \operatorname{cl} A_k^g$; then $q = 1$ and by Proposition A.2.2 there exist $(q_k, a_k, b_k) \in U \cup A_k^g$ for every $k = 1, \ldots, m$ such that

$$(q, a, b) = (1, a, b) = \sum_{k=1}^{m} (q_k, a_k, b_k).$$

Define

$$B := \{k \in \{1, \ldots, m\} : (q_k, a_k, b_k) \in U\}, \quad C := \{k \in \{1, \ldots, m\} : (q_k, a_k, b_k) \in A_k^g \setminus U\}.$$

For each $k \in B$, we have $q_k = a_k = 0$ and $b_k \geq 0$; select any $(x_k, y_k) \in \operatorname{epi} f_k$ and observe that

$$(q_k, q_k x_k, q_k y_k + g_k(q_k)) = 0 = (q_k, a_k, b_k - b_k).$$

Noting that $C \neq \emptyset$ (because $q_k > 0$ for at least one k), define

$$c := \tfrac{1}{|C|} \sum_{k \in B} b_k \geq 0.$$

For each $k \in C$ there exists some $(x_k, y_k') \in \operatorname{epi} f_k$ such that

$$(q_k, a_k, b_k) = (q_k, q_k x_k, q_k y_k' + g_k(q_k)).$$

Define $y_k := y_k' + \frac{c}{q_k} \geq y_k'$; then $(x_k, y_k) \in \operatorname{epi} f_k$ and

$$(q_k, q_k x_k, q_k y_k + g_k(q_k)) = (q_k, a_k, b_k + c)$$

Finally, rearrangement gives that

$$(q, a, b) = \sum_{k=1}^{m} (q_k, a_k, b_k) = \sum_{k \in C}^{m} (q_k, a_k, b_k + c) = \sum_{k=1}^{m} (q_k, q_k x_k, q_k y_k + g_k(q_k)) \in M \cap \sum_{k=1} \operatorname{cl} A_k^g,$$

which establishes (A.7).

Note that $\sum_{k=1}^m A_k^g$ is convex [31, Theorem 3.1]. Furthermore, if $z_k \in 0^+(\operatorname{cl} A_k^g) = U$ for all $k = 1, \dots, m$ satisfies $\sum_{k=1}^m z_k = 0$, then $z_1 = \dots = z_m = 0 \in U \cap (-U)$; this means that

$$\operatorname{cl} \sum_{k=1}^m A_k^g = \sum_{k=1}^m \operatorname{cl} A_k^g \tag{A.8}$$

[31, Corollary 9.1.1]. It remains to show that

$$M \cap \operatorname{ri} \sum_{k=1}^m A_k^g \neq \emptyset, \tag{A.9}$$

because then the closedness E_f follows from (A.7) and (A.8) and

$$M \cap \operatorname{cl} \sum_{k=1}^m A_k^g = \operatorname{cl}\left(M \cap \sum_{k=1}^m A_k^g\right)$$

[31, Corollary 6.5.1].

To establish (A.9), observe that $\operatorname{ri} \sum_{k=1}^m A_k^g \neq \emptyset$ because $\sum_{k=1}^m A_k^g \neq \emptyset$. Thus there exist $q_k \in [0,1]$ and $(x_k, y_k) \in \operatorname{epi} f_k$ for all $k = 1, \dots, m$ such that

$$(q, a, b) := \sum_{k=1}^m (q_k, q_k x_k, q_k y_k + g_k(q_k)) \in \operatorname{ri} \sum_{k=1}^m A_k^g.$$

This can now be used to construct a point $z \in M \cap \operatorname{ri} \sum_{k=1}^m A_k^g$. There are two possibilities, depending on the value of q. If $q \geq 1$, then define $z := \frac{1}{q}(q, a, b)$. Then clearly $z \in M$ and moreover z can be written as the convex combination

$$z = \tfrac{1}{q}(q, a, b) + \left(1 - \tfrac{1}{q}\right)(0, 0, 0) \in \operatorname{ri} \sum_{k=1}^m A_k^g$$

[31, Theorem 6.1]. If $q \in [0,1]$, then define $q'_k := \frac{1}{m}(2 - q) > 0$ for all $k = 1, \dots, m$ and

$$z' := \sum_{k=1}^m (q'_k, q'_k x_k, q'_k y_k + g_k(q'_k)) \in \sum_{k=1}^m A_k^g.$$

Then

$$z := \tfrac{1}{2}(q, a, b) + \tfrac{1}{2} z' \in \operatorname{ri} \sum_{k=1}^m A_k^g$$

[31, Theorem 6.1] and $z \in M$ because

$$\tfrac{1}{2} q + \tfrac{1}{2} \sum_{k=1}^m q'_k = 1.$$

This completes the proof of (A.9). ■

The following result concludes this section.

Proposition A.4. *The function f in (A.2) is continuous on* $\operatorname{dom} f$*, and the infimum in (A.2) is attained for all $x \in \operatorname{dom} f$.*

Proof. It is sufficient to show that $\operatorname{epi} f = E_f$, for then f is lower semicontinuous by Proposition A.3, hence continuous on $\operatorname{dom} f$ because it is a closed bounded interval [31, Theorems 10.2, 20.5]. The fact that the infimum in (A.2) is attained for all $x \in \operatorname{dom} f$ follows from the properties of E_f.

Suppose that $(x, y) \in E_f$. Thus there exist $q_k \in [0, 1]$ and $(x_k, y_k) \in \operatorname{epi} f_k$ for all $k = 1, \ldots, m$ such that

$$\sum_{k=1}^{m} q_k = 1, \qquad \sum_{k=1}^{m} q_k x_k = x, \qquad \sum_{k=1}^{m} (q_k y_k + g_k(q_k)) = y.$$

Then

$$y = \sum_{k=1}^{m} (q_k y_k + g_k(q_k)) \geq \sum_{k=1}^{m} (q_k f_k(x_k) + g_k(q_k)) \geq f(x),$$

and so $(x, y) \in \operatorname{epi} f$.

Conversely, suppose that $(x, y) \in \operatorname{epi} f$. Then $f(x) < \infty$ and so by (A.2) there exists a sequence $(q_{1n}, \ldots, x_{mn}, x_{1n}, \ldots, x_{mn})_{n \geq 1}$ such that for all $n \in \mathbb{N}$ we have $q_{kn} \in [0, 1]$ and $x_{kn} \in [b_k, a_k]$ for all $k = 1, \ldots, m$ and

$$\sum_{k=1}^{m} q_{kn} = 1, \qquad \sum_{k=1}^{m} q_{kn} x_{kn} = 1,$$

and finally

$$f(x) = \lim_{n \to \infty} \sum_{k=1}^{m} (q_{kn} f_k(x_{kn}) + g_k(q_{kn})).$$

For each $n \in \mathbb{N}$ and $k = 1, \ldots, m$ define

$$y_{kn} \coloneqq f_k(x_{kn}) + y - f(x) \geq f_k(x_{kn});$$

then $(x_{kn}, y_{kn}) \in \operatorname{epi} f_k$. Define moreover for all $n \in \mathbb{N}$

$$y_n \coloneqq \sum_{k=1}^{m} (q_{kn} y_{kn} + g_k(q_{kn})) = \sum_{k=1}^{m} (q_{kn} f_k(x_{kn}) + g_k(q_{kn})) + y - f(x);$$

then $(x, y_n) \in E_f$ and

$$\lim_{n \to \infty} y_n = y.$$

This implies that $(x, y) \in \operatorname{cl} E_f = E_f$ by Proposition A.3, which concludes the proof that $\operatorname{epi} f = E_f$. ∎

A.2. Numerical approximation. Computer implementation of the generalised convex hull necessitates a numerical approximation in all but a few special cases. In this section we propose such a numerical approximation, together with error bounds, that will be suitable for use in the dynamic procedure proposed in section 6. It is based on approximation of $f_1, \dots, f_m$ and f by piecewise linear functions. We will refer to this as the *upper approximation* as it approximates the generalised convex hull f from above.

For every $k = 1, \dots, m$, divide $\operatorname{dom} f_k = [b_k, a_k]$ into n_k subintervals. If $b_k = a_k$, then define $\hat{x}_{k0} \coloneqq \hat{x}_{k1} \coloneqq \cdots \coloneqq \hat{x}_{kn_k} \coloneqq a_k$, and if $b_k < a_k$, choose any $(\hat{x}_{kl})_{l=0}^{n_k}$ such that $b_k \eqqcolon \hat{x}_{k0} < \cdots < \hat{x}_{kn_k} \coloneqq a_k$. Define $\hat{f}_k : \mathbb{R} \to \{\infty\}$ as

(A.10)

$$\hat{f}_k(x) \coloneqq \begin{cases} f(\hat{x}_{kl}) & \text{if } x = \hat{x}_{kl} \text{ for some } l = 0, \dots, n_k, \\ \frac{\hat{x}_{kl} - x}{\hat{x}_{kl} - \hat{x}_{k[l-1]}} \hat{f}_k(\hat{x}_{k[l-1]}) + \frac{x - \hat{x}_{k[l-1]}}{\hat{x}_{kl} - \hat{x}_{k[l-1]}} \hat{f}_k(\hat{x}_{kl}) & \text{if } x \in (\hat{x}_{k[l-1]}, \hat{x}_{kl}) \text{ for any } l = 1, \dots, n_k, \\ \infty & \text{if } x \in \mathbb{R} \setminus \operatorname{dom} f_k. \end{cases}$$

Observe that $\hat{f}_k \geq f_k$ by virtue of the convexity of f_k.

Let $\hat{g}$ be the generalised convex hull of $\hat{f}_1, \dots, \hat{f}_m$ and $g_1, \dots g_m$, in other words,

$$\hat{g}(x) \coloneqq \inf \left\{ \sum_{k=1}^{m} (q_k \hat{f}_k(x_k) + g_k(q_k)) : q_k \in [0,1], x_k \in [b_k, a_k] \text{ for all } k = 1, \dots, m, \sum_{k=1}^{m} q_k = 1, \sum_{k=1}^{m} q_k x_k = x \right\}. \tag{A.11}$$

Then $\hat{g} \geq f$ by definition, and it follows from the arguments in the previous subsection that $\hat{g}$ is convex and continuous on its effective domain $\operatorname{dom} \hat{g} = \operatorname{dom} f$, and that the infimum in (A.11) is attained for all $x \in \operatorname{dom} \hat{g} = \operatorname{dom} f$.

In practical applications, one often needs to approximate f on some subinterval $[b, a] \subset \operatorname{dom} f$. Divide this interval into n subintervals, as follows: if $b = a$, then define $\hat{x}_0 \coloneqq \hat{x}_1 \coloneqq \cdots \coloneqq \hat{x}_n \coloneqq a_k$, and if $b < a$, choose $(\hat{x}_l)_{l=0}^{n}$ such that $b \eqqcolon \hat{x}_0 < \cdots < \hat{x}_n \coloneqq a$. Finally, define

$$\hat{f}(x) \coloneqq \begin{cases} \hat{g}(\hat{x}_l) & \text{if } x = \hat{x}_l \text{ for some } l = 0, \dots, n, \\ \frac{\hat{x}_l - x}{\hat{x}_l - \hat{x}_{l-1}} \hat{g}(\hat{x}_{l-1}) + \frac{x - \hat{x}_{l-1}}{\hat{x}_l - \hat{x}_{l-1}} \hat{g}(\hat{x}_l) & \text{if } x \in (\hat{x}_{l-1}, \hat{x}_l) \text{ for any } l = 1, \dots, n, \\ \infty & \text{if } x \in \mathbb{R} \setminus [b, a]. \end{cases} \tag{A.12}$$

Then $\hat{f}$ is piecewise linear on its effective domain, and moreover $\hat{f} \geq \hat{g} \geq f$.

Define the mesh size of the approximation as

$$\Delta \coloneqq \max \left\{ \max_{k=1,\dots,m, l=1,\dots,n_k} (\hat{x}_{kl} - \hat{x}_{k[l-1]}), \max_{l=1,\dots,n} (\hat{x}_l - \hat{x}_{l-1}) \right\}.$$

We now have the following result.

Proposition A.5. *Let f be defined by (A.2), $\hat{f}_k$ for $k = 1, \dots, m$ be defined by (A.10), and $\hat{f}$ be defined by (A.12). If $[b, a] \subseteq \operatorname{ri} \operatorname{dom} f$ and there exists some $c_k \geq 0$ for each $k = 1, \dots, m$ such that*

$$\left| \hat{f}_k(x) - f_k(x) \right| \leq c_k \Delta \text{ for all } x \in \operatorname{dom} f_k, k = 1, \dots, m,$$

then there exists $c \geq 0$ such that

$$\left|\hat{f}(x) - f(x)\right| \leq c\Delta \text{ for all } x \in [a,b].$$

Proof. It is straightforward to show that for any $l = 0, \dots, n$ we have

$$\begin{aligned} 0 \leq \hat{f}(\hat{x}_l) - f(\hat{x}_l) \leq \sup \Bigg\{ & \sum_{k=1}^{m} q_k \big(\hat{f}_k(x_k) - f_k(x_k)\big) : q_k \in [0,1], \\ & x_k \in [b_k, a_k] \text{ for all } k = 1, \dots, m, \sum_{k=1}^{m} q_k = 1, \sum_{k=1}^{m} q_k x_k = \hat{x}_l \Bigg\} \\ \leq \Delta \sup & \left\{ \sum_{k=1}^{m} q_k c_k : q_k \in [0,1] \text{ for all } k = 1, \dots, m, \sum_{k=1}^{m} q_k = 1 \right\} \\ = \Delta & \max\{c_k : k = 1, \dots, m\}. \end{aligned} \tag{A.13}$$

The function f is Lipschitz on $[b,a]$ [31, Theorem 10.4], and so there exists some $d \geq 0$ such that

$$|f(x) - f(y)| \leq d|x - y| \text{ for all } x, y \in [a,b]. \tag{A.14}$$

For any $x \in [b,a]$ such that $\hat{x}_{l-1} < x < \hat{x}_l$ for some $l = 1, \dots, n$, choose $l^* \in \{l-1, l\}$ such that

$$\hat{f}(\hat{x}_{l^*}) = \max\left\{\hat{f}(\hat{x}_{l-1}), \hat{f}(\hat{x}_l)\right\}.$$

Then

$$|\hat{f}(x) - f(x)| \leq |\hat{f}(\hat{x}_{l^*}) - f(x)| \leq |\hat{f}(\hat{x}_{l^*}) - f(\hat{x}_{l^*})| + |f(\hat{x}_{l^*}) - f(x)|$$

by (A.12) and the triangle inequality. Combining this with (A.13) and (A.14) then gives the desired result upon taking $c := d + \max\{c_k : k = 1, \dots, m\}$. ■

The upper approximation $\hat{f}$ depends on $\hat{g}$ only via the values $\hat{g}(\hat{x}_0), \dots, \hat{g}(\hat{x}_n)$. It is possible to calculate these values explicitly in the case where $g_k(q) = q \ln \frac{q}{p_k}$ for $k = 1, \dots, m$ by using standard techniques from calculus. The straightforward (though tedious) details are given in full in Section 4.3 of [39].

The theoretical error bound in Proposition A.5 ensures that the upper approximation $\hat{f}$ will converge uniformly to f on $[b,a]$ if the mesh size converges to zero. However, it relies on the Lipschitz coefficient of f, which is typically unknown in situations that require approximation (and could well be large). We now present a *lower approximation*, which, while slightly less computationally efficient than the upper approximation, can be used in practical applications to estimate the error of the upper approximation.

For each $k = 1, \dots, m$, let $\check{f}_k$ be any convex piecewise linear function with $\operatorname{dom} \check{f}_k = [b_k, a_k]$ and such that $\check{f}_k \leq f_k$. Then let $\check{g}$ be the generalised convex hull of $\check{f}_1, \dots, \check{f}_m$ and $g_1, \dots g_m$, in other words,

$$\check{g}(x) := \inf\left\{\sum_{k=1}^m (q_k\check{f}_k(x_k) + g_k(q_k)) : q_k \in [0,1], x_k \in [b_k, a_k] \text{ for all } k = 1,\dots,m, \sum_{k=1}^m q_k = 1, \sum_{k=1}^m q_k x_k = x\right\}. \tag{A.15}$$

Then $\check{g}$ is clearly convex and continuous on $\operatorname{dom}\check{g} = \operatorname{dom} f$, and the infimum in (A.15) is attained for all $x \in \operatorname{dom}\check{g}$. Furthermore, $\check{g} \leq f \leq \hat{g}$.

If $b = a$, then define

$$\check{f}(x) := \begin{cases} \check{g}(x) & \text{if } x = a, \\ \infty & \text{otherwise;} \end{cases}$$

then clearly $\check{f}(a) \leq f(a) \leq \hat{f}(a)$. Assume for the remainder that $b < a$; this implies that $[b,a] \subset \operatorname{int}\operatorname{dom} f$. Similar to the upper approximation, divide $[b,a]$ into $n-1$ subintervals by choosing $(\breve{x}_l)_{l=1}^n$ such that $b =: \breve{x}_1 < \cdots < \breve{x}_n := a$. Also choose any $\breve{x}_0 \in (\min\operatorname{dom} f, b)$ and $\breve{x}_{n+1} \in (\max\operatorname{dom} f, a)$, and consider the function $\breve{f}$ defined by

$$\breve{f}(x) := \begin{cases} \check{g}(\breve{x}_l) & \text{if } x = \breve{x}_l \text{ for some } l = 0,\dots,n+1, \\ \frac{\breve{x}_l - x}{\breve{x}_l - \breve{x}_{l-1}}\check{g}(\breve{x}_{l-1}) + \frac{x - \breve{x}_{l-1}}{\breve{x}_l - \breve{x}_{l-1}}\check{g}(\breve{x}_l) & \text{if } x \in (\breve{x}_{l-1}, \breve{x}_l) \text{ for any } l = 1,\dots,n+1, \\ \infty & \text{if } x \in \mathbb{R} \setminus [\breve{x}_0, \breve{x}_{n+1}]. \end{cases} \tag{A.16}$$

It is convex, piecewise linear and $\check{g}(x) \leq \breve{f}(x)$ for all $x \in [\breve{x}_0, \breve{x}_{n+1}]$. The graph of $\breve{f}$ consists of $n+1$ line pieces; the l^{th} line piece (where $l = 0,\dots,n$) connects the points $(\breve{x}_l, \check{g}(\breve{x}_l))$ and $(\breve{x}_{l+1}, \check{g}(\breve{x}_{l+1}))$, and has slope $m_l := \frac{\check{g}(\breve{x}_{l+1}) - \check{g}(\breve{x}_l)}{\breve{x}_{l+1} - \breve{x}_l}$. These line pieces are now used to determine the lower approximation $\check{f}$ on $[a,b]$. For $l = 1,\dots,n-1$, determine the point $(\check{x}_l, \check{y}_l)$ by extending the $(l-1)^{\text{th}}$ and $(l+1)^{\text{th}}$ line pieces and finding their intersection, in other words,

$$\check{x}_l := \begin{cases} \frac{m_{l+1}\breve{x}_{l+1} - m_{l-1}\breve{x}_l + \check{g}(\breve{x}_l) - \check{g}(\breve{x}_{l+1})}{m_{l+1} - m_{l-1}} & \text{if } m_{l-1} < m_{l+1}, \\ \frac{1}{2}(\breve{x}_l + \breve{x}_{l+1}) & \text{if } m_{l-1} = m_{l+1}, \end{cases}$$
$$\check{y}_l := m_{l-1}(\check{x}_l - \breve{x}_l) + \check{g}(\breve{x}_l).$$

Finally define

$$\check{x}_0 := \breve{x}_1 = b, \qquad \check{y}_0 := \check{g}(b), \qquad \check{x}_n := \breve{x}_n = a, \qquad \check{y}_n := \check{g}(a);$$

after which the lower approximation is defined as

$$\check{f}(x) := \begin{cases} \check{y}_l & \text{if } x = \check{x}_l \text{ for some } l = 0,\dots,n, \\ \frac{\check{x}_l - x}{\check{x}_l - \check{x}_{l-1}}\check{y}_{l-1} + \frac{x - \check{x}_{l-1}}{\check{x}_l - \check{x}_{l-1}}\check{y}_l & \text{if } x \in (\check{x}_{l-1}, \check{x}_l) \text{ for any } l = 1,\dots,n, \\ \infty & \text{if } x \in \mathbb{R} \setminus [b,a]. \end{cases} \tag{A.17}$$

The lower approximation $\check{f}$ is piecewise linear. It is straightforward to show that it is convex, due to the convexity of $\check{f}$. The fact that $\check{f} \leq \check{g}$ (whence $\check{f} \leq f$) follows from a simple geometric observation: on every interval $[\check{x}_l, \check{x}_{l+1}]$, the graph of $\check{f}$ falls below the extensions of both the $(l-1)^{\text{th}}$ and $(l+1)^{\text{th}}$ line pieces of $\check{f}$, and these extended line pieces in turn fall below the graph of $\check{g}$, due to the convexity of $\check{g}$. See [39, Section 5.4] for full details.

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