**Proceedings Paper:**

Mendoza, F.G., Bauso, D. orcid.org/0000-0001-9713-677X and Konstantopoulos, G. orcid.org/0000-0003-3339-6921 (2019) Online pricing via stackelberg and incentive games in a micro-grid. In: 2019 18th European Control Conference (ECC). 2019 18th European Control Conference (ECC), 25-28 Jun 2019, Naples, Italy. IEEE , pp. 3520-3525. ISBN 9783907144008

10.23919/ECC.2019.8795772

# Online Pricing via Stackelberg and Incentive Games in a Micro-Grid

Fernando Genis Mendoza*, Dario Bauso and George Konstantopoulos

*Abstract*— **This paper deals with the analysis and design of online pricing mechanisms in micro-grids. Two cases are studied in which the market layer is modeled as an open-loop and closed-loop dynamical system respectively. In the case of open-loop market dynamics, the price is generated as equilibrium price of a Stackelberg game with an incentive strategy. In such Stackelberg game, the leader is the energy supplier, the follower is the consumer, and the leader plays an incentive strategy. In the case of closed-loop market dynamics, the price is obtained as a function of the power supplied and the demand. A stability analysis is provided for both cases, which sheds light on the transient and steady-state behavior of the system in terms of the grids time constant, inertial, damping and synchronizing coefficients. Conditions on the parameters that guarantee asymptotic stability are obtained for both open-loop and closed-loop configurations. The findings provide an insight on the impact of the time constant and damping coefficient on the demand and power. The study also elucidates the ways in which the suppliers decisions influence the output values, thus contributing to clarify the interconnection between the market and physical layers in a micro-grid.**

## I. INTRODUCTION

Pricing mechanisms on electrical power systems constitute a viable way to shift the demand peaks and thus to improve efficiency. The underlying assumption is that the consumer and the supplier are rational and try to maximize their profits. Under such an assumption, a change of the price on the part of the independent system operator modifies the consumer's behavior and the analysis of the resulting dynamics is a core element in the literature on online pricing. Online pricing requires the implementation of such incentive mechanisms in real-time to increase the profits of the supplier by charging more when the production costs are higher instead of applying a flat rate. Similarly, incentives can be used to let the consumer know when is more convenient to carry out the more power-consuming tasks. Effective methods to determine the electricity price dynamically present several challenging open problems including: Global optimality for both consumers and supplier, the uncertainties in the consumers' behaviors and preferences, and more importantly, the safe operation of the electrical systems when subjected to such mechanisms.

*Sponsored by Mexico's CONACyT, scholarship number: 440742.

F. Genis Mendoza, and G. Konstantopoulos are with the Department of Automatic Control and Systems Engineering, The University of Sheffield, Mappin Street, S1 3JD Sheffield, United Kingdom {fgenismendoza1, g.konstantopoulos}@sheffield.ac.uk

D. Bauso is with the Jan C. Willems Center for Systems and Control, ENTEG, Fac. Science and Engineering, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands and Dip. dell'Innovazione Industriale e Digitale (DIID), Università di Palermo, 90128 Palermo, Italy. d.bauso@rug.nl

### A. Problem Statement

This paper focuses on bringing together the market and physical dynamics that are involved in determining the functioning of the micro-grid as a whole. Two configurations are studied in which the market layer is modeled as an open-loop and closed-loop dynamical system respectively. In the first configuration, a Stackelberg game is introduced between the supplier and consumer where the supplier plays an incentive strategy to generate an equilibrium price. In the second, the price is obtained as feedback function of the power supplied, the demand and an incentive strategy. However, a detailed stability analysis should be conducted on both configurations to ensure their correct operation.

### B. Main Contributions

As a first result, conditions for stability are obtained and the transient response of a micro-grid system subject to a price which is generated exogenously from a Stackelberg game is studied. The Stackelberg game introduces an incentive problem, which in turn determines the steady-state gain of the open-loop market dynamics. As a second result, a general feedback rule to obtain the price as a function of the power flow and demand is derived. Such a rule is based on an ex-ante price formulation. Stability analysis is performed and the impact of the parameters on the transient dynamics of the micro-grid system is studied. In addition to this, simulations were carried out using both open-loop and closed-loop pricing mechanisms based on data from [1].

### C. Reviewed Literature

The present work is in the same spirit as [2], where a stability analysis of micro-grids together with the study of the effects of damping and inertia for homogeneous micro-grids was conducted. The present paper differs from [2] as we add the market layer to the physical layer of the micro-grid. The approximation we used for the demand response as a first-order system is introduced in [3]; examples of this for households and businesses can be found in [4], [5] and [6]. Transient analysis on coupled oscillators and the relation between damping and inertial coefficients is investigated in [7] and [1]. In this paper we use the swing dynamics to model the transient stability in analogy with the model developed in [7], [8] and [1]. The formulation used in the current study for the ex-ante price, including the supplier and consumer models was first proposed in [9]. We refer to the cost for electricity generation mentioned in [10]. Although the use of incentives on micro-grids have been previously studied in [8], here they are implemented as a reward to the consumer when participating in an online pricing scheme.

The concepts for Stackelberg game and the incentive strategy formulation used in the present work are introduced in [11] and [12], respectively. In [13] the Stackelberg approach is used in conjunction with evolutionary algorithms for online pricing schemes. The existence of equilibrium points using other kinds of games including the Stackelberg game is demonstrated in [14] and [15]. The main difference of our current study to the papers above is our novel inclusion of the incentive strategy when obtaining the price.

This paper is organized as follows. In Section II, we introduce the micro-grid and demand response models, and formulate the Stackelberg game. In Section III, we present the main results. In Section IV, we provide simulations. Finally in Section V, we provide conclusions and discuss future directions.

## II. MICRO-GRID AND ONLINE PRICING MODELS

In this section we introduce the dynamic models for the micro-grid and for the online pricing mechanism in a unified framework.

### A. Micro-Grid Model

A micro-grid connected to the main grid can be modeled combining an integrator dynamics and a swing dynamics. The first equation is associated with the rate of change of the power flow into the grid as function of the deviation between the nominal mains frequency and the frequency of the grid [7]. This is given by

$$\dot{P_{flow}} = T(f_{nom} - f), \tag{1}$$

where $f$ is the operating frequency of the micro-grid, $f_{nom}$ is the nominal frequency, which is considered to be the frequency of the main grid and $T$ is the synchronizing coefficient which is obtained as the power transferred over the transmission line between the micro-grid and the mains [16]. The second dynamics describes the rate of change of frequency as function of the current frequency $f$, the power flow $P_{flow}$, the generated power from the mains $P_{nom}$, the nominal consumed power by the loads $P_{rated}$ and the shiftable demand response $\Delta P_d$ [8]. This second dynamics is given by

$$\dot{f} = -\frac{D}{M} f + \frac{1}{M}(P_{flow} + P_{nom} - P_{rated} - \Delta P_d), \tag{2}$$

where $D$ denotes the damping coefficient of the micro-grid and $M$ is its inertial coefficient, the dynamics for $\Delta P_d$ are explained in the following subsection. The block representation of the market layer and the physical layer of a single micro-grid is shown in Fig. 1. There, the input exogenous to the grid represents the price $\Lambda$ obtained from the online open-loop mechanism.

### B. Demand Response

For a given price, the demand response dynamics can be represented as a first-order system [3]

$$\dot{\Delta P_d} = -\frac{1}{\tau}\Delta P_d + \frac{k}{\tau}\Lambda, \tag{3}$$

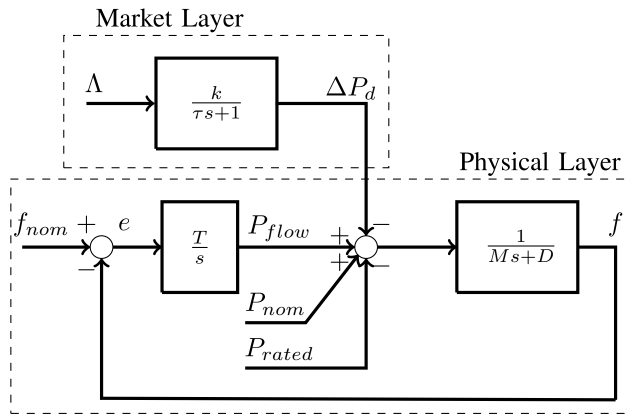


Fig. 1. Block system of a micro-grid with demand response $\Delta P_d$, power $P_{flow}$ and frequency $f$ subject to exogenous price input $\Lambda$.

where $\Delta P_d$ is the demand, $\tau$ the time constant of the market dynamics and $\Lambda$ is the price multiplied by a DC gain $k$. The demand is subtracted from the power available in the grid as shown in Fig. 1 and represents the quantity of electrical energy that is used by the consumer given the price announced by the supplier. The price $\Lambda$ is generated from a Stackelberg game as described in the following section.

### C. Consumer and Supplier Functions

Both the supplier and the consumers are considered to be price-taking, profit-maximizing agents. In particular the supplier wants to maximize the price and the consumers want to consume as much as possible with the minimum prize. The power supplied $P_s$ and power consumed $P_c$ are selected as the quantity that maximizes their respective profit functions [9]:

$$P_s = \arg\max_{x} \max_{\Lambda\in[\underline{\Lambda},\bar{\Lambda}]} \Lambda x - c(x), \tag{4}$$

$$P_c = \arg\max_{x} \min_{\Lambda\in[\underline{\Lambda},\bar{\Lambda}]} v(x) - \Lambda x, \tag{5}$$

where the value function of all the consumers in the grid are denoted by $v(x)$, which represents the value that the consumer obtains by consuming $x$ units of electricity. Analogously, the supplier has a production cost function $c(x)$. We assume that the value and cost functions are concave and convex, respectively [8], [9]. In the maximization problems defined above, we denote by $\bar{\Lambda}$ and $\underline{\Lambda}$ the upper and lower bounds for the price. In other words, we assume that the price lies in the interval $\Lambda \in [\underline{\Lambda}, \bar{\Lambda}]$. The corresponding supply and consumption values obtained from (4) and (5) under the minimum and maximum prices are denoted by $\underline{x}_s$ and $\bar{x}_s$ and $\underline{x}_c$ and $\bar{x}_c$, respectively. This implies that the supply and consumption values lie in the intervals $x_s \in [\underline{x}_s, \bar{x}_s]$ and $x_c \in [\underline{x}_c, \bar{x}_c]$. The optimal supply and consumption values $\bar{x}_s^*$ and $\bar{x}_c^*$ can be obtained by taking the derivative of the objective functions in (4)-(5) and equaling to zero. This corresponds to identifying as optimal those points in which the derivative (slope of the curve) is parallel to the price line, as illustrated in Fig. 2. From the figure we note that $\bar{x}_c$ corresponds to the maximum consumption given the

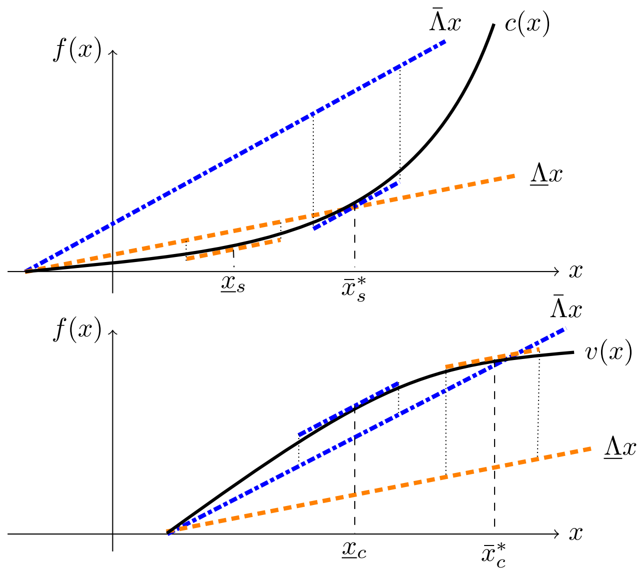


Fig. 2. Supplier and consumer functions and quantities.

lowest price $\underline{\Lambda}$, similar conclusions can be drawn for all other bounds.

In order to steer the solutions to an equilibrium, a Stackelberg game is proposed. The advantage of formulating such a new game is that the incentive strategy arising from such a game no longer depends directly on the cost and value functions, but solely on the Stackelberg equilibrium. Let us define reference points to be employed in the game. Normalizing the optimal solutions to each problem to unitary values, such solutions can be taken equal to

$$(\underline{\Lambda}, \bar{x}_c) = (0, 1), \tag{6}$$

$$(\bar{\Lambda}, \bar{x}_s) = (1, 1), \tag{7}$$

for the consumer and for the supplier respectively. The incentive strategy is illustrated in Section III.

## III. MAIN RESULTS

In this section, the main result of the paper is presented. First for the open-loop configuration we formulate the Stackelberg game, the incentive problem and determine its optimal solution. Secondly we perform the stability analysis and obtain the steady-state gains of the micro-grid model subject to the game-generated price. Thirdly, we show a way to express the price as a function of power and demand and perform both stability and final value analysis on the closed-loop micro-grid model.

### A. Normalized Stackelberg Game Formulation

Assuming that the demand response $\Delta P_d$ of the consumers depends on the price $\Lambda$ set by the supplier, the following Stackelberg game with incentive strategy is proposed. The game provides an incentive strategy and an associated on-line pricing mechanism for the case of open-loop market dynamics. First, denoting the supplier as the "Leader" and the consumers as the "Follower", we introduce $\pi_L(q_L, q_F)$ and $\pi_F(q_L, q_F)$ as their respective profit functions. Both functions depend on the outputs $q_L$ of the leader and $q_F$ of the follower. The output of the supplier is the price $\Lambda$ that will minimize its cost and maximize its profits. The incentive problem is formulated in a way such that the leader selects a price as function of the follower's demand. The following profit functions capture the tension between the supplier and the consumer. Namely the consumer prefers a low price and to consume large quantities of energy, whereas the supplier aims to balancing supply and demand. Let (6) and (7) be the optimal solutions of the optimization problems (4) and (5) respectively. Then the profit function for the leader is given by

$$\pi_L = q_L q_F - \frac{1}{2} q_F^2. \tag{8}$$

Similarly, the profit function for the follower is given by

$$\pi_F = \log q_F + 1 - q_L q_F. \tag{9}$$

We refer to *incentive strategy* as the choice that the leader takes depending on the one of the follower. Namely a function $\Gamma(q_F)$. For the sake of tractability we propose the following assumption, however, whitout loss of generality, other class of strategies can be employed in a similar way.

*Assumption 1:* Strategy $\Gamma(q_F)$ is linear and given by

$$q_L = \gamma q_F. \tag{10}$$

*Theorem 1:* Let Assumption 1 hold true. The Stackelberg game yields the following equilibrium point:

$$q_F^* = \gamma^{-\frac{1}{2}}, \tag{11}$$

$$q_L^* = \gamma^{\frac{1}{2}}. \tag{12}$$

*Proof:* Let the leader maximize (8), and the follower maximize (9). Because of the concavity of (14), the maximum of the follower is obtained by taking the derivative of its profit function and equaling it to zero:

$$\frac{\partial \pi_F}{\partial q_F} = \frac{1}{q_F} - q_L = 0. \tag{13}$$

Under the assumption that the leader is playing according to (10), then (13) can then be rewritten as

$$\frac{1}{q_F} - \gamma q_F = 0. \tag{14}$$

The above yields the optimal solution $q_F^*$ as in (11). Once the follower has chosen its demand, the leader then obtains the price substituting (11) in (10), which leads to (12). ■

The optimal solutions $q_F^*$ and $q_L^*$ determine the equilibrium of the game. Now the leader has to design a proper strategy $\gamma$ to obtain the best equilibrium point. For the supplier the best equilibrium point is the one closest to the optimum $(1, 1)$ as in (7). Figure 3 illustrates the way in which a different choice for strategy $\gamma$ produces different quantities of price and demand at the equilibrium. From the figure it is evident that the output of the leader (the supplier) depends on the quantity selected by the follower (consumer). In particular, the higher the price, the lower the consumption.

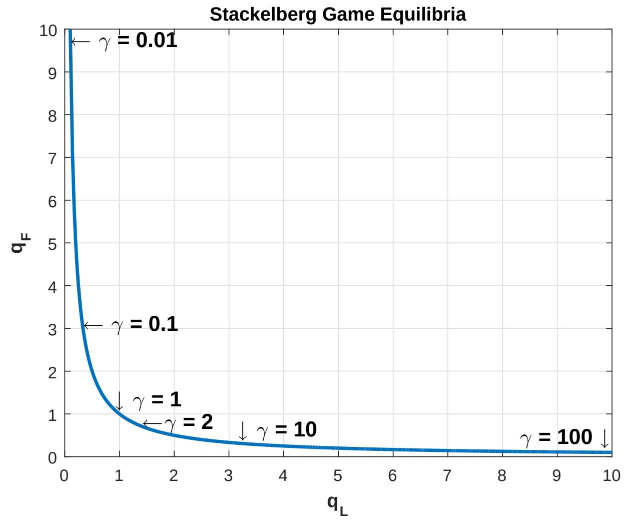


Fig. 3. Stackelberg equilibrium points as a function of $\gamma$.

*Remark:* From the demand response dynamics in (3), the following expression can be obtained at steady-state:

$$\Lambda^{ss} = \frac{1}{k}\Delta P_d^{ss}. \quad (15)$$

Above there is a linear relation between the price and the demand, it can be implied that the incentive $\gamma$ can be treated as a gain, making the choice of the Stackelberg game with a linear $\Gamma(q_F)$ appropriate for the studied case.

### B. Stability of Open Loop Configuration

Now that we have explained the ways in which the price $\Lambda$ is obtained, let us analyze the stability of the system subject to such input. From the system configuration illustrated in Fig. 1 and equations (1)-(3) the following state space representation is derived:

$$\begin{bmatrix} \dot{P}_{flow} \\ \dot{f} \\ \Delta\dot{P}_d \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -T & 0 \\ \frac{1}{M} & -\frac{D}{M} & -\frac{1}{M} \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}}^{A} \begin{bmatrix} P_{flow} \\ f \\ \Delta P_d \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & T & 0 \\ 0 & 0 & \frac{1}{M} \\ \frac{k}{\tau} & 0 & 0 \end{bmatrix}}^{B} \begin{bmatrix} \Lambda \\ f_{nom} \\ P_{nom} - P_{rated} \end{bmatrix}. \quad (16)$$

*Theorem 2:* System (16) is stable for all positive values of parameters $T$, $M$, $D$, and $\tau$.

*Proof:* Consider matrix $A$ in (16). The characteristic polynomial of the entire system can be obtained from the denominator of the system's transfer function, which is expressed by the determinant of $sI - A$:

$$|sI - A| = \left| \begin{bmatrix} s & T & 0 \\ -\frac{1}{M} & s+\frac{D}{M} & \frac{1}{M} \\ 0 & 0 & s+\frac{1}{\tau} \end{bmatrix} \right| = s^3 + (\frac{D}{M} + \frac{1}{\tau})s^2 + (\frac{T}{M} + \frac{D}{\tau M})s + \frac{T}{\tau M}. \quad (17)$$

The roots of the above polynomial, namely the eigenvalues of system (16) are given as follows:

$$s_1 = -\frac{1}{\tau}, \quad s_{2,3} = \frac{-D \pm \sqrt{D^2 - 4MT}}{2M}. \quad (18)$$

From Nyquist stability criterion, for the system to be asymptotically stable the real part of its eigenvalues must be negative, namely the eigenvalues must lie in the left-hand side of the complex plane. From (18) the system is stable if the following conditions on the parameters are met:

$$\tau > 0 \text{ and } MT > 0. \quad (19)$$

The above conditions are always true given that the parameters are strictly positive. ■

*Remark:* The transient of the system is characterized by oscillations, when the eigenvalues have complex part, namely for $D^2 < 4MT$. On the contrary, no oscillations arise when the eigenvalues are real and specifically for $D^2 > 4MT$.

*Theorem 3:* The steady-state-gain from a step input of magnitude $\Lambda_m$, $f_m$ or $P_m = P_{nom} - P_{rated}$ correspondingly, to system (16) is expressed by:

$$P_{flow}^{ss} = k\Lambda_m + Df_m - P_m, \quad (20)$$

$$f^{ss} = f_m, \quad (21)$$

$$\Delta P_d^{ss} = k\Lambda_m. \quad (22)$$

*Proof:* From system (16) a transfer function matrix can be obtained as $Y(s)/U(s) = C(sI - A)^{-1}B$. Since the feedback in the mentioned system is unitary, matrix $C$ is considered to be an identity matrix $I$ of appropriate dimensions. Substituting matrices $A$ and $B$ we obtain

$$\begin{bmatrix} P_{flow}(s) \\ f(s) \\ \Delta P_d(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{12}(s) & G_{22}(s) & G_{23}(s) \\ G_{31}(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda(s) \\ f_{nom}(s) \\ P_{nom}(s) - P_{rated}(s) \end{bmatrix}, \quad (23)$$

where

$$G_{11}(s) = \frac{P(s)}{\Lambda(s)} = \frac{kT}{M(\frac{D}{M}s + s^2 + \frac{T}{M})(\tau s + 1)},$$

$$G_{12}(s) = \frac{P(s)}{f_{nom}(s)} = \frac{T(\frac{D}{M} + s)}{\frac{D}{M}s + s^2 + \frac{T}{M}},$$

$$G_{13}(s) = \frac{P(s)}{P_{nom}(s) - P_{rated}(s)} = -\frac{T}{M(\frac{D}{M}s + s^2 + \frac{T}{M})},$$

$$G_{21}(s) = \frac{f(s)}{\Lambda(s)} = -\frac{ks}{M(\frac{D}{M}s + s^2 + \frac{T}{M})(\tau s + 1)},$$

$$G_{22}(s) = \frac{f(s)}{f_{nom}(s)} = \frac{\frac{T}{M}}{\frac{D}{M}s + s^2 + \frac{T}{M}},$$

$$G_{23}(s) = \frac{f(s)}{P_{nom}(s) - P_{rated}(s)} = \frac{s}{M(\frac{D}{M}s + s^2 + \frac{T}{M})},$$

$$G_{31}(s) = \frac{r(s)}{\Lambda(s)} = \frac{k}{\tau s + 1}.$$

From the final value theorem, we have that the steady-state gain for a system described by a transfer function $F(s)$ and subjected to an input $U(s)$ can be obtained as

$$\lim_{s\to 0} sG(s)U(s). \tag{24}$$

From (24) and (23) and assuming a step input of magnitude $\Lambda_m$, $f_m$ or $P_m$ we obtain:

$$\begin{aligned}
&\lim_{s\to 0} sG_{11}(s)\frac{\Lambda_m}{s} + sG_{12}(s)\frac{f_m}{s} + sG_{13}(s)\frac{P_m}{s} \\
&\quad = \frac{kT}{M\frac{T}{M}}\Lambda_m + \frac{T\frac{D}{M}}{\frac{T}{M}} f_m - \frac{T}{M\frac{T}{M}} P_m, \\
&\lim_{s\to 0} sG_{21}(s)\frac{\Lambda_m}{s} + sG_{22}(s)\frac{f_m}{s} + sG_{23}(s)\frac{P_m}{s} \\
&\quad = -\frac{0}{M\frac{T}{M}}\Lambda_m + \frac{\frac{T}{M}}{\frac{T}{M}} f_m + \frac{0}{M\frac{T}{M}} P_m, \\
&\lim_{s\to 0} sG_{31}(s)\frac{\Lambda_m}{s} = k\Lambda_m.
\end{aligned}$$

Hence, the steady-state values (20)-(22) are obtained. ■

In view of the considerations in subsection III-A, the price $\Lambda$ is the supplier's output from the Stackelberg game and is a function of a selected strategy $\Gamma(q_F)$ that characterizes the equilibrium point in terms of supply and demand as shown in (11). This implies that for every possible equilibrium point we obtain a different steady-state value.

### C. Price as a Function of Power and Demand

To determine a way to express the price $\Lambda$ as a linear function of power and demand while closing the loop of the system in Fig. 1, a few concepts must be introduced in the same spirit as in [9].

An ex-ante price $\Lambda(t)$ can be calculated from an estimated supply $\hat{s}$ which is in turn obtained from the total of a previous demand, namely

$$\hat{s}(t) = P_{rated}(t) + \Delta P_d(t) \tag{25}$$

which essentially represents the balancing of supply and demand. From it and by solving the supplier's cost function in (4) we obtain

$$\Lambda(t) = \frac{d}{dx}c(x)\Big|_{\hat{s}(t)}. \tag{26}$$

A physical interpretation of the above is that the supplier supplies a quantity equal to the demand from a previous period of time. The price is then the one that is optimal for the given supplied quantity. Graphically, the price is identified by the slope of the curve representing the cost evaluated in the point corresponding to the supplied quantity. As mentioned in [10], the supplier cost function is the following:

*Assumption 2:* The supplier has a cost function $c(x)$ of the form:

$$c(x) = \alpha\frac{x^2}{2}, \tag{27}$$

where $x$ is the quantity of supplied power and $\alpha$ is a scalar value. Such cost function has been experimentally validated for thermal generators in [17] and is generally accepted as a sound approximation as seen in [18], [19] and [20].

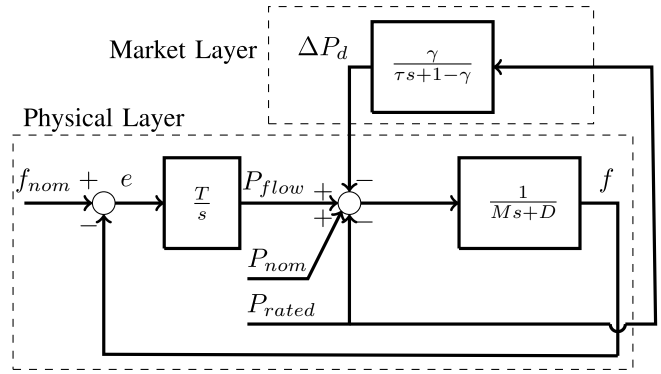


Fig. 4. Block system of the micro-grid with closed-loop online pricing.

Substituting the supply (25) and the cost (27) into (26) yields the following expression for the ex-ante price:

$$\Lambda(t) = \frac{d}{dx}\alpha\frac{x^2}{2}\Big|_{\hat{s}(t)} = \alpha x\Big|_{\hat{s}(t)}. \tag{28}$$

We are ready to establish the following result.

*Lemma 1:* Let Assumption 2 hold, the price is given by

$$\Lambda(t) = \alpha(P_{rated}(t) + \Delta P_d(t)). \tag{29}$$

Now that we have obtained the dependence of price $\Lambda$ on the sum of the nominal consumed power $P_{rated}$ and the demand shift $\Delta P_d$, the block system describing the market dynamics can be rearranged closing the loop as in Fig. 4. The system's dynamics in the case of closed-loop market dynamics can then be written as

$$\dot{P}_{flow} = T(f_{nom} - f), \tag{30}$$

$$\dot{f} = -\frac{D}{M}f + \frac{1}{M}(P_{flow} + P_{nom} - P_{rated} - \Delta P_d), \tag{31}$$

$$\dot{\Delta P}_d = -\frac{1}{\tau}\Delta P_d + \frac{k\alpha}{\tau}(P_{rated} + \Delta P_d). \tag{32}$$

We can freely substitute $k\alpha$ with the incentive $\gamma$ since both are linear relationships to the consumed power and both serve as means of shifting the total demand via $\Delta P_d$. The state space representation of the closed-loop system can be rewritten in matrix form as follows

$$\begin{bmatrix}\dot{P}_{flow}\\ \dot{f}\\ \dot{\Delta P}_d\end{bmatrix} = \overbrace{\begin{bmatrix}0 & -T & 0\\ \frac{1}{M} & -\frac{D}{M} & -\frac{1}{M}\\ 0 & 0 & \frac{1}{\tau}(\gamma-1)\end{bmatrix}}^{A}\begin{bmatrix}P_{flow}\\ f\\ \Delta P_d\end{bmatrix} + \overbrace{\begin{bmatrix}0 & T & 0\\ \frac{1}{M} & 0 & -\frac{1}{M}\\ 0 & 0 & \frac{\gamma}{\tau}\end{bmatrix}}^{B}\begin{bmatrix}P_{nom}\\ f_{nom}\\ P_{rated}\end{bmatrix}, \tag{33}$$

as in Section III-B, the characteristic polynomial can be obtained from $|sI - A|$, yielding

$$|sI - A| = s^3 + (\frac{D}{M} + \frac{1-\gamma}{\tau})s^2 + (\frac{T}{M} - \frac{D(\gamma-1)}{\tau M})s - \frac{T(\gamma-1)}{\tau M}. \tag{34}$$

We are now ready to enunciate the following result.

*Theorem 4:* System (33) is stable for all non-negative values of parameters $T$, $M$, $D$ and $\tau$. Additionally, the incentive strategy $\gamma$ must comply with the condition:

$$0 > \gamma > 1. \tag{35}$$

*Proof:* The role of the parameter conditions for stability are obtained similarly as demonstrated in Theorem 2, the roots of the characteristic polynomial (34) are the following:

$$\begin{aligned} s_1 &= \frac{\gamma - 1}{\tau}, \\ s_{2,3} &= \frac{-D \pm \sqrt{D^2 - 4MT}}{2M}. \end{aligned} \tag{36}$$

which yields the condition $MT > 0$ that is always true. To find the conditions for $\gamma$, the Routh-Hurwitz criteria can be applied to (34), the following conditions are obtained:

$$\begin{aligned} \frac{D}{M} + \frac{1-\gamma}{\tau} &> 0, \\ \frac{T}{M} - \frac{D(\gamma-1)}{\tau M} &> 0, \\ (\frac{D}{M} + \frac{1-\gamma}{\tau})(\frac{T}{M} - \frac{D(\gamma-1)}{\tau M}) &> -\frac{T(\gamma-1)}{\tau M}. \end{aligned} \tag{37}$$

Such conditions together with (36) can be reduced to obtain the range of values (35) for the incentive strategy. ■

The meaning behind (35) is that if the gain is too small this results in a price reduction that will increase the demand, and by trying to maximise their utility function, the consumers will demand power beyond the capabilities of the micro-grid. Therefore, we can conclude that the closed-loop system is stable for certain bounds of $\gamma$, as we will illustrate in section IV-B. Hence the supplier must be aware of the consumption historical patterns in the grid and select an appropriate value for the incentive.

*Remark:* Taking into account the previous statement and the expression of the eigenvalues in (36) we can also derive the two following considerations:

- The system is stable with complex eigenvalues when (38) is met, namely when

$$D^2 < 4MT. \tag{38}$$

- The system is stable with real, distinct and negative eigenvalues when the following inequality holds

$$D^2 > 4MT. \tag{39}$$

As can be seen from the mentioned inequalities, the oscillations in the system's response depend mainly on the value of the damping and inertial parameters $D$ and $M$, which still holds with he findings in [7] despite our new system configuration. Additionally, we can provide the following observations obtained empirically regarding the role of the system's parameters on the transient of the system. The value of the time constant $\tau$ affects directly the settling time of $P_{flow}$ and $\Delta P_d$. The synchronizing coefficient $T$ influences the speed of the oscillations of all states proportionally, and reduces the settling time as well; $T$ also alters the peak values of $P_{flow}$ and $\Delta P_d$. The inertial coefficient $M$ affects oscillation speed on all states and the peak responses of $P_{flow}$ and $\Delta P_d$ without affecting their steady-state values. The damping coefficient $D$ directly increases the settling time for larger values while also modifying the steady-state values of $P_{flow}$ and $f$. The gain $\gamma$ directly changes the magnitude of $P_{flow}$ and $\Delta P_d$, increasing the settling time for larger values. The last two observations can be corroborated by the following result.

*Theorem 5:* The steady-state gain from a corresponding step input of magnitude $P_{nm}$, $f_m$ or $P_{rm}$ to system (33) is expressed by:

$$P_{flow}^{ss} = \frac{1}{1-\gamma} P_{rm} - P_{nm} + D f_m, \tag{40}$$

$$f^{ss} = f_m, \tag{41}$$

$$\Delta P_d^{ss} = \frac{\gamma}{1-\gamma} P_{rm}. \tag{42}$$

*Proof:* From system (33) a transfer function matrix can be obtained. Since the feedback in the mentioned system is unitary, matrix $C$ is considered to be an identity matrix $I$ of appropriate dimensions. Substituting matrices $A$ and $B$ into $Y(s)/U(s) = C(sI - A)^{-1}B$ we obtain

$$\begin{bmatrix} P_{flow}(s) \\ f(s) \\ \Delta P_d(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & G_{22}(s) & G_{23}(s) \\ 0 & 0 & G_{33}(s) \end{bmatrix} \begin{bmatrix} P_{nom}(s) \\ f_{nom}(s) \\ P_{rated}(s) \end{bmatrix} \tag{43}$$

where

$$\begin{aligned} G_{11}(s) &= \frac{P_{flow}(s)}{P_{nom}(s)} = -\frac{T}{s(D+Ms)+T}, \\ G_{12}(s) &= \frac{P_{flow}(s)}{f_{nom}(s)} = \frac{T(D+Ms)}{s(D+Ms)+T}, \\ G_{13}(s) &= \frac{P_{flow}(s)}{P_{rated}(s)} = \frac{sT\tau + T}{(-\gamma + s\tau + 1)(s(D+Ms)+T)}, \\ G_{21}(s) &= \frac{f(s)}{P_{nom}(s)} = \frac{s}{s(D+Ms)+T}, \\ G_{22}(s) &= \frac{f(s)}{f_{nom}(s)} = \frac{T}{s(D+Ms)+T}, \\ G_{23}(s) &= -\frac{f(s)}{P_{rated}(s)} = \frac{\tau s^2 + s}{(-\gamma + s\tau + 1)(s(D+Ms)+T)}, \\ G_{33}(s) &= \frac{\Delta P_d(s)}{P_{rated}(s)} = \frac{\gamma}{\tau s + 1 - \gamma}. \end{aligned}$$

Applying the final value theorem from (24) to (43) and assuming a corresponding step input of magnitude $P_{nm}$, $f_m$ or $P_{rm}$ we obtain:

$$\begin{aligned} &\lim_{s\to 0} sG_{11}(s)\frac{P_{nm}}{s} + sG_{12}(s)\frac{f_m}{s} + sG_{13}(s)\frac{P_{rm}}{s} \\ &\quad = -\frac{T}{T}P_{nm} + \frac{TD}{T}f_m + \frac{T}{(1-\gamma)T}P_{rm}, \\ &\lim_{s\to 0} sG_{21}(s)\frac{P_{nm}}{s} + sG_{22}(s)\frac{f_m}{s} + sG_{23}(s)\frac{P_{rm}}{s} \\ &\quad = -\frac{0}{T}P_{nm} + \frac{T}{T}f_m + \frac{0}{(1-\gamma)T}P_{rm}, \\ &\lim_{s\to 0} sG_{31}(s)\frac{P_{rm}}{s} = \frac{\gamma}{1-\gamma}P_{rm}. \end{aligned}$$

Hence, the steady-state values (40)-(42) are obtained. ■

## IV. SIMULATIONS

Micro-grid parameters were selected based on typical values of a grid with capacity of 60 MVA that is providing 30 MVA of power to the main grid: $T = 30$ MVA, $M = 0.2$ MJ-s/rad and $D = 1$ MJ/rad in accordance to [1]; the simulation time is 60 seconds, initial state values are selected randomly and the grid is subject to step inputs of $f_{nom} = 50$ Hz, $P_{nom} = 50$, $P_{rated} = 20$ MVA. The time constant for the demand response has been selected as $\tau = 3$ s, two justifications are behind this, the first is to show the results more clearly, the second is that in the future, customers might be able to access real time prices in a more immediate way, facilitating the implementation of automated decision making given a price.

### A. Grid with Exogenous Price Input

In addition to the parameters previously mentioned, the gain is selected as $k = 25$ and the price $\Lambda$ is a value in the range of $[0, 1]$, in the simulation, only three different values of $\Lambda$ are selected for illustrative, tractability purposes, and to show the system's response to abrupt changes. Figure 5 shows the open-loop configuration response. Note that the demand $\Delta P_d$ reacts in accordance with the consumer behavior discussed in section II-B. Oscillations arise during the transient of the system, also the sum of powers in the grid $P_{nom} - P_{rated} - \Delta P_d$ does not turn negative, meaning that the increase/decrease of demand does not surpass the power available in the grid. Finally, none of the states exceeds the 60 MVA capacity of the micro-grid, it can be seen that the power flow reacts according to the demand shift. Figure 6 shows the response under the same parameter values with the exception of $D = 2$ MJ/rad, which is sufficiently large to damp the oscillations. Note that when the power flow increases, a larger damping can be chosen but this will in might produce power flow values out of the 60 MVA capacity of the grid.

Selecting large values of $k$ can also result in power flow values larger than the capacity of the grid, Fig. 7 shows the response of the grid under the same parameter values as for the first example except for $k = 250$. Note that the frequency state deviates largely from the desired 50 Hz. These results show that the response can be asymptotically stable but the parameters must be selected in a way that the demand does not exceed the power available.

### B. Grid with Closed Loop Market Dynamics

The following set of simulations adopts the same parameters and frequency inputs as in the previous example. In Fig. 8 with $\gamma = 0.5$ and $D = 1$ MJ/rad, it can be seen that the power flow can take negative values due to oscillations during the transient. However the frequency $f$ has a the same steady-state value. Also the demand shifted because of the incentive $\gamma$ is not larger than the power available. In Fig. 9 the damping is increased to $D = 2$ MJ/rad to eliminate oscillations and as in the open-loop configuration, there is an increase in the power flow state that can ultimately lead to values exceeding the capacity of the grid. As implied in

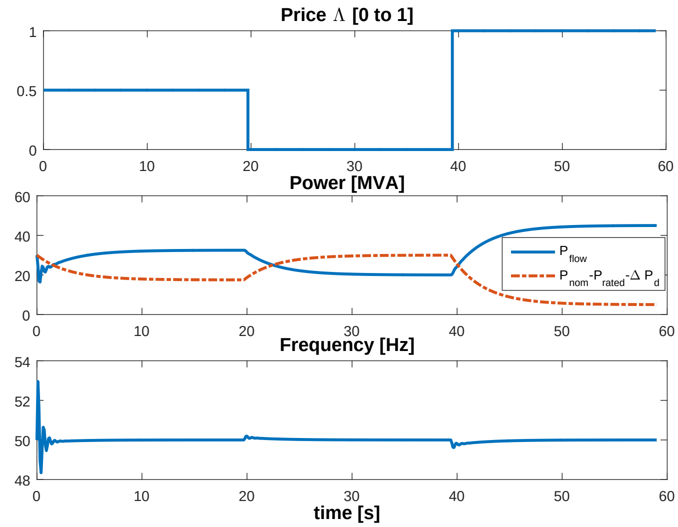


Fig. 5. Time plot of (top) price, (middle) power flow, and (bottom) frequency; oscillations may arise in the frequency and power flow plot.

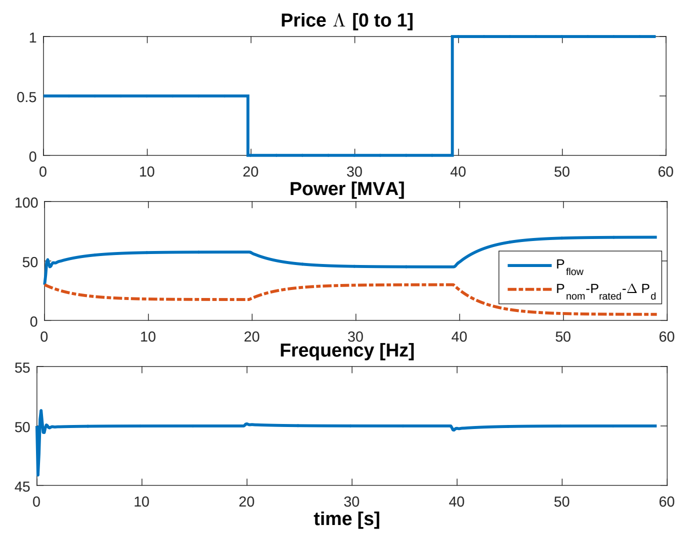


Fig. 6. Time plot of (top) price, (middle) power flow, and (bottom) frequency; by increasing the damping coefficient oscillations are reduced.

the stability analysis, there exists a sufficiently large value of $\gamma$ which compromises the stability of the system. This is illustrated in Fig. 10 where $\gamma = 0.8$ and the power available becomes negative due to increased demand, which is physically impossible.

## V. CONCLUSION

We have explored two ways of implementing online pricing mechanics in a micro-grid, with a novel approach of subjecting the system model to a Stackelberg game with incentive strategies. Furthermore conditions for stability were found, as well as the role of the grid's time constant, DC gain, inertial, damping and synchronizing coefficients in the transient and steady-state behavior of both configurations has been studied. More importantly, we brought closer the market and physical layers in the system by learning the way in

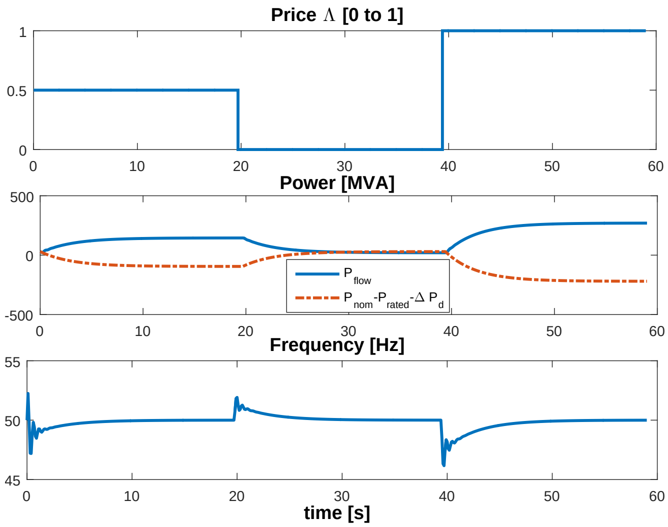


Fig. 7. Time plot of (top) price, (middle) power flow, and (bottom) frequency; by selecting large gain values, the demand exceeds the power available in the grid.

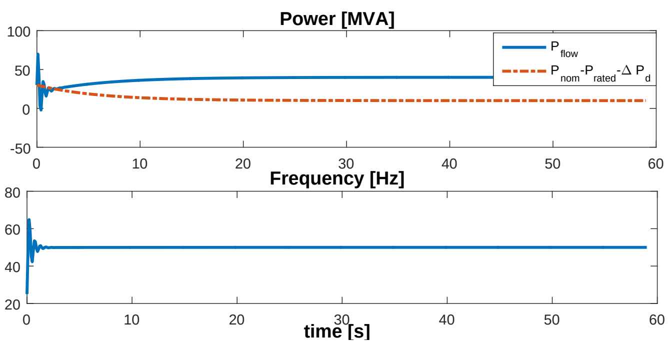


Fig. 8. Time plot of (top) power flow, and (bottom) frequency; oscillations may arise in the frequency and power flow plot.

which the parameters that are chosen by the supplier, being a price, a gain or incentive, affect the system response. Further direction of this work involves the introduction of mean-field models for the demand and the impact on the overall dynamics of the interconnection of multiple grids.

## REFERENCES

[1] D. P. Kothari and I. J. Nagrath. *Modern Power System Analysis*. Tata McGraw-Hill Pub. Co, 2003.

[2] D. Bauso. Nonlinear network dynamics for interconnected micro-grids. *Systems & Control Letters*, 118:8–15, 2018.

[3] D. Materassi, M. Roozbehani, and M. A. Dahleh. Equilibrium price distributions in energy markets with shiftable demand. *Proceedings of the IEEE Conference on Decision and Control*, 2012.

[4] M.M. Gouda, S. Danaher, and C.P. Underwood. Building thermal model reduction using nonlinear constrained optimization. *Building and Environment*, 37(12):1255–1265, 2002.

[5] B. Asare-Bediako, W.L. Kling, and P.F. Ribeiro. Future residential load profiles: Scenario-based analysis of high penetration of heavy loads and distributed generation. *Energy and Buildings*, 75, 2014.

[6] F. Adamek. *Demand Response and Energy Storage for a Cost Optimal Residential Energy Supply with Renewable Generation*. PhD thesis, Eidgenossische Technische Hochschule ETH Zurich, 2011.

[7] F. Dörfler and F. Bullo. Synchronization and Transient Stability in Power Networks and Non-Uniform Kuramoto Oscillators. *SIAM Journal on Control and Optimization*, 50(3):1616–1642, 2012.

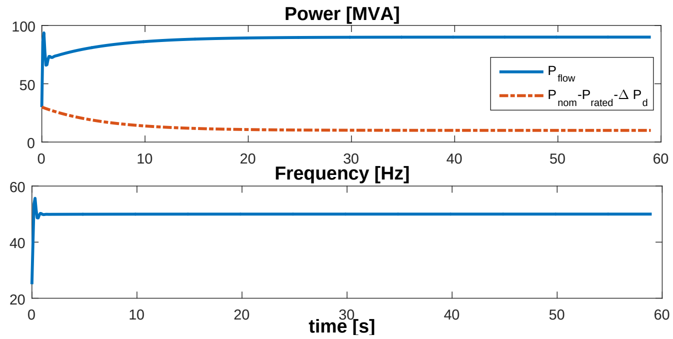


Fig. 9. Time plot of (top) power flow, and (bottom) frequency; by increasing the damping coefficient oscillations are reduced.

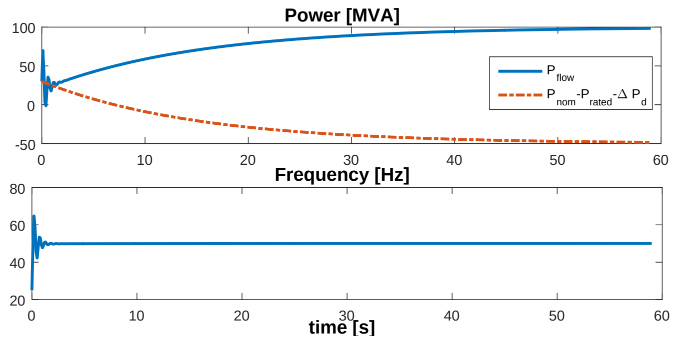


Fig. 10. Time plot of (top) power flow, and (bottom) frequency; by selecting large gain values, the demand exceeds the power available in the grid.

[8] T. Namerikawa, N. Okubo, R. Sato, Y. Okawa, and M. Ono. Real-Time Pricing Mechanism for Electricity Market with Built-In Incentive for Participation. *IEEE Transactions on Smart Grid*, 6(6):2714–2724, 2015.

[9] M. Roozbehani, M. A. Dahleh, and S. K. Mitter. Volatility of Power Grids under Real-Time Pricing. *arXiv preprint:1106.1401*, X:1–15, 2011.

[10] C. Eksin, H. Delic, and A. Ribeiro. Demand Response Management in Smart Grids With Heterogeneous Consumer Preferences. *IEEE Transactions on Smart Grid*, 6(6):3082–3094, 2015.

[11] M. J. Osborne. A course in game theory. *Computers & Mathematics with Applications*, 29(3):115, 1995.

[12] H. Ehtamo and R. P. Hämäläinen. Incentive strategies and equilibria for dynamic games with delayed information. *Journal of Optimization Theory and Applications*, 63(3):355–369, 1989.

[13] A. Belgana, B. P. Rimal, and M. Maier. Open Energy Market Strategies in Microgrids: A Stackelberg Game Approach Based on a Hybrid Multiobjective Evolutionary Algorithm. *IEEE Transactions on Smart Grid*, 6(3):1243–1252, 2015.

[14] J. Lee, J. Guo, J. K. Choi, and M. Zukerman. Distributed energy trading in microgrids: A game-theoretic model and its equilibrium analysis. *IEEE Transactions on Industrial Electronics*, 62, 2015.

[15] K. Ma, S. Hu, J. Yang, C. Dou, and J. M. Guerrero. Energy trading and pricing in microgrids with uncertain energy supply: A three-stage hierarchical game approach. *Energies*, 10(5), 2017.

[16] F. Bullo. Lectures on Network Systems. 2016.

[17] A. J. Wood, B. F. Wollenberg, and G. B. Sheble. *Power generation, operation, and control.* Wiley, third edition, 2013.

[18] A. H. Mohsenian-Rad, V. W. S. Wong, J. Jatskevich, R. Schober, and A. Leon-Garcia. Autonomous Demand-Side Management Based on Game-Theoretic Energy Consumption Scheduling for the Future Smart Grid. *IEEE TRANSACTIONS ON SMART GRID*, 1(3), 2010.

[19] P. Samadi, H. Mohsenian-Rad, R. Schober, and V. W. S. Wong. Advanced Demand Side Management for the Future Smart Grid Using Mechanism Design. *IEEE Transactions on Smart Grid*, 3, 2012.

[20] I. Atzeni, L. G. Ordonez, G. Scutari, D. P. Palomar, and J. R. Fonollosa. Demand-Side Management via Distributed Energy Generation and Storage Optimization. *IEEE Transactions on Smart Grid*, 4(2):866–876, 2013.