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Crepant resolutions and open strings

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Dustin Ross at Ann Arbor

Abstract. In the present paper, we formulate a Crepant Resolution Correspondence for open Gromov–Witten invariants (OCRC) of toric Lagrangian branes inside Calabi–Yau 3-orbifolds by encoding the open theories into sections of Givental’s symplectic vector space. The correspondence can be phrased as the identification of these sections via a linear morphism of Givental spaces. We deduce from this a Bryan–Graber-type statement for disk invariants, which we extend to arbitrary topologies in the Hard Lefschetz case. Motivated by ideas of Iritani, Coates–Corti–Iritani–Tseng and Ruan, we furthermore propose (1) a general form of the morphism entering the OCRC, which arises from a geometric correspondence between equivariant K -groups, and (2) an all-genus version of the OCRC for Hard Lefschetz targets. We provide a complete proof of both statements in the case of minimal resolutions of threefold A_n -singularities; as a necessary step of the proof we establish the all-genus closed Crepant Resolution Conjecture with descendents in its strongest form for this class of examples. Our methods rely on a new description of the quantum D -modules underlying the equivariant Gromov–Witten theory of this family of targets.

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1. Introduction

1.1. Summary of results. This paper proposes an approach to the Crepant Resolution Conjecture for open Gromov–Witten invariants, and supports it with a series of results and verifications about threefold A_n -singularities and their resolutions.

Let \mathcal{Z} be a smooth toric Calabi–Yau Deligne–Mumford stack of dimension three with generically trivial stabilizers and semi-projective coarse moduli space, and let L be an Aganagic–Vafa brane (Section 3.1.1). Fix a Calabi–Yau torus action T on \mathcal{Z} and denote by $\Delta_{\mathcal{Z}}$ the free module over $H(BT)$ spanned by the T -equivariant lifts of orbifold cohomology classes of Chen–Ruan degree at most two. We define (Section 3.1) a family of elements of Givental space,

$$\mathbb{F}_{L,\mathcal{Z}}^{\text{disk}} : H_T(\mathcal{Z}) \rightarrow \mathcal{H}_{\mathcal{Z}} = H_T(\mathcal{Z})((z^{-1})),$$

which we call the *winding neutral disk potential*. Upon appropriate specializations of the variable z , $\mathbb{F}_{L,\mathcal{Z}}^{\text{disk}}$ encodes disk invariants of (\mathcal{Z}, L) at any winding d .

Consider a *crepant resolution diagram* $\mathcal{X} \rightarrow X \leftarrow Y$, where X is the coarse moduli space of \mathcal{X} and Y is a crepant resolution of the singularities of X . A Lagrangian boundary condition L is chosen on \mathcal{X} and we denote by L' its transform in Y . Our version of the open crepant resolution conjecture is a comparison of the (restricted) winding neutral disk potentials.

Proposal 1 (The OCRC). There exists a $\mathbb{C}((z^{-1}))$ -linear map of Givental spaces $\mathbb{O} : \mathcal{H}_{\mathcal{X}} \rightarrow \mathcal{H}_Y$ and analytic functions $\mathfrak{h}_{\mathcal{X}} : \Delta_{\mathcal{X}} \rightarrow \mathbb{C}$, $\mathfrak{h}_Y : \Delta_Y \rightarrow \mathbb{C}$ such that

$$\mathfrak{h}_Y^{\frac{1}{z}} \mathbb{F}_{L',Y}^{\text{disk}}|_{\Delta_Y} = \mathfrak{h}_{\mathcal{X}}^{\frac{1}{z}} \mathbb{O} \circ \mathbb{F}_{L,\mathcal{X}}^{\text{disk}}|_{\Delta_{\mathcal{X}}}$$

upon analytic continuation of quantum cohomology parameters.

Further, we conjecture (Conjecture 3.5) that both \mathbb{O} and \mathfrak{h}_{\bullet} are completely determined by the classical toric geometry of \mathcal{X} and Y . In particular, we give a prediction for the transformation \mathbb{O} depending on a choice of identification of the K -theory lattices of \mathcal{X} and Y .

When \mathcal{X} is a Hard Lefschetz Calabi–Yau orbifold, the OCRC comparison extends to all of $H_T(\mathcal{Z})$. This, together with ideas of Coates–Iritani–Tseng and Ruan (Conjecture 2.4), motivates a comparison for potentials encoding invariants of maps with arbitrary topology.

Proposal 2 (The quantum OCRC). Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a Hard Lefschetz diagram for which the OCRC holds. Defining $\mathbb{O}^{\otimes \ell} = \mathbb{O}(z_1) \otimes \cdots \otimes \mathbb{O}(z_{\ell})$, we have

$$\mathbb{F}_{L',Y}^{g,\ell} = \mathbb{O}^{\otimes \ell} \circ \mathbb{F}_{L,\mathcal{X}}^{g,\ell},$$

where the winding neutral open potential $\mathbb{F}^{g,\ell}$ is the genus- g , ℓ -boundary components analog of \mathbb{F}^{disk} defined in Section 3.3.

Consider now the family of threefold A_n singularities, where $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{C}$ and Y is its canonical minimal resolution.

Main Theorem. *The OCRC, Conjecture 3.5 and the quantum OCRC hold for the A_n -singularities for any choice of Aganagic–Vafa brane on \mathcal{X} .*

Our verification of the OCRC and Conjecture 3.5 in this family of examples follow from Proposition 3.6 and Theorem 4.1. The quantum OCRC is a consequence of the closed string quantum CRC in its strongest version (Conjecture 2.4, Theorem 6.1), which we establish in Section 6. From this, we deduce a series of comparisons of more classical generating functions for open invariants, in the spirit of Bryan–Graber’s formulation of the CRC.

In Section 3.2 we define the *cohomological disk potential* $\mathcal{F}_L^{\text{disk}}$ – a cohomology-valued generating function for disk invariants that “remembers” the twisting and the attaching fixed point of an orbimap. We also consider the coarser *scalar potential* (see Section 3.1.1), which keeps track of the winding of the orbimaps but forgets the twisting and attaching point. There are essentially two different choices for the Lagrangian boundary condition on \mathcal{X} ; the simpler case occurs when L intersects one of the effective legs of the orbifold. In this case we have the following result.

Theorem 4.4 (Effective leg). *Identifying the winding parameters and setting*

$$\mathbb{O}_{\mathbb{Z}}(\mathbf{1}^k) = P^{n+1}$$

for every k , we have

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}).$$

It is immediate to observe that the scalar potentials coincide (Corollary 4.5).

The case when L intersects the ineffective leg of the orbifold is more subtle.

Theorem 4.2 (Ineffective leg). *We exhibit a matrix $\mathbb{O}_{\mathbb{Z}}$ of roots of unity and a specialization of the winding parameters depending on the equivariant weights such that*

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}).$$

The comparison of scalar potentials in this case does not hold anymore. Because of the special form of the matrix $\mathbb{O}_{\mathbb{Z}}$ we deduce in Corollary 4.3 that the scalar disk potential for Y corresponds to the contribution to the potential for \mathcal{X} by the untwisted disk maps. Our proof of the quantum CRC makes it an exercise in book-keeping to extend the statements of Theorems 4.2 and 4.4 to compare generating functions for open invariants with arbitrary genus and number of boundary components, even treating all boundary Lagrangian conditions at the same time. The main tool used in the proof of our main theorem is a new global description of the gravitational quantum cohomology of the A_n geometries, which enjoys a number of remarkable features, and may have an independent interest per se.

Theorem 5.4. *By identifying the A -model moduli space with a genus-zero double Hurwitz space, we construct a global quantum D -module $(\mathcal{F}_{\lambda,\phi}, T\mathcal{F}_{\lambda,\phi}, \nabla^{(g,z)}, H(\cdot, \cdot)_g)$ which is locally isomorphic to $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$ in appropriate neighborhoods of the orbifold and large complex structure points.*

1.2. Context, motivation and further discussion. Open Gromov–Witten (GW) theory intends to study holomorphic maps from bordered Riemann surfaces, where the image of the boundary is constrained to lie in a Lagrangian submanifold of the target. While some general foundational work has been done [66, 74], at this point most of the results in the theory rely on

additional structure. In [19, 21] Lagrangian Floer theory is employed to study the case when the boundary condition is a fiber of the moment map. In the toric context, a mathematical approach [13, 32, 54, 69] to construct operatively a virtual counting theory of open maps is via the use of localization.¹⁾ A variety of striking relations have been verified connecting open GW theory and several other types of invariants, including open B -model invariants and matrix models [3, 4, 8, 40, 60], quantum knot invariants [47, 63], and ordinary Gromov–Witten and Donaldson–Thomas theory via “gluing along the boundary” [2, 61, 64].

Since Ruan’s influential conjecture [70], an intensely studied problem in Gromov–Witten theory has been to determine the relation between GW invariants of target spaces related by a crepant birational transformation (CRC). The most general formulation of the CRC is framed in terms of Givental formalism ([28]; see also [29] for an expository account); the conjecture has been proved in a number of examples [23, 24, 28] and has by now gained folklore status, with a general proof in the toric setting announced for some time [25, 27]. A natural question one can ask is whether similar relations exist in the context of open Gromov–Witten theory. Within the toric realm, physics arguments based on open mirror symmetry [8, 9, 15] have given strong indications that some version of the Bryan–Graber [17] statement of the crepant resolution conjecture should hold at the level of open invariants. This was proven explicitly for the crepant resolution of the Calabi–Yau orbifold $[\mathbb{C}^3/\mathbb{Z}_2]$ in [18]. Around the same time, it was suggested [10, 11] that a general statement of a Crepant Resolution Conjecture for open invariants should have a natural formulation within Givental’s formalism, as in [24, 28]. Some implications of this philosophy were verified in [11] for the crepant resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$ of the orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$.

The OCRC we propose here is a natural extension to open Gromov–Witten theory of the Coates–Iritani–Tseng approach [28] to Ruan’s conjecture. The observation that the disk function of [13, 69] can be interpreted as an endomorphism of Givental space makes the OCRC statement follow almost tautologically from the Coates–Iritani–Tseng/Ruan picture of the ordinary CRC via toric mirror symmetry [28]. The more striking aspect of our conjecture is then that the linear function \mathbb{O} comparing the winding neutral disk potentials is considerably simpler than the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X}, Y}$ in the closed CRC and it is characterized in terms of *purely classical data*: essentially, the equivariant Chern characters of \mathcal{X} and Y . This is intimately related to Iritani’s equation (2.4) that the analytic continuation for the flat sections of the global quantum D -module is realized via the composition of K -theoretic central charges. While Iritani works non-equivariantly on proper targets, his constructions carry through to the equivariant setting, and inspire us to make Conjecture 3.5. We point out that our results do not rely on the validity of Iritani’s proposal, but rather support the fact that an equivariant version of his proposal should hold.

Iritani’s theory is inspired and consistent with the idea of global mirror symmetry, i.e. that there exists a global quantum D -module on the A -model moduli space which locally agrees with the Frobenius structure given by quantum cohomology. In order to verify his proposal in the equivariant setting relevant for this paper, we give a new construction of this global structure: motivated by the connection of the Gromov–Witten theory of A_n -surface singularities to certain integrable systems of Toda type [12], we realize the global A -model quantum D -module as a system of one-dimensional Euler–Pochhammer hypergeometric periods. This

¹⁾ Alternatively, open string invariants in the manifold case can be defined using relative stable morphisms [62].

mirror picture possesses several remarkable properties which enable us to verify in detail our proposals for the open CRC, as well as proving along the way various results of independent interest on Ruan’s conjecture as well as its refinements (Iritani, Coates–Iritani–Tseng) and extensions (the higher genus CRC). First off, the computation of the analytic continuation of flat sections is significantly simplified with respect to the standard toric mirror symmetry methods based on the Mellin–Barnes integral: in particular, the Hurwitz space picture gives closed-form expressions for the analytic continuation of flat sections upon crossing a parametrically large number of walls. A useful consequence for us is Theorem 4.1, which furnishes an explicit form for the morphism $\mathbb{U}_\rho^{\mathcal{X},Y}$ of Givental’s spaces of [24], as well as a verification of Iritani’s proposal [50] in the fully equivariant setting. Furthermore, the monodromy action on branes (and therefore equivariant K -theory) gets identified with the Deligne–Mostow monodromy of hypergeometric integrals, thereby giving a natural action of the pure braid group with $n + 2$ strands on the equivariant K -groups of A_n -resolutions. Finally, proving the strong version of the quantized Crepant Resolution Conjecture (Conjecture 2.4) is reduced by Theorem 5.4 to a calculation in Laplace-type asymptotics. To our knowledge, this provides the first example where a full-descendent version of Ruan’s conjecture is established to all genera.²⁾

1.3. Relation to ongoing and other work. A proof of the all-genus Ruan’s conjecture and of the quantum OCRC for the other case of a Hard Lefschetz Crepant Resolution of toric Calabi–Yau 3-folds – the G -Hilb resolution of $[\mathbb{C}^3/G]$ with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ – will be offered in the companion paper [14], where the OCRC will also be proven for a family of non-Hard Lefschetz targets.

In the current form, the *winding neutral disk potential*, which encodes information about disk invariants, depends on the choice of an Aganagic–Vafa brane incident to one of the torus invariant lines. In other words, we have a different object corresponding to each phase of the open moduli. It would be desirable to have a construction of the winding neutral disk potential that is independent of the choice of Lagrangian, and to obtain the various boundary conditions as specializations, so as to witness more explicitly the phase transitions in the open moduli. We are currently investigating this proposal and have some positive results in the case of target \mathbb{C}^3 .

We have also been made aware of the existence of a number of projects related in various ways to the subject of this paper. In a forthcoming paper, Coates–Iritani–Jiang will establish Iritani’s proposal on the relation between the K -group McKay correspondence and the CRC in the fully equivariant setting for general semi-projective toric varieties. Ke–Zhou [55] have announced a proof of the quantum McKay correspondence for disk invariants on effective outer legs for semi-projective toric Calabi–Yau 3-orbifolds using results of [40]; this is the case where the comparison of cohomological disk potentials of the OCRC is simplified to an identification of the scalar disk potentials. Very recently, a similar statement for scalar potentials was obtained by Chan–Cho–Lau–Tseng in [20] as an application of their construction of a class of non-toric Lagrangian branes inside toric Calabi–Yau 3-orbifolds. This opens up the suggestive hypothesis that our setup for the OCRC may be generalized beyond the toric setting considered here.

1.4. Organization of the paper. This paper is organized as follows. Section 2 is a presentation of various versions of the ordinary (closed string) Crepant Resolution Conjecture

²⁾ For non-descendent invariants, an all-genus Bryan–Graber-type statement for A_n -surface resolutions was proved by Zhou [80]. In an allied context, Krawitz–Shen [56] have established an all-genus LG/CY correspondence for elliptic orbifold lines.

that are addressed in this paper. In Section 3 we present our proposal for the Open Crepant Resolution Conjecture, whose consequences we analyze in detail in Section 4 for the case of A_n -resolutions. Proofs of the statements contained here are offered in Sections 5–6: Section 5 is devoted to the construction of the Hurwitz-space mirror, which is used to verify our prediction on the form of the morphism \mathbb{O} , while in Section 6 the quantum CRC and OCRC are established by combining the tools of Section 5 with Givental’s quantization formalism. Relevant background material on Gromov–Witten theory and quantum D -modules is reviewed in Section A, while Section B collects mostly notational material on the toric geometry concerning our examples. A technical result on the analytic continuation of hypergeometric integrals required in the proof of Theorem 4.1 is discussed in Section C.

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2. Gromov–Witten theory and crepant resolutions: Setup and conjectures

This section collects and ties together various incarnations of the CRC that we wish to focus on. We assume here familiarity with Gromov–Witten theory and Givental’s formalism; relevant background material is collected in Section A.

Consider a toric Gorenstein orbifold \mathcal{X} , and let $X \leftarrow Y$ be a crepant resolution of its coarse moduli space. For $\mathcal{Z} = \mathcal{X}, Y$, fix an algebraic $T \simeq \mathbb{C}^*$ action with zero-dimensional fixed loci such that the resolution morphism is T -equivariant. The equivariant Chen–Ruan cohomology ring $H(\mathcal{Z}) \triangleq H_T^{\text{orb}}(\mathcal{Z})$ of \mathcal{Z} is a rank- $N_{\mathcal{Z}} \triangleq \text{rank}_{\mathbb{C}[\nu]} H(\mathcal{Z})$ free module over $H(BT) \simeq \mathbb{C}[\nu]$, where $\nu = c_1(\mathcal{O}_{BT}(1))$ is the equivariant parameter. Note that the genus-zero Gromov–Witten theory of \mathcal{Z} defines a deformation of the ring structure on $H(\mathcal{Z})$, and equivalently, the existence of a distinguished family of flat structures on its tangent bundle. We shall fix notation as follows:

η	the flat pairing on the space of vector fields $\mathcal{X}(H(\mathcal{Z}))$ induced by the Poincaré pairing on \mathcal{Z}	(A.1)
\circ_{τ}	the quantum product at $\tau \in H(\mathcal{Z})$	(A.4), (A.6)
$\nabla^{(\eta, z)}$	the Dubrovin connection on $\mathcal{X}(H(\mathcal{Z}))$	(A.7)
$\text{QDM}(\mathcal{Z})$	the quantum D -module structure on $\mathcal{X}(\mathcal{Z})$ induced by (η, \circ_{τ})	(A.8)
$\mathcal{S}_{\mathcal{Z}}$	the vector space of horizontal sections of $\nabla^{(\eta, z)}$	
$H(\cdot, \cdot)_{\mathcal{Z}}$	the canonical pairing on $\mathcal{S}_{\mathcal{Z}}$ induced by η	(A.9)
$J^{\mathcal{Z}}$	the big J -function of \mathcal{Z}	(A.11)
$S_{\mathcal{Z}}$	the fundamental solution (S -calibration) of $\text{QDM}(\mathcal{Z})$	(A.10)
$\mathcal{H}_{\mathcal{Z}}$	Givental’s symplectic vector space of \mathcal{Z}	(A.17)–(A.18)

$\mathcal{L}_{\mathcal{Z}}$	the Lagrangian cone associated to $QH(\mathcal{Z})$	(A.20)
$\mathrm{Sp}_{\pm}(\mathcal{H}_{\mathcal{Z}})$	the positive/negative symplectic loop group of $\mathcal{H}_{\mathcal{Z}}$	Section A.2.1
$\Delta_{\mathcal{Z}}$	the free module over $\mathbb{C}[\nu]$ spanned by T -equivariant lifts of orbifold cohomology classes with $\mathrm{deg}^{\mathrm{CR}} \leq 2$.	

2.1. Quantum D -modules and the CRC. Ruan’s Crepant Resolution Conjecture can be phrased as the existence of a *global quantum D -module* underlying the quantum D -modules of \mathcal{X} and Y . This is a 4-tuple $(\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_F)$ given by a connected complex analytic space \mathcal{M}_A and a holomorphic vector bundle $F \rightarrow \mathcal{M}_A$, endowed with a flat $\mathcal{O}_{\mathcal{M}_A}$ -connection ∇ and a non-degenerate ∇ -flat inner product $H(\cdot, \cdot)_F \in \mathrm{End}(F)$.

Conjecture 2.1 (The Crepant Resolution Conjecture). There exist a global quantum D -module $(\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_F)$ such that for open subsets $V_{\mathcal{X}}, V_Y \subset \mathcal{M}_A$ we locally have

$$(2.1) \quad \begin{aligned} (\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_F)|_{V_{\mathcal{X}}} &\simeq \mathrm{QDM}(\mathcal{X}), \\ (\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_F)|_{V_Y} &\simeq \mathrm{QDM}(Y). \end{aligned}$$

In particular, any 1-chain ρ in \mathcal{M}_A with ends in $V_{\mathcal{X}}$ and V_Y gives an analytic continuation map of ∇ -flat sections $\mathbb{U}_{\mathcal{S}, \rho}^{\mathcal{X}, Y} : \mathcal{S}_{\mathcal{X}} \rightarrow \mathcal{S}_Y$, which is an isometry of $H(\cdot, \cdot)_F$ and identifies the quantum D -modules of \mathcal{X} and Y .

Even when equation (2.1) holds, there may be an obstruction to extend the isomorphism of small quantum products to big quantum cohomology which is relevant in our formulation of the OCRC. Locally around the large radius limit point of \mathcal{X} and Y , canonical trivializations of the global flat connection ∇ are given by a system of flat coordinates for the small Dubrovin connection. Generically they are not *mutually flat*: on the overlap $V_{\mathcal{X}} \cap V_Y$, the relation between the two coordinate systems is typically not affine over $\mathbb{C}(\nu)$, and as a result the induced Frobenius structures on $H(\mathcal{X})$ and $H(Y)$ may be inequivalent. In favorable situations, for example when the coarse moduli space Z is semi-projective, the two charts are related by a conformal factor $\mathfrak{h}_Y \mathfrak{h}_{\mathcal{X}}^{-1}$ for local functions $\mathfrak{h}_{\mathcal{X}} \in \mathcal{O}_{V_{\mathcal{X}}}$, $\mathfrak{h}_Y \in \mathcal{O}_{V_Y}$ which are in turn completely determined by the toric combinatorics defining X and Y as GIT quotients (equation (A.16)). A sufficient condition [28] for the two Frobenius structures to coincide (i.e. $\mathfrak{h}_{\mathcal{X}} = \mathfrak{h}_Y$) is given by the Hard Lefschetz criterion for \mathcal{X} ,

$$\mathrm{age}(\theta) - \mathrm{age}(\mathrm{inv}^* \theta) = 0,$$

for any class $\theta \in H(\mathcal{X})$.

2.2. Integral structures and the CRC. In [50], Iritani uses K -groups to define an integral structure in the quantum D -module associated to the Gromov–Witten theory of a smooth and proper Deligne–Mumford stack \mathcal{Z} ; we recall the discussion in [50, 51]. Write $K(\mathcal{Z})$ for the Grothendieck group of topological vector bundles $V \rightarrow \mathcal{Z}$ and consider the map

$$\mathcal{F} : K(\mathcal{Z}) \rightarrow H(\mathcal{Z}) \otimes \mathbb{C}((z^{-1}))$$

given by

$$\mathcal{F}(V) \triangleq (2\pi)^{-\frac{\dim \mathcal{Z}}{2}} z^{-\mu} \bar{\Gamma}_{\mathcal{Z}} \cup (2\pi i)^{\frac{\mathrm{deg}}{2}} \mathrm{inv}^* \mathrm{ch}(V),$$

where $\text{ch}(V)$ is the orbifold Chern character, \cup is the topological cup product on $I\mathcal{Z}$, and

$$(2.2) \quad \begin{aligned} \bar{\Gamma}_{\mathcal{Z}} &\triangleq \bigoplus_v \prod_f \prod_{\delta} \Gamma(1 - f + \delta), \\ \mu &\triangleq \left(\frac{1}{2} \deg(\phi) - \frac{3}{2} \right) \phi; \end{aligned}$$

the sum in equation (2.2) is over all connected components of the inertia stack, the left product is over the eigenbundles in a decomposition of the tangent bundle $T\mathcal{Z}$ with respect to the stabilizer action (with f the rational weight of the action on the eigenspace; note that $1 - f$ is always strictly positive and hence $\bar{\Gamma}_{\mathcal{Z}}$ is an invertible function in a neighborhood of 0), and the right product is over all of the Chern roots δ of the eigenbundle. Via the fundamental solution (A.10) this induces a map to the space of flat sections of $\text{QDM}(\mathcal{Z})$; its image is a lattice [50] in $\mathcal{S}_{\mathcal{Z}}$, which Iritani dubs the *K-theory integral structure* of $QH(\mathcal{Z})$.

Iritani's theory has important implications for the Crepant Resolution Conjecture. At the level of integral structures, the analytic continuation map $\mathbb{U}_{\mathcal{S},\rho}^{\mathcal{X},Y}$ of flat sections should be induced by an isomorphism $\mathbb{U}_{K,\rho}^{\mathcal{X},Y} : K(Y) \rightarrow K(\mathcal{X})$ at the K -group level. The Crepant Resolution Conjecture can then be phrased in terms of the existence of an identification of the integral local systems underlying quantum cohomology, which, according to [50], should take the shape of a natural geometric correspondence between K -groups.

2.3. The symplectic formalism and the CRC. The symplectic geometry of Frobenius manifolds gives the Crepant Resolution Conjecture a natural formulation in terms of morphisms of Givental spaces, as pointed out by Coates–Corti–Iritani–Tseng [24, 28] (see also [29] for a review).

Conjecture 2.2 ([28]). There exists a $\mathbb{C}((z^{-1}))$ -linear symplectic isomorphism of Givental spaces $\mathbb{U}_{\rho}^{\mathcal{X},Y} : \mathcal{H}_{\mathcal{X}} \rightarrow \mathcal{H}_Y$, matching the Lagrangian cones of \mathcal{X} and Y upon a suitable analytic continuation of small quantum cohomology parameters:

$$(2.3) \quad \mathbb{U}_{\rho}^{\mathcal{X},Y}(\mathcal{L}_{\mathcal{X}}) = \mathcal{L}_Y.$$

This version of the CRC is equivalent to the quantum D -module approach via the fundamental solutions, which give a canonical z -linear identification

$$S_{\mathcal{Z}}(\tau, z) : \mathcal{H}_{\mathcal{Z}} \xrightarrow{\cong} \mathcal{S}_{\mathcal{Z}},$$

translating the analytic continuation map $\mathbb{U}_{\mathcal{S},\rho}^{\mathcal{X},Y}$ to a symplectic isomorphism of Givental spaces $\mathbb{U}_{\rho}^{\mathcal{X},Y}$. Iritani's theory of integral structures proposes that the symplectic isomorphism $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ should be induced from a natural equivalence at the level of K lattices of \mathcal{X} and Y , as illustrated in Figure 1:

Proposal 3 ([50]). Inverting the central charge $\mathcal{F}_{\mathcal{X}}$, one obtains

$$(2.4) \quad \mathbb{U}_{\rho}^{\mathcal{X},Y} = \mathcal{F}_Y \circ \mathbb{U}_{K,\rho}^{\mathcal{X},Y} \circ \mathcal{F}_{\mathcal{X}}^{-1}.$$

A case of particular interest for us is the following. Suppose that $c_1(\mathcal{X}) = 0$, $\dim_{\mathbb{C}} \mathcal{X} = 3$ and assume further that the J -functions $J^{\mathcal{Z}}$, for \mathcal{Z} either \mathcal{X} or Y , and $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ admit well-defined

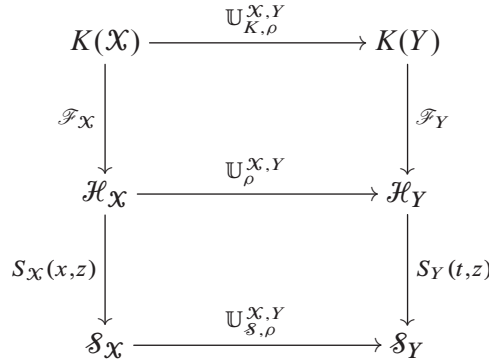


Figure 1. Analytic continuation of flat sections, symplectomorphism of Givental spaces and comparison of integral structures.

non-equivariant limits,

$$J_{\text{n-eq}}^{\mathcal{Z}}(\tau, z) \triangleq \lim_{\nu \rightarrow 0} J^{\mathcal{Z}}(\tau, z), \quad \mathbb{U}_{\rho,0}^{\mathcal{X},Y} \triangleq \lim_{\nu \rightarrow 0} \mathbb{U}_{\rho}^{\mathcal{X},Y}.$$

By the string equation and dimensional constraints, $e^{-\tau^0/z} J_{\text{n-eq}}^{\mathcal{Z}}(\tau, z)$ is a Laurent polynomial of the form [30, Section 10.3.2]

$$J_{\text{n-eq}}^{\mathcal{Z}}(\tau, z) = e^{-\tau^0/z} \left(z + \sum_{i=1}^{N_{\mathcal{Z}}-1} \left(\tau^i + \frac{\mathfrak{f}_i^{\mathcal{Z}}(\tau)}{z} \right) \phi_i + \frac{\mathfrak{g}^{\mathcal{Z}}(\tau)}{z^2} \mathbf{1}_{\mathcal{Z}} \right),$$

where $\mathfrak{f}^{\mathcal{Z}}(\tau)$ and $\mathfrak{g}^{\mathcal{Z}}(\tau)$ are analytic functions around the large radius limit point of \mathcal{Z} . Restricting $J_{\text{n-eq}}^{\mathcal{Z}}(\tau, z)$ to $\Delta_{\mathcal{Z}}$ and picking up a branch ρ of analytic continuation of the quantum parameters, the vector-valued analytic function $\mathcal{J}_{\rho}^{\mathcal{X},Y}$ defined by

$$(2.5) \quad \begin{array}{ccc}
 \mathcal{H}_{\mathcal{X}} & \xrightarrow{\mathfrak{h}_{\mathcal{X}}^{\frac{1}{2}} \mathbb{U}_{\rho,0}^{\mathcal{X},Y} \mathfrak{h}_Y^{-\frac{1}{2}}} & \mathcal{H}_Y \\
 \uparrow J_{\text{n-eq}}^{\mathcal{X}}|_{\Delta_{\mathcal{X}}} & & \uparrow J_{\text{n-eq}}^Y|_{\Delta_Y} \\
 \Delta_{\mathcal{X}} & \xrightarrow{\mathcal{J}_{\rho}^{\mathcal{X},Y}} & \Delta_Y
 \end{array}$$

gives an analytic isomorphism³⁾ between neighborhoods $V_{\mathcal{X}}, V_Y$ of the projections of the large radius points of \mathcal{X} and Y to $\Delta_{\mathcal{X}}$ and Δ_Y . When \mathcal{X} satisfies the Hard–Lefschetz condition, the coefficients of $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ contain only non-positive powers of z [28] and the non-equivariant limit coincides with the $z \rightarrow \infty$ limit; then the isomorphism $\mathcal{J}_{\rho}^{\mathcal{X},Y}$ extends to the full cohomology rings of \mathcal{X} and Y , and induces an affine linear change of variables $\widehat{\mathcal{J}}_{\rho}^{\mathcal{X},Y}$, which gives an isomorphism of Frobenius manifolds.

³⁾ Explicitly, matrix entries $(\mathbb{U}_{\rho,0}^{\mathcal{X},Y})_{ij}$ of $\mathbb{U}_{\rho,0}^{\mathcal{X},Y}$ are monomials in z ; call u_{ij} the coefficient of such monomial. Then (2.5) boils down to the statement that quantum cohomology parameters τ_i^{\bullet} in Δ_{\bullet} for $i = 1, \dots, l_Y$ are identified as

$$(2.6) \quad \tau_i^Y = (\mathcal{J}_{\rho}^{\mathcal{X},Y} \tau^{\mathcal{X}})_i \triangleq u_{i0} + \sum_{j=1}^{l_Y} u_{ij} (\tau^{\mathcal{X}})^j + \sum_{k=l_Y+1}^{N_Y-1} u_{ik} \mathfrak{f}_k^{\mathcal{X}}(\tau^{\mathcal{X}}).$$

Since $\deg(\mathbb{U}_{\rho,0}^{\mathcal{X},Y})_{ij} > 0$ for $j > l_Y$, in the Hard Lefschetz case the condition that the coefficients of $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ are Taylor series in $\frac{1}{z}$ implies that $u_{ik} = 0$ for $k > l_Y$.

2.4. Quantization and the higher genus CRC. Conjecture 2.2 shapes the genus-zero CRC as the existence of a classical canonical transformation identifying the Givental phase spaces of \mathcal{X} and Y . As the higher genus theory is obtained from the genus-zero picture by quantization, it is expected that the full Gromov–Witten partition functions should be identified, upon analytic continuation, via a *quantum* canonical transformation identifying the Fock spaces, and that such quantum transformation is related to the quantization of the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X},Y}$ in equation (2.3).

Conjecture 2.3 ([28, 29]). Let $\mathbb{U}_\rho^{\mathcal{X},Y} = \mathbb{U}_- \mathbb{U}_0 \mathbb{U}_+$ denote the Birkhoff factorization of $\mathbb{U}_\rho^{\mathcal{X},Y}$. Then

$$(2.7) \quad Z_Y = \widehat{\mathbb{U}}_- \widehat{\mathbb{U}}_0 \widehat{\mathbb{U}}_+ Z_{\mathcal{X}}.$$

A much stronger statement stems from equation (2.7) in the Hard Lefschetz case, when $\mathbb{U}_+ = \mathbf{1}_{\mathcal{H}_{\mathcal{X}}}$ (cf. [28, Theorem 5.10]).

Conjecture 2.4 (The Hard Lefschetz quantized CRC). Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a Hard Lefschetz crepant resolution diagram. Then

$$(2.8) \quad Z_Y = \widehat{\mathbb{U}}_\rho^{\mathcal{X},Y} Z_{\mathcal{X}}$$

By [44, Proposition 5.3], equation (2.8) gives, up to quadratic genus-zero terms,

$$(2.9) \quad Z_Y = Z_{\mathcal{X}}|_{t_{\mathcal{X}} = [\mathbb{U}_{\mathcal{X},Y}^{-1} t_Y]_+},$$

where $[f(z)]_+$ denotes the projection $[\cdot]_+ : \mathbb{C}((z)) \rightarrow \mathbb{C}[[z]]$. In other words, Conjecture 2.4 states that the full descendent partition function of Z and Y coincide to all genera, upon analytic continuation and the identification of the Fock space variables dictated by the classical symplectomorphism (2.3).

3. The open crepant resolution conjecture

3.1. Open string maps and Givental’s formalism.

3.1.1. Open Gromov–Witten theory of toric 3-orbifolds. For a three-dimensional toric Calabi–Yau variety, open Gromov–Witten invariants are defined “via localization” in [32, 54]. This theory has been first introduced for orbifold targets in [13] and developed in full generality in [69] (see also [40] for recent results in this context).

Boundary conditions are given by choosing special type of Lagrangian submanifolds introduced by Aganagic–Vafa in [4]. These Lagrangians are defined locally in a formal neighborhood of each torus invariant line: in particular if p is a torus fixed point adjacent to the torus fixed line l , and the local coordinates at p are (z, u, v) , then L is defined to be the fixed points of the anti-holomorphic involution

$$(z, u, v) \mapsto \left(\frac{1}{z}, \overline{zu}, \overline{zv}\right)$$

defined away from $z = 0$. Boundary conditions can then be thought of as “formal” ways of decorating the web diagram of the toric target.

Loci of fixed maps are described in terms of closed curves mapping to the compact edges of the web diagram in the usual way and disks mapping rigidly to the torus invariant lines with Lagrangian conditions. Beside Hodge integrals coming from the contracting curves, the contribution of each fixed locus to the invariants has a factor for each disk, which is constructed as follows. The map from the disk to a neighborhood of its image is viewed as the quotient via an involution of a map of a rational curve to a canonical target. The obstruction theory in ordinary Gromov–Witten theory admits a natural \mathbb{Z}_2 action, and the equivariant Euler class of the involution invariant part of the obstruction theory is chosen as the localization contribution from the disk [13, Section 2.2], [69, Section 2.4]. This construction is encoded via the introduction of a “disk function”, which we now review in the context of cyclic isotropy (see [69, Section 3.3] for the general case of finite abelian isotropy groups).

Let \mathcal{Z} be a three-dimensional CY toric orbifold, p a fixed point such that a neighborhood is isomorphic to $[\mathbb{C}^3/\mathbb{Z}_{n+1}]$, with representation weights (m_1, m_2, m_3) and CY torus weights (w_1, w_2, w_3) . Fix a Lagrangian boundary condition L which we assume to be on the first coordinate axis in this local chart. Define $n_e = (n + 1)/\gcd(m_1, n + 1)$ to be the size of the effective part of the action along the first coordinate axis. There exist a map from an orbifold mapping to the first coordinate axis with winding d and twisting⁴⁾ k if the compatibility condition

$$(3.1) \quad \frac{d}{n_e} - \frac{km_1}{n+1} \in \mathbb{Z}$$

is satisfied. In this case the positively oriented disk function is

$$D_k^+(d; \vec{w}) = \left(\frac{n_e w_1}{d}\right)^{\text{age}(k)-1} \frac{n_e}{d(n+1) \lfloor \frac{d}{n_e} \rfloor!} \frac{\Gamma\left(\frac{dw_2}{n_e w_1} + \left\langle \frac{km_3}{n+1} \right\rangle + \frac{d}{n_e}\right)}{\Gamma\left(\frac{dw_2}{n_e w_1} - \left\langle \frac{km_2}{n+1} \right\rangle + 1\right)}.$$

The negatively oriented disk function is obtained by switching the indices 2 and 3. By renaming the coordinate axes this definition applies to the general boundary condition.

In [69] the disk function is used to construct the GW orbifold topological vertex, a building block for open and closed GW invariants of \mathcal{Z} . The *scalar disk potential* is expressed in terms of the disk and of the J function of \mathcal{Z} . The fixed point basis for the equivariant Chen Ruan cohomology of \mathcal{Z} has $n + 1$ classes supported at the fixed point p , corresponding to all irreducible representations of \mathbb{Z}_{n+1} . For $k = 1, \dots, n$, denote by $\mathbf{1}_{\mathbf{p},k}$ the fundamental class of the twisted sector corresponding to the character k ; we denote $\mathbf{1}_{\mathbf{p},n+1}$ the untwisted class of the fixed point p . Raising indices using the orbifold Poincaré pairing, and extending the disk function to be a cohomology-valued function

$$\mathcal{D}^+(d; \vec{w}) = \sum_{k=1}^{n+1} D_k^+(d; \vec{w}) \mathbf{1}_{\mathbf{p},k}$$

the (genus-zero) *scalar disk potential* is obtained by contraction with the J function:

$$\begin{aligned} F_L^{\text{disk}}(\tau, y, \vec{w}) &\triangleq \sum_d \frac{y^d}{d!} \sum_n \frac{1}{n!} \langle \tau, \dots, \tau \rangle_{0,n}^{L,d} \\ &= \sum_d \frac{y^d}{d!} \left(\mathcal{D}^+(d; \vec{w}), J^{\mathcal{Z}}\left(\tau, \frac{n_e w_1}{d}\right) \right)_{\mathcal{Z}}, \end{aligned}$$

⁴⁾ Here twisting refers to the image of the center of the disk in the evaluation map to the inertia orbifold.

where we denoted by $\langle \tau, \dots, \tau \rangle_{0,n}^{L,d}$ the disk invariants with boundary condition L , winding d and n general cohomological insertions.

Remark 3.1. We may consider the disk potential relative to multiple Lagrangian boundary conditions. In that case, we define the disk function by adding the disk functions for each Lagrangian, and we introduce a winding variable for each boundary condition. Furthermore, it is not conceptually difficult (but book-keeping intensive) to express the general open potential in terms of appropriate contractions of arbitrary copies of these disk functions with the full descendent Gromov–Witten potential of \mathcal{Z} .

3.1.2. The disk function, revisited. We reinterpret the disk function as a symmetric tensor of Givental space. First we homogenize Iritani’s Gamma class (2.2) and make it of degree zero:

$$\bar{\Gamma}_{\mathcal{Z}}(z) \triangleq z^{-\frac{1}{2} \deg \bar{\Gamma}_{\mathcal{Z}}} \triangleq \sum \bar{\Gamma}_{\mathcal{Z}}^k \mathbf{1}_{\mathbf{p},k},$$

where the second equality defines $\bar{\Gamma}_{\mathcal{Z}}^k$ as the $\mathbf{1}_{\mathbf{p},k}$ -coefficient of $\bar{\Gamma}_{\mathcal{Z}}(z)$. With notation as in Section 3.1.1, we define

$$(3.2) \quad \bar{\mathcal{D}}_{\mathcal{Z}}^+(z; \vec{w})(\mathbf{1}_{\mathbf{p},k}) \triangleq \frac{\pi}{w_1(n+1) \sin(\pi(\frac{km_3}{n+1} - \frac{w_3}{z}))} \frac{1}{\bar{\Gamma}_{\mathcal{Z}}^k} \mathbf{1}_{\mathbf{p}}^k.$$

The dual basis of inertia components diagonalizes the tensor $\bar{\mathcal{D}}_{\mathcal{Z}}^+$.

Lemma 3.2. *The k th coefficient of $\bar{\mathcal{D}}_{\mathcal{Z}}^+$ coincides with $D_k^+(d; \vec{w})$ when $z = \frac{n_e w_1}{d}$ and the winding/twisting compatibility condition is met:*

$$\delta_{1, \exp(2\pi i(\frac{d}{n_e} - \frac{km_1}{n+1}))} \left(\bar{\mathcal{D}}_{\mathcal{Z}}^+ \left(\frac{n_e w_1}{d}; \vec{w} \right) (\mathbf{1}_{\mathbf{p},k}), \mathbf{1}_{\mathbf{p},k} \right)_{\mathcal{Z}} = D_k^+(d; \vec{w})$$

Proof. This formula follows from the explicit expression of $\bar{\Gamma}_{\mathcal{Z}}$ in the localization/inertia basis, manipulated via the identity

$$\Gamma(\star) \Gamma(1 - \star) = \frac{\pi}{\sin(\pi \star)}.$$

The Calabi–Yau condition $w_1 + w_2 + w_3 = 0$ is also used. The δ factor encodes the degree/twisting condition. \square

3.1.3. Open crepant resolutions. Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a diagram of toric Calabi–Yau threefolds for which the Coates–Iritani–Tseng/Ruan version of the closed crepant resolution conjecture holds. Choose a Lagrangian boundary condition $L_{\mathcal{X}}$ in \mathcal{X} and denote by L_Y the transform of such condition in Y ; notice that in general this can consist of several Lagrangian boundary conditions.

Proposition 3.3. *There exists a $\mathbb{C}((z^{-1}))$ -linear transformation*

$$\mathbb{O} : \mathcal{H}_{\mathcal{X}} \rightarrow \mathcal{H}_Y$$

of Givental spaces such that

$$\bar{\mathcal{D}}_Y^+ \circ \mathbb{U}_{\rho}^{\mathcal{X},Y} = \mathbb{O} \circ \bar{\mathcal{D}}_{\mathcal{X}}^+.$$

This proposition is trivial, as \mathbb{O} can be constructed as $\overline{\mathcal{D}}_Y^+ \circ \mathbb{U}_\rho^{\mathcal{X},Y} \circ (\overline{\mathcal{D}}_X^+)^{-1}$, where $(\overline{\mathcal{D}}_X^+)^{-1}$ denotes the inverse of $\overline{\mathcal{D}}_X^+$ after restricting to the basis of eigenvectors with non-trivial eigenvalues and \mathbb{O} is defined to be 0 away from these vectors. However, we observe that interesting open crepant resolution statements follow from this simple fact, and that \mathbb{O} is a simpler object than $\mathbb{U}_\rho^{\mathcal{X},Y}$. For a good reason: the disk function almost completely cancels the transcendental part in Iritani’s central charge. We make this precise in the following observation.

Lemma 3.4. *Referring to equations (2.2) and (3.2) for the relevant definitions, we have*

$$(3.3) \quad \Theta_Z(\mathbf{1}_{\mathbf{p},\mathbf{k}}) \triangleq \frac{w_1(n+1)}{z^{\frac{3}{2}}\pi} \overline{\mathcal{D}}_Z^+(\mathbf{1}_{\mathbf{p},\mathbf{k}} \otimes z^{-\mu} \overline{\Gamma}_Z) \mathbf{1}_{\mathbf{p}}^{\mathbf{k}} \\ = \frac{1}{\sin(\pi(\langle \frac{km_3}{n+1} \rangle - \frac{w_3}{z}))} \mathbf{1}_{\mathbf{p}}^{\mathbf{k}}.$$

Combining Lemma 3.4 with Iritani’s equation (2.4), we obtain the following prediction.

Conjecture 3.5. Choose bases for the equivariant CR cohomologies of \mathcal{X} and Y . Consider a set⁵⁾ \mathfrak{B} of equivariant bundles on Z that descend bijectively to bases for $K(\mathcal{X}) \otimes \mathbb{C}$ and $K(Y) \otimes \mathbb{C}$. For $\bullet = \mathcal{X}, Y$, let CH_\bullet denote the matrix of Chern characters in the chosen bases. Denote

$$\overline{\text{CH}}_\bullet = \left(\frac{2\pi i}{z}\right)^{\frac{1}{2} \text{deg}} \text{inv}^* \text{CH}_\bullet.$$

With Θ_\bullet be as in equation (3.3), we have

$$\mathbb{O} = \Theta_Y \circ \overline{\text{CH}}_Y \circ \overline{\text{CH}}_X^{-1} \circ \Theta_X^{-1}.$$

We verify Conjecture 3.5 for the resolution of A_n singularities in Sections 4 and 5. We also note that while we are formulating the statement in the case of cyclic isotropy to keep notation lighter, it is not hard to write an analogous prediction in a completely general toric setting.

3.2. The OCRC. Having modified our perspective on the disk functions, we also update our take on open disk invariants to remember the twisting of the map at the origin of the disk. In correlator notation, denote $\langle \tau, \dots, \tau \rangle_{0,n}^{L,d,k}$ the disk invariants with Lagrangian boundary condition L , winding d , twisting k and n cohomology insertions. We then define the *cohomological disk potential* as a cohomology-valued function, which is expressed as a composition of the J function with the disk function (3.2):

$$(3.4) \quad \mathcal{F}_L^{\text{disk}}(\tau, y, \vec{w}) \triangleq \sum_d \frac{y^d}{d!} \sum_n \frac{1}{n!} \langle \tau, \dots, \tau \rangle_{0,n}^{L,d,k} \mathbf{1}_{\mathbf{p}}^{\mathbf{k}}, \\ = \sum_d \delta_{1, \exp(2\pi i(\frac{d}{n_e} - \frac{km_1}{n+1}))} \frac{y^d}{d!} \overline{\mathcal{D}}_Z^+ \circ J^Z\left(\tau, \frac{n_e w_1}{d}, \vec{w}\right).$$

⁵⁾ In [72] such a set is called a *grade restriction window*. In the hypotheses and notation of Proposition 3.3, note that \mathcal{X} and Y must be related by variation of GIT, and therefore they are quotients of a common space $Z = \mathbb{C}^{l_Y+3}$; the grade restriction window may be chosen from the coordinate axes of Z , thought of as topologically trivial, but not equivariantly trivial, line bundles. See also [5] and [48].

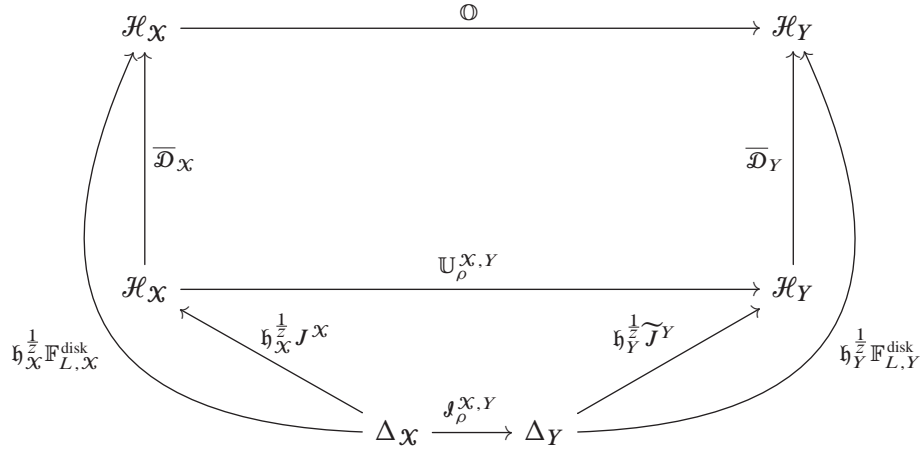


Figure 2. Open potential comparison diagram. In the Hard Lefschetz case this same diagram holds with the \hbar factors omitted, and Δ_\bullet identified with the full cohomologies of either target.

We define a section of Givental space that contains equivalent information to the disk potential:

$$\mathbb{F}_L^{\text{disk}}(t, z, \vec{w}) \triangleq \overline{\mathcal{D}}_Z^+ \circ J^Z(\tau, z; \vec{w}).$$

We call $\mathbb{F}_L^{\text{disk}}(t, z, \vec{w})$ the *winding neutral disk potential*. For any pair of integers k and d satisfying (3.1), the twisting k and winding d part of the disk potential is obtained by substituting $z = \frac{n_e w_1}{d}$. A general “disk crepant resolution statement” that follows from the closed CRC is a comparison of winding-neutral potentials, as illustrated in Figure 2.

Proposition 3.6. *Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a diagram for which the Coates–Iritani–Tseng/Ruan form of the closed crepant resolution conjecture holds and identify quantum parameters in $\Delta_{\mathcal{X}}$ and Δ_Y via $J_\rho^{\mathcal{X},Y}$ as in equation (2.5). Then*

$$\hbar_Y^{1/2} \mathbb{F}_{L,Y}^{\text{disk}}|_{\Delta_Y} = \hbar_X^{1/2} \mathbb{O} \circ \mathbb{F}_{L,\mathcal{X}}^{\text{disk}}|_{\Delta_{\mathcal{X}}}.$$

Assume further that \mathcal{X} satisfies the Hard Lefschetz condition and identify cohomologies via the affine linear change of variables $\widehat{J}_\rho^{\mathcal{X},Y}$. Then

$$\mathbb{F}_{L,Y}^{\text{disk}} = \mathbb{O} \circ \mathbb{F}_{L,\mathcal{X}}^{\text{disk}}.$$

Here we also understand that the winding-neutral disk potential of Y is analytically continued appropriately (we suppressed the tilde to avoid excessive proliferation of superscripts).

Remark 3.7. At the level of cohomological disk potentials, the normalization factors $\hbar_{\mathcal{X}}$ and \hbar_Y enter as a redefinition of the winding number variable y in (3.4) depending on small quantum cohomology parameters; this is the manifestation of the so-called *open mirror map* in the physics literature on open string mirror symmetry [3, 8, 11, 60].

Remark 3.8. The statement of the proposition in principle hinges on the possibility to identify quantum parameters as in (2.5)–(2.6). Restricting to the coordinate hyperplanes of the fundamental class insertions, the existence of the non-equivariant limits of $\mathbb{U}_\rho^{\mathcal{X},Y}$ and the J -functions is guaranteed by the fact that we employ a torus action acting trivially on the canonical bundle of \mathcal{X} and Y ; see e.g. [64].

3.3. The Hard Lefschetz OCRC. In the Hard Lefschetz case the comparison of disk potentials naturally extends to the full open potential. We define the *genus- g , ℓ -holes winding neutral potential*, a function from the space $H(\mathcal{Z})$ to the ℓ th tensor power of Givental space $\mathcal{H}_{\mathcal{Z}}^{\otimes \ell} = H(\mathcal{Z})((z_1^{-1})) \otimes \cdots \otimes H(\mathcal{Z})((z_\ell^{-1}))$:

$$\mathbb{F}_{\mathcal{Z},L}^{g,\ell}(\tau, z_1, \dots, z_\ell, \vec{w}) \triangleq \overline{\mathcal{D}}_{\mathcal{Z}}^{+\otimes \ell} \circ J_{g,\ell}^{\mathcal{Z}}(\tau, z_1, \dots, z_\ell; \vec{w}),$$

where $J_{g,\ell}^{\mathcal{Z}}$ encodes genus- g , ℓ -point descendent invariants:

$$(3.5) \quad J_{g,\ell}^{\mathcal{Z}}(\tau, \mathbf{z}; \vec{w}) \triangleq \left\langle \left\langle \frac{\phi_{\alpha_1}}{z_1 - \psi_1}, \dots, \frac{\phi_{\alpha_\ell}}{z_\ell - \psi_\ell} \right\rangle \right\rangle_{g,\ell} \phi^{\alpha_1} \otimes \cdots \otimes \phi^{\alpha_\ell}.$$

In (3.5), we denoted $\mathbf{z} = (z_1, \dots, z_\ell)$ and a sum over repeated Greek indices is intended. Just as in the disk case, one can now define a winding neutral open potential by summing over all genera g and integers ℓ and a cohomological open potential by introducing winding variables and summing over appropriate specializations of the z variables. For a pair of spaces \mathcal{X} and Y in a Hard Lefschetz CRC diagram the respective potentials can be compared as in Section 3.1 – this all follows from the comparison of the l -hole winding neutral potential, which we now spell out with care.

Theorem 3.9. *Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a Hard Lefschetz diagram for which the higher genus closed Crepant Resolution Conjecture holds (Conjecture 2.4). With all notation as in Proposition 3.6, and $\mathbb{O}^{\otimes \ell} = \mathbb{O}(z_1) \otimes \cdots \otimes \mathbb{O}(z_\ell)$, we have*

$$\mathbb{F}_{L,Y}^{g,\ell} = \mathbb{O}^{\otimes \ell} \circ \mathbb{F}_{L,\mathcal{X}}^{g,\ell}.$$

Proof. The generating function $J_{g,\ell}^{\mathcal{Z}}$ is obtained from the genus- g descendent potential by first ℓ applications of the total differential d , and then restricting to the small phase space variables $\tau = \{\tau^\alpha\} = \{t^{\alpha,0}\}$. Under the natural identification of the i th copy of $T^*\mathcal{H}_{\mathcal{Z}}^+ \cong \mathcal{H}_{\mathcal{Z}}$ with the auxiliary variable z_i , we have

$$(3.6) \quad dt^{\alpha,k} = \frac{\phi^\alpha}{z^{k+1}}.$$

Conjecture 2.4 gives us the equality of the Gromov–Witten partition functions (2.9) after a change of variable given by the linear identification

$$(3.7) \quad \pi_+ \circ \mathbb{U}_\rho^{\mathcal{X},Y} \circ i : \mathcal{H}_{\mathcal{X}}^+ \rightarrow \mathcal{H}_Y^+.$$

If we decompose the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X},Y}$ as a series in $\frac{1}{z}$ of linear maps, i.e. we have $\mathbb{U}_\rho^{\mathcal{X},Y} \triangleq \sum_{n \geq 0} \frac{1}{z^n} \mathbb{U}^n$, then differentiating the change of variable given by (3.7) gives

$$(3.8) \quad dt_Y^{\alpha,k} = \sum_{n=0}^k \mathbb{U}_{\alpha,\mu}^n dt_{\mathcal{X}}^{\mu,k+n},$$

where we denoted by $\mathbb{U}_{\alpha,\mu}^n$ the (α, μ) entry of the matrix representing \mathbb{U}^n after having chosen bases for the cohomologies of \mathcal{X} and Y . Combining (3.6) and (3.8),

$$(3.9) \quad \begin{aligned} \sum_{k=0}^{\infty} \frac{\phi^\alpha}{z^{k+1}} &= \sum_{k=0}^{\infty} dt_Y^{\alpha,k} = \sum_{k=0}^{\infty} \sum_{n=0}^k \mathbb{U}_{\alpha,\mu}^n dt_{\mathcal{X}}^{\mu,k+n} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^k \mathbb{U}_{\alpha,\mu}^n \frac{\phi^\mu}{z^{k+n+1}} = \mathbb{U}_\rho^{\mathcal{X},Y} \left(\sum_{k=0}^{\infty} \frac{\phi^\mu}{z^{k+1}} \right). \end{aligned}$$

Now we differentiate equation (2.9) ℓ times, and restrict to primary variables that are identified via (3.7): such identification reduces to $\widehat{\mathcal{J}}_{\rho}^{\mathcal{X},Y}$. Using (3.9), we obtain

$$J_{g,\ell}^Y = \mathbb{U}_{\rho}^{\mathcal{X},Y \otimes \ell} \circ J_{g,\ell}^{\mathcal{X}}.$$

The statement of the theorem follows by composing by $\overline{\mathcal{D}}_Y^{+\otimes \ell}$ and then using the commutativity of the diagram in Figure 2. \square

4. OCRC for A_n -resolutions

4.1. Equivariant $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ and integral structures. For the pairs

$$(\mathcal{X}, Y) = ([\mathbb{C}^3/\mathbb{Z}_{n+1}], A_n),$$

Proposition 3.6 and Theorem 3.9 imply a Bryan–Graber-type CRC statement comparing the open GW potentials. Notice that since \mathcal{X} is a Hard Lefschetz orbifold, we do not have to deal with the trivializing scalar factors \mathfrak{h}_{\bullet} . In Sections 4.2 and 4.3 we study the two essentially distinct types of Lagrangian boundary conditions.

The reader may find a detailed review of the toric geometry describing our targets in Section B.1, which is summarized by Figure 6. A generic Calabi–Yau torus action is taken on \mathcal{X} and Y , with weights as in Figure 6. We denote by ϕ_j , $j = 1, \dots, n$, the duals of the torus invariant lines $L_j \in H_2(Y)$ and by P_i , $i = 1, \dots, n+1$, the equivariant cohomology classes concentrated on the torus fixed points of the resolution. On the orbifold, we label by $\mathbf{1}_k$, $k = 1, \dots, n+1$, the fundamental classes of the components of the inertia stack $\mathcal{I}\mathcal{X}$ twisted by $e^{2\pi i k/(n+1)}$. A generic point $t \in H(Y)$ is written as $t^{n+1}\mathbf{1}_Y + \sum_j t^j \phi_j$; similarly, we write $x = \sum_{k=1}^{n+1} x^k \mathbf{1}_k$ for $x \in H(\mathcal{X})$.

Let now \mathcal{Y}_{ϵ} be the ball of radius ϵ around the large radius limit point of Y with respect to the Euclidean metric $(ds)^2 = \sum_i (de^{t_i})^2$ in exponentiated flat coordinates e^{t_j} . We define a path ρ in \mathcal{Y}_1 ,

$$(4.1) \quad \rho : [0, 1] \rightarrow \mathcal{Y}_1, \quad s \mapsto (\rho(s))_j = s\omega^{-j}.$$

as the straight line in the coordinates e^{t_j} connecting the large radius point $\text{LR} \triangleq \{e^{t_j} = 0\}$ of Y to the one of \mathcal{X} , which we denote as $\text{OP} \triangleq \{e^{t_j} = \omega^{-j}\}$.

Theorem 4.1. *Let $\widetilde{\mathcal{J}}^Y(z)$ denote the analytic continuation of J^Y along the path ρ to the point $\rho(1)$ composed with the identification (B.16) of quantum parameters. Then the linear transformation*

$$(4.2) \quad \mathbb{U}_{\rho}^{\mathcal{X},Y} \mathbf{1}_k = \sum_i P_i \frac{1}{(n+1)} \frac{\overline{\Gamma}_Y^i}{\overline{\Gamma}_{\mathcal{X}}^k} \left(\sum_{j=0}^{i-1} \omega^{-jk} e^{2\pi i \frac{j\alpha_1}{z}} + \sum_{j=i}^n \omega^{-jk} e^{2\pi i \frac{(n+1-j)\alpha_2}{z}} \right)$$

is an isomorphism of Givental spaces such that

$$\widetilde{\mathcal{J}}^Y = \mathbb{U}_{\rho}^{\mathcal{X},Y} \circ J^{\mathcal{X}}.$$

We prove Theorem 4.1 in the fully equivariant setting in Section 5.3.2 as an application of the one-dimensional mirror construction of Section 5.2.

This result is compatible with Iritani's equation (2.4). We now describe the canonical identification $\mathbb{U}_{K,\rho}^{\mathcal{X},Y}$. Denote by $\mathcal{O}(\lambda_k)$ the geometrically trivial line bundle on \mathbb{C}^{n+3} where the torus $(\mathbb{C}^*)^n$ acts via the k th factor with weight -1 and the torus T acts trivially. We define our grade restriction window $\mathfrak{W} \subset K(\mathbb{C}^{n+3})$ to be the subgroup generated by the $\mathcal{O}(\lambda_k)$. Using the description of the local coordinates in Section B.1, we compute that the quotient (B.2) identifies $\mathcal{O}(\lambda_k)$ with \mathcal{O}_{-k} (with trivial T -action) and the quotient (B.3) identifies $\mathcal{O}(\lambda_k)$ with $\mathcal{O}(\phi_k)$ (with canonical linearization (B.7)). Therefore, we define $\mathbb{U}_{K,\rho}^{\mathcal{X},Y}$ by identifying

$$\mathcal{O}_Y \leftrightarrow \mathcal{O}_{\mathcal{X}}, \quad \mathcal{O}(\phi_k) \leftrightarrow \mathcal{O}_{-k},$$

where the T -linearizations are trivial on the orbifold and canonical on the resolution.

On the orbifold, all of the bundles \mathcal{O}_j are linearized trivially, so the higher Chern classes vanish. The orbifold Chern characters are

$$(2\pi i)^{\frac{\deg}{2}} \text{inv}^* \text{ch}(\mathcal{O}_j) = \sum_{k=1}^{n+1} \omega^{-jk} \mathbf{1}_k.$$

The Γ class is

$$\begin{aligned} z^{-\frac{1}{2} \deg} \bar{\Gamma}_{\mathcal{X}} = \Gamma \left(1 + \frac{\alpha_1 + \alpha_2}{z} \right) & \left[\sum_{k=1}^n \Gamma \left(1 - \frac{k}{n+1} - \frac{\alpha_1}{z} \right) \Gamma \left(\frac{k}{n+1} - \frac{\alpha_2}{z} \right) \frac{\mathbf{1}_k}{z} \right. \\ & \left. + \Gamma \left(1 - \frac{\alpha_1}{z} \right) \Gamma \left(1 - \frac{\alpha_2}{z} \right) \mathbf{1}_{n+1} \right]. \end{aligned}$$

On the resolution, the Chern roots at each P_i are the weights of the action on the fiber above that point:

$$(2\pi i)^{\frac{\deg}{2}} \text{ch}(\mathcal{O}(\phi_j)) = \sum_{i=1}^j e^{2\pi i(n+1-j)\alpha_2} P_i + \sum_{i=j+1}^{n+1} e^{2\pi i j \alpha_1} P_i$$

and

$$(2\pi i)^{\frac{\deg}{2}} \text{ch}(\mathcal{O}) = \sum_{i=1}^{n+1} P_i.$$

The Γ class is

$$z^{-\frac{1}{2} \deg} \bar{\Gamma}_Y = \Gamma \left(1 + \frac{\alpha_1 + \alpha_2}{z} \right) \left[\sum_{i=1}^{n+1} \Gamma \left(1 + \frac{w_i^+}{z} \right) \Gamma \left(1 + \frac{w_i^-}{z} \right) P_i \right].$$

With this information one can compute the symplectomorphism as in equation (2.4) and obtain the formula in Theorem 4.1.

We now derive explicit disk potential CRC statements for the two distinct types of Lagrangian boundary conditions.

4.2. L intersects the ineffective axis. We impose a Lagrangian boundary condition on the gerby leg of the orbifold (the third coordinate axis $-m_3 = 0$); correspondingly there are $n+1$ boundary conditions L' on the resolution, intersecting the horizontal torus fixed lines in Figure 6.

Theorem 4.2. Consider the cohomological disk potentials $\mathcal{F}_{L',Y}^{\text{disk}}(t, y_{P^1}, \dots, y_{P^{n+1}}, \vec{w})$ and $\mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w})$. Choosing the dual bases $\mathbf{1}^k$ and P^i (where k and i both range from 1 to $n+1$), define a linear transformation $\mathbb{O}_{\mathbb{Z}} : H(\mathcal{X}) \rightarrow H(Y)$ by the matrix

$$\mathbb{O}_{\mathbb{Z}}^i = \begin{cases} -\omega^{\left(\frac{1}{2}-i\right)k}, & k \neq n+1, \\ -1, & k = n+1. \end{cases}$$

After the identification of variables from Theorem B.2, and the specialization of winding parameters

$$y_{P^i} = e^{\pi i \left[\frac{w_i^- + (2i-1)\alpha_1}{\alpha_1 + \alpha_2} \right]}_y$$

we have

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}).$$

Proof. From equation (3.2), we have

$$\overline{\mathcal{D}}_{L,\mathcal{X}}^+(z; \vec{w})(\mathbf{1}^k) = \sum_{k=1}^{n+1} \frac{\pi \mathbf{1}^k}{(n+1)(\alpha_1 + \alpha_2) \sin\left(\pi \left(\left\langle \frac{k}{n+1} \right\rangle + \frac{\alpha_2}{z}\right)\right) \overline{\Gamma}_{\mathcal{X}}^k}$$

and

$$\overline{\mathcal{D}}_{L',Y}^+(z; \vec{w})(P_i) = \sum_{i=1}^{n+1} \frac{\pi P^i}{(\alpha_1 + \alpha_2) \sin\left(\pi \left(-\frac{w_i^-}{z}\right)\right) \overline{\Gamma}_Y^i}.$$

The transformation \mathbb{O} is now obtained as $\overline{\mathcal{D}}_Y^+ \circ \mathbb{U}_{\rho}^{\mathcal{X},Y} \circ (\mathcal{D}_{\mathcal{X}}^+)^{-1}$:

$$\mathbb{O}(\mathbf{1}^k) = \sum_{i=1}^{n+1} \left[\frac{\sin\left(\pi \left(\left\langle \frac{k}{n+1} \right\rangle + \frac{\alpha_2}{z}\right)\right)}{\sin\left(\pi \left(-\frac{w_i^-}{z}\right)\right)} \left(\sum_{j=0}^{i-1} \omega^{-jk} e^{2\pi i \frac{j\alpha_1}{z}} + \sum_{j=i}^n \omega^{-jk} e^{2\pi i \frac{(n+1-j)\alpha_2}{z}} \right) \right] P^i.$$

We now specialize $z = \frac{\alpha_1 + \alpha_2}{d}$, for $d \in \mathbb{Z}$. The i, k coefficient for $k \neq n+1$, after some gymnastics with telescoping sums, becomes

$$\mathbb{O}_k^i = (-1) e^{d\pi i \left[n-i+2+(2i-n-2)\frac{\alpha_1}{\alpha_1+\alpha_2} \right]} \omega^{\left(\frac{1}{2}-i\right)k}.$$

For $k = n+1$,

$$\mathbb{O}_{n+1}^i = (-1) e^{d\pi i \left[n-i+2+(2i-n-2)\frac{\alpha_1}{\alpha_1+\alpha_2} \right]}.$$

It is now immediate to see that we can incorporate the part of the transformation that depends multiplicatively on d into a specialization of the winding variables, and that the remaining linear map is precisely $\mathbb{O}_{\mathbb{Z}}$. \square

From this formulation of the disk CRC one can deduce a statement about scalar disk potentials which essentially says that the scalar potential of the resolution compares with the untwisted disk potential on the orbifold.

Corollary 4.3. With all notation as in Theorem 4.2:

$$\left(\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}), \sum_{i=1}^{n+1} P_i \right)_Y = -\frac{1}{n+1} \left(\mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}), \mathbf{1}_{n+1} \right)_{\mathcal{X}}.$$

Proof. This statement amounts to the fact that the coefficients of all but the last column of the matrix $\mathbb{O}_{\mathbb{Z}}$ add to zero. \square

4.3. L intersects the effective axis. We impose our boundary condition L on the first coordinate axis, which is an effective quotient of \mathbb{C} with representation weight $m_1 = -1$ and torus weight $-\alpha_1$. We can obtain results for the boundary condition on the second axis by switching α_1 with α_2 , m_1 with m_2 and $+$ with $-$ in the orientation of the disks. In this case there is only one corresponding boundary condition L' on the resolution, which intersects the (diagonal) non-compact leg incident to P_{n+1} in Figure 6.

Theorem 4.4. *Consider the two cohomological disk potentials $\mathcal{F}_{L',Y}^{\text{disk}}(t, y_{P_{n+1}}, \vec{w})$ and $\mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w})$. Choosing the bases $\mathbf{1}^k$ and P^i (where k and i both range from 1 to $n+1$), define $\mathbb{O}_{\mathbb{Z}}(\mathbf{1}^k) = P^{n+1}$ for every k . After the identification of variables from Theorem B.2, and the identification of winding parameters $y = y_{P_{n+1}}$ we have*

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}).$$

We obtain as an immediate corollary a comparison among scalar potentials.

Corollary 4.5. *Setting $y = y_{P_{n+1}}$, we have*

$$F_{L',Y}^{\text{disk}}(t, y, \vec{w}) = F_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}).$$

Proof. The orbifold disk endomorphism is

$$\overline{\mathcal{D}}_{\mathcal{X}}^+(z; \vec{w})(\mathbf{1}^k) = \frac{\pi}{-\alpha_1(n+1) \sin\left(\pi\left(-\frac{\alpha_1+\alpha_2}{z}\right)\right) \overline{\Gamma}_{\mathcal{X}}^k} \mathbf{1}^k,$$

The resolution disk endomorphism is

$$\overline{\mathcal{D}}_{\mathcal{X}}^+(z; \vec{w})(P_i) = \frac{\pi}{-(n+1)\alpha_1 \sin\left(\pi\left(-\frac{\alpha_1+\alpha_2}{z}\right)\right) \overline{\Gamma}_Y^{n+1}} \delta_{i,n+1} P^{n+1}.$$

We can now compute \mathbb{O} :

$$\mathbb{O}(\mathbf{1}^k) = \frac{1}{n+1} \left(\sum_{j=0}^n \omega^{-jk} e^{2\pi i \frac{j\alpha_1}{z}} \right) P^{n+i}.$$

Specializing $z = \frac{-(n+1)\alpha_1}{d}$ for any positive integer d , we obtain

$$\mathbb{O}_k^{n+1} = \frac{1}{n+1} \sum_{j=0}^n \omega^{-jk} e^{2\pi i j \frac{-d}{n+1}} = \delta_{k, -d \bmod n+1},$$

which implies the statement of the theorem. \square

5. One-dimensional mirror symmetry

It is known that the quantum D -modules associated to the equivariant Gromov–Witten theory of the A_n -singularity \mathcal{X} and its resolution Y admit a Landau–Ginzburg description in terms of n -dimensional oscillating integrals [6, 28, 42, 49]. We provide here an alternative description via one-dimensional twisted periods of a genus-zero double Hurwitz space $\mathcal{F}_{\lambda, \phi}$.

5.1. Weak Frobenius structures on double Hurwitz spaces.

Definition 5.1. Let $\vec{x} \in \mathbb{Z}^{n+3}$ be a vector of integers adding to 0. The genus-zero double Hurwitz space $\mathcal{H}_{\vec{x}} \triangleq \mathcal{M}_0(\mathbb{P}^1; \mathbf{x})$ parameterizes isomorphism classes of covers λ of the projective line by a smooth genus-zero curve C , with marked ramification profile over 0 and ∞ specified by \vec{x} . This means that the principal divisor of λ is of the form

$$(\lambda) = \sum x_i q_i.$$

We denote by π and λ the universal family and universal map, and by Σ_i the sections marking the i th point in (λ) :

$$\begin{array}{ccc} \mathbb{P}^1 & \hookrightarrow & \mathcal{U} \xrightarrow{\lambda} \mathbb{P}^1 \\ \left(\begin{array}{c} \downarrow \\ P_i \end{array} \right) & & \left(\begin{array}{c} \downarrow \\ \Sigma_i \end{array} \right) \pi \\ [\lambda] & \xrightarrow{\text{pt}} & \mathcal{H}_{\vec{x}}. \end{array}$$

A genus-zero double Hurwitz space is naturally isomorphic to $\mathbb{C}^* \times M_{0,n+3}$, and is therefore an open set in affine space \mathbb{A}^{n+1} . The genus-zero case is the only case we consider in this paper and it may seem overly sophisticated to use the language of moduli spaces to work on such a simple object: we choose to do so in order to connect to the work of Dubrovin [35,36] and Romano [67] (after Saito [71]; see also [57]), who studied existence and construction of Frobenius structures on double Hurwitz spaces for arbitrary genus.

Write $\text{supp}(\lambda) = \{q_i \in C\}_i$ for the set of points (λ) is supported on; let $\phi \in \Omega_{\mathbb{C}}^1(\log(\lambda))$ be a meromorphic one-form having simple poles at $\text{supp}(\lambda)$ with constant residues. We call (λ, ϕ) respectively the *superpotential* and the *primitive differential* of $\mathcal{H}_{\vec{x}}$. Borrowing the terminology from [67, 68], we say that an analytic Frobenius manifold structure $(\mathcal{F}, \circ, \eta)$ on a complex manifold \mathcal{F} is weak if

- (1) the \circ -multiplication gives a commutative and associative unital \mathcal{O} -algebra structure on the space of holomorphic vector fields on \mathcal{F} ,
- (2) the metric η provides a flat pairing which is Frobenius with respect to \circ ,
- (3) the algebra structure admits a potential, meaning that the 3-tensor

$$R(X, Y, Z) \triangleq \eta(X, Y \circ Z)$$

satisfies the integrability condition

$$(\nabla^{(\eta)} R)_{[\alpha\beta]\gamma\delta} = 0.$$

In particular, this encompasses non-quasihomogeneous solutions of WDVV, and solutions without a flat identity element.

By choosing the last three sections to be the constant sections 0, 1, ∞ , we can realize $\mathcal{H}_{\vec{x}} \simeq \mathbb{C}^* \times M_{0,n+3}$ as an open subset of \mathbb{A}^{n+1} and trivialize the universal family. In homogeneous coordinates $[u_0 : \dots : u_n]$ for \mathbb{P}^n ,

$$(5.1) \quad \begin{aligned} \mathcal{H}_{\vec{x}} &= \mathbb{C}^* \times \mathbb{P}^n \setminus \text{discr } \mathcal{H}_{\vec{x}}, \\ \text{discr } \mathcal{H}_{\vec{x}} &\triangleq \text{Proj} \frac{\mathbb{C}[u_0, \dots, u_n]}{\langle \prod_{i=0}^n u_i \prod_{j < k} (u_j - u_k) \rangle}. \end{aligned}$$

Theorem 5.1 ([38, 67]). *For vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{H}_{\tilde{x}})$, define the non-degenerate symmetric pairing g and quantum product \star as*

$$(5.2) \quad g(X, Y) \triangleq \sum_{P \in \text{supp}(\lambda)} \text{Res}_P \frac{X(\log \lambda)Y(\log \lambda)}{d_\pi \log \lambda} \phi^2,$$

$$(5.3) \quad g(X, Y \star Z) \triangleq \sum_{P \in \text{supp}(\lambda)} \text{Res}_P \frac{X(\log \lambda)Y(\log \lambda)Z(\log \lambda)}{d_\pi \log \lambda} \phi^2,$$

where d_π denotes the relative differential with respect to the universal family (i.e. the differential in the fiber direction). Then the triple $\mathcal{F}_{\lambda, \phi} = (\mathcal{H}_{\tilde{x}}, \star, g)$ endows $\mathcal{H}_{\tilde{x}}$ with a holomorphic weak Frobenius manifold structure. The embedding $\mathcal{H}_{\tilde{x}} \hookrightarrow \mathbb{C}^\star \times \mathbb{P}^n$ induces uniquely a meromorphic weak Frobenius structure on $\mathbb{P}^1 \times \mathbb{P}^n$.

Equations (5.2)–(5.3) are the Dijkgraaf–Verlinde–Verlinde formulae [33] for a topological Landau–Ginzburg model on a sphere with $\log \lambda(q)$ as its superpotential. The case in which $\lambda(q)$ itself is used as the superpotential gives rise to a *different* Frobenius manifold structure, which is the case originally analyzed by Dubrovin in his study of Frobenius structures on Hurwitz spaces [36, Lecture 5]; the situation at hand is its dual in the sense of [38], where g plays the role of the intersection form and \star the dual product, whose poles coincide with the discriminant ideal in the Zariski closure (5.1) of $\mathcal{H}_{\tilde{x}}$.

Remark 5.2. Since λ is a genus-zero covering map, in an affine chart parametrized by $q \in \mathbb{C}$ its logarithm takes the form

$$(5.4) \quad \log \lambda = \sum_i x_i \log(q - q_i) + y,$$

where $x_i, y \in \mathbb{Z}$. In fact, the existence of the weak Frobenius structure (5.2)–(5.3) carries through unscathed [68] to the case where $d_\pi \log \lambda$ is a meromorphic differential on C upon identifying $\text{supp}(\lambda) = \{q_i\}$; this in particular encompasses the case where $x_i, y \in \mathbb{C}$ in equation (5.4). The locations q_i of the punctures provide a special type of local coordinates on $\mathcal{H}_{\tilde{x}}$: by the general theory of double Hurwitz spaces [67], for suitable choices of ϕ their logarithms are flat coordinates for the pairing g in equation (5.2).

5.1.1. Twisted homology and the quantum differential equation. Let

$$C_\lambda \triangleq C \setminus \text{supp}(\lambda)$$

and denote by $\pi : \tilde{C}_\lambda \rightarrow C_\lambda$ its universal covering space. Fix $z \in \mathbb{C}$ and pick the canonical principal branch for $\lambda^{\frac{1}{z}} = \exp(z^{-1} \log \lambda)$ in equation (5.4), defined as

$$\lambda^{\frac{1}{z}}(q) = \prod_{i=1} |q - q_i|^{\xi_i} e^{i \xi_i \arg_i(q)},$$

where $\xi_i := \frac{x_i}{z}$ and $\arg_i(q) \in [0, 2\pi)$ is the angle formed by $q - q_i$ with the real axis. Then we have a monodromy representation

$$\rho_\lambda : \pi_1(C_\lambda) \rightarrow L_\lambda \simeq \mathbb{C}$$

on the complex line L_λ parametrized by $\lambda^{\frac{1}{z}}$, a simple loop l_{q_i} around q_i resulting in multiplication by

$$q_i := \rho_\lambda(l_{q_i}) = e^{2\pi i \sum_{j=1}^i \xi_j}.$$

We denote by $H_\bullet(C_\lambda, L_\lambda)$ (resp. $H^\bullet(C_\lambda, L_\lambda)$) the homology (resp. cohomology) groups of C_λ twisted by the set of local coefficients determined by q_i . Integration over $\gamma \in H_1(C_\lambda, L_\lambda)$ of $\lambda^{\frac{1}{2}}\phi \in H^1(C_\lambda, L_\lambda)$ defines the *twisted period mapping*

$$(5.5) \quad \pi_{\lambda, \phi} : H_1(C_\lambda, L_\lambda) \rightarrow \mathcal{O}(\mathcal{H}_{\vec{x}}), \quad \gamma \mapsto \int_\gamma \lambda^{\frac{1}{2}}\phi.$$

Let now $\nabla^{(g,z)} : \mathcal{X}(\mathcal{H}_{\vec{x}}) \rightarrow \mathcal{X}(\mathcal{H}_{\vec{x}}) \otimes \Omega^1(\mathcal{H}_{\vec{x}})$ be the Dubrovin connection associated to $\mathcal{F}_{\lambda, \phi}$

$$(5.6) \quad \nabla_X^{(g,z)}(Y, z) \triangleq \nabla_X^{(g)}Y + z^{-1}X \star Y$$

and write $\text{Sol}_{\lambda, \phi}$ for its $\mathbb{C}(\xi_1, \dots, \xi_n)$ -vector space of parallel sections

$$\text{Sol}_{\lambda, \phi} = \{s \in \mathcal{X}(\mathcal{H}_{\vec{x}}) : \nabla^{(g,z)}s = 0\}.$$

The following statement [38] is a *verbatim* application of the arguments of [37] for the ordinary Hurwitz case.

Proposition 5.3. *The solution space of the quantum differential equations of $\mathcal{F}_{\lambda, \phi}$ is generated by gradients of the twisted periods (5.5)*

$$\text{Sol}_{\lambda, \phi} = \text{span}_{\mathbb{C}(a_1, \dots, a_n)} \{\text{grad}_g \pi_{\lambda, \phi}(\gamma)\}_{\gamma \in H_1(C_\lambda, L_\lambda)}.$$

In other words, twisted periods are a flat coordinate frame for the Dubrovin connection on $T\mathcal{F}_{\lambda, \phi}$.

5.2. A one-dimensional Landau–Ginzburg mirror. We now fix the ramification profile

$$\vec{x} = ((n+1)\alpha_1, -\alpha_1 - \alpha_2, (n+1)\alpha_2, \underbrace{-\alpha_1 - \alpha_2, \dots, -\alpha_1 - \alpha_2}_n).$$

Define $\mathcal{M}_A \triangleq \mathcal{H}_{\vec{x}}$. We pick global coordinates on it as follows: we write κ_0 for an (exponentiated-linear) coordinate in the first factor of $\mathcal{M}_A \simeq \mathbb{C}^\star \times M_{0, n+3}$, and we pick $\kappa_i = \frac{u_i}{u_0}$, $i = 1, \dots, n$, as a set of global coordinates on $M_{0, n+3}$. As before, we write q to denote an affine coordinate on the fibers of the universal family. We give \mathcal{M}_A the structure of a one-parameter family of double Hurwitz spaces as follows:

$$(5.7) \quad \lambda(\kappa_0, \dots, \kappa_n, q) = \prod_{j=0}^n \kappa_j^{\alpha_1} \frac{q^{(n+1)\alpha_1}}{(1-q)^{\alpha_1 + \alpha_2} \prod_{k=1}^n (1 - q\kappa_k)^{\alpha_1 + \alpha_2}},$$

$$(5.8) \quad \phi(q) = \frac{1}{\alpha_1 + \alpha_2} \frac{dq}{q}.$$

The Frobenius structure on \mathcal{M}_A determined by equations (5.2), (5.3), (5.7), (5.8) is denoted by $\mathcal{F}_{\lambda, \phi}$. By Remark 5.2, and since both the metric and the associative product in equations (5.2), (5.3), (5.7), (5.8) depend rationally on (α_1, α_2) , we will in the following consider them as complex parameters.

We claim that there exist neighborhoods $V_{\mathcal{X}}, V_Y \subset \mathcal{M}_A$ such that $\mathcal{F}_{\lambda, \phi}$ is locally isomorphic to the quantum cohomologies of $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{n+1}]$ and its canonical resolution Y . The ultimate justification of this statement resides in the relation of the Gromov–Witten theory of \mathcal{X} and Y with integrable systems, and notably the two-dimensional Toda hierarchy; the details of this connection can be found in [12]. For the purposes of this paper, it is enough to offer a direct proof of the existence of said local isomorphisms.

Theorem 5.4. *The following statements hold.*

(1) *With notation as at the beginning of Section 4.1, let*

$$(5.9) \quad \kappa_0 = e^{\frac{t_{n+1} + \delta_Y}{\alpha_1}},$$

$$(5.10) \quad \kappa_j = \prod_{i=j}^n e^{t_i}, \quad 1 \leq j \leq n,$$

where δ_Y is an arbitrary constant. Then, in a neighborhood V_Y of $\text{LR} = \{e^{t_i} = 0\}$,

$$\mathcal{F}_{\lambda, \phi} \simeq QH(Y).$$

(2) *Let*

$$(5.11) \quad \kappa_0 = e^{\frac{x_{n+1} + \delta_X}{\alpha_1}},$$

$$(5.12) \quad \kappa_j = \exp \left[-\frac{2i}{n+1} \left(\pi j + \sum_{k=1}^n e^{-\frac{i\pi k(j-1)}{n+1}} \sin \left(\frac{\pi j k}{n+1} \right) x_k \right) \right], \quad 1 \leq k \leq n,$$

where δ_X is an arbitrary constant. Then, in a neighborhood V_X of $\text{OP} = \{x_i = 0\}$,

$$\mathcal{F}_{\lambda, \phi} \simeq QH(X).$$

Proof. The proof of the theorem is a direct computation from the Landau–Ginzburg formulae (5.2)–(5.3).

(i) Consider the three-point correlator $R(\kappa_i \partial_i, \kappa_j \partial_j, \kappa_k \partial_k)$, where $\partial_k \triangleq \frac{\partial}{\partial \kappa_k}$, and define

$$R_{i,j,k}^{(l)} \triangleq \text{Res}_{q=\kappa_l^{-1}} \frac{\kappa_i \frac{\partial \ln \lambda}{\partial \kappa_i} \kappa_j \frac{\partial \ln \lambda}{\partial \kappa_j} \kappa_k \frac{\partial \ln \lambda}{\partial \kappa_k} dq}{(\alpha_1 + \alpha_2)^2 q \frac{\partial \ln \lambda}{\partial q}}.$$

Inspection shows that $R_{ijk}^{(l)} = 0$ unless $l = i = j$, $l = i = k$ or $l = j = k$. Assume without loss of generality $l = j = i$, and suppose that $i, k > 0$. We compute

$$(5.13) \quad R_{i,i,k}^{(i)} = \frac{\kappa_i}{\kappa_k - \kappa_i} + \frac{\alpha_2}{\alpha_1 + \alpha_2},$$

$$(5.14) \quad R_{i,i,i}^{(i)} = \frac{(n-1)\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2} + \sum_{l \neq i}^{n+1} \frac{\kappa_l}{\kappa_i - \kappa_l}, \quad R_{0,i,i}^{(i)} = -\frac{1}{\alpha_1 + \alpha_2}.$$

Moreover, for all i, j and k we have

$$(5.15) \quad R_{i,j,k}^{(0)} \triangleq \text{Res}_{q=0} \frac{\kappa_i \frac{\partial \ln \lambda}{\partial \kappa_i} \kappa_j \frac{\partial \ln \lambda}{\partial \kappa_j} \kappa_k \frac{\partial \ln \lambda}{\partial \kappa_k} dq}{(\alpha_1 + \alpha_2)^2 q \frac{\partial \ln \lambda}{\partial q}} = \frac{\alpha_1^{2-\delta_{i,n+1}-\delta_{j,n+1}-\delta_{k,n+1}}}{(n+1)(\alpha_1 + \alpha_2)^2},$$

$$(5.16) \quad R_{i,j,k}^{(\infty)} \triangleq \text{Res}_{q=\infty} \frac{\kappa_i \frac{\partial \ln \lambda}{\partial \kappa_i} \kappa_j \frac{\partial \ln \lambda}{\partial \kappa_j} \kappa_k \frac{\partial \ln \lambda}{\partial \kappa_k} dq}{(\alpha_1 + \alpha_2)^2 q \frac{\partial \ln \lambda}{\partial q}} = -\frac{(-\alpha_2)^{2-\delta_{i,n+1}-\delta_{j,n+1}-\delta_{k,n+1}}}{(n+1)(\alpha_1 + \alpha_2)^2}.$$

It is immediate to see that (5.13)–(5.14) under the identification (5.10) imply that the quantum part of the three-point correlator $R(\partial_{t_1} \partial_{t_2} \partial_{t_3})$ coincides with that of $\langle\langle \phi_{i_1}, \phi_{i_2}, \phi_{i_3} \rangle\rangle_0^Y$ in equation (B.15). A tedious, but straightforward computation shows that (5.13)–(5.16) yield the expressions (B.10)–(B.13) for the classical triple intersection numbers of Y .

(ii) This part is obtained by composing the computation above with the Coates–Corti–Iritani–Tseng isomorphism of quantum cohomologies (Theorem B.2). \square

Remark 5.5. The freedom of shift by $\delta_{\mathcal{X}}$ and δ_Y respectively along $H^0(\mathcal{X})$ and $H^0(Y)$ in equations (5.9) and (5.11) is a consequence of the restriction of the String Axiom to the small phase space. We set $\delta_{\mathcal{X}} = \delta_Y = 0$ throughout this section, but it will be useful to reinstate the shifts in the computations of Section 5.3.2.

5.3. The global quantum D -module. Theorems 5.4 and 5.3 together imply the existence of a global quantum D -module $(\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_g)$ interpolating between $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$. Let $F \triangleq T\mathcal{F}_{\lambda, \phi}$ be endowed with the family of connections $\nabla^{(g, z)}$, as in equation (5.6) and for $\nabla^{(g, z)}$ -flat sections s_1, s_2 define

$$H(s_1, s_2)_g = g(s_1(\kappa, -z), s_2(\kappa, z)).$$

With notation as in Section 4.1, let $V_{\mathcal{X}}$ and V_Y be neighborhoods of OP and LR, respectively. Then Theorems 5.4 and 5.3 imply that

$$\begin{aligned} (\mathcal{F}_{\lambda, \phi}, T\mathcal{F}_{\lambda, \phi}, \nabla^{(g, z)}, H(\cdot, \cdot)_g)|_{V_{\mathcal{X}}} &\simeq \text{QDM}(\mathcal{X}), \\ (\mathcal{F}_{\lambda, \phi}, T\mathcal{F}_{\lambda, \phi}, \nabla^{(g, z)}, H(\cdot, \cdot)_g)|_{V_Y} &\simeq \text{QDM}(Y). \end{aligned}$$

In particular, choosing a basis of integral twisted 1-cycles yields a global flat frame for the quantum differential equations of \mathcal{X} and Y upon analytic continuation in the κ -variables, and

$$(5.17) \quad \text{Sol}_{\lambda, \phi}|_{V_{\mathcal{X}}} = \mathfrak{S}_{\mathcal{X}}, \quad \text{Sol}_{\lambda, \phi}|_{V_Y} = \mathfrak{S}_Y.$$

Representatives of one such integral basis can be constructed as follows. For generic monodromy weights, the monodromy representation $\rho_{\lambda}(l_{q_i})$ factors through a faithful representation $\tilde{\rho}: H_1(C_{\lambda}, \mathbb{Z}) \rightarrow V_{\lambda}$. Then in this case the twisted homology coincides with the integral homology of the Riemannian covering [78] of C_{λ}

$$H^{\bullet}(C_{\lambda}, L_{\lambda}) \simeq H^{\bullet}(\tilde{C}_{\lambda}/[\pi_1(C_{\lambda}), \pi_1(C_{\lambda})], \mathbb{Z}).$$

In particular, compact loops in the kernel of the abelianization morphism

$$h_*: \pi_1(C_{\lambda}) \rightarrow H_1(C_{\lambda}, \mathbb{Z})$$

may have non-trivial lifts to $H_1(C_{\lambda}, L_{\lambda})$. One such basis is given explicitly [78, 79] by the Pochhammer double loop contours $\{\gamma_i\}_{i=1}^{n+1}$: these are compact loops encircling the points $q = 0$ and $q = \kappa_i^{-1}$, $i = 1, \dots, n+1$, as in Figure 3 (that is $\gamma_i = [l_0, l_{\kappa_i^{-1}}]$, where the l_q are simple oriented loops around each of the punctures). Then the twisted periods

$$(5.18) \quad \Pi_i \triangleq \frac{za\pi_{\lambda, \phi}(\gamma_i)}{(1 - e^{2\pi ia})(1 - e^{-2\pi ib})},$$

where we defined

$$\begin{aligned} a &\triangleq \frac{(n+1)\alpha_1}{z}, & \mathfrak{q}_a &\triangleq e^{2\pi ia}, \\ b &\triangleq \frac{\alpha_1 + \alpha_2}{z}, & \mathfrak{q}_b &\triangleq e^{2\pi ib}, \end{aligned}$$

are a $\mathbb{C}(\mathfrak{q}_a, \mathfrak{q}_b)$ -basis of $\text{Sol}_{\lambda, \phi}$.

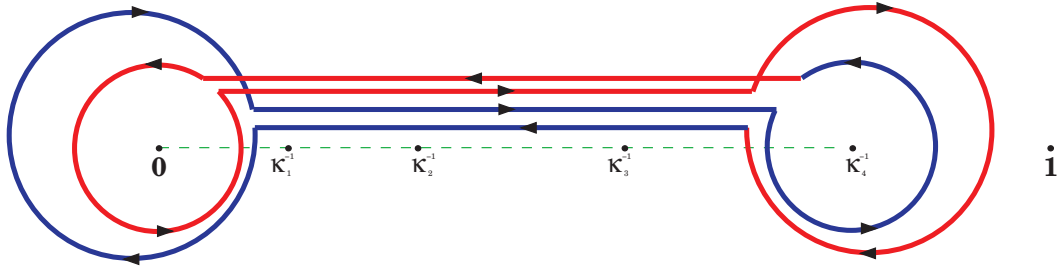


Figure 3. The double loop contour γ_4 for $n = 4$.

Remark 5.6. We have a natural isomorphism with the homology of the complex line relative to the punctures

$$\mathfrak{P} : H_1(C_\lambda, L_\lambda) \xrightarrow{\sim} H_1(\mathbb{P}^1, \text{supp}(\lambda)), \quad \gamma_i \mapsto (1 - \alpha_a)(1 - \alpha_b)[0, \kappa_i^{-1}]$$

obtained by associating to any Pochhammer contour the path in C_λ that it encircles. The choice of coefficient reflects the existence [79], when $\Re e(a) > 0, \Re e(b) < 1$, of an Euler-type integral representation: namely, a factorization of the period mapping

$$(5.19) \quad \begin{array}{ccc} H_1(C_\lambda, L_\lambda) & & \\ \mathfrak{P} \downarrow & \searrow \pi_{\lambda, \phi} & \\ H_1(\mathbb{P}^1, (\lambda)) & \xrightarrow{f \lambda^{\frac{1}{2}} \phi} & \mathbb{C}(\alpha_a, \alpha_b) \end{array}$$

which reduces (5.18) to convergent line integrals of $z a \lambda^{\frac{1}{2}} \phi$ over the interval $\mathfrak{P}(\gamma_i)$.

By the above remark, the period integrals of equation (5.18) are a multi-variable generalization of the classical Euler representation for the Gauss hypergeometric function. Explicitly, they take the form [39]

$$(5.20) \quad \begin{aligned} \Pi_i(\kappa, z) &= \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \kappa_i^{-a} \prod_{j=0}^n \kappa_j^{\frac{\alpha_j}{z}} \\ &\times \Phi^{(n)}\left(a, b, 1+a-b; \frac{\kappa_1}{\kappa_i}, \dots, \frac{\kappa_{i-1}}{\kappa_i}, \frac{1}{\kappa_i}, \frac{\kappa_{i+1}}{\kappa_i}, \dots, \frac{\kappa_n}{\kappa_i}\right) \end{aligned}$$

for $1 \leq i \leq n$ and

$$(5.21) \quad \Pi_{n+1}(\kappa, z) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \left(\prod_{j=0}^n \kappa_j^{\frac{\alpha_j}{z}} \right) \Phi^{(n)}(a, b, 1+a-b; \kappa_1, \dots, \kappa_n),$$

where we defined

$$(5.22) \quad \Phi^{(m)}(a, b, c, w_1, \dots, w_m) \triangleq F_D^{(m)}(a; b, \dots, b; c; w_1, \dots, w_m),$$

and $F_D^{(m)}(a; b_1, \dots, b_m; c; w_1, \dots, w_m)$ in equation (5.22) is the Lauricella function of type D (cf. [58]):

$$(5.23) \quad F_D^{(m)}(a; b_1, \dots, b_m; c; w_1, \dots, w_m) \triangleq \sum_{i_1, \dots, i_m} \frac{(a)_{\sum_j i_j}}{(c)_{\sum_j i_j}} \prod_{j=1}^m \frac{(b_j)_{i_j} w_j^{i_j}}{i_j!}.$$

In (5.23), we used the Pochhammer symbol $(x)_m$ to denote the ratio

$$(x)_m = \frac{\Gamma(x+m)}{\Gamma(x)}.$$

5.3.1. Example: $n = 1$ and the Gauss system. In this case $\tilde{\mathcal{F}}_{\lambda,\phi}$ has dimension 2. The equations for the flat coordinates $\tilde{t}(\kappa_0, \kappa_1, z)$ of the Dubrovin connection, equation (5.6), reduce to the classical Gauss hypergeometric system for a function $f(\kappa_1, z)$ such that

$$(5.24) \quad \tilde{t}(\kappa_0, \kappa_1, z) = (\kappa_0 \kappa_1)^{-\frac{a}{2}} f(\kappa_1, z),$$

where

$$(5.25) \quad \kappa_1(\theta+a)(\theta+b)f = \theta(\theta+a-b)f, \quad \theta = \kappa_1 \partial_{\kappa_1}.$$

When $n = 1$, we have from (5.20)–(5.21) that

$$\begin{aligned} \Pi_1(\kappa, z) &= \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \kappa_0^{\frac{a}{2}} \kappa_1^{-\frac{a}{2}} {}_2F_1\left(a, b, 1+a-b, \frac{1}{\kappa_1}\right), \\ \Pi_2(\kappa, z) &= \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} (\kappa_0 \kappa_1)^{\frac{a}{2}} {}_2F_1(a, b, 1+a-b, \kappa_1). \end{aligned}$$

These are immediately seen to satisfy (5.24)–(5.25).

Remark 5.7. Equivariant mirror symmetry for toric Deligne–Mumford stacks implies that flat sections of $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$ take the form of generalized hypergeometric functions in so-called B -model variables; see [24, Appendix A] for the case under study here. Less expected, however, is the fact that they are hypergeometric functions in *exponentiated flat variables* for the Poincaré pairing, that is, in A -model variables. This is a consequence of the particular form (equations (B.15), (5.13), (5.14)) of the quantum product: its rational dependence⁶⁾ on the variables κ gives the quantum differential (A.8) the form of a generalized hypergeometric system in exponentiated flat coordinates. The explicit equivalence between twisted periods and solutions of the Picard–Fuchs equations of \mathcal{X} and Y , which is a consequence of Theorem 5.4 here and [24, Proposition A.3], should follow by comparing⁷⁾ the respective Barnes integral representations [7, 39]. A significant advantage of the Hurwitz-space picture is that sections of the quantum D -modules have one-dimensional integral representations, as opposed to the n -fold Mellin–Barnes integrals of [7]; this drastically reduces the complexity of computing the analytic continuation from the large radius to the orbifold chamber, as we now show.

We are almost ready to compute the analytic continuation map $\mathbb{U}_\rho^{\mathcal{X}, Y} : \mathcal{H}_\mathcal{X} \rightarrow \mathcal{H}_Y$ that identifies the corresponding flat frames and Lagrangian cones upon analytic continuation along the path ρ in (4.1). The main missing technical tool is provided by the following lemma.

⁶⁾ From the vantage point of mirror symmetry, the rational dependence of the A -model three-point correlators on the quantum parameters is an epiphenomenon of the Hard Lefschetz condition, which ensures that the inverse mirror map is a rational function of exponentiated A -model variables.

⁷⁾ Equivalence between the two types of hypergeometric functions can be derived from the quadratic transformations for the Gauss function for $n = 1$, and from a generalized Bayley identity for $n = 2$; the higher rank case appears to be non-trivial.

Lemma 5.8. *In \mathbb{C}^m with coordinates (w_1, \dots, w_m) , where $m \geq 1$, let χ_i , for every $i = 1, \dots, m$, be any path in $\mathbb{C}^m \setminus \{w_k \neq w_l, 0, 1\}$, up to homotopy, that connects the origin with the point at infinity W_i^∞ ,*

$$W_i^\infty \triangleq (\overbrace{0, \dots, 0}^{i-1 \text{ times}}, \overbrace{\infty, \dots, \infty}^{m-i+1 \text{ times}}),$$

and has zero winding number along the hyperplanes $w_k = w_l$ ($k \neq l$) and $w_k = 0, 1$. Denote $\tilde{\Phi}_i^{(m)}(a, b, c; w_1, \dots, w_m)$ the analytic continuation of $\Phi^{(m)}(a, b, c; w_1, \dots, w_m)$ in (5.22) along χ_i to the neighborhood

$$(5.26) \quad \begin{aligned} |w_l| < 1, & \quad l < i, \\ |w_l^{-1}| < 1, & \quad l = i, \\ |w_l^{-1}| < 1, \quad |w_l| < |w_k|, & \quad l > k \geq i, \end{aligned}$$

of W_i^∞ . Then we have

$$(5.27) \quad \begin{aligned} & \tilde{\Phi}_i^{(m)}(a, b, c; w_1, \dots, w_m) \\ & \sim \sum_{j=0}^{m-i} \frac{\Gamma(c)\Gamma(a-jb)\Gamma((j+1)b-a)}{\Gamma(a)\Gamma(b)\Gamma(c-a)} \\ & \quad \times \prod_{k=1}^j (-w_{m-k+1})^{-b} (-w_{m-j})^{-a+jb} (1 + \mathcal{O}(w)) \\ & \quad + \prod_{j=i}^m (-w_j)^{-b} \frac{\Gamma(c)\Gamma(a-(m-i+1)b)}{\Gamma(a)\Gamma(c-(m-i+1)b)} (1 + \mathcal{O}(w)) \end{aligned}$$

around W_i^∞ in the region of (5.26).

Proof. The statement of the lemma follows from computing the analytic continuation along χ_i of the Lauricella function $F_D^{(m)}(a, b_1, \dots, b_m, c, w_1, \dots, w_{i-1}, w_i^{-1}, \dots, w_m^{-1})$ from an open ball centered on W_i^∞ to the origin $W_{m+1}^\infty = (0, \dots, 0)$ in the sector where $w_k \ll 1$ for $k < i$, $w_i \ll 1$, $w_k/w_j \ll 1$ for $k > j \geq i$. One possible way to do this is to perform the continuation in each individual variable w_j , $j > i$ appearing in equation (5.23) through an iterated use of Kummer's identity, equation (C.2). This is done in Section C, to which we refer the reader for the details of the derivation; the final result is equation (C.5), from which equation (5.27) follows by equation (5.22). \square

5.3.2. Proof of Theorem 4.1. We recall here the notation we used in Sections 4.1 and 5.3: we write P_i for the equivariant class concentrated on the i th-fixed point of Y , $\mathbf{1}_k$ for the fundamental class of the k th-twisted sector of \mathcal{X} , $i, k = 1, \dots, n+1$, and V_Y and $V_{\mathcal{X}}$ for the neighborhoods of the large radius point (LR) and the orbifold point (OP), respectively, such that the isomorphisms of equation (5.17) hold. We also let ρ be the path in $QH(Y) \simeq QH(\mathcal{X})$ connecting the large radius point LR to the orbifold point OP as spelled out in equation (4.1) and we write

$$J^{\mathcal{X}}(x, z) = \sum_{k=1}^{n+1} \tilde{J}_k^{\mathcal{X}}(x, z) \mathbf{1}_k, \quad J^Y(t, z) = \sum_{i=1}^{n+1} J_i^Y(t, z) P_i$$

for the decomposition of the J -function of the orbifold and the resolution in the bases above.

The String Equation for \mathcal{X} and Y and (A.10)–(A.11) in Section A together imply that the power series $\{J_k^{\mathcal{X}}\}_{k=1}^{n+1}$, and $\{J_i^Y\}_{i=1}^{n+1}$ give systems of flat coordinates of $\nabla^{(g,z)}$ locally around OP and LR, respectively. Also, by Theorem 5.3 and Theorem 5.4, the twisted periods $\{\Pi_j\}_{j=1}^{n+1}$ yield a system of *global* flat coordinates of $\nabla^{(g,z)}$; we here single out the principal branch of (5.20)–(5.21) obtained by analytically continuing along ρ . This means that, upon restriction to the neighborhood V_\bullet , the gradients of $\{\Pi_j\}_j$ and $\{J_i^\bullet\}_i$ are a priori different linear bases of the *same* vector space. This entails the existence of invertible, $\mathbb{C}[[a, b]]$ -linear maps $\mathcal{A} \in \text{Hom}(\text{Sol}_{\lambda, \phi}, \mathcal{S}_Y)$, $\mathcal{B} \in \text{Hom}(\mathcal{S}_X, \text{Sol}_{\lambda, \phi})$,

$$(5.28) \quad \begin{aligned} \mathcal{A} \text{grad}_{\eta_Y} \pi_{\lambda, \phi}|_{\mathcal{S}_Y} : H_1(C_\lambda, L_\lambda) &\rightarrow \text{Sol}_{\lambda, \phi}|_{V_Y} \simeq \mathcal{S}_Y, \\ \mathcal{B}^{-1} \text{grad}_{\eta_X} \pi_{\lambda, \phi}|_{\mathcal{S}_X} : H_1(C_\lambda, L_\lambda) &\rightarrow \text{Sol}_{\lambda, \phi}|_{V_X} \simeq \mathcal{S}_X \end{aligned}$$

such that

$$\mathcal{A}\{\Pi_j\}_{j=1}^{n+1} = \{J_i^Y\}_{i=1}^{n+1}$$

and

$$\mathcal{B}\{J_k^{\mathcal{X}}\}_{k=1}^{n+1} = \{\Pi_j\}_{j=1}^{n+1}.$$

In particular, the sought-for identification of J -functions factorizes as

$$\mathbb{U}_\rho^{\mathcal{X}, Y} = \mathcal{A}\mathcal{B}.$$

To compute \mathcal{A} , notice that the components J_i^Y of J^Y in the localized basis $\{P_i\}_{i=1}^{n+1}$ of $H(Y)$ are eigenvectors of the monodromy around LR (see equation (A.14)), generically with distinct eigenvalues. \mathcal{A} can thus be computed by determining the monodromy decomposition of the twisted periods, (5.20)–(5.21), from their asymptotic behavior around LR. For each $1 \leq j \leq n+1$, consider the principal branch of Π_j given by the integral expression of (5.18). The unit polydisk $|e^{t_j}| < 1$ centered at LR coincides with the region of (5.26) for the arguments

$$w_k \triangleq \begin{cases} \frac{\kappa_k}{\kappa_i}, & k \neq i, \\ \kappa_i^{-1}, & k = i, \end{cases}$$

of (5.20) by virtue of (5.10). This puts squarely the problem of analytic continuation of $\{\Pi_j\}_j$ to LR within the setup of Lemma 5.8: by (5.20) and (5.21), for each $1 \leq j \leq n$, the analytic continuation problem of $\Pi_j(\kappa, z)$ to LR along ρ in the κ -variables translates to the analytic continuation of a generalized hypergeometric function $\Phi^{(n)}(a, b, 1+a-b, w_1, \dots, w_n)$ to W_j^∞ along χ_j in the w variables of the lemma. Applying the final result, equation (5.27), entails (compare with equations (B.8), (B.9) and (A.14))

$$\Pi_i = \sum_{j=1}^{n+1} \mathcal{A}_{ij}^{-1} J_j^Y,$$

where

$$(5.29) \quad \mathcal{A}_{ij} = \begin{cases} e^{\pi i(n-i+1)b} \frac{z\Gamma(1+a-(n-i+2)b)}{\Gamma(1-b)\Gamma(a-(n-i+1)b)}, & i = j, \\ e^{-i\pi(a-b(2n-2j+3))} \frac{z \sin(\pi b)\Gamma(1-a+b(n+1-i))\Gamma(1+a-b(n-i+2))}{\pi\Gamma(1-b)}, & j > i, \\ 0, & j < i. \end{cases}$$

Consider now the situation at the orbifold point $\text{OP} = \{\kappa_j = \omega^{-j}\}$. Since

$$\begin{aligned} J^{\mathcal{X}}(0, z) &= z \mathbf{1}_0, \\ \frac{\partial J^{\mathcal{X}}}{\partial x_k}(0, z) &= \mathbf{1}_k, \end{aligned}$$

to compute the operator \mathcal{B} in (5.28) it suffices to evaluate the expansion of the Lauricella functions (5.20)–(5.21) at OP to linear order in x_k , $k = 0, \dots, n$. A remarkable feature here, by equations (5.12) and (5.18), is that the Lauricella function of equation (5.22) at these roots of unity reduces to Euler's Beta integral, a statement whose easy verification we leave to the reader. Explicitly,

$$(5.30) \quad \Pi_j(\kappa, z)|_{x=0} = \omega^{(j-\frac{n}{2})a} \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \Phi(a, b, 1+a-b; \omega, \dots, \omega^n)$$

$$(5.31) \quad = \frac{\omega^{(j-\frac{n}{2})a}}{n+1} B\left(\frac{a}{n+1}, 1-b\right)$$

$$(5.32) \quad = \frac{\omega^{(j-\frac{n}{2})a}}{n+1} \frac{\Gamma(\frac{a}{n+1})\Gamma(1-b)}{\Gamma(1-b+\frac{a}{n+1})}, \quad j = 1, \dots, n+1.$$

Similarly, a short computation shows that

$$(5.33) \quad \kappa_k \frac{\partial \Pi_j}{\partial \kappa_k}(0, z) = \frac{b\omega^{(j-\frac{n}{2})a}}{n+1} \sum_{l=1}^n \omega^{(j-k)l} B\left(\frac{a+l}{n+1}, -b\right),$$

$$(5.34) \quad \left. \frac{\partial \Pi_j}{\partial x_k}(\kappa, z) \right|_{x=0} = \frac{b\omega^{(j-\frac{n}{2})a-jk+\frac{k}{2}}}{n+1} B\left(1+\frac{a+l}{n+1}, -b\right)$$

$$(5.35) \quad = -\frac{\omega^{(j-\frac{n}{2})a-jk+\frac{k}{2}}}{n+1} \frac{\Gamma(\frac{a-k}{n+1}+1)\Gamma(1-b)}{\Gamma(\frac{a-k}{n+1}+1-b)}.$$

In matrix form we have

$$(5.36) \quad \Pi = \mathcal{B} J^{\mathcal{X}} = D_1 V D_2 J^{\mathcal{X}},$$

where

$$(5.37) \quad (D_1)_{jk} = \omega^{(j-\frac{n}{2})a} \delta_{jk},$$

$$(5.38) \quad (D_2)_{jk} = \delta_{jk} \begin{cases} -\omega^{\frac{k}{2}} \frac{\Gamma(\frac{a-k}{n+1}+1)\Gamma(1-b)}{\Gamma(\frac{a-k}{n+1}+1-b)} & \text{for } 1 \leq k \leq n, \\ \frac{\Gamma(\frac{a}{n+1})\Gamma(1-b)}{z\Gamma(1-b+\frac{a}{n+1})} & \text{for } k = n+1, \end{cases}$$

$$(5.39) \quad V_{jk} = \frac{\omega^{-jk}}{n+1}.$$

Piecing (5.29), (5.36), (5.37), (5.38) and (5.39) together yields⁸⁾ equation (4.2), up to a scalar factor of α_a . By Remark 5.5, this corresponds to our freedom of a String Equation shift along either of $H^0(\mathcal{X})$ and $H^0(Y)$. Setting $\delta_{\mathcal{X}} - \delta_Y = 2\pi i \alpha_1$ in (5.9) and (5.11) concludes the proof.

⁸⁾ This amounts to a rather tedious exercise in telescoping sums and additions of roots of unity. The computation can be made available upon request.

5.3.3. Monodromy and pure braids. The expression (4.2) for the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X},Y}$ was obtained for the analytic continuation path ρ of equation (4.1). Fixing a reference point $m_0 = (\tilde{\kappa}_1, \dots, \tilde{\kappa}_n) \in \mathcal{M}_A$, for a general path $\rho \circ \sigma$ with $[\sigma] \in \pi_1(\mathcal{M}_A, m_0)$ we get a composition

$$\mathbb{U}_{\rho \circ \sigma}^{\mathcal{X},Y} = \mathbb{U}_\rho^{\mathcal{X},Y} M_\sigma,$$

where

$$(5.40) \quad M_\sigma : \pi_1(\mathcal{M}_A, m_0) \rightarrow \text{Aut}(\text{Sol}_{\lambda,\phi})$$

is the monodromy representation of the fundamental group of \mathcal{M}_A in the space of solutions of the Lauricella system $F_D^{(n)}$.

By definition (5.1), \mathcal{M}_A is the configuration space of n distinct points in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Therefore, its fundamental group is the pure braid group in $n + 2$ strands

$$\pi_1(\mathcal{M}_A) \simeq \text{PB}_{n+2};$$

with the monodromy action (5.40) given by the reduced Gassner representation (cf. [31, 53]) of PB_{n+2} . Writing $\tilde{\kappa}_i = 0, 1, \infty$ for $i = n + 1, n + 2$ and $n + 3$, respectively, generators P_{ij} , $i = 1, \dots, n + 3$, $j = 1, \dots, n$, of PB_{n+2} are in bijection with paths $\sigma_{ij} : [0, 1] \rightarrow \mathcal{M}_A$ given by lifts to \mathcal{M}_A of closed contours in the j th affine coordinate plane that start at $\kappa_j = \tilde{\kappa}_j$, turn counterclockwise around $\tilde{\kappa}_i$ (and around no other point) and then return to their original position, as in Figure 4.

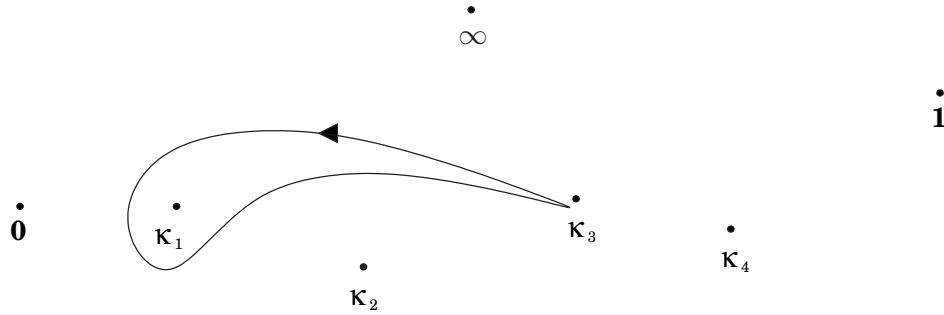


Figure 4. The path σ_{13} in $\pi_1(\mathcal{M}_A)$ for $n = 4$.

The image of the period map (5.5), by Theorem 5.3, is a lattice in $\text{Sol}_{\lambda,\phi}$:

$$\text{Sol}_{\lambda,\phi} = \nabla^{(g)} \pi_{\lambda,\phi}(H_1(C_\lambda, L_\lambda)) \otimes_{\mathbb{Z}(\mathfrak{q}_a, \mathfrak{q}_b)} \mathbb{C}(\mathfrak{q}_a, \mathfrak{q}_b),$$

and by (5.29), (5.36), (5.37), (5.38) and (5.39) the induced morphism

$$H_1(C_\lambda, L_\lambda) \simeq K(Y)$$

is a lattice isomorphism. The monodromy action on $\text{Sol}_{\lambda,\phi}$, at the level of equivariant K -groups, is given by lattice automorphisms

$$\pi_1(\mathcal{M}_A) \rightarrow \text{Aut}_{\mathbb{Z}(\mathfrak{q}_a, \mathfrak{q}_b)} K(Y);$$

this can be verified explicitly from the form of the monodromy matrices in the twisted period basis [65]. For example, when $n = 1$, the action on $K(Y)$ is given by the classical monodromy

•
LR2

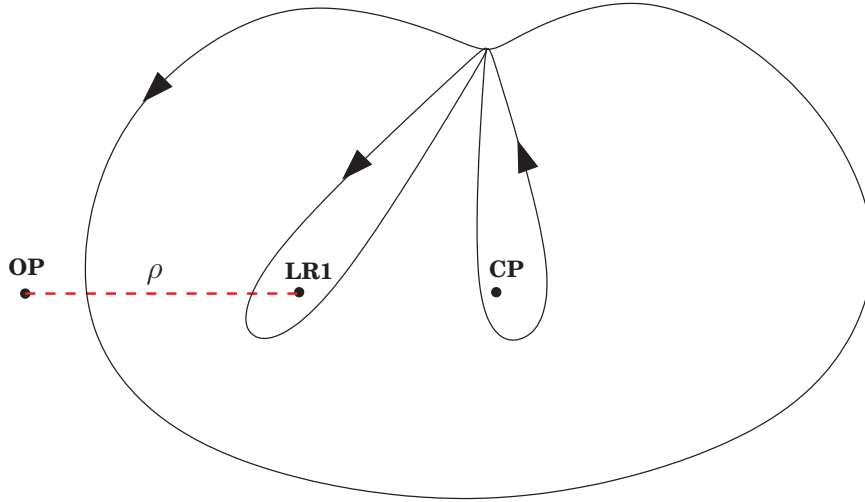


Figure 5. The Kähler moduli space of the A_1 singularity in A -model coordinates. LR1 and LR2 indicate the large radius points $\kappa = 0, \infty$, respectively, CP is the conifold point, and OP is the orbifold point. Circuits around LR1-2 and CP generate the monodromy group of the global quantum D -module. The dashed segment depicts the analytic continuation path ρ of Theorems B.2 and 4.1.

of the Gauss system for $c = a - b + 1$. With reference to Figure 5, we have in the standard basis $\{\mathcal{O}_Y, \mathcal{O}_Y(1)\}$ for $K(Y)$,

$$M_{\text{LR1}} = \begin{pmatrix} e^{-ia\pi} (e^{2ia\pi} + e^{2ib\pi}) & e^{2ib\pi} \\ -1 & 0 \end{pmatrix},$$

$$M_{\text{CP}} = \begin{pmatrix} 1 & -2ie^{-i(a-b)\pi} \sin(b\pi) \\ -2ie^{-i(a+b)\pi} \sin(b\pi) & 1 - 4e^{-2ia\pi} \sin^2(b\pi) \end{pmatrix},$$

$$M_{\text{LR2}} = \begin{pmatrix} 2\cos(a\pi) & 1 - 2ie^{i\pi(b-2a)} \sin(b\pi) \\ -1 & 2ie^{-i(a-b)\pi} \sin(b\pi) \end{pmatrix},$$

for the large radius and the conifold monodromy of $\text{QDM}(Y)$. It is straightforward to check that they induce symplectic automorphisms of \mathcal{H}_Y .

Remark 5.9. In the non-equivariant setting, representations of the braid group B_n have a natural place in the derived context where they correspond to elements of $\text{Auteq}(D^b(Y))$ generated by spherical twists [73]. From a quantum D -module perspective, the interpretation of flat sections as B -branes identifies this braid group action with the monodromy action. Recently, different flavors of braid group actions, including mixed and pure braids, have been shown to arise upon taking deformations of the Seidel–Thomas setup [34]; the D -module picture of equations (5.7) and (5.18) indicates that the lift to the equivariant theory should naturally provide another such extension, whose origin in the derived context would be fascinating to trace in detail.

6. Quantization

The goal of this section is to prove the following

Theorem 6.1. *The Hard Lefschetz quantized CRC (Conjecture 2.4) holds for the pairs (\mathcal{X}, Y) , where $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{C}$ has the threefold A_n singularity as a coarse space and Y is its crepant resolution.*

We outline the proof of Theorem 6.1: Givental's quantization formula (6.1) for \mathcal{X} and Y and the Hard Lefschetz condition are used in Lemma 6.3 to show that Theorem 6.1 follows from appropriately comparing the canonical R -calibrations for \mathcal{X} and Y . The existence of the canonical R -calibrations is a consequence of Teleman's reconstruction theorem [75]. Work of Jarvis–Kimura and the orbifold quantum Riemann–Roch theorem of Tseng (Section 6.1) computes the Gromov–Witten R -calibration for \mathcal{X} at the orbifold point. In Section 6.3, we verify that this agrees with the R -calibration of Y upon analytic continuation, concluding the proof.

Givental's quantization formalism for semi-simple quantum cohomology [44, 45, 75] gives an expression for the all-genus GW partition function of a target \mathcal{Z} with semi-simple quantum product of rank $N_{\mathcal{Z}}$ as the action of a sequence of differential operators on n -copies of the partition function of a point. The operators in question are obtained through Weyl-quantization of infinitesimal symplectomorphisms determined by the genus-zero Gromov–Witten theory of \mathcal{Z} – the S - and R -calibrations of $QH(\mathcal{Z})$, defined in Sections A.1 and A.2.1, respectively. Here we assume familiarity with this story and standard notation, and review them in Section A.2.

Givental's formula at a semi-simple point $\tau \in QH(\mathcal{Z})$ reads

$$(6.1) \quad Z_{\mathcal{Z}}(t_{\tau}) = e^{C_{\mathcal{Z}}(\tau)} \widehat{S}_{\mathcal{Z}}^{-1} \widehat{\Psi}_{\mathcal{Z}} \widehat{R}_{\mathcal{Z}} e^{\widehat{u}/z} \prod_{i=1}^{N_{\mathcal{Z}}} Z_{\text{pt}}(q^i),$$

where

$$(6.2) \quad C_{\mathcal{Z}}(\tau) \triangleq \sum_{i=1}^{N_{\mathcal{Z}}} \int (R_{\mathcal{Z}}^{(1)})_i^i(\tau) du^i, \quad R_{\mathcal{Z}}^{(1)}(\tau) = \partial_{\mathcal{Z}} R_{\mathcal{Z}}(\tau, 0)$$

and the shifted descendent times t_{τ} are defined in equation (A.21). In equation (6.1), $S_{\mathcal{Z}}$ and $R_{\mathcal{Z}}$ are the canonical Gromov–Witten S - and R -calibrations of $QH(\mathcal{Z})$, viewed as morphisms of Givental's symplectic space $\mathcal{H}_{\mathcal{Z}}$.

Remark 6.2. The existence of canonical R -calibrations such that equation (6.1) holds is a consequence of Teleman's theorem [75, Theorem 2]. In the conformal case, their form is uniquely determined by homogeneity. In the non-conformal case, the lack of an Euler vector field constrains the form of asymptotic solutions of the quantum differential equation only up to right multiplication by a constant diagonal matrix in odd powers of z . Therefore, in order to verify that a given R -calibration is equal to the canonical R -calibration guaranteed by Teleman's theorem, we need only check equation (6.1) at a single semi-simple point. The specialization of $R_{\mathcal{X}}$ to the orbifold point will be the focus of Section 6.1.

Lemma 6.3. *Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a resolution diagram of Hard Lefschetz targets with generically semi-simple quantum cohomology. Conjecture 2.4 holds if and only if the canoni-*

cal R -calibrations coincide on the semi-simple locus,

$$(6.3) \quad R_{\mathcal{X}} = R_Y.$$

Remark 6.4. The local independence of Givental's formula on the choice of a base point [44] implies that it suffices to check equation (6.3) at any given semi-simple point.

Proof. We start by observing that (6.3) implies $Z_Y = \widehat{\mathbb{U}}_{\rho}^{\mathcal{X},Y} Z_{\mathcal{X}}$. It is immediate that $C_{\mathcal{X}} = C_Y$. Moreover, equation (6.1) gives

$$(6.4) \quad \begin{aligned} Z_Y &= e^{C_Y} \widehat{S}_Y^{-1} \widehat{\psi}_Y \widehat{R}_Y e^{u/z} \prod_{i=1}^{N_Y} Z_{\text{pt},i} \\ &= e^{C_Y} \widehat{S}_Y^{-1} \widehat{\mathbb{U}}_0 \widehat{\psi}_{\mathcal{X}} \widehat{R}_{\mathcal{X}} e^{u/z} \prod_{i=1}^{N_{\mathcal{X}}} Z_{\text{pt},i} \\ &= \widehat{\mathbb{U}}_{\rho}^{\mathcal{X},Y} e^{C_{\mathcal{X}}} \widehat{S}_{\mathcal{X}}^{-1} \widehat{\psi}_{\mathcal{X}} \widehat{R}_{\mathcal{X}} e^{u/z} \prod_{i=1}^{N_{\mathcal{X}}} Z_{\text{pt},i} = \widehat{\mathbb{U}}_{\rho}^{\mathcal{X},Y} Z_{\mathcal{X}}. \end{aligned}$$

We have made essential use of the HL condition twice: to identify S -calibrations via $\widehat{\mathbb{U}}_{\rho}^{\mathcal{X},Y}$, which is only true if the analytic continuation of the quantum product gives an isomorphism in big quantum cohomology, and to ensure that no cocycle is generated in the quantization of products. To see the reverse implication, we note that for the string of equations (6.4) to hold,

$$(6.5) \quad (e^{C_{\mathcal{X}}} \widehat{R}_{\mathcal{X}} - e^{C_Y} \widehat{R}_Y) \prod_{i=1}^{n+1} Z_{\text{pt},i} = 0.$$

Equation (6.5) implies

$$e^{C_{\mathcal{X}}} \widehat{R}_{\mathcal{X}} = e^{C_Y} \widehat{R}_Y e^{\widehat{\mathcal{D}}}$$

for some quantized quadratic Hamiltonian $\widehat{\mathcal{D}}$ such that $\widehat{\mathcal{D}} \in \bigoplus_{i=1}^n \mathcal{B}(\text{Vir}_i)$, the Borel subalgebra of level $k \geq -1$ Virasoro constraints acting on the product of Witten–Kontsevich tau functions $\prod_{i=1}^{n+1} Z_{\text{pt},i}$. Imposing that $e^{\widehat{\mathcal{D}}} \in \text{Sp}_+(\mathcal{H}_Y)$ then sets $\widehat{\mathcal{D}} = 0$ and $C_{\mathcal{X}} = C_Y$. \square

6.1. R -calibrations in orbifold Gromov–Witten theory. Equation (6.3) reduces the quantum CRC to a comparison between asymptotic expression of horizontal sections of the global quantum D -module, given by the Gromov–Witten R -calibrations of \mathcal{X} and Y . In the context of toric orbifolds, the R -calibration is uniquely constructed [52, 77] in terms of group theoretic and toric data.

Lemma 6.5. *Consider the orbifold $\mathcal{X} = [\mathbb{C}^m/G]$ given by a diagonal representation V of a finite abelian group G , together with a compatible torus action. Then the canonical R -calibration $R_{\mathcal{X}}$ is uniquely determined locally around the large radius point of \mathcal{X} by (6.9).*

Proof. Denote by $N_{\mathcal{X}}$ the number of elements of G , which is also the rank of the Chen–Ruan cohomology of \mathcal{X} , and by $\mathbf{1}_g$ the fundamental class of the component of the inertia orbifold labeled by g . Jarvis–Kimura [52, Proposition 4.3] establish that the partition

function Z_{BG} agrees with $N_{\mathcal{X}}$ copies of Z_{pt} after a change of variables given by the character table χ_G of G , cf. [52, Proposition 4.1]. In operator notation,

$$Z_{BG} = \widehat{\chi_G^{-1}} \prod_{i=1}^{N_{\mathcal{X}}} Z_{\text{pt},i}.$$

On the other hand, the Gromov–Witten theory of \mathcal{X} is the twisted Gromov–Witten theory of BG with twisting class the inverse Euler class of the representation V , thought of as a vector bundle on the classifying stack. Since the representation is diagonal, $V \cong \bigoplus_{i=1}^m V_i$ is the sum of m orbifold line bundles. Denote by $\{w_i\}_{i=1}^m$ the weights of the torus action on each line bundle. Tseng, in [77], constructs a symplectomorphism $\mathcal{D}_{\mathcal{X}} \in \text{Aut}_+(\mathcal{H}_{\mathcal{X}})$ defined by

$$(6.6) \quad \mathcal{D}_{\mathcal{X}} \mathbf{1}_g \triangleq \sqrt{\frac{1}{e^{\text{eq}}(V^{(0)})}} \exp\left(\sum_{i=1}^m \sum_{k \geq 0} s_{i,k} \frac{B_{k+1}(l_i(g)/|g|)}{(k+1)!} z^k\right) \mathbf{1}_g,$$

where $s_{i,0} = -\ln(w_i)$, $s_{i,k} = (-w_i)^{-k} (k-1)!$, $V^{(0)}$ is the trivial part of the representation V , $B_k(x)$ is the order k -Bernoulli polynomial

$$\frac{e^{xy} y}{e^y - 1} = \sum_{k \geq 0} \frac{B_k(x) y^k}{k!},$$

and the integer $l_i(g) \in [0, N_{\mathcal{X}} - 1]$ is defined by $g v_i = e^{2\pi i l_i(g)/|g|} v_i$ for $v_i \in V_i$. The orbifold Quantum Riemann–Roch theorem of [77] then asserts that, upon quantization, $\widehat{\mathcal{D}_{\mathcal{X}}}$ acts on Z_{BG} to return the GW partition function for \mathcal{X} , up to a scalar prefactor $\mathcal{E}_{\mathcal{X}}$, whose precise form will not concern us, and a rescaling of the Darboux coordinates by $\sqrt{e^{\text{eq}}(V^{(0)})}$, cf. [77, Section 1.2]. Then,

$$(6.7) \quad Z_{\mathcal{X}} = \mathcal{E}_{\mathcal{X}} \widehat{(e^{\text{eq}}(V^{(0)}))^{-\frac{1}{2}}} \widehat{\mathcal{D}_{\mathcal{X}}} \widehat{\chi_G^{-1}} \prod_{i=1}^{N_{\mathcal{Z}}} Z_{\text{pt},i}.$$

To compare equation (6.7) with equation (6.1), we fix the integration constant in equation (6.2) so that $\mathcal{E}_{\mathcal{X}}|_{\text{OP}} = e^{C_{\mathcal{X}}}$; notice that this is always possible, since OP is a regular point for the Dubrovin connection of \mathcal{X} . Moreover, at the large radius point for \mathcal{X} we have

$$(6.8) \quad (S_{\mathcal{X}}^{-1})_{\alpha}^{\beta}|_{\text{OP}} = \delta_{\alpha}^{\beta}$$

in flat coordinates for the orbifold Poincaré pairing of \mathcal{X} . Define now $R_{\mathcal{X}}(\tau, z)$ locally around OP by parallel transporting the symplectomorphism $(e^{\text{eq}}(V^{(0)}))^{-\frac{1}{2}} \mathcal{D}_{\mathcal{X}} \chi_G^{-1} \in \text{Sp}_+(\mathcal{H}_{\mathcal{X}})$: in other words $(\psi R_{\mathcal{X}}(\tau, z) e^{\frac{\mu}{z}})_{\alpha j}$ is a matrix whose columns are horizontal sections for the Dubrovin connection such that

$$(6.9) \quad \psi_{\mathcal{X}} R_{\mathcal{X}}|_{\text{OP}} = (e^{\text{eq}}(V^{(0)}))^{-\frac{1}{2}} \mathcal{D}_{\mathcal{X}} \chi_G^{-1}.$$

Altogether, equations (6.9), (6.7) and (6.8) imply that Givental’s formula, equation (6.1), holds by construction at OP with the R -calibration determined by equation (6.9). This verifies that $R_{\mathcal{X}}(\tau, z)$ is the canonical R -calibration guaranteed by Teleman’s theorem. \square

Remark 6.6. Pinning down the canonical R -calibration for an arbitrary toric orbifold \mathcal{X} can be achieved by localization. Choose a basis for equivariant cohomology supported on the fixed points: naturally vectors supported on different fixed points are mutually orthogonal. The R -calibration is then computed as a block matrix by applying Lemma 6.5 to the local geometry

of each fixed point. In particular, when $\mathcal{X} = Y$ is a toric manifold and denoting by LR the large radius limit point for Y , equation (6.9) becomes (cf. [44, Theorem 9.1])

$$(6.10) \quad (R_Y)_{ij} \Big|_{\text{LR}} = (\mathcal{D}_Y)_i \delta_{ij}.$$

We now turn our attention to the specific geometries we are investigating in depth: $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{n+1}]$ and Y its crepant resolution.

6.2. Prolegomena on asymptotics and analytic continuation. In view of (6.3), Conjecture 2.4 can be formulated as an identification of bases of horizontal 1-forms of $\nabla^{(g,z)}$ upon analytic continuation to some chosen semi-simple point. In our case, the proof of Theorem 4.1 in Section 5.3.2 contains already most of the technical ingredients to compute the analytic continuation of flat coordinates of $\nabla^{(g,z)}$ from LR to OP; however, a few substantive details in the formulation of equation (6.3), particularly in what concerns the continuation of asymptotic series, are worth spelling out with care.

6.2.1. Global canonical coordinates. First of all, the reasoning leading to (6.3) assumed implicitly a choice of global canonical coordinates $\{u^i \in \mathcal{O}(\mathcal{M}_A)\}_{i=1}^{n+1}$ on \mathcal{M}_A – or at least, two consistent choices of canonical coordinates for both $QH(Y)$ and $QH(\mathcal{X})$; recall that two such sets of coordinates may differ by permutations and shifts by constants. A natural way to fix this ambiguity is to define globally u^i as the critical values

$$(6.11) \quad u^i = \log \lambda(q_i^{\text{cr}})$$

of the Hurwitz space superpotential (5.7), where the critical points q_i^{cr} of $\lambda(q)$ are the roots of the polynomial equation

$$(6.12) \quad \frac{a}{q_i^{\text{cr}}} + b \sum_{j=1}^{n+1} \frac{\kappa_j}{1 - \kappa_j q_i^{\text{cr}}} = 0.$$

The leftover permutation ambiguity is fixed upon ordering the set of critical points such that

$$(6.13) \quad \left. \frac{\partial}{\partial u^i} \right|_{\kappa=0} \simeq P_i$$

under the identification $T_{\text{LR}} \tilde{\mathcal{F}}_{\lambda, \phi} \simeq H_T(Y_T)$.

6.2.2. Sectors, thimbles, walls. A second aspect pertains to the nature of $R_\bullet(\kappa, z)$ as a *formal asymptotic series* in z (see Section A.2.1). Since $z = 0$ is an irregular singularity for the global D -module, asymptotic expansions of components of horizontal 1-forms at $z = 0$ depend on a choice of Stokes sector, namely, a choice of phase for z , as well as for the other external parameters α_1, α_2 and κ in the asymptotic analysis. Picking one such choice poses no restriction for the purpose of proving equation (6.3): as individual Gromov–Witten correlators depend analytically on a and b , it is enough for us to prove equation (6.3) in a wedge of parameter space. A particularly convenient choice is to pick the Stokes sector \mathfrak{S}_+ defined by

$$(6.14) \quad \mathfrak{S}_+ \triangleq \left\{ (a, b, \kappa) : \Re e(a) > 0, \Re e(b) < 0, \arg(\kappa_j) = -\frac{2\pi i j}{n+1} \right\}$$

where the phase of the quantum cohomology parameters in equation (6.14) is fixed by our choice of path ρ in equation (4.1). This choice turns out to trigger two favorable consequences.

First off, when $(a, b, \kappa) \in \mathcal{S}_+$, we can employ the factorization of the twisted period mapping through the line integral representation (5.19) to obtain an interpretation of the twisted periods as a sum of steepest descent integrals (see Remark 5.6). In detail, note that throughout \mathcal{S}_+ the superpotential has algebraic zeroes at $q = \{0, 1\} \cup \{\kappa_i^{-1}\}_{i=1}^n$. Upon regarding $\Re(\log \lambda(q))$ as a perfect Morse function, the Lefschetz thimbles \mathfrak{L}_i emerging from each of the critical points q_i^{cr} give a canonical basis of the relative homology group $H_1(\mathbb{P}^1, (\lambda))$, with the negative infinity of each downward gradient flow coinciding with the log-divergences of $\log \lambda(q)$ at the zeroes of the superpotential. In this basis, the Laplace expansion at small z gives asymptotic solutions $\Sigma_i(\kappa, z)$ for the flat coordinates of $\nabla^{(g,z)}$ in the form

$$(6.15) \quad \Sigma_i(\kappa, z) \triangleq \int_{\mathfrak{L}_i} \lambda^{\frac{1}{z}} \phi \simeq e^{\frac{u_i}{z}} \mathcal{Q}_i(\kappa, z),$$

where $\mathcal{Q}_i(\kappa, z) \in \mathcal{O}(\mathcal{M}_A) \otimes \mathbb{C}[[z]]$, and the equivalence sign is to be intended in the sense of classical (Poincaré) asymptotics.

The second useful consequence of the choice of parameters (6.14) relates to the nature of the canonical R -calibrations as asymptotic series. In light of the representation (6.15) of R -operators as the (perturbative) Laplace expansion of a steepest descent integral around a saddle, in proving (6.3) we are supposed to *discard* any exponentially suppressed (non-perturbative) contribution from neighboring critical points that may arise in the process of analytic continuation (see Remark 6.7). Now, throughout $\mathcal{S}_+ \setminus \{|\kappa| = 1\}$ we have

$$(6.16) \quad \Re\left(\frac{u_i}{z}\right) > \Re\left(\frac{u_j}{z}\right), \quad i < j,$$

which means that, away from $|\kappa| = 1$, the i th-saddle is exponentially dominant over saddles q_j^{cr} with $j > i$. In the following, we repeatedly exploit the fact that terms of the form

$$e^{z^{-1}[u_j(\rho(s)) - u_i(\rho(s))]}, \quad j > i, s \in [0, 1),$$

are exponentially suppressed, and therefore invisible, in the classical small z -asymptotics inside this region.

Remark 6.7. One potential source of such exponential contributions is due to the Stokes phenomenon. Since we are dealing with the analytic continuation of asymptotic series of the form (6.15), a complication that may occur when varying (6.15) along \mathcal{M}_A is given by the possibility of a non-trivial “monodromy” of the Lefschetz thimbles along the analytic continuation path ρ in (4.1): when $\Im\left(\frac{u_i}{z}\right) = \Im\left(\frac{u_j}{z}\right)$ for some $j > i$, the i th Lefschetz thimble passes through a sub-dominant saddle point, and in turn an exponentially subleading contribution in the asymptotic expansion of the i th-period integral appears. Such a jump in the asymptotics arises across *walls* – and not just *divisors* – in moduli space, and it may affect in principle⁹⁾ the continuation of (6.15) along the trivial path ρ in (4.1). The existence of Stokes walls may be all the more delicate in light of the fact that the orbifold point belongs to the maximal anti-Stokes

⁹⁾ Generically, there is no Stokes phenomenon for $n = 1$, where we can compute $\Im(u_1 - u_2) = 2i\pi\alpha_1$ identically in \mathcal{S}_+ . For higher n , however, the possibility of the existence of the Stokes phenomenon can be tested numerically. A little experimentation shows that ρ *does* indeed cross one or more Stokes walls for $n > 1$ and fairly generic a, b .

submanifold $\{p \in \mathcal{M}_A : \Re e(\frac{u^i(p)}{z}) = \Re e(\frac{u^j(p)}{z}) \text{ for all } (i, j)\}$ – see equation (6.20) below. In the following, we must be wary of the possible generation of exponentially suppressed terms generated when crossing a wall, as they are no longer subdominant when they are continued all the way up to $|\kappa| = 1$, where their contribution should be included in the asymptotics.¹⁰⁾

6.3. Proof of Theorem 6.1.

6.3.1. R-normalizations for \mathcal{X} and Y . Let us first compute from Lemma 6.5 the canonical R -calibration for \mathcal{X} , thought of as an equivariant vector bundle over the classifying stack:

$$\begin{aligned} \mathcal{X} &= \mathcal{O}_{-1}^{-\alpha_1} \oplus \mathcal{O}_1^{-\alpha_2} \oplus \mathcal{O}^{\alpha_1 + \alpha_2} \rightarrow B\mathbb{Z}_{n+1} \\ &\triangleq V_1 \oplus V_2 \oplus V_3 \rightarrow B\mathbb{Z}_{n+1}. \end{aligned}$$

Note that $V^{(0)}$ is an equivariant vector bundle on the inertia stack; it agrees with the whole three-dimensional bundle on the component of the identity, and to the line bundle corresponding to the untwisted direction in all twisted sectors. Therefore,

$$e^{\text{eq}}(V^{(0)}) = (\alpha_1 + \alpha_2) \sum_{j=1}^n \mathbf{1}_j + \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \mathbf{1}_{n+1}.$$

For $i = 1, 2, 3$ and $j \in \mathbb{Z}_{n+1}$, the integers $l_i(j)$ are $(n+1) - j, j, 0$. Then, using that

$$B_k(x) = (-1)^k B_k(1-x), \quad B_{2k+1} = 0, \quad n > 0,$$

equation (6.6) gives

$$\begin{aligned} (6.17) \quad \mathcal{D}_{\mathcal{X}} &= \sum_{j=1}^n \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{2} - \frac{j}{n+1}} \exp \left[\sum_{k>0} \left(-\frac{B_{k+1} \binom{j}{n+1}}{(-\alpha_1)^k} + \frac{B_{k+1} \binom{j}{n+1}}{\alpha_2^k} \right. \right. \\ &\quad \left. \left. - \frac{B_{k+1}}{(\alpha_1 + \alpha_2)^k} \right) \frac{z^k}{k(k+1)} \right] \mathbf{1}_j \\ &\quad + \exp \left[\sum_{k>0} \left(\frac{1}{\alpha_1^{2k-1}} + \frac{1}{\alpha_2^{2k-1}} - \frac{1}{(\alpha_1 + \alpha_2)^{2k-1}} \right) \frac{B_{2k} z^{2k-1}}{2k(2k-1)} \right] \mathbf{1}_{n+1}. \end{aligned}$$

As far as Y is concerned, by Remark 6.6, we apply Lemma 6.5 to the local geometry of each fixed point p_i . Then, denoting by $(w_i^-, w_i^+, \alpha_1 + \alpha_2)$ the characters of the torus action on the tangent space T_{p_i} at the i th fixed point, as in equation (B.6), we have from equation (6.6) that

$$(6.18) \quad \mathcal{D}_Y p_i = \exp \left[- \sum_{k>0} \frac{B_{2k} z^{2k-1}}{2k(2k-1)} \left((w_i^+)^{1-2k} + (w_i^-)^{1-2k} + (\alpha_1 + \alpha_2)^{1-2k} \right) \right] p_i.$$

¹⁰⁾ A typical example of this phenomenon the reader may be familiar with is the appearance of subleading exponentials in the asymptotics of the Airy integral along its anti-Stokes ray, that is, for large negative values of the argument.

6.3.2. Analytic continuation. To compare the classical R -operators in equation (6.3), we avoid troubles with the Stokes phenomenon as follows: we fix the R -calibration *first* at OP, where pinning down the contribution of each critical point is potentially delicate, and then compute its continuation to $|\kappa| < 1$ where the classical asymptotics are controlled by the leading saddle. Then, equation (6.16) grants us the right to safely ignore any possible issues stemming from the generation of subleading exponential terms by analytic continuation through a wall when $|\kappa| < 1$.

At the orbifold point $\kappa_j = \omega^{-j}$, equation (6.12) gives

$$(6.19) \quad q_i^{\text{cr}}|_{\text{OP}} = \omega^{\sigma(i)} \left(\frac{a}{a - (1+n)b} \right)^{\frac{1}{n+1}}$$

for some permutation $\sigma \in S_{n+1}$, which by continuity is locally constant in (a, b) . Noting that the roots of (6.12) admit a smooth limit at $b = 0$, where

$$q_i^{\text{cr}}|_{b=0} = \kappa_i^{-1},$$

and comparing to equations (6.11) and (6.13) sets $\sigma = \text{id}$. Therefore,

$$(6.20) \quad u^j|_{\text{OP}} = \alpha_1 \log(\alpha_1) + \alpha_2 \log(-\alpha_2) - (\alpha_1 + \alpha_2) \log(-\alpha_1 - \alpha_2) + 2i\alpha_1 \left(j - \frac{n}{2} \right).$$

Note that $\Re e(\frac{u^i}{z})|_{\text{OP}} = \Re e(\frac{u^j}{z})|_{\text{OP}}$ for all (i, j) , as anticipated in Remark 6.7. Since we have $\arg \lambda(q)|_{\text{OP}} = \alpha_1 \arg q$ by (5.7), at the orbifold point the constant phase/steepest descent contour \mathfrak{L}_j must be contained in the straight line through the origin making an angle of $\frac{2\pi j}{n+1}$ with the positive semi-axis. In particular, since $|q_i^{\text{cr}}| < 1$ in $\mathfrak{S}_+ \cap \text{OP}$ by equation (6.19), the union of the downward gradient lines emanating from $|q_j^{\text{cr}}| < 1$ is given by the segment $[0, \omega^j]$. Then, by equation (5.19), twisted periods coincide with line integrals over steepest descent paths for $\lambda(q)^{\frac{1}{2}} \phi|_{\text{OP}}$, and we have

$$(6.21) \quad \Pi_i(\kappa, z)|_{\text{OP}} = \int_{\mathfrak{L}_i} \lambda^{\frac{1}{2}} \phi \sim e^{\frac{u^i}{z}}.$$

Now, in flat coordinates x^α for the orbifold quantum product, the R -calibration must satisfy by equation (6.21)

$$(6.22) \quad (\psi_{\mathfrak{X}} R_{\mathfrak{X}} e^{\frac{u}{z}})_{\alpha, i} dx^\alpha = d\Pi_i \mathcal{N}_i^{\mathfrak{X}}$$

in a neighborhood of OP for some constant normalization factor $\mathcal{N}^{\mathfrak{X}} \in H(\mathfrak{X}) \otimes \mathbb{C}[[z]]$. The left-hand side is uniquely determined by (6.9) and (6.17). For the right-hand side, we have already computed the differential of the twisted period map at the orbifold point in equations (5.32)–(5.35). Putting it all together, we obtain¹¹⁾

$$(6.23) \quad \sum_{\alpha=1}^{n+1} (\mathcal{B}^{-1})_{i, \alpha} (e^{\text{eq}(V^{(0)})} V^{-\frac{1}{2}} \mathcal{D}_{\mathfrak{X}})_{\alpha} (\chi^{-1})_{\alpha, j} e^{\frac{u^j}{z}}|_{\text{OP}} \simeq \mathcal{N}_i^{\mathfrak{X}} \delta_{ij}.$$

¹¹⁾ Notice that this is a severely overconstrained system for the unknown \mathcal{N}_i , as is apparent from the fact that the left-hand side has no *a priori* reason to be diagonal in the indices i, j (see also Remark 6.2). Existence of solutions is a non-trivial statement about the boundary values at OP of the twisted periods and their derivatives, (5.32) and (5.35).

The small $|z|$ -asymptotics of the left-hand side, by (5.31), (5.32) and (5.35), is computed by the steepest descent asymptotics of the Beta integral in (5.31) and (5.34). With our choice of sector \mathfrak{S}_+ , as all the Γ -functions appearing in equation (5.38) have arguments with large and positive real part for small $|z|$, the latter is determined by the generalized Stirling formula:

$$(6.24) \quad \Gamma(x+y)x^{-x}e^x x^{\frac{1}{2}-y} \simeq \sqrt{2\pi} \exp\left(\sum_{k>0} \frac{B_{k+1}(1-y)}{k(k+1)} x^k\right), \quad \Re(x) \gg 0.$$

Keeping track judiciously of the (rather massive) cancellations occurring upon plugging equations (6.17), (6.24), (6.20), (5.36), (5.37), (5.38) and (5.39) into equation (6.23), we get that (6.23) admits the unique solution

$$(6.25) \quad \mathcal{N}_i^{\mathcal{X}} = -b^{-1} \sqrt{\frac{z}{2\pi}}.$$

Let us now analytically continue (6.22) and (6.25) to LR along ρ . By (5.28) and (6.16), $\rho \cap (\mathfrak{S}_+ \setminus \{|\kappa| < 1\})$ does not contain anti-Stokes points. The classical asymptotics around the i th-saddle of the continuation of equation (6.22) is therefore computed unambiguously as a formal power series in z from the classical asymptotics of $e^{-u^i/z} d\Pi_i$. Denote by

$$(\widetilde{R_{\mathcal{X}}})_{ij} du^j \in \Omega^1(\mathcal{M}_A)[[z]]$$

the formal series obtained for every $i = 1, \dots, n$ from the analytic continuation along ρ of the coefficients of

$$\mathcal{N}_i^{\mathcal{X}} e^{-\frac{u^i}{z}} d\Pi_i = (\psi_{\mathcal{X}} R_{\mathcal{X}})_{\alpha,i} dx^\alpha$$

to $\kappa \sim 0$. From the discussion above, we isolate for each i the contributions from the leading saddle to obtain

$$(6.26) \quad (\widetilde{R_{\mathcal{X}}})_{ij} du^j \simeq e^{-\frac{u^i}{z}} \mathcal{A}_{ij}^{-1} \mathcal{N}_i^{\mathcal{X}} dJ_i^Y(u, z)$$

as one-form-valued formal series in z ; notice that the off-diagonal terms of \mathcal{A}_{ij}^{-1} have become invisible in the asymptotics after projecting out subleading exponentials. Expressing the components of (6.26) in normalized canonical coordinates and taking the $\kappa \rightarrow 0$ limit, by (6.27) and (A.14), we have

$$J_i^Y(\kappa, z) = ze^{\frac{u_{\text{cl}}^i}{z}} (1 + \mathcal{O}(\kappa)),$$

where u_{cl}^i are coordinates for the idempotents of the classical T -equivariant cohomology ring of Y defined by $u_{\text{cl}}^j P_j \triangleq i_j^*(t^\mu \phi_\mu)$ in terms of the localization (B.8)–(B.9) of $\phi_\mu \in H^2(Y)$ to the T -fixed points. Explicitly,

$$(6.27) \quad u_{\text{cl}}^i = t_0 + \alpha_2 \sum_{j \geq i} t_j (n+1-j) + \alpha_1 \sum_{j < i} j t_j \\ = t_0 + z((n+1-i)b - a) \ln \kappa_i + \frac{za}{n+1} \sum_{j=1}^n \ln \kappa_j - zb \sum_{j=i+1}^n \ln \kappa_j.$$

By the discussion of Section 6.2.1 and (6.13), the limit $\log \epsilon^i := \lim_{\kappa \rightarrow 0} (u^i - u_{\text{cl}}^i)$ must be finite. A direct calculation from equation (5.7) gives

$$\epsilon_i^{\frac{1}{z}} = \left(\frac{-w_i^+}{z}\right)^{-\frac{w_i^+}{z}} \left(\frac{w_i^-}{z}\right)^{-\frac{w_i^-}{z}} (-b)^{-b} e^{-i\pi(n+1-i)b}.$$

Then

$$\lim_{\kappa \rightarrow 0} e^{-\frac{u^j}{z}} \partial_i J_j^Y = \sqrt{\Delta_i(\kappa)}|_{\kappa=0} \epsilon_i^{-\frac{1}{z}} \delta_{ij} = i \sqrt{(\alpha_1 + \alpha_2) w_i^+ w_i^-} \delta_{ij},$$

where $\Delta_i(\kappa)$ is the Poincaré square-norm of ∂_{u^i} at κ , $\partial_i \triangleq \partial_{\tilde{u}^i} = \sqrt{\Delta_i} \partial_{u^i}$, and we pick the positive determination for the square root for all i . This sets

$$(\widetilde{R\mathcal{X}})_{ij}|_{\text{LR}} = \sqrt{\frac{w_i^+ w_i^-}{2\pi b}} \frac{1}{\epsilon_i^{\frac{1}{z}} \mathcal{A}_{ii}}.$$

With our choice (6.14) of Stokes sector, all arguments of the Γ -functions appearing in the diagonal of (5.29) have large positive real part for small $|z|$. Making use again of Stirling's formula, equation (6.24), yields

$$\mathcal{A}_{ii} = z e^{\pi i(n-i+1)b} \frac{\Gamma(1 + \frac{w_i^-}{z})}{\Gamma(1-b)\Gamma(-\frac{w_i^+}{z})} \simeq \frac{1}{\epsilon_i^{\frac{1}{z}} (\mathcal{D}_Y)_i} \sqrt{\frac{w_i^- w_i^+}{2\pi b}},$$

so that

$$(\widetilde{R\mathcal{X}})_{ij}|_{\text{LR}} = (\mathcal{D}_Y)_i \delta_{ij}$$

and therefore $\widetilde{R\mathcal{X}} = R_Y$ near LR by equation (6.10), which concludes the proof.

A. Gromov–Witten theory background

This appendix reviews and synthesizes key aspects of Gromov–Witten theory, for the benefit of the non-expert reader. Let \mathcal{Z} be a smooth Deligne–Mumford stack with coarse moduli space Z and suppose that \mathcal{Z} carries an algebraic $T \simeq \mathbb{C}^*$ action with zero-dimensional fixed loci. Write $I\mathcal{Z}$ for the inertia stack of \mathcal{Z} , $\text{inv} : I\mathcal{Z} \rightarrow I\mathcal{Z}$ for its canonical involution and $i : I\mathcal{Z}^T \hookrightarrow I\mathcal{Z}$ for the inclusion of the T -fixed loci into $I\mathcal{Z}$. The equivariant Chen–Ruan cohomology ring $H(\mathcal{Z}) \triangleq H_T^{\text{orb}}(\mathcal{Z})$ of \mathcal{Z} is a finite rank free module over the T -equivariant cohomology of a point $H(BT) \simeq \mathbb{C}[\nu]$, where $\nu = c_1(\mathcal{O}_{BT}(1))$; we define

$$N_{\mathcal{Z}} \triangleq \text{rank}_{\mathbb{C}[\nu]} H(\mathcal{Z})$$

and denote by $\Delta_{\mathcal{Z}}$ the free module over $\mathbb{C}[\nu]$ spanned by the T -equivariant lifts of Chen–Ruan cohomology classes having age-shifted degree at most two. We assume that odd cohomology groups vanish in all degrees.

A.1. Quantum D -modules in GW theory. The T -action on \mathcal{Z} gives a non-degenerate inner product on $H(\mathcal{Z})$ via the equivariant orbifold Poincaré pairing

$$(A.1) \quad \eta(\theta_1, \theta_2)_{\mathcal{Z}} \triangleq \int_{I\mathcal{Z}^T} \frac{i^*(\theta_1 \cup \text{inv}^* \theta_2)}{e(N_{I\mathcal{Z}^T/I\mathcal{Z}})},$$

and this pairing induces a torus action on the moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{Z}, \beta)$ of degree β twisted stable maps [1, 22] from genus- g orbicurves to \mathcal{Z} . For classes $\theta_1, \dots, \theta_n \in H(\mathcal{Z})$ and integers

$r_1, \dots, r_n \in \mathbb{N}$, the Gromov–Witten invariants of \mathcal{Z}

$$(A.2) \quad \langle \sigma_{r_1}(\theta_1) \dots \sigma_{r_n}(\theta_n) \rangle_{g,n,\beta}^{\mathcal{Z}} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{Z},\beta)]_T^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^* \theta_i \psi_i^{r_i},$$

$$(A.3) \quad \langle \theta_1 \dots \theta_n \rangle_{g,n,\beta}^{\mathcal{Z}} \triangleq \langle \sigma_0(\theta_1) \dots \sigma_0(\theta_n) \rangle_{g,n,\beta}^{\mathcal{Z}}$$

define a sequence of multi-linear functions on $H(\mathcal{Z})$ with values in the field of fractions $\mathbb{C}(\nu)$ of $H(BT)$ (the integrals in (A.2) are defined by localization). The correlators (A.3) (respectively, (A.2) with $r_i > 0$) are the *primary* (respectively, *descendent*) Gromov–Witten invariants of \mathcal{Z} .

Fix a basis $\{\phi_i\}_{i=0}^{N_{\mathcal{Z}}-1}$ of $H(\mathcal{Z})$ such that $\phi_0 = \mathbf{1}_{\mathcal{Z}}$ and $\phi_j, 1 \leq j \leq b_2(\mathcal{Z})$, are untwisted T -equivariant divisors in \mathcal{Z} . Denote by $\{\phi^i\}_{i=0}^{N_{\mathcal{Z}}-1}$ the dual basis with respect to the pairing. Let $\tau = \sum \tau^i \phi_i$ denote a general point of $H(\mathcal{Z})$. The WDVV equation for primary Gromov–Witten invariants (A.3) defines a family of associative deformations \circ_{τ} of the T -equivariant Chen–Ruan cohomology ring of \mathcal{Z} via

$$(A.4) \quad \eta(\theta_1 \circ_{\tau} \theta_2, \theta_3)_{\mathcal{Z}} \triangleq \langle \langle \theta_1, \theta_2, \theta_3 \rangle \rangle_{0,3}^{\mathcal{Z}}(\tau),$$

where

$$(A.5) \quad \langle \langle \theta_1, \dots, \theta_k \rangle \rangle_{0,k}^{\mathcal{Z}}(\tau) \triangleq \sum_{\beta} \sum_{n \geq 0} \frac{\langle \theta_1, \dots, \theta_k, \overbrace{\tau, \tau, \dots, \tau}^{n \text{ times}} \rangle_{0,n+k,\beta}^{\mathcal{Z}}}{n!} \in \mathbb{C}((\nu)),$$

and the index β ranges over the semigroup of effective curves $\text{Eff}(\mathcal{Z}) \subset H_2(\mathcal{Z}, \mathbb{Q})$; we denote by $l_{\mathcal{Z}} \triangleq l_{\mathcal{Z}}$ its rank. Applying the Divisor Axiom [1], equation (A.5) can be rewritten as

$$(A.6) \quad \eta(\theta_1 \circ_{\tau} \theta_2, \theta_3)_{\mathcal{Z}} = \sum_{\beta \in \text{Eff}(\mathcal{Z}), n \geq 0} \frac{\langle \theta_1, \theta_2, \theta_3, \overbrace{\tau', \tau', \dots, \tau'}^{n \text{ times}} \rangle_{0,n+3,\beta}^{\mathcal{Z}}}{n!} e^{\tau_{0,2} \cdot \beta},$$

where we have decomposed $\tau = \sum_{i=0}^{N_{\mathcal{Z}}-1} \tau^i \phi_i = \tau_{0,2} + \tau'$ as

$$\begin{aligned} \tau_{0,2} &= \sum_{i=1}^{l_{\mathcal{Z}}} \tau^i \phi_i, \\ \tau' &= \tau^0 \mathbf{1}_{\mathcal{Z}} + \sum_{i=l_{\mathcal{Z}}+1}^{N_{\mathcal{Z}}-1} \tau^i \phi_i. \end{aligned}$$

The quantum product (A.6) is a formal Taylor series in $(\tau', e^{\tau_{0,2}})$. Suppose that it is actually *convergent* in a contractible open set $U \ni (0, 0)$; this is the case for many toric orbifolds [26, 43] and for all the examples of Section 4. The quantum product \circ_{τ} is an analytic deformation of the Chen–Ruan cup product \cup_{CR} , to which it reduces in the limit $\tau' \rightarrow 0$, $\Re e(\tau_{0,2}) \rightarrow -\infty$. Thus, the holomorphic family of rings $H(\mathcal{Z}) \times U \rightarrow U$, together with the equivariant Poincaré pairing and the associative product (A.6), gives U the structure of a (non-conformal) Frobenius manifold $QH(\mathcal{Z}) \triangleq (U, \eta, \circ_{\tau})$, cf. [36]; this is the *quantum cohomology ring* of \mathcal{Z} . We refer to the Chen–Ruan limit $\tau' \rightarrow 0, \Re e(\tau_{0,2}) \rightarrow -\infty$ as the *large radius limit point* of \mathcal{Z} .

Assigning a Frobenius structure on U amounts to endowing the trivial cohomology bundle $TU \simeq H(\mathcal{Z}) \times U \rightarrow U$ with a flat pencil of affine connections [36, Lecture 6]. Denote by $\nabla^{(\eta)}$ the Levi–Civita connection associated to the Poincaré pairing on $H(\mathcal{Z})$; in Cartesian coordinates for $U \subset H(\mathcal{Z})$ this reduces to the ordinary de Rham differential $\nabla^{(\eta)} = d$. The one-parameter family of covariant derivatives on TU

$$(A.7) \quad \nabla_X^{(\eta,z)} \triangleq \nabla_X^{(\eta)} + z^{-1} X \circ_\tau$$

is called the *Dubrovin connection*. The fact that the quantum product is commutative, associative and integrable implies that $R_{\nabla^{(\eta,z)}} = T_{\nabla^{(\eta,z)}} = 0$ identically in z ; this statement is equivalent to the WDVV equations for the genus-zero Gromov–Witten potential. The equation for the horizontal sections of $\nabla^{(\eta,z)}$,

$$(A.8) \quad \nabla^{(\eta,z)} \omega = 0,$$

is a rank- $N_{\mathcal{Z}}$ holonomic system of coupled linear PDEs. We denote by $\mathcal{S}_{\mathcal{Z}}$ the vector space of solutions of equation (A.8): a $\mathbb{C}((z))$ -basis of $\mathcal{S}_{\mathcal{Z}}$ is by definition given by the gradient of a flat frame $\tilde{\tau}(\tau, z)$ for the deformed connection $\nabla^{(\eta,z)}$. The Poincaré pairing induces a non-degenerate inner product $H(s_1, s_2)_{\mathcal{Z}}$ on $\mathcal{S}_{\mathcal{Z}}$ via

$$(A.9) \quad H(s_1, s_2)_{\mathcal{Z}} \triangleq \eta(s_1(\tau, -z), s_2(\tau, z))_{\mathcal{Z}}.$$

The triple $\text{QDM}(\mathcal{Z}) \triangleq (U, \nabla^{(\eta,z)}, H(\cdot, \cdot)_{\mathcal{Z}})$ defines a *quantum D -module* structure on U , and system (A.8) is the *quantum differential equation* (in short, QDE) of \mathcal{Z} .

Remark A.1. The assumption that the quantum product (A.6) is analytic in $(\tau', e^{\tau_0, 2})$ around the large radius limit point translates into the statement that the QDE (A.8) has a Fuchsian singularity along $\bigcup_{i=1}^{l_{\mathcal{Z}}} \{q_i \triangleq e^{\tau^i} = 0\}$.

In the same way in which the genus-zero primary theory of \mathcal{Z} defines a quantum D -module structure on $H(\mathcal{Z}) \times U$, the genus-zero gravitational invariants (A.2) furnish a basis of horizontal sections of $\nabla^{(\eta,z)}$, cf. [42]. For every $\theta \in H(\mathcal{Z})$, a flat section of the D -module is given by an $\text{End}(H(\mathcal{Z}))$ -valued function $S_{\mathcal{Z}}(\tau, z) : H(\mathcal{Z}) \rightarrow \mathcal{S}_{\mathcal{Z}}$ defined as

$$(A.10) \quad S_{\mathcal{Z}}(\tau, z)\theta \triangleq \theta - \sum_{k=0}^{N_{\mathcal{Z}}-1} \phi^k \left\langle \left\langle \phi_k, \frac{\theta}{z + \psi} \right\rangle \right\rangle_{0,2}^{\mathcal{Z}}(\tau),$$

where ψ is a cotangent line class and we expand the denominator as a geometric series in $-\frac{\psi}{z}$. The pair $(\text{QDM}(\mathcal{Z}), S_{\mathcal{Z}})$ is called the *S -calibration* of the Frobenius structure (U, \circ_τ, η) .

The flows of coordinate vectors for the flat frame of $TH(\mathcal{Z})$ induced by $S_{\mathcal{Z}}(\tau, z)$ give a basis of flat coordinates of $\nabla^{(\eta,z)}$, which is defined uniquely up to an additive z -dependent constant. A canonical basis is obtained upon applying the String Axiom: define the *J -function* $J^{\mathcal{Z}}(\tau, z) : U \times \mathbb{C} \rightarrow H(\mathcal{Z})$ by

$$(A.11) \quad J^{\mathcal{Z}}(\tau, z) \triangleq z S_{\mathcal{Z}}(\tau, -z)^{\dagger} \mathbf{1}_{\mathcal{Z}},$$

where $S_{\mathcal{Z}}(\tau, z)^{\dagger}$ denotes the adjoint of $S_{\mathcal{Z}}(\tau, z)$ under $\eta(-, -)_{\mathcal{Z}}$. Explicitly,

$$(A.12) \quad J^{\mathcal{Z}}(\tau, z) = (z + \tau^0) \mathbf{1}_{\mathcal{Z}} + \tau^1 \phi_1 + \cdots + \tau^{N_{\mathcal{Z}}} \phi_{N_{\mathcal{Z}}} + \sum_{k=0}^{N_{\mathcal{Z}}-1} \phi^k \left\langle \left\langle \frac{\phi_k}{z - \psi_{n+1}} \right\rangle \right\rangle_{0,1}^{\mathcal{Z}}(\tau).$$

Components of $J^{\mathcal{Z}}(\tau, -z)$ in the ϕ -basis give flat coordinates of equation (A.7); this is a consequence of equation (A.11) combined with the String Equation. From equation (A.12), the undeformed flat coordinate system is obtained in the limit $z \rightarrow \infty$ as

$$\lim_{z \rightarrow \infty} (J^{\mathcal{Z}}(\tau, -z) + z \mathbf{1}_{\mathcal{Z}}) = \tau.$$

By Remark A.1, a loop around the origin in the variables $q_i = e^{\tau^i}$ gives a non-trivial monodromy action on the J -function. Setting $\tau' = 0$ in equation (A.12) and applying the Divisor Axiom then gives (cf. [30, Proposition 10.2.3])

$$\begin{aligned} \text{(A.13)} \quad J^{\mathcal{Z}, \text{sm}}(\tau_{0,2}, z) &\triangleq J^{\mathcal{Z}}(\tau, z)|_{\tau'=0} \\ &= z e^{\tau^1 \frac{\phi_1}{z}} \dots e^{\tau^{l_{\mathcal{Z}}} \frac{\phi_{l_{\mathcal{Z}}}}{z}} \\ &\quad \times \left(\mathbf{1}_{\mathcal{Z}} + \sum_{\beta, k} e^{\tau^1 \beta_1} \dots e^{\tau^{l_{\mathcal{Z}}} \beta_{l_{\mathcal{Z}}}} \phi^k \left\langle \frac{\phi_k}{z(z - \psi_1)} \right\rangle_{0,1,\beta}^{\mathcal{Z}} \right). \end{aligned}$$

In our situation where the T -action has only zero-dimensional fixed loci $\{P_i\}_{i=1}^{N_{\mathcal{Z}}}$, write

$$\phi_i \rightarrow \sum_{j=1}^{N_{\mathcal{Z}}} c_{ij}(v) P_j, \quad i = 1, \dots, l_{\mathcal{Z}},$$

for the image of $\{\phi_i \in H^2(\mathcal{Z}, \mathbb{C})\}_{i=1}^{l_{\mathcal{Z}}}$ under the Atiyah–Bott isomorphism. The image of each ϕ_i is concentrated on the fixed point cohomology classes with trivial isotropy which are idempotents of the classical Chen–Ruan cup product on $H(\mathcal{Z})$. Therefore, the components of the J -function in the fixed points basis

$$J^{\mathcal{Z}, \text{sm}}(\tau_{0,2}, z) =: \sum_{j=1}^{N_{\mathcal{Z}}} J_j^{\mathcal{Z}, \text{sm}}(\tau_{0,2}, z) P_j$$

satisfy

$$\text{(A.14)} \quad J_j^{\mathcal{Z}, \text{sm}}(\tau_{0,2}, z) = z e^{\sum_{i=1}^{l_{\mathcal{Z}}} \tau^i \frac{c_{ij}}{z}} (1 + \mathcal{O}(e^{\tau_{0,2}})),$$

where the $\mathcal{O}(e^{\tau_{0,2}})$ term on the right-hand side is an analytic power series around $e^{\tau_{0,2}} = 0$ by equation (A.13) and the assumption of convergence of the quantum product. The localized basis $\{P_j\}_{j=1}^{N_{\mathcal{Z}}}$ therefore diagonalizes the monodromy around large radius: by (A.14), each $J_j^{\mathcal{Z}, \text{sm}}(\tau_{0,2}, z)$ is an eigenvector of the monodromy around a loop in the q_i -plane encircling the large radius limit of \mathcal{Z} with eigenvalue $e^{2\pi i c_{ij}/z}$.

A.1.1. Toric data and trivializations. Suppose that $c_1(\mathcal{Z}) \geq 0$ and that the coarse moduli space Z is a semi-projective toric variety given by a GIT quotient of $\mathbb{C}^{\dim \mathcal{Z} + n_{\mathcal{Z}}}$ by $(\mathbb{C}^*)^{n_{\mathcal{Z}}}$. In this setting, the global quantum D -module arises naturally in the form of the Picard–Fuchs system associated to \mathcal{Z} , cf. [7, 25, 43]. The scaling factor $\hbar_{\mathcal{Z}}^{1/z}$ then measures the discrepancy between the small J -function and the canonical basis-vector of solutions of the Picard–Fuchs system (the I -function), restricted to zero twisted insertions:¹²⁾

$$\text{(A.15)} \quad \hbar_{\mathcal{Z}}^{1/z}(\tau_{0,2}) J^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z) = I^{\mathcal{Z}}(\alpha(\tau_{0,2}), z),$$

¹²⁾ See [23] for a discussion of why in equivariant Gromov–Witten theory $I^{\mathcal{Z}}$ and $J^{\mathcal{Z}, \text{small}}$ could a priori differ, even in the semi-positive case, by a uniquely determined scaling factor induced by the String Equation.

where $\alpha(\tau_{0,2})$ is the inverse mirror map. As a consequence of equation (A.15), the scaling factor $\mathfrak{h}_{\mathcal{Z}}$ is determined by the toric data defining \mathcal{Z} , cf. [25, 28, 43]. Let $\Xi_i \in H^2(\mathcal{Z})$ be the T -equivariant Poincaré dual of the reduction to the quotient of the i th coordinate hyperplane in $\mathbb{C}^{\dim \mathcal{Z} + n_{\mathcal{Z}}}$ and write $\zeta_i^{(j)} = \text{Coeff}_{\phi_j} \Xi_i \in \mathbb{C}[\nu]$ for the coefficient of the projection of Ξ_i along $\phi_j \in H(\mathcal{Z})$ for $j = 0, \dots, n_{\mathcal{Z}}$. Defining, for every β ,

$$D_i(\beta) \triangleq \int_{\beta} \Xi_i$$

and

$$J_{\beta}^{\pm} \triangleq \{j \in \{1, \dots, \dim \mathcal{Z} + n_{\mathcal{Z}}\} : \pm D_j(\beta) > 0\},$$

we have

$$\tau^l = \log \alpha_l + \sum_{\beta \in \text{Eff}(\mathcal{Z})} \alpha^{\beta} \frac{\prod_{j \in J_{\beta}^{-}} (-1)^{D_{j-}(\beta)} |D_{j-}(\beta)|!}{\prod_{j \in J_{\beta}^{+}} D_{j+}(\beta)!} \sum_{k \in J_{\beta}^{-}} \frac{-\zeta_{k-}^{(l)}}{D_{k-}(\beta)}, \quad l = 1, \dots, n_{\mathcal{Z}},$$

and

$$(A.16) \quad \mathfrak{h}_{\mathcal{Z}} = \exp \left[\sum_{\beta \in \text{Eff}(\mathcal{Z})} \alpha^{\beta} \frac{\prod_{j \in J_{\beta}^{-}} (-1)^{D_{j-}(\beta)} |D_{j-}(\beta)|!}{\prod_{j \in J_{\beta}^{+}} D_{j+}(\beta)!} \sum_{k \in J_{\beta}^{-}} \frac{-\zeta_{k-}^{(0)}}{D_{k-}(\beta)} \right].$$

A.2. Givental's symplectic structures and quantization. Let $(\mathcal{H}_{\mathcal{Z}}, \Omega_{\mathcal{Z}})$ be a pair given by the infinite-dimensional vector space

$$(A.17) \quad \mathcal{H}_{\mathcal{Z}} \triangleq H(\mathcal{Z}) \otimes \mathcal{O}(\mathbb{C}^*)$$

endowed with the symplectic form

$$(A.18) \quad \Omega_{\mathcal{Z}}(f, g) \triangleq \text{Res}_{z=0} \eta(f(-z), g(z))_{\mathcal{Z}}.$$

A general point of $\mathcal{H}_{\mathcal{Z}}$ can be written in Darboux coordinates for (A.18); as

$$\sum_{k \geq 0} \sum_{\alpha=0}^{N_{\mathcal{Z}}-1} q^{\alpha, k} \phi_{\alpha} z^k + \sum_{l \geq 0} \sum_{\beta=0}^{N_{\mathcal{Z}}-1} p_{\beta, l} \phi_{\beta} (-z)^{-k-1}.$$

We call $\mathcal{H}_{\mathcal{Z}}^+$ the Lagrangian subspace spanned by $q^{\alpha, k}$.

The genus-zero Gromov–Witten theory of \mathcal{Z} can be compactly codified through the symplectic geometry of $\mathcal{H}_{\mathcal{Z}}$ as follows [46]. The generating function of genus-zero descendent Gromov–Witten invariants of \mathcal{Z} ,

$$(A.19) \quad \mathcal{F}_0^{\mathcal{Z}} \triangleq \sum_{n=0}^{\infty} \sum_{\beta \in \text{Eff}(\mathcal{Z})} \sum_{\substack{a_1, \dots, a_n \\ r_1, \dots, r_n}} \frac{\prod_{i=1}^n t^{a_i, r_i}}{n!} \langle \sigma_{r_1}(\phi_{a_1}) \dots \sigma_{r_n}(\phi_{a_n}) \rangle_{0, n, \beta}^{\mathcal{Z}},$$

is the germ of an analytic function on $\mathcal{H}_{\mathcal{Z}}^+$ upon identifying $t^{0,1} = q^{0,1} + 1$, $t^{\alpha, n} = q^{\alpha, n}$; under the assumption of convergence of the quantum product, the coefficients of $t^{\alpha_1, n_1} \dots t^{\alpha_r, n_r}$ with $n_i(\deg \phi_{\alpha_i} - 2) \neq 0$ are analytic functions of $e^{\iota_{2,0}}$ in a neighborhood of the origin; the mirror

theorem of [25] guarantees that this is the case when \mathcal{Z} is semi-positive. As is often common [59], in writing equation (A.19) and in the following we chose to dispose altogether of the Novikov variables; there is no loss of information however here about the degree of the curves by virtue of the Divisor Axiom; see e.g. [28, Remarks 5.3 and 5.5] for a discussion of both the primary and the descendent theory.

The graph $\mathcal{L}_{\mathcal{Z}}$ of the differential of (A.19),

$$(A.20) \quad \mathcal{L}_{\mathcal{Z}} = \left\{ (q, p) \in \mathcal{H}_{\mathcal{Z}} : p_{l,\beta} = \frac{\partial \mathcal{F}_0^{\mathcal{Z}}}{\partial q^{l,\beta}} \right\},$$

is by design a formal germ of a Lagrangian submanifold. This is a ruled cone [46], as a consequence of the genus-zero Gromov–Witten axioms, depending analytically on the small quantum cohomology variables $t^{0,2}$ around the large radius limit point of \mathcal{Z} . By the equations defining the cone, the J -function $J^{\mathcal{Z}}(\tau, -z)$ yields a family of elements of $\mathcal{L}_{\mathcal{Z}}$ parameterized by $\tau \in H(\mathcal{Z})$, which is uniquely determined by its large z asymptotics

$$J(\tau, -z) = -z + \tau + \mathcal{O}(z^{-1}).$$

Conversely, the genus-zero topological recursion relations imply that $\mathcal{L}_{\mathcal{Z}}$ can be reconstructed entirely from $J^{\mathcal{Z}}(\tau, z)$.

A.2.1. The R -calibration and quantization. When \mathcal{Z} is a manifold, Givental’s theorem for equivariant Gromov–Witten invariants [44, 45] asserts that the higher genus theory of \mathcal{Z} is obtained through Weyl-quantization of a pair $(S_{\mathcal{Z}}, R_{\mathcal{Z}})$ of symplectic automorphisms of $(\mathcal{H}_{\mathcal{Z}}, \Omega_{\mathcal{Z}})$, both of which are determined *from genus-zero data alone*. More in detail, the Gromov–Witten canonical S -calibration satisfies

$$\eta(S_{\mathcal{Z}}(\tau, -z)\theta, S_{\mathcal{Z}}(\tau, z)\theta')_{\mathcal{Z}} = \eta(\theta, \theta')_{\mathcal{Z}}$$

for arbitrary cohomology classes $\theta, \theta' \in H(\mathcal{Z})$, as can be readily seen upon differentiating the left-hand side with respect to τ . Allowing θ, θ' to be formal cohomology-valued Laurent series in z , this implies that for fixed τ , $S_{\mathcal{Z}}(\tau, z)$ belongs to the *negative symplectic loop group* $\mathrm{Sp}_{-}(\mathcal{H}_{\mathcal{Z}})$ of \mathcal{Z} : an element of $\mathrm{GL}(H(\mathcal{Z}))[[z^{-1}]]$ which is a symplectic automorphism of $(\mathcal{H}_{\mathcal{Z}}, \Omega_{\mathcal{Z}})$.

Notice that the $R_{\mathcal{Z}}$ -calibration is instead an element of the *positive* symplectic loop group $\mathrm{Sp}_{+}(\mathcal{H}_{\mathcal{Z}}) \cap \mathrm{GL}(H(\mathcal{Z}))[[z]]$, constructed as follows. Let $\tilde{\tau} \in U$ be such that the Frobenius algebra on $T_{\tilde{\tau}}H(\mathcal{Z})$ is semi-simple for τ in a neighborhood $V_{\tilde{\tau}}$ of $\tilde{\tau}$. Then [36] there exist local coordinates $\{u^i\}_{i=1}^{N_{\mathcal{Z}}}$ such that their coordinate vector fields $\partial_{u^i} \in \mathcal{X}(V_{\tilde{\tau}})$ are a basis of idempotents of \circ_{τ} . Denoting by Δ_i their squared norm with respect to the flat pairing, we obtain a normalized orthonormal system $\tilde{u}^i \triangleq u^i / \sqrt{\Delta_i}$ of local coordinates. Then an R -calibration for \mathcal{Z} is a choice of a $\mathbb{C}((z))$ -basis of horizontal 1-forms on $V_{\tilde{\tau}}$

$$(R_{\mathcal{Z}})_i^j(\tau, z) e^{\frac{u^j}{z}} d\tilde{u}^i, \quad i = 1, \dots, N_{\mathcal{Z}},$$

where $(R_{\mathcal{Z}})_i^j(\tau, z)$ is an asymptotic $\mathrm{End}(T_{\tilde{\tau}}H)$ -valued series in z satisfying

$$\sum_j (R_{\mathcal{Z}})_i^j(\tau, z) (R_{\mathcal{Z}})_k^j(\tau, -z) = \delta_i^k.$$

If we let Ψ be the differential of \tilde{u} , the above implies that $\Psi_{\mathcal{Z}} R_{\mathcal{Z}} e^{\frac{u}{z}}$ is a holomorphic family of symplectomorphisms parameterized by $\tau \in V_{\tilde{\tau}}$.

Givental's quantization formalism assigns quantum operators to the S - and R -calibrations of \mathcal{Z} . The *Fock space* \mathbb{F}_τ of \mathcal{Z} at τ is the space of functions $f(\lambda, t_\tau)$ of the form

$$f(\lambda, t_\tau) = \sum_{g \in \mathbb{Z}} \lambda^{g-1} f_g(t_\tau),$$

where in terms of coordinates $\{q^{i,\alpha}\}$ we define

$$(A.21) \quad t_\tau^{\alpha,l} \triangleq q^{\alpha,l} - \delta^{l0} \tau^\alpha + \delta^{l1} \delta^{\alpha 0}$$

and

$$f_g(t_\tau) \in \mathbb{C}[[t_\tau]].$$

To each symplectomorphism connected with the identity $\mathcal{Q} \in \text{Aut}_0(\mathcal{H}_\mathcal{Z}, \Omega_\mathcal{Z})$, we associate a quantized operator $\widehat{\mathcal{Q}}$ acting on \mathbb{F}_τ via

$$\widehat{\mathcal{Q}} = e^{\widehat{\log \mathcal{Q}}},$$

where we define the quantization of an infinitesimal symplectomorphism to be the quantization of its quadratic Hamiltonian with the normal-ordering prescriptions

$$\begin{aligned} \widehat{p_{\alpha,k} p_{\beta,l}} &= \lambda \frac{\partial^2}{\partial q^{\alpha,k} \partial q^{\beta,l}}, \\ \widehat{p_{\alpha,k} q^{\beta,l}} &= \widehat{q^{\beta,l} p_{\alpha,k}} = q^{\beta,l} \frac{\partial}{\partial q^{\alpha,k}}, \\ \widehat{q^{\alpha,k} q^{\beta,l}} &= \lambda^{-1} q^{\alpha,k} q^{\beta,l}. \end{aligned}$$

Let now $Z_\mathcal{Z} \in \mathbb{F}_\tau$ be the generating function of disconnected Gromov–Witten invariants of \mathcal{Z} ,

$$Z_\mathcal{Z} = \exp \sum_{g \geq 0} \lambda^{g-1} \mathcal{F}_g^\mathcal{Z},$$

in the shifted variables (A.21), and denote likewise by $Z_{\text{pt}} \in (-q^1)^{-1/24} \mathbb{C}[[1/q^1, q^{k>1}]][[q^0]]$ the generating function of disconnected Gromov–Witten invariants of the point in dilation-unshifted variables. In the context of fixed point localization for toric orbifolds, knowledge of the T -action fixes uniquely a canonical choice for the symplectomorphism $R_\mathcal{Z}$ (cf. [44, 77]), which we call *the Gromov–Witten R -calibration*. The Givental–Teleman theorem [44, 45, 75] then asserts that $Z_\mathcal{Z}$ can be obtained from Z_{pt} via the action of operators $\widehat{S}_\mathcal{Z}$ and $\widehat{R}_\mathcal{Z}$ obtained by quantizing the canonical S - and R -calibrations of $QH(\mathcal{Z})$ at τ , giving formula (6.1).

B. A_n -resolutions

B.1. GIT quotients. Here we review the relevant toric geometry concerning our targets. Let $\mathcal{X} \triangleq [\mathbb{C}^3/\mathbb{Z}_{n+1}]$ be the 3-fold A_n singularity and Y its resolution. The toric fan for \mathcal{X} has rays $(0, 0, 1)$, $(1, 0, 0)$, and $(1, n+1, 0)$, while the fan for Y is obtained by adding the rays $(1, 1, 0)$, $(1, 2, 0)$, \dots , $(1, n, 0)$. The divisor class group is described by the short exact sequence

$$(B.1) \quad 0 \longrightarrow \mathbb{Z}^n \xrightarrow{M^T} \mathbb{Z}^{n+3} \xrightarrow{N} \mathbb{Z}^3 \longrightarrow 0,$$

where

$$M = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & 2 & \dots & n+1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Both \mathcal{X} and Y are GIT quotients:

$$(B.2) \quad \mathcal{X} = \left[\frac{\mathbb{C}^{n+3} \setminus V(x_1 \cdots x_n)}{(\mathbb{C}^*)^n} \right],$$

$$(B.3) \quad Y = \frac{\mathbb{C}^{n+3} \setminus V(I_1, \dots, I_{n+1})}{(\mathbb{C}^*)^n},$$

where

$$I_i = \prod_{j=0, j \neq i-1, i}^{n+1} x_j,$$

and the torus action is specified by M . From the quotient (B.2), we can compute coordinates on the orbifold

$$(B.4) \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_0 x_1^{\frac{n}{n+1}} x_2^{\frac{n-1}{n+1}} \cdots x_n^{\frac{1}{n+1}} \\ x_1^{\frac{1}{n+1}} x_2^{\frac{2}{n+1}} \cdots x_n^{\frac{n}{n+1}} x_{n+1} \\ x_{n+2} \end{bmatrix}.$$

These coordinates are only defined up to a choice of $(n+1)$ st root of unity for each x_i . This accounts for a residual $\mathbb{Z}_{n+1} \subset (\mathbb{C}^*)^n$ acting with dual representations on the first two coordinates. We identify this residual \mathbb{Z}_{n+1} as the subgroup generated by $(\omega, \omega^2, \dots, \omega^n) \in (\mathbb{C}^*)^n$, where $\omega = e^{\frac{2\pi i}{n+1}}$. This realizes the quotient (B.2) as the 3-fold A_n singularity where the group $\mathbb{Z}_{n+1} = \langle \omega \rangle$ acts by $\omega \cdot (z_1, z_2, z_3) = (\omega z_1, \omega^{-1} z_2, z_3)$.

Remark B.1. The weights of the \mathbb{Z}_{n+1} action on the corresponding fibers of $T\mathcal{X}$ are inverse to the weights on the local coordinates because a local trivialization of the tangent bundle is given by $\frac{\partial}{\partial z^\alpha}$, where z^α are the local coordinates.

The geometry of the space Y is captured by the toric web diagram in Figure 6. In particular, Y has $n+1$ torus fixed points (corresponding to the $n+1$ three-dimensional cones in the fan) and a chain of n torus invariant lines connecting these points. We label the points p_1, \dots, p_{n+1} , where p_i corresponds to the cone spanned by $(0, 0, 1)$, $(1, i-1, 0)$, and $(1, i, 0)$ and we label the torus invariant lines by L_1, \dots, L_n , where L_i connects p_i to p_{i+1} . We also denote by L_0 and L_{n+1} the torus invariant (affine) lines corresponding to the two-dimensional

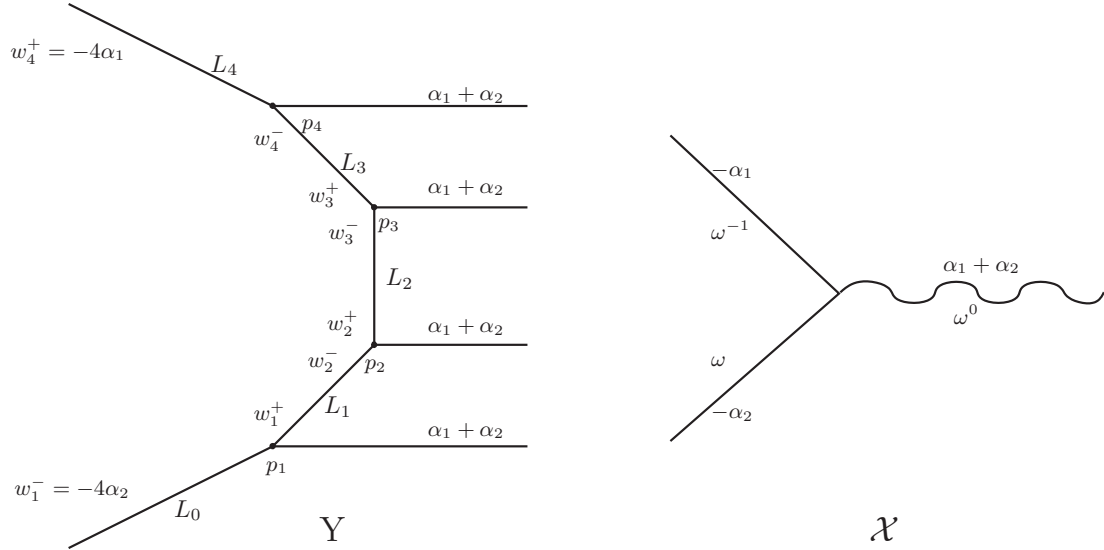


Figure 6. The toric web diagrams for Y and \mathcal{X} for $n = 3$. Fixed points and invariants lines are labelled, together with the relevant torus and representation weights.

cones spanned by the rays $(1, 0, 0)$, $(0, 0, 1)$ and $(1, n, 0)$, $(0, 0, 1)$, respectively. From the quotient (B.3) we compute homogeneous coordinates on the line L_i

$$(B.5) \quad \begin{bmatrix} x_0^i x_1^{i-1} \cdots x_{i-1} \\ x_{n+1}^{n+1-i} x_{n-1}^{n-i} \cdots x_{i+1} \end{bmatrix},$$

where $p_i \leftrightarrow [0 : 1]$ and $p_{i+1} \leftrightarrow [1 : 0]$.

Note that $H_2(Y)$ is generated by the torus invariant lines L_i . Define $\phi_i \in H^2(Y)$ to be dual to L_i . The ϕ_i form a basis of $H^2(Y)$; denote the corresponding line bundles by $\mathcal{O}(\phi_i)$. Note that $\mathcal{O}(\phi_i)$ restricts to $\mathcal{O}(1)$ on L_i and \mathcal{O} on L_j if $j \neq i$ and this uniquely determines the line bundle $\mathcal{O}(\phi_i)$. On the orbifold, line bundles correspond to \mathbb{Z}_{n+1} equivariant line bundles on \mathbb{C}^3 . We denote \mathcal{O}_k the line bundle where \mathbb{Z}_{n+1} acts on fibers with weight ω^k ; then, for example, $T_{\mathcal{X}} = \mathcal{O}_{-1} \oplus \mathcal{O}_1 \oplus \mathcal{O}_0$, where the subscripts are computed modulo $n + 1$ (cf. (B.1)).

B.2. Classical equivariant geometry. Given that we are working with non-compact targets, all of our quantum computations utilize Atiyah–Bott localization with respect to an additional $T = \mathbb{C}^*$ action on our spaces. Let T act on \mathbb{C}^{n+3} with the following weights: $(\alpha_1, 0, \dots, 0, \alpha_2, -\alpha_1 - \alpha_2)$. Then the induced action on the orbifold and resolution can be read off from the local coordinates in (B.4) and (B.5). In particular, the three weights on the fibers of $T_{\mathcal{X}}$ are $-\alpha_1, -\alpha_2, \alpha_1 + \alpha_2$. As a $H_T(\text{pt})$ -module, the T -equivariant Chen–Ruan cohomology $H^\bullet(\mathcal{X})$ of X is by definition the T -equivariant cohomology of the inertia stack $\mathcal{I}\mathcal{X}$. The latter has components $\mathcal{X}_1, \dots, \mathcal{X}_n, \mathcal{X}_{n+1}$, the last being the untwisted sector:¹³⁾

$$\begin{aligned} \mathcal{X}_k &= [\mathbb{C}/\mathbb{Z}_{n+1}], \quad 1 \leq k \leq n, \\ \mathcal{X}_{n+1} &= [\mathbb{C}^3/\mathbb{Z}_{n+1}]. \end{aligned}$$

¹³⁾ While it is more common to index the untwisted sector by 0, we make this choice of notation for the sake of the computations of Section 5, where certain matrices are triangular with this ordering.

Writing $\mathbf{1}_k$, $k = 1, \dots, n+1$, for the fundamental class of \mathcal{X}_k we obtain a $\mathbb{C}(v)$ basis of $H(\mathcal{X})$; the age-shifted grading assigns degree zero to the fundamental class of the untwisted sector, and degree one to every twisted sector. The Atiyah–Bott localization isomorphism is trivial, i.e. the fundamental class on each twisted sector is identified with the unique T -fixed point on that sector. We abuse notation and use $\mathbf{1}_k$ to also denote the fixed point basis. The equivariant Chen–Ruan pairing in orbifold cohomology is

$$\eta(\mathbf{1}_i, \mathbf{1}_j)_{\mathcal{X}} = \frac{\delta_{i,n+1}\delta_{j,n+1} + \alpha_1\alpha_2\delta_{i+j,n+1}}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)(n+1)}.$$

On the resolution Y , the three weights on the tangent bundle at p_i are

$$(B.6) \quad (w_i^-, w_i^+, \alpha_1 + \alpha_2) \triangleq ((i-1)\alpha_1 + (-n+i-2)\alpha_2, -i\alpha_1 + (n+1-i)\alpha_2, \alpha_1 + \alpha_2).$$

Moreover, $\mathcal{O}(\phi_j)$ is canonically linearized via the homogeneous coordinates in (B.4). The weight of $\mathcal{O}(\phi_j)$ at the fixed point p_i is

$$(B.7) \quad \begin{cases} (n+1-j)\alpha_2, & i \leq j, \\ j\alpha_1, & i > j. \end{cases}$$

Denote by $\{P_i\}_{i=1}^{n+1}$ the equivariant cohomology classes corresponding to the fixed points of Y . Choosing the canonical linearization given in (B.7), the Atiyah–Bott localization isomorphism on Y is given by

$$(B.8) \quad \phi_j \mapsto \sum_{i \leq j} (n+1-j)\alpha_2 P_i + \sum_{i > j} j\alpha_1 P_i, \quad j \neq n+1,$$

$$(B.9) \quad \phi_{n+1} \mapsto \sum_{i=1}^{n+1} P_i.$$

where ϕ_{n+1} is the fundamental class on Y . Genus zero, degree zero GW invariants are given by equivariant triple intersections on Y ,

$$\langle \phi_i, \phi_j, \phi_k \rangle_{0,3,0}^Y = \int_Y \phi_i \cup \phi_j \cup \phi_k.$$

With $i \leq j \leq k < n+1$, (B.8)–(B.9) yield

$$(B.10) \quad \langle \phi_{n+1}, \phi_{n+1}, \phi_{n+1} \rangle_{0,3,0}^Y = \frac{1}{(n+1)\alpha_1\alpha_2(\alpha_1 + \alpha_2)},$$

$$(B.11) \quad \langle \phi_{n+1}, \phi_{n+1}, \phi_i \rangle_{0,3,0}^Y = 0,$$

$$(B.12) \quad \langle \phi_{n+1}, \phi_i, \phi_j \rangle_{0,3,0}^Y = \frac{i(n+1-j)}{-(n+1)(\alpha_1 + \alpha_2)},$$

$$(B.13) \quad \langle \phi_i, \phi_j, \phi_k \rangle_{0,3,0}^Y = -\frac{ij(n+1-k)\alpha_1 + i(n+1-j)(n+1-k)\alpha_2}{(n+1)(\alpha_1 + \alpha_2)}.$$

The T -equivariant pairing $\eta(\phi_i, \phi_j)_Y$ is given by equation (B.12) and diagonalizes in the fixed point basis:

$$\eta(P_i, P_j)_Y = \frac{\delta_{i,j}}{w_i^- w_i^+ (\alpha_1 + \alpha_2)}.$$

B.3. Quantum equivariant geometry. We compute the genus-zero GW invariants of Y via localization (extending the computations of [16, Section 2] to a more general torus action):

$$(B.14) \quad \langle \phi_{i_1}, \dots, \phi_{i_l} \rangle_{0,l,\beta} = \begin{cases} -d^{l-3} & \text{if } \beta = d(L_j + \dots + L_k) \text{ with } j \leq \min\{i_\alpha\} \leq \max\{i_\alpha\} \leq k, \\ 0 & \text{else.} \end{cases}$$

Denote by $\Phi = \sum_{i=1}^{n+1} t_i \phi_i$ a general cohomology class $\Phi \in H(Y)$. The equivariant three-point correlators used to define the quantum cohomology can be computed from (B.10), (B.11), (B.12), (B.13) and (B.14) (with $1 \leq i \leq j \leq k < n+1$):

$$(B.15) \quad \langle \langle \phi_i, \phi_j, \phi_k \rangle \rangle_{0,3}^Y(t) = \int_Y \phi_i \cup \phi_j \cup \phi_k - \sum_{l \leq i \leq k \leq m} \frac{e^{t_l + \dots + t_m}}{1 - e^{t_l + \dots + t_m}}.$$

The equivariant quantum cohomology of \mathcal{X} can then be computed from the following result, which is proved in the appendix of [24].

Proposition B.2. *Let $\rho : [0, 1] \rightarrow H^2(Y)$ be as in (4.1). Then upon analytic continuation in the quantum parameters t_i along ρ , the quantum products for \mathcal{X} and Y coincide after the affine-linear change of variables*

$$(B.16) \quad t_i = (\widehat{j}_\rho^{\mathcal{X},Y} x)_i = \begin{cases} \frac{2\pi i}{n+1} + \sum_{k=1}^n \frac{\omega^{-ik}(\omega^{\frac{k}{2}} - \omega^{-\frac{k}{2}})}{n+1} x_k, & 0 < i < n+1, \\ x_{n+1}, & i = n+1, \end{cases}$$

and the linear isomorphism $\mathbb{U}_{\rho,0}^{\mathcal{X},Y} : H(\mathcal{X}) \rightarrow H(Y)$ given by

$$\mathbf{1}_k \mapsto \sum_{i=1}^n \frac{\omega^{-ik}(\omega^{\frac{k}{2}} - \omega^{-\frac{k}{2}})}{n+1} \phi_i, \quad 1 \leq k \leq n, \\ \mathbf{1}_{n+1} \mapsto \phi_{n+1}.$$

Furthermore, $\mathbb{U}_{\rho,0}^{\mathcal{X},Y}$ preserves the equivariant Poincaré pairings of \mathcal{X} and Y .

C. Analytic continuation of Lauricella $F_D^{(N)}$

Consider the Lauricella function

$$F_D^{(M+N)}(a; b_1, \dots, b_{M+N}; c; z_1, \dots, z_M, w_1, \dots, w_N)$$

around $P = (0, 0, \dots, \infty, \dots, \infty)$. We are interested in the leading terms of the asymptotics of this function in the region Ω_{M+N} defined as

$$\Omega_{M+N} \triangleq B(P, \epsilon) \bigcap_{i < j} H_{ij}$$

given by the intersection of the ball $B(P, \epsilon)$ with the interior of the real hyperquadrics

$$H_{ij} \triangleq \left\{ (z, w) \in \mathbb{C}^{M+N} : \left| \frac{w_i}{w_j} \right| < \epsilon \right\}.$$

As our interest is confined to the *leading* asymptotics only, we can assume without loss of generality that $M = 0$.

Following [39, Chapter 6], start from the power series expression (5.23) and perform the sum with respect to w_N :

$$(C.1) \quad F_D^{(N)}(a; b_1, \dots, b_N; c; w_1, \dots, w_N) \\ = \sum_{i_1, \dots, i_{N-1}} \frac{(a)_{\sum_{j=1}^{N-1} i_j}}{(c)_{\sum_{j=1}^{N-1} i_j}} \prod_{j=1}^{N-1} \frac{(b_j)_{i_j} w_j^{i_j}}{i_j!} \\ \times {}_2F_1\left(a + \sum_{j=1}^{N-1} i_j, b_N, c + \sum_{j=1}^{N-1} i_j, w_N\right).$$

The main idea then is to apply the connection formula for the inner Gauss function

$$(C.2) \quad {}_2F_1(a, b; c; z) = \frac{(-z)^{-a} \Gamma(c) \Gamma(b-a) {}_2F_1(a, a-c+1; a-b+1; \frac{1}{z})}{\Gamma(b) \Gamma(c-a)} \\ + \frac{(-z)^{-b} \Gamma(c) \Gamma(a-b) {}_2F_1(b, b-c+1; -a+b+1; \frac{1}{z})}{\Gamma(a) \Gamma(c-b)}$$

to analytically continue it to $|z| = |w_N| > 1$; in doing so, we fix a path of analytic continuation by choosing the principal branch for both the power functions $(-z)^{-a}$ and $(-z)^{-b}$ in (C.2) and continue ${}_2F_1(a, b; c; z)$ to $|z| > 1$ along a path that has winding number zero around the Fuchsian singularity at $z = 1$. As a power series in w_N the analytic continuation of (C.1) around $w_N = \infty$ then reads

$$(C.3) \quad F_D^{(N)}(a; b_1, \dots, b_N; c; w_1, \dots, w_N) \\ = (-w_N)^{-a} \Gamma \begin{bmatrix} c, & b_N - a \\ b_N, & c - a \end{bmatrix} \\ \times F_D^{(N)}\left(a; b_1, \dots, b_{N-1}, 1 - c + a; 1 - b_N + a, \frac{w_1}{w_N}, \dots, \frac{1}{w_N}\right) \\ + (-w_N)^{-b_N} \Gamma \begin{bmatrix} c, & a - b_N \\ a, & c - b_N \end{bmatrix} \\ \times C_N^{(N-1)}\left(b_1, \dots, b_N, 1 - c + b_N; a - b_N, -w_1, -w_2, \dots, \frac{1}{w_N}\right),$$

where we defined (cf. [39, Chapter 3])

$$C_N^{(k)}(b_1, \dots, b_N, a; a', x_1, \dots, x_N) \triangleq \sum_{i_1, \dots, i_N} (a)_{\alpha_N^{(k)}(\mathbf{i})} (a')_{-\alpha_N^{(k)}(\mathbf{i})} \prod_{j=1}^N \frac{(b_j)_{i_j} w_j^{i_j}}{i_j!}$$

and

$$\alpha_N^{(k)}(\mathbf{i}) \triangleq \sum_{j=k+1}^N i_j - \sum_{j=1}^k i_j, \quad \Gamma \begin{bmatrix} a_1, & \dots, & a_m \\ b_1, & \dots, & b_n \end{bmatrix} \triangleq \frac{\prod_{i=1}^m \Gamma(a_i)}{\prod_{i=1}^n \Gamma(b_i)}.$$

Notice that the function $F_D^{(N-1)}$ on the right-hand side of equation (C.3) is analytic in Ω_N ; there is nothing more that should be done there. The analytic continuation of the function $C_N^{(N-1)}$ is instead much more involved (see [39] for a complete treatment of the case $N = 3$);

but as all we are interested in is the leading term of the expansion around P in Ω_N we isolate the $\mathcal{O}(1)$ term in its $\frac{1}{w_N}$ expansion to find

$$(C.4) \quad C_N^{(N-1)}\left(b_1, \dots, b_N, 1 - c + b_N; a - b_N, -w_1, -w_2, \dots, \frac{1}{w_N}\right) \\ = F_D^{(N-1)}\left(a - b_N, b_1, \dots, b_{N-1}, c - b_N; w_1, \dots, w_{N-1}\right) + \mathcal{O}\left(\frac{1}{w_N}\right).$$

We are done: by (C.4), the form of the leading terms in the expansion of $F_D^{(N)}$ inside Ω_N can be found recursively by iterating N times the procedure we have followed in (C.1)–(C.4); as at each step equations (C.2)–(C.4) generate one additional term, we end up with a sum of $N + 1$ monomials each having power-like monodromy around P . Explicitly:

$$(C.5) \quad F_D^{(N)}(a; b_1, \dots, b_N; c; w_1, \dots, w_N) \\ \sim \sum_{j=0}^{N-1} \Gamma \left[\begin{matrix} c, & a - \sum_{i=N-j+1}^N b_i, & \sum_{i=N-j}^N b_i - a \\ a, & b_{N-j}, & c - a \end{matrix} \right] \\ \times \prod_{i=1}^j (-w_{N-i+1})^{-b_{N-i+1}} (-w_{N-j})^{-a + \sum_{i=N-j+1}^N b_i} \\ + \prod_{i=1}^N (-w_i)^{-b_i} \Gamma \left[\begin{matrix} c, & a - \sum_{i=1}^N b_j \\ a, & c - \sum_{i=1}^N b_j \end{matrix} \right].$$

Remark C.1. The analytic continuation to some other sectors of the ball $B(P, \epsilon)$ is straightforward. In particular, we can replace the condition $\frac{w_i}{w_j} \sim 0$ for $j > i$ by its reciprocal $\frac{w_j}{w_i} \sim 0$; this amounts to relabeling $b_i \rightarrow b_{N-i+1}$ in equation (C.5).

Remark C.2. When $a = -d$ for $d \in \mathbb{Z}^+$, the function $F_D^{(N)}$ reduces to a polynomial in w_1, \dots, w_N . In this case the arguments above reduce to a formula of Toscano [76] for Lauricella polynomials:

$$F_D^{(N)}(-d; b_1, \dots, b_N; c; w_1, \dots, w_N) \\ = (-w_N)^d \frac{(b)_d}{(c)_d} F_D^{(N)}\left(-d; b_1, b_2, \dots, b_{N-1}; 1 - d - c, 1 - d - b_N, \frac{w_1}{w_N}, \dots, \frac{1}{w_N}\right).$$

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